On the Units and the Structure of the 3-Sylow Subgroups of the Ideal Class Groups of Pure Bicubic Fields and their Normal Closures

Alberto Pablo Chalmeta

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Dr. Charles Parry (Chair)
Dr. Ezra Brown
Dr. Edward Green
Dr. Peter Linnell
Dr. Michael Renardy

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ABSTRACT

Let $\mathbb{Q}(\sqrt[3]{m})$ and $\mathbb{Q}(\sqrt[3]{m}, \sqrt[3]{n})$, where m and n are cube free rational integers, be called a cubic and a bicubic field respectively. The number theoretic invariants for the cubic fields and their normal closures are well known. Some work has been done on the units, classnumbers and other invariants of the bicubic fields and their normal closures by Parry but no method is available for calculating those invariants. This dissertation provides an algorithm for calculating the number theoretic invariants of the bicubic fields and their normal closure. Among these invariants are the discriminant, an integral basis, a set of fundamental units, the class number and the rank of the 3-class group.

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Contents

List of Tables

Chapter 1

Statement of Problem

A field $K = \mathbb{Q}(\sqrt[3]{m_1}, \sqrt[3]{m_2})$ where m_1 and m_2 are positive integers and $[K : \mathbb{Q}] = 9$ will be called a bicubic field. The objective of this dissertation is to compute the number theoretic invariants of these fields and their normal closure. Among these invariants are the discriminant, an integral basis, a set of fundamental units, the class number and the rank of the 3-class group. The discriminant and integral basis are determined in chapter 3. The determination of a set of fundamental units and class numbers of these fields requires a knowledge of the same invariants for the pure cubic subfields and their normal closure. Williams, et al [12] describe a method for determining the fundamental units of a cubic field using Vornoi's algorithm. Once this is known the class number of a cubic field can be determined by estimating the zeta function. Barrucand and Cohn in [1] and [2] give a method for determining a set of fundamental units and class numbers of the normal closure of a cubic field, knowing these invariants for the cubic field. Parry [9] describes relationships between a set of fundamental units and the class number of a bicubic field and its cubic subfields, as well as similar relations for the normal closure of a bicubic field. Using ideas from this article we develop algorithms for determining a set of fundamental units and the class number of a bicubic field and its normal closure in chapters 4 - 6. In chapter 7, we would like to determine the rank of the 3-class group of a bicubic field and its normal closure. Using methods of Gerth [5] we are able to do this when the bicubic field has a cubic subfield with class number relatively prime to 3.

Chapter 2

Notation

The following notation will be used throughout this article.

$\zeta = e^{2\pi i/3}$

$\omega = e^{2\pi i/9}$

$k_i = \mathbb{Q}(\sqrt[3]{m_i})$ where $i = 1, 2, 3, 4$

$K = \mathbb{Q}(\sqrt[3]{m_1}, \sqrt[3]{m_2})$

$K_i = \mathbb{Q}(\zeta, \sqrt[3]{m_i})$ where $i = 1, 2, 3, 4$

$L = \mathbb{Q}(\zeta, \sqrt[3]{m_1}, \sqrt[3]{m_2})$

$M_i = \mathbb{Q}(\omega, \sqrt[3]{m_i})$ where $i = 1, 2, 3, 4$

O_F: the ring of integers of a field F.

ϵ_i: the fundamental unit of k_i where $i = 1, 2, 3, 4$

u_i: unit of K_i such that $\{\epsilon_i, u_i\}$ form a fundamental set of units of K_i

$G = Gal(L/\mathbb{Q})$: Galois group of $L/\mathbb{Q}$

σ_i: nontrivial element of G that fixes K_i for $i = 1, 2, 3, 4$

τ: nontrivial element of G that fixes K

$N_{M/F}$: norm function for the field extension M/F

B_i: the unique primitive integer of K_i such that $\epsilon_i = \frac{B_i}{B_i^\sigma}$ where $\sigma = \sigma_1$ for $i \neq 1$ and $\sigma = \sigma_2$ for $i = 1$.

A_i: the unique primitive integer of K_i such that $B_i = \frac{A_i}{A_i^\sigma}$ (Defined only when $N_{K_i/k}(B_i) = 1$) or A_i is the unique primitive integer of M_i such that $\omega^j B_i = \frac{A_i}{A_i^\sigma}$ $j = 1$ or 2 (Defined when $N_{K_i/k}(B_i) = \zeta$ or ζ^2)

$H,\ h$: class number of L, K respectively.

$H_i,\ h_i$: class number of K_i, k_i respectively.

$\hat{E}$: group of units of L.

$\hat{\epsilon}$: subgroup of units of $\hat{E}$ generated by the units of K_1, K_2, K_3, K_4.

$\hat{\epsilon}_0$: subgroup of units of $\hat{\epsilon}$ generated by the units of k_1, k_2, k_3, k_4 and their congugates.

$\hat{\epsilon}_i$: subgroup of units of $\hat{E}$ generated by the units of K, K^{σ_i}, K_i.

$\hat{e}$: group of units of K.

$\hat{e}_0 = \hat{e} \cap \hat{\epsilon}_0$: subgroup of $\hat{e}$ generated by the units of k_1, k_2, k_3, k_4.

$\hat{U}_i$ ($i = 1$, 2, 3,4): group of units of K_i.

$\hat{u}_i$ ($i = 1$, 2, 3,4): subgroup of $\hat{U}_i$ generated by the units of k_i and its conjugates.

Chapter 3

Integral Basis for K_i, K and L

Let M be an an algebraic number field of degree n over $\mathbb{Q}$ and let O_M be the rings of algebraic integers for M. A field basis $B = \{\beta_1, \beta_2, \ldots, \beta_n\}$ for $M/\mathbb{Q}$ is an integral basis if every $\alpha \in O_L$ is uniquely representable in the form $a_1\beta_1 + a_2\beta_2 + \cdots + a_n\beta_n$ with $a_n \in \mathbb{Z}$. It is well known [8] that for any field basis consisting of integers $A = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and and integral basis $B = \{\beta_1, \beta_2, \ldots, \beta_n\}$ of a number ring O_M that the discriminants are related by

$$disc(A) = N^2 disc(B)$$

where $N \in \mathbb{Z}$.

Any two integral bases have the same discriminant and this discriminant can be considered an invariant of the ring O_M so $\delta_{M/\mathbb{Q}} = disc(B)$. We will use this to find integral bases for the rings of integers of $L = \mathbb{Q}(\zeta, \sqrt[3]{m_1}, \sqrt[3]{m_2})$ and all its subfields. If we can find a basis which has the same discriminant as the ring then we have found an integral basis. Marcus [8] provides a basis for the ring of integers of the pure cubic field $\mathbb{Q}(\sqrt[3]{m})$ where $m = ab^2$:

$$\begin{cases} 1, \sqrt[3]{m}, \frac{\sqrt[3]{m}^2}{b} & \text{if } m \not\equiv \pm 1 \pmod 9 \\ 1, \sqrt[3]{m}, \frac{(b^2 \pm b^2 \sqrt[3]{m_i} + \sqrt[3]{m_i}^2)}{3b} & \text{if } m \equiv \pm 1 \pmod 9 \end{cases}.$$

The theorems in this section will provide an integral basis for the rings of integers of K_i, K and L.

Theorem 3.1 *Let $m_i \in \mathbb{Z}$ with $m_i = ab^2$ where a and b are relatively prime and square free. Let O_{K_i} be the ring of integers of $K_i = \mathbb{Q}(\sqrt[3]{m_i}, \zeta)$. Then an integral basis for O_{K_i} is*

$$\begin{cases} A_1 & \text{if } m_i \equiv 2, 4, 5 \text{ or } 7 \pmod 9 \\ A_2 & \text{if } m_i \equiv 0 \pmod 3 \\ A_3 & \text{if } m_i \equiv \pm 1 \pmod 9 \end{cases},$$

The A_i's are defined as follows:

$$A_1 = \{1, \zeta, \sqrt[3]{m_i}, (1-\zeta)\frac{(1\mp 2\sqrt[3]{m_i}+\sqrt[3]{m_i}^2/b)}{3}, \zeta\sqrt[3]{m_i}, \frac{1-\zeta\pm 4\sqrt[3]{m_i}\pm 2\zeta\sqrt[3]{m_i}+\sqrt[3]{m_i}^2/b+2\zeta\sqrt[3]{m_i}^2/b}{3}\}$$

where $\pm$ or $\mp$ correspond as $\begin{cases} \text{top sign} & \text{if } m_i \equiv 4 \text{ or } 7 \pmod 9 \\ \text{bottom sign} & \text{if } m_i \equiv 2 \text{ or } 5 \pmod 9 \end{cases}$

$$A_2 = \{1, \zeta, \sqrt[3]{m_i}, (1-\zeta)\frac{\sqrt[3]{m_i}^2}{3b}, \zeta\sqrt[3]{m_i}, \frac{\sqrt[3]{m_i}^2+2\zeta\sqrt[3]{m_i}^2}{3b}\}$$

$$A_3 = \{1, \zeta, \sqrt[3]{m_i}, \frac{1\pm\sqrt[3]{m_i}+\sqrt[3]{m_i}^2/b}{3}, \frac{1}{3}(2+\zeta\pm\sqrt[3]{m_i}\mp\zeta\sqrt[3]{m_i}), \zeta\left(\frac{1\pm\sqrt[3]{m_i}+\sqrt[3]{m_i}^2/b}{3}\right)\}$$

where $\pm$ or $\mp$ correspond as $m_i \equiv \pm 1 \pmod 9$.

Proof: To show that the A_i's are integral bases for the ring of integers we will use the relationship between the different, $\Delta_{K_i/\mathbb{Q}}$, and the discriminant, $\delta_{K_i/\mathbb{Q}}$, which is given by result P. in section 13.2 of Ribenboim [10]:

$$N_{K_i/\mathbb{Q}}(\Delta_{K_i/\mathbb{Q}}) = \delta_{K_i/\mathbb{Q}}.$$

To calculate the different we need to know which primes are ramified from $\mathbb{Q}$ to K_i. Then we know from result O. in section 13.2 of [10] that $\Delta_{K_i/\mathbb{Q}} = \prod Q^{s_Q}$ where Q is any nonzero prime ideal of O_{K_i}. Let e_Q be the ramification index of Q in $K_i/\mathbb{Q}$ then $s_Q \geq e_Q - 1$. Moreover, $s_Q = e_Q - 1$ if and only if $q = Q \bigcap \mathbb{Z}$ does not divide the ramification index e_Q.

We begin by looking at those primes p which are ramified in K_i but are not 3. If $p \equiv 2 \pmod 3$ then in k_i we can factor $(p) = \mathfrak{p}^3$ with $e = 3$ and $f = 1$. In K_i, $(p) = P^3$ with $e = 3$ and $f = 2$. If we look at norms we get that $N_{K_i/\mathbb{Q}}(P) = p^2$. The relation on the different $P^{3-1} \mid \Delta_{K_i/\mathbb{Q}}$ follows from the formula and applying norms we get that $N_{K_i/\mathbb{Q}}(P) \mid \delta_{K_i/\mathbb{Q}} \implies p^4 \mid \delta_{K_i/\mathbb{Q}}$

We can perform a similar analysis on $p \equiv 1 \pmod 3$. In k_i we can factor $(p) = \mathfrak{p}^3$ with $e = 3$ and $f = 1$ and in K_i it factors as $(p) = (P_1 P_2)^3$ with $e = 3$ and $f = 1$. If we look at norms we get that $N_{K_i/\mathbb{Q}}(P_1) = N_{K_i/\mathbb{Q}}(P_2) = p$. The relation on the different $(P_1 P_2)^{3-1} \mid \Delta_{K_i/\mathbb{Q}}$ is clear and applying norms we get that $N_{K_i/\mathbb{Q}}(P_1^2 P_2^2) \mid \delta_{K_i/\mathbb{Q}} \implies p^4 \mid\mid \delta_{K_i/\mathbb{Q}}$.

For the prime 3 we need to consider 3 different cases. If $m_i \equiv \pm 1 \pmod 9$ then in k_i we can factor $(3) = \mathfrak{p}_1^2\mathfrak{p}_2$. In K_i $(3) = (P_1 P_2 P_3)^2$ with $e = 2$ and $f = 1$. If we look at norms we get that $N_{K_i/\mathbb{Q}}(P_i) = 3$ for $i = 1$, 2, and 3. Since the ramification index is relatively prime to the characteristic of O_{K_i}/P_j we have that $(P_1 P_2 P_3)^{2-1} \mid \Delta_{K_i/\mathbb{Q}}$ and applying norms we get that $N_{K_i/\mathbb{Q}}(P_1 P_2 P_3) \mid \delta_{K_i/\mathbb{Q}} \implies 3^3 \mid \delta_{K_i/\mathbb{Q}}$. Since $disc(A_1) = 3^3 \prod q_i^4$, where the q_i's are the ramified primes relatively prime to 3, then A_1 is an integral basis for K_i.

If $m_i \equiv 2,\ 4,\ 5$ or $7 \pmod 9$ then in k_i we can factor $(3) = \mathfrak{p}^3$ and in K_i as $(3) = P^6$ with $e = 6$, $f = 1$ and $N_{K_i/\mathbb{Q}}(P) = 3$. Since the ramification index from K_i to $\mathbb{Q}$ is not relatively prime to the characteristic of O_{K_i}/P we will look at intermediate fields. From k_i to K_i we can factor $(\mathfrak{p}) = P^2$ and the ramification index is relatively prime to the characteristic so $(P)^{2-1} \mid \Delta_{K_i/k_i}$. We know that $\Delta_{k_i/\mathbb{Q}} = (3A^2)$, where $A = (\sqrt[3]{ab^2}, \sqrt[3]{a^2b})$, so we can use the product formula for differents to get that $\Delta_{K_i/\mathbb{Q}} = \Delta_{K_i/k_i} \cdot \Delta_{k_i/\mathbb{Q}} \implies P \cdot 3A^2 \mid \Delta_{K_i/\mathbb{Q}}$. Applying norms we get that $N_{K_i/\mathbb{Q}}(P \cdot 3A^2) \mid \delta_{K_i/\mathbb{Q}} \implies 3^7 \mid \delta_{K_i/\mathbb{Q}}$. Since $disc(A_2) = 3^7 \prod q_i^4$ and then A_2 is an integral basis for K_i.

If $m_i \equiv 0 \pmod 3$ then the factorization of 3 in the fields is the same as the previous case. The only thing which changes is $\Delta_{k_i/\mathbb{Q}}$. In this case we can always choose m_i such that $m_i = 3m^*$ where $3 \nmid m^*$ because if $9 \mid m_i$ then we can choose $m_i = m_i^2/27$. We then get that $\Delta_{k_i/\mathbb{Q}} = (3A^2)$, where $A = (\sqrt[3]{3ab^2}, \sqrt[3]{9a^2b})$, and when we use the product formula $\Delta_{K_i/\mathbb{Q}} = \Delta_{K_i/k_i} \cdot \Delta_{k_i/\mathbb{Q}}$ we get $P \cdot 3A^2 \mid \Delta_{K_i/\mathbb{Q}} \implies N_{K_i/\mathbb{Q}}(P \cdot 3A^2) \mid \delta_{K_i/\mathbb{Q}} \implies 3 \cdot 3^6 \cdot 3^4 = 3^{11} \mid \delta_{K_i/\mathbb{Q}}$. Since $disc(A_3) = 3^{11} \prod q_i^4$ then A_3 is an integral basis for K_i.

Theorem 3.2 *Let $m_1, m_2 \in \mathbb{Z}$ with $m_i = a_i b_i^2$ where a_i and b_i are relatively prime and square free. Let O_K be the ring of integers of $K = \mathbb{Q}(\sqrt[3]{m_1}, \sqrt[3]{m_2})$. Then an integral basis for O_K is*

$$\begin{cases} A_1 & \text{if } m_1 \equiv 1 \pmod 9 \text{ and } m_2 \equiv \pm 1 \pmod 9 \\ A_2 & \text{if } m_1 \equiv 1 \pmod 9 \text{ and } m_2 \equiv 2,4,5, \text{ or } 7 \pmod 9 \\ A_3 & \text{if } m_1 \equiv -1 \pmod 9 \text{ and } m_2 \equiv 2,4,5, \text{ or } 7 \pmod 9 \\ A_4 & \text{if } m_1 \equiv \pm 1 \pmod 9 \text{ and } m_2 \equiv 0 \pmod 3 \\ A_5 & \text{if } m_1 \equiv 2,4,5, \text{ or } 7 \pmod 9 \text{ and } m_2 \equiv 0 \pmod 3. \end{cases}$$

Let $d_1 = GCD(a_1, a_2), d_2 = GCD(a_1, b_2), d_3 = GCD(b_1, a_2),$ and $d_4 = GCD(b_1, b_2)$ and the A_i's are defined as follows:

$$A_1 = \{1, \sqrt[3]{m_1}, \tfrac{1}{3}\left(1 + \sqrt[3]{m_1} + \frac{\sqrt[3]{m_1}^2}{b_1}\right), \sqrt[3]{m_2}, \tfrac{1}{3}\left(1 \pm \sqrt[3]{m_2} + \frac{\sqrt[3]{m_2}^2}{b_2}\right), \tfrac{1}{3}\left(1 - \sqrt[3]{m_1} \mp \sqrt[3]{m_2} \pm \frac{\sqrt[3]{m_1 m_2}}{d_2 d_3 d_4}\right),$$
$$\tfrac{1}{3}\left(\sqrt[3]{m_2} + \frac{\sqrt[3]{m_1 m_2}}{d_2 d_3 d_4} + \frac{\sqrt[3]{m_1^2 m_2}}{b_1 d_1 d_2 d_4}\right), \tfrac{1}{3}\left(\sqrt[3]{m_1} + \frac{\sqrt[3]{m_1 m_2}}{d_2 d_3 d_4} + \frac{\sqrt[3]{m_1 m_2^2}}{b_2 d_1 d_3 d_4}\right),$$
$$\tfrac{1}{9}\left(1 + \sqrt[3]{m_1} + \frac{\sqrt[3]{m_1}^2}{b_1} + \sqrt[3]{m_2} + \frac{\sqrt[3]{m_1 m_2}}{d_2 d_3 d_4} + \frac{\sqrt[3]{m_1^2 m_2}}{b_1 d_1 d_2 d_4} + \frac{\sqrt[3]{m_2}^2}{b_2} + \frac{\sqrt[3]{m_1 m_2^2}}{b_2 d_1 d_3 d_4} + \frac{\sqrt[3]{m_1^2 m_2^2}}{b_1 b_2 d_1 d_2 d_3}\right)\}$$

where $\pm$ or $\mp$ correspond as $m_2 \equiv \pm 1 \pmod 9$.

$$A_2 = \{1, \sqrt[3]{m_1}, \tfrac{1}{3}\left(1 + \sqrt[3]{m_1} + \frac{\sqrt[3]{m_1}^2}{b_1}\right), \sqrt[3]{m_2}, \frac{\sqrt[3]{m_2}^2}{b_2}, \tfrac{1}{3}\left(\sqrt[3]{m_2} + \frac{\sqrt[3]{m_1 m_2}}{d_2 d_3 d_4} + \frac{\sqrt[3]{m_1^2 m_2}}{b_1 d_1 d_2 d_4}\right),$$
$$\frac{\sqrt[3]{m_1 m_2^2}}{b_2 d_1 d_3 d_4}, \tfrac{1}{3}\left(\frac{\sqrt[3]{m_2}^2}{b_2} + \frac{\sqrt[3]{m_1 m_2^2}}{b_2 d_1 d_3 d_4} + \frac{\sqrt[3]{m_1^2 m_2^2}}{b_1 b_2 d_1 d_2 d_3}\right), \alpha\}$$

$$\text{where } \alpha = \begin{cases} \frac{1}{3}\left(2 - 2\sqrt[3]{m_1} - 2\sqrt[3]{m_2} - \frac{\sqrt[3]{m_1^2 m_2}}{b_1 d_1 d_2 d_4} - \frac{\sqrt[3]{m_1 m_2^2}}{b_2 d_1 d_3 d_4} - \frac{\sqrt[3]{m_1^2 m_2^2}}{b_1 b_2 d_1 d_2 d_3}\right) & \text{if } m_2 \equiv 4 \text{ or } 7 \pmod 9 \\ \frac{1}{3}\left(2 - 2\sqrt[3]{m_1} - \sqrt[3]{m_2} + \frac{\sqrt[3]{m_1^2 m_2}}{b_1 d_1 d_2 d_4} - \frac{\sqrt[3]{m_1 m_2^2}}{b_2 d_1 d_3 d_4} - \frac{\sqrt[3]{m_1^2 m_2^2}}{b_1 b_2 d_1 d_2 d_3}\right) & \text{if } m_2 \equiv 2 \text{ or } 5 \pmod 9 \end{cases}$$

$$A_3 = \{1, \sqrt[3]{m_1}, \frac{1}{3}\left(1 - \sqrt[3]{m_1} + \frac{\sqrt[3]{m_1}^2}{b_1}\right), \sqrt[3]{m_2}, \frac{\sqrt[3]{m_2}^2}{b_2}, \frac{1}{3}\left(\sqrt[3]{m_2} - \frac{\sqrt[3]{m_1 m_2}}{d_2 d_3 d_4} + \frac{\sqrt[3]{m_1^2 m_2}}{b_1 d_1 d_2 d_4}\right),$$
$$\frac{\sqrt[3]{m_1 m_2^2}}{b_2 d_1 d_3 d_4}, \frac{1}{3}\left(\frac{\sqrt[3]{m_2}^2}{b_2} - \frac{\sqrt[3]{m_1 m_2^2}}{b_2 d_1 d_3 d_4} + \frac{\sqrt[3]{m_1^2 m_2^2}}{b_1 b_2 d_1 d_2 d_3}\right), \alpha\}$$

$$\text{where } \alpha = \begin{cases} \frac{1}{3}\left(-\sqrt[3]{m_1} - \frac{\sqrt[3]{m_1}^2}{b_1} + \sqrt[3]{m_2} + \frac{\sqrt[3]{m_1 m_2}}{d_2 d_3 d_4} - \frac{\sqrt[3]{m_2}^2}{b_2} + \frac{\sqrt[3]{m_1^2 m_2^2}}{b_1 b_2 d_1 d_2 d_3}\right) & \text{if } m_2 \equiv 4 \text{ or } 7 \pmod 9 \\ \frac{1}{3}\left(\sqrt[3]{m_1} + \frac{\sqrt[3]{m_1}^2}{b_1} + \sqrt[3]{m_2} + \frac{\sqrt[3]{m_1 m_2}}{d_2 d_3 d_4} + \frac{\sqrt[3]{m_2}^2}{b_2} - \frac{\sqrt[3]{m_1^2 m_2^2}}{b_1 b_2 d_1 d_2 d_3}\right) & \text{if } m_2 \equiv 2 \text{ or } 5 \pmod 9 \end{cases}$$

$$A_4 = \{1, \sqrt[3]{m_1}, \frac{1}{3}\left(1 \pm \sqrt[3]{m_1} + \frac{\sqrt[3]{m_1}^2}{b_1}\right), \sqrt[3]{m_2}, \frac{\sqrt[3]{m_2}^2}{b_2}, \frac{\sqrt[3]{m_1 m_2}}{d_2 d_3 d_4}, \frac{1}{3}\left(\sqrt[3]{m_2} \pm \frac{\sqrt[3]{m_1 m_2}}{d_2 d_3 d_4} + \frac{\sqrt[3]{m_1^2 m_2}}{b_1 d_1 d_2 d_4}\right),$$
$$\frac{1}{3}\left(\mp\frac{\sqrt[3]{m_2}^2}{b_2} + \frac{\sqrt[3]{m_1 m_2^2}}{b_2 d_1 d_3 d_4}\right), \frac{1}{3}\left(\mp\frac{\sqrt[3]{m_1 m_2^2}}{b_2 d_1 d_3 d_4} + \frac{\sqrt[3]{m_1^2 m_2^2}}{b_1 b_2 d_1 d_2 d_3}\right)\}$$

$$\text{where } \pm \text{ or } \mp \text{ correspond as} \begin{cases} \text{top sign} & \text{if } m_1 \equiv 1 \pmod 9 \\ \text{bottom sign} & \text{if } m_1 \equiv 8 \pmod 9 \end{cases}$$

$$A_5 = \{1, \sqrt[3]{m_1}, \frac{\sqrt[3]{m_1}^2}{b_1}, \sqrt[3]{m_2}, \frac{\sqrt[3]{m_2}^2}{b_2}, \frac{\sqrt[3]{m_1 m_2}}{d_2 d_3 d_4}, \frac{1}{3}\left(\frac{\sqrt[3]{m_2}^2}{b_2} \pm \frac{\sqrt[3]{m_1 m_2^2}}{b_2 d_1 d_3 d_4}\right),$$
$$\frac{1}{3}\left(1 \pm \frac{2\sqrt[3]{m_1 m_2}}{d_2 d_3 d_4} + \frac{\sqrt[3]{m_1^2 m_2}}{b_1 d_1 d_2 d_4}\right), \frac{1}{3}\left(\frac{\sqrt[3]{m_2}^2}{b_2} \pm \frac{2\sqrt[3]{m_1 m_2^2}}{b_2 d_1 d_3 d_4} + \frac{\sqrt[3]{m_1^2 m_2^2}}{b_1 b_2 d_1 d_2 d_3}\right)\}$$

$$\text{where } \pm \text{ corresponds as} \begin{cases} + & \text{if } m_2 \equiv 2 \text{ or } 5 \pmod 9 \\ - & \text{if } m_2 \equiv 4 \text{ or } 7 \pmod 9 \end{cases}.$$

Proof:

Note: If neither m_1 nor $m_2 \equiv 0 \pmod 3$ then at least one of m_1 or $m_2 \equiv 1$ or $8 \pmod 9$. If both m_1 and $m_2 \equiv 8 \pmod 9$ then, without loss of generality, we can choose $m_1 \equiv m_1 m_2 \equiv 1 \pmod 9$.

We will first show that all the terms in the basis elements are integers. Certainly $\sqrt[3]{m_i^2} = \sqrt[3]{a_i^2 b_i^4} = b_i \sqrt[3]{a_i^2 b_i}$, so $\frac{\sqrt[3]{m_i^2}}{b_i}$ is an algebraic integer in the ring of integers of K for $i = 1, 2$. Similarly $\sqrt[3]{m_1 m_2} = \sqrt[3]{a_1 b_1^2 a_2 b_2^2}$ and $d_2 = GCD(a_1, b_2), d_3 = GCD(b_1, a_2)$, and $d_4 = GCD(b_1, b_2)$ are all perfect cubes so $\frac{\sqrt[3]{m_1 m_2}}{d_2 d_3 d_4}$ is an algebraic integer of K. We can similarly show that for $\frac{\sqrt[3]{m_1^2 m_2}}{b_1 d_1 d_2 d_4}$, $\frac{\sqrt[3]{m_1 m_2^2}}{b_2 d_1 d_3 d_4}$ and $\frac{\sqrt[3]{m_1^2 m_2^2}}{b_1 b_2 d_1 d_2 d_3}$ the denominators are those elements that are perfect cubes in their respective products and hence the fractions are algebraic integers of K.

To show that the A_i's are integral bases for the ring of integers we will use the relationship between the different, $\Delta_{K/\mathbb{Q}}$, and the discriminant, $\delta_{K/\mathbb{Q}}$, which is given by result P. in section 13.2 of [10]:

$$N_{K/\mathbb{Q}}(\Delta_{K/\mathbb{Q}}) = \delta_{K/\mathbb{Q}}.$$

To calculate the different we only need to know which primes are ramified from $\mathbb{Q}$ to K. Then we know from result O. in section 13.2 of [10] that $\Delta_{K/\mathbb{Q}} = \prod Q^{s_Q}$ where Q is any nonzero prime ideal of O_K. Let e_Q be the ramification index of Q in $K/\mathbb{Q}$ then $s_Q \geq e_Q - 1$. Moreover, $s_Q = e_Q - 1$ if and only if the characteristic of O_K/Q does not divide the ramification index e_Q.

This last statement tells us for any prime $Q \nmid 3$ that $s_Q = e_Q - 1$ because $e_Q = 3^j,\ j = 0$ or 1 as Q is unramified or ramified respectively. Then for those primes that are unramified $Q^{e_Q-1} = Q^{1-1} = 1$. For those primes $Q \in O_K$ that are ramified there will be exactly one of the cubic subfields k_i where $N_{K/k_i}(Q)$ is unramified. Without loss of generality let that field be k_1 and we will consider the three cases. If $q \equiv 2 \pmod 3$ is a rational prime then in k_1 it splits as $q = \mathfrak{q}_1\mathfrak{q}_2$ and in K it ramifies as $q = (Q_1Q_2)^3$ where the degree of Q_1 is 2 and of Q_2 is 1. Then $Q_j^{e_{Q_j}-1} = Q_j^{3-1} = Q_j^2$ for $j = 1$ and 2 where $N_{K/\mathbb{Q}}(Q_1) = q^2$ and $N_{K/\mathbb{Q}}(Q_2) = q$, thus $q^6 \mid \delta_{K/\mathbb{Q}}$. If $q \equiv 1 \pmod 3$ then either q stays prime in k_1 or splits completely. If q stays prime then $q = Q^3$ in K and $Q^{e_Q-1} = Q^{3-1} = Q^2$ where $N_{K/\mathbb{Q}}(Q) = q^3$. If q splits completely in k_1 then $q = (Q_1Q_2Q_3)^3$ in K and $Q_j^{e_{Q_j}-1} = Q_j^{3-1} = Q_j^2$ where $N_{K/\mathbb{Q}}(Q_j) = q$ for $j = 1$, 2 and 3. Thus $q^6 \mid \delta_{K/\mathbb{Q}}$ for the last two cases as well.

For $Q = 3$ the problem is more complicated and we will look at different cases.

Case 1: m_1 and $m_2 \equiv \pm 1 \pmod 9$.

In this case m_3 and $m_4 \equiv \pm 1 \pmod 9$ as well. In k_i, 3 factors as $(3) = \mathfrak{p}_{i_a}^2\mathfrak{p}_{i_b}$ for $i = 1, 2, 3, 4$ and in K as $(3) = P_1^2P_2^2P_3^2P_4^2P_5$. Since the ramification indices 2 and 1 are relatively prime to the characteristic of O_K/P_j then

$$P_1^{2-1}P_2^{2-1}P_3^{2-1}P_4^{2-1}P_5^{1-1} \mid \Delta_{K/\mathbb{Q}} \implies P_1P_2P_3P_4 \mid \Delta_{K/\mathbb{Q}}$$

and $N_{K/\mathbb{Q}}(P_i) = 3$ for $i = 1$, 2, 3, 4. Using the norm relationship between the different and the discriminant gives us that

$$N_{K/\mathbb{Q}}(P_1P_2P_3P_4) \mid N_{K/\mathbb{Q}}(\Delta_{K/\mathbb{Q}}) \implies 3^4 \mid \delta_{K/\mathbb{Q}}.$$

Then the lower bound on the discriminant is $\delta_{K/\mathbb{Q}} \geq 3^4 \prod q_i^6$ where the q_i's are the primes that ramify in K but are not 3. Since $disc(A_1) = 3^4 \prod q_i^6$ then A_1 must be an integral basis for O_K.

Case 2: $m_1 \equiv \pm 1 \pmod 9$ and $m_2 \equiv 2, 4, 5$, or $7 \pmod 9$.

The factorization of 3 in the real subfields is different for k_1 and k_2 in this case. In k_1, $(3) = \mathfrak{p}_1^2\mathfrak{p}_2$ and in k_2, $(3) = \mathfrak{p}^3$. In K we have $(3) = P_1^6P_2^3$ where P_1 lies over $\mathfrak{p}_1$, P_2 lies over $\mathfrak{p}_2$, $\mathfrak{p} = P_1^2P_2$ and $N_{K/\mathbb{Q}}(P_i) = 3$.

It is well known that $3 \mid\mid \Delta_{k_2/\mathbb{Q}}$. Since $\mathfrak{p} = P_1^2 P_2$ in K and the ramification index is relatively prime to 3 then we have that

$$P_1^{2-1}\mathfrak{p}_2^{1-1} \mid\mid \Delta_{K/k_2} \implies P_1 \mid\mid \Delta_{K/k_2}.$$

Using the multiplicative property of the different we get that $3P_1 \mid\mid \Delta_{K/\mathbb{Q}}$. Using the norm relationship between the different and the discriminant gives us that

$$N_{K/\mathbb{Q}}(3P_1) \mid\mid N_{K/\mathbb{Q}}(\Delta_{K/\mathbb{Q}}) \implies 3^{10} \mid\mid \delta_{K/\mathbb{Q}}.$$

Since $disc(A_2) = disc(A_3) = 3^{10} \prod q_i^6$ then A_2 and A_3 are integral bases for O_K when $m_1 \equiv 1 \pmod 9$ or $m_1 \equiv 8 \pmod 9$ respectively.

Case 3: $m_1 \equiv \pm 1 \pmod 9$ and $m_2 \equiv 0 \pmod 3$.

In this case we can factor (3) in k_1 as $(3) = \mathfrak{p}_1^2\mathfrak{p}_2$, in k_2 as $(3) = \mathfrak{p}^3$ and in K as $(3) = P_1^6 P_2^3$. From k_1 to K, $(\mathfrak{p}_i) = P_i^3$ for $i = 1, 2$ and from k_2 to K, $(\mathfrak{p}) = P_1^2 P_2$ with $N_{K/\mathbb{Q}}(P_1) = 3$. We will calculate the different of K/k_2 since 3 is relatively prime to the ramification index of $\mathfrak{p}$. Again for $k_2/\mathbb{Q}$ we know that $3\sqrt[3]{3}^2 \mid\mid \Delta_{k_2/\mathbb{Q}}$ and for K/k_2 we have that $P_1^{2-1}\mathfrak{p}_2^{1-1} \mid\mid \Delta_{K/k_2} \implies P_1 \mid\mid \Delta_{K/k_2}$ so using the multiplicative property we get

$$N_{K/\mathbb{Q}}(3\sqrt[3]{3}^2 P_1) \mid\mid N_{K/\mathbb{Q}}(\Delta_{K/\mathbb{Q}}) \implies 3^{16} \mid\mid \delta_{K/\mathbb{Q}}.$$

Since $disc(A_4) = 3^{16} \prod q_i^6$ then A_4 is an integral basis for O_K.

Case 4: $m_1 \equiv 2, 4, 5,$ or $7 \pmod 9$ and $m_2 \equiv 0 \pmod 3$.

Since 3 is totally ramified in k_1, k_2 and K then $(3) = \mathfrak{p}_i^3$ in k_i for $i = 1, 2$ and $(3) = P^9$ in K. For K/k_2, $\mathfrak{p}_2 = P^3$ and for K/k_1, $\mathfrak{p}_1 = P^3$ so 3 is wildly ramified in both cases. Here we have that $N_{K/\mathbb{Q}}(P) = 3$.

Using k_2 as the intermediate field we will calculate: $\Delta_{K/\mathbb{Q}} = \Delta_{K/k_2} \cdot \Delta_{k_2/\mathbb{Q}}$. We already know that $\Delta_{k_2/\mathbb{Q}} = 3\sqrt[3]{m_2}^2$ and since $3 \mid m_2$ then we have that $3\sqrt[3]{3}^2 \mid \Delta_{k_2/\mathbb{Q}}$. For K/k_2 we know that $\mathfrak{p}_2$ is wildly ramified so at least $P^3 \mid \Delta_{K/k_2}$. We get the relationship:

$$P^3 \cdot 3\sqrt[3]{3}^2 \mid \Delta_{K/k_2} \cdot \Delta_{k_2/\mathbb{Q}}.$$

The norm relationship between the different and the discriminant gives us that

$$N_{K/\mathbb{Q}}(P^3 \cdot 3\sqrt[3]{3}^2) \mid N_{K/\mathbb{Q}}(\Delta_{K/\mathbb{Q}}) \implies 3^{18} \mid \delta_{K/\mathbb{Q}}$$

and the discriminant of the basis A_5 is $3^{18} \prod q_i^6$ so A_5 must be an integral basis for O_K.

Theorem 3.3 *Let m_1 and m_2 be as in Theorem 3.2. Let O_L be the ring of integers of $L = \mathbb{Q}(\zeta, \sqrt[3]{m_1}, \sqrt[3]{m_2})$ then an integral basis for O_L is*

$$\begin{cases} A_1 & \text{if } m_1 \equiv 1 \pmod 9 \text{ and } m_2 \equiv \pm 1 \pmod 9 \\ A_2 & \text{if } m_1 \equiv 1 \pmod 9 \text{ and } m_2 \equiv 2,4,5, \text{ or } 7 \pmod 9 \\ A_3 & \text{if } m_1 \equiv -1 \pmod 9 \text{ and } m_2 \equiv 2,4,5, \text{ or } 7 \pmod 9 \\ A_4 & \text{if } m_1 \equiv \pm 1 \pmod 9 \text{ and } m_2 \equiv 0 \pmod 3 \\ A_5 & \text{if } m_1 \equiv 2 \text{ or } 5 \pmod 9 \text{ and } m_2 \equiv 0 \pmod 3 \\ A_6 & \text{if } m_1 \equiv 4 \text{ or } 7 \pmod 9 \text{ and } m_2 \equiv 0 \pmod 3 \end{cases}$$

where $d_1 = GCD(a_1, a_2), d_2 = GCD(a_1, b_2), d_3 = GCD(b_1, a_2),$ *and* $d_4 = GCD(b_1, b_2)$ *and the* A_i*'s are as follows:*

$$A_1 = \{1, \sqrt[3]{m_1}, \tfrac{1}{3}(1+\sqrt[3]{m_1}+\tfrac{\sqrt[3]{m_1}^2}{b_1}), \sqrt[3]{m_2}, \tfrac{1\pm\sqrt[3]{m_2}+\sqrt[3]{m_2}^2/b_2}{3}, \tfrac{1}{3}(1-\sqrt[3]{m_1}\mp\sqrt[3]{m_2}\pm\tfrac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}),$$
$$\tfrac{1}{3}(\sqrt[3]{m_2}+\tfrac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}+\tfrac{\sqrt[3]{m_1^2m_2}}{b_1d_1d_2d_4}), \tfrac{1}{3}(\sqrt[3]{m_1}+\tfrac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}+\tfrac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}),$$
$$\tfrac{1}{9}(1+\sqrt[3]{m_1}+\tfrac{\sqrt[3]{m_1}^2}{b_1}+\sqrt[3]{m_2}+\tfrac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}+\tfrac{\sqrt[3]{m_1^2m_2}}{b_1d_1d_2d_4}+\tfrac{\sqrt[3]{m_2}^2}{b_2}+\tfrac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}+\tfrac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3})$$
$$\zeta, \tfrac{1}{3}(1-\zeta)(1-\sqrt[3]{m_1}), \tfrac{1}{3}\zeta(1+\sqrt[3]{m_1}+\tfrac{\sqrt[3]{m_1}^2}{b_1}), \tfrac{1}{3}(1-\zeta)(1\mp\sqrt[3]{m_2}), \zeta\tfrac{1}{3}(1\pm\sqrt[3]{m_2}+\tfrac{\sqrt[3]{m_2}^2}{b_2}),$$
$$\zeta\tfrac{1}{3}(1-\sqrt[3]{m_1}\mp\sqrt[3]{m_2}\pm\tfrac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}), \tfrac{1}{9}(1-\zeta)(1+\sqrt[3]{m_1}+\tfrac{\sqrt[3]{m_1}^2}{b_1}\mp\sqrt[3]{m_2}\mp\tfrac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}\mp\tfrac{\sqrt[3]{m_1^2m_2}}{b_1d_1d_2d_4}),$$
$$\tfrac{1}{9}(1-\zeta)(1-\sqrt[3]{m_1}\pm\sqrt[3]{m_2}\mp\tfrac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}+\tfrac{\sqrt[3]{m_2}^2}{b_2}-\tfrac{\sqrt[3]{m_1m_2^2}}{b_1d_1d_3d_4}),$$
$$\tfrac{1}{9}\zeta(1+\sqrt[3]{m_1}+\tfrac{\sqrt[3]{m_1}^2}{b_1}\pm\sqrt[3]{m_2}\pm\tfrac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}\pm\tfrac{\sqrt[3]{m_1^2m_2}}{b_1d_1d_2d_4})+\tfrac{\sqrt[3]{m_2}^2}{b_2}+\tfrac{\sqrt[3]{m_1m_2^2}}{b_1d_1d_3d_4})+\tfrac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3})\}$$

where $\pm$ *or* $\mp$ *correspond as* $m_2 \equiv \pm 1 \pmod 9$*.*

$$A_2 = \{1, \sqrt[3]{m_1}, \tfrac{1}{3}(1+\sqrt[3]{m_1}+\tfrac{\sqrt[3]{m_1}^2}{b_1}), \sqrt[3]{m_2}, \tfrac{\sqrt[3]{m_2}^2}{b_2}, \tfrac{1}{3}(\mp\sqrt[3]{m_1}\pm\tfrac{\sqrt[3]{m_1}^2}{b_1}-\sqrt[3]{m_2}+\tfrac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}\mp\tfrac{\sqrt[3]{m_2}^2}{b_2}\pm\tfrac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}),$$
$$\tfrac{1}{3}(\sqrt[3]{m_2}+\tfrac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}+\tfrac{\sqrt[3]{m_1^2m_2}}{b_1d_1d_2d_4}), \tfrac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}, \tfrac{1}{3}(\tfrac{\sqrt[3]{m_2}^2}{b_2}+\tfrac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}+\tfrac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3}), \zeta, \tfrac{1}{3}(1-\zeta-\sqrt[3]{m_1}+\zeta\sqrt[3]{m_1}),$$
$$\zeta\tfrac{1}{3}(1+\sqrt[3]{m_1}+\tfrac{\sqrt[3]{m_1}^2}{b_1}), \zeta\sqrt[3]{m_2}, \tfrac{1}{3}(1-\zeta)(1\pm\sqrt[3]{m_2}+\tfrac{\sqrt[3]{m_2}^2}{b_2}), \zeta\tfrac{1}{3}(\sqrt[3]{m_2}+\tfrac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}+\tfrac{\sqrt[3]{m_1^2m_2}}{b_1d_1d_2d_4}),$$
$$\zeta\tfrac{1}{3}(\mp\sqrt[3]{m_1}\pm\tfrac{\sqrt[3]{m_1}^2}{b_1}-\sqrt[3]{m_2}+\tfrac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}\mp\tfrac{\sqrt[3]{m_2}^2}{b_2}\pm\tfrac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}, \alpha, \beta\}$$

where $\pm$ *and* $\mp$ *corresponds as* $\begin{cases} \text{top sign} & \text{if } m_2 \equiv 4 \text{ or } 7 \pmod 9 \\ \text{bottom sign} & \text{if } m_2 \equiv 2 \text{ or } 5 \pmod 9 \end{cases}$

and $\alpha =$

$$\begin{cases} \tfrac{1}{3}(1-\zeta+2\sqrt[3]{m_1}-2\zeta\sqrt[3]{m_1}+\tfrac{\sqrt[3]{m_1}^2}{b_1}-\zeta\tfrac{\sqrt[3]{m_1}^2}{b_1}+2\tfrac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}-2\zeta\tfrac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}+2\tfrac{\sqrt[3]{m_1^2m_2}}{b_1d_1d_2d_4}- \\ 2\zeta\tfrac{\sqrt[3]{m_1^2m_2}}{b_1d_1d_2d_4}+\tfrac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3})-\zeta\tfrac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3}) & \text{if } m_2 \equiv 4,7 \pmod 9 \\ \tfrac{1}{3}(1-\zeta-\sqrt[3]{m_1}+\zeta\sqrt[3]{m_1}+\tfrac{\sqrt[3]{m_1}^2}{b_1}-\zeta\tfrac{\sqrt[3]{m_1}^2}{b_1}+\tfrac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}-\zeta\tfrac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}+\tfrac{\sqrt[3]{m_1^2m_2}}{b_1d_1d_2d_4}-\zeta\tfrac{\sqrt[3]{m_1^2m_2}}{b_1d_1d_2d_4}+ \\ \tfrac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3})-\zeta\tfrac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3}) & \text{if } m_2 \equiv 2,5 \pmod 9 \end{cases}$$

and $\beta =$

$$\begin{cases} \frac{1}{9}(1-\zeta+\sqrt[3]{m_1}-\zeta\sqrt[3]{m_1}+\frac{\sqrt[3]{m_1}^2}{b_1}-\zeta\frac{\sqrt[3]{m_1}^2}{b_1}+\sqrt[3]{m_2}-\zeta\sqrt[3]{m_2}+\frac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}-\zeta\frac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}+\frac{\sqrt[3]{m_1^2m_2}}{b_1d_1d_2d_4}- \\ \zeta\frac{\sqrt[3]{m_1^2m_2}}{b_1d_1d_2d_4}+\frac{\sqrt[3]{m_2}^2}{b_2}-\zeta\frac{\sqrt[3]{m_2}^2}{b_2}+\frac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}-\zeta\frac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}+\frac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3})-\zeta\frac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3}) \\ \qquad \text{if } m_2\equiv 4,7 \pmod 9 \\ \frac{1}{9}(1-\zeta-\sqrt[3]{m_1}+\zeta\sqrt[3]{m_1}+\frac{\sqrt[3]{m_1}^2}{b_1}-\zeta\frac{\sqrt[3]{m_1}^2}{b_1}-\sqrt[3]{m_2}+\zeta\sqrt[3]{m_2}-\frac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}+\zeta\frac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}-\frac{\sqrt[3]{m_1^2m_2}}{b_1d_1d_2d_4}+ \\ \zeta\frac{\sqrt[3]{m_1^2m_2}}{b_1d_1d_2d_4}+\frac{\sqrt[3]{m_2}^2}{b_2}-\zeta\frac{\sqrt[3]{m_2}^2}{b_2}+\frac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}-\zeta\frac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}+\frac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3})-\zeta\frac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3}) \\ \qquad \text{if } m_2\equiv 2,5 \pmod 9 \end{cases}.$$

$$A_3=\{1,\sqrt[3]{m_1},\tfrac{1}{3}(1-\sqrt[3]{m_1}+\tfrac{\sqrt[3]{m_1}^2}{b_1}),\sqrt[3]{m_2},\tfrac{\sqrt[3]{m_2}^2}{b_2},\tfrac{1}{3}(-\sqrt[3]{m_1}-\tfrac{\sqrt[3]{m_1}^2}{b_1}\pm\sqrt[3]{m_2}\pm\tfrac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}+\tfrac{\sqrt[3]{m_2}^2}{b_2}+\tfrac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4},$$
$$\tfrac{1}{3}(\sqrt[3]{m_2}-\tfrac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}+\tfrac{\sqrt[3]{m_1^2m_2}}{b_1d_1d_2d_4}),\tfrac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4},\tfrac{1}{3}(\tfrac{\sqrt[3]{m_2}^2}{b_2}-\tfrac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}+\tfrac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3}),\zeta,\tfrac{1}{3}(2+\zeta+2\sqrt[3]{m_1}+\zeta\sqrt[3]{m_1}),$$
$$\zeta\tfrac{1}{3}(1-\sqrt[3]{m_1}+\tfrac{\sqrt[3]{m_1}^2}{b_1}),\zeta\sqrt[3]{m_2},\tfrac{1}{3}(1-\zeta)(1\pm\sqrt[3]{m_2}+\tfrac{\sqrt[3]{m_2}^2}{b_2}),\zeta\tfrac{1}{3}(\sqrt[3]{m_2}-\tfrac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}+\tfrac{\sqrt[3]{m_1^2m_2}}{b_1d_1d_2d_4}),$$
$$\zeta\tfrac{1}{3}(-\sqrt[3]{m_1}-\tfrac{\sqrt[3]{m_1}^2}{b_1}\pm\sqrt[3]{m_2}\pm\tfrac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}+\tfrac{\sqrt[3]{m_2}^2}{b_2}+\tfrac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4},\alpha,\beta\}$$

where $\pm$ *and* $\mp$ *corresponds as* $\begin{cases} \textit{top sign} & \text{if } m_2\equiv 4 \text{ or } 7 \pmod 9 \\ \textit{bottom sign} & \text{if } m_2\equiv 2 \text{ or } 5 \pmod 9 \end{cases}$

and $\alpha =$

$$\begin{cases} \frac{1}{3}(1-\zeta-2\sqrt[3]{m_1}+2\zeta\sqrt[3]{m_1}+\frac{\sqrt[3]{m_1}^2}{b_1}-\zeta\frac{\sqrt[3]{m_1}^2}{b_1}-2\frac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}+2\zeta\frac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}+2\frac{\sqrt[3]{m_1^2m_2}}{b_1d_1d_2d_4}+ \\ 2\zeta\frac{\sqrt[3]{m_1^2m_2}}{b_1d_1d_2d_4}+\frac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3})-\zeta\frac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3}) \qquad \text{if } m_2\equiv 4,7 \pmod 9 \\ \frac{1}{3}(1-\zeta+\sqrt[3]{m_1}-\zeta\sqrt[3]{m_1}+\frac{\sqrt[3]{m_1}^2}{b_1}-\zeta\frac{\sqrt[3]{m_1}^2}{b_1}-\frac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}+\zeta\frac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}+\frac{\sqrt[3]{m_1^2m_2}}{b_1d_1d_2d_4}- \\ \zeta\frac{\sqrt[3]{m_1^2m_2}}{b_1d_1d_2d_4}+\frac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3})-\zeta\frac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3}) \qquad \text{if } m_2\equiv 2,5 \pmod 9. \end{cases}$$

and $\beta =$

$$\begin{cases} \frac{1}{9}(2-2\zeta+\sqrt[3]{m_1}-\zeta\sqrt[3]{m_1}+2\frac{\sqrt[3]{m_1}^2}{b_1}+2\zeta\frac{\sqrt[3]{m_1}^2}{b_1}+2\sqrt[3]{m_2}-2\zeta\sqrt[3]{m_2}+\frac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}-\zeta\frac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}+2\frac{\sqrt[3]{m_1^2m_2}}{b_1d_1d_2d_4}- \\ 2\zeta\frac{\sqrt[3]{m_1^2m_2}}{b_1d_1d_2d_4}+2\frac{\sqrt[3]{m_2}^2}{b_2}-2\zeta\frac{\sqrt[3]{m_2}^2}{b_2}+\frac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}-\zeta\frac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}+2\frac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3})-2\zeta\frac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3}) \\ \qquad \text{if } m_2\equiv 4,7 \pmod 9 \\ \frac{1}{9}(1-\zeta+5\sqrt[3]{m_1}-5\zeta\sqrt[3]{m_1}+\frac{\sqrt[3]{m_1}^2}{b_1}-\zeta\frac{\sqrt[3]{m_1}^2}{b_1}-\sqrt[3]{m_2}+\zeta\sqrt[3]{m_2}-5\frac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}+5\zeta\frac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}-\frac{\sqrt[3]{m_1^2m_2}}{b_1d_1d_2d_4}+ \\ \zeta\frac{\sqrt[3]{m_1^2m_2}}{b_1d_1d_2d_4}+\frac{\sqrt[3]{m_2}^2}{b_2}-\zeta\frac{\sqrt[3]{m_2}^2}{b_2}+5\frac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}-5\zeta\frac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}+\frac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3})-\zeta\frac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3}) \\ \qquad \text{if } m_2\equiv 2,5 \pmod 9 \end{cases}.$$

$$A_4=\{1,\sqrt[3]{m_1},\tfrac{1}{3}(1\pm\sqrt[3]{m_1}+\tfrac{\sqrt[3]{m_1}^2}{b_1}),\sqrt[3]{m_2},\tfrac{\sqrt[3]{m_2}^2}{b_2},\tfrac{\sqrt[3]{m_1m_2}}{d_2d_3d_4},\tfrac{1}{3}(\sqrt[3]{m_2}\pm\tfrac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}+\tfrac{\sqrt[3]{m_1^2m_2}}{b_1d_1d_2d_4}),$$

$$\tfrac{1}{3}(\mp\tfrac{\sqrt[3]{m_2}^2}{b_2}+\tfrac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}),\ \tfrac{1}{3}(\mp\tfrac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}+\tfrac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3}),\ \zeta,\ \zeta\sqrt[3]{m_2},\ \tfrac{1}{3}(1-w)\tfrac{\sqrt[3]{m_2}^2}{b_2},\ \tfrac{1}{3}(1-w\mp\sqrt[3]{m_1}\pm\zeta\sqrt[3]{m_1}),$$
$$\zeta\tfrac{1}{3}(1\pm\sqrt[3]{m_1}+\tfrac{\sqrt[3]{m_1}^2}{b_1}),\ \zeta\tfrac{\sqrt[3]{m_1m_2}}{d_2d_3d_4},\ \zeta\tfrac{1}{3}(\sqrt[3]{m_2}\pm\tfrac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}+\tfrac{\sqrt[3]{m_1^2m_2}}{b_1d_1d_2d_4}),$$
$$\tfrac{1}{9}(1-\zeta)(\sqrt[3]{m_2}\mp 2\tfrac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}+\tfrac{\sqrt[3]{m_1^2m_2}}{b_1d_1d_2d_4}),\ \alpha\}$$

where $\alpha =$

$$\begin{cases}\tfrac{1}{9}(10\tfrac{\sqrt[3]{m_2}^2}{b_2}-10\zeta\tfrac{\sqrt[3]{m_2}^2}{b_2}+\tfrac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}-\zeta\tfrac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}-2\tfrac{\sqrt[3]{m_1^2m_2}}{b_1d_1d_2d_4}+2\zeta\tfrac{\sqrt[3]{m_1^2m_2}}{b_1d_1d_2d_4}) \\ \qquad \text{if } m_1\equiv 1 \pmod 9 \\ \tfrac{1}{9}(17\tfrac{\sqrt[3]{m_2}^2}{b_2}-17\zeta\tfrac{\sqrt[3]{m_2}^2}{b_2}+\tfrac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}-\zeta\tfrac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}+2\tfrac{\sqrt[3]{m_1^2m_2}}{b_1d_1d_2d_4}-2\zeta\tfrac{\sqrt[3]{m_1^2m_2}}{b_1d_1d_2d_4}) \\ \qquad \text{if } m_1\equiv -1 \pmod 9\end{cases}.$$

$$A_5=\{1,\ \sqrt[3]{m_1},\ \tfrac{1}{3}(1-\zeta)(1+2\sqrt[3]{m_1}+\tfrac{\sqrt[3]{m_1}^2}{b_1}),\ \sqrt[3]{m_2},\ \tfrac{1}{3}(1-\zeta)\tfrac{\sqrt[3]{m_2}^2}{b_2},\ \tfrac{1}{3}(1-\zeta)(\sqrt[3]{m_2}+\tfrac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}),$$
$$\tfrac{1}{3}(\tfrac{\sqrt[3]{m_2}^2}{b_2}+\tfrac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}),\ \tfrac{1}{3}(\sqrt[3]{m_2}+\tfrac{2\sqrt[3]{m_1m_2}}{d_2d_3d_4}+\tfrac{\sqrt[3]{m_1^2m_2}}{b_1d_1d_2d_4}),\ \tfrac{1}{3}(\tfrac{\sqrt[3]{m_2}^2}{b_2}+\tfrac{2\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}+\tfrac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3}),\ \zeta,\ \zeta\sqrt[3]{m_2},$$
$$\tfrac{1}{3}(\tfrac{\sqrt[3]{m_2}^2}{b_2}+\tfrac{2\zeta\sqrt[3]{m_2}^2}{b_2}),\ \zeta\sqrt[3]{m_1},\ \tfrac{1}{3}(1+2\sqrt[3]{m_1}+\tfrac{\sqrt[3]{m_1}^2}{b_1}+\zeta(2+4\sqrt[3]{m_1}+\tfrac{2\sqrt[3]{m_1}^2}{b_1})),$$
$$\tfrac{1}{3}(\sqrt[3]{m_2}+\tfrac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}+\zeta(2\sqrt[3]{m_2}+\tfrac{2\sqrt[3]{m_1m_2}}{d_2d_3d_4})),\ \tfrac{1}{3}(\zeta(\tfrac{\sqrt[3]{m_2}^2}{b_2}+\tfrac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4})),$$
$$\tfrac{1}{3}(\zeta(\sqrt[3]{m_2}+\tfrac{2\sqrt[3]{m_1m_2}}{d_2d_3d_4}+\tfrac{\sqrt[3]{m_1^2m_2}}{b_1d_1d_2d_4})),\ \alpha\}$$

where $\alpha =$

$$\begin{cases}\tfrac{1}{3}(6+3\zeta+6\sqrt[3]{m_1}+3\zeta\sqrt[3]{m_1}+6\sqrt[3]{m_2}+3\zeta\sqrt[3]{m_2}+5\tfrac{\sqrt[3]{m_2}^2}{b_2}+\zeta\tfrac{\sqrt[3]{m_2}^2}{b_2}+\tfrac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}+2\zeta\tfrac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}+ \\ 5\tfrac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3})+\zeta\tfrac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3}) \qquad \text{if } m_2\equiv 3 \pmod 9 \text{ and } m_1\equiv 2 \pmod 9 \\ \tfrac{1}{3}(6+3\zeta+6\sqrt[3]{m_1}+3\zeta\sqrt[3]{m_1}+3\sqrt[3]{m_2}+6\zeta\sqrt[3]{m_2}+5\tfrac{\sqrt[3]{m_2}^2}{b_2}+\zeta\tfrac{\sqrt[3]{m_2}^2}{b_2}+\tfrac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}+2\zeta\tfrac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}+ \\ 5\tfrac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3})+\zeta\tfrac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3}) \qquad \text{if } m_2\equiv 6 \pmod 9 \text{ and } m_1\equiv 2 \pmod 9 \\ \tfrac{1}{3}(3+6\zeta+3\sqrt[3]{m_1}+6\zeta\sqrt[3]{m_1}+6\sqrt[3]{m_2}+3\zeta\sqrt[3]{m_2}+5\tfrac{\sqrt[3]{m_2}^2}{b_2}+\zeta\tfrac{\sqrt[3]{m_2}^2}{b_2}+\tfrac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}+2\zeta\tfrac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}+ \\ 5\tfrac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3})+\zeta\tfrac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3}) \qquad \text{if } m_2\equiv 3 \pmod 9 \text{ and } m_1\equiv 5 \pmod 9 \\ \tfrac{1}{3}(3+6\zeta+3\sqrt[3]{m_1}+6\zeta\sqrt[3]{m_1}+3\sqrt[3]{m_2}+6\zeta\sqrt[3]{m_2}+5\tfrac{\sqrt[3]{m_2}^2}{b_2}+\zeta\tfrac{\sqrt[3]{m_2}^2}{b_2}+\tfrac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}+2\zeta\tfrac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}+ \\ 5\tfrac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3})+\zeta\tfrac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3}) \qquad \text{if } m_2\equiv 6 \pmod 9 \text{ and } m_1\equiv 5 \pmod 9\end{cases}.$$

and $A_6=\{1,\ \sqrt[3]{m_1},\ \tfrac{\sqrt[3]{m_1}^2}{b_1},\ \sqrt[3]{m_2},\ \tfrac{\sqrt[3]{m_2}^2}{b_2},\ \tfrac{\sqrt[3]{m_1m_2}}{d_2d_3d_4},\ \tfrac{1}{3}(\tfrac{\sqrt[3]{m_2}^2}{b_2}+\tfrac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}),\ \tfrac{1}{3}(\sqrt[3]{m_2}-\tfrac{2\sqrt[3]{m_1m_2}}{d_2d_3d_4}+\tfrac{\sqrt[3]{m_1^2m_2}}{b_1d_1d_2d_4}),$

$$\tfrac{1}{3}(\tfrac{\sqrt[3]{m_2}^2}{b_2}-\tfrac{2\sqrt[3]{m_1}\sqrt[3]{m_2}^2}{b_2d_1d_3d_4}+\tfrac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3}),\ \zeta,\ \zeta\sqrt[3]{m_2},\ \tfrac{1}{3}(1-\zeta)\tfrac{\sqrt[3]{m_2}^2}{b_2},\ \zeta\sqrt[3]{m_1},$$
$$\tfrac{1}{3}(1-\zeta)(1-2\sqrt[3]{m_1}+\tfrac{\sqrt[3]{m_1}^2}{b_1}),\ \tfrac{1}{3}(1-\zeta)(-\sqrt[3]{m_2}+\tfrac{\sqrt[3]{m_1m_2}}{d_2d_3d_4}),\ \tfrac{1}{3}(\zeta)(\sqrt[3]{m_2}^2-\tfrac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}),$$
$$\tfrac{1}{3}\zeta(\sqrt[3]{m_2}-\tfrac{2\sqrt[3]{m_1m_2}}{d_2d_3d_4}+\tfrac{\sqrt[3]{m_1^2m_2}}{b_1d_1d_2d_4}),\ \alpha\}$$

were $\alpha =$

$$\begin{cases} \frac{1}{3}(6+3\zeta+3\sqrt[3]{m_1}+6\zeta\sqrt[3]{m_1}+6\sqrt[3]{m_2}+3\zeta\sqrt[3]{m_2}+5\frac{\sqrt[3]{m_2}^2}{b_2}+4\zeta\frac{\sqrt[3]{m_2}^2}{b_2}+8\frac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}+4\zeta\frac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}+ \\ 5\frac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3})+\zeta\frac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3}) \qquad \textit{if } m_2 \equiv 3 \pmod 9 \textit{ and } m_1 \equiv 7 \pmod 9 \\ \frac{1}{3}(6+3\zeta+3\sqrt[3]{m_1}+6\zeta\sqrt[3]{m_1}+3\sqrt[3]{m_2}+6\zeta\sqrt[3]{m_2}+5\frac{\sqrt[3]{m_2}^2}{b_2}+4\zeta\frac{\sqrt[3]{m_2}^2}{b_2}+8\frac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}+4\zeta\frac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}+ \\ 5\frac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3})+\zeta\frac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3}) \qquad \textit{if } m_2 \equiv 6 \pmod 9 \textit{ and } m_1 \equiv 7 \pmod 9 \\ \frac{1}{3}(3+6\zeta+6\sqrt[3]{m_1}+3\zeta\sqrt[3]{m_1}+3\sqrt[3]{m_2}+6\zeta\sqrt[3]{m_2}+5\frac{\sqrt[3]{m_2}^2}{b_2}+4\zeta\frac{\sqrt[3]{m_2}^2}{b_2}+8\frac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}+4\zeta\frac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}+ \\ 5\frac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3})+\zeta\frac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3}) \qquad \textit{if } m_2 \equiv 6 \pmod 9 \textit{ and } m_1 \equiv 4 \pmod 9 \\ \frac{1}{3}(3+6\zeta+6\sqrt[3]{m_1}+3\zeta\sqrt[3]{m_1}+6\sqrt[3]{m_2}+3\zeta\sqrt[3]{m_2}+5\frac{\sqrt[3]{m_2}^2}{b_2}+4\zeta\frac{\sqrt[3]{m_2}^2}{b_2}+8\frac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}+4\zeta\frac{\sqrt[3]{m_1m_2^2}}{b_2d_1d_3d_4}+ \\ 5\frac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3})+\zeta\frac{\sqrt[3]{m_1^2m_2^2}}{b_1b_2d_1d_2d_3}) \qquad \textit{if } m_2 \equiv 3 \pmod 9 \textit{ and } m_1 \equiv 4 \pmod 9 \end{cases}$$

Proof: The proof follows much the same as the basis for K. We will determine the discriminant of the number field $L/\mathbb{Q}$ by first finding the different $\Delta_{L/\mathbb{Q}}$ and then applying $N_{L/\mathbb{Q}}(\Delta_{L/\mathbb{Q}}) = \delta_{L/\mathbb{Q}}$.

We will define our subfields in the usual way: $L = \mathbb{Q}(\zeta, \sqrt[3]{m_1}, \sqrt[3]{m_2})$, $K_i = \mathbb{Q}(\zeta, \sqrt[3]{m_i})$, $k_i = \mathbb{Q}(\sqrt[3]{m_i})$ and $k = \mathbb{Q}(\zeta)$ where m_i is cube free for $i = 1$, 2, 3 and 4, and then we will look at the factorization of all ramified primes in L. We can write

$$\delta_{L/\mathbb{Q}} = 3^a \prod p_l^b \tag{3.1}$$

where $p_l \neq 3$. We begin with those primes $p \neq 3$ and we will show that $b = 12$ by considering the different cases based on the congruence of $p \pmod 3$. For some m_i we know that $p \nmid m_i$ so without loss of generality we will always choose that m_i to be m_1

Case 1: $p \equiv 2 \pmod 3$

We can factor p in each of the subfields as follows:

in k	$(p) = \pi$	where $f = 2$ and $e = 1$,
in k_1	$(p) = \mathfrak{p}_1\mathfrak{p}_2$	where $f_1 = 2$ and f_2=1 and $e = 1$,
in K_1	$(p) = P_1P_2P_3$	where $f_j = 2$ for $j = 1$, 2, 3 and $e = 1$,
in L	$(p) = (\mathfrak{P}_1\mathfrak{P}_2\mathfrak{P}_3)^3$	where $f_j = 2$ for $j = 1$, 2, 3 and $e = 3$,

where $N_{L/\mathbb{Q}}(\mathfrak{P}_j) = p^2$ for $j = 1$, 2, 3. In this case the characteristic p does not divide $e_p = 3$ so we have that

$$(\mathfrak{P}_1\mathfrak{P}_2\mathfrak{P}_3)^{3-1} \mid\mid \Delta_{L/\mathbb{Q}} \implies N_{L/\mathbb{Q}}[(\mathfrak{P}_1\mathfrak{P}_2\mathfrak{P}_3)^2] = p^{12} \mid\mid \delta_{L/\mathbb{Q}}.$$

For $p \equiv 1 \pmod 3$ we have to consider two possibilities, either p stays prime in k_1 or p splits completely.

Case 2: $p \equiv 1 \pmod 3$ and stays prime in k_1.

We can factor p in each of the subfields as follows:

in k_1 $(p) = \mathfrak{p}$ where $f = 3$ and $e = 1$,

in K_1 $(p) = P_1P_2$ where $f_j = 3$ for $j = 1, 2$ and $e = 1$,

in L $(p) = (\mathfrak{P}_1\mathfrak{P}_2)^3$ where $f_j = 3$ for $j = 1, 2$ and $e = 3$,

where $N_{L/\mathbb{Q}}(\mathfrak{P}_j) = p^3$ for $j = 1, 2$. In this case p does not divide $e_p = 3$ so we have that

$$(\mathfrak{P}_1\mathfrak{P}_2)^{3-1} \mid\mid \Delta_{L/\mathbb{Q}} \implies N_{L/\mathbb{Q}}[(\mathfrak{P}_1\mathfrak{P}_2)^2] = p^{12} \mid\mid \delta_{L/\mathbb{Q}}.$$

Case 3: $p \equiv 1 \pmod 3$ and splits completely in k_1.

We can factor p in each of the subfields as follows:

in k_1 $(p) = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$ where $f_j = 1$ and $e_i = 1$ for $j = 1, 2, 3$,

in K_1 $(p) = P_1P_2P_3P_4P_5P_6$ where $f_j = 1$ and $e_j = 1$ for $j = 1, 2, \ldots, 6$

in L $(p) = (\mathfrak{P}_1\mathfrak{P}_2\ldots\mathfrak{P}_6)^3$ where $f_j = 1$ and $e_i = 3$ for $j = 1, 2, \ldots, 6$,

where $N_{L/\mathbb{Q}}(\mathfrak{P}_j) = p$ for $j = 1, 2, \ldots, 6$. In this case p does not divide $e_p = 3$ so we have that

$$(\mathfrak{P}_1\mathfrak{P}_2\mathfrak{P}_3\mathfrak{P}_4\mathfrak{P}_5\mathfrak{P}_6)^{3-1} \mid\mid \Delta_{L/\mathbb{Q}} \implies N_{L/\mathbb{Q}}[(\mathfrak{P}_1\mathfrak{P}_2\mathfrak{P}_3\mathfrak{P}_4\mathfrak{P}_5\mathfrak{P}_6)^2] = p^{12} \mid\mid \delta_{L/\mathbb{Q}}.$$

In each case we have that $p^{12} \mid\mid \delta_{L/\mathbb{Q}}$ so $b = 12$ in equation (3.1).

For $p = 3$ we have to consider 4 different cases based on the congruences of m_1 and $m_2 \pmod 9$. In all these cases we will let q_i be those primes that ramify in L but are different from 3.

Case 1: m_1 and $m_2 \equiv \pm 1 \pmod 9$.

In this case m_3 and $m_4 \equiv \pm 1 \pmod 9$ as well. We can factor (3) in each of the subfields as follows:

in k_i $(3) = \mathfrak{p}_1^2\mathfrak{p}_2$ where $f_j = 1$ for $j = 1, 2$,

in K_i $(3) = (P_1P_2P_3)^2$ where $f_j = 1$ for $j = 1, 2$,

in L $(3) = (\mathfrak{P}_1\mathfrak{P}_2\ldots\mathfrak{P}_9)^2$ where $f_j = 1$ and $e_i = 2$ for $j = 1, 2, \ldots, 9$,

where $N_{L/\mathbb{Q}}(\mathfrak{P}_j) = 3$ for $j = 1, 2, \ldots, 9$. In this case 3 does not divide $e_p = 2$ so we have that

$$(\mathfrak{P}_1\mathfrak{P}_2\ldots\mathfrak{P}_9)^{2-1} \mid\mid \Delta_{L/\mathbb{Q}} \implies N_{L/\mathbb{Q}}[(\mathfrak{P}_1\mathfrak{P}_2\ldots\mathfrak{P}_9)] = 3^9 \mid\mid \delta_{L/\mathbb{Q}}$$

Since $disc(A_1) = 3^9 \prod q_i^{12}$ then A_1 must be an integral basis for O_L.

Case 2: $m_1 \equiv \pm 1 \pmod 9$ and $m_2 = 2, 4, 5$, or $7 \pmod 9$.

We can factor (3) in each of the subfields as follows:

in k_2 $(3) = \mathfrak{p}^3$ where $f = 1$ and $e = 3$,

in K_2 $(3) = (P)^6$ where $f = 1$ and $e = 3$,

in L $(3) = (\mathfrak{P}_1\mathfrak{P}_2\mathfrak{P}_3)^6$ where $f = 1$ and $e = 6$,

where $N_{L/\mathbb{Q}}(\mathfrak{P}_i) = 3$. In this case 3 is wildly ramified in L and 3 does not divide m_2 so we can apply the theorem to get a lower bound for the discriminant:

$$(\mathfrak{P}_1\mathfrak{P}_2\mathfrak{P}_3)^6 \mid \Delta_{L/\mathbb{Q}} \implies N_{L/\mathbb{Q}}[(\mathfrak{P}_1\mathfrak{P}_2\mathfrak{P}_3)^6] = 3^{18} \mid \delta_{L/\mathbb{Q}}$$

Since $disc(A_2) = disc(A_3) = 3^{18}\prod q_i^{12}$ then A_2 and A_3 must be integral bases for O_L.

Case 3: $m_1 \equiv \pm 1 \pmod 9$ and $m_2 = 0 \pmod 3$.

We can factor (3) in each of the subfields as follows:

in k_1 $(3) = \mathfrak{p}_1^2\mathfrak{p}_2$ where $f_j = 1$ for $j = 1, 2$ and $e = 1$,

in K_1 $(3) = (P_1P_2P_3)^2$ where $f = 1$ and $e = 2$,

in k_2 $(3) = \mathfrak{p}^3$ where $f = 1$ and $e = 3$,

in K_2 $(3) = (P)^6$ where $f = 1$ and $e = 3$,

in L $(3) = (\mathfrak{P}_1\mathfrak{P}_2\mathfrak{P}_3)^6$ where $f = 1$ and $e = 6$,

where $N_{L/\mathbb{Q}}(\mathfrak{P}_i) = 3$, $N_{L/K_2}(P) = P^3$ and $N_{L/\mathbb{Q}}(P) = 3^3$. In this case 3 is wildly ramified in L so we will use the product rule using K_2 as the intermediate field to get $\Delta_{L/K_2} \cdot \Delta_{K_2/\mathbb{Q}} = \Delta_{L/\mathbb{Q}}$. The prime P of K_2 is unramified in L so Δ_{L/K_2} is relatively prime to 3 and in the proof of Theorem 3.1 it was shown that $p^{11} \mid\mid \Delta_{K_2/\mathbb{Q}}$ so $p^{11} \mid\mid \Delta_{L/\mathbb{Q}}$. Then the norm relation implies that

$$N_{L/\mathbb{Q}}[P^{11}] = 3^{33} \mid \delta_{L/\mathbb{Q}}$$

and since $disc(A_4) = 3^{33}\prod q_i^{12}$ then A_4 must be an integral basis for O_L.

Case 4: $m_1 \equiv 2, 4, 5$, or $7 \pmod 9$ and $m_2 = 0 \pmod 3$.

We can factor (3) in each of the subfields as follows:

in k_1 $(3) = \mathfrak{p}^3$ where $f = 1$ and $e = 3$,

in K $(3) = P^9$ where $f = 1$ and $e = 9$,

in L $(3) = (\mathfrak{P})^{18}$ where $f = 1$ and $e = 18$,

in L $(P) = \mathfrak{P}^2$ where $f = 1$ and $e = 2$,

and $N_{L/\mathbb{Q}}(\mathfrak{P}) = 3$; $N_{L/\mathbb{Q}}(P) = 3^2$ and $N_{L/\mathbb{Q}}(\mathfrak{Q}) = 3$.

We know from the proof of Theorem 3.2 that $P^{18} \mid\mid \Delta_{K/\mathbb{Q}}$ and from K to L we know that $\mathfrak{P} \mid \Delta_{L/K}$.

By the product rule we get

$$\mathfrak{P} \cdot P^{18} \mid \Delta_{L/K} \cdot \Delta_{K/\mathbb{Q}} \implies \mathfrak{P}^{37} \mid \Delta_{L/\mathbb{Q}} \implies N_{L/\mathbb{Q}}[\mathfrak{P}^{37}] = 3^{37} \mid \delta_{L/\mathbb{Q}}$$

and since $disc(A_5) = disc(A_6) = 3^{37} \prod q_i^{12}$ then A_5 and A_6 must be integral bases for O_L.

Chapter 4

Unit Group of K_i

4.1 Types of Cubic Fields

Let $k_i = \mathbb{Q}(\sqrt[3]{m_i})$ be a pure cubic field with unit group $< \epsilon_i >$. Vornoi's algorithm [12] will quickly generate the element ϵ_i. In [1] Barrucand and Cohn give a classification of the pure cubic fields and their normal closures. In [9] Parry showed that Type II fields do not exist and consequently simplified the definitions of the remaining Types. Since $N_{k_i/\mathbb{Q}}(\epsilon_i) = 1$ we can write $\epsilon_i = \frac{B_i}{B_i^{\sigma^2}}$, where B_i is a primitive integer in K_i, then the fields k_i and K_i are of

Type I if $N(B_i) = 1$

Type III if $N(B_i)$ is not a unit

Type IV if $N(B_i) = \zeta^a$ where $a = 1$ or 2.

An integer B_i of K_i is said to be primitive if it is not divisible by any integer of k other than roots of unity.

To find the Type of the subfield it is necessary to calculate the element B_i. Since the calculation of B_i is the same in all the fields we will drop the subscript and simply write B, ϵ and σ.

4.2 Calculation of B

Let $k_i = \mathbb{Q}(\sqrt[3]{m_i})$ and let ϵ be the fundamental unit for k_i then it is known from Hilbert's Theorem 90 that there exists B, a unique (up to multiplication by roots of unity in k) primitive integer of K_i, such that $\epsilon = \frac{B}{B^\sigma}$. To find B we start with a solution of the form $\beta = 1 + \epsilon + \epsilon^\sigma \epsilon$ which gives us $\beta = B\alpha$ where $\alpha \in k$ and $B \in K_i$ is primitive. In Barrucand and Cohn [1] it is shown that $N_{K_i/\mathbb{Q}}(\beta) = [3 + tr(\epsilon) + tr(1/\epsilon)]^3$

where tr is the trace for $k_i/\mathbb{Q}$. Since $\beta = B\alpha$ then $b_1^3 = N_{K_i/\mathbb{Q}}(\beta) = N_{K_i/\mathbb{Q}}(B)(N_{k/\mathbb{Q}}(\alpha))^3$ and hence $N_{K_i/\mathbb{Q}}(B)$ is a cube in $\mathbb{Z}$. Let $b_1^3 = b_r b_k$ where b_r is divisible only by ramified primes and b_k is divisible by no ramified primes. We know that B is divisible only by ramified primes so $b_k = N_{K_i/\mathbb{Q}}(\alpha_1)$, where $\alpha_1 \mid \alpha$. To find α_1 we use the gcd algorithm for the third cyclotomic field on $N_{K_i/k}(\beta) = N_{K_i/k}(B)(\alpha^3)$ and b_k. We can now easily find $B^* = \frac{\beta}{\alpha_1}$ with $B^* \in K_i$. At this point B^* will be divisible by units of K_i and may be divisible by ramified primes. If $N_{K_i/k}(B^*) = 1,\ \zeta,\ \zeta^2$ and the field K_i is either Type I or IV and $B = B^*$ is a unit of K_i. If B^* is not a unit then more reduction may be needed.

Let $\gamma = N_{K_i/k}(B^*)$ and write $\gamma = 3^{a_0}\pi_1^{a_1}\pi_2^{a_2}\cdots\pi_v^{a_v}p_{v+1}^{a_{v+1}}\cdots p_n^{a_n}$ where π_j for $1 \le j \le v$ is a prime divisor in k of a rational prime congruent to 1 modulo 3 and $p_j \equiv 2 \pmod 3$ for $v+1 \le j \le n$. It was shown above that $N_{K_i/\mathbb{Q}}(B)$ is a cube in $\mathbb{Z}$. Let $p \equiv 2 \pmod 3$ with $p^a \mid\mid N_{K_i/\mathbb{Q}}(B^*)$ and P the prime divisor of p in K_i. Since $N_{K_i/\mathbb{Q}}(P) = p^2$ if follows that $6 \mid a$. Suppose that $P^b \mid\mid B^*$ then $p^{2b} \mid\mid N_{K_i/\mathbb{Q}}(B^*)$ so $2b = a = 6c \implies b = 3c$ and so $p^c = P^{3c} \mid\mid B^*$. Hence all primes $p_j \equiv 2 \pmod 3$ can be removed from B^* by dividing B^* by $p_j^{a_j/3}$.

The same is true for 3 since a prime divisor of 3 will only divide B if $m \not\equiv \pm 1 \pmod 9$. Hence 3 has only one prime divisor P_3 in K_i and $(1-\zeta)^2 = (3) = P_3^6$. As above if $P_3^b \mid\mid B^*$ then b is divisible by 3 so $(1-\zeta)^{b/3} \mid B^*$ and can be removed.

For a prime $p \equiv 1 \pmod 3$ let $p = \pi\bar{\pi}$ in k then $(\pi) = P^3$ and $(\bar{\pi}) = \bar{P}^3$ for distinct primes P and $\bar{P}$ of K_i. If $P^b \mid\mid B^*$ then $\pi^{\lfloor b/3 \rfloor} \mid B^*$ and can be removed. Hence we can assume that the powers of π and $\bar{\pi}$ dividing $N_{K_i/k}(B^*)$ are both less than or equal to 2. Since the power of p dividing $N_{K_i/\mathbb{Q}}(B^*)$ is a multiple of 3 then if either π or $\bar{\pi}$ divides $N_{K_i/\mathbb{Q}}(B^*)$ that norm must be exactly divisible by $\pi\bar{\pi}^2$ or $\pi^2\bar{\pi}$. Now B^* is primitive so $B = B^*$.

Now there are 3 cases. If $B = \prod_{j=1}^{v} \pi_j^{a_j}\bar{\pi}_j^{b_j}$ where $a_j + b_j = 3$ and $a_j, b_j \neq 0$ then K_i is a Type III field. If $N_{K_i/k}(B) = \zeta$ or ζ^2 then K_i is Type IV and if $N_{K_i/k}(B) = 1$ then K_i is Type I.

4.3 Unit group for K_i

If K_i is Type III then the only units in K_i are those units in the real subfield k_i and their conjugates so the unit group of K has basis $\{\epsilon_i, \epsilon_i^\sigma\}$.

If K_i is Type I or IV then we can find B_i as above where $N_{K_i/k}(B_i) = 1,\ \zeta,$ or ζ^2 so B_i is a unit in K_i which is not a product of the the units of the subfields. Let $u_i = B_i$ then the unit group of K_i has the basis $\{\epsilon_i, u_i\}$. Moreover $\frac{\epsilon_i}{\epsilon_i^{\sigma^2}} = \frac{u_i^3}{N_{K_i/k}(u_i)}$ so $u_i^3 = \zeta^a \frac{\epsilon_i}{\epsilon_i^{\sigma^2}}$ for some $a = 0$, 1 or 2.

Chapter 5

Unit group of K

Let $K = \mathbb{Q}(\sqrt[3]{m_1}, \sqrt[3]{m_2})$ and $\hat{e}$ be the unit group of K. Then K has 4 fundamental units, $e_1,\ e_2,\ e_3,\ e_4$, and a basis for $\hat{e}$ can be chosen in one of four possible ways as described in Theorem VI of [9]. These bases will distinguish K into 4 Kinds.

(1) $(\hat{e} : \hat{e_0}) = 27$ and $e_1 = \epsilon_1,\ e_2^3 = \epsilon_1^{a_1}\epsilon_2,\ e_3^3 = \epsilon_1^{b_1}\epsilon_3,\ e_4^3 = \epsilon_1^{c_1}\epsilon_4$

(2) $(\hat{e} : \hat{e_0}) = 9$ and $e_1 = \epsilon_1,\ e_2 = \epsilon_2,\ e_3^3 = \epsilon_1^{a_1}\epsilon_2^{a_2}\epsilon_3,\ e_4^3 = \epsilon_1^{b_1}\epsilon_2^{b_2}\epsilon_4$

(3) $(\hat{e} : \hat{e_0}) = 3$ and $e_1 = \epsilon_1,\ e_2 = \epsilon_2,\ e_3 = \epsilon_3,\ e_4^3 = \epsilon_1^{a_1}\epsilon_2^{a_2}\epsilon_3^{a_3}\epsilon_4$

(4) $(\hat{e} : \hat{e_0}) = 1$ and $e_1 = \epsilon_1,\ e_2 = \epsilon_2,\ e_3 = \epsilon_3,\ e_4 = \epsilon_4$

To find a basis for the unit group of K we will be able to use the different Types of subfields to "construct" the basis elements. Theorem VI of [9] tells us the form of the basis elements based on the Kind of K and Theorem XII and XIII tell us the conditions required for each Kind to occur. The units in K are dependent on the Type of the subfields and thus we need to look at three cases based on those Types.

To find a new unit in K from Type I fields is the easiest of the three cases. By [9] Theorem IX we know that for a Type I field k_i that we can write the fundamental unit $\epsilon_i = \frac{A_i^3}{r_i}$ with $A_i \in k_i$, $r_i \in \mathbb{Z}$ and $r_i \mid 9m_i^2$. Since $B_i \in K_i$ where $\epsilon_i = \frac{B_i}{B_i^{\sigma^2}}$ is a unit of norm 1 we can calculate A_i by using Hilbert's Theorem 90 we can write $B_i = \frac{A_i}{A_i^{\sigma}}$ as we did for the units of K_i. Using this method we see that

$$\epsilon_i = \frac{B_i}{B_i^{\sigma^2}} = \frac{A_i^3}{(A_i A_i^{\sigma^2} A_i^{\sigma})} = \frac{A_i^3}{N(A_i)}$$

which implies that $r_i = N(A_i)$.

Using Barracund and Cohn's [1] terminology we will define a principal factor of the discriminant Δ_{k_i} to be an element $\alpha_i \in \mathbb{Z}$ such that $\alpha_i \mid \Delta_{k_i}$ and there exists $A \in k_i$ such that $N_{k_i/\mathbb{Q}}(A) = \alpha_i$. For brevity we will refer to the principal factors of the discriminant simply as "principal divisors".

In section 7 of [1] Barrucand and Cohn show that a basis for the principal divisors of $k_i = \mathbb{Q}(\sqrt[3]{m_i})$ can be constructed from the elements $N(A_i)$ from $\epsilon_i = \frac{A_i^3}{N(A_i)}$. Let $m_i = ab^2$ with $(a, b) = 1$ for square-free a and b then principal divisors for are all of the form

$$1,\ ab^2,\ a^2b,\ d,\ dab^2,\ da^2b,\ d^2,\ d^2ab^2,\ d^2a^2b.$$

where $d = N(A_i)$.

5.1 Units in K from Type I fields

We will look for products of the principal divisors of the form $\alpha_1^{a_1}\alpha_2^{a_2}\alpha_3^{a_3}\alpha_4^{a_4} = \alpha_p$ with $0 \le a_i \le 2$ where $\frac{A_i^3}{\alpha_i} = \epsilon_i$ such that $\alpha_p = m^3$ where $m \in \mathbb{Z}$. Then

$$A = A_1^{a_1}A_2^{a_2}A_3^{a_3}A_4^{a_4} \ \text{ with } \ 0 \le a_i \le 2,\ i = 1,\ 2,\ 3,\ 4$$

then

$$\epsilon_1^{a_1}\epsilon_2^{a_2}\epsilon_3^{a_3}\epsilon_4^{a_4} = \frac{A^3}{\alpha_p} = \left(\frac{A}{m}\right)^3. \tag{5.1}$$

This gives us a new unit in K, $e = \sqrt[3]{\epsilon_1^{a_1}\epsilon_2^{a_2}\epsilon_3^{a_3}\epsilon_4^{a_4}}$ and we choose the largest subscript i with $a_i \neq 0$ such that $a_i = 1$.

It will be advantageous in the calculation of the units of L to choose the unit basis in K such that whenever possible $a_l = 0$, for some $l = 1, 2, 3$, or 4 the class number of k_l is relatively prime to 3. Choose a maximal independent set elements from the list of all solutions to equation (5.1); this will provide at most 3 elements depending on the kind of K. Since $\hat{e}$ has four elements we can choose the remaining non-cube elements as described at the beginning of this chapter.

5.2 Example Type I units in K

Example 1 $K = \mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{5})$

The cubic subfields $k_1 = \mathbb{Q}(\sqrt[3]{2})$, $k_2 = \mathbb{Q}(\sqrt[3]{5})$, $k_3 = \mathbb{Q}(\sqrt[3]{10})$, and $k_4 = \mathbb{Q}(\sqrt[3]{20})$ are all Type I so for each we can calculate A_i, and the principal divisor α_i where $\epsilon_i = \frac{A_i^3}{\alpha_i}$.

$A_1 = 1 + \sqrt[3]{2}$, $\alpha_1 = 3$ and $\epsilon_1 = 1 + \sqrt[3]{2} + \sqrt[3]{4}$

$A_2 = 4 + 2\sqrt[3]{5} + \sqrt[3]{25}$, $\alpha_2 = 9$ and $\epsilon_2 = 41 + 24\sqrt[3]{5} + 14\sqrt[3]{25}$

$A_3 = \frac{1}{3}(4 + \sqrt[3]{10} + \sqrt[3]{100})$, $\alpha_3 = 2$ and $\epsilon_3 = \frac{1}{3}(23 + 11\sqrt[3]{10} + 5\sqrt[3]{100})$

$A_4 = 2 + \sqrt[3]{20} + \sqrt[3]{50}$, $\alpha_4 = 18$ and $\epsilon_4 = 11 + 4\sqrt[3]{20} + 3\sqrt[3]{50}$

Then to find the units in K we look for products of the α's that are cubes in K.

$\alpha_1\alpha_2 = 3^3$ so $\mathfrak{e}_1 = \sqrt[3]{\epsilon_1\epsilon_2} = A_1A_2/3 = \frac{1}{3}(4 + 4\sqrt[3]{2} + 2\sqrt[3]{5} + 2\sqrt[3]{10} + \sqrt[3]{25} + \sqrt[3]{50})$

$\alpha_3 = (\sqrt[3]{2})^3$ so $\mathfrak{e}_2 = \sqrt[3]{\epsilon_3} = A_3/\sqrt[3]{2} = \frac{1}{3}(2\sqrt[3]{4} + \sqrt[3]{5} + \sqrt[3]{50})$

$\alpha_1\alpha_4 = (3\sqrt[3]{2})^3$ so $\mathfrak{e}_3 = \sqrt[3]{\epsilon_1\epsilon_4} = A_1A_4/(3\sqrt[3]{2}) = \frac{1}{3}(2 + \sqrt[3]{4} + \sqrt[3]{10} + \sqrt[3]{20} + \sqrt[3]{25} + \sqrt[3]{50})$

$[\hat{e} : \hat{e}_0] = 3^3$ and $\hat{e} =< \epsilon_1, \mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3 >$.

Example 2 $K = \mathbb{Q}(\sqrt[3]{10}, \sqrt[3]{42})$

The cubic subfields $k_1 = \mathbb{Q}(\sqrt[3]{10})$, $k_2 = \mathbb{Q}(\sqrt[3]{42})$, and $k_3 = \mathbb{Q}(\sqrt[3]{420})$ are all Type I so for each we can calculate A_i, and the principal divisor α_i and $k_4 = \mathbb{Q}(\sqrt[3]{525})$ is Type III.

$A_1 = \frac{1}{3}(4 + \sqrt[3]{10} + \sqrt[3]{100})$, $\alpha_1 = 2$ and $\epsilon_1 = \frac{1}{3}(23 + 11\sqrt[3]{10} + 5\sqrt[3]{100})$

$A_2 = 49 + 14\sqrt[3]{42} + 4\sqrt[3]{1764}$, $\alpha_2 = 49$ and $\epsilon_2 = 21169 + 6090\sqrt[3]{42} + 1752\sqrt[3]{1764}$

$A_3 = 12 + 15\sqrt[3]{420} + 4\sqrt[3]{22050})$, $\alpha_3 = 28$ and $\epsilon_3 = \frac{1}{3}(453610 + 60570\sqrt[3]{420} + 16176\sqrt[3]{22050})$

Then to find the units in K we look for products of the α's that are cubes in K.

$\alpha_1\alpha_2\alpha_3 = (14)^3$ so $\mathfrak{e}_1 = \sqrt[3]{\epsilon_1\epsilon_2\epsilon_3} = A_1A_2A_3/14 = \frac{1}{3}(4208+1952\sqrt[3]{10}+908\sqrt[3]{100}+1213\sqrt[3]{42}+562\sqrt[3]{420}+ 1042\sqrt[3]{525} + 116\sqrt[3]{1764} + 108\sqrt[3]{2205} + 50\sqrt[3]{22050})$

No other products produce a cube so $[\hat{e} : \hat{e}_0] = 3$ and $\hat{e} =< \epsilon_1, \epsilon_2, \epsilon_4, \mathfrak{e}_1 >$.

5.3 Units in K from Type IV fields

If k_i is a Type IV field then $\epsilon_i = \frac{B}{B^{\sigma^2}}$ where $N(B) = \zeta^k$ with $k = 1$ or 2. We would like to apply Hilbert's Theorem 90 as we did in the case for the Type I fields but we need an element of norm 1. We can do this by adjoining the 9^{th} roots of unity to K_i.

Let ω be a root of $x^6 + x^3 + 1 = 0$, $M_i = K_i(\omega)$ and $F = \mathbb{Q}(\omega)$ where ω is chosen so $\omega^3 = \zeta$, then

$$N_{M_i/F}(\omega^j B) = \omega^j B \cdot \omega^j B^{\sigma} \cdot \omega^j B^{\sigma^2} = \omega^{3j} N_{K_i/k}(B) = \zeta^j \zeta^k$$

If $N(B) = \zeta^k$ then choose $j = 3 - k$ then $N_{M_i/F}(\omega^j B) = 1$. Then Hilbert's Theorem 90 can be applied and $\omega^j B = \frac{A}{A^{\sigma}}$ where $A \in M_i$ is unique up to multiplication by an element $\alpha \in \mathbb{Q}(\omega)$.

Let $A_0 = 1+\omega^j B+(\omega^j B)\sigma(\omega^j B)$ then $\omega^j B = \frac{A_0}{A_0^{\sigma}}$ and it can be shown that $N_{M_i/K_i}(A_0) = \epsilon\epsilon'(3-tr(\epsilon))$ where tr is the trace from k_i to $\mathbb{Q}$ and $N_{K_i/k}[\epsilon\epsilon'(3 - tr(\epsilon))] = (3 - tr(\epsilon))^3 = N_{M_i/k}(A_0) = a^3m^3$ with $gcd(a, m_i) = 1$ and $gcd(a, m) = 1$. Similarly we can calculate that $a_0 = N_{M_i/F}(A_0) = 6+tr(\epsilon)+3tr(B)\omega^2+ 3tr(B \cdot B^{\sigma})\omega^4$ and $N_{F/k}[a_0] = (3 - tr(\epsilon))^3$.

Now $(A_0) = A_1A_2$ where A_1 is an ideal of F and A_2 is an ideal of M_i that is divisible by no nontrivial ideals of F. Since $h_F = 1$ then $A_1 = (A_{t_1})$ is principal and hence $A_2 = (A_{t_2})$ is principal and we can write $A_0 = A_{t_1}A_{t_2}$ as integers. We will say that A_{t_2} is primitive. Note that since $A_{t_1} \in F$ then $A_{t_1} = A_{t_1}^{\sigma}$ and $\omega^j B = \frac{A_0}{A_0^{\sigma}} = \frac{A_{t_1}A_{t_2}}{A_{t_1}^{\sigma}A_{t_2}^{\sigma}} = \frac{A_{t_2}}{A_{t_2}^{\sigma}}$ so $(A_{t_2}) = \left(A_{t_2}^{\sigma}\right)$. Since A_{t_2} is primitive it can only be divisible by prime ideals of R that are ramified over F with exponent at most 2. To calculate A_{t_2} we use the GCD function in the 9th cyclotomic field on $N_{M_i/F}(A_0)$ and $N_{M_i/k}(A_0)/m^3$ to find A_{t_1}. Then $A_{t_2} = A_0/A_{t_1}$ and A_{t_2} is divisible by no prime ideals of F and

$$\epsilon_i = \frac{B}{B^{\sigma^2}} = \frac{\omega^j B}{\omega^j B^{\sigma^2}} = \frac{A_{t_2}^3}{N_{M_i/F}(A_{t_2})}.$$

Lemma 5.1 *Let A be a primitive integer of M_i such that $\epsilon = A^3/N_{M_i/F}(A)$ and P be a prime ideal of F with $P^a \mid\mid N_{M_i/F}(A)$ and $P \bigcap \mathbb{Q} = p \equiv 1 \pmod 9$ then $\bar{P}^a \mid\mid N_{M_i/F}(A)$ where $\bar{P} = P^{\tau}$.*

Proof: (We will abbreviate $N_{M_i/F}(\alpha) = N(\alpha)$) Since $\epsilon = \bar{\epsilon}$ then $A^3/N(A) = (A^{\tau})^3/N(A^{\tau})$ and we can rearrange the expression to be $(A/A^{\tau})^3 = N(A)/N(A^{\tau})$. Since $\sqrt[3]{N(A)/N(A^{\tau})}$ is in M_i, Kummer Theory says that $N(A)/N(A^{\tau}) = m_i^j\beta^3$ for some $\beta \in F$ and $j = 0$, 1, or 2. So we have that $N(A) = N(A^{\tau})m_i^j\beta^3$.

Suppose that $\bar{P}^b \mid\mid N(A)$ then

$$a \equiv b + jc \pmod 3 \text{ and } b \equiv a + jc \pmod 3$$

where $p^c \mid\mid m_i$. Adding the two congruences together gives

$$a + b \equiv a + b + 2jc \pmod 3 \implies 0 \equiv 2jc \pmod 3$$

so

$$a \equiv b \pmod 3.$$

and since $0 \leq a, b \leq 2$ then $a = b$

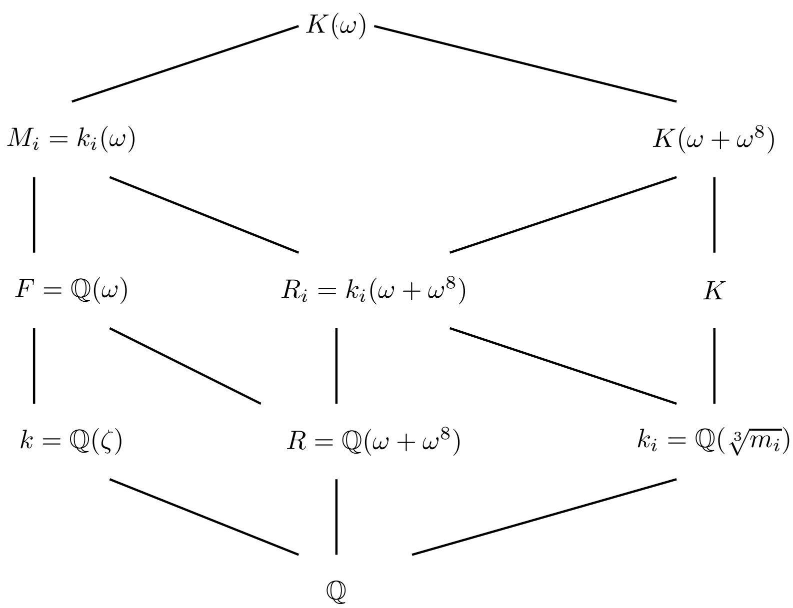
Theorem 5.2 *Let A be a primitive integer of M_i such that $\epsilon_i = \frac{A^3}{N_{M_i/F}(A)}$ and let R be the maximal real subfield of F. Then (A) is an ideal of $R_i = R(\sqrt[3]{m_i})$. If $p \neq 3$ is a prime that divides $N_{M_i/\mathbb{Q}}(A)$ then $(p) = P_1P_2P_3$ in R and $P_j = \mathfrak{p}_j^3$ in R_i. Moreover, subscripts can be assigned so that $\mathfrak{p}_1^2\mathfrak{p}_2 \mid\mid (A)$ and $\mathfrak{p}_3 \nmid (A)$.*

Proof: Since k_i is Type IV, $p \equiv \pm 1 \pmod 9$ so p splits completely in R. Suppose $\mathfrak{P}$ is a prime ideal of M_i lying over p with $\mathfrak{P}^a \mid\mid (A)$. If $p \equiv 8 \pmod 9$ then $\mathfrak{P} = \mathfrak{p}$ is a prime ideal of R_i so assume $p \equiv 1 \pmod 9$. Then $P^a = N_{M_i/F}(\mathfrak{P}^a) \mid\mid N_{M_i/F}(A)$, by Lemma 5.1 $\bar{P}^a \mid\mid N_{M_i/F}(A)$, hence $\bar{\mathfrak{P}}^a \mid\mid (A)$. Since $p \equiv 1 \pmod 9$ and $\mathfrak{P} \neq \bar{\mathfrak{P}}$ then $(\mathfrak{P}\bar{\mathfrak{P}})^a = (\mathfrak{p})^a \mid\mid (A)$ and hence (A) is an ideal of R_i.

Let $a_0 = N_{M_i/F}(A)$ and $p \neq 3$ be a prime divisor of m_i with $(p) = P_1P_2P_3$ in R, $p \mid N_{F/\mathbb{Q}}(a_0)$ and $\epsilon_i \equiv a + b\sqrt[3]{m_i} + c\sqrt[3]{m_i}^2 \pmod 3$ where $a,\ b,\ c \in \mathbb{Z}$. Then the constant term of a_0, which we know from above to be $6 + tr(\epsilon_i)$, must be divisible by p so we have that $6 + 3a \equiv 0 \pmod{p} \implies a \equiv -2 \pmod{p}$. Since $p \mid N_{F/\mathbb{Q}}(a_0)$ then we have that $P_1^{b_1} P_2^{b_2} P_3^{b_3} \mid a_0$ with $b_1 + b_2 + b_3 \equiv 0 \pmod{3}$ since $N(a_0)$ is a cube. If none of the b_i's are equal to zero then we have that $p \mid a_0$ but we know from above that $N_{k_i/\mathbb{Q}}(\epsilon_i) = 1 \equiv a^3 \pmod{p}$ so $(-2)^3 \equiv 1 \pmod{p} \implies 9 \equiv 0 \pmod{p}$. Clearly this is impossible for $p \neq 3$ and since A is primitive all the exponents on P_i's must be less than 3 so for some choice of subscripts $\mathfrak{p}_1^2 \mathfrak{p}_2 \mid\mid (A)$.

We know from [9] Theorem XI that if $e^3 = \epsilon_1^a \epsilon_2^b \epsilon_3^c$ with $1 \leq a,\ b,\ c \leq 2$ has a solution in K then $k_1,\ k_2,\ k_3$ are Type IV fields and if $e^3 = \epsilon_1^a \epsilon_2^b \epsilon_3^c \epsilon_4^d$ with $1 \leq a,\ b,\ c,\ d \leq 2$ has a solution in K then exactly three of $k_1,\ k_2,\ k_3,\ k_4$ are Type IV fields and the remaining field is Type I. We will consider cases which fit these criteria.

To begin we present the field diagram for the Type IV field calculations:



For each k_i, $i = 1, 2, 3, 4$ that is Type IV choose A_i to be a primitive integer satisfying the conditions of Theorem 5.2. If k_i is Type I choose A_i to be a primitive integer of k_i such that $\epsilon_i = \frac{A_i^3}{N(A_i)}$ and if k_i is Type III choose $A_i = 1$. We will also define $a_i = N(A_i)$ where $N(A_i) = N_{M_i/F}(A_i)$ for k_i Type IV and $N(A_i) = N_{K_i/k}(A_i)$ otherwise. Since at least three of the fields are of Type IV we know that for p a prime and $p \mid m_i$ then either $p \equiv \pm 1 \pmod{9}$ or $p = 3$ so the prime factors of $m_1 m_2$ can be written $3, p_1, p_2, \cdots, p_t$. Hence each p_j factors as $P_{j_1} P_{j_2} P_{j_3}$ where the P_{j_k}'s are prime ideals of R.

Lemma 5.3 *Let k_1, k_2, k_3 be Type IV fields, k_4 Type I or IV and suppose*

$$A_e^3 = \epsilon_1^{b_1}\epsilon_2^{b_2}\epsilon_3^{b_3}\epsilon_4^{b_4} \tag{5.2}$$

has a solution $A_e \in K$ where $1 \le b_i \le 2$ for $i =$ 1, 2, 3 and $0 \le b_4 \le 2$. If

$$\epsilon_1^{b_1}\epsilon_2^{b_2}\epsilon_3^{b_3} = \frac{A_0^3}{a_0} \tag{5.3}$$

where $a_0 \in R$ and $A_0 \in M_4$ then $a_4^{b_4}a_0 = m_4^l\alpha^3$ for some $l =$ 0, 1, or 2 and $\alpha \in R$.

Proof: Combining equations (5.2) and (5.3) gives $(\frac{A_0}{A_e})^3 = a_0\epsilon_4^{-b_4}$. It follows from [9] Theorem XI that if k_4 is Type IV then $b_4 = 0$ and if k_4 is Type I then $\epsilon_4 = \frac{A_4^3}{a_4}$. So $(\frac{A_0}{A_e})^3 = a_0(\frac{a_4}{A_4^3})^{b_4} \implies (\frac{A_4^{b_4}A_0}{A_e})^3 = a_0a_4^{b_4}$ so $\frac{A_4^{b_4}A_0}{A_e} = \sqrt[3]{a_0a_4^{b_4}}$ is a cube root of an element of R. Then $a_0a_4^{b_4} = m_4^l\alpha^3$ for some $\alpha \in F$ and $l =$ 0, 1, or 2. Since $\alpha^3 \in R$ and $[F : R] = 2$ then one of α, $\zeta\alpha$ or $\zeta^2\alpha$ is in R and hence the equation $a_0a_4^{b_4} = m_4^l\alpha^3$ has a solution with $\alpha \in R$ and the theorem is proved.

Let $N_{K(\omega+\omega^8)/R_4} = N$, P be a prime divisor lying over 3 in R and $e_1 = \omega + \omega^8$ and $e_2 = \omega^4 + \omega^5$ be fundamental units of R. If we can find a product $A_0 = A_1^{b_1}A_2^{b_2}A_3^{b_3}$ such that, for each prime $p_j \ne 3$ that divides m_1m_2 with p_j as above, the prime ideal factorization of $N(A_0)$ has the form $\left(\prod_{j=1}^{t} P_{j_1}^{c_{j_1}}P_{j_2}^{c_{j_2}}P_{j_3}^{c_{j_3}}\right)P^c$ and $c_{j_1} \equiv c_{j_2} \equiv c_{j_3} \pmod 3$ then let $a_0 = N(A_0)$. By Theorem 5.2 $N(A_i) \in R$ for $i =$ 1, 2, 3 so $a_0 \in R$. Furthermore if $a_0a_4^{b_4} = m_4^l\alpha^3$ as described in Lemma 5.3 and suppose that $p_j \mid a_0a_4^{b_4}$ where p_j^a exactly divides a_4 and p_j^b exactly divides m_4 then

$$N(A_e) = N(A_1^{b_1}A_2^{b_2}A_3^{b_3}A_4^{b_4}) = a_oa_4^{b_4} = m_4^l\alpha^3$$

and the exponents on the P_{j_k}'s are $(c_{j_1}, c_{j_2}, c_{j_3}) + b_4(a, a, a) \equiv l(b, b, b) + (0, 0, 0) \pmod 3 \implies c_{j_k} \equiv l \cdot b - b_4 \cdot a \pmod 3$ for each k. So if there exists a solution to (5.2) then there has to exist at least one choice of b_i's so that $c_{j_1} \equiv c_{j_2} \equiv c_{j_3} \equiv c_j \pmod 3$ for all $j = 1, \ldots, t$. In addition Lemma 5.3 shows that if (5.2) has a solution in K then the b_i's can be chosen so that $c \equiv 0 \pmod 3$. We would like to choose $0 \le b_4 \le 2$ so that we can find

$$\epsilon_1^{b_1}\epsilon_2^{b_2}\epsilon_3^{b_3}\epsilon_4^{b_4} = \frac{(A_1^{b_1}A_2^{b_2}A_3^{b_3}A_4^{b_4})^3}{a_0a_4^{b_4}} = \left(\frac{A_0A_4^{b_4}}{\alpha\sqrt[3]{m_4^l}}\right)^3.$$

At this point we have that $a_0a_4^{b_4} = ne\gamma_1^3$ for some $n \in \mathbb{Z}$, $\gamma_1 \in R$ and e a unit of R. We will show that unless $\frac{n}{m_4^l}$ is a power of 3 times a cube of a rational number then (5.2) will have no solution for this choice of $\mathbf{b} =< b_1, b_2, b_3, b_4 >$.

Assume that (5.2) has a solution for this choice of $\mathbf{b}$ then $m_4^l\alpha^3 = ne\gamma_1^3$ for some $\gamma_1 \in R$ and e a unit of R of the form $e = \pm e_1^ue_2^v$, where $\{e_1, e_2\}$ form a fundamental set of units of R, so

$$e = e_1^ue_2^v = \frac{m_4^l}{n}\left(\frac{\alpha}{\gamma_1}\right)^3. \tag{5.4}$$

Now let φ be the element of $Gal(F/\mathbb{Q})$ with $\varphi(\omega) = \omega^4$ then φ fixes $k = \mathbb{Q}(\omega^3)$ and $N(e_1) = e_1 e_1^{\varphi} e_1^{\varphi^2} = 1$ and $e_1^{\varphi} = e_2$. By Hilbert's Theorem 90 we can find $\beta_1 \in F$, in fact $\beta_1 = 1 + e_1 + e_1 e_2$, such that $e_1 = \frac{\beta_1}{\beta_1^{\varphi}}$, $N_{F/k}(\beta_1) = 9$ and

$$\frac{e_1}{e_2} = \frac{N_{F/k}(\beta_1)}{\beta_1^3} = \frac{9}{\beta_1^3}$$

so $e_2 = \frac{e_1 \beta_1^3}{9}$. Now we can rewrite (5.4) and obtain

$$e = e_1^u \left(\frac{e_1 \beta_1^3}{9} \right)^v = e_1^{u+v} \frac{(\beta_1^v)^3}{9^v} = \frac{m_4^l}{n} \left(\frac{\alpha}{\gamma_1} \right)^3$$

so $e_1^{u+v} = q\gamma_2^3$ where $q \in \mathbb{Q}$ and $\gamma_2 \in R$. This says that

$$\left(\frac{e_1}{e_2} \right)^{u+v} = \left(\frac{\gamma_2}{\gamma_2^{\varphi}} \right)^3 = \left(\frac{9}{\beta_1^3} \right)^{u+v}$$

or

$$9^{u+v} = \left(\frac{\gamma_2 \beta_1^{u+v}}{\gamma_2^{\varphi}} \right)^3$$

is a cube of an element of R. But this is false unless $u + v \equiv 0 \pmod 3$ and thus $v \equiv 2u \pmod 3$. Hence $e = (e_1 e_2^2)^u = s\gamma_3^3$ for some $s \in \mathbb{Q}$, $\gamma_3 \in R$. But $e_1 e_2^2 = 9\beta_2^3$ for some $\beta_2 \in R$ so $\frac{m_4^l}{n} \left(\frac{\alpha}{\gamma_1} \right)^3 = e = (e_1 e_2^2)^u = 9^u \beta_2^{3u}$ which can be simplified to $\frac{m_4^l}{9^u n} = \left(\frac{\beta_2 \gamma_1}{\alpha} \right)^3 = \gamma_4^3$ for some $\gamma_4 \in R$. Since $\gamma_4^3 \in \mathbb{Q}$ then $\gamma_4 \in \mathbb{Q}$ so $m_4^l = 9^u n \gamma_4^3$. Thus n differs from m_4^l by a power of 3 times a rational cube, so $n = 3^r m_4^l \gamma_0^3$ for $r \in \mathbb{Z}$ and $\gamma_0 \in \mathbb{Q}$. Thus for a fixed vector $\mathbf{b}$ either n satisfies this condition or no solution exists for this value of $\mathbf{b}$. Thus we assume this condition holds and since 3 differs from a unit of R by a cube in R we have that $a_0 a_4^{b_4} = ne\gamma_1^3 = 3^r m_4^l (\gamma_0 \gamma_1)^3 e$ so

$$\epsilon_1^{b_1} \epsilon_2^{b_2} \epsilon_3^{b_3} \epsilon_4^{b_4} = \left(\frac{A_0 A_4^{b_4}}{a_0 a_4^{b_4}} \right)^3 = \left(\frac{A_0 A_4^{b_4}}{\sqrt[3]{m_4^l}} \right)^3 \cdot e \cdot \gamma^3$$

for some unit e of R and $\gamma \in R$.

If e is a cube of a unit in R, we are done. Suppose $e = (e_1 e_2^2)^u$ and either $\sqrt[3]{3}$ is in K or k_4 is Type I and 3 is a principal divisor in k_4 so that $3 = \alpha^3 \epsilon_4^i$ for $\alpha \in k_4$ and i =0, 1, 2. In the first case $e_1 e_2^2 = \left(\sqrt[3]{9} \beta_2 \right)^3$ is a cube in $K(\omega + \omega^8)$ and in the second case $e_1 e_2^2 = \left(\alpha^2 \beta_2 \right)^3 \epsilon_4^{2i}$ which gives us that $e_1 e_2^2 \epsilon_4^i = (\alpha_0)^3$ for some $\alpha_0 \in K(\omega + \omega^8)$. In either case we get a solution to

$$A_e^3 = \epsilon_1^{b_1} \epsilon_2^{b_2} \epsilon_3^{b_3} \epsilon_4^{b_4}$$

for some integer b_4. In the case where neither $\sqrt[3]{3}$ is in K nor is 3 a principal divisor, Lemma 5.3 shows that no solution exists.

5.4 Example Type IV units in K

Example 3 $K = \mathbb{Q}(\sqrt[3]{3}, \sqrt[3]{17})$

Then the cubic subfields are $k_1 = \mathbb{Q}(\sqrt[3]{3})$, $k_2 = \mathbb{Q}(\sqrt[3]{17})$, and $k_4 = \mathbb{Q}(\sqrt[3]{153})$ are all Type IV so for each we calculate $B_i \in K_i$ and $A_i \in M_i$ and $k_3 = \mathbb{Q}(\sqrt[3]{51})$ is Type I.
$B_1 = -1 + \sqrt[3]{3} + \zeta\sqrt[3]{3} - \sqrt[3]{3}^2/3 - 2\zeta\sqrt[3]{3}^2/3$

$A_1 = 73/3 - 17\sqrt[3]{3}/3 - 7\sqrt[3]{3}^2/3 + (-21 + 3\sqrt[3]{3}^2)\omega + (47/3 + 17\sqrt[3]{3}/3 - 4\sqrt[3]{3}^2)\omega^2 + (47/3 - 15\sqrt[3]{3} + 5\sqrt[3]{3}^2/3)\omega^3 + (-21 + 37\sqrt[3]{3}/3)\omega^4 + (73/3 - 28\sqrt[3]{3}/3 - 5\sqrt[3]{3}^2/3)\omega^5$

$B_2 = \frac{1}{3}\left[-7 + 13\sqrt[3]{17} - 4\sqrt[3]{17}^2 + w(28 + 2\sqrt[3]{17} - 5\sqrt[3]{17}^2)\right]$

$A_2 = -10 - \sqrt[3]{17}/3 + 5\sqrt[3]{17}^2/3 + (43/3 - 7\sqrt[3]{17}^2/3)\omega + (31/3 + \sqrt[3]{17}/3 + 5\sqrt[3]{17}^2/3)\omega^2 + (-31/3 + 11\sqrt[3]{17}/3)\omega^3 + (43/3 - 17\sqrt[3]{17}/3)\omega^4 + (-10 + 4\sqrt[3]{17})\omega^5$

$B_4 = 28 + 19\zeta - 5\sqrt[3]{153}/3 - 16/3\zeta\sqrt[3]{153} - 2\sqrt[3]{867} + \zeta\sqrt[3]{867}$

$A_4 = -1403/3 + 98\sqrt[3]{153}/3 + 98\sqrt[3]{867}/3 + (400 - 131/3\sqrt[3]{867})\omega + (-898/3 - 98/3\sqrt[3]{153} + 51\sqrt[3]{867})\omega^2 + (-898/3 + 91\sqrt[3]{153} - 55/3\sqrt[3]{867})\omega^3 + (400 - 78\sqrt[3]{153})\omega^4 + (-1403/3 + 175/3\sqrt[3]{153} + 55/3\sqrt[3]{867})\omega^5$

Let $A_0 = \frac{A_1 A_2 A_4 * e_1^6 * e_2^{13}}{\sqrt[3]{17}^2}$ then $N(A_0) = e_2 e_3^2$. We can calculate $e_1^2 e_2 = \left[\frac{\sqrt[3]{3}}{3}(-\omega + \omega^2 + \omega^4 + 2\omega^5)\right]^3$ then $A = A_0 * \frac{\sqrt[3]{3}}{3}(-\omega + \omega^2 + \omega^4 + 2\omega^5)$ and to get a unit in K we multiply A by its complex conjugate to get $\mathfrak{e}_1 = AA^\tau = 3 - 4\sqrt[3]{3}/3 - 7\sqrt[3]{3}^2/3 - \sqrt[3]{17} + 4\sqrt[3]{51}/3 - \sqrt[3]{867}/3 + \sqrt[3]{2601}/3$ and $e_1 = \sqrt[3]{\epsilon_1\epsilon_2\epsilon_4}$.

It turns out that in this case since k_3 is Type I with $A_3 = 1513 - 408\sqrt[3]{51} - 408\zeta\sqrt[3]{51} + 110\zeta\sqrt[3]{51}^2$, $\alpha_3 = (\sqrt[3]{17}^2)^3$ and $\epsilon_3 = 107846641 + 29081484\sqrt[3]{51} + 7841994\sqrt[3]{51}^2$ then $\mathfrak{e}_2 = \sqrt[3]{\epsilon_3} = \frac{A_3}{(\sqrt[3]{17})^2} = 110\sqrt[3]{9} + 89\sqrt[3]{17} + 24\sqrt[3]{867}$.

Then $[\hat{e} : \hat{e}_0] = 3^2$ and the basis for $\hat{e}$ can be chosen $< \sqrt[3]{\epsilon_1\epsilon_2\epsilon_4},\ \sqrt[3]{\epsilon_3},\ \epsilon_1,\ \epsilon_2 >$.

Example 4 $K = \mathbb{Q}(\sqrt[3]{17}, \sqrt[3]{19})$

Then the cubic subfields are $k_1 = \mathbb{Q}(\sqrt[3]{17})$, $k_3 = \mathbb{Q}(\sqrt[3]{323})$, and $k_4 = \mathbb{Q}(\sqrt[3]{5491})$ are all Type IV so for each we calculate $B_i \in K_i$ and $A_i \in M_i$ and $k_2 = \mathbb{Q}(\sqrt[3]{19})$ is Type III.
$B_1 = \frac{1}{3}\left[-7 + 13\sqrt[3]{17} - 4\sqrt[3]{17}^2 + \zeta(28 + 2\sqrt[3]{17} - 5\sqrt[3]{17}^2)\right]$

$A_1 = \frac{1}{3}\left[-30 - \sqrt[3]{17} + 5\sqrt[3]{17}^2 + (43 - 7\sqrt[3]{17}^2)\omega + (-31 + \sqrt[3]{17} + 5\sqrt[3]{17}^2)\omega^2 + (-31 + 11\sqrt[3]{17})\omega^3 + (43 - 17\sqrt[3]{17})\omega^4 + (-30 + 12\sqrt[3]{17})\omega^5\right]$

where $N_{K_1/k}(B_1) = -\zeta^2$ and $\omega B_1 = \frac{A_1}{A_1^{\sigma^2}}$

$B_3 = -52177936089095795/3 - 43275181296929777\zeta/3 + 1297548413554459\sqrt[3]{323}/3$
$+2534923169868269\zeta\sqrt[3]{323} + 306419189646737\sqrt[3]{323}^2 - 189113586167644\zeta\sqrt[3]{323}^2/3$

$A_3 = -5482098754152 + 5482098754152\omega^2 + 7800620169767\omega^3/3 + 6788827068112\omega^4$
$+24246916432223\omega^5/3 + \sqrt[3]{323}(-1136915770715/3 - 989450119403\omega - 1177970515295\omega^2$
$-1177970515295\omega^3/3 - 989450119403\omega^4 - 1136915770715\omega^5/3) + \sqrt[3]{323}^2(515056454665/3$
$+144209232164\omega + 55233961812\omega^2 + 349354569229\omega^3/3 + 349354569229\omega^5/3)$

where $N_{K_3/k}(B_3) = \zeta^2$ and $\omega B_3 = \frac{A_3}{A_3^{\sigma^2}}$

$B_4 = -57735118463347/3 + 828970907077\zeta/3 + 3319562986196\sqrt[3]{5491}/3 + 3272574786271\zeta\sqrt[3]{5491}/3 - 45278001347\sqrt[3]{6137}/3 - 3198743037625\zeta\sqrt[3]{6137}/3$

$A_4 = -12674882642/3 + 106264169\sqrt[3]{5491}/3 + 196633197\sqrt[3]{6137} + 12674882642\omega/3$
$-612180730\omega\sqrt[3]{5491}/3 - 102396542\omega\sqrt[3]{6137}/3 + 18498534619\omega^2/3 - 1010381267\omega^2\sqrt[3]{6137}/3$
$-10800158675\omega^3/3 + 239481633\omega^3\sqrt[3]{5491} - 102396542\omega^3\sqrt[3]{6137}/3 + 624907989\omega^4$
$-239481633\omega^4\sqrt[3]{5491} + 196633197\omega^4\sqrt[3]{6137} + 18498534619\omega^5/3 - 1048544449/3\omega^5\sqrt[3]{5491}$

where $N_{K_4/k}(B_4) = \zeta$ and $\omega^2 B_4 = \frac{A_4}{A_4^{\sigma^2}}$

The prime divisors of 17 in R are $p_{17a} = 3 + \omega - \omega^2 - \omega^5$, $p_{17b} = p_{17a}^{\varphi} = 3 + \omega^4 + \omega^5$ and $p_{17c} = p_{17a}^{\varphi^2} = 3 - \omega + \omega^2 - \omega^4$ so $A_0 = A_1 A_3 A_4 \sqrt[3]{19} * p_{17a}^2 p_{17b} * e_1^7 e_3^6$ and $N_{K(\omega)/M_2}(A_0) = \omega^3(17^2 \cdot 19)^3$. Then $A = \frac{A_0}{17^2 \cdot 19}$ and $\mathfrak{e}_1 = AA^{\tau} = 10070766629 - 5253124558\sqrt[3]{17}/3 - 2250284651\sqrt[3]{17}^2/3 + 11216410025\sqrt[3]{19}/3 - 2009801662\sqrt[3]{17 \cdot 19}/3 - 859315960\sqrt[3]{17^2 \cdot 19}/3 + 1342037855\sqrt[3]{19}^2 - 274047206\sqrt[3]{17 \cdot 19^2} - 348851731\sqrt[3]{17^2 \cdot 19^2}/3$ where $\mathfrak{e}_1 = \sqrt[3]{(\epsilon_1 \epsilon_3 \epsilon_4)}$. Here $\epsilon_1 = 18 - 7\sqrt[3]{17}$, $\epsilon_3 = -4167355395831946/3 + 124966834144486621\sqrt[3]{17 \cdot 19}/3 - 1732828610414410\sqrt[3]{17 \cdot 19}^2/3$, and
$\epsilon_4 = 20614589130834 + 1027551896964\sqrt[3]{17^2 \cdot 19} - 2116111936999\sqrt[3]{17 \cdot 19^2}$.

Then $[\hat{e} : \hat{e}_0] = 3$ and the basis for $\hat{e}$ can be chosen $< \sqrt[3]{\epsilon_1 \epsilon_3 \epsilon_4},\ \epsilon_1,\ \epsilon_2,\ \epsilon_3 >$.

5.5 Units in K from Type III Fields

In order to get a unit in $\hat{e}$ that is not in $\hat{e}_0$ and involves units from Type III subfields, it is shown in [9] Theorem XII that K is not Kind 1. The same theorem shows that K can be of Kind 2 if exactly three of the subfields are Type III and that K can be of Kind 3 if at least 3 are of Type III. It can be shown that the first case is not possible.

Definition. If α and β are in k we shall say α and β are *equivalent* and write $\alpha \sim \beta$ if $\alpha = \beta$ or $\alpha = \overline{\beta}$.

Theorem 5.4 *If k_1, k_2 and k_3 are all of Type III and k_4 is of Type I then K is not of Kind 2. Furthermore, if k_1 and k_2 are of Type III such that $N(B_1) \sim N(B_2)$ and k_3 and k_4 are of Type I then K is also not of Kind 2.*

Proof: Suppose K is of Kind 2 and $k_i = \mathbb{Q}(\sqrt[3]{m_i})$ for $i = 1, 2, 3, 4$ and k_1, k_2 and k_3 are all of Type III and k_4 is of Type I. Then since k_1, k_2 and k_3 are all of Type III then $\sqrt[3]{\epsilon_1^a \epsilon_2^b \epsilon_3^c} \in K$ with $1 \leq a, b, c \leq 2$ and since k_4 is of Type I then $\sqrt[3]{\epsilon_4} \in K$. Thus $K = k_4(\sqrt[3]{\epsilon_4})$ and only prime divisors of 3 can ramify from k_4 to K. All other primes that ramify in K must also be ramified in k_4 over $\mathbb{Q}$ so m_4 must be divisible by all ramified primes, except possibly 3. We know that m_1, m_2 and m_3 must all have a common prime divisor $p \equiv 1 \pmod 3$ by Corollary III of Theorem X of [9]. But any prime that divides m_1, m_2 and m_3 can not divide m_4 which is a contradiction. Hence K is not of Kind 2.

Suppose K is of Kind 2 and k_1 and k_2 are of Type III such that $N(B_1) \sim N(B_2)$ and k_3 and k_4 are of Type I, then $\sqrt[3]{\epsilon_3}$, $\sqrt[3]{\epsilon_4} \in K$. Thus $K = k_3(\sqrt[3]{\epsilon_3}) = k_4(\sqrt[3]{\epsilon_4})$ and only prime divisors of 3 can ramify from k_3 to K or from k_4 to K. As above m_3 and m_4 must be divisible by all ramified primes, except possibly 3. Since $N(B_1) \sim N(B_2)$ then there exists a prime $p \neq 3$ such that $p \mid m_1$ and $p \mid m_2$ so p divides exactly one of m_3 or m_4 but this is not possible unless $p = 3$. Thus K is not of Kind 2 and the theorem is proved.

Let k_1, k_2, k_3 by Type III fields and k_4 be of any Type. We know from Corollaries III and IV of Theorem X of [9] that to find a unit in K from Type III fields we will look for units of the form $\sqrt[3]{\epsilon_1^a \epsilon_2^b \epsilon_3^c \epsilon_4^d}$ with $1 \leq a, b, c \leq 2$ and $0 \leq d \leq 2$. Corollary IV of [9] had a minor notational error which we correct here as Theorem 5.5. First we will note that the significance of *Remark* A from [9] is that if we wish to replace ϵ_i with ϵ_i^2 in any part of [9] Theorem X, its corollaries or Theorem 5.5 then we should replace $N(B_i)$ with $\overline{N(B_i)}$.

Theorem 5.5 *If k_1 is Type III and $e^3 = \epsilon_1^a \epsilon_2^b \epsilon_3^c \epsilon_4^d$ has a solution e in K where $1 \leq a, b, c, d \leq 2$ then at least three of the fields k_1, k_2, k_3 and k_4 are of Type III. If k_4 is not of Type III then $N(B_4) = \zeta^t$ with $t = 0$, 1 or 2 and $\left\{N(B_1) = \zeta^{s \cdot t} N(B_2) \text{ or } N(B_1) = \zeta^{s \cdot t} \overline{N(B_2)}\right\}$ and $\left\{N(B_1) = \zeta^{2s \cdot t} N(B_3) \text{ or } N(B_1) = \zeta^{2s \cdot t} \overline{N(B_3)}\right\}$ where $s = 1$ or 2 and m_1, m_2, m_3 must have a common prime divisor $p \equiv 1 \pmod 3$.*

Proof: The proof of the first statement of the Theorem is correct in [9] so we will only prove the last part of the Theorem. To do so we will first look at the case where all the exponents are 1 and then see how the exponents on the ϵ_i's effect the relations between the norms. Assume that $e^3 = \epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4$ and that k_4 is not of

Type III then $N(B_4) = \zeta^t$ with $t = 0$, 1 or 2. We know from the proof of the first statement that in this case $N(B_1) = \zeta^{2t}N(B_2)$ and $N(B_1) = \zeta^t\overline{N(B_3)}$. Now suppose that $e^3 = \epsilon_1^2\epsilon_2\epsilon_3\epsilon_4$ then from *Remark* A we can replace $N(B_1)$ with $\overline{N(B_1)}$ in the previous case which gives $\overline{N(B_1)} = \zeta^{2t}N(B_2) \implies N(B_1) = \zeta^t\overline{N(B_2)}$ and $\overline{N(B_1)} = \zeta^t\overline{N(B_3)} \implies N(B_1) = \zeta^{2t}N(B_3)$. Suppose that $e^3 = \epsilon_1\epsilon_2^2\epsilon_3\epsilon_4$ then $N(B_1) = \zeta^{2t}\overline{N(B_2)}$ and $N(B_1) = \zeta^t\overline{N(B_3)}$. Suppose that $e^3 = \epsilon_1\epsilon_2\epsilon_3^2\epsilon_4$ then $N(B_1) = \zeta^{2t}N(B_2)$ and $N(B_1) = \zeta^tN(B_3)$. Suppose that $e^3 = \epsilon_1\epsilon_2\epsilon_3\epsilon_4^2$ then we replace $N(B_4) = \zeta^t$ with $N(B_4) = \zeta^{2t}$ so we get $N(B_1) = \zeta^tN(B_2)$ and $N(B_1) = \zeta^{2t}\overline{N(B_3)}$. By combining these cases we get all possible exponents on the ϵ_i's and it is clear that if the exponent on ζ is $s \cdot t$ for the $N(B_2)$ term it will be $2s \cdot t$ on the $N(B_3)$ term. That m_1, m_2 and m_3 have a common prime divisor is shown in [9].

To find a unit in K we first find a product, A, of powers of three B_i's and/or $\overline{B}_i$'s for i =1, 2, 3 such that the norm of A is a cube of some element $\alpha \in k$. We will apply Hilbert's Theorem 90 to A/α. We know from the corollaries to Theorem X in [9] and Theorem 5.5 that we need to have that $N(B_1) \sim \zeta^uN(B_2) \sim \zeta^vN(B_3)$ with $0 \le u,\ v \le 2$. We can find the exponents by considering the cases in the proof of the Theorem 5.5. Suppose that $e^3 = \epsilon_1\epsilon_2\epsilon_3\epsilon_4^d$ where $0 \le d \le 2$ has a solution e in K. Then $N(B_1B_2\overline{B_3}) = N(B_1)N(B_2)\overline{N(B_3)} = N(B_1)\zeta^tN(B_1)\zeta^{2t}N(B_1) = N(B_1)^3$. Then if $A_0 = B_1B_2B_3^\tau$ then $N_{L/K_4}(A_0) = \alpha^3$ where $\alpha = N(B_1) \in k$. Similarly if $e^3 = \epsilon_1^2\epsilon_2\epsilon_3\epsilon_4^d$ where $0 \le d \le 2$ has a solution e in K then $N(B_1^2B_2B_3^\tau) = N(B_1)^2N(B_2)\overline{N(B_3)} = N(B_1)^2\zeta^t\overline{N(B_1)}\zeta^{2t}\overline{N(B_1)} = (N(B_1)\overline{N(B_1)})^2$. Since $N(B_1)\overline{N(B_1)}$ is the cube of a rational integer then for $A_0 = B_1^2B_2B_3^\tau$ we have that $N_{L/K_4}(A_0) = \alpha^3$ for some $\alpha \in \mathbb{Z}$. A similar product can be found for the other 2 cases so let $A = \frac{A_0}{\alpha}$ then $N_{L/K_4}(A) = 1$.

Let $\sigma = \sigma_4$, since $N_{L/K_4}(A) = 1$ we can use Hilbert's Theorem 90 to find an element $E_0 \in K$ such that $A = \frac{E_0}{E_0^\sigma}$. Without loss of generality let $A = \frac{B_1B_2\overline{B}_3}{\alpha}$ and $\epsilon_1\epsilon_2\epsilon_3 = \frac{B_1B_2B_3}{(B_1B_2B_3)^\sigma}$ then

$$\epsilon_1\epsilon_2\epsilon_3 = \frac{A}{A^\sigma} = \frac{E_0^{\sigma^2}E_0}{(E_0^\sigma)^2} = \frac{N(E_0)}{(E_0^\sigma)^3}. \tag{5.5}$$

Taking the complex conjugate of (5.5) and multiplying we obtain

$$(\epsilon_1\epsilon_2\epsilon_3)^2 = \frac{N(E_0)\overline{N(E_0)}}{(E_0^\sigma E_0^{\sigma\tau})^3}$$

where $\rho = N(E_0)\overline{N(E_0)} \in k_4$. To find a unit in K we need ρ to be a cube times a unit in k_4 so we want $\rho\epsilon_4^d = \beta^3$ with $\beta \in K$. By Kummer Theory this is true if and only if $\rho\epsilon_4^dm_1^l = \gamma^3$ where $\gamma^3 \in k_4$ and $0 \le l \le 2$. Since $\gamma \in K_4$ and $[K_4 : k_4] = 2$, $\zeta^i\gamma \in k_4$ for some $i = 0,\ 1,\ 2$. Use the cube root function (section 5.6) to test if $\rho\epsilon_4^dm_1^l$ is a cube in k_4 for $d, l \in \{0, 1, 2\}$. If there is a solution for β then

$$\epsilon_4^d(\epsilon_1\epsilon_2\epsilon_3)^2 = \frac{\rho\epsilon_4^d}{(E_0^\sigma E_0^{\sigma\tau})^3} = \frac{\beta^3m_1^l}{(E_0^\sigma E_0^{\sigma\tau})^3} = \left(\frac{\beta\sqrt[3]{m_1^l}}{(E_0^\sigma E_0^{\sigma\tau})}\right)^3 = E^3$$

and $E^3 = \epsilon_1^{d_1}\epsilon_2^{d_2}\epsilon_3^{d_3}\epsilon_4^{d_4}$ where $0 \le d_4 \le 2$.

5.6 Cube Root Function

Given an integer $\beta_0 \in K_1$ we would like to be able to solve the equation $\alpha_0^3 = \beta_0$ for some α_0 an integer of K_1 if such a solution exists. If $m_1 = ab^2$ where a and b are relatively prime and square free and β_0 and α_0 are both integers of K_1 then we can express them as $\alpha_0 = \mathfrak{a}_1 + \mathfrak{a}_2\zeta + \mathfrak{a}_3\sqrt[3]{m_1} + \mathfrak{a}_4\zeta\sqrt[3]{m_1} + \mathfrak{a}_5\sqrt[3]{m_1^2} + \mathfrak{a}_6\zeta\sqrt[3]{m_1^2}$ and $3^3b^3\beta_0 = \mathfrak{b}_1 + \mathfrak{b}_2\zeta + \mathfrak{b}_3\sqrt[3]{m_1} + \mathfrak{b}_4\zeta\sqrt[3]{m_1} + \mathfrak{b}_5\sqrt[3]{m_1^2} + \mathfrak{b}_6\zeta\sqrt[3]{m_1^2}$ where $\mathfrak{a}_j, \mathfrak{b}_j \in \mathbb{Z}$ for all j. The advantage of multiplying by 3^3b^3 is that this will eliminate all the denominators in β_0 and then solutions with integer coefficients can be found. Since α_0 has 6 unknowns then we will need 6 equations. We can find these equations by conjugating with the elements of $Gal(K_1/\mathbb{Q})$ to get $\beta_1 = \beta_0^\sigma$, $\beta_2 = \beta_0^{\sigma^2}$, $\beta_3 = \beta_0^\tau$, $\beta_4 = \beta_0^{\sigma\tau}$ and $\beta_5 = \beta_0^{\sigma^2\tau}$ and similarly $\alpha_1, \dots, \alpha_5$ are the conjugates of α_0. To solve the equation $\alpha_0^3 = \beta_0$ we replace ζ and $\sqrt[3]{m_1}$ with a numerical approximation in each β_j and α_j and we get a set of 6 equations of the form

$$\alpha_j = 3b\sqrt[3]{\beta_j} = C_j + D_j\sqrt{-1}, \quad j = 1, \dots, 6, \quad C_j, D_j \in \mathbb{R}.$$

When a solution for α_0 is produced this way, the coefficients will have real values. To find integer solutions we see if there is a solution to the equation where the coefficients are close to integers. It is easy to check if the solution is correct by verifying $\alpha_0^3 = 3^3b^3\beta_0$. If a solution exists in $\mathbb{Z}$ then $\left(\frac{\alpha_0}{3b}\right)^3 = \beta_0$.

5.7 Example Type III units in K

Example 5 $K = \mathbb{Q}(\sqrt[3]{7}, \sqrt[3]{19})$

Then the cubic subfields, $k_1 = \mathbb{Q}(\sqrt[3]{7})$, $k_2 = \mathbb{Q}(\sqrt[3]{19})$, $k_3 = \mathbb{Q}(\sqrt[3]{133})$, and $k_4 = \mathbb{Q}(\sqrt[3]{931})$ are all Type III so for each ϵ_i we calculate $B_i \in K_i$ with $\epsilon_i = \frac{B_i}{B_i^\sigma}$. The prime 7 factors as $(3+2\zeta)(3+2\zeta^2) = \pi_7\overline{\pi_7}$ and 19 factors as $(5+2\zeta)(5+2\zeta^2) = \pi_{19}\overline{\pi_{19}}$ in k. Let $\sigma = \sigma_1$ and $N = N_{K_i/k}$ for $1 \le i \le 4$ then
$B_1 = \frac{1}{3}(7 + 14\zeta + \sqrt[3]{7} - 4\zeta\sqrt[3]{7} - 2\sqrt[3]{49} - \zeta\sqrt[3]{49})$ and $N(B_1) = \zeta^2\pi_7^2\overline{\pi_7}$
$B_2 = \frac{1}{3}(19 - 3\sqrt[3]{19} - 5\zeta\sqrt[3]{19} - \sqrt[3]{361} + 2\zeta\sqrt[3]{361})$ and $N(B_2) = -\zeta\pi_{19}^2\overline{\pi_{19}}$
$B_3 = \frac{1}{3}(18297 + 5871\zeta - 2436\sqrt[3]{133} - 3585\zeta\sqrt[3]{133} - 225\sqrt[3]{17689} + 477\zeta\sqrt[3]{17689})$ and $N(B_3) = \pi_{19}^2\overline{\pi_{19}}$
$B_4 = \frac{1}{3}(-855 - 114\zeta + 75\sqrt[3]{931} + 87\zeta\sqrt[3]{931} + 9\sqrt[3]{2527} - 54\zeta\sqrt[3]{2527})$ and $N(B_4) = \pi_{19}^2\overline{\pi_{19}}$

Let $A_0 = B_2B_3B_4$, then $N_{L/K_1}(A_0) = \pi_{19}^6\overline{\pi_{19}}^3 = \alpha^3$ and $A = \frac{A_0}{\alpha} = (945269 + 253083\zeta + 493886\sqrt[3]{7} + 132373\zeta\sqrt[3]{7} + 258362\sqrt[3]{49} + 69331\zeta\sqrt[3]{49} - 259218\sqrt[3]{19} - 354292\zeta\sqrt[3]{19} - 135610\sqrt[3]{133} - 185186\zeta\sqrt[3]{133} - 70822\sqrt[3]{931} - 96758\zeta\sqrt[3]{931} - 35564\sqrt[3]{361} + 97126\zeta\sqrt[3]{361} - 18628\sqrt[3]{2527} + 50782\zeta\sqrt[3]{2527} - 9712\sqrt[3]{17689} + 26566\zeta\sqrt[3]{17689})/3$ with $N_{L/K_1}(A) = 1$.

Using Hilbert's Theorem 90 on A we get that $A = \frac{E_0}{E_0^\sigma}$ where $E_0 = 229816/3 + 40064\sqrt[3]{7} + 20948\sqrt[3]{49} + 86182\sqrt[3]{19}/3 + 15007\sqrt[3]{133} + 7849\sqrt[3]{931} + 32290\sqrt[3]{361}/3 + 5628\sqrt[3]{2527} + 2940\sqrt[3]{17689}$ and $N_{K/k_1}(E_0) =$

$(9 + 7\sqrt[3]{7} + 4\sqrt[3]{49})^3 = \beta^3$. The new unit in K is then $E = \frac{E_0}{\beta} = 6226/3 + 3242\sqrt[3]{7}/3 + 1694\sqrt[3]{49}/3 + 2323\sqrt[3]{19}/3 + 1220\sqrt[3]{133}/3 + 635\sqrt[3]{931}/3 + 871\sqrt[3]{361}/3 + 455\sqrt[3]{2527}/3 + 239\sqrt[3]{17689}/3$ and $E^3 = \epsilon_2\epsilon_3\epsilon_4$.

Then $[\hat{e} : \hat{e}_0] = 3$ and the basis for $\hat{e}$ can be chosen $< \sqrt[3]{\epsilon_2\epsilon_3\epsilon_4},\ \epsilon_1,\ \epsilon_2,\ \epsilon_3 >$.

Chapter 6

Unit Group of L

Once we know the basis for the group of units of K we would like to be able to determine the basis for the group of units of L. The units in L are dependent on the Kind of K and the Types of the cubic subfields. In this section we will provide some criteria for when there can be units in L that are not products of the units of the subfields and what their form will be. We will also give a method for computing the unit basis of L.

6.1 Criteria for Units in L

For this section we will define a new equivalence relation. This equivalence relation says that the units in L are equivalent up to multiplication by a cube of an element in L.

Definition 1 *: If e_1 and e_2 are units in L we shall say e_1 and e_2 are equivalent and write $e_1 \approx e_2$ if and only if $e_1 = \zeta^f e^3 e_2$ for some $e \in L$ and $f = 0, 1,$ or 2.*

Using this equivalence relation we can define some new relationships for the units of Type I and IV fields. These relationships are summarized below with $f = 0, 1, \text{or } 2$.

For $i = 1$ let $\sigma = \sigma_2$

For $i > 1$ let $\sigma = \sigma_1$

$\epsilon_1^{\sigma^2} = \epsilon'' \approx \epsilon^2(\epsilon')^2$

$\epsilon_1^{\sigma} = \epsilon_1' = u_1^3 \epsilon_1^{-2} \zeta^f \approx \epsilon_1.$

$\epsilon_2^{\sigma} = \epsilon_2' = u_2^3 \epsilon_2^{-2} \zeta^f \approx \epsilon_2.$

$\epsilon_3^\sigma = \epsilon_3' = u_3^3\epsilon_3^{-2}\zeta^f \approx \epsilon_3.$

$\epsilon_4^\sigma = \epsilon_4' = u_4^3\epsilon_4^{-2}\zeta^f \approx \epsilon_4.$

$u_i^\sigma = \epsilon_i^{-1}u_i$

$u_i^{\sigma^2} = \epsilon_i u_i^{-2} \approx \epsilon_i u_i$

$u_i^\tau = \overline{u}_i = \epsilon_i u_i^{-1}$

$(\epsilon')^\tau = \epsilon''$

$\frac{u}{\overline{u}} = \frac{u^2}{\epsilon} = \sqrt[3]{\epsilon(\epsilon'')^2}$

Lemma 6.1 *Let K_1 be of Type I or IV and ζ be a cube root of unity, possibly 1. If ϵ_1 and u_1 form a fundamental system of units for K_1 then the equation*

$$e^3 = \zeta^i\epsilon_1^a u_1^b$$

has no solutions $e \in L$ when $b \equiv 1$ or 2 (mod 3).

Proof: Assume $e^3 = \zeta^i\epsilon_1^a u_1^b$ has a solution $e \in L$ with $b \equiv 1$ or 2 (mod 3). Then $(e^{1-\tau})^3 = \zeta^{-i}(\frac{u_1}{\overline{u_1}})^b = \zeta^{-i}\sqrt[3]{\epsilon_1(\epsilon_1'')^2}^b$. Then by changing notation we may assume $e^3 = \zeta^i\sqrt[3]{\epsilon_1(\epsilon_1'')^2}$. Thus if $L = K_1(\sqrt[3]{n})$ for an integer n then

$$n^j = \alpha^3\zeta^i\sqrt[3]{\epsilon_1(\epsilon_1'')^2} \text{ where } j = 1 \text{ or } 2 \text{ and } \alpha \in K_1.$$

By taking complex conjugates and multiplying we get

$$(n^j)^{1+\tau} = n^{2j} = (\alpha^{1+\tau})^3\sqrt[3]{\epsilon_1^2(\epsilon_1')^2(\epsilon_1'')^2} = (\alpha^{1+\tau})^3,$$

but then $\sqrt[3]{n} \in K_1$ which is a contradiction. Hence the equation has no solution.

Lemma 6.2 *Let K_1 be a Type III field then the equation*

$$e^3 = \zeta^a\epsilon_1^b(\epsilon_1')^c$$

has no solution in L unless $a \equiv b \equiv c \equiv 0 (\text{mod } 3)$.

Proof: Suppose the equation $e^3 = \zeta^a\epsilon_1^b(\epsilon_1')^c$ had a solution with not all of $a, b, c \equiv 0$ (mod 3). If b and $c \equiv 0$ (mod 3) then the 9^{th} roots of unity would be in L so it must be that one of b or c is not divisible by 3. Assume without loss of generality that b is not divisible by 3. Then

$$(e^{1-\sigma_2\tau})^3 = \zeta^{2a}\left(\frac{\epsilon_1}{\epsilon_1''}\right)^b,$$

but by Theorem X of [9] this equation can have no solution when K_1 is Type III, contradiction. Thus $a \equiv b \equiv c \equiv 0$ (mod 3).

Lemma 6.3 *Let K_1 and K_2 be of Type I and $\sqrt[3]{\epsilon_1}$, $\sqrt[3]{\epsilon_2} \in K$. If the equation*

$$e^3 = \zeta^i \sqrt[3]{\epsilon_1}^a \sqrt[3]{\epsilon_1'}^b \sqrt[3]{\epsilon_2}^c \sqrt[3]{\epsilon_2'}^d$$

has a solution in L then none of a, b, c, and d are divisible by 3 and $b \equiv 2a, d \equiv 2c \pmod 3$

Proof: We know from Corollary I of Theorem III of [9] that $\hat{E}^3 \subset \hat{\epsilon}$. Then, since $e \in \hat{E}$, we know that

$$e^3 = \zeta^i \epsilon_1^r u_1^s \epsilon_2^t u_2^v \tag{6.1}$$

and by hypothesis $e^3 = \zeta^i \sqrt[3]{\epsilon_1}^a \sqrt[3]{\epsilon_1'}^b \sqrt[3]{\epsilon_2}^c \sqrt[3]{\epsilon_2'}^d$ so

$$e^3 = \zeta^i \sqrt[3]{\epsilon_1}^a u_1^{-b} \sqrt[3]{\epsilon_1}^b \sqrt[3]{\epsilon_2}^c u_2^{-d} \sqrt[3]{\epsilon_2}^d = \zeta^i \sqrt[3]{\epsilon_1}^{a+b} u_1^{-b} \sqrt[3]{\epsilon_2}^{c+d} u_2^{-d}. \tag{6.2}$$

Then equations (6.1) and (6.2) must be equal to each other so the exponents must be equivalent mod 3, thus $a+b = 3r$ and $c+d = 3t \implies b \equiv 2a \pmod 3$ and $d \equiv 2c \pmod 3$ and the result is proved.

Lemma 6.4 *Let K_1 and K_2 be of Type I or IV. If $e^3 = e_1 e_2$ has a solution in L but not in K where $e_i \in K_i$ and e_i not a cube in K_i for $i = 1$ or 2, then $\sqrt[3]{\epsilon_1}$, $\sqrt[3]{\epsilon_2} \in K$.*

Proof: Note that for $e_1 \in K_1$ we can write $e_1 = \zeta^i u_1^a \epsilon_1^b$ for some integers $0 \le a, b \le 2$ (not both zero). Now

$$(e^{1-\sigma_2})^3 = e_1^{1-\sigma_2} = \frac{u_1^a \epsilon_1^{a+b}}{u_1^a {\epsilon_1'}^b} = \epsilon_1^a (\epsilon_1^2 \epsilon_1'')^b = \epsilon_1^a u_1^{3b}.$$

Thus the equation $E^3 = \epsilon_1^a$ has a solution in L and then $\sqrt[3]{\epsilon_1} \in L$ so $\sqrt[3]{\epsilon_1} \in K$.

Assume $a = 0$. We can write $e_2 = \zeta^i u_2^c \epsilon_2^d$ for some integers $0 \le c, d \le 2$ (not both zero). If $c \ne 0$ then by considering $e^{1-\sigma_1}$ we see that $\sqrt[3]{\epsilon_2} \in K$.

If $c = 0$ then $e^3 = \zeta^i \epsilon_1^b \epsilon_2^d$ so $(e^{1+\tau})^3 = \epsilon_1^{2b} \epsilon_2^{2d}$. Hence $e^3 = e_1 e_2$ has a solution in K, contradictory to hypothesis so $c \ne 0$ and we have $e^3 = \zeta^i \epsilon_1^b u_2^c \epsilon_2^d = \zeta^i \epsilon_1^b u_2^c \sqrt[3]{\epsilon_2}^{3d}$. Moving all cubes to the left of the equation we see that $E^3 = \zeta^i \epsilon_1^b u_2^c$ has a solution in L. Conjugating with τ and multiplying we get $(E^{1+\tau})^3 = \epsilon_1^{2b} \epsilon_2^c = \epsilon_1^{2b} \sqrt[3]{\epsilon_2}^{3c}$ so $\sqrt[3]{\epsilon_1} \in K$. Thus $\sqrt[3]{\epsilon_1}$, $\sqrt[3]{\epsilon_2} \in K$

Theorem 6.5 *Under the hypotheses of Lemma 6.4, either $e^3 = \zeta^t u_1 u_2$ or $e^3 = \zeta^t u_1 u_2^2$ with $t = 0$, 1 or 2 has a solution in L.*

Proof: Since $\sqrt[3]{\epsilon_1}$ and $\sqrt[3]{\epsilon_2}$ are in K then $e^3 = \zeta^t u_1^a u_2^b$ with $0 \le$ a,b ≤ 2 and not both zero. Lemma 6.1 shows that neither can be zero, so the theorem follows.

There is a special ideal in the integers of L, called the different of O_L (with respect to O_{K_i}) which is divisible by exactly those primes which are ramified over O_L (see [8]). The term "different" comes from the idea of the derivative: If $L = K_i(\sqrt[3]{\epsilon_i})$ then $f(x) = x^3 - \epsilon_i$ is the irreducible polynomial in $K_i[x]$ for the extension and its derivative is $f'(x) = 3x^2$. The number different of ϵ_i for the extension L/K_i is then $f'(\sqrt[3]{\epsilon_i}) = 3\sqrt[3]{\epsilon_i^2}$ and the different for L/K_i is the GCD of all the number differents of integers in L.

Lemma 6.6 *If* $\sqrt[3]{\epsilon_1}, \sqrt[3]{\epsilon_2}, \sqrt[3]{\epsilon_3} \in K$ *then* $\sqrt[3]{3} \in K$

Proof: Since $\sqrt[3]{\epsilon_1}, \sqrt[3]{\epsilon_2}, \sqrt[3]{\epsilon_3} \in K$ then $L = \mathbb{Q}(\sqrt[3]{m_1}, \sqrt[3]{m_2}, \zeta) = K_i(\sqrt[3]{\epsilon_i})$ for $i = 1,\ 2,\ 3$. The only primes that ramify are those that divide the different of L over K_i. Since the different of L over K_i divides 3 then only prime divisors of 3 can ramify in the extension L/K_i.

Suppose $p|m_1$, $p \nmid m_2$ and $p \neq 3$ then P must ramify from K_2 to L where P is a prime divisor of p in K_2 but as shown only divisors of 3 can ramify from K_i to L so P does not exist. Thus $p|m_1$ and $p|m_2$ and hence m_1 and m_2 have the same prime divisors, except 3. A similar argument will show that m_3 will have the same prime divisors, except 3, as m_1 and m_2. Since any prime divides exactly three of $m_1,\ m_2,\ m_3$ and m_4 or none of them, then $p \nmid m_4$ so $\sqrt[3]{3} \in K$.

Lemma 6.7 *If* K_1*,* K_2 *and* K_3 *are Type IV and* $e = \sqrt[3]{\epsilon_1\epsilon_2^a\epsilon_3^b} \in K$ *then we can write* $e = \frac{B}{B^{\sigma_4}}$ *with* $B \in L$ *and* $N_{L/K_4}[B]$ *is not a unit in* K_4*.*

Proof: Since K_1, K_2, K_3 are Type IV $1 \leq a, b \leq 2$.

Case 1: K_4 Type I or IV

Suppose $N[B] \approx \{\text{unit in } K_4\} \approx \epsilon_4^c u_4^d$ then using the usual technique we can get $\frac{e}{e^{\sigma_4^2}} = \frac{B^3}{N[B]}$. So we would be able to get a solution to

$$E^3 \approx \frac{e}{e^{\sigma_4^2}}\epsilon_4^c u_4^d \approx \sqrt[3]{\left(\frac{\epsilon_1}{\epsilon_1'}\right)\left(\frac{\epsilon_2}{\epsilon_2'}\right)^a\left(\frac{\epsilon_3}{\epsilon_3''}\right)^b}\epsilon_4^c u_4^d \approx u_1^2 u_2^{2a} u_3^b \epsilon_4^c u_4^d. \tag{6.3}$$

Then if we conjugate with σ_1

$$(E^3)^{\sigma_1} \approx u_1^2(u_2^{2a})^{\sigma_1}(u_3^b)^{\sigma_1}(\epsilon_4')^c(u_4^d)^{\sigma_1} \approx u_1^2\epsilon_2^{-2a}(u_2^{2a})\epsilon_3^{-b}(u_3^b)(\epsilon_4)^c\epsilon_4^{-d}(u_4^d)$$

and take the quotient then we get

$$\left(\frac{E}{E^{\sigma_1}}\right)^3 \approx \epsilon_2^{2a}\epsilon_3^b\epsilon_4^d.$$

Thus there is no solution unless K_4 is of Type IV.

If K_4 is of Type IV we show in Case (3A) of Theorem 6.9 and in Corollary 6.10.2 that $N(B)$ is not a unit.

Case 2: Suppose K_4 is Type III. Then $N[B] \approx \{\text{unit in } K_4\} \approx \epsilon_4^c(\epsilon_4')^d$ and equation (6.3) becomes:

$$E^3 \approx u_1^2 u_2^{2a} u_3^b \epsilon_4^c(\epsilon_4')^d.$$

Then if we conjugate with σ_2 (Note: $\sigma_2 = \sigma_1^2$ on K_4.)

$$(E^3)^{\sigma_2} \approx u_1^2 \epsilon_1^{-2}(u_2^{2a})\epsilon_3^{-b}(u_3^b)(\epsilon_4'')^c(\epsilon_4)^d \approx u_1^2 \epsilon_1^{-2}(u_2^{2a})\epsilon_3^{-b}(u_3^b)(\epsilon_4)^{2c+d}(\epsilon_4')^c$$

and take the quotient

$$\left(\frac{E}{E^{\sigma_2}}\right)^3 \approx \epsilon_1^2\epsilon_3^b \frac{\epsilon_4^c {\epsilon_4'}^d}{(\epsilon_4')^c \epsilon_4^{2c+d}} \approx \epsilon_1^2\epsilon_3^b \epsilon_4^{-(c+d)} {\epsilon_4'}^{d-c}.$$

By Theorem X of [9], $c + d \equiv 0 \pmod 3$ and $d - c \equiv 0 \pmod 3$ but $(E^3) = \epsilon_1^2\epsilon_3^b$ has no solution so then N[B] $\not\approx$ {unit in K_4}.

Lemma 6.8 *If K_1, K_2 and K_3 are Type III and $e = \sqrt[3]{\epsilon_1\epsilon_2^a\epsilon_3^b} \in K$ then we can write $e = \frac{B}{B^{\sigma_4}}$ with $B \in L$ and $N_{L/K_4}[B]$ not a unit in K_4.*

Proof: Since K_1, K_2, K_3 are Type III $1 \leq a, b \leq 2$.

Case 1: K_4 Type I or IV

Suppose $N[B] \approx$ {unit in K_4} $\approx \epsilon_4^c u_4^d$. Following a similar procedure to the proof of Lemma 6.7 we can find a solution to:

$$E^3 \approx \sqrt[3]{\left(\frac{\epsilon_1}{\epsilon_1'}\right)\left(\frac{\epsilon_2}{\epsilon_2'}\right)^a\left(\frac{\epsilon_3}{\epsilon_3''}\right)^b}\,\epsilon_4^c u_4^d.$$

As before we conjugate with σ_1

$$(E^3)^{\sigma_1} \approx \sqrt[3]{\left(\frac{\epsilon_1}{\epsilon_1'}\right)\left(\frac{\epsilon_2'}{\epsilon_2''}\right)^a\left(\frac{\epsilon_3'}{\epsilon_3}\right)^b}\,(\epsilon_4')^c(u_4')^d$$

and take the quotient to get

$$\left(\frac{E}{E^{\sigma_1}}\right)^3 \approx \sqrt[3]{\left(\frac{\epsilon_2\epsilon_2''}{{\epsilon_2'}^2}\right)^a\left(\frac{\epsilon_3^2}{\epsilon_3''\epsilon_3'}\right)^b}\,\frac{\epsilon_4^c u_4^d}{{\epsilon_4'}^c {u_4'}^d} \approx (\epsilon_2')^{2a}\epsilon_3^b\epsilon_4^d.$$

This has no solution by Corollary III to Theorem X of [9].

Case 2: Suppose K_4 is Type III. Then $N[B] \approx$ {unit in K_4} $\approx \epsilon_4^c(\epsilon_4')^d$ and with the same procedure we get that:

$$\left(\frac{E}{E^{\sigma_1}}\right)^3 \approx (\epsilon_2')^{2a}\epsilon_3^b\epsilon_4^{c+d}(\epsilon_4')^{2d-c}$$

Which has no solution by Corollary III of Theorem X of [9].

We would like to be able to identify when we can find units in L which are not products of units in the subfields. In [9] there are criteria outlined for when a unit in L, which is not a product of units in the subfields, can exist for Type III fields and the following theorem will more specifically define the criteria for all Types of subfields.

Theorem 6.9 *Let $\hat{E}_0 = \hat{\epsilon} \prod_{i=1}^{4} \hat{\epsilon}_i$ then $[\hat{E} : \hat{E}_0] = 3^{b^*}$ where $b^* \leq 2$. Furthermore b^* can not be 2 when K is Kind 2. Moreover, if K is Kind 3 or 4 and $b^* = 2$ then at least three of the cubic subfields are Type III or $\sqrt[3]{\epsilon_i} \in K$ for some i.*

Proof: The group of units generated by $\hat{E}_0 = \hat{\epsilon} \prod_{i=1}^{4} \hat{\epsilon}_i$ represent all the units in L that are products of units in the subfields of L. Let $e_1, e_2, \cdots, e_8$ be a basis for $\hat{E}_0$.

We are concerned with the number of solutions to an equation of the form $e^3 = \prod_{i=1}^{n} e_i$, where $e_i \in \hat{E}_0$ and $1 \leq n \leq 8$. The basis $\hat{E}_0$ is dependent on the Type of the four real subfields so it is necessary to consider the possible cases. Theorem IV of [9] gives us 4 possible choices for the basis $(\hat{e} : \hat{e}_0)$ (the Kind of K) and Theorem XII of [9] tells us the subcases for each Kind.

Let $\{E_1, E_2, \cdots, E_8\}$ be a basis for $\hat{E}$. We know from Corollary I of Theorem III in [9] that $\hat{E}^3 \subset \hat{\epsilon}$ and $\hat{E}^3 \subset \hat{\epsilon}_i$ for $i = 1, 2, 3, 4$ so we can write

$(e_1, e_2, \cdots, e_8) = (E_1, E_2, \cdots, E_8)A$ where A is an 8x8 matrix (the exponents for the E_i's). We can put A in upper triangular form with elementary row operations that act as a change of basis for $\hat{E}$. We can assume that A is in upper triangular form with diagonal entries either 1 or 3.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{18} \\ 0 & a_{22} & a_{23} & \cdots & \vdots \\ 0 & 0 & a_{33} & \cdots & \vdots \\ \vdots & \vdots & 0 & \ddots & a_{78} \\ 0 & \cdots & \cdots & 0 & a_{88} \end{pmatrix} \qquad [\hat{E} : \hat{E}_0] = det(A) = 3^{b^*} \text{ where } b^* \leq 8$$

We need to solve the system of 8 equations but many of our entries are 0. In all cases $e_1 = \epsilon_1$ and $e_2 = \epsilon_1'$ or u_1 as K_1 is Type III or not. Our first equation is $e_1 = E_1^{a_{11}}$ but Theorem V in [9] shows us that this can only have a solution when $a_{11} = 1$ because no noncube of a unit of K_1 is in $\hat{E}^3$. Similarly $a_{22} = 1$ so we know that $E_1 = e_1$ and $E_2 = e_2$.

Note: Once we know that $a_{22} = 1$ we can let $a_{12} = 0$ by row reduction.

We now proceed to consider the individual cases. For the basis $\hat{E}_0$ we will always assume that the basis elements are are in the order $\{e_1, e_2, \cdots, e_8\}$.

Case 1: Kind 1: $(\hat{e} : \hat{e}_0)$**= 27**

All fields are Type I or IV: $\hat{E}_0 =< \epsilon_1, u_1, \sqrt[3]{\epsilon_1^a \epsilon_2}, \sqrt[3]{\epsilon_1^a \epsilon_2'}, \sqrt[3]{\epsilon_1^b \epsilon_3}, \sqrt[3]{\epsilon_1^c \epsilon_4}, \sqrt[3]{\epsilon_1^b \epsilon_3'}, \sqrt[3]{\epsilon_1^c \epsilon_4'}, >$

We already have the first two columns of A from above:

$$A = \begin{pmatrix} 1 & 0 & a_{13} & \cdots & a_{18} \\ 0 & 1 & a_{23} & \cdots & \vdots \\ 0 & 0 & a_{33} & \cdots & \vdots \\ \vdots & \vdots & 0 & \ddots & a_{78} \\ 0 & \cdots & \cdots & 0 & a_{88} \end{pmatrix}$$

For convenience of notation when we are looking at an equation of the form $e_i = e_1^{a_{1i}} e_2^{a_{2i}} \cdots e_{j-1}^{a_{(j-1)i}} E_i^{a_{ji}}$ we will drop the second subscript and write $e_i = e_1^{a_1} e_2^{a_2} \cdots e_{j-1}^{a_{j-1}} E_i^{a_j}$. Also, when we do calculations we will be looking at the equation as $E_i^{a_j} = \ldots$ but we will continue to write the exponents as positive.

Suppose $a_{33} = 3$. Then

$$E_3^3 = e_1^{a_1} e_2^{a_2} e_3 = \epsilon_1^{a_1} u_1^{a_2} \sqrt[3]{\epsilon_1^a \epsilon_2}$$

but Lemma 6.4 says that if the equation has a solution then $\sqrt[3]{\epsilon_1} \in K$ but it is not. So $a_{33} = 1$ and $E_3 = e_3$. A similar argument shows that $a_{44} = 1$ and $E_4 = e_4$.

Suppose $a_{55} = 3$. Then

$$E_5^3 = e_1^{a_1} e_2^{a_2} e_3^{a_3} e_4^{a_4} e_5 = \epsilon_1^{a_1} u_1^{a_2} \sqrt[3]{\epsilon_1^a \epsilon_2}^{a_3} \sqrt[3]{\epsilon_1^a \epsilon_2'}^{a_4} \sqrt[3]{\epsilon_1^b \epsilon_3}. \tag{6.4}$$

Multiplying (6.4) by its complex conjugate gives

$$\begin{aligned}
(E_5\overline{E_5})^3 &= \epsilon_1^{2a_1} u_1^{a_2} \overline{u}_1^{a_2} \sqrt[3]{\epsilon_1^a \epsilon_2}^{2a_3} \sqrt[3]{\epsilon_1^a \epsilon_2'}^{a_4} \sqrt[3]{\epsilon_1^a \epsilon_2''}^{a_4} \sqrt[3]{\epsilon_1^b \epsilon_3}^{2} \\
&= \epsilon_1^{2a_1+a_2} \sqrt[3]{\epsilon_1^a \epsilon_2}^{2a_3} \sqrt[3]{\epsilon_1^{2a} \epsilon_2' \epsilon_2''}^{a_4} \sqrt[3]{\epsilon_1^b \epsilon_3}^{2} \qquad (\overline{u}_i = \epsilon_i u_i^{-1}) \\
&= \epsilon_1^{2a_1+a_2} \sqrt[3]{\epsilon_1^a \epsilon_2}^{2a_3-a_4} \sqrt[3]{\epsilon_1^{3a} \epsilon_2 \epsilon_2' \epsilon_2''}^{a_4} \sqrt[3]{\epsilon_1^b \epsilon_3}^{2} \\
&= \epsilon_1^{2a_1+a_2+a\cdot a_4} \sqrt[3]{\epsilon_1^a \epsilon_2}^{2a_3-a_4} \sqrt[3]{\epsilon_1^b \epsilon_3}^{2}
\end{aligned}$$

All exponents must be divisible by 3 but 2 is clearly not, therefore $a_{55} \neq 3$ and so $a_{55} = 1$ and $E_5 = e_5$.

Suppose $a_{66} = 3$, then

$$E_6^3 = \epsilon_1^{a_1} u_1^{a_2} \sqrt[3]{\epsilon_1^a \epsilon_2}^{a_3} \sqrt[3]{\epsilon_1^a \epsilon_2'}^{a_4} \sqrt[3]{\epsilon_1^b \epsilon_3}^{a_5} \sqrt[3]{\epsilon_1^c \epsilon_4}. \tag{6.5}$$

Multiplying (6.5) by its complex conjugate gives $(E_6\overline{E_6})^3 = \epsilon_1^{2a_1+a_2+a\cdot a_4} \sqrt[3]{\epsilon_1^a \epsilon_2}^{2a_3-a_4} \sqrt[3]{\epsilon_1^b \epsilon_3}^{2a_5} \sqrt[3]{\epsilon_1^b \epsilon_4}^{2}$. All exponents must be divisible by 3, therefore $a_{66} \neq 3$. So $a^{66} = 1$ and $E_6 = e_6$.

So $A = \begin{pmatrix} 1 & 0 & 0 & \cdots & a_{18} \\ 0 & 1 & 0 & \cdots & \vdots \\ 0 & 0 & \ddots & \cdots & \vdots \\ \vdots & \vdots & 0 & a_{77} & a_{78} \\ 0 & \cdots & \cdots & 0 & a_{88} \end{pmatrix}$ and $\det(\mathrm{A}) = 3^{b^*}$ where $b^* \leq 2$.

Case 2: Kind 2: $(\hat{e} : \hat{e}_0)$**= 9**

(A) All fields are Type I or IV: $\hat{E}_0 =< \epsilon_1, u_1, \epsilon_2, u_2, \sqrt[3]{\epsilon_1^{c_1}\epsilon_2^{c_2}\epsilon_3}, \sqrt[3]{\epsilon_1^{c_1}{\epsilon'_2}^{c_2}\epsilon'_3}, \sqrt[3]{\epsilon_1^{b_1}\epsilon_2^{b_2}\epsilon_4}, \sqrt[3]{\epsilon_1^{b_1}{\epsilon'_2}^{b_2}\epsilon'_4} >$

It is clear from Lemma 6.4 that $a_{11} = a_{22} = a_{33} = a_{44} = 1$ so suppose $a_{55} = 3$. Then $E_5^3 = \epsilon_1^{a_1}u_1^{a_2}\epsilon_2^{a_3}u_2^{a_4}\sqrt[3]{\epsilon_1^{c_1}\epsilon_2^{c_2}\epsilon_3}$ and by taking the complex conjugates and multiplying we get

$$(E_5\overline{E}_5)^3 = \epsilon_1^{2a_1+a_2}\epsilon_2^{2a_3+a_4}\sqrt[3]{\epsilon_1^{c_1}\epsilon_2^{c_2}\epsilon_3}^{\,2}.$$

All exponents must be divisible by 3 but 2 is clearly not, therefore $a_{55} \neq 3$ and so $a_{55} = 1$ and $E_5 = e_5$.

Suppose $a_{66} = 3$. Then

$$E_6^3 = \epsilon_1^{a_1}u_1^{a_2}\epsilon_2^{a_3}u_2^{a_4}\sqrt[3]{\epsilon_1^{c_1}\epsilon_2^{c_2}\epsilon_3}^{\,a_5}\sqrt[3]{\epsilon_1^{c_1}{\epsilon'_2}^{c_2}\epsilon'_3} \tag{6.6}$$

and multiplying by the complex conjugate gives

$$\begin{aligned}(E_6\overline{E}_6)^3 &= \epsilon_1^{2a_1+a_2}\epsilon_2^{2a_3+a_4}\sqrt[3]{\epsilon_1^{c_1}\epsilon_2^{c_2}\epsilon_3}^{\,2a_5}\sqrt[3]{\epsilon_1^{c_1}{\epsilon'_2}^{c_2}\epsilon'_3}\sqrt[3]{\epsilon_1^{c_1}{\epsilon''_2}^{c_2}\epsilon''_3}\\ &= \epsilon_1^{2a_1+a_2}\epsilon_2^{2a_3+a_4}\sqrt[3]{\epsilon_1^{c_1}\epsilon_2^{c_2}\epsilon_3}^{\,2a_5-1}\sqrt[3]{\epsilon_1^{3c_1}(\epsilon_2\epsilon'_2\epsilon''_2)^{c_2}(\epsilon_3\epsilon'_3\epsilon''_3)}\\ &= \epsilon_1^{2a_1+a_2+c_1}\epsilon_2^{2a_3+a_4}\sqrt[3]{\epsilon_1^{c_1}\epsilon_2^{c_2}\epsilon_3}^{\,2a_5-1}\end{aligned}$$

$$\begin{array}{llll}\implies & 2a_5 - 1 \equiv 0 \pmod 3 & 2a_1 + a_2 + c_1 \equiv 0 \pmod 3 & \text{and } \ 2a_3 + a_4 \equiv 0 \pmod 3\\ \text{so} & a_5 \equiv 2 \pmod 3 & a_2 \equiv a_1 + 2c_1 \pmod 3 & a_3 \equiv a_4 \pmod 3.\end{array}$$

Note: $\sqrt[3]{\epsilon_1^{c_1}\epsilon_2^{c_2}\epsilon_3}^{\,2}\sqrt[3]{\epsilon_1^{c_1}{\epsilon'_2}^{c_2}\epsilon'_3} = \sqrt[3]{\epsilon_1^{3c_1}(\epsilon_2^2\epsilon'_2)^{c_2}(\epsilon_3^2\epsilon'_3)} = \epsilon_1^{c_1}u_2^{c_2}u_3$.

So $E_6^3 = \epsilon_1^{a_1}u_1^{a_1+2c_1}\epsilon_2^{a_3}u_2^{a_3}\epsilon_1^{c_1}u_2^{c_2}u_3$ and conjugating with σ_1 and dividing gives

$$(E_6^{1-\sigma_1})^3 = \frac{\epsilon_2^{a_3}u_2^{a_3}u_2^{c_2}u_3}{{\epsilon'_2}^{a_3}{u'_2}^{a_3}{u'_2}^{c_2}u'_3} \approx \epsilon_2^{a_3+c_2}\epsilon_3$$

From the basis we know that $e^3 = \epsilon_1^{c_1}\epsilon_2^{c_2}\epsilon_3 \in K$ so we can take the quotient and rename to get

$$E^{*3} = \epsilon_1^{-c_1}\epsilon_2^{a_3}$$

where all exponents must be divisible by 3 so $c_1 \equiv a_3 \equiv 0 \pmod 3$. Then (6.6) becomes

$$E_6^3 = \epsilon_1^{a_1}u_1^{a_1}u_2^{c_2}u_3 \tag{6.7}$$

and by conjugating with σ_2 and dividing we get $(E_6^{1-\sigma_2})^3 = \frac{\epsilon_1^{a_1}u_1^{a_1}u_2^{c_2}u_3}{{\epsilon'_1}^{a_1}{u'_1}^{a_1}u_2^{c_2}u'_3} \approx \epsilon_1^{a_1}\epsilon_3$. We know that $e^3 = \epsilon_2^{c_2}\epsilon_3 \in K$ so we can take the quotient and rename to get $(E^*)^3 = \epsilon_1^{a_1}\epsilon_2^{-c_2}$ where all exponents must be divisible by 3 so $a_1 \equiv c_2 \equiv 0 \pmod 3$. Then (6.7) becomes

$$E_6^3 = \epsilon_3^{a_5}u_3$$

and this equation can have no solution by Lemma 6.1 so $a_{66} = 1$ and $E_6 = e_6$.

Suppose $a_{77} = 3$, then

$$E_7^3 = e_1^{a_1} e_2^{a_2} e_3^{a_3} e_4^{a_4} e_5^{a_5} e_6^{a_6} \sqrt[3]{\epsilon_1^{b_1} \epsilon_2^{b_2} \epsilon_4}. \tag{6.8}$$

Multiplying (6.8) by its complex conjugate gives $(E_7\overline{E}_7)^3 = e_1^{2a_1+a_2+a_6} e_3^{2a_3+a_4} e_5^{2a_5-a_6} \sqrt[3]{\epsilon_1^{b_1}\epsilon_2^{b_2}\epsilon_4}^{\,2}$. All exponents must be divisible by 3 so we get a contradiction. Thus $a_{77} = 1$ and $E_7 = e_7$.

So $A = \begin{pmatrix} 1 & 0 & 0 & \cdots & a_{18} \\ 0 & 1 & 0 & \cdots & \vdots \\ 0 & 0 & \ddots & \cdots & \vdots \\ \vdots & \vdots & 0 & 1 & a_{78} \\ 0 & \cdots & \cdots & 0 & a_{88} \end{pmatrix}$ and $\det(A) = 3^{b^*}$ where $b^* \leq 1$.

(B) Three Type I and one Type III: $\hat{E}_0 =< \epsilon_1, u_1, \sqrt[3]{\epsilon_1^b \epsilon_2}, \sqrt[3]{\epsilon_1^b \epsilon_2'}, \sqrt[3]{\epsilon_1^c \epsilon_3}, \sqrt[3]{\epsilon_1^c \epsilon_3'}, \epsilon_4, \epsilon_4' >$

Here we let k_4 be Type III.

Clearly we can take $a_{11} = a_{22} = a_{33} = a_{44} = a_{55} = a_{66} = 1$. Suppose $a_{77} = 3$. Then $e_0 = \epsilon_1^{a_1} u_1^{a_2} \sqrt[3]{\epsilon_1^b\epsilon_2}^{a_3} \sqrt[3]{\epsilon_1^b\epsilon_2'}^{a_4} \sqrt[3]{\epsilon_1^c\epsilon_3}^{a_5} \sqrt[3]{\epsilon_1^c\epsilon_3'}^{a_6}$ and $E_7^3 = e_0\epsilon_4$, so $(E_7\overline{E}_7)^3 = e_0\overline{e}_0\epsilon_4^2$. All exponents must be divisible by 3 so $a_{77} = 1$ and $E_7 = e_7$. The matrix A is the same as in case(**2A**) so $\det(A) = 3^{b^*}$ where $b^* \leq 1$.

(C) Two Type I and two Type III:

In this case k_1 and k_2 are Type III.

Suppose $N(B_1) \nsim N(B_2)$**:** $\hat{E}_0 =< \epsilon_1, \epsilon_1', \epsilon_2, \epsilon_2', \sqrt[3]{\epsilon_3}, u_3, \sqrt[3]{\epsilon_4}, u_4 >$

We can take $a_{11} = a_{22} = a_{33} = a_{44} = a_{55} = a_{66} = 1$. Suppose $a_{77} = 3$. Let $e_0 = \epsilon_1^{a_1}{\epsilon_1'}^{a_2}\epsilon_2^{a_3}{\epsilon_2'}^{a_4}\sqrt[3]{\epsilon_3}^{a_5}u_3^{a_6}$ and $E_7^3 = e_0\sqrt[3]{\epsilon_4}$, so $(E_7\overline{E}_7)^3 = e_0\overline{e}_0\sqrt[3]{\epsilon_4}^2$. All exponents must be divisible by 3 so $a_{77} = 1$ and $E_7 = e_7$.

Suppose $a_{88} = 3$. Then $E_8^3 = \epsilon_1^{a_1}{\epsilon_1'}^{a_2}\epsilon_2^{a_3}{\epsilon_2'}^{a_4}\sqrt[3]{\epsilon_3}^{a_5}u_3^{a_6}\sqrt[3]{\epsilon_4}^{a_7}u_4$ and

$$(E_8\overline{E}_8)^3 = \epsilon_1^{2a_1-a_2}\epsilon_2^{2a_3-a_4}\sqrt[3]{\epsilon_3}^{2a_5+3a_6}\sqrt[3]{\epsilon_4}^{2a_7+3}.$$

All the exponents must be divisible by 3 so the following relations are clear

$$a_2 \equiv 2a_1 \pmod 3, \quad a_4 \equiv 2a_3 \pmod 3, a_5 \equiv a_7 \equiv 0 \pmod 3$$

and $E_8^3 = \epsilon_1^{a_1}{\epsilon_1'}^{2a_1}\epsilon_2^{a_3}{\epsilon_2'}^{2a_3}u_3^{a_6}u_4$ so $\det(A) = 3^{b^*}$ where $b^* \leq 1$.

Suppose $N(B_1) \sim N(B_2)$**:** This case does not exist by Theorem 5.4.

Case 3: Kind 3: $(\hat{e} : \hat{e}_0) = 3$

(A) All fields are Type I or IV: $\hat{E}_0 =< \epsilon_1, u_1, \epsilon_2, u_2, \epsilon_3, u_3, \sqrt[3]{\epsilon_1^{c_1}\epsilon_2^{c_2}\epsilon_3^{c_3}\epsilon_4}, \sqrt[3]{\epsilon_1^{c_1}{\epsilon_2'}^{c_2}{\epsilon_3'}^{c_3}\epsilon_4'} >$

We know that $a_{11} = a_{22} = a_{33} = a_{44} = 1$ by previous work and $a_{55} = 1$ by a similar proof to Case 2. Suppose $a_{66} = 3$, then

$$E_6^3 = \epsilon_1^{a_1} u_1^{a_2} \epsilon_2^{a_3} u_2^{a_4} \epsilon_3^{a_5} u_3. \tag{6.9}$$

Conjugating (6.9) with σ_1 and dividing gives $(E_6^{1-\sigma_1})^3 = \frac{\epsilon_2^{a_3} u_2^{a_4} \epsilon_3^{a_5} u_3}{\epsilon_2'^{a_3} u_2'^{a_3} \epsilon_3'^{a_5} u_3'} \approx \epsilon_2^{a_4}\epsilon_3$. Once again we need all exponents divisible by 3 but the exponent on ϵ_3 is not. This is a contradiction so $a_{66} = 1$ and $E_6 = e_6$.

Suppose $a_{77} = 3$. Then

$$E_7^3 = \epsilon_1^{a_1} u_1^{a_2} \epsilon_2^{a_3} u_2^{a_4} \epsilon_3^{a_5} u_3^{a_6} \sqrt[3]{\epsilon_1^{c_1}\epsilon_2^{c_2}\epsilon_3^{c_3}\epsilon_4}$$

and

$$(E_7\overline{E}_7)^3 = \epsilon_1^{2a_1+a_2}\epsilon_2^{2a_3+a_4}\epsilon_3^{2a_5+a_6} \sqrt[3]{\epsilon_1^{c_1}\epsilon_2^{c_2}\epsilon_3^{c_3}\epsilon_4}^{\,2}.$$

All exponents must be divisible by 3 so we have a contradiction. Thus $a_{77} = 1$ and $E_7 = e_7$.

Suppose $a_{88} = 3$. Then

$$E_8^3 = g_1 g_2 g_3 \sqrt[3]{\epsilon_1^{c_1}\epsilon_2^{c_2}\epsilon_3^{c_3}\epsilon_4}^{\,a_7} \sqrt[3]{\epsilon_1^{c_1}\epsilon_2'^{c_2}\epsilon_3'^{c_3}\epsilon_4'}$$

where $g_i = \epsilon_i^{a_{2i-1}} u_i^{a_{2i}}$ and $g_i\overline{g}_i = \epsilon_i^{2a_{2i-1}+a_{2i}}$ and

$$\begin{aligned}(E_8\overline{E}_8)^3 &= g_1\overline{g}_1 g_2\overline{g}_2 g_3\overline{g}_3 \sqrt[3]{\epsilon_1^{c_1}\epsilon_2^{c_2}\epsilon_3^{c_3}\epsilon_4}^{\,2a_7} \sqrt[3]{\epsilon_1^{c_1}\epsilon_2'^{c_2}\epsilon_3'^{c_3}\epsilon_4'} \sqrt[3]{\epsilon_1^{c_1}\epsilon_2''^{c_2}\epsilon_3''^{c_3}\epsilon_4''} \\ &= \epsilon_1^{2a_1+a_2+c_1}\epsilon_2^{2a_3+a_4}\epsilon_3^{2a_5+a_6} \sqrt[3]{\epsilon_1^{c_1}\epsilon_2^{c_2}\epsilon_3^{c_3}\epsilon_4}^{\,2a_7-1}\end{aligned}$$

$$\begin{aligned} 2a_7 - 1 &\equiv 0 \pmod 3 \\ a_7 &\equiv 2 \pmod 3 \\ a_3 &\equiv a_4 \pmod 3 \\ a_5 &\equiv a_6 \pmod 3 \\ 2a_1 + a_2 + c_1 &\equiv 0 \pmod 3 \end{aligned}$$

So $E_8^3 = g_1 g_2 g_3 \sqrt[3]{\epsilon_1^{3c_1}(\epsilon_2^2\epsilon_2')^{c_2}(\epsilon_3^2\epsilon_3')^{c_3}(\epsilon_4^2\epsilon_4')}$ and, since $\sqrt[3]{\epsilon_i^2\epsilon_i'} = u_i$ for $i = 2, 3, 4$, then $E_8^3 = g_1 g_2 g_3 \epsilon_1^{c_1} u_2^{c_2} u_3^{c_3} u_4$. Also, if $g_i = \epsilon_i^{s_i} u_i^{t_i}$ then the quotient after conjugating with σ_1 is $\frac{g_i}{g_i^{\sigma_1}} = \frac{g_i}{g_i'} \approx \epsilon_i^{t_i}$ for $i = 2, 3$. Then

$$E_8^3 = \epsilon_1^{a_1+c_1} u_1^{a_1+2c_1} \epsilon_2^{a_3} u_2^{a_3+c_2} \epsilon_3^{a_5} u_3^{a_5+c_3} u_4$$

and conjugating with σ_1 and dividing gives

$$(E_8^{1-\sigma_1})^3 = \frac{\epsilon_2^{a_3} u_2^{a_3+c_2} \epsilon_3^{a_5} u_3^{a_5+c_3} u_4}{\epsilon_2'^{a_3} u_2'^{a_3+c_2} \epsilon_3'^{a_5} u_3'^{a_5+c_3} u_4'} \approx \epsilon_2^{a_3+c_2}\epsilon_3^{a_5+c_3}\epsilon_4.$$

Since $e^3 = \epsilon_1^{c_1}\epsilon_2^{c_2}\epsilon_3^{c_3}\epsilon_4 \in \hat{e}$ we divide both sides by e^3 to and rename to get $E^{*3} = \epsilon_1^{-c_1}\epsilon_2^{a_3}\epsilon_3^{a_5}$, where all exponents must be divisible by 3 so $c_1 \equiv a_3 \equiv a_5 \equiv 0 \pmod 3$. The equation reduces to $E_8^3 =$

$\epsilon_1^{a_1} u_1^{a_1} u_2^{c_2} u_3^{c_3} u_4$ and by conjugating with σ_2 and dividing we get

$$(E_8^{1-\sigma_2})^3 = \frac{\epsilon_1^{a_1} u_1^{a_1} u_2^{c_2} u_3^{c_3} u_4}{\epsilon_1'^{a_1} u_1'^{a_1} u_2'^{c_2} u_3'^{c_3} u_4'} \approx \epsilon_1^{a_1} \epsilon_3^{c_3} \epsilon_4.$$

As above we get that $c_3 \equiv a_1 \equiv 0 \pmod 3$.

If we conjugate with σ_3 we get that $c_2 \equiv 0 \pmod 3$ so $E_8^3 = u_4$. Lemma 6.1 says this can have no solutions so $a_{88} = 1$ and $E_8 = e_8$.

$$\text{So } A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \vdots \\ 0 & 0 & \ddots & \cdots & \vdots \\ \vdots & \vdots & 0 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \quad \text{and } \det(A) = 3^{b^*} \text{ where } b^* = 0.$$

(B) One Type III: $\hat{E}_0 =< \epsilon_1, \epsilon_1', \epsilon_2, u_2, \epsilon_3, u_3, \sqrt[3]{\epsilon_2^{c_2} \epsilon_3^{c_3} \epsilon_4}, \sqrt[3]{\epsilon_2'^{c_2} \epsilon_3'^{c_3} \epsilon_4'} >$

We know that $a_{11} = a_{22} = a_{33} = a_{44} = a_{55} = a_{66} = a_{77} = 1$ from the previous case so suppose $a_{88} = 3$. Then

$$E_8^3 = \epsilon_1^{a_1} \epsilon_1'^{a_2} \epsilon_2^{a_3} u_2^{a_4} \epsilon_3^{a_5} u_3^{a_6} \sqrt[3]{\epsilon_2^{c_2} \epsilon_3^{c_3} \epsilon_4}^{a_7} \sqrt[3]{\epsilon_2'^{c_2} \epsilon_3'^{c_3} \epsilon_4'} \tag{6.10}$$

and $(E_8 \overline{E}_8)^3 = \epsilon_1^{2a_1 - a_2} \epsilon_2^{2a_3 + a_4} \epsilon_3^{2a_5 + a_6} \sqrt[3]{\epsilon_2^{c_2} \epsilon_3^{c_3} \epsilon_4}^{2a_7 - 1}$ so the exponents must all be divisible by 3 and

$$a_2 \equiv 2a_1 \pmod 3, a_4 \equiv a_3 \pmod 3, a_6 \equiv a_5 \pmod 3 \text{ and } a_7 \equiv 2 \pmod 3$$

Then equation (6.10) becomes $E_8^3 = \epsilon_1^{a_1} \epsilon_1'^{2a_1} \epsilon_2^{a_3} u_2^{a_3} \epsilon_3^{a_5} u_3^{a_5} \sqrt[3]{\epsilon_2^{c_2} \epsilon_3^{c_3} \epsilon_4}^2 \sqrt[3]{\epsilon_2'^{c_2} \epsilon_3'^{c_3} \epsilon_4'}$ and

$$(E_8^{1-\sigma_1})^3 = \frac{\epsilon_2^{a_3} u_2^{a_3} \epsilon_3^{a_5} u_3^{a_5} \sqrt[3]{\epsilon_2^{c_2} \epsilon_3^{c_3} \epsilon_4}^2 \sqrt[3]{\epsilon_2'^{c_2} \epsilon_3'^{c_3} \epsilon_4'}}{\epsilon_2'^{a_3} u_2'^{a_3} \epsilon_3'^{a_5} u_3'^{a_5} \sqrt[3]{\epsilon_2'^{c_2} \epsilon_3'^{c_3} \epsilon_4'}^2 \sqrt[3]{\epsilon_2''^{c_2} \epsilon_3''^{c_3} \epsilon_4''}} \approx \epsilon_2^{a_3 + c_2} \epsilon_3^{a_5 + c_3} \epsilon_4.$$

Since $e^3 = \epsilon_2^{c_2} \epsilon_3^{c_3} \epsilon_4 \in \hat{e}$ we divide both sides by e^3 to and rename to get $E^{*3} = \epsilon_2^{a_3} \epsilon_3^{a_5}$, where all exponents must be divisible by 3 so $a_3 \equiv a_5 \equiv 0 \pmod 3$. Equation (6.10) reduces to

$$E_8^3 = \epsilon_1^{a_1} \epsilon_1'^{2a_1} \sqrt[3]{\epsilon_2^{c_2} \epsilon_3^{c_3} \epsilon_4}^2 \sqrt[3]{\epsilon_2'^{c_2} \epsilon_3'^{c_3} \epsilon_4'}$$

and by conjugating with σ_2 and dividing we get

$$(E_8^{1-\sigma_2})^3 \approx \epsilon_3^{c_3} \epsilon_4,$$

which has a solution if $c_2 \equiv 0 \pmod 3$. If we conjugate with σ_3 we see that $c_3 \equiv 0 \pmod 3$ so $E_8^3 = \epsilon_1^{a_1} \epsilon_1'^{2a_1} u_4$. Then $\det(A) = 3^{b^*}$ where $b^* \leq 1$.

(C) Two Type III and two Type I or IV:

Suppose $N(B_1) \nsim N(B_2)$**:** $\hat{E}_0 =< \epsilon_1, \epsilon_1', \epsilon_2, \epsilon_2', \epsilon_3, u_3, \sqrt[3]{\epsilon_3^a \epsilon_4}, u_4 >$

In this case k_1 and k_2 are Type III, k_4 is a Type I field and k_3 is Type I if $a \neq 0$.

We know from [9] Theorem X that $E^3 = \epsilon_1^{a_1} {\epsilon_1'}^{a_2} \epsilon_2^{a_3} {\epsilon_2'}^{a_4}$ has no solution in L so $a_{11} = a_{22} = a_{33} = a_{44} = 1$. Suppose that $a_{55} = 3$. Then

$$E_5^3 = \epsilon_1^{a_1} {\epsilon_1'}^{a_2} \epsilon_2^{a_3} {\epsilon_2'}^{a_4} \epsilon_3$$

and if we multiply E_5^3 by it's complex conjugate we get

$$\left(E_5 \overline{E}_5\right)^3 = \epsilon_1^{2a_1 - a_2} \epsilon_2^{2a_3 - a_4} \epsilon_3^2.$$

All exponents must be divisible by 3 so this equation has no solution and $a_{55} = 1$.

We will number the basis elements of $\hat{E}_0$ as e_1 to e_8 and we can find some relations between the conjugates.

$$e_1\overline{e}_1 = e_1^2, \quad e_2\overline{e}_2 = \epsilon_1'\epsilon_1'' = \epsilon_1^{-1} = e_1^{-1}, \quad e_3\overline{e}_3 = e_3^2, \quad e_4\overline{e}_4 = \epsilon_2'\epsilon_2'' = \epsilon_2^{-1} = e_3^{-1},$$
$$e_5\overline{e}_5 = e_5^2, \quad e_6\overline{e}_6 = u_3\bar{u}_3 = \epsilon_3 = e_5, \quad e_7\overline{e}_7 = e_7^2, \quad e_8\overline{e}_8 = u_4\overline{u}_4 = \epsilon_4 = e_7^3 e_5^{-a}$$

Suppose $a_{66} = 3$. Then

$$E_6^3 = \epsilon_1^{a_1} {\epsilon_1'}^{a_2} \epsilon_2^{a_3} {\epsilon_2'}^{a_4} \epsilon_3^{a_5} u_3$$

and

$$(E_6\overline{E}_6)^3 = \epsilon_1^{2a_1 - a_2} \epsilon_2^{2a_3 - a_4} \epsilon_3^{2a_5 + 1}.$$

All exponents must be divisible by 3 so we get the following relations

$$a_2 \equiv 2a_1 \pmod 3, \quad a_4 \equiv 2a_3 \pmod 3, \quad a_5 \equiv 1 \pmod 3.$$

Then

$$E_6^3 = \epsilon_1^{a_1} {\epsilon_1'}^{2a_1} \epsilon_2^{a_3} {\epsilon_2'}^{2a_3} \epsilon_3 u_3.$$

We can also find some relations by conjugating with σ_1.

$$e_3^{1-\sigma_1} = \tfrac{\epsilon_2}{\epsilon_2'} = e_3 e_4^{-1}, \ \ e_4^{1-\sigma_1} = \tfrac{\epsilon_2'}{\epsilon_2''} = e_3 e_4^2, \quad e_5^{1-\sigma_1} = \tfrac{\epsilon_3}{\epsilon_3'} = e_6^3, \quad e_6^{1-\sigma_1} = \epsilon_3 = e_5,$$
$$e_7^{1-\sigma_1} = \sqrt[3]{\frac{\epsilon_3^a\epsilon_4}{{\epsilon_3'}^a\epsilon_4'}} = u_3^a u_4 = e_6^a e_8$$

Then $E_6^{1-\sigma_1} = \epsilon_2^{3a_3} {\epsilon_2'}^{3a_3} u_3^3 \epsilon_3$, which has no solution, so $a_{66} = 1$.

Suppose $a_{77} = 3$. Then

$$E_7^3 = \epsilon_1^{a_1} {\epsilon_1'}^{a_2} \epsilon_2^{a_3} {\epsilon_2'}^{a_4} \epsilon_3^{a_5} u_3^{a_6} \sqrt[3]{\epsilon_3^a \epsilon_4}$$

and

$$(E_7\overline{E}_7)^3 = \epsilon_1^{2a_1 - a_2} \epsilon_2^{2a_3 - a_4} \epsilon_3^{2a_5 + a_6} \sqrt[3]{\epsilon_3^a \epsilon_4}^2.$$

All exponents must be divisible by 3 so $a_{77} = 1$. Suppose $a_{88} = 3$. Then

$$E_8^3 = \epsilon_1^{a_1} {\epsilon'_1}^{a_2} \epsilon_2^{a_3} {\epsilon'_2}^{a_4} \epsilon_3^{a_5} u_3^{a_6} \sqrt[3]{\epsilon_3^a \epsilon_4}^{a_7} u_4$$

and

$$(E_8 \overline{E}_8)^3 = \epsilon_1^{2a_1 - a_2} \epsilon_2^{2a_3 - a_4} \epsilon_3^{2a_5 + a_6 - a} \sqrt[3]{\epsilon_3^a \epsilon_4}^{2a_7+3}.$$

All exponents must be divisible by 3 so the following relations are clear

$$a_2 \equiv 2a_1 \pmod 3, \quad a_4 \equiv 2a_3 \pmod 3, \quad a_6 \equiv a_5 + a \pmod 3 \quad a_7 \equiv 0 \pmod 3.$$

Now for brevity we change to the basis notation e_i and we have that

$$E_8^3 = e_1^{a_1} e_2^{2a_1} e_3^{a_3} e_4^{2a_3} e_5^{a_5} e_6^{a_5+a} e_8$$

and we can conjugate with $1 - \sigma_1$ to get

$$(E_8^{1-\sigma_1})^3 = e_3^{3a_3} e_4^{3a_3} e_6^{3a_5} e_5^{a_5} e_7^3.$$

Then moving all cubes to the left and renaming we get $(E_8^*)^3 = e_5^{a_5} = \epsilon_3^{a_5}$ so $a_5 \equiv 0 \pmod 3$. Now $E_8^3 = e_1^{a_1} e_2^{2a_1} e_3^{a_3} e_4^{2a_3} e_6^{a} e_8$ so if we consider

$$(E_8^{1-\sigma_2})^3 = e_1^{3a_1} e_2^{3a_1} e_5^{-2a} e_6^{3a_5} e_7^3$$

then by moving the cubes to the left and renaming we get $(E_8^*)^3 = e_5^{-2a} = \epsilon_3^{-2a}$ so $a \equiv 0 \pmod 3$. So $E_8^3 = \epsilon_1^{a_1} {\epsilon'_1}^{2a_1} \epsilon_2^{a_3} {\epsilon'_2}^{2a_3} u_4$ and $b^* \leq 1$.

Suppose $N(B_1) \sim N(B_2)$**:** $\hat{E}_0 =< \epsilon_1, \epsilon'_1, \epsilon_2, \epsilon'_2, \epsilon_3, u_3, \sqrt[3]{\epsilon_3^a \epsilon_4}, u_4 >$

In this case k_1 and k_2 are Type III, k_4 is a Type I field and k_3 is Type I if $a \neq 0$.

We know from [9] Theorem X that $e^3 = \zeta^a(\epsilon_1\epsilon_2/\epsilon'_1\epsilon''_2)$ has a solution $e \in L$ so b^* is at least 1. Let $\hat{E}_1 =< \epsilon_1, \epsilon'_1, \epsilon_2, \sqrt[3]{\frac{\epsilon_1\epsilon_2}{\epsilon'_1\epsilon''_2}}, \epsilon_3, u_3, \sqrt[3]{\epsilon_3^a\epsilon_4}, u_4 >$. Then $\hat{E}_0 \subset \hat{E}_1$ and we will show that $\left[\hat{E} : \hat{E}_1\right] \leq 3$, thus $\left[\hat{E} : \hat{E}_0\right] = \left[\hat{E} : \hat{E}_1\right]\left[\hat{E}_1 : \hat{E}_0\right] \leq 3^2$.

We know from above that $a_{11} = a_{22} = a_{33} = 1$. Suppose that $a_{44} = 3$. Then $E_4^3 = \epsilon_1^{a_1} {\epsilon'_1}^{a_2} \epsilon_2^{a_3} \sqrt[3]{\frac{\epsilon_1\epsilon_2}{\epsilon'_1\epsilon''_2}}$ and conjugating with σ_1 gives:

$$(E_4^{1-\sigma_1})^3 = \frac{\epsilon_2^{a_3+1}}{{\epsilon'_2}^{a_3}}$$

which has no solution by [9] Theorem X.

Suppose that $a_{55} = 3$. Then $E_5^3 = \epsilon_1^{a_1} {\epsilon'_1}^{a_2} \epsilon_2^{a_3} \sqrt[3]{\frac{\epsilon_1\epsilon_2}{\epsilon'_1\epsilon''_2}}^{a_4} \epsilon_3$ and multiplying by the complex conjugate gives

$$(E_5 \bar{E}_5)^3 = \epsilon_1^{2a_1 - a_2 + a_4} \epsilon_2^{2a_3 + a_4} \epsilon_3^2$$

which has no solution because all exponents must be divisible by 3 so $a_{55} = 1$.

Suppose that $a_{66} = 3$. Then $E_6^3 = \epsilon_1^{a_1}{\epsilon_1'}^{a_2}\epsilon_2^{a_3}\sqrt[3]{\frac{\epsilon_1\epsilon_2}{\epsilon_1'\epsilon_2''}}^{a_4}\epsilon_3^{a_5}u_3$. Multiplying by the complex conjugate gives

$$(E_6\bar{E}_6)^3 = \epsilon_1^{2a_1-a_2+a_4}\epsilon_2^{2a_3+a_4}\epsilon_3^{2a_5+1}$$

which provides the following relations:

$$a_2 \equiv 2a_1 + a_3 \pmod 3, \quad a_4 \equiv a_3 \pmod 3, \; a_5 \equiv 1 \pmod 3.$$

Now $E_6^3 = \epsilon_1^{a_1}{\epsilon_1'}^{2a_1+a_3}\epsilon_2^{a_3}\sqrt[3]{\frac{\epsilon_1\epsilon_2}{\epsilon_1'\epsilon_2''}}^{a_3}\epsilon_3 u_3$ and we conjugate with σ_1 to get

$$(E_6^{\sigma_1})^3 = \epsilon_1^{a_1}{\epsilon_1'}^{2a_1+a_3}{\epsilon_2'}^{a_3}\sqrt[3]{\frac{\epsilon_1\epsilon_2'}{\epsilon_1'\epsilon_2}}^{a_3}\epsilon_3'u_3'.$$

Taking the quotient gives

$$(E_6^{1-\sigma_1})^3 = \epsilon_2^{2a_3}{\epsilon_2'}^{-a_3}\epsilon_3 u_3^3$$

and all exponents must be divisible by 3 because $\sqrt[3]{\epsilon_3} \notin \hat{E}_1$ so $a_{66} = 1$.

Suppose that $a_{77} = 3$. Then $E_7^3 = \epsilon_1^{a_1}{\epsilon_1'}^{a_2}\epsilon_2^{a_3}\sqrt[3]{\frac{\epsilon_1\epsilon_2}{\epsilon_1'\epsilon_2''}}^{a_4}\epsilon_3^{a_5}u_3^{a_6}\sqrt[3]{\epsilon_3^a\epsilon_4}$. Multiplying by the complex conjugate gives

$$(E_7\bar{E}_7)^3 = \epsilon_1^{2a_1-a_2+a_4}\epsilon_2^{2a_3+a_4}\epsilon_3^{2a_5+a_6}\sqrt[3]{\epsilon_3^a\epsilon_4}^2$$

which has no solution since the exponents must be divisible by 3. Thus $a_{77} = 1$.

Suppose that $a_{88} = 3$. Then $E_8^3 = \epsilon_1^{a_1}{\epsilon_1'}^{a_2}\epsilon_2^{a_3}\sqrt[3]{\frac{\epsilon_1\epsilon_2}{\epsilon_1'\epsilon_2''}}^{a_4}\epsilon_3^{a_5}u_3^{a_6}\sqrt[3]{\epsilon_3^a\epsilon_4}^{a_7}u_4$. Multiplying by the complex conjugate gives

$$(E_8\bar{E}_8)^3 = \epsilon_1^{2a_1-a_2+a_4}\epsilon_2^{2a_3+a_4}\epsilon_3^{2a_5+a_6-a}\sqrt[3]{\epsilon_3^a\epsilon_4}^{2a_7+3}$$

which provides the following relations:

$$a_2 \equiv 2a_1 + a_3 \pmod 3, \quad a_4 \equiv a_3 \pmod 3, \; a_6 \equiv a_5 + a \pmod 3 \; a_7 \equiv 0 \pmod 3.$$

Now $E_8^3 = \epsilon_1^{a_1}{\epsilon_1'}^{2a_1+a_3}\epsilon_2^{a_3}\sqrt[3]{\frac{\epsilon_1\epsilon_2}{\epsilon_1'\epsilon_2''}}^{a_3}\epsilon_3^{a_5}u_3^{a_5+a}u_4$ and we conjugate with σ_1 to get $(E_8^{\sigma_1})^3 = \epsilon_1^{a_1}{\epsilon_1'}^{2a_1+a_3}{\epsilon_2'}^{a_3}\sqrt[3]{\frac{\epsilon_1\epsilon_2'}{\epsilon_1'\epsilon_2}}^{a_3}{\epsilon_3'}^{a_5}{u_3'}^{a_5+a}u_4'$. Taking the quotient and simplifying gives

$$(E_8^{1-\sigma_1})^3 = \epsilon_1^{a_3}{\epsilon_1'}^{-a_3}\epsilon_2^{4a_3}\epsilon_3^{a_5}$$

so $a_3 \equiv a_5 \equiv 0 \pmod 3$.

We are left with $E_8^3 = \epsilon_1^{a_1}{\epsilon_1'}^{2a_1}u_3^a u_4$ and if we conjugate with σ_3 and take the quotient and simplify then

$$(E_8^{1-\sigma_3})^3 = \sqrt[3]{\epsilon_3^a\epsilon_4}^{-3}\epsilon_3^a$$

so $a \equiv 0 \pmod 3$. Thus $E_8^3 = \epsilon_1^{a_1}{\epsilon_1'}^{2a_1} u_4$ and $b^* \leq 2$.

(D) Three Type III and k_4 Type I or IV:

Suppose $N(B_1) \sim \zeta^{c_1} N(B_2) \sim \zeta^{c_2} N(B_3)$**:**

There are two possibilities for this case. There can be a unit in K of the form $\sqrt[3]{\epsilon_4}$ from a Type I field or the three Type III fields produce $\sqrt[3]{\epsilon_1\epsilon_2\epsilon_3}$. The first case doesn't occur because if the principal divisor of k_4 is d then $d \mid 9m_4^2$ and $K = \mathbb{Q}(\sqrt[3]{d}, \sqrt[3]{m_4})$. Since the norms of the B_i's are similar they must all share a common prime divisor $p \equiv 1 \pmod 3$ but p must also divide d which implies that $p \mid m_4$ which is not possible.

Consider the case where the unit in K comes from the Type III fields. Then
$\hat{E}_0 =< \epsilon_1, \epsilon_1', \epsilon_2, \epsilon_2', \sqrt[3]{\epsilon_1\epsilon_2\epsilon_3}, \epsilon_3', \epsilon_4, u_4 >$

We know from [9] Theorem X that $e^3 = \zeta^a(\epsilon_1\epsilon_2/\epsilon_1'\epsilon_2'')$ and $e^3 = \zeta^c(\epsilon_1\epsilon_3/\epsilon_1'\epsilon_3')$ both have solutions $e \in L$ so b^* is at least 2. Let $\hat{E}_1 =< \epsilon_1, \epsilon_1', \epsilon_2, \sqrt[3]{\epsilon_1\epsilon_2\epsilon_3}, \sqrt[3]{\frac{\epsilon_1\epsilon_2}{\epsilon_1'\epsilon_2''}}, \sqrt[3]{\frac{\epsilon_1\epsilon_3}{\epsilon_1'\epsilon_3'}}, \epsilon_4, u_4 >$. Then $\hat{E}_0 \subset \hat{E}_1$ and we will show that $\left[\hat{E} : \hat{E}_1\right] = 1$, thus $\left[\hat{E} : \hat{E}_0\right] = \left[\hat{E} : \hat{E}_1\right]\left[\hat{E}_1 : \hat{E}_0\right] = 3^2$.

The same reasoning as in Case 3 (B) gives us that $a_{11} = a_{22} = a_{33} = a_{44} = 1$. We will again number the basis elements of $\hat{E}_1$ as e_1 to e_8 and we can find some relations between the conjugates.

$$e_1\overline{e}_1 = e_1^2, \qquad e_2\overline{e}_2 = \epsilon_1'\epsilon_1'' = \epsilon_1^{-1} = e_1^{-1}, \qquad e_4\overline{e}_4 = e_4^2,$$
$$e_3\overline{e}_3 = e_3^2, \qquad e_5\overline{e}_5 = \sqrt[3]{\frac{\epsilon_1^2\epsilon_2^2}{\epsilon_1'\epsilon_1''\epsilon_2'\epsilon_2''}} = \epsilon_1\epsilon_2 = e_1e_3, \qquad e_6\overline{e}_6 = e_1^{-1}e_3^{-1}e_4^3,$$
$$e_7\overline{e}_7 = e_7^2, \qquad e_8\overline{e}_8 = u_4\overline{u}_4 = u_4u_4^{-1}\epsilon_4 = e_7$$

Suppose $a_{55} = 3$. Then $E_5^3 = e_1^{a_1}e_2^{a_2}e_3^{a_3}e_4^{a_4}e_5$ and taking the product with the complex conjugate gives:

$$\begin{aligned}(E_5\overline{E}_5)^3 &= e_1^{2a_1}e_1^{-a_2}e_3^{2a_3}(e_4)^{2a_4}e_1e_3 \\ &= e_1^{2a_1-a_2+1}e_3^{2a_3}e_4^{2a_4}.\end{aligned}$$

All exponents must be divisible by 3 so we get the following relations

$$a_2 \equiv 2a_1 + 1 \pmod 3, \quad a_3 \equiv a_4 \equiv 0 \pmod 3.$$

So $E_5^3 = e_1^{a_1}e_2^{2a_1+1}e_5 = \epsilon_1^{a_1}{\epsilon_1'}^{2a_1+1}\sqrt[3]{\frac{\epsilon_1\epsilon_2}{\epsilon_1'\epsilon_2''}}$ which can have no solution in L since $\hat{E}^3 \subset \hat{\epsilon}$ so $a_{55} = 1$. Similarly $a_{66} = 1$.

Suppose $a_{77} = 3$. Then $E_7^3 = e_1^{a_1}e_2^{a_2}e_3^{a_3}e_4^{a_4}e_5^{a_5}e_6^{a_6}e_7$ and taking the product with the complex conjugate gives:

$$(E_7\overline{E}_7)^3 = e_1^{2a_1}e_1^{-a_2}e_3^{2a_3}(e_4)^{2a_4}(e_1e_3)^{a_5}(e_4^3e_3^{-1})^{a_6}e_7^2$$

$$= \epsilon_1^{2a_1-a_2+a_5}\epsilon_2^{2a_3+a_5-a_6}\sqrt[3]{\epsilon_1\epsilon_2\epsilon_3}^{2a_4+3a_6}\epsilon_4^2.$$

Since all exponents must be divisible by 3 so $a_{77} = 1$.

Suppose that $a_{88} = 3$ then $E_8^3 = e_1^{a_1}e_2^{a_2}e_3^{a_3}e_4^{a_4}e_5^{a_5}e_6^{a_6}e_7^{a_7}e_8$. Multiplying by the complex conjugate gives:

$$(E_8\overline{E}_8)^3 = e_1^{2a_1}e_1^{-a_2}e_3^{2a_3}(e_4)^{2a_4}(e_1e_3)^{a_5}(e_4^3e_3^{-1})^{a_6}e_7^{2a_7}e_8$$
$$= \epsilon_1^{2a_1-a_2+a_5}\epsilon_2^{2a_3+a_5-a_6}\sqrt[3]{\epsilon_1\epsilon_2\epsilon_3}^{2a_4+3a_6}\epsilon_4^{2a_7+1}.$$

Then the following relations are clear

$$a_4 \equiv 0 \pmod 3, \quad a_2 \equiv 2a_1 + a_5 \pmod 3, \quad a_6 \equiv 2a_3 + a_5 \pmod 3, \quad a_7 \equiv 1 \pmod 3)$$

and $E_8^3 = e_1^{a_1}e_2^{2a_1+a_5}e_3^{a_3}e_5^{a_5}e_6^{2a_3+a_5}e_7e_8$. We will now conjugate with σ_1 and we find the following relations

$$e_3^{1-\sigma_1} = \tfrac{\epsilon_2}{\epsilon_2'} = e_1e_2^{-1}e_3^3e_5^{-1}, \qquad e_5^{1-\sigma_1} = \epsilon_2 = e_3, \qquad e_6^{1-\sigma_1} = {\epsilon_3'}^{-1} = e_2e_3e_4^{-3}e_6^3,$$
$$e_7^{1-\sigma_1} = \tfrac{\epsilon_4}{\epsilon_4'} = u_4^3 = e_8^3, \qquad e_8^{1-\sigma_1} = \epsilon_4 = e_7.$$

Now $(E_8^{1-\sigma_1})^3 = \left(e_1e_2^{-1}e_3^3e_5^{-3}\right)^{a_5} e_3^{a_5}\left(e_2e_3e_4^{-3}e_6^3\right)^{2a_3+a_5} e_7e_8^3$ and we can move all the cubes to the left at rename to get

$$(E_8^*)^3 = e_1^{a_5}e_2^{2a_3}e_3^{2a_3+2a_5}e_7$$
$$= \epsilon_1^{a_5}{\epsilon_1'}^{2a_3}\epsilon_2^{2a_3+2a_5}\epsilon_4.$$

All exponents must be divisible by 3 so $a_{88} = 1$ and $b^* = 1$.

Suppose $N(B_1)N(B_2) \sim \zeta^a N(B_3)$: $\hat{E}_0 =< \epsilon_1, \epsilon_1', \epsilon_2, \epsilon_2', \epsilon_3, \epsilon_3', \sqrt[3]{\epsilon_4}, u_4 >$

In this case we have a solution in K from the Type I field.

We know from [9] Theorem X that $e_a^3 = \zeta^a \frac{\epsilon_1\epsilon_2\epsilon_3}{\epsilon_1'\epsilon_2'\epsilon_3''}$ has a solution e_a in L so b^* is at least 1. Let $\hat{E}_1 =< \epsilon_1, \epsilon_1', \epsilon_2, \epsilon_2', \epsilon_3, \sqrt[3]{\frac{\epsilon_1\epsilon_2\epsilon_3}{\epsilon_1'\epsilon_2'\epsilon_3''}}, \sqrt[3]{\epsilon_4}, u_4 >$. Then $\hat{E}_0 \subset \hat{E}_1$ and we will show that $\left[\hat{E} : \hat{E}_1\right] \leq 3$, thus $\left[\hat{E} : \hat{E}_0\right] = \left[\hat{E} : \hat{E}_1\right]\left[\hat{E}_1 : \hat{E}_0\right] = 3^{b^*}$ where $b^* \leq 2$.

The same reasoning as in the previous case gives us that $a_{11} = a_{22} = a_{33} = a_{44} = a_{55} = 1$. Suppose $a_{66} = 3$, then

$$E_6^3 = \epsilon_1^{a_1}{\epsilon_1'}^{a_2}\epsilon_2^{a_3}{\epsilon_2'}^{a_4}\epsilon_3^{a_5}\sqrt[3]{\frac{\epsilon_1\epsilon_2\epsilon_3}{\epsilon_1'\epsilon_2'\epsilon_3''}}.$$

Taking complex conjugates and multiplying we get

$$(E_6\overline{E}_6)^3 = \epsilon_1^{2a_1}(\epsilon_1'\epsilon_1'')^{a_2}\epsilon_2^{2a_3}(\epsilon_2'\epsilon_2'')^{a_4}\epsilon_3^{2a_5}\sqrt[3]{\frac{\epsilon_1^2\epsilon_2^2\epsilon_3^2}{\epsilon_1'\epsilon_1''\epsilon_2'\epsilon_2''\epsilon_3'\epsilon_3''}}$$
$$= \epsilon_1^{2a_1-a_2-1}\epsilon_2^{2a_3-a_4-1}\epsilon_3^{2a_5+1}.$$

Then the following relations are clear

$$a_2 \equiv 2a_1 + 1 \pmod 3, \quad a_4 \equiv 2a_3 + 1 \pmod 3, \quad a_5 \equiv 1 \pmod 3$$

and $E_6^3 = \epsilon_1^{a_1} {\epsilon'_1}^{2a_1+1} \epsilon_2^{a_3} {\epsilon'_2}^{2a_3+1} \epsilon_3 \sqrt[3]{\frac{\epsilon_1\epsilon_2\epsilon_3}{\epsilon'_1\epsilon'_2\epsilon''_3}}$. We will now conjugate with σ_1 and we find the following relations

$$e_3^{1-\sigma_1} = \frac{\epsilon_2}{\epsilon'_2} = e_3 e_4^{-1}, \quad e_4^{1-\sigma_1} = \epsilon_2 {\epsilon'_2}^2 = e_3 e_4^2, \quad e_5^{1-\sigma_1} = \epsilon_3^2 \epsilon''_3 = e_1 e_2^{-1} e_3 e_4^{-1} e_5^3 e_6^{-3}, \quad e_6^{1-\sigma_1} = \frac{\epsilon_3}{\epsilon'_2} = u_4^{-1} e_5.$$

Then $(E_6^{1-\sigma_1})^3 = (e_3 e_4^{-1})^{a_3} (e_3 e_4^2)^{2a_3+1} (e_1 e_2^{-1} e_3 e_4^{-1} e_5^3 e_6^{-3}) e_4^{-1} e_5$ and by moving all cubes to the left and renaming we get

$$(E_6^*)^3 = \epsilon_1 \epsilon'_1 \epsilon_2^2 \epsilon_3$$

which has no solution so $a_{66} = 1$.

Suppose $a_{77} = 3$, then $E_7^3 = \epsilon_1^{a_1} {\epsilon'_1}^{a_2} \epsilon_2^{a_3} {\epsilon'_2}^{a_4} \epsilon_3^{a_5} \sqrt[3]{\frac{\epsilon_1\epsilon_2\epsilon_3}{\epsilon'_1\epsilon'_2\epsilon''_3}}^{a_6} \sqrt[3]{\epsilon_4}$. Taking complex conjugates and multiplying we get

$$\begin{aligned}(E_7\overline{E}_7)^3 &= \epsilon_1^{2a_1} (\epsilon'_1\epsilon''_1)^{a_2} \epsilon_2^{2a_3} (\epsilon'_2\epsilon''_2)^{a_4} \epsilon_3^{2a_5} \sqrt[3]{\frac{\epsilon_1^2\epsilon_2^2\epsilon_3^2}{\epsilon'_1\epsilon''_1\epsilon'_2\epsilon''_2\epsilon'_3\epsilon''_3}}^{a_6} \sqrt[3]{\epsilon_4}^2 \\ &= \epsilon_1^{2a_1-a_2+a_6} \epsilon_2^{2a_3-a_4+a_6} \epsilon_3^{2a_5+a_6} \sqrt[3]{\epsilon_4}^2.\end{aligned}$$

This can have no solution in K unless all exponents are divisible by 3 so $a_{77} \neq 3$ which implies that $a_{77} = 1$.

Suppose $a_{88} = 3$. Then a similar calculation results in

$$E_8^3 = \epsilon_1^{a_1} {\epsilon'_1}^{2a_1} \epsilon_2^{a_3} {\epsilon'_2}^{2a_3} u_4$$

so $\det(A) = 3^{b^*}$ where $b^* \leq 2$.

Suppose $N(B_1) \sim \zeta^a N(B_2) \nsim N(B_3)$**:** $\hat{E}_0 =< \epsilon_1, \epsilon'_1, \epsilon_2, \epsilon'_2, \epsilon_3, \epsilon'_3, \sqrt[3]{\epsilon_4}, u_4 >$

We know from [9] Theorem X that $e^3 = \zeta^a(\epsilon_1\epsilon_2)/(\epsilon'_1\epsilon''_2)$ has solutions e in L so b^* is at least 1. Let $\hat{E}_1 =< \epsilon_1, \epsilon'_1, \epsilon_2, \sqrt[3]{\frac{\epsilon_1\epsilon_2}{\epsilon'_1\epsilon''_2}}, \epsilon_3, \epsilon'_3, \sqrt[3]{\epsilon_4}, u_4 >$. Then $\hat{E}_0 \subset \hat{E}_1$ and we will show that $\left[\hat{E} : \hat{E}_1\right] \leq 3$, thus $\left[\hat{E} : \hat{E}_0\right] = \left[\hat{E} : \hat{E}_1\right] \left[\hat{E}_1 : \hat{E}_0\right] = 3^{b^*}$ where $b^* \leq 2$.

The same reasoning as in the previous case gives us that $a_{11} = a_{22} = a_{33} = a_{44} = a_{55} = a_{66} = 1$. We will again number the basis elements e_1 to e_8 and we can find some relations between the conjugates.

$$e_1\overline{e}_1 = e_1^2, \quad e_2\overline{e}_2 = e_1^{-1}, \quad e_3\overline{e}_3 = e_3^2, \quad e_4\overline{e}_4 = e_1 e_3, \quad e_5\overline{e}_5 = e_5^2, \quad e_6\overline{e}_6 = e_5^{-1}, \quad e_7\overline{e}_7 = e_7^2$$

Suppose $a_{77} = 3$. Then $E_7^3 = e_1^{a_1} e_2^{a_2} e_3^{a_3} e_4^{a_4} e_5^{a_5} e_6^{a_6} e_7$ and multiplying by the conjugate gives:

$$\begin{aligned}(E_7\overline{E}_7)^3 &= e_1^{2a_1} e_1^{-a_2} e_3^{2a_3} (e_1e_3)^{a_4} e_5^{2a_5} e_5^{-a_6} e_7^2 \\ &= e_1^{2a_1-a_2+a_4} e_3^{2a_3+a_4} e_5^{2a_5-a_6} e_7^2\end{aligned}$$

$$=\epsilon_1^{2a_1-a_2+a_4}\epsilon_2^{2a_3+a_4}\epsilon_3^{2a_5-a_6}\sqrt[3]{\epsilon_4}^{2}$$

All exponents must be divisible by 3 so so $a_{77} \neq 3$ which implies that $a_{77} = 1$.

Suppose $a_{88} = 3$. Then a similar calculation results in

$$E_8^3 = \epsilon_1^{a_1}\epsilon_1'^{\,2a_1}\epsilon_3^{a_5}\epsilon_3'^{\,2a_5}u_4$$

so $\det(A) = 3^{b^*}$ where $b^* \leq 2$.

Suppose $N(B_1) \nsim N(B_2) \nsim N(B_3) \nsim N(B_1)$ and $N(B_1)N(B_2) \nsim N(B_3)$: Then $\hat{E}_0 =< \epsilon_1, \epsilon_1', \epsilon_2, \epsilon_2', \epsilon_3, \epsilon_3', \sqrt[3]{\epsilon_4}, u_4 >$

The same reasoning as in the previous case gives us that $a_{11} = a_{22} = a_{33} = a_{44} = a_{55} = a_{66} = a_{77} = 1$. Suppose $a_{88} = 3$. Then a similar calculation as above results in

$$E_8^3 = \epsilon_1^{a_1}\epsilon_1'^{\,2a_1}\epsilon_2^{a_3}\epsilon_2'^{\,2a_3}\epsilon_3^{a_5}\epsilon_3'^{\,2a_5}u_4$$

so $\det(A) = 3^{b^*}$ where $b^* \leq 1$.

(E)Four Type III:

We can have at most 3 Type three fields of similar norm so the proof of this case is the same as the proof of **case(3D)** Three Type III fields.

Case 4: Kind 4: $(\hat{e} : \hat{e}_0) = 1$

(A) All fields are Type I or IV: $\hat{E}_0 =< \epsilon_1, u_1, \epsilon_2, u_2, \epsilon_3, u_3, \epsilon_4, u_4 >$

We know that $a_{11} = a_{22} = a_{33} = a_{44} = a_{55} = a_{66} = 1$ by a similar proof to Case 3 so suppose $a_{77} = 3$. Then $E_7^3 = \epsilon_1^{a_1}u_1^{a_2}\epsilon_2^{a_3}u_2^{a_4}\epsilon_3^{a_5}u_3^{a_6}\epsilon_4$ and multiplying by the complex conjugate gives:

$$(E_7\overline{E_7})^3 = \epsilon_1^{2a_1}(u_1\overline{u}_1)^{a_2}\epsilon_2^{2a_3}(u_2\overline{u}_2)^{a_4}\epsilon_3^{2a_5}(u_3\overline{u}_3)^{a_6}\epsilon_4^2 \approx \epsilon_1^{2a_1+a_2}\epsilon_2^{2a_3+a_4}\epsilon_3^{2a_5+a_6}\epsilon_4^2.$$

This clearly has no solution because K is Kind 4 all exponents must be divisible by 3 so $a_{77} = 1$.

Suppose $a_{88} = 3$, then $E_8^3 = \epsilon_1^{a_1}u_1^{a_2}\epsilon_2^{a_3}u_2^{a_4}\epsilon_3^{a_5}u_3^{a_6}\epsilon_4^{a_7}u_4$ and conjugating with σ_1 and dividing we get

$$(E_8^{1-\sigma_1})^3 = \left(\frac{\epsilon_2}{\epsilon_2'}\right)^{a_3}\left(\frac{u_2}{u_2'}\right)^{a_4}\left(\frac{\epsilon_3}{\epsilon_3'}\right)^{a_5}\left(\frac{u_3}{u_3'}\right)^{a_6}\left(\frac{\epsilon_4}{\epsilon_4'}\right)^{a_7}\left(\frac{u_4}{u_4'}\right) \approx \epsilon_2^{a_4}\epsilon_3^{a_6}\epsilon_4.$$

Again all the exponents must be divisible by three hence $a_{88} \neq 3 \implies a_{88} = 1$. As with the other cases we then have $\det(A) = 1$ and $b^* = 0$.

(B) One Type III: $\hat{E}_0 =< \epsilon_1, \epsilon_1', \epsilon_2, u_2, \epsilon_3, u_3, \epsilon_4, u_4 >$

The proof for this case is identical to the previous one.

(C) Two Type III and two Type I or IV:

Suppose $N(B_1) \sim \zeta^a N(B_2)$**:** $\hat{E}_0 =< \epsilon_1, \epsilon_1', \epsilon_2, \epsilon_2', \epsilon_3, u_3, \epsilon_4, u_4 >$

We know from [9] Theorem X that $e^3 = \zeta^a(\epsilon_1\epsilon_2/\epsilon_1'\epsilon_2')$ has a solutions e in L so b^* is at least 1. Let $\hat{E}_1 =< \epsilon_1, \epsilon_1', \epsilon_2, \sqrt[3]{\frac{\epsilon_1\epsilon_2}{\epsilon_1'\epsilon_2'}}, \epsilon_3, u_3, \epsilon_4, u_4 >$. Then $\hat{E}_0 \subset \hat{E}_1$ and we will show that $\left[\hat{E} : \hat{E}_1\right] = 1$, thus $\left[\hat{E} : \hat{E}_0\right] = \left[\hat{E} : \hat{E}_1\right]\left[\hat{E}_1 : \hat{E}_0\right] = 3^1$.

The proof follows much as the previous ones. The same reasoning as in the previous case gives us that $a_{11} = a_{22} = a_{33} = a_{44} = a_{55} = a_{66} = a_{77} = 1$. We will again number the basis elements e_1 to e_8 and find some relations between the conjugates.

$e_1\overline{e}_1 = e_1^2, \quad e_2\overline{e}_2 = e_1^{-1}, \quad e_3\overline{e}_3 = e_3^2, \quad e_4\overline{e}_4 = e_1e_3, \quad e_5\overline{e}_5 = e_5^2, \quad e_6\overline{e}_6 = e_5, \quad e_7\overline{e}_7 = e_7^2, \quad e_8\overline{e}_8 = e_7$

Suppose $a_{88} = 3$. Then $E_8^3 = e_1^{a_1}e_2^{a_2}e_3^{a_3}e_4^{a_4}e_5^{a_5}e_6^{a_6}e_7^{a_7}e_8$ and multiplying by the complex conjugate:

$(E_8\overline{E}_8)^3 = e_1^{2a_1-a_2+a_4}e_3^{2a_3+a_4}e_5^{2a_5+a_6}e_7^{2a_7+1} = \epsilon_1^{2a_1-a_2+a_4}\epsilon_2^{2a_3+a_4}\epsilon_3^{2a_5+a_6}\epsilon_4^{2a_7+1}$

For this to have a solution in K we need

$$2a_1 - a_2 + a_4 \equiv 0 \pmod 3, \quad 2a_3 + a_4 \equiv 0 \pmod 3, \quad 2a_5 + a_6 \equiv 0 \pmod 3, \quad 2a_7 + 1 \equiv 0 \pmod 3.$$

So $E_8^3 = \epsilon_1^{2a_2+a_3}{\epsilon_1'}^{a_2}\epsilon_2^{a_3}\sqrt[3]{\frac{\epsilon_1\epsilon_2}{\epsilon_1'\epsilon_2'}}^{a_3}\epsilon_3^{a_5}u_3^{a_5}\epsilon_4u_4$ and conjugating with σ_1 and taking the quotient:

$$(E_8^{1-\sigma_1})^3 = \left(\frac{\epsilon_2}{\epsilon_2'}\right)^{a_3}\sqrt[3]{\frac{\epsilon_2''\epsilon_2}{{\epsilon_2'}^2}}^{a_3}\left(\frac{\epsilon_3}{\epsilon_3'}\right)^{a_5}\left(\frac{u_3}{u_3'}\right)^{a_5}\frac{\epsilon_4}{\epsilon_4'}\frac{u_4}{u_4'} = \left(\frac{\epsilon_2}{\epsilon_2'}\right)^{a_3}(\epsilon_2')^{-a_3}\epsilon_3^{3a_5}u_3^{-3a_5}\epsilon_3^{a_5}\epsilon_4^3{u_4^{-3}}\epsilon_4.$$

Then renaming and moving cubes to the left gives

$$(E^*)^3 = \epsilon_2^{a_3}{\epsilon_2'}^{a_3}\epsilon_3^{a_5}\epsilon_4.$$

Since $a_{11} = \ldots = a_{77} = 1$ all the exponents must be divisible by 3. Thus $a_{88} \neq 3 \implies a_{88} = 1$ and $b^* = 1$.

Suppose $N(B_1) \nsim N(B_2)$**:** $\hat{E}_0 =< \epsilon_1, \epsilon_1', \epsilon_2, \epsilon_2', \epsilon_3, u_3, \epsilon_4, u_4 >$

The proof follows much as the previous ones. The same reasoning as in the previous case gives us that $a_{11} = a_{22} = a_{33} = a_{44} = a_{55} = a_{66} = a_{77} = 1$. We will again number the basis elements e_1 to e_8 and find some relations between the conjugates.

$e_1\overline{e}_1 = e_1^2, \quad e_2\overline{e}_2 = e_1^{-1}, \quad e_3\overline{e}_3 = e_3^2, \quad e_4\overline{e}_4 = e_3{-1}, \quad e_5\overline{e}_5 = e_5^2, \quad e_6\overline{e}_6 = e_5, \quad e_7\overline{e}_7 = e_7^2, \quad e_8\overline{e}_8 = e_7$

Suppose $a_{88} = 3$. Then $E_8^3 = e_1^{a_1}e_2^{a_2}e_3^{a_3}e_4^{a_4}e_5^{a_5}e_6^{a_6}e_7^{a_7}e_8$ and multiplying by the complex conjugate gives:

$(E_8\overline{E}_8)^3 = e_1^{2a_1-a_2}e_3^{2a_3-a_4}e_5^{2a_5+a_6}e_7^{2a_7+1}$

For this to have a solution in K we need

$$2a_1 - a_2 \equiv 0 \pmod 3$$

$$2a_3 - a_4 \equiv 0 \pmod 3$$
$$2a_5 + a_6 \equiv 0 \pmod 3$$
$$2a_7 + 1 \equiv 0 \pmod 3$$

So we can rewrite $E_8^3 = \epsilon_1^{a_1}{\epsilon_1'}^{2a_1}\epsilon_2^{a_3}{\epsilon_2'}^{2a_3}\epsilon_3^{a_5}u_3^{a_5}\epsilon_4 u_4$, conjugating with σ_1 and take the quotient to get:

$(E_8^{1-\sigma_1})^3 = \frac{\epsilon_2^{a_3}{\epsilon_2'}^{2a_3}\epsilon_3^{a_5}u_3^{a_5}\epsilon_4 u_4}{{\epsilon_2'}^{a_3}{\epsilon_2''}^{2a_3}{\epsilon_3'}^{a_5}{u_3'}^{a_5}\epsilon_4' u_4'} = (\epsilon_2'')^{-3a_3}\epsilon_3^{3a_5}u_3^{-3a_5}\epsilon_3^{a_5}\epsilon_4^3 u_4^{-3}\epsilon_4 \approx \epsilon_3^{a_5}\epsilon_4.$

This can have no solution unless all exponents are divisible by 3, contradiction. Thus $a_{88} \neq 3 \implies a_{88} = 1$. As with the other cases we then have $\det(A) = 1$ and $b^* = 0$.

(D) Three Type III and one Type I or IV:

Suppose $N(B_1) \nsim N(B_2) \nsim N(B_3) \nsim N(B_1)$ and $N(B_1)N(B_2) \nsim N(B_3)$: then $\hat{E}_0 =< \epsilon_1, \epsilon_1', \epsilon_2, \epsilon_2', \epsilon_3, \epsilon_3', \epsilon_4, u_4 >$

The same reasoning as in the previous case gives us that $a_{11} = a_{22} = a_{33} = a_{44} = a_{55} = a_{66} = a_{77} = 1$. We will again number the basis elements e_1 to e_8 and find some relations between the conjugates.

$e_1\overline{e}_1 = e_1^2, \quad e_2\overline{e}_2 = e_1^{-1}, \quad e_3\overline{e}_3 = e_3^2, \quad e_4\overline{e}_4 = e_3^{-1}, \quad e_5\overline{e}_5 = e_5^2, \quad e_6\overline{e}_6 = e_5^{-1}, \quad e_7\overline{e}_7 = e_7^2, \quad e_8\overline{e}_8 = e_7$

Suppose $a_{88} = 3$. Then $E_8^3 = e_1^{a_1}e_2^{a_2}e_3^{a_3}e_4^{a_4}e_5^{a_5}e_6^{a_6}e_7^{a_7}e_8$ and multiplying by the complex conjugate gives:

$(E_8\overline{E}_8)^3 = e_1^{2a_1-a_2}e_3^{2a_3-a_4}e_5^{2a_5-a_6}e_7^{2a_7+1}$

So the following relations on the exponents are clear: $a_2 \equiv 2a_1 \pmod 3$, $a_4 \equiv 2a_3 \pmod 3$, $a_6 \equiv 2a_5 \pmod 3$, and $a_7 \equiv 0 \pmod 3$ and the equation can be rewritten:

$E_8^3 = e_1^{a_1}e_2^{2a_1}e_3^{a_3}e_4^{2a_3}e_5^{a_5}e_6^{2a_5}e_7e_8.$

We will conjugate with σ_1 to get some relations.

$e_3^{1-\sigma_1} = e_3e_4^{-1}, \quad e_4^{1-\sigma_1} = e_3e_4^2, \quad e_5^{1-\sigma_1} = e_5e_6^{-1}, \quad e_6^{1-\sigma_1} = e_5e_6^2, \quad e_7^{1-\sigma_1} = e_7^3e_8^{-3}, \quad e_8^{1-\sigma_1} = e_7$

Then $(E_8^{1-\sigma_1})^3 = (e_3e_4^{-1})^{a_3}e_3^{a_3}(e_5e_6^{-1})^{a_5}(e_5e_6^2)^{2a_5}e_7^3e_8^{-3}e_7$. Combining like terms, moving all cubes to the left, and renaming gives $(E_8')^3 = e_7 = u_4$ and this clearly has no solution hence $a_{88} \neq 3 \implies a_{88} = 1$. As with the other cases we then have $\det(A) = 1$ and $b^* = 0$.

Suppose $N(B_1) \sim \zeta^a N(B_2) \nsim N(B_3)$**:** $\hat{E}_0 =< \epsilon_1, \epsilon_1', \epsilon_2, \epsilon_2', \epsilon_3, \epsilon_3', \epsilon_4, u_4 >$

We know from [9] Theorem X that $e_a^3 = \zeta^a(\epsilon_1\epsilon_2/\epsilon_1'\epsilon_2'')$ has solutions e_a in L so b^* is at least 1. Let $\hat{E}_1 =< \epsilon_1, \epsilon_1', \epsilon_2, \sqrt[3]{\frac{\epsilon_1\epsilon_2}{\epsilon_1'\epsilon_2''}}, \epsilon_3, \epsilon_3', \epsilon_4, u_4 >$. Then $\hat{E}_0 \subset \hat{E}_1$ and we will show that $\left[\hat{E} : \hat{E}_1\right] = 1$, thus $\left[\hat{E} : \hat{E}_0\right] = \left[\hat{E} : \hat{E}_1\right]\left[\hat{E}_1 : \hat{E}_0\right] = 3^1$.

The same reasoning as in the previous case gives us that $a_{11} = a_{22} = a_{33} = a_{44} = a_{55} = a_{66} = a_{77} = 1$. We will again number the basis elements e_1 to e_8 and find some relations between the conjugates.

$e_1\overline{e}_1 = e_1^2, \quad e_2\overline{e}_2 = e_1^{-1}, \quad e_3\overline{e}_3 = e_3^2, \quad e_4\overline{e}_4 = e_1e_3, \quad e_5\overline{e}_5 = e_5^2, \quad e_6\overline{e}_6 = e_5, \quad e_7\overline{e}_7 = e_7^2, \quad e_8\overline{e}_8 = e_7$

Suppose $a_{88} = 3$. Then $E_8^3 = e_1^{a_1}e_2^{a_2}e_3^{a_3}e_4^{a_4}e_5^{a_5}e_6^{a_6}e_7^{a_7}e_8$ and multiplying by the complex conjugate gives: $(E_8\overline{E}_8)^3 = e_1^{2a_1-a_2+a_4}e_3^{2a_3+a_4}e_5^{2a_5-a_6}e_7^{2a_7+1}$. So the following relations on the exponents are apparent: $a_2 \equiv 2a_1 + a_4 \pmod 3$, $a_4 \equiv a_3 \pmod 3$, $a_6 \equiv 2a_5 \pmod 3$, and $a_7 \equiv 1 \pmod 3$ and the equation can be rewritten:

$E_8^3 = e_1^{a_1}e_2^{2a_1+a_3}e_3^{a_3}e_4^{a_3}e_5^{a_5}e_6^{2a_5}e_7e_8$.

Conjugating with σ_1 gives the following relations:

$e_3^{1-\sigma_1} = e_1^{-1}e_2e_3^3e_4^{-3}, \quad e_4^{1-\sigma_1} = e_1^{-1}e_2e_3^{-1}e_4^3, \quad e_5^{1-\sigma_1} = e_5e_6^{-1}, \quad e_6^{1-\sigma_1} = e_5e_6^2, \quad e_7^{1-\sigma_1} = e_7^3e_8^3, \quad e_8^{1-\sigma_1} = e_7$

Thus $(E_8^{1-\sigma_1})^3 = (e_1^{-1}e_2e_3^3e_4^{-3})^{a_3}(e_1^{-1}e_2e_3^{-1}e_4^3)^{a_3}(e_5e_6^{-1})^{a_5}(e_5e_6^2)^{2a_5}e_7^3e_8^3e_7$.

Combining like terms, moving the cubes to the left and renaming $(E_8^{1-\sigma_1})^3$

$(E_8^*)^3 = \frac{{\epsilon'_1}^{2a_3}{\epsilon'_3}^{3a_5}\epsilon_4}{\epsilon_1^{2a_3}\epsilon_2^{a_3}\epsilon_3^{3a_5}}$

This can only have solutions for Type III fields with similar norm hence $a_{88} \neq 3 \implies a_{88} = 1$ and $b^* = 1$.

Suppose $N(B_1) \sim \zeta^a N(B_2) \sim \zeta^c N(B_3)$**:** $\hat{E}_0 =< \epsilon_1, \epsilon'_1, \epsilon_2, \epsilon'_2, \epsilon_3, \epsilon'_3, \epsilon_4, u_4 >$

We know from [9] Theorem X that $e_a^3 = \zeta^a(\epsilon_1\epsilon_2/\epsilon'_1\epsilon'_2)$ and $e_c^3 = \zeta^c(\epsilon_1\epsilon_3/\epsilon'_1\epsilon'_3)$ both have solutions e_a and e_c in L so b^* is at least 2. Let $\hat{E}_1 =< \epsilon_1, \epsilon'_1, \epsilon_2, \sqrt[3]{\frac{\epsilon_1\epsilon_2}{\epsilon'_1\epsilon'_2}}, \epsilon_3, \sqrt[3]{\frac{\epsilon_1\epsilon_3}{\epsilon'_1\epsilon'_3}}, \epsilon_4, u_4 >$. Then $\hat{E}_0 \subset \hat{E}_1$ and we will show that $\left[\hat{E} : \hat{E}_1\right] = 1$, thus $\left[\hat{E} : \hat{E}_0\right] = \left[\hat{E} : \hat{E}_1\right]\left[\hat{E}_1 : \hat{E}_0\right] = 3^2$

The same reasoning as in the previous case gives us that $a_{11} = a_{22} = a_{33} = a_{44} = a_{55} = a_{66} = a_{77} = 1$. We will again number the basis elements e_1 to e_8 and find some relations between the conjugates.

$e_1\overline{e}_1 = e_1^2, \quad e_2\overline{e}_2 = e_1^{-1}, \quad e_3\overline{e}_3 = e_3^2, \quad e_4\overline{e}_4 = e_1e_3, \quad e_5\overline{e}_5 = e_5^2, \quad e_6\overline{e}_6 = e_1e_5, \quad e_7\overline{e}_7 = e_7^2, \quad e_8\overline{e}_8 = e_7$

Suppose $a_{88} = 3$, then $E_8^3 = e_1^{a_1}e_2^{a_2}e_3^{a_3}e_4^{a_4}e_5^{a_5}e_6^{a_6}e_7^{a_7}e_8$ and multiplying by the complex conjugate gives: $(E_8\overline{E}_8)^3 = e_1^{2a_1-a_2+a_4+a_6}e_3^{2a_3+a_4}e_5^{2a_5+a_6}e_7^{2a_7+1}$

We get the following relations on the exponents: $a_2 \equiv 2a_1 + a_4 + a_6 \pmod 3$, $a_4 \equiv a_3 \pmod 3$, $a_6 \equiv a_5 \pmod 3$, and $a_7 \equiv 1 \pmod 3$ and the equation becomes $E_8^3 = e_1^{a_1}e_2^{2a_1+a_3+a_5}e_3^{a_3}e_4^{a_3}e_5^{a_5}e_6^{a_5}e_7^2e_8$.

We will again take conjugates with σ_1.

$e_3^{1-\sigma_1} = e_1^{-1}e_2e_3^3e_4^{-3}, \quad e_4^{1-\sigma_1} = e_1^{-1}e_2e_3^{-1}e_4^3, \quad e_5^{1-\sigma_1} = e_1^{-1}e_2e_6^3, \quad e_6^{1-\sigma_1} = e_1^{-1}e_2e_5^{-1}e_6^3, \quad e_7^{1-\sigma_1} = e_7^3e_8^{-3},$
$e_8^{1-\sigma_1} = e_7$

Taking the quotient gives $(E_8^{1-\sigma_1})^3 = (e_1^{-1}e_2e_3^3e_4^{-3})^{a_3}(e_1^{-1}e_2e_3^{-1}e_4^3)^{a_3}(e_1^{-1}e_2e_6^3)^{a_5}(e_1^{-1}e_2e_5^{-1}e_6^3)^{a_5}e_7^3e_8^{-3}e_7$.

And combining like terms, moving the cubes to the left and renaming $(E_8^{1-\sigma_1})^3$:

$(E_8^*)^3 = e_1^{-2a_3-2a_5}e_2^{2a_3+2a_5}e_3^{-a_3}e_5^{-a_5}e_7 = \frac{(\epsilon'_1)^{2a_3+2a_5}\epsilon_4}{\epsilon_1^{2a_3+2a_5}\epsilon_2^{a_3}\epsilon_3^{a_5}}$

This can have no solution because the exponent on ϵ_4 is not divisible by 3 hence $a_{88} \neq 3 \implies a_{88} = 1$ so $b^* = 2$.

Suppose $N(B_1)N(B_2) \sim \zeta^a N(B_3)$: $\hat{E}_0 =< \epsilon_1, \epsilon_1', \epsilon_2, \epsilon_2', \epsilon_3, \epsilon_3', \epsilon_4, u_4 >$

We know from [9] Theorem X that $e_a^3 = \zeta^a \frac{\epsilon_1\epsilon_2\epsilon_3}{\epsilon_1'\epsilon_2'\epsilon_3''}$ has a solution e_a in L so b^* is at least 1. Let $\hat{E}_1 =< \epsilon_1, \epsilon_1', \epsilon_2, \epsilon_2', \epsilon_3, \sqrt[3]{\frac{\epsilon_1\epsilon_2\epsilon_3}{\epsilon_1'\epsilon_2'\epsilon_3''}}, \epsilon_4, u_4 >$. Then $\hat{E}_0 \subset \hat{E}_1$ and we will show that $\left[\hat{E} : \hat{E}_1\right] = 1$, thus $\left[\hat{E} : \hat{E}_0\right] = \left[\hat{E} : \hat{E}_1\right]\left[\hat{E}_1 : \hat{E}_0\right] = 3^1$

The same reasoning as in the previous case gives us that $a_{11} = a_{22} = a_{33} = a_{44} = a_{55} = a_{66} = a_{77} = 1$. We will again number the basis elements e_1 to e_8 and find some relations between the conjugates.

$e_1\overline{e}_1 = e_1^2, \quad e_2\overline{e}_2 = e_1^{-1}, \quad e_3\overline{e}_3 = e_3^2, \quad e_4\overline{e}_4 = e_3^{-1}, \quad e_5\overline{e}_5 = e_5^2, \quad e_6\overline{e}_6 = e_1e_3e_5, \quad e_7\overline{e}_7 = e_7^2, \quad e_8\overline{e}_8 = e_7$

Suppose $a_{88} = 3$. Then $E_8^3 = e_1^{a_1}e_2^{a_2}e_3^{a_3}e_4^{a_4}e_5^{a_5}e_6^{a_6}e_7^{a_7}e_8$ and multiplication with the complex conjugate gives:

$$(E_8\overline{E}_8)^3 = e_1^{2a_1-a_2+a_6}e_3^{2a_3-a_4+a_6}e_5^{2a_5+a_6}e_7^{2a_7+1}$$

We get the following relations on the exponents: $a_2 \equiv 2a_1 + a_6 \pmod 3$, $a_4 \equiv 2a_3 + a_6 \pmod 3$, $a_6 \equiv a_5 \pmod 3$, and $a_7 \equiv 1 \pmod 3$ and the equation can be rewritten:

$$E_8^3 = e_1^{a_1}e_2^{2a_1+a_5}e_3^{a_3}e_4^{2a_3+a_5}e_5^{a_5}e_6^{a_5}e_7e_8.$$

We will again take conjugates with σ_1.

$e_3^{1-\sigma_1} = e_3e_4^{-1}, \quad e_4^{1-\sigma_1} = e_3e_4^2, \quad e_5^{1-\sigma_1} = e_1e_2^{-1}e_3e_4^{-1}e_5^3e_6^{-3}, \quad e_6^{1-\sigma_1} = e_4^{-1}e_5, \quad e_7^{1-\sigma_1} = e_7^3e_8^{-3},$
$e_8^{1-\sigma_1} = e_7.$

Taking the quotient gives $(E_8^{1-\sigma_1})^3 = (e_3e_4^{-1})^{a_3}(e_3e_4^2)^{2a_3+a_5}(e_4^{-1}e_5)^{a_5}(e_1e_2^{-1}e_3e_4^{-1}e_5^3e_6^{-3})^{a_5}e_7^3e_8^{-3}e_7$ and combining like terms, moving the cubes to the left and renaming $(E_8^{1-\sigma_1})^3 = (E_8')^3$

$$(E_8')^3 = e_1^{a_5}e_2^{-a_5}e_3^{3a_3+2a_5}e_4^{3a_3}e_5^{a_5}e_7 = \frac{(\epsilon_1)^{a_5}\epsilon_2^{2a_5}\epsilon_3^{a_5}\epsilon_4}{\epsilon_1'^{a_5}}.$$

This can have no solution because the exponent on ϵ_4 is not divisible by 3, hence $a_{88} \neq 3 \implies a_{88} = 1$ so $b^* = 1$.

(E) Four Type III:

Suppose no norms or products of norms are similar: $\hat{E}_0 =< \epsilon_1, \epsilon_1', \epsilon_2, \epsilon_2', \epsilon_3, \epsilon_3', \epsilon_4, \epsilon_4' >$

The same reasoning as in the previous case gives us that $a_{11} = a_{22} = a_{33} = a_{44} = a_{55} = a_{66} = a_{77} = 1$. We will again find the following relations between the conjugates: $\epsilon_{2i-1}\epsilon_{2i-1}' = \epsilon_{2i-1}^2$ and $\epsilon_{2i}\epsilon_{2i}' = \epsilon_{2i-1}^{-1}$.

Suppose $a_{88} = 3$. Then $E_8^3 = \epsilon_1^{a_1}\epsilon_1'^{a_2}\epsilon_2^{a_3}\epsilon_2'^{a_4}\epsilon_3^{a_5}\epsilon_3'^{a_6}\epsilon_4^{a_7}\epsilon_4'$ and multiplying by the complex conjugate gives:

$$(E_8\overline{E}_8)^3 = \epsilon_1^{2a_1-a_2}\epsilon_2^{2a_3-a_4}\epsilon_3^{2a_5-a_6}\epsilon_4^{2a_7-1}$$

The following relations on the exponents are apparent: $a_2 \equiv 2a_1 \pmod 3$, $a_4 \equiv 2a_3 \pmod 3$, $a_6 \equiv 2a_5 \pmod 3$, and $a_7 \equiv 2 \pmod 3$ and the equation can be rewritten:

$$E_8^3 = \epsilon_1^{a_1}{\epsilon'_1}^{2a_1}\epsilon_2^{a_3}{\epsilon'_2}^{2a_3}\epsilon_3^{a_5}{\epsilon'_3}^{2a_5}\epsilon_4^2\epsilon'_4.$$

We know that this can have no solutions in L by [9] because no product of the norms is a cube hence $b^* = 0$

Suppose $N(B_1) \sim \zeta^a N(B_2)$ and no other norms or products of norms are similar:

$\hat{E}_0 =< \epsilon_1, \epsilon'_1, \epsilon_2, \epsilon'_2, \epsilon_3, \epsilon'_3, \epsilon_4, \epsilon'_4 >$

We know from [9] Theorem X that $e_a^3 = \zeta^a(\epsilon_1\epsilon_2/\epsilon'_1\epsilon'_2)$ has solutions e_a in L so b^* is at least 1. Let $\hat{E}_1 =< \epsilon_1, \epsilon'_1, \epsilon_2, \sqrt[3]{\frac{\epsilon_1\epsilon_2}{\epsilon'_1\epsilon'_2}}, \epsilon_3, \epsilon'_3, \epsilon_4, \epsilon'_4 >$. Then $\hat{E}_0 \subset \hat{E}_1$ and we will show that $\left[\hat{E} : \hat{E}_1\right] = 1$, thus $\left[\hat{E} : \hat{E}_0\right] = \left[\hat{E} : \hat{E}_1\right]\left[\hat{E}_1 : \hat{E}_0\right] = 3^1$.

The same reasoning as in the previous case gives us that $a_{11} = a_{22} = a_{33} = a_{44} = a_{55} = a_{66} = a_{77} = 1$. We will again number the basis elements e_1 to e_8 and find some relations between the conjugates.

$e_1\overline{e}_1 = e_1^2$, $e_2\overline{e}_2 = e_1^{-1}$, $e_3\overline{e}_3 = e_3^2$, $e_4\overline{e}_4 = e_1e_3$, $e_5\overline{e}_5 = e_5^2$, $e_6\overline{e}_6 = e_5^{-1}$, $e_7\overline{e}_7 = e_7^2$, $e_8\overline{e}_8 = e_7^{-1}$

Suppose $a_{88} = 3$. Then $E_8^3 = e_1^{a_1}e_2^{a_2}e_3^{a_3}e_4^{a_4}e_5^{a_5}e_6^{a_6}e_7^{a_7}e_8$ and multiplying by the complex conjugate gives:

$$(E_8\overline{E}_8)^3 = e_1^{2a_1-a_2+a_4}e_3^{2a_3+a_4}e_5^{2a_5-a_6}e_7^{2a_7-1} = \epsilon_1^{2a_1-a_2+a_4}\epsilon_2^{2a_3+a_4}\epsilon_3^{2a_5-a_6}\epsilon_4^{2a_7-1}.$$

The following relations on the exponents are apparent: $a_2 \equiv 2a_1 + a_4 \pmod 3$, $a_4 \equiv a_3 \pmod 3$, $a_6 \equiv 2a_5 \pmod 3$, and $a_7 \equiv 2 \pmod 3$ and the equation can be rewritten:

$$E_8^3 = e_1^{a_1}e_2^{2a_1+a_3}e_3^{a_3}e_4^{a_3}e_5^{a_5}e_6^{2a_5}e_7^2e_8.$$

Conjugating with σ_1 gives the following relations:

$$e_3^{1-\sigma_1} = e_1^{-1}e_2e_4^3,\ \ e_4^{1-\sigma_1} = e_1^{-1}e_2e_3^{-1}e_4^3,\ \ e_5^{1-\sigma_1} = e_5e_6^{-1},\ \ e_6^{1-\sigma_1} = e_5e_6^2,\ \ e_7^{1-\sigma_1} = e_7e_8^{-1},\ \ e_8^{1-\sigma_1} = e_7e_8^2.$$

Thus $(E_8^{1-\sigma_1})^3 = (e_1^{-1}e_2e_4^3)^{a_3}(e_1^{-1}e_2e_3^{-1}e_4^3)^{a_3}(e_5e_6^{-1})^{a_5}(e_5e_6^2)^{2a_5}(e_7e_8^{-1})^2e_7e_8^2$. Combining like terms, moving the cubes to the left and renaming $(E_8^{1-\sigma_1})^3$ gives

$$(E_8^*)^3 = \frac{{\epsilon'_1}^{2a_3}}{\epsilon_1^{2a_3}\epsilon_2^{a_3}}$$

and all exponents must be divisible by 3 so $a_3 \equiv 0 \pmod 3$. Now

$$E_8^3 = (\epsilon_1{\epsilon'_1}^2)^{a_1}(\epsilon_3{\epsilon'_3}^2)^{a_5}\epsilon_4^2\epsilon'_4$$

which has no solution in L unless $N(B_1) \sim \zeta^a N(B_2) \sim \zeta^c N(B_4)$ hence $a_{88} \neq 3 \implies a_{88} = 1$ and $b^* = 1$.

Suppose $N(B_1) \sim \zeta^a N(B_2) \sim \zeta^c N(B_3)$**:** $\hat{E}_0 =< \epsilon_1, \epsilon_1', \epsilon_2, \epsilon_2', \epsilon_3, \epsilon_3', \epsilon_4, \epsilon_4' >$

Any three of the norms can be similar in this case but the fourth will always be different.

We know from [9] Theorem X that $e_a^3 = \zeta^a(\epsilon_1\epsilon_2/\epsilon_1'\epsilon_2')$ and $e_c^3 = \zeta^c(\epsilon_1\epsilon_3/\epsilon_1'\epsilon_3')$ both have solutions e_a and e_c in L so b^* is at least 2. Let $\hat{E}_1 =< \epsilon_1, \epsilon_1', \epsilon_2, \sqrt[3]{\frac{\epsilon_1\epsilon_2}{\epsilon_1'\epsilon_2'}}, \epsilon_3, \sqrt[3]{\frac{\epsilon_1\epsilon_3}{\epsilon_1'\epsilon_3'}}, \epsilon_4, \epsilon_4' >$. Then $\hat{E}_0 \subset \hat{E}_1$ and we will show that $\left[\hat{E} : \hat{E}_1\right] = 1$, thus $\left[\hat{E} : \hat{E}_0\right] = \left[\hat{E} : \hat{E}_1\right]\left[\hat{E}_1 : \hat{E}_0\right] = 3^2$

This case is similar to Case 3 with three Type III fields so we know that $a_{11} = a_{22} = a_{33} = a_{44} = a_{55} = a_{66} = a_{77} = 1$. We will again number the basis elements e_1 to e_8 and find some relations between the conjugates.

$e_i\overline{e}_i = e_i^2$ for $i = 1, 3, 5, 7$, $\quad e_2\overline{e}_2 = e_1^{-1}$, $\quad e_4\overline{e}_4 = e_1e_3$, $\quad e_6\overline{e}_6 = e_1e_5$, $\quad e_8\overline{e}_8 = e_7^{-1}$

Suppose $a_{88} = 3$. Then $E_8^3 = e_1^{a_1}e_2^{a_2}e_3^{a_3}e_4^{a_4}e_5^{a_5}e_6^{a_6}e_7^{a_7}e_8$ and multiplying by the complex conjugate gives: $(E_8\overline{E}_8)^3 = e_1^{2a_1-a_2+a_4+a_6}e_3^{2a_3+a_4}e_5^{2a_5+a_6}e_7^{2a_7+1}$.

This gives the following relations on the exponents: $a_2 \equiv 2a_1 + a_4 + a_6 \pmod 3$, $a_4 \equiv a_3 \pmod 3$, $a_6 \equiv a_5 \pmod 3$, and $a_7 \equiv 2 \pmod 3$ and the equation can be rewritten: $E_8^3 = e_1^{a_1}e_2^{2a_1+a_3+a_5}e_3^{a_3}e_4^{a_3}e_5^{a_5}e_6^{a_5}e_7^2e_8$.

We will again take conjugates with σ_1.

$e_3^{1-\sigma_1} = e_1e_2^{-1}e_3^3e_4^{-3}$, $\quad e_4^{1-\sigma_1} = e_1^{-1}e_2e_3^{-1}e_4^3$, $\quad e_5^{1-\sigma_1} = e_1^{-1}e_2e_6^3$, $\quad e_6^{1-\sigma_1} = e_1^{-1}e_2e_5^{-1}e_6^3$, $\quad e_7^{1-\sigma_1} = e_7e_8^{-1}$, $e_8^{1-\sigma_1} = e_7e_8^2$

Then $(E_8^{1-\sigma_1})^3 = (e_1e_2^{-1}e_3^3e_4^{-3})^{a_3}(e_1^{-1}e_2e_3^{-1}e_4^3)^{2a_3+a_5}(e_1^{-1}e_2e_6^3)^{a_5}(e_1^{-1}e_2e_5^{-1}e_6^3)^{a_5}(e_7e_8^{-1})^2e_7^2e_8^2$.

Combining like terms, moving the cubes to the left and renaming $(E_8^{1-\sigma_1})^3 = (E_8^*)^3$

$$(E_8^*)^3 = e_1^{-a_5}e_2^{a_5}e_3^{-a_3} = \frac{{\epsilon_1'}^{a_5}}{\epsilon_1^{a_5}\epsilon_2^{a_3}}.$$

Then $a_5 = a_3 = 0$ so $E_8^3 = e_1^{a_1}e_2^{2a_1}e_7^2e_8 = \epsilon_1^{a_1}{\epsilon_1'}^{2a_1}\epsilon_4^2\epsilon_4'$ This can have no solution because $N(B_1) \nsim N(B_4)$ hence $a_{88} \neq 3 \implies a_{88} = 1$ and $b^* = 2$.

Suppose $N(B_1) \sim N(B_2) \nsim N(B_3) \sim N(B_4)$**:** $\hat{E}_0 =< \epsilon_1, \epsilon_1', \epsilon_2, \epsilon_2', \epsilon_3, \epsilon_3', \epsilon_4, \epsilon_4' >$

We know from [9] Theorem X that $e^3 = \zeta^a(\epsilon_1\epsilon_2/\epsilon_1'\epsilon_2')$ and $e^3 = \zeta^c(\epsilon_3\epsilon_4/\epsilon_3'\epsilon_4')$ both have solutions in L so b^* is at least 2. Let $\hat{E}_1 =< \epsilon_1, \epsilon_1', \epsilon_2, \sqrt[3]{\frac{\epsilon_1\epsilon_2}{\epsilon_1'\epsilon_2'}}, \epsilon_3, \epsilon_3', \epsilon_4, \sqrt[3]{\frac{\epsilon_3\epsilon_4}{\epsilon_3'\epsilon_4'}} >$. Then $\hat{E}_0 \subset \hat{E}_1$ and we will show that $\left[\hat{E} : \hat{E}_1\right] = 1$, thus $\left[\hat{E} : \hat{E}_0\right] = \left[\hat{E} : \hat{E}_1\right]\left[\hat{E}_1 : \hat{E}_0\right] = 3^2$

This case is similar to Case 3 with three Type III fields so we know that $a_{11} = a_{22} = a_{33} = a_{44} = a_{55} = a_{66} = a_{77} = 1$. We will again number the basis elements e_1 to e_8 and find some relations between the conjugates.

$e_1\overline{e}_1 = e_1^2$, $\quad e_2\overline{e}_2 = e_1^{-1}$, $\quad e_3\overline{e}_3 = e_3^2$, $\quad e_4\overline{e}_4 = e_1e_3$, $\quad e_5\overline{e}_5 = e_5^2$, $\quad e_6\overline{e}_6 = e_5^{-1}$, $\quad e_7\overline{e}_7 = e_7^2$, $\quad e_8\overline{e}_8 = e_5e_7$

Suppose $a_{88} = 3$. Then $E_8^3 = e_1^{a_1} e_2^{a_2} e_3^{a_3} e_4^{a_4} e_5^{a_5} e_6^{a_6} e_7^{a_7} e_8$ and the product with the complex conjugate is $(E_8 \overline{E}_8)^3 = e_1^{2a_1 - a_2 + a_4} e_3^{2a_3 + a_4} e_5^{2a_5 - a_6 + 1} e_7^{2a_7 + 1}$

We get the following relations on the exponents: $a_2 \equiv 2a_1 + a_4 \pmod 3$, $a_4 \equiv a_3 \pmod 3$, $a_6 \equiv 2a_5 + 1 \pmod 3$, and $a_7 \equiv 1 \pmod 3$ and the equation can be rewritten:

$E_8^3 = e_1^{a_1} e_2^{2a_1 + a_4} e_3^{a_3} e_4^{a_3} e_5^{a_5} e_6^{2a_5 + 1} e_7 e_8$.

We will again take conjugates with σ_1.

$e_3^{1-\sigma_1} = e_1^{-1} e_2 e_4^3$, $\quad e_4^{1-\sigma_1} = e_1^{-1} e_2 e_3^{-1} e_4^3$, $\quad e_5^{1-\sigma_1} = e_5 e_6^{-1}$, $\quad e_6^{1-\sigma_1} = e_5 e_6^2$, $\quad e_7^{1-\sigma_1} = e_5^{-1} e_6 e_8^3$,
$e_8^{1-\sigma_1} = e_5^{-1} e_7^{-1} e_8^3$

Then $(E_8^{1-\sigma_1})^3 = (e_1^{-1} e_2 e_4^3)^{a_3} (e_1^{-1} e_2 e_3^{-1} e_4^3)^{a_3} (e_5 e_6^{-1})^{a_5} (e_5 e_6^2)^{2a_5 + 1} (e_5^{-1} e_6 e_8^3)(e_5^{-1} e_7^{-1} e_8^3)$

Combining like terms, moving the cubes to the left and renaming $(E_8^{1-\sigma_1})^3$ we get

$$(E_8^*)^3 = e_1^{-2a_3} {e_2}^{2a_3} e_3^{-a_3} e_5^{-1} e_7^{-1}$$

This can only have no solutions since $a_{11} = \ldots = a_{77} = 1$ hence $a_{88} \neq 3 \implies a_{88} = 1$ and $b^* = 2$.

Suppose $N(B_1)N(B_2) \sim \zeta^a N(B_3)$ **and** $N(B_4) \nsim N(B_i)$ **for** $\mathbf{1 \le i \le 3}$**:**

$\hat{E}_0 =< \epsilon_1, \epsilon_1', \epsilon_2, \epsilon_2', \epsilon_3, \epsilon_3', \epsilon_4, \epsilon_4' >$

We know from [9] Theorem X that $e_a^3 = \zeta^a \frac{\epsilon_1 \epsilon_2 \epsilon_3}{\epsilon_1' \epsilon_2' \epsilon_3''}$ has a solution e_a in L so b^* is at least 1. Let $\hat{E}_1 =< \epsilon_1, \epsilon_1', \epsilon_2, \epsilon_2', \epsilon_3, \sqrt[3]{\frac{\epsilon_1 \epsilon_2 \epsilon_3}{\epsilon_1' \epsilon_2' \epsilon_3''}}, \epsilon_4, \epsilon_4' >$. Then $\hat{E}_0 \subset \hat{E}_1$ and we will show that $\left[\hat{E} : \hat{E}_1\right] = 1$, thus $\left[\hat{E} : \hat{E}_0\right] = \left[\hat{E} : \hat{E}_1\right]\left[\hat{E}_1 : \hat{E}_0\right] = 3^1$.

The same reasoning as in the previous case gives us that $a_{11} = a_{22} = a_{33} = a_{44} = a_{55} = a_{66} = a_{77} = 1$. Suppose $a_{88} = 3$ then $E_8^3 = \epsilon_1^{a_1} {\epsilon_1'}^{a_2} \epsilon_2^{a_3} {\epsilon_2'}^{a_4} \epsilon_3^{a_5} \sqrt[3]{\frac{\epsilon_1 \epsilon_2 \epsilon_3}{\epsilon_1' \epsilon_2' \epsilon_3''}}^{a_6} \epsilon_4^{a_7} \epsilon_4'$. Using the usual technique we multiply by the complex conjugate and simplify to get

$$E_8^3 = \epsilon_1^{2a_1 - a_2 + a_6} \epsilon_2^{2a_3 - a_4 + a_6} \epsilon_3^{2a_5 + a_6} \epsilon_4^{2a_7 - 1}.$$

The following relations on the exponents are clear: $a_2 \equiv 2a_1 + a_5 \pmod 3$, $a_4 \equiv 2a_3 + a_5 \pmod 3$, $a_6 \equiv a_5 \pmod 3$, and $a_7 \equiv 2 \pmod 3$ and the equation can be rewritten

$$E_8^3 = \epsilon_1^{a_1} {\epsilon_1'}^{2a_1 + a_5} \epsilon_2^{a_3} {\epsilon_2'}^{2a_3 + a_5} \epsilon_3^{a_5} \sqrt[3]{\frac{\epsilon_1 \epsilon_2 \epsilon_3}{\epsilon_1' \epsilon_2' \epsilon_3''}}^{a_5} \epsilon_4^2 \epsilon_4'.$$

We can conjugate with $1 - \sigma_1$ and simplify to get

$$(E_8^{1-\sigma_1})^3 = {\epsilon_1}^{a_5} {\epsilon_1'}^{-a_5} \epsilon_2^{2a_5} \epsilon_3^{a_5} \epsilon_4^3$$

so $a_5 = 0$ and $E_8^3 = \epsilon_1^{a_1} {\epsilon_1'}^{2a_1} \epsilon_2^{a_3} {\epsilon_2'}^{2a_3} \epsilon_4^2 \epsilon_4'$. We know this can have no solution because the norms are not similar so $b^* = 1$.

Suppose $N(B_1)N(B_2) \sim \zeta^a N(B_3)$ **and** $N(B_4) \sim N(B_1)$**:** $\hat{E}_0 =< \epsilon_1, \epsilon_1', \epsilon_2, \epsilon_2', \epsilon_3, \epsilon_3', \epsilon_4, \epsilon_4' >$

We know from [9] Theorem X that $e_a^3 = \zeta^a \frac{\epsilon_1\epsilon_2\epsilon_3}{\epsilon_1'\epsilon_2'\epsilon_3''}$ and $e_b^3 = \zeta^b \frac{\epsilon_1\epsilon_4}{\epsilon_1'\epsilon_4'}$ have a solutions in L so b^* is at least 2. Let $\hat{E}_1 =< \epsilon_1, \epsilon_1', \epsilon_2, \epsilon_2', \epsilon_3, \sqrt[3]{\frac{\epsilon_1\epsilon_2\epsilon_3}{\epsilon_1'\epsilon_2'\epsilon_3''}}, \epsilon_4, \sqrt[3]{\frac{\epsilon_1\epsilon_4}{\epsilon_1'\epsilon_4'}} >$. Then $\hat{E}_0 \subset \hat{E}_1$ and we will show that $\left[\hat{E} : \hat{E}_1\right] = 1$, thus $\left[\hat{E} : \hat{E}_0\right] = \left[\hat{E} : \hat{E}_1\right]\left[\hat{E}_1 : \hat{E}_0\right] = 3^2$.

The same reasoning as in the previous case gives us that $a_{11} = a_{22} = a_{33} = a_{44} = a_{55} = a_{66} = a_{77} = 1$. Suppose $a_{88} = 3$ then $E_8^3 = \epsilon_1^{a_1}{\epsilon_1'}^{a_2}\epsilon_2^{a_3}{\epsilon_2'}^{a_4}\epsilon_3^{a_5}\sqrt[3]{\frac{\epsilon_1\epsilon_2\epsilon_3}{\epsilon_1'\epsilon_2'\epsilon_3''}}^{a_6}\epsilon_4^{a_7}\sqrt[3]{\frac{\epsilon_1\epsilon_4}{\epsilon_1'\epsilon_4'}}$. Using the usual technique we multiply by the complex conjugate and simplify to get

$$E_8^3 = \epsilon_1^{2a_1 - a_2 + a_6 + 1}\epsilon_2^{2a_3 - a_4 + a_6}\epsilon_3^{2a_5 + a_6}\epsilon_4^{2a_7 + 1}.$$

The following relations on the exponents are clear: $a_2 \equiv 2a_1 + a_5 + 1 \pmod 3$, $a_4 \equiv 2a_3 + a_5 \pmod 3$, $a_6 \equiv a_5 \pmod 3$, and $a_7 \equiv 1 \pmod 3$ and the equation can be rewritten

$$E_8^3 = \epsilon_1^{a_1}{\epsilon_1'}^{2a_1 + a_5 + 1}\epsilon_2^{a_3}{\epsilon_2'}^{2a_3 + a_5}\epsilon_3^{a_5}\sqrt[3]{\frac{\epsilon_1\epsilon_2\epsilon_3}{\epsilon_1'\epsilon_2'\epsilon_3''}}^{a_5}\epsilon_4\sqrt[3]{\frac{\epsilon_1\epsilon_4}{\epsilon_1'\epsilon_4'}}.$$

Now we conjugate with σ_1 and take the quotient to get

$$(E_8^{1-\sigma_1})^3 \approx {\epsilon_1}^{a_5 - 2}{\epsilon_1'}^{-a_5 + 2}\epsilon_3^{a_5}\epsilon_4^{-1}$$

which has no solution because the exponents are not all divisible by 3. Thus $a_{88} = 1$ and $b^* = 2$.

Suppose $N(B_1)N(B_2)N(B_3) \sim \zeta^a N(B_4)$ **or** $N(B_1)N(B_2) \sim \zeta^a N(B_3)N(B_4)$**:**

$\hat{E}_0 =< \epsilon_1, \epsilon_1', \epsilon_2, \epsilon_2', \epsilon_3, \epsilon_3', \epsilon_4, \epsilon_4' >$

We know that $e_a^3 = \zeta^a \frac{\epsilon_1\epsilon_2\epsilon_3\epsilon_4}{\epsilon_1'\epsilon_2''\epsilon_3'\epsilon_4'}$ has a solutions in L so b^* is at least 1. Let $\hat{E}_1 =< \epsilon_1, \epsilon_1', \epsilon_2, \epsilon_2', \epsilon_3, \epsilon_3', \epsilon_4, \sqrt[3]{\frac{\epsilon_1\epsilon_2\epsilon_3\epsilon_4}{\epsilon_1'\epsilon_2''\epsilon_3'\epsilon_4'}} >$. Then $\hat{E}_0 \subset \hat{E}_1$ and we will show that $\left[\hat{E} : \hat{E}_1\right] = 1$, thus $\left[\hat{E} : \hat{E}_0\right] = \left[\hat{E} : \hat{E}_1\right]\left[\hat{E}_1 : \hat{E}_0\right] = 3^1$.

The same reasoning as in the previous case gives us that $a_{11} = a_{22} = a_{33} = a_{44} = a_{55} = a_{66} = a_{77} = 1$. Suppose $a_{88} = 3$ then $E_8^3 = \epsilon_1^{a_1}{\epsilon_1'}^{a_2}\epsilon_2^{a_3}{\epsilon_2'}^{a_4}\epsilon_3^{a_5}{\epsilon_3'}^{a_6}\epsilon_4^{a_7}\sqrt[3]{\frac{\epsilon_1\epsilon_2\epsilon_3\epsilon_4}{\epsilon_1'\epsilon_2''\epsilon_3'\epsilon_4'}}$. Using the usual technique we multiply by the complex conjugate and simplify to get

$$E_8^3 = \epsilon_1^{2a_1 - a_2 + 1}\epsilon_2^{2a_3 - a_4 + 1}\epsilon_3^{2a_5 - a_6 + 1}\epsilon_4^{2a_7 + 1}.$$

The following relations on the exponents are clear: $a_2 \equiv 2a_1 + 1 \pmod 3$, $a_4 \equiv 2a_3 + 1 \pmod 3$, $a_6 \equiv 2a_5 + 1 \pmod 3$, and $a_7 \equiv 1 \pmod 3$ and the equation can be rewritten

$$E_8^3 = \epsilon_1^{a_1}{\epsilon_1'}^{2a_1 + 1}\epsilon_2^{a_3}{\epsilon_2'}^{2a_3 + 1}\epsilon_3^{a_5}{\epsilon_3'}^{2a_5 + 1}\epsilon_4\sqrt[3]{\frac{\epsilon_1\epsilon_2\epsilon_3\epsilon_4}{\epsilon_1'\epsilon_2''\epsilon_3'\epsilon_4'}}.$$

Now we conjugate with σ_1 and take the quotient and simplify to get

$$(E_8^*)^3 = \epsilon_1{}^{-1}\epsilon_1'\epsilon_2'\epsilon_4^{-1}$$

which has no solution because the exponents are not all divisible by 3. Thus $a_{88} = 1$ and $b^* = 1$ and the theorem is proved.

We can see from Theorem 6.9 that to find units in L that are not contained in any of the subfields we need to have at least two of the cubic subfields to be Type III or for K to be of Kind 1 or 2. We will now describe a procedure for calculating the units in L as well as the basis $\hat{E}$ given that we have a basis $\hat{e}$ for K. The units from the Type III fields will be discussed near the end of this section. Theorem 6.10 will show that we can use Hilbert's Theorem 90 to find units in L when K is of Kind 1 or 2.

Theorem 6.10 *Let $E_L \in \hat{E}$ and $E_L \notin \hat{e}\hat{\epsilon}_i$ for any i and K be of Kind 1 or 2. If $\{e_1, e_2, e_3, e_4\}$ is a basis for $\hat{e}$ then $\frac{E_L}{E_L^\sigma} = e_1^{c_1}e_2^{c_2}e_3^{c_3}e_4^{c_4}$ for some σ where $0 \leq c_i \leq 2$ for i = 1, 2, 3, 4.*

Proof: For K being of Kind 1 or 2 the basis elements for the units of K can be chosen with elements of the form $e = \epsilon_k$ or $e^3 = \epsilon_i^{b_i}\epsilon_j^{b_j}\epsilon_k$ where $0 \leq b_i, b_j \leq 2$. So for each unit in K there is some base field k_l such that the norm $N_{K/k_l}(e) = 1$ and by renumbering our fields we can always choose $k_l = k_1$. We will then write $\sigma = \sigma_1$ and $N(\alpha) = N_{K/k_1}(\alpha)$ for $\alpha \in K$.

We would like to show that the units of L, which are not contained in any of the subfields, can be constructed from units of K using Hilbert's Theorem 90. We will then have to consider the different Kinds of K as separate cases since they have distinct bases.

Case 1: Suppose K is of Kind 1. Then a basis for $\hat{e}$ can be chosen as $e_1 = \epsilon_1$, $e_2^3 = \epsilon_1^a\epsilon_2$, $e_3^3 = \epsilon_1^b\epsilon_3$, $e_4^3 = \epsilon_1^c\epsilon_4$ and $\hat{\epsilon}_1 =< \epsilon_1, u_1, e_2, e_2', e_3, e_3', e_4, e_4', >$.

In this case we have that $\hat{e}\hat{\epsilon}_1 = \hat{\epsilon}_1$ because $\hat{e} \subseteq \hat{\epsilon}_1$ since $u_i = \sqrt[3]{\frac{\epsilon_i}{\epsilon_i'}}$ for $i = 2, 3, 4$ and clearly $\hat{\epsilon}_1 \subseteq \hat{e}\hat{\epsilon}_1$. Since $\hat{E}^3 \subseteq \hat{\epsilon}_1$ then we can write

$$E_L^3 = \epsilon_1^{a_1}u_1^{a_2}e_2^{a_3}e_2'^{\,a_4}e_3^{a_5}e_3'^{\,a_6}e_4^{a_7}e_4'^{\,a_8}. \tag{6.11}$$

Note: $e_2e_2'e_2'' = \sqrt[3]{\epsilon_1^{3a}\epsilon_2\epsilon_2'\epsilon_2''} = \epsilon_1^a$ and similarly $e_3e_3'e_3'' = \epsilon_1^b$ and $e_4e_4'e_4'' = \epsilon_1^c$
Then if we multiply E_L^3 by its complex conjugate we get

$$\begin{aligned}(E_L^{1+\tau})^3 &= \epsilon_1^{2a_1+a_2}e_2^{2a_3}e_2'^{\,a_4}e_2''^{a_4}e_3^{2a_5}e_3'^{\,a_6}e_3''^{a_6}e_4^{2a_7}e_4'^{\,a_8}e_4''^{a_8} \\ &= \epsilon_1^{2a_1+a_2+a\cdot a_4+b\cdot a_6+c\cdot a_8}e_2^{2a_3-a_4}e_3^{2a_5-a_6}e_4^{2a_7-a_8}\end{aligned}$$

Since $E_L^{1+\tau} \in K$ then all the exponents must be divisible by 3 which gives the following relations:

$$a_4 \equiv 2a_3 \pmod 3$$
$$a_6 \equiv 2a_5 \pmod 3$$
$$a_8 \equiv 2a_7 \pmod 3$$
$$a_2 \equiv a_1 + a \cdot a_3 + b \cdot a_5 + c \cdot a_7 \pmod 3$$

And equation (6.11) can be written as

$$E_L^3 = \epsilon_1^{a_1} u_1^{a_2} e_2^{a_3} {e'_2}^{2a_3} e_3^{a_5} {e'_3}^{2a_5} e_4^{a_7} {e'_4}^{2a_7} \tag{6.12}$$

Note: ${e'_i}^{\sigma\tau} = {e_i}^{\sigma^2\tau} = (e_i^\tau)^\sigma = e_i^\sigma = e'_i$ and ${e_i}^{\sigma\tau} = (e_i^\tau)^{\sigma^2} = e_i^{\sigma^2} = e''_i$ for i = 2, 3, 4.

Then

$$\begin{aligned}(E_L^{1+\sigma\tau})^3 &= \epsilon_1^{2a_1+a_2}(e_2 {e'_2}^4 e''_2)^{a_3}(e_3 {e'_3}^4 e''_3)^{a_5}(e_4 {e'_4}^4 e''_4)^{a_7} \\ &= \epsilon_1^{2a_1+a_2+a\cdot a_3+b\cdot a_5+c\cdot a_7}(e'_2)^{3a_3}(e'_3)^{3a_5}(e'_4)^{3a_7}\end{aligned}$$

So we now have two relationships for the exponent of ϵ_1

$$\begin{aligned}2a_1 + a_2 + a \cdot a_3 + b \cdot a_5 + c \cdot a_7 &\equiv 0 \pmod 3 \\ 2a_1 + a_2 + 2(a \cdot a_3 + b \cdot a_5 + c \cdot a_7) &\equiv 0 \pmod 3,\end{aligned}$$

which we can add together to get

$$\begin{aligned}4a_1 + 2a_2 &\equiv 0 \pmod 3 \\ a_2 &\equiv a_1 \pmod 3.\end{aligned}$$

So equation (6.12) becomes

$$E_L^3 = \epsilon_1^{a_1} u_1^{a_1} e_2^{a_3} {e'_2}^{2a_3} e_3^{a_5} {e'_3}^{2a_5} e_4^{a_7} {e'_4}^{2a_7}$$

and

$$a \cdot a_3 + b \cdot a_5 + c \cdot a_7 \equiv 0 \pmod 3 \tag{6.13}$$

To use Hilbert's Theorem 90 we need to find $e \in K$ such that $N(e) = 1$. We have that $a \cdot a_3 + b \cdot a_5 + c \cdot a_7 = 3t$ so there are 2 possible cases for the solutions to the equation which are based on whether a, b, or c are zero or not. If $\{a, b, c\} = \{0, 0, 0\}$ then there are three independent solutions for $\{a_3, a_5, a_7\}$, otherwise there are two independent solutions for $\{a_3, a_5, a_7\}$.

For each case let $\mathfrak{e}_i = e_2^{a_{3_i}} e_3^{a_{5_i}} e_4^{a_{7_i}} \epsilon_1^{-t_i}$ where $\{a_{3_i}, a_{5_i}, a_{7_i}\}$ is a solution to $a \cdot a_{3_i} + b \cdot a_{5_i} + c \cdot a_{7_i} = 3t_i$ and $t_i = (a \cdot a_{3_i} + b \cdot a_{5_i} + c \cdot a_{7_i})/3$ where i = 1, 2 in the case of two solutions and i = 1, 2, 3 if there are three. Then $N(\mathfrak{e}_\mathfrak{i}) = \epsilon_1^{a \cdot a_{3_i} + b \cdot a_{5_i} + c \cdot a_{7_i}} \epsilon_1^{-3t_i} = 1$ and for each i we can write $B_i^{1-\sigma} = \mathfrak{e}_i$ and $\frac{B_i^3}{N(B_i)} = \frac{\mathfrak{e}_i}{\mathfrak{e}_i^{\sigma^2}}$.

Without loss of generality suppose there are two independent solutions to (6.13). If $N(B_1^{d_1}B_2^{d_2}) = \alpha_j^3$ for some $\alpha_j \in K_1$ has solution(s) $\{d_1, d_2\}$, not both zero, then there are units in L that are not products of the units of the subfields, otherwise there are none. Let $\{d_1, d_2\}$ be one of those solutions, then

$$\begin{aligned} \mathfrak{e}_1^{d_1}\mathfrak{e}_2^{d_2} &= (e_2^{a_{3_1}}e_3^{a_{5_1}}e_4^{a_{7_1}}\epsilon_1^{-t_1})^{d_1}(e_2^{a_{3_2}}e_3^{a_{5_2}}e_4^{a_{7_2}}\epsilon_1^{-t_2})^{d_2} \\ &= e_2^{a_{3_1}d_1+a_{3_2}d_2}e_3^{a_{5_1}d_1+a_{5_2}d_2}e_4^{a_{7_1}d_1+a_{7_2}d_2}\epsilon_1^{-t_1d_1-t_2d_2} \end{aligned}$$

and we can rename the exponents to get $\mathfrak{e}_1^{d_1}\mathfrak{e}_2^{d_2} = e_2^{a_3}e_3^{a_5}e_4^{a_7}\epsilon_1^{-t}$. Now,

$$\left(\frac{B_1^{d_1}B_2^{d_2}}{\alpha_j}\right)^3 = \frac{\mathfrak{e}_1^{d_1}\mathfrak{e}_2^{d_2}}{(\mathfrak{e}_1^{d_1}\mathfrak{e}_2^{d_2})^{\sigma^2}} = \frac{e_2^{a_3}e_3^{a_5}e_4^{a_7}\epsilon_1^{-t}}{e_2''^{a_3}e_3''^{a_5}e_4''^{a_7}\epsilon_1^{-t}} = \frac{e_2^{2a_3}e_2'^{\,a_3}e_3^{2a_5}e_3'^{\,a_5}e_4^{2a_7}e_4'^{\,a_7}}{N(e_2^{a_3}e_3^{a_5}e_4^{a_7})} = e_2^{2a_3}e_2'^{\,a_3}e_3^{2a_5}e_3'^{\,a_5}e_4^{2a_7}e_4'^{\,a_7}\epsilon_1^{3t}$$

Then we can square both sides, let $\left(\frac{B_1^{d_1}B_2^{d_2}}{\alpha_j}\right)^2 = E_j$ and reduce modulo cubes to get

$$(E_j)^3 = e_2^{a_3}e_2'^{\,2a_3}e_3^{a_5}e_3'^{\,2a_5}e_4^{a_7}e_4'^{\,2a_7}$$

so $E_j = E_L$.

If there are three solutions to (6.13) then there may be three solution for $N(B_1^{d_1}B_2^{d_2}B_3^{d_3}) = \alpha_j^3$ for some $\alpha_j \in K_1$ but Theorem 6.9 shows there will be at most two independent solutions for E_L.

Case 2: Suppose K is of Kind 2. Then a basis for $\hat{e}$ can be chosen in two ways depending on whether or not one of the subfields is Type III. If all the subfields are Type I or IV then the basis can be chosen as $e_1 = \epsilon_1$, $e_2 = \epsilon_2$, $e_3^3 = \epsilon_1^a\epsilon_2^b\epsilon_3$, $e_4^3 = \epsilon_1^c\epsilon_2^d\epsilon_4$ and $\hat{\epsilon}\hat{\epsilon}_1 =< \epsilon_1, u_1, \epsilon_2, u_2, e_3, e_3', e_4, e_4', >$. Since $\hat{E}^3 \subseteq \hat{\epsilon}\hat{\epsilon}_1$ then we can write

$$E_L^3 = \epsilon_1^{a_1}u_1^{a_2}\epsilon_2^{a_3}u_2^{a_4}e_3^{a_5}e_3'^{\,a_6}e_4^{a_7}e_4'^{\,a_8} \tag{6.14}$$

Note: $e_3e_3'e_3'' = \sqrt[3]{\epsilon_1^{3a}(\epsilon_2\epsilon_2'\epsilon_2'')^b\epsilon_3\epsilon_3'\epsilon_3''} = \epsilon_1^a$ and similarly $e_4e_4'e_4'' = \epsilon_1^c$

Then if we multiply E_L^3 by its complex conjugate we get

$$\begin{aligned} (E_L^{1+\tau})^3 &= \epsilon_1^{2a_1+a_2}\epsilon_2^{2a_3+a_4}e_3^{2a_5}e_3'^{\,a_6}e_3''^{a_6}e_4^{2a_7}e_4'^{\,a_8}e_4''^{a_8} \\ &= \epsilon_1^{2a_1+a_2+a\cdot a_6+c\cdot a_8}\epsilon_2^{2a_3+a_4}e_3^{2a_5-a_6}e_4^{2a_7-a_8} \end{aligned}$$

Since $E_L^{1+\tau} \in K$ then all the exponents must be divisible by 3 which gives the following relations:

$$a_6 \equiv 2a_5 \pmod 3, \quad a_8 \equiv 2a_7 \pmod 3, \quad a_4 \equiv a_3 \pmod 3, \quad 0 \equiv 2a_1 + a_2 + a\cdot 2a_5 + c\cdot 2a_7 \pmod 3.$$

Then we have that

$$E_L^3 = \epsilon_1^{a_1}u_1^{a_2}\epsilon_2^{a_3}u_2{}^{a_3}e_3^{a_5}e_3'^{\,2a_5}e_4^{a_7}e_4'^{\,2a_7}$$

Note: ${e'_i}^{\sigma\tau} = e'_i$ and $e_i^{\sigma\tau} = e''_i$ for $i = 3$, 4 and $u_2^{1+\sigma\tau} = \epsilon_2^{1+\sigma\tau} = {\epsilon'_2}^{-1}$.

Then

$$\begin{aligned}(E_L^{1+\sigma\tau})^3 &= \epsilon_1^{2a_1+a_2}{\epsilon'_2}^{-2a_3}(e_3{e'_3}^4 e''_3)^{a_5}(e_4{e'_4}^4 e''_4)^{a_7} \\ &= \epsilon_1^{2a_1+a_2+a\cdot a_5+c\cdot a_7}{\epsilon'_2}^{-2a_3}{e'_3}^{3a_5}{e'_4}^{3a_7}.\end{aligned}$$

This shows that $a_3 = 0$ and provides a second congruence for the exponent on ϵ_1.

$$\begin{aligned}2a_1 + a_2 + a\cdot 2a_5 + c\cdot 2a_7 &\equiv 0 \pmod 3) \\ 2a_1 + a_2 + a\cdot a_5 + c\cdot a_7 &\equiv 0 \pmod 3).\end{aligned}$$

These two together show that

$$\begin{aligned}a_2 &\equiv a_1 \pmod 3) \\ 0 &\equiv a\cdot a_5 + c\cdot a_7 \pmod 3)\end{aligned} \tag{6.15}$$

and equation (6.14) becomes

$$E_L^3 = \epsilon_1^{a_1}u_1^{a_1}e_3^{a_5}{e'_3}^{2a_5}e_4^{a_7}{e'_4}^{2a_7}.$$

To use Hilbert's Theorem 90 we need to find $e \in K$ such that $N(e) = 1$. We have that $a\cdot a_5 + c\cdot a_7 = 3t$ so there are 2 possible cases for the solutions to the equation which are based on whether a, or c are zero or not. If $\{a, c\} = \{0, 0\}$ then there are two independent solutions for $\{a_5, a_7\}$, otherwise there is one solutions for $\{a_5, a_7\}$.

For each case let $\mathfrak{e}_i = e_3^{a_{5_i}}e_4^{a_{7_i}}\epsilon_1^{-t_i}$ where $\{a_{5_i}, a_{7_i}\}$ is a solution to $a\cdot a_{5_i} + c\cdot a_{7_i} = 3t_i$ and $t_i = (a\cdot a_{5_i} + c\cdot a_{7_i})/3$ where $i = 1$ in the case of one solution and $i = 1, 2$ if there are two. Then $N(\mathfrak{e}_\mathfrak{i}) = \epsilon_1^{a\cdot a_{5_i}+c\cdot a_{7_i}}\epsilon_1^{-3t_i} = 1$ and for each i we can write $B_i^{1-\sigma} = \mathfrak{e}_i$ and $\frac{B_i^3}{N(B_i)} = \frac{\mathfrak{e}_i}{\mathfrak{e}_i^{\sigma^2}}$.

Without loss of generality suppose there are two independent solutions to (6.15). If $N(B_1^{d_1}B_2^{d_2}) = \alpha_j^3$ for some $\alpha_j \in K_1$ has solution(s) $\{d_1, d_2\}$, not both zero, then there are units in L that are not products of the units of the subfields, otherwise there are none. Let $\{d_1, d_2\}$ be one of those solutions, then

$$\begin{aligned}\mathfrak{e}_1^{d_1}\mathfrak{e}_2^{d_2} &= (e_3^{a_{5_1}}e_4^{a_{7_1}}\epsilon_1^{-t_1})^{d_1}(e_3^{a_{5_2}}e_4^{a_{7_2}}\epsilon_1^{-t_2})^{d_2} \\ &= e_3^{a_{5_1}d_1+a_{5_2}d_2}e_4^{a_{7_1}d_1+a_{7_2}d_2}\epsilon_1^{-t_1d_1-t_2d_2}\end{aligned}$$

and we can rename the exponents to get $\mathfrak{e}_1^{d_1}\mathfrak{e}_2^{d_2} = e_3^{a_5}e_4^{a_7}\epsilon_1^{-t}$. Now,

$$\left(\frac{B_1^{d_1}B_2^{d_2}}{\alpha_j}\right)^3 = \frac{\mathfrak{e}_1^{d_1}\mathfrak{e}_2^{d_2}}{(\mathfrak{e}_1^{d_1}\mathfrak{e}_2^{d_2})^{\sigma^2}} = \frac{e_3^{a_5}e_4^{a_7}\epsilon_1^{-t}}{{e''_3}^{a_5}{e''_4}^{a_7}\epsilon_1^{-t}} = \frac{e_3^{2a_5}{e'_3}^{a_5}e_4^{2a_7}{e'_4}^{a_7}}{N(e_3^{a_5}e_4^{a_7})} = e_3^{2a_5}{e'_3}^{a_5}e_4^{2a_7}{e'_4}^{a_7}\epsilon_1^{3t}$$

Then we can square both sides, let $\left(\frac{B_1^{d_1}B_2^{d_2}}{\alpha_j}\right)^2 = E_j$ and reduce modulo cubes to get

$$(E_j)^3 = e_3^{a_5}{e'_3}^{2a_5}e_4^{a_7}{e'_4}^{2a_7}$$

so $E_j = E_L$.

If there are two solutions to (6.15) then there may be two solution for $N(B_1^{d_1}B_2^{d_2}) = \alpha_j^3$ for some $\alpha_j \in K_1$ but Theorem 6.9 shows there will be at most one independent solution for E_L.

If one of the subfields is Type III then let k_2 be that field and the basis can be chosen as $e_1 = \epsilon_1$, $e_2 = \epsilon_2$, $e_3^3 = \epsilon_1^a\epsilon_3$, $e_4^3 = \epsilon_1^b\epsilon_4$ and $\hat{\epsilon}\hat{\epsilon}_1 =< \epsilon_1, u_1, \epsilon_2, \epsilon'_2, e_3, e'_3, e_4, e'_4, >$. Since $\hat{E}^3 \subseteq \hat{\epsilon}\hat{\epsilon}_1$ then we can write

$$E_L^3 = \epsilon_1^{a_1}u_1^{a_2}\epsilon_2^{a_3}{\epsilon'_2}^{a_4}e_3^{a_5}{e'_3}^{a_6}e_4^{a_7}{e'_4}^{a_8}. \tag{6.16}$$

Note: $e_3e'_3e''_3 = \epsilon_1^a$ and similarly $e_4e'_4e''_4 = \epsilon_1^b$

Proceeding the same way as before:

$$\begin{aligned}(E_L^{1+\tau})^3 &= \epsilon_1^{2a_1+a_2}\epsilon_2^{2a_3}{\epsilon'_2}^{a_4}{\epsilon''_2}^{a_4}e_3^{2a_5}{e'_3}^{a_6}{e''_3}^{a_6}e_4^{2a_7}{e'_4}^{a_8}{e''_4}^{a_8}\\ &= \epsilon_1^{2a_1+a_2+a\cdot a_6+b\cdot a_8}\epsilon_2^{2a_3-a_4}e_3^{2a_5-a_6}e_4^{2a_7-a_8}.\end{aligned}$$

Since $E_L^{1+\tau} \in K$ then all the exponents must be divisible by 3 which gives the following relations:

$$a_4 \equiv 2a_3 \pmod 3,\quad a_6 \equiv 2a_5 \pmod 3,\quad a_8 \equiv 2a_7 \pmod 3,\quad 0 \equiv 2a_1 + a_2 + a\cdot 2a_5 + b\cdot 2a_7 \pmod 3.$$

And we have that

$$E_L^3 = \epsilon_1^{a_1}u_1^{a_2}\epsilon_2^{a_3}{\epsilon'_2}^{2a_3}e_3^{a_5}{e'_3}^{2a_5}e_4^{a_7}{e'_4}^{2a_7}.$$

Note: ${e'_i}^{\sigma\tau} = e'_i$ and $e_i^{\sigma\tau} = e''_i$ for $i = 3, 4$ and $\epsilon_2^{\sigma\tau} = \epsilon''_2$.

Then

$$\begin{aligned}(E_L^{1+\sigma\tau})^3 &= \epsilon_1^{2a_1+a_2}(\epsilon_2{\epsilon'_2}^4\epsilon_2)^{a_3}(e_3{e'_3}^4e''_3)^{a_5}(e_4{e'_4}^4e''_4)^{a_7}\\ &= \epsilon_1^{2a_1+a_2+a\cdot a_5+b\cdot a_7}{\epsilon'_2}^{3a_3}{e'_3}^{3a_5}{e'_4}^{3a_7}.\end{aligned}$$

This provides a second equation for the exponent on ϵ_1.

$$\begin{aligned}2a_1 + a_2 + a\cdot 2a_5 + b\cdot 2a_7 &\equiv 0 \pmod 3\\ 2a_1 + a_2 + a\cdot a_5 + b\cdot a_7 &\equiv 0 \pmod 3.\end{aligned}$$

These two together show that

$$\begin{aligned}a_2 &\equiv a_1 \pmod 3\\ 0 &\equiv a\cdot a_5 + b\cdot a_7 \pmod 3.\end{aligned} \tag{6.17}$$

Then equation (6.16) becomes

$$E_L^3 = \epsilon_1^{a_1} u_1^{a_1} \epsilon_2^{a_3} \epsilon_2'^{\,2a_3} e_3^{a_5} e_3'^{\,2a_5} e_4^{a_7} e_4'^{\,2a_7}.$$

To use Hilbert's Theorem 90 we need to find $e \in K$ such that $N(e) = 1$. We have that $a \cdot a_5 + b \cdot a_7 = 3t$ so there are 2 possible cases for the solutions to the equation which are based on whether a, or b are zero or not. If $\{a, b\} = \{0, 0\}$ then there are two independent solutions for $\{a_5, a_7\}$, otherwise there is one solutions for $\{a_5, a_7\}$. The value of a_3 is determined by the calculation below.

For each case let $\mathfrak{e}_i = \epsilon_2^{a_{3_i}} e_3^{a_{5_i}} e_4^{a_{7_i}} \epsilon_1^{-t_i}$ where $0 \leq a_{3_i} \leq 2$ and $\{a_{5_i}, a_{7_i}\}$ is a solution to $a \cdot a_{5_i} + b \cdot a_{7_i} = 3t_i$ and $i = 1$ in the case of one solutions and $i = 1$, 2 if there are two. Then $N(\mathfrak{e}_i) = \epsilon_1^{a \cdot a_{5_i} + b \cdot a_{7_i}} \epsilon_1^{-3t} = 1$. For each choice of a_{3_i} and $\{a_{5_i}, a_{7_i}\}$ we can write $B_i^{1-\sigma} = \mathfrak{e}_i$ and $\frac{B_i^3}{N(B_i)} = \frac{\mathfrak{e}_i}{\mathfrak{e}_i^{\sigma^2}}$.

Without loss of generality suppose there are two independent solutions to (6.17). If $N(B_1^{d_1} B_2^{d_2}) = \alpha_j^3$ for some $\alpha_j \in K_1$ has solution(s) $\{d_1, d_2\}$, not both zero, then there are units in L that are not products of the units of the subfields, otherwise there are none. Let $\{d_1, d_2\}$ be one of those solutions, then

$$\begin{aligned} \mathfrak{e}_1^{d_1} \mathfrak{e}_2^{d_2} &= (\epsilon_2^{a_{3_1}} e_3^{a_{5_1}} e_4^{a_{7_1}} \epsilon_1^{-t_1})^{d_1} (\epsilon_2^{a_{3_2}} e_3^{a_{5_2}} e_4^{a_{7_2}} \epsilon_1^{-t_2})^{d_2} \\ &= \epsilon_2^{a_{3_1} d_1 + a_{3_2} d_2} e_3^{a_{5_1} d_1 + a_{5_2} d_2} e_4^{a_{7_1} d_1 + a_{7_2} d_2} \epsilon_1^{-t_1 d_1 - t_2 d_2} \end{aligned}$$

and we can rename the exponents to get $\mathfrak{e}_1^{d_1} \mathfrak{e}_2^{d_2} = \epsilon_2^{a_3} e_3^{a_5} e_4^{a_7} \epsilon_1^{-t}$. Now,

$$\left(\frac{B_1^{d_1} B_2^{d_2}}{\alpha_j} \right)^3 = \frac{\mathfrak{e}_1^{d_1} \mathfrak{e}_2^{d_2}}{(\mathfrak{e}_1^{d_1} \mathfrak{e}_2^{d_2})^{\sigma^2}} = \frac{\epsilon_2^{a_3} e_3^{a_5} e_4^{a_7} \epsilon_1^{-t}}{\epsilon_2''^{a_3} e_3''^{a_5} e_4''^{a_7} \epsilon_1^{-t}} = \frac{\epsilon_2^{2a_3} \epsilon_2'^{\,a_3} e_3^{2a_5} e_3'^{\,a_5} e_4^{2a_7} e_4'^{\,a_7}}{N(\epsilon_2^{a_3} e_3^{a_5} e_4^{a_7})} = \epsilon_2^{2a_3} \epsilon_2'^{\,a_3} e_3^{2a_5} e_3'^{\,a_5} e_4^{2a_7} e_4'^{\,a_7} \epsilon_1^{3t}$$

Then we can square both sides, let $\left(\frac{B_1^{d_1} B_2^{d_2}}{\alpha_j} \right)^2 = E_j$ and reduce modulo cubes to get

$$(E_j)^3 = \epsilon_2^{a_3} \epsilon_2'^{\,2a_3} e_3^{a_5} e_3'^{\,2a_5} e_4^{a_7} e_4'^{\,2a_7}$$

so $E_j = E_L$.

Corollary 6.10.1 will show that our algorithm which will be presented in the next section will produce the new units in L. Corollary 6.10.2 will show that in the case where three of the cubic subfields are of Type IV there can be no new unit in L.

Corollary 6.10.1 *Let K and L be as in Theorem 6.10 and $\{e_1, e_2, e_3, e_4\}$ be a basis for $\hat{e}$.*

(1) Suppose K is of Kind 1,

$$E^3 = \epsilon_1^{a_1} u_1^{a_1} e_2^{a_3} e_2'^{\,2a_3} e_3^{a_5} e_3'^{\,2a_5} e_4^{a_7} e_4'^{\,2a_7}$$

has a solution in L and $a \cdot a_3 + b \cdot a_5 + c \cdot a_7 = 3t$. If $B^{1-\sigma} = e_2^{a_3} e_3^{a_5} e_4^{a_7} \epsilon_1^{-t}$ then $\epsilon_1^d u_1^d N(B) = m_2^j \alpha^3$ for some $\alpha \in K_1$ and $0 \leq j, d \leq 2$.

(2) Suppose K is of Kind 2, all subfields are Type I or IV,

$$E^3 = \epsilon_1^{a_1} u_1^{a_1} e_3^{a_5} {e'_3}^{2a_5} e_4^{a_7} {e'_4}^{2a_7}$$

has a solution in L and $a \cdot a_5 + c \cdot a_7 = 3t$. If $B^{1-\sigma} = e_3^{a_5} e_4^{a_7} \epsilon_1^{-t}$ then $\epsilon_1^d u_1^d N(B) = m_2^j \alpha^3$ for some $\alpha \in K_1$ and $0 \leq j, d \leq 2$.

(3) Suppose K is of Kind 2, k_2 is of Type III,

$$E^3 = \epsilon_1^{a_1} u_1^{a_1} \epsilon_2^{a_3} {\epsilon'_2}^{2a_3} e_3^{a_5} {e'_3}^{2a_5} e_4^{a_7} {e'_4}^{2a_7}$$

has a solution in L and $a \cdot a_5 + c \cdot a_7 = 3t$. If $B^{1-\sigma} = \epsilon_2^{a_3} e_3^{a_5} e_4^{a_7} \epsilon_1^{-t}$ then $\epsilon_1^d u_1^d N(B) = m_2^j \alpha^3$ for some $\alpha \in K_1$ and $0 \leq j, d \leq 2$.

Proof: Since

$$E^3 = \epsilon_1^{a_1} u_1^{a_1} e_2^{a_3} {e'_2}^{2a_3} e_3^{a_5} {e'_3}^{2a_5} e_4^{a_7} {e'_4}^{2a_7}$$

then

$$E^3 \epsilon_1^{-a_1} u_1^{-a_1} = e_2^{a_3} {e'_2}^{2a_3} e_3^{a_5} {e'_3}^{2a_5} e_4^{a_7} {e'_4}^{2a_7}.$$

As in the proof of Theorem 6.10 we have that

$$\frac{(B)^3}{N(B)} = \gamma^3 e_2^{a_3} {e'_2}^{2a_3} e_3^{a_5} {e'_3}^{2a_5} e_4^{a_7} {e'_4}^{2a_7}$$

and by combining these last two equations we get

$$\frac{(B)^3}{N(B)} = \gamma^3 E^3 \epsilon_1^{-a_1} u_1^{-a_1}.$$

Moving all the cubed terms to one side of the equation shows that $\left(\frac{B}{\gamma E}\right)^3 = N(B)\epsilon_1^{-a_1} u_1^{-a_1}$ where $N(B)\epsilon_1^{-a_1} u_1^{-a_1} \in K_1$. Since $N(B)\epsilon_1^{-a_1} u_1^{-a_1}$ is also a cube in K_1,

$$K_1\left(\sqrt[3]{N(B)\epsilon_1^d u_1^d}\right) \subseteq L = K_1\left(\sqrt[3]{m_2}\right)$$

and hence Kummer Theory says that $N(B)\epsilon_1^d u_1^d = m_2^j \alpha^3$ where $\alpha \in K_1$ and $j = 0$, 1, or 2.

The proofs of the other two cases are the same.

Corollary 6.10.2 *Let K be of Kind 2 with at least three of the subfields Type IV then $\hat{E} = \hat{\epsilon}\hat{\epsilon}_i$ for any i.*

Proof: If $E_L \in \hat{E}$ then we know from Theorem 6.10 that

$$E_L^3 = \epsilon_1^{a_1} u_1^{a_1} e_3^{a_5} {e'_3}^{2a_5} e_4^{a_7} {e'_4}^{2a_7} \tag{6.18}$$

with $e_3^3 = \epsilon_1^a \epsilon_2^b \epsilon_3$, $e_4^3 = \epsilon_1^c \epsilon_2^d \epsilon_4$, ${e'_3}^3 = \epsilon_1^a {\epsilon'_2}^b \epsilon'_3$, ${e'_4}^3 = \epsilon_1^c {\epsilon'_2}^d \epsilon'_4$, and $a \cdot a_5 + c \cdot a_7 \equiv 0 \pmod 3$. There are two cases to consider, either one of the subfields is Type I or not. In the first case we will assume without loss of generality that k_3 is Type I and $e_3^3 = \epsilon_3$. Then $a = 0$ so $c \cdot a_7 \equiv 0 (\text{mod } 3) \implies a_7 \equiv 0 (\text{mod } 3)$ since $c \neq 0$ by [9] Corollary I to Theorem X so then (6.18) becomes

$$E_L^3 = \epsilon_1^{a_1} u_1^{a_1} e_3^{a_5} {e'_3}^{2a_5}$$

and we know that has no solution from the proof of Theorem 6.9 Case 2(A).

Suppose all four subfields are Type IV then $abcd \neq 0$ so $a \cdot a_5 + c \cdot a_7 \equiv 0 \pmod 3$ has a solution where a_5 and a_7 are not divisible by 3.
Note: $(\epsilon_1)^{\sigma_2 \tau} = u_1^3 \epsilon_1^{-2}$, $(u_1)^{\sigma_2 \tau} = u_1^2 \epsilon_1^{-1}$, $(e_3)^{\sigma_2 \tau} = \sqrt[3]{{\epsilon''_1}^a \epsilon_2^b \epsilon''_3} = (e_3 e'_3)^{-1} u_1^a \epsilon_2^b u_2^{-b}$, $(e'_3)^{\sigma_2 \tau} = \sqrt[3]{{\epsilon''_1}^a {\epsilon''_2}^b \epsilon'_3} = e'_3 u_1^a \epsilon_1^{-a} u_2^{2b} \epsilon_2^{-b}$, $(e_4)^{\sigma_2 \tau} = \sqrt[3]{{\epsilon''_1}^c \epsilon_2^d \epsilon'_4} = e'_4 \epsilon_1^{-c} u_1^c u_2^d$, and $(e'_4)^{\sigma_2 \tau} = \sqrt[3]{{\epsilon''_1}^c {\epsilon''_2}^d \epsilon_4} = e_4 \epsilon_1^{-c} u_1^c \epsilon_2^{-d} u_2^d$

Then $N_{L/K^{\sigma_2}}(E_L^3) = (E_L^3)^{1+\sigma_2 \tau}$,

$$(E_L^3)^{1+\sigma_2\tau} = \epsilon_1^{-2a_1 - 2a\cdot a_5 - 3c\cdot a_7} u_1^{6a_1 + 3a\cdot a_5 + 3c \cdot a_7} \epsilon_2^{-b\cdot a_5 - 2d\cdot a_7} u_2^{3b\cdot a_5 + 3d\cdot a_7} {e'_3}^{3a_5} e_4^{3a_7} {e'_4}^{3a_7}$$

and some simplification gives

$$(E_L^3)^{1+\sigma_2\tau} = {\epsilon'_1}^{-(2a_1 + a\cdot a_5 + c\cdot a_7)} {\epsilon'_3}^{a_5} {\epsilon_4}^{a_7} {\epsilon'_4}^{a_7}.$$

Now $(E_L)^{1+\sigma_2\tau}$ is in $K^{\sigma_2} = k'_1 k_2$, a field of index 2 which is isomorphic to K, so we can write $(E_L)^{1+\sigma_2\tau}$ in terms of the basis for K^{σ_2}, which is $< \epsilon'_1, \epsilon_2, e_3^*, e_4^* >$ where $(e_3^*)^3 = {\epsilon'_1}^a \epsilon_2^b \epsilon'_3$ and $(e_4^*)^3 = {\epsilon'_1}^c \epsilon_2^d \epsilon''_4$. So

$$(E_L^3)^{1+\sigma_2\tau} = {\epsilon'_1}^{-2a_1 - 2a\cdot a_5} \epsilon_2^{-b\cdot a_5 + d\cdot a_7} (e_3^*)^{3a_5} (e_4^*)^{-3a_7}$$

and since $[E : K^{\sigma_2}] = 2$ then all the exponents must be divisible by 3 so we get the following congruences:

$$\begin{aligned} 2a_1 + 2a \cdot a_5 &\equiv 0 \pmod 3 \\ -b \cdot a_5 + d \cdot a_7 &\equiv 0 \pmod 3. \end{aligned}$$

So we know that $a_5 \equiv a \cdot a_1 \pmod 3$ and $a_7 \equiv abd \cdot a_1 \pmod 3$ so our initial condition $a \cdot a_5 + c \cdot a_7 \equiv 0 \pmod 3$ becomes $(1 + abcd)a_1 \equiv 0 \pmod 3$ and

$$E_L^3 = \epsilon_1^{a_1} u_1^{a_1} e_3^{a\cdot a_1} {e'_3}^{2a\cdot a_1} e_4^{a\cdot b\cdot d\cdot a_1} {e'_4}^{2a\cdot b\cdot d\cdot a_1}.$$

To find another relation on the exponents we norm to K^{σ_4}.

Note: $(\epsilon_1)^{\sigma_4\tau} = u_1^3\epsilon_1^{-2}$, $(u_1)^{\sigma_4\tau} = u_1^2\epsilon_1^{-1}$, $(e_3)^{\sigma_4\tau} = \sqrt[3]{\epsilon_1''^a\epsilon_2''^b\epsilon_3'} = (e_3')u_1^a\epsilon_1^{-a}\epsilon_2^{-b}u_2^{2b}$, $(e_3')^{\sigma_4\tau} = \sqrt[3]{\epsilon_1''^a{\epsilon_2'}^b\epsilon_3} = e_3u_1^a\epsilon_1^{-a}u_2^{-b}$, $(e_4)^{\sigma_4\tau} = \sqrt[3]{\epsilon_1''^c\epsilon_2''^d\epsilon_4} = e_4\epsilon_1^{-c}u_1^cu_2^d\epsilon_2^{-d}$, and $(e_4')^{\sigma_4\tau} = \sqrt[3]{\epsilon_1''^c{\epsilon_2'}^d\epsilon_4''} = (e_4e_4')^{-1}u_1^c\epsilon_2^du_2^{-2d}$.

Then $N_{L/K^{\sigma_4}}(E_L^3) = (E_L^3)^{1+\sigma_4\tau}$,

$$(E_L^3)^{1+\sigma_4\tau} = \epsilon_1^{-5a_1-abcd\cdot a_1}u_1^{9a_1+3abcd\cdot a_1}u_2^{-3ab\cdot a_1}e_3^{3a\cdot a_1}{e_3'}^{3a\cdot a_1}$$

which can be simplified to

$$(E_L^3)^{1+\sigma_4\tau} = {\epsilon_1'}^{-(3a_1+abcd\cdot a_1)}{\epsilon_2'}^{2ab\cdot a_1}(\epsilon_3\epsilon_3')^{a\cdot a_1}.$$

It is again necessary to write this in terms of the basis for $K^{\sigma_4\tau}$, which is $< \epsilon_1', \epsilon_2', e_3^{**}, e_4^{**} >$ where $(e_3^{**})^3 = {\epsilon_1'}^a{\epsilon_2'}^b\epsilon_3''$ and $(e_4^{**})^3 = {\epsilon_1'}^c{\epsilon_2'}^d\epsilon_4$. So we have

$$(E_L^3)^{1+\sigma_4\tau} = {\epsilon_1'}^{-2a_1-abcd\cdot a_1}\epsilon_2^{3ab\cdot a_1}(e_3^{**})^{-3a\cdot a_1}$$

and that gives another relation on the exponents:

$$-2a_1 - abcd\cdot a_1 \equiv 0 \pmod 3$$

Now we have two congruences, one from each norm,

$$(1+abcd)a_1 \equiv 0 \pmod 3$$
$$(2+abcd)a_1 \equiv 0 \pmod 3.$$

Since $a, b, c, d \neq 0$ then adding these two together gives that $a_1 \equiv 0 \pmod 3 \implies a_5 \equiv a_7 \equiv 0 \pmod 3$ so $E_L^3 = \epsilon_1^{a_1}u_1^{a_1}e_3^{a_5}{e_3'}^{2a_5}e_4^{a_7}{e_4'}^{2a_7}$ has no nontrivial solution in L so $E_L = \hat{\epsilon}\hat{\epsilon}_1$. It is easy to see that $\hat{\epsilon}\hat{\epsilon}_i = \hat{\epsilon}\hat{\epsilon}_1$ for $i = 2, 3, 4$.

Now we know very specifically the cases where a new unit can be found in L and we have shown that we can find those units using Hilbert's Theorem 90 when K is of Kind 1 or 2. The next sections will outline the procedure for calculating the units in L which are not contained in any of the subfields.

6.2 Units in L from Type I Subfields

Let $\{e_1, e_2, e_3, e_4\}$ be a basis for $\hat{e}$ where $[\hat{e} : \hat{e}_0] \geq 9$. Suppose E_0 is a unit in L such that $E_0 \notin \hat{E}_0$ where $\hat{E}_0 = \hat{\epsilon}\prod_{i=1}^{4}\hat{\epsilon}_i$, then we know from Theorem 6.10 that

$$\frac{E_0}{E_0^\sigma} = e_1^{b_1}e_2^{b_2}e_3^{b_3}e_4^{b_4}$$

so we can apply Hilbert's Theorem 90 to the basis elements of $\hat{e}$ to find E_0.

Let K be of Kind 1 and $\hat{e} =< \epsilon_1, e_2, e_3, e_4 >$ where $e_i^3 = \epsilon_1^{a_i}\epsilon_i$ with $0 \le a_i \le 2$ for $i = 2, 3, 4$. Consider the set $\mathfrak{E} = \{e_2^{c_2}, e_3^{c_3}, e_4^{c_4}\}$, where $c_i = 0$ or 1 for each i and as many c_i's as possible are 1, subject to the condition $N_{L/K_l}(e_i^{c_i}) = 1$ for some $l = 1, 2, 3,$ or 4. Let $\sigma = \sigma_l$ and $N = N_{L/K_l}$ then for each $e_i^{c_i} \in \mathfrak{E}$ there exists $B_i \in L$ such that $e_i^{c_i} = \frac{B_i}{B_i^\sigma}$ and $\left(\frac{e_i}{e_i^{\sigma^2}}\right)^{c_i} = \frac{B_i^3}{N(B_i)}$ where $N(B_i) = \beta_i \in K_l$. We want to find a product $B_\alpha = B_2^{b_2}B_3^{b_3}B_4^{b_4}$, with $0 \le b_2,\ b_3,\ b_4 \le 2$ and $B_\alpha \ne 1$, such that $N(B_\alpha) = \beta_2^{b_2}\beta_3^{b_3}\beta_4^{b_4} = (\alpha)^3$ for some $\alpha \in L$ and we will choose $b_i = 0$ when $c_i = 0$. If such a product exists then we let $E_s = \frac{B_\alpha}{\alpha}$ so that

$$E_s^3 = \left(\frac{B_2^{b_2}B_3^{b_3}B_4^{b_4}}{\alpha}\right)^3 = \frac{e_2^{b_2}e_3^{b_3}e_4^{b_4}}{(e_2^{b_2}e_3^{b_3}e_4^{b_4})^\sigma}\zeta^a\epsilon_l^b u_l^c. \tag{6.19}$$

To find this product we need to be able to solve the equation $N(B_\alpha) = \beta_2^{b_2}\beta_3^{b_3}\beta_4^{b_4} = (\alpha)^3$ for the exponents b_i for $i = 2, 3,$ and 4. Since α is unique up to multiplication by units in the base field we will look for elements of the form

$$\beta = \zeta^a\epsilon_l^b u_l^c\beta_2^{b_2}\beta_3^{b_3}\beta_4^{b_4} = \alpha^3 \tag{6.20}$$

where $\{\epsilon_l,\ u_l\}$ is a fundamental system of units of K_l.

Let $p_r \in \mathbb{Z}$ be any prime such that p_r is not ramified in K_l and $p_r \nmid N_{K_l/\mathbb{Q}}(B_\alpha)$. If p_r splits as $P_{r_1}\cdots P_{r_t}$ in K_l then for $P = P_{r_j}$ with $1 \le r_j \le r_t$ we define the map $\phi_P : K_l^* \to \mathbb{Z}_3$ (where $K_l^* = K_l - \{0\}$) by

$$\phi_P(z) = v$$

where

$$\left(\frac{z}{P}\right)_3 = \zeta^v$$

where $\left(\frac{z}{P}\right)_3$ is the cubic power residue symbol over K_l.

It is well known that $\left(\frac{z}{P}\right)_3 = z^{(N(P)-1)/3} \equiv a \pmod{P}$. Let $z \in K_l$ then $z = \mathfrak{a}_1 + \mathfrak{a}_2\zeta + \mathfrak{a}_3\sqrt[3]{m_l} + \mathfrak{a}_4\zeta\sqrt[3]{m_l} + \mathfrak{a}_5\sqrt[3]{m_l^2} + \mathfrak{a}_6\zeta\sqrt[3]{m_l^2} \pmod{P}$. The value of $N(P)$ depends on the congruence of p_r modulo 3. For $p_r \equiv 2 \pmod 3$ we know that p_r factors in K_l as $P_{r_1}P_{r_2}P_{r_3}$ with $N(P_{r_j}) = p_r^2$ for $j = 1, 2, 3$. If we solve the equation $x^3 - m_l \equiv 0 \pmod{p_r}$ for $x \equiv m_l^{(2p_r-1)/3} \equiv b \pmod{p_r}$ then we know, for some ordering of the P_{r_j}'s, that $\sqrt[3]{m_l} \equiv \zeta^{j-1}b \pmod{P_{r_j}}$. So for $P = P_{r_j}$

$$\left(\frac{z}{P}\right)_3 \equiv (\mathfrak{a}_1 + \mathfrak{a}_2\zeta + \mathfrak{a}_3\zeta^{j-1}b + \mathfrak{a}_4\zeta(\zeta^{j-1}b) + \mathfrak{a}_5(\zeta^{j-1}b)^2 + \mathfrak{a}_6\zeta(\zeta^{j-1}b)^2)^{(p_r^2-1)/3} \pmod{P}$$

and then

$$\left(\frac{z}{P}\right)_3 \equiv \zeta^v \pmod{P}$$
$$\left(\frac{z}{P}\right)_3 = \zeta^v$$

where $v = 0$, 1 or 2.

For $p_r \equiv 1 \pmod 3$ and $x^3 - m_l \equiv 0 \pmod{p_r}$ solvable we know that p_r factors in K_l as $P_{r_1} \cdots P_{r_6}$ with $N(P_{r_j}) = p_r$ for all j so we need to solve two equations

$$x^3 - m_l \equiv 0 \pmod{p_r}$$

and

$$w^2 + w + 1 \equiv 0 \pmod{p_r}.$$

The first equation factors as $(x - d_1)(x - d_2)(x - d_3) \equiv 0 \pmod{p_r}$ and the second as $(w - f_1)(w - f_2) \equiv 0 \pmod{p_r}$. Each P is generated by a distinct pair $(\sqrt[3]{m_l} - d_q, \zeta - f_s)$ for $q = 1$, 2, or 3 and $s = 1$ or 2. So for each P we can calculate

$$\left(\frac{z}{P}\right)_3 \equiv (\mathfrak{a}_1 + \mathfrak{a}_2 f_s + \mathfrak{a}_3 d_q + \mathfrak{a}_4 f_s d_q + \mathfrak{a}_5 d_q^2 + \mathfrak{a}_6 f_s d_q^2)^{(p_r-1)/3} \pmod P$$
$$\left(\frac{z}{P}\right)_3 \equiv f_s^v \pmod P$$

and then

$$\left(\frac{z}{P}\right)_3 = \zeta^v$$

where $v = 0$, 1 or 2.

For $p_r \equiv 1 \pmod 3$ and $x^3 - m_l \equiv 0 \pmod{p_r}$ not solvable we know that p_r factors in K_l as $P_{r_1} P_{r_2}$ with $N(P_{r_j}) = p_r^3$ for all $j = 1$, or 2 so we need to solve

$$w^2 + w + 1 \equiv 0 \pmod{p_r}.$$

This equation factors as $(w - f_1)(w - f_2) \equiv 0 \pmod{p_r}$ so each P is generated by $(\zeta - f_s)$ for $s = 1$ or 2. Then for each P we can calculate

$$\left(\frac{z}{P}\right)_3 \equiv (\mathfrak{a}_1 + \mathfrak{a}_2 f_s + \mathfrak{a}_3 \sqrt[3]{m_l} + \mathfrak{a}_4 f_s \sqrt[3]{m_l} + \mathfrak{a}_5 \sqrt[3]{m_l}^2 + \mathfrak{a}_6 f_s \sqrt[3]{m_l}^2)^{(p_r^3-1)/3} \pmod P$$
$$\left(\frac{z}{P}\right)_3 \equiv f_s^v \pmod P$$

and then

$$\left(\frac{z}{P}\right)_3 = \zeta^v$$

where $v = 0$, 1 or 2.

For each P in a set of n prime ideals, where n is yet to be specified, construct the matrix with rows

$$W = [\phi_P(\zeta), \phi_P(\epsilon_l), \phi_P(u_l), \phi_P(\beta_2), \phi_P(\beta_3), \phi_P(\beta_4)] \tag{6.21}$$

and solve for vector $\mathfrak{w} = [a, b, c, b_2, b_3, b_4]$ such that $W \cdot \mathfrak{w} = [0]$. This gives a solution to (6.20) that is a cube modulo every prime in our set. We need to choose n sufficiently large so that we find at most 2 nontrivial solutions for $\mathfrak{w}$. By nontrivial we need that at least one of b_2, b_3 or b_4 is non zero modulo 3 for the entries where the corresponding $c_j \neq 0$.

Once we have a possible solution $\mathfrak{w}$ it is necessary to see if α can be calculated by using the numerical cube root function in K_l. If there is no solution to $\sqrt[3]{\beta}$ then that choice of $\mathfrak{w}$ does not produce a new unit in L. If there is a solution then we have solved for $\sqrt[3]{\beta} = \alpha$ and (6.19) has a solution in L.

This method is probabilistic since it is possible for β to be a cube modulo a large number of consecutive primes and not be cube of an element of K_l. However, if β is not the cube of an element of K_l then the equation $x^3 - \beta \equiv 0 \pmod{P}$ has a solution for only 1/3 of the primes $P \in K_l$ so the probability of being a cube modulo n primes and not have a solution in K_l is approximately $1/3^n$.

For the case where K is of Kind 2 then the basis for K can be chosen as $\hat{e} =< \epsilon_1, \epsilon_2, e_3, e_4 >$ where $e_3^3 = \epsilon_1^{a_1}\epsilon_2^{a_2}\epsilon_3$ and $e_4^3 = \epsilon_1^{a_3}\epsilon_2^{a_4}\epsilon_4$ where $0 \leq a_i \leq 2$ for $1 \leq i \leq 4$. Then to find the units in L we follow the same procedure as for Kind 1 but in this case it may be that e_3 and e_4 do not both have norm 1 to the same base field. In that case we simply follow our procedure for each of the units individually. In the case of Kind 2 we will solve $W \cdot \mathfrak{w} = [0]$ for only one non trivial solution because there can be only one unit in L by Theorem 6.9.

If for a particular set $\mathfrak{E}$ there is more than one choice of l such that $N_{L/K_l}(e_i^{c_i}) = 1$ for each $e_i \in \mathfrak{E}$ then it is necessary to run the algorithm for all choices for l.

6.3 Example Type I units in L

Example 6

The fields $k_1 = \mathbb{Q}(\sqrt[3]{2})$, $k_2 = \mathbb{Q}(\sqrt[3]{5})$, $k_3 = \mathbb{Q}(\sqrt[3]{10})$, and $k_4 = \mathbb{Q}(\sqrt[3]{20})$ are all Type I and in Example 1 we found that $[\hat{e} : \hat{e}_0] = 3^3$ and $\hat{e} =< \epsilon_1, \mathfrak{e}_2, \mathfrak{e}_3, \mathfrak{e}_4 >$ where $\mathfrak{e}_2 = \sqrt[3]{\epsilon_3}$, $\mathfrak{e}_3 = \sqrt[3]{\epsilon_1\epsilon_2}$ and $\mathfrak{e}_4 = \sqrt[3]{\epsilon_1\epsilon_4}$. Let $\mathfrak{E} = \{\mathfrak{e}_2, \mathfrak{e}_3, 1\}$ and $\sigma = \sigma_4$ then using Hilbert's Theorem 90 we can find $B_i \in L$ such that $\mathfrak{e}_\mathfrak{i} = \frac{B_i}{B_i^\sigma}$ and $\beta_i = N_{L/K_4}(B_i)$ for $i = 2$ and 3

$B_2 = \frac{1}{3}(3 - \sqrt[3]{2} - \zeta\sqrt[3]{2} + 2\sqrt[3]{4} + \sqrt[3]{5} + \sqrt[3]{20} - \zeta\sqrt[3]{25} + \sqrt[3]{50})$

$\beta_2 = 11 + 3\zeta + 3\sqrt[3]{20} + 3\sqrt[3]{50}$

$B_3 = \frac{1}{3}(9 + 6\sqrt[3]{2} + 2\zeta\sqrt[3]{2} + 2\zeta\sqrt[3]{4} + 3\sqrt[3]{5} + \zeta\sqrt[3]{5} + 2\sqrt[3]{10} + \zeta\sqrt[3]{10} - \sqrt[3]{20} + \sqrt[3]{25} + \sqrt[3]{50})$

$\beta_3 = 18 - 7\zeta - 7\sqrt[3]{20} - 4\zeta\sqrt[3]{20} + 3\sqrt[3]{50} - 2\zeta\sqrt[3]{50}$

Using the primes $p =$ 7, 11, 17, 19, 23, 29, 41, 47, 53, 59 and 61, the row-reduced matrix for W is

$$W = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Then $\mathfrak{w} = \{2, 1, 1, 2, 1\}$ so we solve for $\alpha^3 = \zeta^2 \epsilon_4 u_4 \beta_2^2 \beta_3$ where

$$\zeta^2 \epsilon_4 u_4 \beta_2^2 \beta_3 = 1620881 + 1667320\zeta + 597121\sqrt[3]{20} + 614210\zeta\sqrt[3]{20} + 439989\sqrt[3]{50} + 452568\zeta\sqrt[3]{50}.$$

Using the cube root function we find that

$$\alpha = \frac{1}{3}(200 + 73\zeta + 70\sqrt[3]{20} + 26\zeta\sqrt[3]{20} + 52\sqrt[3]{50} + 17\zeta\sqrt[3]{50})$$

where $\alpha^3 = \zeta^2 \epsilon_4 u_4 \beta_2^2 \beta_3$. Now we can solve for the unit in L

$$E_1 = \frac{B_2^2 B_3}{\alpha} = \frac{1}{9}\Big(1 - \zeta + 8\sqrt[3]{2} - 2\zeta\sqrt[3]{2} - 2\sqrt[3]{4} - \zeta\sqrt[3]{4} - 7\sqrt[3]{5} - 5\zeta\sqrt[3]{5} + \sqrt[3]{10} + 5\zeta\sqrt[3]{10} + 2\sqrt[3]{20} - 2\zeta\sqrt[3]{20} + \sqrt[3]{5}^2 - 4\zeta\sqrt[3]{5}^2 - \sqrt[3]{50} + \zeta\sqrt[3]{50} + \sqrt[3]{100} + 2\zeta\sqrt[3]{100}\Big)$$

and

$$E_1 = \sqrt[3]{\frac{u_1 u_2 \epsilon_3^2}{u_3^2 u_4 \epsilon_4}}.$$

6.4 Units in L from Type III Subfields

For a Type III field, K_i we can write $\epsilon_i = \frac{B_i}{B_i^\sigma}$ where $N_{K_i/k}(B_i) = \pi^a \bar{\pi}^b$ with $1 \le a, b \le 2$ and $a + b = 3$. Suppose without loss of generality that K_i for i =1, 2 are Type III fields and the other cubic fields can be of any Type. To find a unit in L we can use the criteria outlined in [9]:

Theorem 6.11 *Let k_1 be a Type III field. Then:*

(a) $e^3 = \zeta^a \epsilon_1 / \epsilon_i'$ *has no solution e in L.*

(b) $e^3 = \zeta^a \frac{\epsilon_1 \epsilon_2}{\epsilon_1' \epsilon_2''}$ *has a solution e in L if and only if k_2 is a Type III and $\zeta^a N(B_1) = N(B_2)$.*

(c) $e^3 = \zeta^a \frac{\epsilon_1 \epsilon_2}{\epsilon_1' \epsilon_2'}$ *has a solution e in L if and only if k_2 is a Type III and $\zeta^a N(B_1) = \overline{N(B_2)}$.*

(d) $e^3 = \zeta^a \frac{\epsilon_1 \epsilon_2 \epsilon_3}{\epsilon_1' \epsilon_2' \epsilon_3''}$ *has a solution e in L if and only if*

$$\alpha^3 = \frac{\zeta^a N(B_1) N(B_2)}{N(B_3)}$$

has a solution $\alpha \in k$. Thus either k_2 or k_3 is of Type III.

Note: For $\alpha^3 = \frac{\zeta^a N(B_1)N(B_2)}{N(B_3)}$ if B_3 is not Type III then $N(B_3) = 1, \zeta, \zeta^2$ so case (d) reduces to case (b) or (c). For units of the form in (d) we will only consider the case where k_3 is Type III.

To find units in L of the form of (b) or (c) in Theorem 6.11 it is necessary that $\zeta^a N(B_1) \sim N(B_2)$. If $\zeta^a N(B_1) = N(B_2)$ then the quotient of B_1 and B_2 can be expressed as

$$e^3 = \left(\frac{B_1}{B_2}\right)^3 = \frac{\zeta^a B_1^3 N(B_2)}{N(B_1)B_2^3} = \frac{\zeta^a \epsilon_1 \epsilon_2^{-1}}{\epsilon_1^{\sigma^2} \epsilon_2^{-\sigma^2}}.$$

If $\zeta^a N(B_1) = \overline{N(B_2)} = N(B_2)^\tau$ then since $\sigma\tau = \tau\sigma^2$ we get

$$e^3 = \left(\frac{B_1}{B_2^\tau}\right)^3 = \frac{\zeta^a \epsilon_1 \epsilon_2^{-1}}{\epsilon_1^{\sigma^2} \epsilon_2^{-\sigma}}.$$

Similarly we can extend this idea to products of three B_i's where $\zeta^a N(B_1)N(B_2) \sim N(B_3)$ or $\zeta^a N(B_1)\overline{N(B_2)} \sim N(B_3)$ and we generate units in L of the form of (d) in Theorem 6.11 where

$$e^3 = \left(\frac{B_1 B_2}{B_3}\right)^3 = \frac{\zeta^a \epsilon_1 \epsilon_2 \epsilon_3^{-1}}{\epsilon_1^{\sigma^2} \epsilon_2^{\sigma^2} \epsilon_3^{-\sigma^2}}$$

or

$$e^3 = \left(\frac{B_1 B_2^\tau}{B_3}\right)^3 = \frac{\zeta^a \epsilon_1 \epsilon_2 \epsilon_3^{-1}}{\epsilon_1^{\sigma^2} \epsilon_2^{\sigma} \epsilon_3^{-\sigma^2}}.$$

Using these quotients we can find new units in L which are not in any of the subfields.

6.5 Example Type III units in L

Example 7

Let $L = \mathbb{Q}(\zeta, \sqrt[3]{7}, \sqrt[3]{19})$. In Example 5 we saw that $N(B_2) \sim N(B_3) \sim N(B_4)$ so we can find two new units in L

$E_1 = \left(\frac{B_2}{B_3}\right) = 2033 + 1957\zeta/3 - 2819\sqrt[3]{7}/3 - 2773\zeta\sqrt[3]{7}/3 + 31\sqrt[3]{49} + 315\zeta\sqrt[3]{49} + 1169\sqrt[3]{19}/3 - 127\zeta\sqrt[3]{19}/3 - 812\sqrt[3]{133}/3 - 1195\zeta\sqrt[3]{133}/3 + 3\sqrt[3]{931} + 184\zeta\sqrt[3]{931} + 745\sqrt[3]{361}/3 - 4\zeta\sqrt[3]{361} - 254\sqrt[3]{2527}/3 - 229/3\zeta\sqrt[3]{2527} - 25\sqrt[3]{17689} + 53\zeta\sqrt[3]{17689}$

and

$E_2 = \left(\frac{B_2}{B_4}\right) = -95 - 38\zeta/3 - 9\sqrt[3]{7} - 32\zeta\sqrt[3]{7} + 79\sqrt[3]{49}/3 + 62\zeta\sqrt[3]{49}/3 - 49\sqrt[3]{19}/3 + 20\zeta\sqrt[3]{19}/3 - 4\sqrt[3]{133} - 19\zeta\sqrt[3]{133} + 25\sqrt[3]{931}/3 + 29/3\zeta\sqrt[3]{931} - 32\sqrt[3]{361}/3 + 3\zeta\sqrt[3]{361} + \sqrt[3]{2527} - 6\zeta\sqrt[3]{2527} + 7\sqrt[3]{17689}/3 + 5/3\zeta\sqrt[3]{17689}$

where

$$E_1^3 = \epsilon_2^{-2}\epsilon_2^{-\sigma}\epsilon_3^2\epsilon_3^\sigma$$

and

$$E_2^3 = \epsilon_2^{-2}\epsilon_2^{-\sigma}\epsilon_4^2\epsilon_4^\sigma$$

We are now in a position to describe the basis for the units of L

6.6 Basis for the Unit Group of L

Theorem 6.12 *The basis for $\hat{E}$ can be chosen in one of 41 possible ways. If $e_1,\ e_2, \cdots,\ e_8$ is a basis for $\hat{E}$ and $\epsilon_i,\ u_i$, where $u_i = \sqrt[3]{\epsilon_i^2\epsilon_i'}$ for K_i Type I or IV and $u_i = \epsilon_i'$ for k_i Type III, is a basis for $\hat{U}_i$ then the basis will depend on the Kind of K and the Types of the subfields.*

Case 1: *If K is of Kind 1 and*

(A) $[\hat{E} : \hat{E}_0] = 3^2$ then

$e_1 = \epsilon_1,\ e_2^3 = \epsilon_1^a\epsilon_2,\ e_3^3 = \epsilon_1^b\epsilon_3,\ e_4^3 = \epsilon_1^c\epsilon_4,\ e_5 = u_1,\ e_6 = u_2,$

$e_7^3 = \epsilon_1^{a_1}u_1^{a_1}u_2^{a_2}u_3$, *where* $0 \leq\ a_1, a_2\ \leq 2$ *and* $a_1 + a_2 > 0$,

$e_8^3 = \epsilon_1^{b_1}u_1^{b_1}u_2^{b_2}u_4$, *where* $0 \leq\ b_1, b_2\ \leq 2$ *and* $b_1 + b_2 > 0$.

(B) $[\hat{E} : \hat{E}_0] \leq 3$ then

$e_1 = \epsilon_1,\ e_2^3 = \epsilon_1^a\epsilon_2,\ e_3^3 = \epsilon_1^b\epsilon_3,\ e_4^3 = \epsilon_1^c\epsilon_4,\ e_5 = u_1,\ e_6 = u_2,\ e_7 = u_3,$

$$\begin{cases} e_8^3 = \epsilon_1^{a_1}u_1^{a_1}u_2^{a_2}u_3^{a_3}u_4, \text{ where } 0 \leq a_1, a_2, a_3 \leq 2 \text{ and } a_1 + a_2 + a_3 > 0 & \text{if } [\hat{E} : \hat{E}_0] = 3 \\ e_8 = u_4 & \text{if } [\hat{E} : \hat{E}_0] = 1 \end{cases}$$

Case 2: *If K is of Kind 2 and*

(A) K_1 and K_2 are Type I or IV and K_3 and K_4 are Type I:

$e_1 = \epsilon_1,\ e_2 = \epsilon_2,\ e_3^3 = \epsilon_1^a\epsilon_2^b\epsilon_3,\ e_4^3 = \epsilon_1^c\epsilon_2^d\epsilon_4,\ e_5 = u_1,\ e_6 = u_2,\ e_7 = u_3$

$$\begin{cases} e_8^3 = \epsilon_1^{a_1}u_1^{a_1}u_3^{a_2}u_4, \text{ where } 0 \leq a_1, a_2 \leq 2 \text{ and } a_1 + a_2 > 0 & \text{if } [\hat{E} : \hat{E}_0] = 3 \\ e_8 = u_4 & \text{if } [\hat{E} : \hat{E}_0] = 1 \end{cases}$$

(B) K_1 is Type I or IV, K_2 is Type III and K_3 and K_4 are Type I:

$e_1 = \epsilon_1,\ e_2 = \epsilon_2,\ e_3^3 = \epsilon_1^a\epsilon_3,\ e_4^3 = \epsilon_1^b\epsilon_4,\ e_5 = u_1,\ e_6 = \epsilon_2',\ e_7 = u_3,$

$$\begin{cases} e_8^3 = \epsilon_1^{a_1}u_1^{a_1}\epsilon_2^{a_2}\epsilon_2'^{2a_2}u_3^{a_3}u_4, \text{ where } 0 \leq a_1, a_2, a_3 \leq 2 \text{ and } a_1 + a_3 > 0 & \text{if } [\hat{E} : \hat{E}_0] = 3 \\ e_8 = u_4 & \text{if } [\hat{E} : \hat{E}_0] = 1 \end{cases}$$

(C) Three of the subfields are Type IV and K_4 is Type I or IV:

Note: *If k_4 is Type I then $c = d = 0$.*

$e_1 = \epsilon_1,\ e_2 = \epsilon_2,\ e_3^3 = \epsilon_1^a\epsilon_2^b\epsilon_3,\ e_4^3 = \epsilon_1^c\epsilon_2^d\epsilon_4,\ e_5 = u_1,\ e_6 = u_2,\ e_7 = u_3,\ e_8 = u_4$

(D) K_1 and K_2 are Type III and two Type I:

$\hat{E} =< \epsilon_1, \epsilon_1', \epsilon_2, \epsilon_2', \sqrt[3]{\epsilon_3}, u_3, \sqrt[3]{\epsilon_4}, u_4 >$

$N(B_1) \nsim N(B_2):\quad \hat{E} =< \epsilon_1, \epsilon_1', \epsilon_2, \epsilon_2', \sqrt[3]{\epsilon_3}, u_3, \sqrt[3]{\epsilon_4}, e_8 >$ *where*

$$\begin{cases} e_8^3 = \epsilon_1^a {\epsilon'_1}^{2a} \epsilon_2^b {\epsilon'_2}^{2b} u_3^c u_4, \text{ where } 0 \le a,b,c \le 2 \text{ and } a+b+c>0 & \text{if } [\hat{E}:\hat{E}_0]=3 \\ e_8 = u_4 & \text{if } [\hat{E}:\hat{E}_0]=1 \end{cases}$$

Case 3: *If K is of Kind 3 and*

(A) All fields are Type I or IV:

$$\hat{E} =< \epsilon_1, u_1, \epsilon_2, u_2, \epsilon_3, u_3, \sqrt[3]{\epsilon_1^{c_1}\epsilon_2^{c_2}\epsilon_3^{c_3}\epsilon_4}, \sqrt[3]{\epsilon_1^{c_1}{\epsilon'_2}^{c_2}{\epsilon'_3}^{c_3}\epsilon'_4} >$$

(B) K_1 Type III, K_2 and K_3 are Type I or IV and K_4 is Type I:

$$\hat{E} =< \epsilon_1, \epsilon'_1, \epsilon_2, u_2, \epsilon_3, u_3, \sqrt[3]{\epsilon_2^{c_2}\epsilon_3^{c_3}\epsilon_4}, e_8 > \text{ where}$$

$$\begin{cases} e_8^3 = \epsilon_1^a {\epsilon'_1}^{2a} u_4, \text{ where } c_2=c_3=0 \text{ and } a>0 & \text{if } [\hat{E}:\hat{E}_0]=3 \\ e_8 = \sqrt[3]{{\epsilon'_2}^{c_2}{\epsilon'_3}^{c_3}\epsilon'_4} & \text{if } [\hat{E}:\hat{E}_0]=1 \end{cases}$$

(B) K_1 and K_2 are Type III, K_3 is Type I or IV and K_4 is Type I:

$$N(B_1) \nsim N(B_2): \quad \hat{E} =< \epsilon_1, \epsilon'_1, \epsilon_2, \epsilon'_2, \epsilon_3, u_3, \sqrt[3]{\epsilon_4}, e_8 > \text{ where}$$

$$\begin{cases} e_8^3 = \epsilon_1^a {\epsilon'_1}^{2a} \epsilon_2^b {\epsilon'_2}^{2b} u_4, \text{ where } 0 \le a,b \le 2 \text{ and } a+b>0 & \text{if } [\hat{E}:\hat{E}_0]=3 \\ e_8 = u_4 & \text{if } [\hat{E}:\hat{E}_0]=1 \end{cases}$$

$$N(B_1) \sim N(B_2): \quad \hat{E} =< \epsilon_1, \epsilon'_1, \epsilon_2, \sqrt[3]{\tfrac{\epsilon_1\epsilon_2}{\epsilon'_1\epsilon''_2}}, \epsilon_3, u_3, \sqrt[3]{\epsilon_4}, e_8 > \text{ where}$$

$$\begin{cases} e_8^3 = \epsilon_1^a {\epsilon'_1}^{2a} u_4, \text{ where } a>0 & \text{if } [\hat{E}:\hat{E}_0]=9 \\ e_8 = u_4 & \text{if } [\hat{E}:\hat{E}_0]=3 \end{cases}$$

(C) Three Type III and K_4 Type I or IV and

$$N(B_1) \sim N(B_2) \sim N(B_3): \quad \hat{E} =< \epsilon_1, \epsilon'_1, \epsilon_2, \sqrt[3]{\epsilon_1\epsilon_2\epsilon_3}, \sqrt[3]{\tfrac{\epsilon_1\epsilon_2}{\epsilon'_1\epsilon''_2}}, \sqrt[3]{\tfrac{\epsilon_1\epsilon_3}{\epsilon'_1\epsilon'_3}}, \epsilon_4, u_4 >$$

$$N(B_1) \sim N(B_2) \nsim N(B_3): \quad \hat{E} =< \epsilon_1, \epsilon'_1, \epsilon_2, \sqrt[3]{\tfrac{\epsilon_1\epsilon_2}{\epsilon'_1\epsilon''_2}}, \epsilon_3, \epsilon'_3, \sqrt[3]{\epsilon_4}, e_8 > \text{ where}$$

$$\begin{cases} e_8^3 = \epsilon_1^a {\epsilon'_1}^{2a} \epsilon_3^b {\epsilon'_3}^{2b} u_4, \text{ where } 0 \le a,b \le 2 \text{ and } a+b>0 & \text{if } [\hat{E}:\hat{E}_0]=9 \\ e_8 = u_4 & \text{if } [\hat{E}:\hat{E}_0]=3 \end{cases}$$

$$N(B_1)N(B_2) \sim \zeta^a N(B_3): \quad \hat{E} =< \epsilon_1, \epsilon'_1, \epsilon_2, \epsilon'_2, \epsilon_3, \sqrt[3]{\tfrac{\epsilon_1\epsilon_2\epsilon_3}{\epsilon'_1\epsilon'_2\epsilon''_3}}, \sqrt[3]{\epsilon_4}, e_8 > \text{ where}$$

$$\begin{cases} e_8^3 = \epsilon_1^a {\epsilon'_1}^{2a} \epsilon_2^b {\epsilon'_2}^{2b} u_4, \text{ where } 0 \le a,b \le 2 \text{ and } a+b>0 & \text{if } [\hat{E}:\hat{E}_0]=9 \\ e_8 = u_4 & \text{if } [\hat{E}:\hat{E}_0]=3 \end{cases}$$

$$N(B_1) \nsim N(B_2) \nsim N(B_3) \nsim N(B_1) \text{ and } N(B_1)N(B_2) \nsim N(B_3): \quad \hat{E} =< \epsilon_1, \epsilon'_1, \epsilon_2, \epsilon'_2, \epsilon_3, \epsilon'_3, \sqrt[3]{\epsilon_4}, e_8 >$$

where

$$\begin{cases} e_8^3 = \epsilon_1^a {\epsilon'_1}^{2a} \epsilon_2^b {\epsilon'_2}^{2b} \epsilon_3^c {\epsilon'_3}^{2c} u_4, \text{ where } 0 \le a,b,c \le 2 \text{ and } a+b+c>0 & \text{if } [\hat{E}:\hat{E}_0]=3 \\ e_8 = u_4 & \text{if } [\hat{E}:\hat{E}_0]=1 \end{cases}$$

(D) Four Type III :

$$N(B_1) \sim N(B_2) \sim N(B_3): \quad \hat{E} =< \epsilon_1, \epsilon'_1, \epsilon_2, \sqrt[3]{\epsilon_1\epsilon_2\epsilon_3}, \sqrt[3]{\tfrac{\epsilon_1\epsilon_2}{\epsilon'_1\epsilon''_2}}, \sqrt[3]{\tfrac{\epsilon_1\epsilon_3}{\epsilon'_1\epsilon'_3}}, \epsilon_4, \epsilon'_4 >$$

Case 4: *If K is of Kind 4 and*

(A) All fields are Type I or IV:

$$\hat{E} =< \epsilon_1, u_1, \epsilon_2, u_2, \epsilon_3, u_3, \epsilon_4, u_4 >$$

(B) K_1 Type III:

$\hat{E} =< \epsilon_1, \epsilon_1', \epsilon_2, u_2, \epsilon_3, u_3, \epsilon_4, u_4 >$

(C) K_1, K_2 Type III and two Type I or IV and

$N(B_1) \sim N(B_2): \quad \hat{E} =< \epsilon_1, \epsilon_1', \epsilon_2, \sqrt[3]{\frac{\epsilon_1\epsilon_2}{\epsilon_1'\epsilon_2'}}, \epsilon_3, u_3, \epsilon_4, u_4 >$

$N(B_1) \nsim N(B_2): \quad \hat{E} =< \epsilon_1, \epsilon_1', \epsilon_2, \epsilon_2', \epsilon_3, u_3, \epsilon_4, u_4 >$

(D) Three Type III and K_4 Type I or IV and

$N(B_1) \nsim N(B_2) \nsim N(B_3) \nsim N(B_1)$ *and* $N(B_1)N(B_2) \nsim N(B_3): \quad \hat{E} =< \epsilon_1, \epsilon_1', \epsilon_2, \epsilon_2', \epsilon_3, \epsilon_3', \epsilon_4, u_4 >$

$N(B_1) \sim N(B_2) \nsim N(B_3): \quad \hat{E} =< \epsilon_1, \epsilon_1', \epsilon_2, \sqrt[3]{\frac{\epsilon_1^a\epsilon_2^b}{\epsilon_1'^a\epsilon_2'^b}}, \epsilon_3, \epsilon_3', \epsilon_4, u_4 >$

$N(B_1) \sim N(B_2) \sim N(B_3): \quad \hat{E} =< \epsilon_1, \epsilon_1', \epsilon_2, \sqrt[3]{\frac{\epsilon_1\epsilon_2}{\epsilon_1'\epsilon_2'}}, \epsilon_3, \sqrt[3]{\frac{\epsilon_1\epsilon_3}{\epsilon_1'\epsilon_3'}}, \epsilon_4, u_4 >$

$N(B_1)N(B_2) \sim N(B_3): \quad \hat{E} =< \epsilon_1, \epsilon_1', \epsilon_2, \epsilon_2', \epsilon_3, \sqrt[3]{\frac{\epsilon_1^a\epsilon_2^b\epsilon_3}{\epsilon_1'^a\epsilon_2'^b\epsilon_3''}}, \epsilon_4, \epsilon_4' >$

Note: *For the next case with 4 Type III fields only the norms that are similar will be mentioned. Any other norms or products of norms are assumed to be dissimilar.*

(E) Four Type III and

No norms are similar: $\quad \hat{E} =< \epsilon_1, \epsilon_1', \epsilon_2, \epsilon_2', \epsilon_3, \epsilon_3', \epsilon_4, \epsilon_4' >$

$N(B_1) \sim \zeta^a N(B_2): \quad \hat{E}_0 =< \epsilon_1, \epsilon_1', \epsilon_2, \sqrt[3]{\frac{\epsilon_1\epsilon_2}{\epsilon_1'\epsilon_2'}}, \epsilon_3, \epsilon_3', \epsilon_4, \epsilon_4' >$

$N(B_1) \sim N(B_2) \sim N(B_3): \quad \hat{E} =< \epsilon_1, \epsilon_1', \epsilon_2, \sqrt[3]{\frac{\epsilon_1\epsilon_2}{\epsilon_1'\epsilon_2'}}, \epsilon_3, \sqrt[3]{\frac{\epsilon_1\epsilon_3}{\epsilon_1'\epsilon_3'}}, \epsilon_4, \epsilon_4' >$

$N(B_1) \sim N(B_2) \nsim N(B_3) \sim N(B_4): \quad \hat{E} =< \epsilon_1, \epsilon_1', \epsilon_2, \sqrt[3]{\frac{\epsilon_1\epsilon_2}{\epsilon_1'\epsilon_2'}}, \epsilon_3, \epsilon_3', \epsilon_4, \sqrt[3]{\frac{\epsilon_3\epsilon_4}{\epsilon_3'\epsilon_4'}} >$

$N(B_1)N(B_2) \sim N(B_3): \quad \hat{E} =< \epsilon_1, \epsilon_1', \epsilon_2, \epsilon_2', \epsilon_3, \sqrt[3]{\frac{\epsilon_1\epsilon_2\epsilon_3}{\epsilon_1'\epsilon_2'\epsilon_3''}}, \epsilon_4, \epsilon_4' >$

$N(B_1)N(B_2) \sim \zeta^a N(B_3)$ *and* $N(B_4) \sim N(B_1)$: $\quad \hat{E} =< \epsilon_1, \epsilon_1', \epsilon_2, \epsilon_2', \epsilon_3, \sqrt[3]{\frac{\epsilon_1\epsilon_2\epsilon_3}{\epsilon_1'\epsilon_2'\epsilon_3''}}, \epsilon_4, \sqrt[3]{\frac{\epsilon_1\epsilon_4}{\epsilon_1'\epsilon_4'}} >$

$N(B_1)N(B_2)N(B_3) \sim N(B_4)$ *or* $N(B_1)N(B_2) \sim N(B_3)N(B_4)$:

$\hat{E} =< \epsilon_1, \epsilon_1', \epsilon_2, \epsilon_2', \epsilon_3, \epsilon_3', \epsilon_4, \sqrt[3]{\frac{\epsilon_1\epsilon_2\epsilon_3\epsilon_4}{\epsilon_1'\epsilon_2''\epsilon_3'\epsilon_4'}} >$

Proof: Let $\hat{E}_1$ be the group generated by $< e_1, e_2, \cdots, e_8 >$. We will show that $\hat{E}_1 = \hat{E}$. Clearly $\hat{E}_1 \subseteq \hat{E}$ and we know that the rank of $\hat{E} = 8$. The rank of $\hat{E}_1 \leq 8$ since it is generated by 8 elements. To show it has rank 8 we will show that it contains the subgroup of rank 8 generated by $< \epsilon_i, u_i >$ where $i =$ 1, 2, 3 and 4.

First we consider the case when K is of Kind 1 then clearly ϵ_1 and $u_1 \in \hat{E}_1$. $\frac{e_2}{e_6} = \sqrt[3]{\frac{\epsilon_1^a\epsilon_2}{\epsilon_1^a\epsilon_2'}} = \sqrt[3]{\epsilon_2^2\epsilon_2''} = u_2$ and $e_2^3\epsilon_1^{-a} = \epsilon_2$ so ϵ_2 and $u_2 \in \hat{E}_1$. Similarly we find that ϵ_3 and $\epsilon_4 \in \hat{E}_1$.

If $\left[\hat{E} : \hat{E}_0\right] = 3^2$ then $e_7^3\epsilon_1^{-a_1}u_1^{-a_1}u_2^{-a_2} = u_3$ and $e_8^3\epsilon_1^{-b_1}u_1^{-b_1}u_2^{-b_2} = u_4$ so u_3 and $u_4 \in \hat{E}_1$ and the rank of $\hat{E}_1 = 8$. It is known from Theorem 6.10 that e_7 and e_8 are not in $\hat{E}_0$ but are in $\hat{E}_1$ so $\left[\hat{E}_1 : \hat{E}_0\right] = 3^2$. Thus since $\hat{E}_1 \subseteq \hat{E}$ then it must be that $\hat{E}_1 = \hat{E}$.

If $\left[\hat{E} : \hat{E}_0\right] = 3^1$ then $\frac{e_3}{e_7} = u_3$ and $e_8^3\epsilon_1^{-a_1}u_1^{-a_1}u_2^{-a_2}u_3^{-a_3} = u_4$ so u_3 and $u_4 \in \hat{E}_1$ and the rank of $\hat{E}_1 = 8$.

If K is of Kind 2 then again clearly ϵ_i and $u_i \in \hat{E}_1$ for $i = 1$ and 2. If the subfields are all Type I or IV then $e_3^3\epsilon_1^{-a}\epsilon_2^{-b} = \epsilon_3$, $e_4^3\epsilon_1^{-c}\epsilon_2^{-d} = \epsilon_4$ and $\frac{e_3}{e_7}u_2^{-b} = u_3$ so ϵ_3, ϵ_4 and $u_3 \in \hat{E}_1$. $e_8^3\epsilon_1^{-a_1}u_1^{-a_1}u_3^{-a_2} = u_4$ so $u_4 \in \hat{E}_1$ and the rank of $\hat{E}_1 = 8$.

If K_2 is Type III then replace e_2 with ϵ_2' above and $e_3^3\epsilon_1^{-a} = \epsilon_3$, $e_4^3\epsilon_1^{-b} = \epsilon_4$ and $\frac{e_3}{e_7} = u_3$ so ϵ_3, ϵ_4 and $u_3 \in \hat{E}_1$. $e_8^3\epsilon_1^{-a_1}u_1^{-a_1}\epsilon_2^{-a_2}{\epsilon_2'}^{-2a_2}u_3^{-a_3} = u_4$ so $u_4 \in \hat{E}_1$ and the rank of $\hat{E}_1 = 8$. Thus the rank $\hat{E}_1 = 8$ for all the cases.

For the last three cases it is known from Theorem 6.10 that e_7 is not in $\hat{E}_0$ but is in $\hat{E}_1$ so $\left[\hat{E}_1 : \hat{E}_0\right] = 3^1$. Thus since $\hat{E}_1 \subseteq \hat{E}$ then it must be that $\hat{E}_1 = \hat{E}$.

For the cases when K is of Kind 3 or 4 if we choose $\hat{E}_1 = e_1, e_2, \cdots, e_8$ it was shown in the proof of Theorem 6.9 that $\left[\hat{E} : \hat{E}_1\right] = 1$ so $\hat{E} = \hat{E}_1$.

Chapter 7

Rank of the Class Group of K and L

7.1 Class numbers of L and all its subfields

The class number formula for an algebraic number field, F, is well known and is

$$h = \frac{2\sqrt{\mid disc(F) \mid}}{2^{r+s}\pi^s reg(F)} \prod_p \frac{1 - 1/p}{\prod_{P|p}(1 - N(P))}$$

where r is the number of real embeddings and s is half the number of non-real embeddings of F into $\mathbb{C}$. The regulator of F, $reg(F)$, is an $(r+s-1) \times (r+s-1)$ determinant which depends on fundamental units of F. Then for a pure cubic field k_i, $reg(k_i) = log(\epsilon_i)$ where $\epsilon_i > 1$ and the product is taken over all rational primes p. Since the class number is an integer it is sufficient to take the product over a large enough number of primes so that the value of the right side remains close to an integer. We can calculate ϵ_i using Vornoi's algorithm so implementing the formula is straight forward. The class number h of K satisfies the relation from Theorem XIV in [9]

$$3^3 h = (\hat{e} : \hat{e}_0) h_1 h_2 h_3 h_4$$

and $(\hat{e} : \hat{e}_0) = 3^r$ where r is the number of new units in K calculated in chapter 5. The class number relations H_i and H for K_i and L respectively are given in Theorem I of [9] and are summarized here.

Theorem 7.1 *The following class numbers relations hold*

(1) $3^5 H = (\hat{E} : \hat{\epsilon}) H_1 H_2 H_3 H_4$

(2) $3 H_i = (\hat{U}_i : \hat{u}_i) h_i^2$

Here $(\hat{U}_i : \hat{u}_i) = 3^r$ and $r = 0$ or 1 as k_i is type III or not and $(\hat{E} : \hat{\epsilon})$ is the group index.

Using the notation from Gerth [5] let G be an abelian 3-group. Then G may be viewed as a module over $\mathbb{Z}_3$ and we define $G^+ = \{a \in G \mid a^\tau = a\}$ and $G^- = \{a \in G \mid a^\tau = a^{-1}\}$ so that $G = G^+ \times G^-$. Let M be any finite algebraic extension field of $\mathbb{Q}$, C_M denote the ideal class group of M, S_M be the Sylow 3-subgroups of C_M. The rank of S_M is the number of cyclic factors in the decomposition of S_M.

We say that an ideal class $a \in C_L$ is an ambiguous ideal class of the extension L/K_1 if $a^\sigma = a$, where $< \sigma >= Gal(L/K_1)$. Let $C_L^{(\sigma)} = \{a \in C_L \mid a^\sigma = a\}$ and $S_L^{(\sigma)} = \{a \in S_L \mid a^\sigma = a\}$. Then $S_L^{(\sigma)}$ is called the group of ambiguous ideals of S_L. Let $C^{1-\sigma} = \left\{a^{1-\sigma} \mid a \in C\right\}$ for any abelian group C on which σ acts.

In [3] Gerth talks about strong ambiguous classes. If $a \in C_L^{(\sigma)}$ then there exists $\mathfrak{a} \in a$ such that $\mathfrak{a}^{1-\sigma} = (x)$ for some $x \in L^* = L - 0$. We say that $a \in C_L$ is a strong ambiguous ideal class if there exists a representative $\mathfrak{a} \in a$ such that $\mathfrak{a}^{1-\sigma} = (1)$. If an ambiguous ideal class is not strongly ambiguous it is said to be a weak ambiguous ideal class.

We want to calculate the rank of S_K and S_L. From [4] Theorem 3.1 we know that if $S_{K_i} = \{1\}$ for some $1 \leq i \leq 4$ then the rank $S_L = 2t - s$ where t is the rank of the group of ambiguous ideal classes $S_L^{(\sigma)}$ in S_L and

$$s = \operatorname{rank}(S_L^{(\sigma)} \cdot S_L^{1-\sigma})/S_L^{1-\sigma}. \tag{7.1}$$

The calculations for t and s will be shown in sections 7.3 and 7.4 respectively. In [4] it is also shown that the rank of $S_K = t - s_1$ where

$$s_1 = \operatorname{rank}((S_L^{(\sigma)} \cdot S_L^{1-\sigma})/S_L^{1-\sigma})^-. \tag{7.2}$$

The calculation of s_1 will be shown in 7.4.

The calculations for t and s depend on the cubic Hilbert symbol so we will begin by discussing how to calculate the symbol.

7.2 Calculation of the cubic Hilbert symbol for divisors of 3

Let $\mathfrak{Q} = \{\pi_1, \ldots, \pi_w, \ldots, \pi_d\}$ be the primes that ramify from K_1 to L such that $\pi_1, \ldots, \pi_w$ are divisors of 3. For primes $\pi_j \in K_1$ that are not divisors of 3 and $\pi_j \nmid \beta$ the cubic Hilbert symbol $\left(\frac{\beta, m_2}{\pi_j}\right)$ reduces to the power symbol $\left(\frac{\beta}{\pi_j}\right)_3^l$ where $\pi_j^l \mid\mid m_2$ and the calculation proceeds as it did in section 6.2. If π_1 is the only divisor of 3 then we can calculate $\left(\frac{\beta, m_2}{\pi_1}\right)$ using the fact that the cubic Hilbert symbol is multiplicative. Since $1 = \prod_{\pi_j} \left(\frac{\beta, m_2}{\pi_j}\right)$ then

$$\left(\frac{\beta, m_2}{\pi_1}\right) = \prod_{\substack{\pi_j \\ j>1}} \left(\frac{\beta, m_2}{\pi_j}\right)^{-1}.$$

When we have three divisors of 3 we have to take into account two different cases, either $3 \mid m_2$ or not.

From Hasse [6] we have a formula for calculating the symbol for $\mathfrak{L} \in K_1$ a divisor of 3 in our basefield. We define e and e^* to be $\mathfrak{L}^e \mid\mid 3$ and $\mathfrak{L}^{e^*} \mid\mid 1-\zeta$ respectively. For this case $e = 2$ and $e^* = e/2 = 1$

$$\left(\frac{\beta, \alpha}{\mathfrak{L}}\right) = \zeta^{2\left(\frac{\alpha-1}{3}\frac{\beta-1}{1-\zeta}\right)},$$

where $\alpha \equiv 1 \pmod{\mathfrak{L}^e}$ and $\beta \equiv 1 \pmod{\mathfrak{L}^{e^*}}$.

We will begin with the case were $m_1 \equiv \pm 1 \pmod 9$ and $3 \nmid m_2$ and we let $\mathfrak{L} = \pi_j$ for $1 \le j \le 3$. Since $m_1 \equiv \pm 1 \pmod 9$ then $\frac{(\sqrt[3]{m_1} \mp 1)^2}{3}$ is an integer in k_1. The factorization of 3 is as follows: $(3) = \mathfrak{p}_1\mathfrak{p}_2^2$ in k_1, $\mathfrak{p}_1 = \pi_1^2$ and $\mathfrak{p}_2 = \pi_2\pi_3$ in K_1 so $\mathfrak{p}_1\mathfrak{p}_2 \mid (\sqrt[3]{m_1} \mp 1)$ and $\pi_1^2\pi_2\pi_3 \mid (\sqrt[3]{m_1} \mp 1) \implies \sqrt[3]{m_1} \equiv \pm 1 \pmod{1-\zeta}$ and

$$\sqrt[3]{m_1} \equiv \pm 1 \pmod{\mathfrak{L}} \tag{7.3}$$

Similarly $(1-\zeta) = \pi_1\pi_2\pi_3 \implies$

$$\zeta \equiv 1 \pmod{\mathfrak{L}} \tag{7.4}$$

for all $\mathfrak{L}$.

For $\beta \in K_1$ where $\mathfrak{L} \nmid \beta$ we will calculate $\left(\frac{\beta, m_2^*}{\mathfrak{L}}\right)$ where $m_2^* = \pm m_2$ such that $m_2^* \equiv 1 \pmod 3$ and $(m_2^* - 1)/3 \in \mathbb{Z}$. We need to have $\beta \equiv 1 \pmod{\mathfrak{L}}$. We know that $\beta \equiv b \pmod{\mathfrak{L}}$ where $b = 1$ or 2. To calculate the value of b we need to find a representative for $\beta \pmod{\mathfrak{L}}$ which has no denominators of 3 since $3 \equiv 0 \pmod{\mathfrak{L}}$, otherwise we will be dividing by zero.

Lemma 7.2 *Let $m_i \equiv \pm 1 \pmod 9$, $m_i = ac^2$ where $c \equiv 1 \pmod 3$, $\beta \in O_{K_i}$, $(3) = (\pi_1\pi_2\pi_3)^2 \in K_i$. Then for j = 1, 2, 3, there exists $\gamma_j \in O_{K_i}$ such that $\beta - \gamma_j\pi_j \equiv \mathfrak{b}_1 + \mathfrak{b}_2\zeta + \mathfrak{b}_3\sqrt[3]{m_i} + \mathfrak{b}_4\zeta\sqrt[3]{m_i} + \mathfrak{b}_5\sqrt[3]{m_i^2} + \mathfrak{b}_6\zeta\sqrt[3]{m_i^2} \pmod{\pi_j}$ where $\mathfrak{b}_l \in \mathbb{Z}$ for all l.*

Proof:

Since $\beta \in O_{K_i}$ then β can be expressed as a $\mathbb{Z}$ linear combination of the basis for O_{K_i} defined in Theorem 3.1, $A_3 = \{1, \zeta, \sqrt[3]{m_i}, \frac{1 \pm \sqrt[3]{m_i} + \sqrt[3]{m_i}^2/c}{3}, \frac{1}{3}(2 + \zeta \pm \sqrt[3]{m_i} \mp \zeta\sqrt[3]{m_i}), \zeta(\frac{1 \pm \sqrt[3]{m_i} + \sqrt[3]{m_i}^2/c}{3})\}$ where $\pm$ or $\mp$ correspond as $m_i \equiv \pm 1 \pmod 9$. For some ordering of the π_i's we can choose $\rho_1 = (\sqrt[3]{m_i} \mp 1)(1-\zeta)/3$, $\rho_2 = \rho_1^\sigma$ and $\rho_3 = \rho_2^\sigma$ where ρ_i is in π_i but not in π_i^2. For each π_j it is sufficient to show that we can find γ's such that each basis element can be expressed with no 3 in the denominator. Let $\rho = \rho_1$, then

$$\frac{1 \pm \sqrt[3]{m_i} + \sqrt[3]{m_i}^2/c}{3} + \zeta^2\rho^2 \equiv \left(\frac{\sqrt[3]{m_i}^2}{c}\right)\left(\frac{1-c}{3}\right) \pm \sqrt[3]{m_i} \pmod{\pi_j}$$

$$\frac{1}{3}(2 + \zeta \pm \sqrt[3]{m_i} \mp \zeta\sqrt[3]{m_i}) \pm 2\rho \equiv \zeta + \sqrt[3]{m_i} - \zeta\sqrt[3]{m_i} \pmod{\pi_j}.$$

So the lemma is proved.

Using Lemma 7.2 we can find $\gamma \in K_1$ such that $(\beta) - \gamma\rho = (\mathfrak{b}_1 + \mathfrak{b}_2\zeta + \mathfrak{b}_3\sqrt[3]{m_1} + \mathfrak{b}_4\zeta\sqrt[3]{m_1} + \mathfrak{b}_5\sqrt[3]{m_1^2} + \mathfrak{b}_6\zeta\sqrt[3]{m_1^2})$ where $\mathfrak{b}_j \in \mathbb{Z}$ for all j. To find the numerical value for b substitute ± 1 for $\sqrt[3]{m_1}$ and 1 for ζ in $\beta - \gamma\rho$ to get $\beta - \gamma\rho \equiv b \pmod{\mathfrak{L}}$. If $b = 2$ then let $\beta = -\beta$. Now $\beta \equiv 1 \pmod{\mathfrak{L}}$ so $(\beta-1)(1-\zeta^2)/3 \equiv b_1 \pmod{\mathfrak{L}}$ where $b_1 = 0$, 1, or 2 then, as before, find γ_1 such that $\mathfrak{c} = (\beta-1)(1-\zeta^2)/3 - \gamma_1\rho$ has no denominator of 3. We know that $\mathfrak{c} = \mathfrak{a}_1 + \mathfrak{a}_2\zeta + \mathfrak{a}_3\sqrt[3]{m_1} + \mathfrak{a}_4\zeta\sqrt[3]{m_1} + \mathfrak{a}_5\sqrt[3]{m_1^2} + \mathfrak{a}_6\zeta\sqrt[3]{m_1^2}$ where $\mathfrak{a}_j \in \mathbb{Z}$ for all j so we can substitute ± 1 for $\sqrt[3]{m_1}$ and 1 for ζ in $\mathfrak{c}$ to get

$$\mathfrak{c} \equiv b_1 \pmod{\mathfrak{L}}.$$

Now we can calculate

$$\left(\frac{\beta, m_2^*}{\mathfrak{L}}\right) = \zeta^{2b_1\left(\frac{m_2^*-1}{3}\right)} = \zeta^{\alpha_i}.$$

For the case where $m_2 = 3^a n_2$ with $a = 1$ or 2 and $3 \nmid n_2$ we will never have that π_i is relatively prime to m_2 so since $(3) = (\pi_1\pi_2\pi_3)^2$ we can use the product formula. Without loss of generality let $\mathfrak{L} = \pi_1$, then for $\beta \neq \pi_j$ for $j = 1, 2, 3$

$$\left(\frac{\beta, 3^a n_2}{\mathfrak{L}}\right) = \left(\frac{\beta, n_2(\pi_2\pi_3)^{2a}}{\pi_1}\right)\left(\frac{\beta, \pi_1}{\pi_1}\right)^{2a}.$$

For the second term we use the product formula again

$$1 = \left(\frac{\beta, \pi_1}{\pi_1}\right)\left(\frac{\beta, \pi_1}{\pi_2}\right)\left(\frac{\beta, \pi_1}{\pi_3}\right)\prod_{j=1}^{t}\left(\frac{\beta, \pi_1}{p_j}\right) \quad \text{where } p_j \mid \beta$$

so

$$\left(\frac{\beta, \pi_1}{\pi_1}\right)^2 = \left(\frac{\beta, \pi_1}{\pi_2}\right)\left(\frac{\beta, \pi_1}{\pi_3}\right)\prod_{j=1}^{t}\left(\frac{\beta, \pi_1}{p_j}\right).$$

We need to calculate each piece separately.

To find $\left(\frac{\beta,\ \alpha}{\pi_1}\right)$, when $\alpha \not\equiv 0 \pmod{\pi_1}$, we need $\beta \equiv 1 \pmod{\pi_1}$ and $\alpha \equiv 1 \pmod{\pi_1^2}$. For the first condition note that $\beta^* = \pm\beta \equiv 1 \pmod{\pi_1}$. For the second we know that $\pm\alpha \equiv 1 \pmod{\pi_1}$ so $\pm\alpha \equiv \zeta^{2j} \pmod{\pi_1^2}$ for $j = 0$, 1 or 2 and $\pm\zeta^j\alpha \equiv 1 \pmod{\pi_1^2}$. Then

$$\left(\frac{\beta, \alpha}{\pi_1}\right) = \left(\frac{\beta^*, \pm\zeta^j\alpha}{\pi_1}\right)\left(\frac{\zeta^j, \beta^*}{\pi_1}\right)$$

where $\left(\frac{\beta^*, \pm\zeta^j\alpha}{\pi_1}\right) = \zeta^{\mathfrak{c}}$ with $\mathfrak{c} \equiv \frac{(\beta^*-1)}{1-\zeta}\frac{(\pm\zeta^j\alpha-1)}{3} \pmod{\pi_1}$. To calculate the value of $\mathfrak{c}$ we need to have that $\mathfrak{c}$ is an integer in K_1 and has no 3's in the denominator. Find a linear combination of π_1 and $\pi_2\pi_3$ such that $r_1(\pi_1) + r_2(\pi_2\pi_3) = 1$ then $r_2(\pi_2\pi_3) \equiv 1 \pmod{\pi_1}$. Now rationalizing the denominator of $\mathfrak{c}$ gives

$$\mathfrak{c} \equiv \frac{(\beta^*-1)(\pm\zeta^j\alpha-1)(1-\zeta^2)}{9} \pmod{\pi_1}$$

then

$$\mathfrak{c} \equiv \frac{(\beta^* - 1)(\pm\zeta^j \alpha - 1)(1 - \zeta^2)(r_2^4 \pi_2^4 \pi_3^4)}{(\pi_1 \pi_2 \pi_3)^4} \pmod{\pi_1} \tag{7.5}$$

and we know that $\pi_1^2 \mid (\pm\zeta^j \alpha - 1)$, $\pi_1 \mid (\pm\beta - 1)$ and $\pi_1 \mid (1 - \zeta^2)$ so the numerator of (7.5) is divisible by $(\pi_1 \pi_2 \pi_3)^4$ so $\mathfrak{c}$ is an integer in K_1. As in the previous case we can eliminate the denominators which are 3 by finding γ such that $\mathfrak{c}' = \mathfrak{c} + \gamma\rho \equiv b \pmod{\pi_1}$ and then substitute ± 1 for $\sqrt[3]{m_1}$ and 1 for ζ in $\mathfrak{c}'$ to get

$$\mathfrak{c} \equiv \mathfrak{c}' \equiv b \pmod{\pi_1}.$$

We perform a similar calculation to find $\left(\frac{\zeta^j, \beta}{\pi_1}\right) = \left(\frac{\zeta^j, \pm\zeta^i \beta}{\pi_1}\right)\left(\frac{\zeta^i, \zeta^{2j}}{\pi_1}\right)$ where $\left(\frac{\zeta^i, \zeta^{2j}}{\pi_1}\right) = 1$.

If $\beta \mid 3$ then the calculation proceeds as above except for the case of $\left(\frac{\beta, \pi_j}{\pi_1}\right)$. Without loss of generality let $\beta = \pi_1$ and we will show how to calculate the symbol $\left(\frac{\pi_1, \pi_2}{\pi_1}\right)$. We would like to be able to use formula (4) from [6]:

$$\left(\frac{\beta, \alpha}{\mathfrak{p}}\right) = \left(\frac{\alpha}{\mathfrak{p}}\right)^{-l}, \quad \begin{array}{l}\text{if } \mathfrak{p} \text{ is unramified in the extension } K_1(\sqrt[3]{\alpha}) \\ \text{and divides } \beta \text{ exactly to the } l \text{ power.}\end{array} \tag{7.6}$$

and the product formula. Hence we need to find α such that π_1 is unramified in the extension $K_1(\sqrt[3]{\alpha})$ and $\pi_2 \mid \alpha$ so that

$$\left(\frac{\pi_1, \alpha}{\pi_1}\right) = \left(\frac{\pi_1, \alpha/\pi_2}{\pi_1}\right)\left(\frac{\pi_1, \pi_2}{\pi_1}\right). \tag{7.7}$$

To do this we use Theorem 119 from Hecke [7] which states that if $\pi_1 \mid (1 - \zeta)$ and $\pi_1 \nmid \alpha$ then π_1 is unramified in $K_1(\sqrt[3]{\alpha})$ if the congruence

$$\alpha \equiv \pm 1 \pmod{\pi_1^3}$$

can be solved. We know that $\pi_2 \equiv \pm 1 \pmod{\pi_1}$ and $\pi_2 \equiv \pm\zeta^j \pmod{\pi_1^2}$. There are 18 reduced residues modulo π_1^3 and those can all be represented by $\pm\zeta^j A$ where $A = 1,\ 4,\ 7$ and $j = 0,\ 1,\ 2$. To see that these residues are distinct we suppose that

$$\pm\zeta^i A \equiv \pm\zeta^j B \pmod{\pi_1^3}$$

then, since $A \equiv B \equiv 1 \pmod{\pi_1^2}$,

$$\pm\zeta^i \equiv \pm\zeta^j \pmod{\pi_1^2}$$

so $i = j$ and the sign is the same. Then $A \equiv B \pmod{\pi_1^3} \implies A = B$ and the 18 residues are distinct.

Thus we can choose $\alpha = \pm\zeta^j 4^l \pi_2 \equiv 1 \pmod{\pi_1^3}$ where $0 \le j,\ l \le 2$ and

$$\left(\frac{\pi_1, \alpha}{\pi_1}\right) = \left(\frac{\alpha}{\pi_1}\right)^2 \equiv (\alpha)^4 \equiv 1 \pmod{\pi_1^2}. \tag{7.8}$$

We can use the product rule to find

$$\left(\frac{\pi_1, \alpha/\pi_2}{\pi_1}\right) = \left(\frac{\pi_1, \pm\zeta^j 4^l}{\pi_1}\right) = \left(\frac{\pi_1, \pm\zeta^j 4^l}{\pi_2}\right)^2 \left(\frac{\pi_1, \pm\zeta^j 4^l}{\pi_3}\right)^2 \left[\prod_{\mathfrak{p}|2} \left(\frac{\pi_1, \pm\zeta^j 4^l}{\mathfrak{p}}\right)\right]^2$$

and each Hilbert symbol on the right can be calculated by one of the methods above. Then combining equations (7.7) and (7.8) gives the desired result

$$\left(\frac{\pi_1, \pi_2}{\pi_1}\right) = \left(\frac{\pi_1, \pm\zeta^j 4^l}{\pi_1}\right)^2.$$

7.3 Calculation of ${}_N B$

By [3] the rank of $S_L^{(\sigma)}$ is

$$t = \text{rank } S_L^{(\sigma)} = d + q^* - (r + 1 + o)$$

where

d = number of ramified primes in L/K_1

r = 2 (the rank of the free abelian part of the group of units $\hat{U}_1$ of K_1).

o = 1 since K_1 contains the cube root of unity.

q^* is defined by $[V_{K_1}^* : \hat{U}_1^3] = 3^{q^*}$, where $V_{K_1}^* = \left\{x \in \hat{U}_1 \mid x = N_{L/K_1}(y),\ y \in L^*\right\}$. Here $\hat{U}_1^3 = \left\{x^3 \mid x \in \hat{U}_1\right\}$, and $L^* = L - \{0\}$.

To calculate d we need to know some information about the primes ramified from K_1 to L. Let $\lambda = 1 - \zeta$ be a prime element in k dividing 3 and let P_3 be a prime element in K_1 dividing 3. Using a different ordering than in the previous section, suppose that $\{\pi_1, \ldots, \pi_g\}$ are the primes, different from divisors of 3, that ramify from K_1 to L. The total number of ramified primes, d, depends on the ramification of the divisors of 3 from K_1 to L which will depend on the subfields and the base field. If $m_1 m_2 \not\equiv 0 \pmod 3$ then there are two cases. If at least one of $m_i \equiv \pm 1 \pmod 9$ for $i = 2$, 3 or 4 then 3 is unramified from K_1 to L so $d = g$. If $m_1 \equiv \pm 1 \pmod 9$ and $m_2 \not\equiv \pm 1 \pmod 9$ then 3 has three divisors in K_1 and each of those divisors ramifies from K_1 to L so $d = g + 3$. If $m_1 \equiv 0 \pmod 3$ then the divisor of 3 in K_1 is unramified if $m_i \equiv \pm 1 \pmod 9$ for i =2, 3 or 4 so $d = g$ and it is ramified otherwise so $d = g + 1$. If $m_2 \equiv 0 \pmod 3$ and $m_1 \equiv \pm 1 \pmod 9$ then there are three divisors of 3 in K_1 which are ramified so $d = g + 3$ and if $m_1 \not\equiv \pm 1 \pmod 9$ there is only one divisor of 3 to ramify so $d = g + 1$. These results are summarized in the following table.

$$d = \begin{cases} g, & \text{if } m_i \equiv \pm 1 \pmod 9 \text{ for some } i = 2,\ 3,\ 4 \\ g+1, & \text{if } m_i \not\equiv \pm 1 \pmod 9 \text{ for any } i \\ g+3, & \text{if } m_1 \equiv \pm 1 \pmod 9 \text{ and } m_i \not\equiv \pm 1 \pmod 9 \text{ for any } i \geq 2 \end{cases}$$

To find q^* we will use the cubic Hilbert symbol. For K_1, $L = K_1(\sqrt[3]{m_2})$ and $u \in \hat{U}_1$ it is known that $u \in N_{L/K_1}(L^*) \Leftrightarrow \left(\frac{u,m_2}{P}\right) = 1$ for all prime ideals $P \in K_1$. For any prime ideal P which is unramified from K_1 to L we know that $\left(\frac{u,m_2}{P}\right) = 1$ so we need only consider the ramified primes. Let $\mathfrak{Q} = \{\pi_1, \dots, \pi_d\}$ be the primes that ramify from K_1 to L. We will calculate the matrix, ${}_NB$ with rows

$$(\alpha_i)\,, \quad 1 \leq i \leq d$$

where $\alpha_i \in \mathbb{Z}_3$ is defined by

$$\left(\frac{u, m_2}{\pi_i}\right) = \zeta^{\alpha_i}$$

for each $u \in \{\zeta, \epsilon_1, u_1\}$ and $\pi_i \in \mathfrak{Q}$ where $\left(\frac{,m_2}{\pi_i}\right)$ is the cubic Hilbert symbol which can be calculated as in Section 7.2. Since $U_{K_1} =< \zeta, \epsilon_1, u_1 >$ then $q^* = 3 - \text{rank}\ {}_NB$.

7.4 Calculation of ${}_ND$

Let $K_1 = \mathbb{Q}(\sqrt[3]{m_1}, \zeta)$ be a subfield of L such that H_1 (The class number of K_1) is relatively prime to 3. Theorem 2.7 in [5] defines ${}_ND$ and ${}_ND^-$ and we restate it here.

Let $\pi_1, \dots, \pi_d$ be the prime ideals of k which ramify in L. Let $X = \mathbb{Z}_3 \times \cdots \times \mathbb{Z}_3$ (a product of $d-1$ copies of $\mathbb{Z}_3$). For $1 \leq i \leq d-1$, we define a map $\psi_i : K_1^* \to \mathbb{Z}_3$ (where $K_1^* = K_1 - 0$) by

$$\psi_i(z) = v_i \tag{7.9}$$

where

$$(z, L/\pi_\mathrm{i}) = \zeta^{v_i}$$

and $(\ , L/\pi_\mathrm{i}) = \left(\frac{,m_2}{\pi_i}\right)$ is the cubic Hilbert symbol described above. Then we define $\psi : K_1^* \to X$ by

$$\psi(z) = (\psi_1(z), \dots, \psi_{d-1}(z)). \tag{7.10}$$

Theorem 7.3 *Let $\sigma = \sigma_1$ be the generator of the cyclic group Gal(L/K_1), and let τ be the generator of Gal(L/K). Let S_L (resp. S_K, resp. S_{K_1}) be the 3-class group of L (resp. K, resp. K_1). Assume $S_{K_1} = \{1\}$. Now let t denote the number of ambiguous ideals in L/K_1. If t =0 or 1 then $s_1 = 0$. If $t \geq 2$,* *let $a_1, \ldots, a_t$ be norms of ideals chosen from a basis for the ambiguous ideal classes for L/K_1. Let $\hat{U}_1$ be the group of units of K_1, and ψ be the map defined by equations (7.9) and (7.10). If*

$$s = rank\ \{[\psi_i(\mathfrak{a}_j)] \pmod{\psi(\hat{U}_1)}\},$$

where $[\psi_i(\mathfrak{a}_j)]$ is the $t \times d-1$ matrix (over $\mathbb{Z}_3$) whose $ij-th$ element is defined by equation (7.9), then rank $S_L = 2t - s$. Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_u$ be ideals of L whose ideal classes generate $S_L^{(\sigma)^-}$ and $N_{L/K_1}(\mathfrak{A}_\mathfrak{j}) = (y_j)$, then if

$$s_1 = rank\ \{[\psi_i(y_j)] \pmod{\psi(\hat{U}_1)}\},$$

then rank $S_K = t - s_1$.

The matrices for the calculation of s and s_1 can be constructed as follows. Let $\pi_1, \ldots, \pi_t$ be the prime ideals in K_1 that ramify in L then we can construct the matrix $\left[\left(\frac{\pi_j,\ m_2}{\pi_i}\right)\right]$ for $1 \le i,\ j \le\ t$. The ideals $\mathfrak{A}_1, \ldots, \mathfrak{A}_u$ can be constructed from taking appropriate products of the π_j's. A rational prime p factors in K_l as $\mathfrak{p}_1 \cdots \mathfrak{p}_r$ where $r = 3$ if $p \equiv 2 \pmod 3$, $r = 6$ if $p \equiv 1 \pmod 3$ and $x^3 - m_l \equiv 0 \pmod p$ is solvable and $r = 2$ otherwise. In L each factor is totally ramified so $(p) = (\mathfrak{p}_1 \cdots \mathfrak{p}_r)^3 = \mathfrak{P}_1, \cdots \mathfrak{P}_\mathfrak{r}$ so if $a \in S_L^{\sigma-}$ then $a^{1+\tau} = \prod_{1 \le i \le s} \mathfrak{P}_\mathfrak{i} = \prod_{1 \le i \le r} {\mathfrak{p}_\mathfrak{i}}^3$. If $p = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$ the subscripts can be choses so that ${\mathfrak{p}_2}^\tau = \mathfrak{p}_3$ then $(\mathfrak{p}_2{\mathfrak{p}_3}^2)^{1+\tau} = ({\mathfrak{p}_2}^3{\mathfrak{p}_3}^3)$ and

$$\mathfrak{A}_j = \mathfrak{p}_2{\mathfrak{p}_3}^2 \in S_L^{\sigma-}.$$

If $p = \mathfrak{p}_1 \cdots \mathfrak{p}_6$ then the subscripts can be chosen so that ${\mathfrak{p}_\mathfrak{i}}^\tau = \mathfrak{p}_{\mathfrak{i}+3}$ for $i = 1,\ 2,\ 3$ then $(\mathfrak{p}_\mathfrak{i}{\mathfrak{p}_{\mathfrak{i}+3}}^2)^{1+\tau} = (\mathfrak{p}_\mathfrak{i}{\mathfrak{p}_\mathfrak{i}}^{2\tau})^{1+\tau} = (\mathfrak{p}_\mathfrak{i}{\mathfrak{p}_\mathfrak{i}}^\tau)^3$ and

$$\mathfrak{A}_j = \mathfrak{p}_\mathfrak{i}{\mathfrak{p}_{\mathfrak{i}+3}}^2 \in S_L^{\sigma-}$$

for i =1, 2, 3. If $p = \mathfrak{p}_1\mathfrak{p}_2$ then $(\mathfrak{p}_1{\mathfrak{p}_2}^2)^{1+\tau} = (\mathfrak{p}_1\mathfrak{p}_2)^3$ so

$$\mathfrak{A}_j = \mathfrak{p}_1{\mathfrak{p}_2}^2 \in S_L^{\sigma-}.$$

Note: In the calculation for ${}_N D$ it is necessary for the class number to be relatively prime to 3 so that a representatives for the ideal classes can be found. If there are weak ambiguous ideal classes we have no way of finding a representative and the rank of ${}_N D$ may be too large. It is possible to find the number of weak ambiguous classes if they exist. If we let $S_{L,s}^{(\sigma)}$ be the subgroup of $S_L^{(\sigma)}$ containing the strong ambiguous classes, then [3] shows that rank of $S_{L,s}^{(\sigma)}$ is

$$\text{rank } S_{L,s}^{(\sigma)} = d + q - 4$$

where

q is defined by $[V_{K_1} : \hat{U}_1^3] = 3^q$, where $V_{K_1} = \left\{x \in \hat{U}_1 \mid x = N_{L/K_1}(y),\ y \in \hat{E}\right\}$. So to find q it is sufficient to count the number of units of K_1 that are norms of integers of L. If $q < q^*$ then there are weak ambiguous ideal classes and the calculation of the rank can be smaller by the the value of $q^* - q$.

7.5 Rank of the 3-Class Group Example

Example 8

Let $L = \mathbb{Q}(\zeta, \sqrt[3]{2}, \sqrt[3]{31})$ then $\{m_1, m_2, m_3, m_4\} = \{2, 31, 62, 124\}$ and we know the following information about the cubic subfields $k_i = \mathbb{Q}(\sqrt[3]{m_i})$ and their normal closures $K_i = \mathbb{Q}(\sqrt[3]{m_i}, \zeta)$

Subfield	k_1	k_2	k_3	k_4
Type	I	III	I	III
h_i	1	3	3	9
H_i	1	3	9	27

Using the techniques in Chapters 5 and 6 we can easily find $\hat{e} = \{\epsilon_1, \epsilon_2, \epsilon_4, \sqrt[3]{\epsilon_3}\}$ and $\hat{E} = \left\{\epsilon_1, u_1, \epsilon_2, \epsilon_2', \sqrt[3]{\epsilon_3}, u_3, \epsilon_4, \sqrt[3]{\frac{\epsilon_2^2\epsilon_2'}{\epsilon_4^2\epsilon_4'}}\right\}$ so $[\hat{e} : \hat{e}_0] = 3$ and $\left[\hat{E} : \hat{\epsilon}\right] = 3^2$. Then using the class number relations we see that $h = h_1h_2h_3h_4\,[\hat{e} : \hat{e}_0]\,/3^3 = 9$ and $H = H_1H_2H_3H_4\left[\hat{E} : \hat{\epsilon}\right]/3^5 = 27$.

Since $\gcd(h_1, 3) = 1$ we can use K_1 as the base field for the calculation of the cubic Hilbert symbols and the rank. From K_1 to L only prime divisors of $p = 31$ are ramified and $(31) = P_1P_2P_3P_4P_5P_6$ where $P_i \in K_1$ and $P_1 = \frac{1}{3}(-1+\zeta+4\sqrt[3]{2}+2\zeta\sqrt[3]{2}+2\sqrt[3]{4}+\zeta\sqrt[3]{4})$, $P_2 = P_1^{\sigma}$, $P_3 = P_1^{\sigma^2}$, $P_4 = P_1^{\tau}$, $P_5 = P_2^{\tau}$ and $P_6 = P_3^{\tau}$ so so $d = 6$. To find q^* we need to calculate the matrix ${}_NB = [\alpha_{ji}]$ where $\left(\frac{\mu_j, 31}{P_i}\right) = \zeta^{\alpha_{ij}}$, $1 \le j \le 3$, $1 \le i \le 6$ and $\mu_j \in \{\zeta, \epsilon_1, u_1\}$. We will show the calculation of $\left(\frac{u_1, 31}{P_1}\right)$ where $u_1 = \frac{1}{3}(1+2\zeta+2\sqrt[3]{2}+\zeta\sqrt[3]{2}+\sqrt[3]{4}+2\zeta\sqrt[3]{4})$ and the rest of the entries for the ${}_NB$ matrix are similar.

To calculate the symbol we need to know which solutions to $X^3 - 2 \equiv (X-4)(X-7)(X-20) \equiv 0 \pmod{31}$ and $\zeta^2 + \zeta + 1 \equiv (\zeta - 25)(\zeta - 5) \equiv 0 \pmod{31}$ generate the ideal P_1. It is easy to see that $P_1 \mid (\sqrt[3]{2} - 20)$ and $P_1 \mid (\zeta - 25)$ so $\sqrt[3]{2} \equiv 20 \pmod{P_1}$ and $\zeta \equiv 25 \pmod{P_1}$ so

$$\begin{aligned}
\left(\frac{u_1, 31}{P_1}\right) &\equiv u_1^{(31-1)/3} \pmod{P_1} \\
&\equiv 3^3\frac{1}{3}(1 + 2(25) + 2(20) + (25)(20) + 20^2 + 2(25)20^2)^{10} \pmod{P_1} \\
&\equiv 25 \equiv \zeta^1 \pmod{P_1}.
\end{aligned}$$

Then

$${}_NB = [\alpha_{ji}] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 2 & 2 & 1 & 0 & 0 \end{bmatrix}$$

and $\text{rank}({}_NB) = 3$ and $q^* = 3 - 3 = 0$ so there are no weak ambiguous ideal classes in this example. Now we have that $t = d + q^* - 4 = 2$.

We also want to calculate $[\psi_i(P_j)] =_N D$. We will show the calculations for $\left(\frac{P_2,31}{P_1}\right)$ and $\left(\frac{P_1,31}{P_1}\right)$ and the rest of the entries will be similar. We begin with $\left(\frac{P_2,31}{P_1}\right)$ since it is the same as the calculations for ${}_NB$.

$$\left(\frac{P_2,31}{P_1}\right) \equiv P_2^{(31-1)/3} \equiv 25 \equiv \zeta^1 \pmod{P_1}.$$

The diagonal element $\left(\frac{P_1,31}{P_1}\right)$ is calculated by the product rule. Since

$$\left(\frac{31,31}{P_1}\right) = \left(\frac{P_1,31}{P_1}\right)\left(\frac{P_2\cdots P_6,31}{P_1}\right) = 1$$

then we can calculate

$$\left(\frac{P_1,31}{P_1}\right) = \left(\frac{P_2\cdots P_6,31}{P_1}\right)^2$$

where $P_2\cdots P_6 = \frac{1}{3}(-26 - \zeta + 8\sqrt[3]{2} - 14\zeta\sqrt[3]{2} + 19\sqrt[3]{4} - 10\zeta\sqrt[3]{4})$. So

$$\left(\frac{P_2\cdots P_6,31}{P_1}\right) \equiv (P_2\cdots P_6)^{(31-1)/3} \equiv 25 \equiv \zeta^1 \pmod{P_1}.$$

We can perform similar calculations on the rest of the entries to get

$${}_ND = \begin{bmatrix} 1 & 0 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 2 \\ 1 & 2 & 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 2 & 2 & 2 \\ 2 & 2 & 0 & 2 & 1 & 2 \\ 1 & 2 & 2 & 2 & 0 & 2 \end{bmatrix}$$

and by taking sums of the rows we can find the entries of ${}_ND^-$. An entry in the ${}_ND^-$ matrix comes from the symbol $\left(\frac{P_jP_{j+3}^2,31}{P_i}\right) = \left(\frac{P_j,31}{P_i}\right)\left(\frac{P_{j+3},31}{P_i}\right)^2$ for $1 \le j \le 3$, $1 \le i \le 6$ which can be calculated by taking the sum of the j^{th} row and twice the $\{j+3\}^{th}$ row which gives

$${}_ND^- = \begin{bmatrix} 1 & 1 & 0 & 0 & 2 & 2 \\ 2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 & 1 & 1 \end{bmatrix}.$$

Then $s = \text{rank}[{}_ND \pmod{{}_NB}] = 1$ and $s_1 = \text{rank}[{}_ND^- \pmod{{}_NB}] = 0$ so $\text{rank}S_L = 2t - s = 3$ and $\text{rank}S_K = t - s_1 = 2$. Then $S_L \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and $S_K \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

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Appendix A

Units of Cubic Fields and their Normal Closures

Let $M = jk^2$ and $N = j^2k$ where $(j,k) = 1$ then the following table provides the fundamental units in $k_i = \mathbb{Q}(\sqrt[3]{M})$ and $K_i = \mathbb{Q}(\sqrt[3]{M}, \zeta)$. Let

$$\epsilon = \frac{1}{d_1}(a + bM + cN)$$

be the fundamental unit of k_i and

$$u = \frac{1}{d_2}(d + e\zeta + fM + g\zeta M + hN + i\zeta N)$$

the fundamental unit of K_i where $uu^\tau = \epsilon$ when k_i is Type I or IV and $u = \epsilon^\sigma$ if k_i is Type III. Tables A.1 and A.2 contain both ϵ and u as well as the Type of the cubic subfileds where $M \leq 495$. Table A.2 contains those fields where the units are too large to fit in the format of Table A.1.

Table A.1: Units of $k_i = \mathbb{Q}(\sqrt[3]{M})$ and $K_i = k_i(\zeta)$ where $M < 495$

M	a	b	c	d_1	d	e	f	g	h	i	d_2	Type
2	-1	1	0	1	-1	- 2	- 2	- 1	2	1	3	I
3	-2	0	1	1	-3	0	3	3	-1	-2	3	IV
5	1	- 4	2	1	-13	-8	4	8	2	-2	3	I
6	1	-6	3	1	-7	-4	2	4	1	-1	1	I
7	2	-1	0	1	2	0	0	-1	0	0	1	III
10	-7	-1	2	3	3	5	1	-2	-1	0	3	I
11	1	4	-2	1	-11	-19	-4	4	4	2	3	I
12	1	3	-3	1	5	2	-1	-2	-1	1	1	I
13	-4	-3	2	1	-4	0	0	-3	-2	-2	1	III
14	1	2	-1	1	11	4	-2	-4	-1	1	3	I
15	1	-30	12	1	25	-24	-20	-10	4	8	1	I
17	18	-7	0	1	-7	28	13	2	-4	-5	3	IV
19	2	2	-1	3	2	0	0	2	1	1	3	III
20	1	1	-1	1	-2	5	2	1	-1	-2	3	I
21	-47	6	4	1	-47	0	0	6	-4	-4	1	III
22	23	3	-4	1	-14	41	19	5	-5	-7	3	I
23	-41399	-3160	6230	1	21272	-67853	-31340	-7480	8390	11020	3	I
26	3	-1	0	1	3	0	0	-1	0	0	1	III
28	-1	-1	1	3	-1	0	0	-1	-1	-1	3	III
30	1	9	-3	1	-9	10	6	3	-1	-2	1	I

continued on next page

Table A.1: *continued*

M	a	b	c	d_1	d	e	f	g	h	i	d_2	Type
31	-367	54	20	1	-367	0	0	54	-20	-20	1	III
33	3742201	97392	-394098	1	-511512	1956481	769436	159472	-190166	-239884	1	I
34	613	-24	-51	1	-52	305	110	16	-29	-34	1	I
35	-22	10	-1	3	-22	0	0	10	1	1	3	III
37	10	-3	0	1	10	0	0	-3	0	0	1	III
38	-151	55	-3	1	326	247	-23	-97	-22	7	3	I
39	-23	0	2	1	-23	0	0	0	-2	-2	1	III
42	1	-42	12	1	48	97	14	-14	-8	-4	1	I
43	-7	2	0	1	-7	0	0	2	0	0	1	III
44	113	-2	-17	3	69	11	-16	-19	-2	9	3	I
45	1081	66	-312	1	-591	-776	-52	166	184	44	1	I
46	-4139	48	309	1	2092	2529	122	-584	-197	-34	1	I
47	-592199	-69704	64786	1	404203	-1049416	-402796	-112004	80578	111614	3	I
51	-11015	2592	102	1	6408	5473	-252	-1728	-398	68	1	I
52	1	-4	2	1	13	29	4	-4	-4	-2	3	I
53	-344340	4517	23202	1	-344627	-672283	-87229	91747	47647	23222	3	IV
55	6597361	-1012254	-189996	1	3250985	-122584	-887092	-854858	8476	233264	1	I
57	1084	57	-88	1	1317	-1098	-627	-342	74	163	3	IV
58	1	-8	2	1	-61	-32	8	16	2	-2	3	I
60	1	-12	6	1	16	-15	-8	-4	2	4	1	I

continued on next page

Table A.1: *continued*

M	a	b	c	d_1	d	e	f	g	h	i	d_2	Type
61	1	-16	4	1	64	-61	-32	-16	4	8	3	I
62	1	-24	6	1	32	-31	-16	-8	2	4	1	I
63	4	-1	0	1	4	0	0	-1	0	0	1	III
65	-4	1	0	1	-4	0	0	1	0	0	1	III
66	1	24	-6	1	-32	33	16	8	-2	-4	1	I
67	1	16	-4	1	-67	-131	-16	16	8	4	3	I
68	1	12	-6	1	33	16	-4	-8	-2	2	1	I
69		Table A.2					Table A.2					I
70	1	8	-2	1	-32	35	16	8	-2	-4	3	I
73	154	-87	12	1	154	0	0	-87	-12	-12	1	III
74	-961	-23	60	1	1514	1921	97	-361	-109	-23	3	I
76	1	4	-2	1	-16	19	8	4	-2	-4	3	I
77	-40232807	-7113592	3894986	1	-120026200	-38759501	19102156	28212776	2141498	-4490060	3	I
78	-2134079	841944	-80154	1	-643839	-2042912	-327448	150688	111906	76638	1	I
79	292	95	-38	1	292	0	0	95	38	38	1	III
82	4290653	201605	-273730	3	-2333553	-3053803	-165785	537130	161795	38160	3	I
84	379	12	-45	1	-57	202	59	13	-21	-27	1	I
85		Table A.2					Table A.2					I
86	-7	6	-1	1	-7	0	0	6	1	1	1	III
89	62570	-2330	-2617	3	37019	28326	-1947	-8291	-1421	436	3	IV

continued on next page

Table A.1: *continued*

M	a	b	c	d_1	d	e	f	g	h	i	d_2	Type
90	1	-54	36	1	-161	-81	18	36	12	-12	1	I
91	9	-2	0	1	9	0	0	-2	0	0	1	III
92	-8279	-737	1139	1	5494	-14539	-4438	-1217	1427	1966	3	I
93	-16022	-64428	15001	1	-16022	0	0	-64428	-15001	-15001	1	III
94	-1128751	107457	30965	1	1672357	1722926	11122	-367807	-83339	-2446	3	I
95	1867321	-419488	2246	1	683357	3563317	631168	-149764	-171148	-138326	3	I
102	-123929	71883	-9708	1	-118387	76211	41649	25338	-3491	-8914	1	I
105	30241	-5844	-120	1	15135	-2849	-3812	-3208	128	808	1	I
106	-8585	3177	-288	1	8533	5778	-582	-1803	-258	123	1	I
109	10945	-1890	-84	1	10945	0	0	-1890	84	84	1	III
110	1	-5	1	1	22	47	5	-5	-2	-1	3	I
114	61561	-13758	219	1	-40604	-32187	1736	8374	1369	-358	1	I
115	-19825999	1437980	542670	1	-31095997	-1955000	5992460	6394480	82670	-1232270	3	I
116	75169	5385	-8529	1	58666	15543	-8842	-12029	-1307	3626	1	I
117	412	-50	-21	1	412	0	0	-50	21	21	1	III
118	46647527	-363175	-1864939	3	-23600059	-28562133	-1011663	4811558	1187233	206257	3	I
122	1	-25	5	1	122	247	25	-25	-10	-5	3	I
124	5	-1	0	1	5	0	0	-1	0	0	1	III
126	-5	1	0	1	-5	0	0	1	0	0	1	III
129	25978	10401	-3076	1	25978	0	0	10401	3076	3076	1	III

continued on next page

Table A.1: *continued*

M	a	b	c	d_1	d	e	f	g	h	i	d_2	Type
130	1	15	-3	1	51	25	-5	-10	-1	1	1	I
132	-224531	-22959	26340	1	57113	-137764	-38274	-11217	10628	15034	1	I
134	-270703	97717	-8758	3	-270703	0	0	97717	8758	8758	3	III
138	25429951	13776309	-3618144	1	-24530283	33090283	11150247	4746894	-1239123	-2157702	1	I
140	1	5	-2	1	-28	-53	-5	5	4	2	3	I
141	253801	15288	-12306	1	-143633	-193641	-9608	27596	7148	1846	1	I
142	14059	-5199	480	1	10153	-5253	-2953	-1946	193	566	1	I
148	1	259	-98	1	2741	1369	-259	-518	-98	98	3	I
150	1	3	-3	1	-6	-11	-1	1	2	1	1	I
153	-50	4	3	1	84	57	-5	-16	-6	3	3	IV
154	-91627	10973	1142	3	-2901	-47806	-8378	541	1664	1563	3	I
156	-42119	-501	3093	1	21893	27443	1031	-4067	-1894	-383	1	I
158	10010881	-1625176	-41914	1	-17520544	-14935901	478096	3240872	511046	-88436	3	I
164	329	22	-30	1	779	208	-104	-142	-14	38	3	I
165	1	-66	12	1	120	241	22	-22	-8	-4	1	I
166	1	-242	44	1	-2659	-1331	242	484	44	-44	3	I
170	454411	-15021	-12096	1	-223991	-256581	-5883	40434	8361	1062	1	I
171	58	-16	3	1	93	156	11	-17	-15	-6	3	IV
172	-427	74	1	3	-46	-261	-39	8	17	14	3	I
174	15661	-2688	-21	1	7917	-1652	-1714	-1418	53	307	1	I

continued on next page

Table A.1: *continued*

M	a	b	c	d_1	d	e	f	g	h	i	d_2	Type
175	281	-48	-2	1	88	515	76	-16	-82	-68	3	I
182	-17	3	0	1	-17	0	0	3	0	0	1	III
186	-36827	-2676	1599	1	21668	29945	1450	-3796	-919	-254	1	I
198	1	-6	3	1	12	-11	-4	-2	1	2	1	I
204	1	-9	3	1	18	-17	-6	-3	1	2	1	I
206	-24067	-20033	4082	3	-24067	0	0	-20033	-4082	-4082	3	III
207	1	-12	6	1	23	47	4	-4	-4	-2	1	I
210	1	-18	3	1	35	71	6	-6	-2	-1	1	I
212	1	-27	9	1	-107	-54	9	18	3	-3	1	I
214	1	-54	9	1	-215	-108	18	36	3	-3	1	I
218	1	54	-9	1	217	108	-18	-36	-3	3	1	I
220	1	27	-9	1	109	54	-9	-18	-3	3	1	I
222	1	18	-3	1	73	36	-6	-12	-1	1	1	I
228	1	9	-3	1	-18	19	6	3	-1	-2	1	I
230	-140759	-3954	4395	1	75716	98165	3664	-12358	-2615	-598	1	I
234	1	6	-3	1	-12	13	4	2	-1	-2	1	I
236	1889	-695	126	1	-7372	-4661	439	1193	244	-142	3	I
238	3095	-549	8	1	-6185	-4879	211	998	127	-34	3	I
244	25755	-1344	-889	1	25755	0	0	-1344	889	889	1	III
252	1	3	-3	1	-7	-13	-1	1	2	1	1	I

continued on next page

Table A.1: *continued*

M	a	b	c	d_1	d	e	f	g	h	i	d_2	Type
260	77221	-3291	-2760	1	-37804	-41755	-619	5923	2050	194	1	I
268	2681	-793	117	1	-8710	-5759	458	1351	277	-142	3	I
275	1	338	-260	1	-2200	-4397	-338	338	520	260	3	I
276	-119231	-19092	11490	1	89953	138576	7468	-13816	-6538	-2294	1	I
284	-4728599	-775803	454938	1	-5596363	-1979836	550198	851399	91646	-167408	1	I
285	61561	-9552	30	1	-39064	-31615	1132	5936	730	-172	1	I
292	-18710629	-1638012	1344019	1	-50518631	-15237877	5317942	7614779	692414	-1603169	3	I
306	-88127	30504	-7758	1	-43775	-117792	-10984	6496	7782	4890	1	I
308	-769	124	-3	1	-319	-1526	-179	47	67	53	3	I
315	-106989119	9581742	2708292	1	-52569041	2606199	8109126	7726092	-168884	-3575404	1	I
316	-135	-113	39	1	-135	0	0	-113	-39	-39	1	III
325	263	14	-38	3	-55	153	30	8	-16	-22	3	I
340	1	-49	14	1	343	-340	-98	-49	14	28	3	I
342	7	-1	0	1	7	0	0	-1	0	0	1	III
348	48859201	-5115420	-520590	1	23926625	-2594224	-3770460	-3401640	104870	1072090	1	I
350	1	21	-15	1	99	49	-7	-14	-5	5	1	I
356	247829401	-74788635	11237145	1	108324125	302228426	27359245	-15284170	-12033685	-7720595	1	I
364	1	7	-2	1	-49	52	14	7	-2	-4	3	I
369	98549569	20930412	-14501130	1	-54812777	88473327	19976924	7641988	-5159204	-8355538	1	I
380	-1709	312	-21	1	1311	970	-47	-181	-37	13	1	I

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Table A.1: *continued*

M	a	b	c	d_1	d	e	f	g	h	i	d_2	Type
388	379137	-29430	-6184	1	379137	0	0	-29430	6184	6184	1	III
396	1	99	-81	1	-242	243	66	33	-27	-54	1	I
412	4441361	-1129537	143169	1	9484034	-4831727	-1923898	-1274561	174529	517106	3	I
414	-26081	-7233	4320	1	27829	46827	2549	-3734	-2529	-1026	1	I
420	1	-90	24	1	-449	-225	30	60	8	-8	1	I
423	1	150	-60	1	-375	376	100	50	-20	-40	1	I
425	1	54	-36	1	271	135	-18	-36	-12	12	1	I
436	17659	1800	-1089	1	-5778	11989	2343	762	-417	-618	1	I
460	3272671	-112368	-80727	1	-172661	1602549	229966	22367	-53786	-59581	1	I
468	-95	20	-6	1	-95	0	0	20	6	6	1	III
476	368164819	-39992352	-1834125	1	182236923	-29109983	-27068254	-23339993	954994	6933533	1	I
477	-79976	26597	-6282	1	15996	322155	39184	-2047	-15831	-15045	3	IV
490	1	1	-1	1	-7	10	2	1	-1	-2	3	I
492	-1436885	252624	-17889	1	1148942	836645	-39559	-145538	-26849	10022	1	I
495	-166319	17244	1434	1	-12056	-94071	-10368	1524	4510	3932	1	I

Table A.2: Units of k_i and K_i not on Table A.1

M / Type	a / d / g	b / e / h	c / f / i	d_1 / d_2
29	-322461439	103819462	370284	1
I	-109399864	-602078573	-160360646	
	35608262	63785384	52195348	3
41	-211991370839	305478475184	-70761183382	1
I	-1069595287171	856448462821	558562183216	
	310187906564	-72029764756	-161985786938	3
59	46334227393	-42285555004	7804684934	1
I	169839753749	-122976557075	-75216177860	
	-43626999772	8114360948	19320895738	3
69	13753611475894008059401	-5630668308465438120720	555253697459615284770	1
I	4123411326168946471551	13116586408449629284375	2192619900890712138320	
	-1005326111256288955940	-779689042418751871660	-534581166917834122790	1
71	1386226224	-309672529	-6061266	1
IV	-523451755	2009741824	611761669	
	126412652	-117210910	-147739337	3
83	1146072952401913	454318149188752	-164383888363874	1
I	-3304820485678901	-5439894130636600	-489463331928460	
	757626534412996	285894007747822	112208941302002	3

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Table A.2: *continued*

M / Type	a / d / g	b / e / h	c / f / i	d_1 / d_2
85	445707361	-213091974	25409628	1
I	-533782711	-343954216	43173948	
	121401726	17791860	-9819336	1
87	-2025487074437153495	2295582499903407744	-414906405399771342	1
I	-3441714711709488360	2762956548473276257	1400271852147510068	
	776727087034554784	-140721747620176586	-316013722193375668	1
97	-12891251368	7987833890	-1127854887	1
III	-12891251368	0	0	
	7987833890	1127854887	1127854887	1
99	-50708057399	10025456082	606946152	1
I	-4813203273	-30205758737	-5489018294	
	1040453012	4234357684	3559624616	1
101	-7591749839	4748284228	-669551396	1
I	38665759603	23196431744	-3321729928	
	-8302701452	-1069564388	713275316	3
103	-239972695	-293511054	73536248	1
I	1628928520	3017205371	296162686	
	-347501182	-137313548	-63180724	3

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Table A.2: *continued*

M / Type	a / d / g	b / e / h	c / f / i	d_1 / d_2
107	638744633822	-28576936259	-22320903094	3
IV	387694430043	171097280053	-45623802503	
	-81663558863	-7591377479	9610151172	3
111	43525986334	-12634237836	744387361	1
IV	45278508885	-62670266316	-22461739845	
	-9421450917	2713394171	4673788618	3
113	2462929921	-147738928	-74815168	1
I	-347714563	3615652861	819794224	
	71922272	-154691968	-169568576	3
119	1712946929	-244884818	-21015420	1
I	-2748335971	-2514474539	47545522	
	558754244	103931996	-9666308	3
123	-589540519295	9801566106	21865679148	1
I	295378811087	351721655975	11329317154	
	-59394236092	-14220963932	-2278078564	1
146	1392658961	-4998461411	899032809	1
I	26345529761	-24952414265	-9742019789	
	-5003293546	899934017	1850112151	3

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Table A.2: *continued*

M / Type	a / d / g	b / e / h	c / f / i	d_1 / d_2
159	-17294170949567	-2947566357324	1133335052934	1
I	-18891295049295	-6431331798152	2299946426724	
	3487086251304	219130501634	-424540061570	1
178	1558656289	-413710656	24288330	1
I	384361184	1330251137	168151448	
	-68328128	-42039122	-29892388	1
188	-21849781589	595631892	1123645827	1
I	1705391523	-10808591927	-2184454282	
	-297694961	658709434	762641333	1
190	117911153	-89330395	11971022	3
I	135091922	309773397	30385215	
	-23498755	-9372923	-5285399	3
194	2807954442181	-406402766104	-13586300107	1
I	4163583020909	-645613479887	-830753052656	
	-719228108902	19265107478	143506432807	3
195	-5072427359	2018952168	-197318508	1
I	6704467809	4223193664	-427887776	
	-1156163992	-125588900	73787876	1

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Table A.2: *continued*

M / Type	a / d / g	b / e / h	c / f / i	d_1 / d_2
202	3790812432089244251	-524090932863673845	-20789634078252400	1
I	-6402534796362796625	-5602031576267739622	136430863148706970	
	1091192798079789905	162721387764947950	-23252099369807575	3
226	-564900116420417725483	142879003022457802169	-8231322666505614370	3
I	-146890335321056935593	-496276578298684186939	-57359187076940357717	
	24115174517659893922	13375750312464972611	9416731220118818184	3
246	-1300032932448067653781430999	120087865458900330831652725	13947403785035954184145185	1
I	-22790736227929793803239249	-663798133335677666780021750	-102302129076326535659215050	
	3637307228987051755842375	16907495819986173195588395	16326996631819173907149730	1
255	6917784652738931219641	-1154384867582077845114	10010485940314078200	1
I	-4491073115396420102415	-3589734305195677262024	142137283602281136228	
	708223063133729402802	89269311470743927048	-22414443005439607132	1
258	135225343601162542061713	-12415709061055583290698	-1386371500055886366024	1
I	2542881466280207691425	69169516105572037451408	10465853276620428660068	
	-399441221517607543502	-1706743536431341914584	-1643998460987114831260	1
261	-35657633489660100071	4127420997953573658	681717778886150760	1
I	-1920290481447809432	-19384539036145942503	-2732769915217795694	
	300483129276579182	1423911708415379476	1282854750857941292	1

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Table A.2: *continued*

M / Type	a / d / g	b / e / h	c / f / i	d_1 / d_2
279	-1463404502	232270591	-3817272	1
III	-1463404502	0	0	
	232270591	3817272	3817272	1
282	2606837131937809	-532875098916735	20639996084577	1
I	-2023122042970869	-1490409222248147	81235048901381	
	308512225914202	34658209584211	-12387787403083	1
318	-85199430998501	-222813638828235	34472441430732	1
I	507458291473817	-535867461161821	-152854262495421	
	-74346063724074	11501980846081	22394180824154	1
330	-23418509399	11750712930	-1210033749	1
I	48045379249	27972819324	-2904678810	
	-6952595760	-585769759	420332983	1
332	-2982473893363	961613263390	-153340198943	3
I	-2415979450702	1389403635219	549564944493	
	348910367956	-57956173519	-158733889018	3
333	7309383190	-288532187	-331545960	1
IV	5506089603	-7527428976	-1880385661	
	-794380418	470043912	813867003	3

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Table A.2: *continued*

M / Type	a / d / g	b / e / h	c / f / i	d_1 / d_2
372	-14204107237112240759	1604368313324222241	103067231355020337	1
I	-1003626357733526979	-8001452356436296877	-973005943013757547	
	139548255532646767	309388043461278722	270581339208947101	1
387	2599095562	-2533876181	896298960	1
III	2599095562	0	0	
	-2533876181	-896298960	-896298960	1
404	3280344238926020701	-598335692703660915	41825587903242060	1
I	677422112510481651	2558570782722640375	254465747129185565	
	-91635885409508630	-93635353655122730	-68843911687970620	1
428	240885506522274203773	-101690663296223364813	18504621678700224252	1
I	184294445525157668292	450057706605053745517	35265254404009571187	
	-24454811700766739715	-15849017773574567490	-9358992383869364598	1
444	563095091251	15491781300	-23411741025	1
I	-95375761249	306866281250	52726199825	
	12501928975	-10545257425	-13822782575	1
452	76611901591	-36813000572	6992115857	1
I	-504152787170	-290784848569	27802446677	
	65692535974	9874371887	-7245475082	3

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Table A.2: *continued*

M / Type				
M	a	b	c	d_1
Type	d	e	f	
	g	h	i	d_2
475	3087924841	-786868522	250629632	1
I	3625654648	10470332315	877246810	
	-464681338	-859939828	-562160852	3

Appendix B

Some invariants of K and L where $m_i \leq 500$ for all i

Let $K = \mathbb{Q}(m_1, m_2)$, $L = \mathbb{Q}(m_1, m_2, \zeta)$ then $m_3 = m_1 m_2$ and $m_4 = m_1^2 m_2$ where m_3 and m_4 are cube free. Then we can find the fundamental units for k_i and K_i using Tables A.1 and A.2. Table B.1 provides a basis for the the fundamental units of K, $\hat{e}$ and L, $\hat{E}$ where $m_i \leq 500$ for $1 \leq i \leq 4$. The basis for the units of K has 4 elements and all of them are specified in the table. The basis for the units of L has 8 elements so the only elements listed in the table are those which are not products of units in the subfields. The remaining basis elements are specified in Theorem 6.12. Table B.1 also lists the class numbers, h_i, of the cubic subfields k_i, $1 \leq i \leq 4$ as well as the class numbers of K and L (h and H respectively). The last column of Table B.1 gives the rank of the 3-class group of $\{h, H\}$. The entries marked with $^{(v)}$ indicate that the rank may be smaller than indicated by v because of the presence of weak ambiguous classes. The value of v is the difference between the number of units that are norms and the number of units of K_i, where i is the base field used for the calculation, that have Hilbert symbol 1. Since there are three units in K_i then $v \leq 3$.

Table B.1: Unit Basis, Class Numbers and Rank of the 3-Class Group for K and L

$\{m_1, m_2, m_3, m_4\}$	$\{h_1, h_2, h_3, h_4\}$	Type				$\hat{e}$	$\hat{E}$	$\{h, H\}$	ranks
{3, 2, 6, 12}	{1, 1, 1, 1}	IV	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_2}, \sqrt[3]{\epsilon_3}, \sqrt[3]{\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_2^2\epsilon_4}{u_4}}, \sqrt[3]{u_2^2u_3}\right\}$	{1, 1}	{0, 0}
{2, 5, 10, 20}	{1, 1, 1, 3 }	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1\epsilon_2}\}, \sqrt[3]{\epsilon_3}, \sqrt[3]{\epsilon_1\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1u_3\epsilon_4}{u_2^2\epsilon_2^2u_4}}\right\}$	{3, 3}	{1, 1}
{2, 7, 14, 28}	{1, 3, 3, 3}	I	III	I	III	$\{\epsilon_1, \epsilon_2, \epsilon_4, \sqrt[3]{\epsilon_1\epsilon_3}\}$	$\left\{\sqrt[3]{\frac{\epsilon_2^2\epsilon_2'}{\epsilon_4^2\epsilon_4'}}\right\}$	{3, 3}	{1, 1}
{2, 11, 22, 44}	{1, 2, 3, 1}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1\epsilon_2}, \sqrt[3]{\epsilon_1^2\epsilon_3}, \sqrt[3]{\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1u_3^2\epsilon_4}{u_2u_4\epsilon_2}}\right\}$	{6, 12}	{1, 1}
{2, 13, 26, 52}	{1, 3, 3, 3}	I	III	III	I	$\{\epsilon_1, \epsilon_2, \epsilon_3, \sqrt[3]{\epsilon_1\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{\epsilon_2^2\epsilon_2'\epsilon_3'}{\epsilon_3}}\right\}$	{3, 3}	{1, 1}
{2, 15, 30, 60}	{1, 2, 3, 3}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1^2\epsilon_2}, \sqrt[3]{\epsilon_1\epsilon_3}, \sqrt[3]{\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1u_3}{u_2^2\epsilon_2^2}}\right\}$	{18, 108}	{2, 3}
{2, 17, 34, 68}	{1, 1, 3, 3}	I	IV	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_3}, \sqrt[3]{\epsilon_4}\}$	$\left\{\sqrt[3]{u_3^2u_4}\right\}$	{3, 9}	{1, 2}
{2, 19, 38, 76}	{1, 3, 3, 6}	I	III	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_1^2\epsilon_3}, \sqrt[3]{\epsilon_1\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1^2u_3^2\epsilon_4}{u_4}}\right\}$	{18, 108}	{1, 2}
{2, 21, 42, 84}	{1, 3, 3, 3}	I	III	I	I	$\{\epsilon_1, \epsilon_2, \sqrt{\epsilon_1^2\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1u_3\epsilon_4}{\epsilon_2'u_4\epsilon_2^2}}\right\}$	{9, 27}	$\{2, 3^{(1)}\}$
{2, 23, 46, 92}	{1, 1, 1, 3}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1\epsilon_2}, \sqrt[3]{\epsilon_3}, \sqrt[3]{\epsilon_1\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1u_3^2\epsilon_4}{u_2^2u_4\epsilon_2^2}}\right\}$	{3, 3}	{1, 1}
{2, 29, 58, 116}	{1, 1, 6, 1}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1^2\epsilon_2}, \sqrt[3]{\epsilon_1\epsilon_3}, \sqrt[3]{\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1u_3\epsilon_4}{u_2^2u_4\epsilon_2^2}}\right\}$	{6, 12}	{1, 1}
{2, 31, 62, 124}	{1, 3, 3, 9}	I	III	I	III	$\{\epsilon_1, \epsilon_2, \epsilon_4, \sqrt[3]{\epsilon_3}\}$	$\left\{\sqrt[3]{\frac{\epsilon_2^2\epsilon_2'}{\epsilon_4^2\epsilon_4'}}\right\}$	{9, 27}	{2, 2}
{2, 33, 66, 132}	{1, 1, 6, 3}	I	I	I	I	$\{\epsilon_1 \sqrt[3]{\epsilon_1^2\epsilon_2}, \sqrt[3]{\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1^2\epsilon_4}{u_2^2u_4\epsilon_2^2}}\right\}$	{18, 108}	{2, 3}
{2, 35, 70, 140}	{1, 3, 9, 9}	I	III	I	I	$\{\epsilon_1, \epsilon_2, \epsilon_4, \sqrt[3]{\epsilon_1\epsilon_3}\}$	{ }	{27, 243}	$\{3, 5^{(1)}\}$
{2, 37, 74, 148}	{1, 3, 3, 6}	I	III	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_1^2\epsilon_3}, \sqrt[3]{\epsilon_1\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1^2u_3^2\epsilon_4}{u_4}}\right\}$	{18, 108}	{1, 2}
{2, 39, 78, 156}	{1, 6, 3, 3}	I	III	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_1^2\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1u_3\epsilon_4}{\epsilon_2'\epsilon_2^2u_4}}\right\}$	{18, 108}	$\{2, 3^{(1)}\}$

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Table B.1: *continued*

$\{m_1, m_2, m_3, m_4\}$	$\{h_1, h_2, h_3, h_4\}$	Type				$\hat{e}$	$\hat{E}$	$\{h, H\}$	ranks
{2, 41, 82, 164}	{1, 1, 1, 6}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1\epsilon_2}, \sqrt[3]{\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1^2u_3^2\epsilon_4}{u_2\epsilon_2u_4}}\right\}$	{6, 12}	{1, 1}
{2, 43, 86, 172}	{1, 12, 9, 3}	I	III	III	I	$\{\epsilon_1, \epsilon_2, \epsilon_3, \sqrt[3]{\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{\epsilon_2^2\epsilon_2'}{\epsilon_3^2\epsilon_3'}}\right\}$	{36, 432}	{2, 3}
{2, 45, 90, 150}	{1, 1, 3, 3}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1\epsilon_2}, \sqrt[3]{\epsilon_1^2\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1^2u_3}{u_2^2\epsilon_2^2}}\right\}$	{9, 27}	{2, 3}
{2, 47, 94, 188}	{1, 2, 3, 1}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1\epsilon_2}, \sqrt[3]{\epsilon_1^2\epsilon_3}, \sqrt[3]{\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1^2u_3}{u_2^2\epsilon_2^2}}\right\}$	{6, 12}	{1, 1}
{2, 51, 102, 204}	{1, 3, 3, 12}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1^2\epsilon_2}, \sqrt[3]{\epsilon_1^2\epsilon_3}, \sqrt[3]{\epsilon_1\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_3\epsilon_4}{u_2^2\epsilon_2^2u_4}}\right\}$	{108, 3888}	$\{2, 3^{(1)}\}$
{2, 53, 106, 212}	{1, 1, 6, 6}	I	IV	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_3}, \sqrt[3]{\epsilon_4}\}$	$\left\{\sqrt[3]{u_3^2u_4}\right\}$	{12, 144}	{1, 2}
{2, 55, 110, 220}	{1, 1, 9, 9}	I	I	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_1\epsilon_2^2\epsilon_3}, \sqrt[3]{\epsilon_4}\}$	{ }	{27, 243}	{3, 4}
{2, 57, 114, 228}	{1, 6, 3, 9}	I	IV	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_1\epsilon_3}, \sqrt[3]{\epsilon_1\epsilon_4}\}$	{ }	{54, 972}	$\{2, 3^{(1)}\}$
{2, 59, 118, 236}	{1, 1, 2, 6}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1\epsilon_2}, \sqrt[3]{\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1^2u_3^2\epsilon_4}{u_2\epsilon_2u_4}}\right\}$	{12, 48}	{1, 1}
{2, 61, 122, 244}	{1, 6, 12, 3}	I	I	I	III	$\{\epsilon_1, \epsilon_4, \sqrt[3]{\epsilon_1\epsilon_2}, \sqrt[3]{\epsilon_1\epsilon_3}\}$	$\left\{\sqrt[3]{\frac{u_1^2u_2\epsilon_3}{u_3}}\right\}$	{72, 1728}	{1, 2}
{2, 63, 126, 252}	{1, 6, 9, 6}	I	III	III	I	$\{\epsilon_1, \epsilon_2, \epsilon_3, \sqrt[3]{\epsilon_1^2\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{\epsilon_2^2\epsilon_2'\epsilon_3'}{\epsilon_3}}\right\}$	{36, 432}	$\{2, 3^{(1)}\}$
{2, 65, 130, 260}	{1, 18, 9, 3}	I	III	I	I	$\{\epsilon_1, \epsilon_2, \epsilon_3, \sqrt[3]{\epsilon_3\epsilon_4}\}$	{ }	{54, 972}	$\{3, 5^{(2)}\}$
{2, 67, 134, 268}	{1, 6, 3, 6}	I	I	III	I	$\{\epsilon_1, \epsilon_3, \sqrt[3]{\epsilon_1\epsilon_2}, \sqrt[3]{\epsilon_1^2\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1\epsilon_2^2u_4}{u_2^2}}\right\}$	{36, 144}	{1, 2}
{2, 69, 138, 276}	{1, 1, 3, 3}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1\epsilon_2}, \sqrt[3]{\epsilon_1^2\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1^2u_3}{u_2^2\epsilon_2^2}}\right\}$	{9, 27}	{2, 3}
{2, 71, 142, 284}	{1, 1, 6, 3}	I	IV	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_3}, \sqrt[3]{\epsilon_4}\}$	$\left\{\sqrt[3]{u_3u_4}\right\}$	{6, 36}	{1, 2}
{2, 73, 146, 292}	{1, 3, 3, 3}	I	III	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_1^2\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1u_3\epsilon_4}{u_4}}\right\}$	{9, 27}	{1, 2}
{2, 77, 154, 308}	{1, 3, 3, 9}	I	I	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_1\epsilon_2\epsilon_3}, \sqrt[3]{\epsilon_2\epsilon_4}\}$	{ }	{27, 243}	$\{3, 5^{(1)}\}$
{2, 79, 158, 316}	{1, 6, 3, 6}	I	III	I	III	$\{\epsilon_1, \epsilon_2, \epsilon_4, \sqrt[3]{\epsilon_1\epsilon_3}\}$	$\left\{\sqrt[3]{\frac{\epsilon_2^2\epsilon_2'\epsilon_4'}{\epsilon_4}}\right\}$	{12, 48}	{1, 1}
{2, 83, 166, 332}	{1, 2, 6, 1}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1\epsilon_2}, \sqrt[3]{\epsilon_1\epsilon_3}, \sqrt[3]{\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1u_3}{u_2\epsilon_2}}\right\}$	{12, 48}	{1, 1}

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Table B.1: *continued*

$\{m_1, m_2, m_3, m_4\}$	$\{h_1, h_2, h_3, h_4\}$	Type				$\hat{e}$	$\hat{E}$	$\{h, H\}$	ranks
{2, 85, 170, 340}	{1, 3, 3, 9}	I	I	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_2\epsilon_3}, \sqrt[3]{\epsilon_1\epsilon_4}\}$	$\{\ \}$	{27, 243}	{3, 4}
{2, 87, 174, 348}	{1, 1, 12, 3}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1^2\epsilon_2}, \sqrt[3]{\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1^2\epsilon_4}{u_2^2\epsilon_2^2u_4}}\right\}$	{36, 432}	{2, 3}
{2, 89, 178, 356}	{1, 2, 3, 6}	I	IV	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_3}, \sqrt[3]{\epsilon_4}\}$	$\{\sqrt[3]{u_3u_4}\}$	{12, 144}	{1, 2}
{2, 91, 182, 364}	{1, 9, 27, 18}	I	III	III	I	$\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$	$\{\ \}$	{162, 8748}	$\{3, 6^{(2)}\}$
{2, 93, 186, 372}	{1, 3, 6, 3}	I	III	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_3}, \sqrt[3]{\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_3\epsilon_4}{\epsilon_2'^2\epsilon_2u_4}}\right\}$	{18, 108}	{2, 3}
{2, 95, 190, 380}	{1, 3, 3, 9}	I	I	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_1^2\epsilon_2^2\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_2^2\epsilon_4}\}$	$\{\ \}$	{27, 243}	$\{3, 5^{(1)}\}$
{2, 97, 194, 388}	{1, 3, 3, 3}	I	III	I	III	$\{\epsilon_1, \epsilon_2, \epsilon_4, \sqrt[3]{\epsilon_1\epsilon_3}\}$	$\left\{\sqrt[3]{\frac{\epsilon_2^2\epsilon_2'\epsilon_4'}{\epsilon_4}}\right\}$	{3, 3}	{1, 1}
{2, 99, 198, 396}	{1, 1, 6, 3}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1^2\epsilon_2}, \sqrt[3]{\epsilon_1^2\epsilon_3}, \sqrt[3]{\epsilon_1\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1\epsilon_4}{u_2\epsilon_2u_4}}\right\}$	{18, 108}	{2, 3}
{2, 101, 202, 404}	{1, 2, 3, 1}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1\epsilon_2}, \sqrt[3]{\epsilon_1^2\epsilon_3}, \sqrt[3]{\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1^2u_3\epsilon_4}{u_2^2\epsilon_2^2u_4}}\right\}$	{6, 12}	{1, 1}
{2, 103, 206, 412}	{1, 3, 3, 6}	I	I	III	I	$\{\epsilon_1, \epsilon_3, \sqrt[3]{\epsilon_1^2\epsilon_2}, \sqrt[3]{\epsilon_1^2\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1^2\epsilon_4}{u_2^2\epsilon_2^2u_4}}\right\}$	{18, 108}	{1, 2}
{2, 105, 210, 420}	{1, 6, 18, 9}	I	I	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_1\epsilon_3}, \sqrt[3]{\epsilon_2^2\epsilon_4}\}$	$\{\ \}$	{324, 34992}	$\{4, 6^{(1)}\}$
{2, 107, 214, 428}	{1, 1, 12, 3}	I	IV	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_3}, \sqrt[3]{\epsilon_4}\}$	$\left\{\sqrt[3]{u_3^2u_4}\right\}$	{12, 144}	{1, 2}
{2, 109, 218, 436}	{1, 3, 18, 9}	I	III	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_3}, \sqrt[3]{\epsilon_4}\}$	$\left\{\sqrt[3]{u_3^2u_4}\right\}$	{162, 8748}	{2, 5}
{2, 111, 222, 444}	{1, 3, 18, 3}	I	IV	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_1\epsilon_3}, \sqrt[3]{\epsilon_1\epsilon_4}\}$	$\{\ \}$	{54, 972}	$\{2, 3^{(1)}\}$
{2, 113, 226, 452}	{1, 4, 1, 3}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1\epsilon_2}, \sqrt[3]{\epsilon_3}, \sqrt[3]{\epsilon_1\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1\epsilon_4}{u_2^2\epsilon_2^2u_4}}\right\}$	{12, 48}	{1, 1}
{2, 115, 230, 460}	{1, 3, 9, 3}	I	I	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_1^2\epsilon_2}, \sqrt[3]{\epsilon_3\epsilon_4}\}$	$\{\ \}$	{27, 243}	{3, 4}
{2, 117, 234, 468}	{1, 3, 6, 9}	I	III	I	III	$\{\epsilon_1, \epsilon_2, \epsilon_4, \sqrt[3]{\epsilon_1^2\epsilon_3}\}$	$\left\{\sqrt[3]{\frac{\epsilon_2^2\epsilon_2'}{\epsilon_4^2\epsilon_4'}}\right\}$	{18, 108}	$\{2, 3^{(1)}\}$
{2, 119, 238, 476}	{1, 3, 9, 3}	I	I	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_1^2\epsilon_3}, \sqrt[3]{\epsilon_1\epsilon_2^2\epsilon_4}\}$	$\{\ \}$	{27, 243}	$\{3, 5^{(1)}\}$
{2, 123, 246, 492}	{1, 2, 3, 6}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1^2\epsilon_2}, \sqrt[3]{\epsilon_1\epsilon_3}, \sqrt[3]{\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1u_3}{u_2^2\epsilon_2^2}}\right\}$	{36, 432}	{2, 3}

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Table B.1: *continued*

$\{m_1, m_2, m_3, m_4\}$	$\{h_1, h_2, h_3, h_4\}$	Type				$\hat{e}$	$\hat{E}$	$\{h, H\}$	ranks
{2, 175, 350, 490}	{1, 3, 3, 9}	I	I	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_1^2\epsilon_2\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_2^2\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1^2\epsilon_2^2\epsilon_4}{u_2^2u_3\epsilon_3u_4}}\right\}$	{27, 729}	{3, 5}
{3, 5, 15, 45}	{1, 1, 2, 1}	IV	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_2}, \sqrt[3]{\epsilon_3}, \sqrt[3]{\epsilon_4}\}$	$\left\{\sqrt[3]{u_2u_3}, \sqrt[3]{u_2^2u_4}\right\}$	{2, 4}	{0, 0}
{3, 7, 21, 63}	{1, 3, 3, 6}	IV	III	III	III	$\{\epsilon_1, \epsilon_2, \epsilon_4, \sqrt[3]{\epsilon_1\epsilon_2\epsilon_3\epsilon_4^2}\}$	$\left\{\sqrt[3]{\frac{\epsilon_2^2\epsilon_2'}{\epsilon_3^2\epsilon_3'}}, \sqrt[3]{\frac{\epsilon_2^2\epsilon_2'\epsilon_4'}{\epsilon_4}}\right\}$	{6, 12}	{1, 1}
{3, 10, 30, 90}	{1, 1, 3, 3}	IV	I	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_3}, \sqrt[3]{\epsilon_4}\}$	$\{\sqrt[3]{u_3u_4}\}$	{3, 9}	{1, 2}
{3, 11, 33, 99}	{1, 2, 1, 1}	IV	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_2}, \sqrt[3]{\epsilon_3}, \sqrt[3]{\epsilon_4}\}$	$\{\sqrt[3]{u_2u_4}, \sqrt[3]{u_2u_3}\}$	{2, 4}	{0, 0}
{3, 13, 39, 117}	{1, 3, 6, 3}	IV	III	III	III	$\{\epsilon_1, \epsilon_2, \epsilon_3, \sqrt[3]{\epsilon_1\epsilon_2\epsilon_3^2\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{\epsilon_2^2\epsilon_2'\epsilon_3'}{\epsilon_3}}, \sqrt[3]{\frac{\epsilon_2^2\epsilon_2'}{\epsilon_4^2\epsilon_4'}}\right\}$	{6, 12}	{1, 1}
{3, 14, 42, 126}	{1, 3, 3, 9}	IV	I	I	III	$\{\epsilon_1, \epsilon_2, \epsilon_4, \sqrt[3]{\epsilon_2\epsilon_3}\}$	{ }	{9, 27}	{2, 3}
{3, 17, 51, 153}	{1, 1, 3, 9}	IV	IV	I	IV	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_3}\,\sqrt[3]{\epsilon_1\epsilon_2\epsilon_4}\}$	{ }	{9, 27}	{1, 2}
{3, 19, 57, 171}	{1, 3, 6, 6}	IV	III	IV	IV	$\{\epsilon_1, \epsilon_2, \epsilon_3, \sqrt[3]{\epsilon_1^2\epsilon_3\epsilon_4}\}$	{ }	{12, 48}	{1, 1}
{3, 20, 60, 150}	{1, 3, 3, 3}	IV	I	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_2^2\epsilon_3}, \sqrt[3]{\epsilon_2^2\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_2u_3\epsilon_4}{u_4}}\right\}$	{9, 81}	{2, 4}
{3, 22, 66, 198}	{1, 3, 6, 6}	IV	I	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_2\epsilon_3}, \sqrt[3]{\epsilon_2\epsilon_4}\}$	$\left\{\sqrt[3]{u_2^2u_3u_4}\right\}$	{36, 1296}	{2, 4}
{3, 23, 69, 207}	{1, 1, 1, 8}	IV	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_2}, \sqrt[3]{\epsilon_3}, \sqrt[3]{\epsilon_4}\}$	$\left\{\sqrt[3]{u_2^2u_3}, \sqrt[3]{u_2u_4}\right\}$	{8, 64}	{0, 0}
{3, 26, 78, 234}	{1, 3, 3, 6}	IV	III	I	I	$\{\epsilon_1, \epsilon_2, \epsilon_3, \sqrt[3]{\epsilon_3^2\epsilon_4}\}$	{ }	{6, 12}	{1, 1}
{3, 28, 84, 252}	{1, 3, 3, 6}	IV	III	I	I	$\{\epsilon_1, \epsilon_2, \epsilon_3, \sqrt[3]{\epsilon_3^2\epsilon_4}\}$	{ }	{6, 12}	{1, 1}
{3, 29, 87, 261}	{1, 1, 1, 1}	IV	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_2}, \sqrt[3]{\epsilon_3}, \sqrt[3]{\epsilon_4}\}$	$\left\{\sqrt[3]{u_2^2u_3}, \sqrt[3]{u_2^2u_4}\right\}$	{1, 1}	{0, 0}
{3, 31, 93, 279}	{1, 3, 3, 3}	IV	III	III	III	$\{\epsilon_1, \epsilon_2, \epsilon_3, \sqrt[3]{\epsilon_1\epsilon_2\epsilon_3\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{\epsilon_2^2\epsilon_2'}{\epsilon_3^2\epsilon_3'}}\,\sqrt[3]{\frac{\epsilon_2^2\epsilon_2'}{\epsilon_4^2\epsilon_4'}}\right\}$	{3, 3}	{1, 1}
{3, 34, 102, 306}	{1, 3, 3, 3}	IV	I	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_2^2\epsilon_3}, \sqrt[3]{\epsilon_2^2\epsilon_4}\}$	$\left\{\sqrt[3]{u_2u_3u_4}\right\}$	{9, 81}	{2, 4}
{3, 35, 105, 315}	{1, 3, 6, 3}	IV	III	I	I	$\{\epsilon_1, \epsilon_2, \epsilon_3, \sqrt[3]{\epsilon_3^2\epsilon_4}\}$	{ }	{6, 12}	{1, 1}
{3, 37, 111, 333}	{1, 3, 3, 3},	IV	III	IV	IV	$\{\epsilon_1, \epsilon_2, \epsilon_3, \sqrt[3]{\epsilon_1\epsilon_3\epsilon_4}\}$	{ }	{3, 3}	{1, 1}
{3, 38, 114, 342}	{1, 3, 3, 27}	IV	I	I	III	$\{\epsilon_1, \epsilon_2, \epsilon_4, \sqrt[3]{\epsilon_2\epsilon_3}\}$	{ }	{27, 243}	{2, 3}

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Table B.1: *continued*

$\{m_1, m_2, m_3, m_4\}$	$\{h_1, h_2, h_3, h_4\}$	Type				$\hat{e}$	$\hat{E}$	$\{h, H\}$	ranks
{3, 41, 123, 369}	{1, 1, 2, 4}	IV	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_2}, \sqrt[3]{\epsilon_3}, \sqrt[3]{\epsilon_4}\}$	$\left\{\sqrt[3]{u_2u_3}, \sqrt[3]{u_2^2u_4}\right\}$	{8, 64}	{0, 0}
{3, 43, 129, 387}	{1, 12, 6, 3}	IV	III	III	III	$\{\epsilon_1, \epsilon_2, \epsilon_3, \sqrt[3]{\epsilon_1\epsilon_2\epsilon_3^2\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{\epsilon_2^2\epsilon_2'\epsilon_3'}{\epsilon_3}} \sqrt[3]{\frac{\epsilon_2^2\epsilon_2'}{\epsilon_4^2\epsilon_4'}}\right\}$	{24, 192}	{1, 1}
{3, 44, 132, 396}	{1, 1, 3, 3}	IV	I	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_3}, \sqrt[3]{\epsilon_4}\}$	$\left\{\sqrt[3]{u_3u_4}\right\}$	{3, 9}	{1, 2}
{3, 46, 138, 414}	{1, 1, 3, 6}	IV	I	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_3}, \sqrt[3]{\epsilon_4}\}$	$\left\{\sqrt[3]{u_3^2u_4}\right\}$	{6, 36}	{1, 2}
{3, 47, 141, 423}	{1, 2, 8, 7}	IV	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_2}, \sqrt[3]{\epsilon_3}, \sqrt[3]{\epsilon_4}\}$	$\left\{\sqrt[3]{u_2^2u_3}, \sqrt[3]{u_2u_4}\right\}$	{112, 12544}	{0, 0}
{3, 52, 156, 468}	{1, 3, 3, 9}	IV	I1	I	III	$\{\epsilon_1, \epsilon_2, \epsilon_4, \sqrt[3]{\epsilon_2\epsilon_3}\}$	{ }	{9, 27}	{2, 3}
{3, 53, 159, 477}	{1, 1, 3, 9}	IV	IV	I	IV	$\{\epsilon_1, \epsilon_4, \sqrt[3]{\epsilon_3}, \sqrt[3]{\epsilon_1\epsilon_2\epsilon_4}\}$	{ }	{9, 27}	{1, 2}
{3, 55, 165, 495}	{1, 1, 9, 9}	IV	I	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_2\epsilon_3}, \sqrt[3]{\epsilon_2^2\epsilon_4}\}$	{ }	{27, 243}	{2, 4}
{5, 6, 30, 150}	{1, 1, 3, 3}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1\epsilon_2}, \sqrt[3]{\epsilon_1^2\epsilon_3}, \sqrt[3]{\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1^2u_3}{u_2^2\epsilon_2^2}}\right\}$	{9, 27}	{2, 3}
{5, 7, 35, 175}	{1, 3, 3, 3}	I	III	III	I	$\{\epsilon_1, \epsilon_2, \epsilon_3, \sqrt[3]{\epsilon_1\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{\epsilon_2^2\epsilon_2'}{\epsilon_3^2\epsilon_3'}}\right\}$	{3, 3}	{1, 1}
{5, 11, 55, 275}	{1, 2, 1, 3}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1^2\epsilon_2}, \sqrt[3]{\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1^2\epsilon_4}{u_2^2\epsilon_2^2u_4}}\right\}$	{6, 12}	{1, 1}
{5, 12, 60, 90}	{1, 1, 3, 3}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1\epsilon_2}, \sqrt[3]{\epsilon_1\epsilon_3}, \sqrt[3]{\epsilon_1\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1u_4}{u_2^2\epsilon_2^2}}\right\}$	{9, 27}	{2, 3}
{5, 13, 65, 325}	{1, 3, 18, 3}	I	III	III	I	$\{\epsilon_1, \epsilon_2, \epsilon_3, \sqrt[3]{\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{\epsilon_2^2\epsilon_2'}{\epsilon_3^2\epsilon_3'}}\right\}$	{18, 108}	{2, 3}
{5, 14, 70, 350}	{1, 3, 9, 3}	I	I	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_2^2\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_2\epsilon_4}\}$	{ }	{27, 243}	$\{3, 5^{(1)}\}$
{5, 17, 85, 425}	{1, 1, 3, 6}	I	IV	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_3}, \sqrt[3]{\epsilon_4}\}$	$\left\{\sqrt[3]{u_3u_4}\right\}$	{6, 36}	{1, 2}
{5, 19, 95, 475}	{1, 3, 3, 3}	I	III	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_1^2\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1u_3\epsilon_4}{u_4}}\right\}$	{9, 27}	{1, 2}
{5, 28, 140, 490}	{1, 3, 9, 9}	I	III	I	I	$\{\epsilon_1, \epsilon_2, \epsilon_4, \sqrt[3]{\epsilon_1^2\epsilon_3}\}$	{ }	{27, 243}	$\{3, 5^{(1)}\}$
{6, 7, 42, 252}	{1, 3, 3, 6}	I	III	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_3}, \sqrt[3]{\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_3\epsilon_4}{\epsilon_2'\epsilon_2^2u_4}}\right\}$	{18, 108}	{2, 3}
{6, 10, 60, 45}	{1, 1, 3, 1}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1\epsilon_2}, \sqrt[3]{\epsilon_1\epsilon_3}, \sqrt[3]{\epsilon_1\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1u_3}{u_2\epsilon_2}}\right\}$	{3, 3}	{1, 1}

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Table B.1: *continued*

$\{m_1, m_2, m_3, m_4\}$	$\{h_1, h_2, h_3, h_4\}$	Type				$\hat{e}$	$\hat{E}$	$\{h, H\}$	ranks
{6, 11, 66, 396}	{1, 2, 6, 3}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1\epsilon_2}, \sqrt[3]{\epsilon_1^2\epsilon_3}, \sqrt[3]{\epsilon_1\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1\epsilon_4}{u_2^2\epsilon_2^2u_4}}\right\}$	{36, 432}	{2, 3}
{6, 13, 78, 468}	{1, 3, 3, 9}	I	III	I	III	$\{\epsilon_1, \epsilon_2, \epsilon_4, \sqrt[3]{\epsilon_1\epsilon_3}\}$	$\left\{\sqrt[3]{\frac{\epsilon_2^2\epsilon_2'}{\epsilon_4^2\epsilon_4'}}\right\}$	{9, 27}	$\{2, 3^{(1)}\}$
{6, 14, 84, 63}	{1, 3, 3, 6}	I	I	I	III	$\{\epsilon_1, \epsilon_4, \sqrt[3]{\epsilon_2}, \sqrt[3]{\epsilon_3}\}$	$\left\{\sqrt[3]{\frac{u_2^2\epsilon_3}{u_3}}\right\}$	{18, 36}	{2, 3}
{6, 15, 90, 20}	{1, 2, 3, 3}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1^2\epsilon_2}, \sqrt[3]{\epsilon_1^2\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1^2u_3}{u_2\epsilon_2}}\right\}$	{18, 108}	{2, 3}
{6, 21, 126, 28}	{1, 3, 9, 3}	I	III	III	III	$\{\epsilon_1, \epsilon_2, \epsilon_3, \sqrt[3]{\epsilon_2\epsilon_3\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{\epsilon_2^2\epsilon_2'}{\epsilon_3^2\epsilon_3'}}, \sqrt[3]{\frac{\epsilon_2^2\epsilon_2'}{\epsilon_4^2\epsilon_4'}}\right\}$	{9, 27}	{2, 3}
{6, 22, 132, 99}	{1, 3, 3, 1}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_2}, \sqrt[3]{\epsilon_1^2\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1u_3\epsilon_4}{u_4}}\right\}$	{9, 27}	{2, 3}
{6, 26, 156, 117}	{1, 3, 3, 3}	I	III	I	III	$\{\epsilon_1, \epsilon_2, \epsilon_4, \sqrt[3]{\epsilon_1\epsilon_3}\}$	$\left\{\sqrt[3]{\frac{\epsilon_2^2\epsilon_2'\epsilon_4'}{\epsilon_4}}\right\}$	{3, 3}	{1, 1}
{6, 33, 198, 44}	{1, 1, 6, 1}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1^2\epsilon_2}, \sqrt[3]{\epsilon_1\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_3\epsilon_4}{u_2\epsilon_2u_4}}\right\}$	{6, 12}	{1, 1}
{6, 34, 204, 153}	{1, 3, 12, 9}	I	I	I	IV	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_1\epsilon_2}, \sqrt[3]{\epsilon_1^2\epsilon_3}\}$	{ }	{108, 3888}	{2, 3}
{6, 38, 228, 171}	{1, 3, 9, 6}	I	I	I	IV	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_1\epsilon_2}, \sqrt[3]{\epsilon_1^2\epsilon_3}\}$	{ }	{54, 972}	$\{2, 3^{(1)}\}$
{6, 39, 234, 52}	{1, 6, 6, 3}	I	III	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_1\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1u_3^2\epsilon_4}{\epsilon_2'\epsilon_2^2u_4}}\right\}$	{36, 432}	$\{2, 3^{(1)}\}$
{6, 46, 276, 207}	{1, 1, 3, 8}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1\epsilon_2}, \sqrt[3]{\epsilon_1\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_3\epsilon_4}{u_2^2\epsilon_2^2u_4}}\right\}$	{24, 192}	{1, 1}
{6, 51, 306, 68}	{1, 3, 3, 3}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1^2\epsilon_2}, \sqrt[3]{\epsilon_1^2\epsilon_3}, \sqrt[3]{\epsilon_1\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_3\epsilon_4}{u_2^2\epsilon_2^2u_4}}\right\}$	{27, 243}	$\{2, 3^{(1)}\}$
{6, 57, 342, 76}	{1, 6, 27, 6}	I	IV	III	I	$\{\epsilon_1, \epsilon_2, \epsilon_3, \sqrt[3]{\epsilon_1^2\epsilon_4}\}$	{ }	{108, 3888}	$\{2, 3^{(1)}\}$
{6, 58, 348, 261}	{1, 6, 3, 1}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_2}, \sqrt[3]{\epsilon_1^2\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1u_3\epsilon_4}{u_4}}\right\}$	{18, 108}	{2, 3}
{6, 62, 372, 279}	{1, 3, 3, 3}	I	I	I	III	$\{\epsilon_1, \epsilon_4, \sqrt[3]{\epsilon_1^2\epsilon_1}, \sqrt[3]{\epsilon_1\epsilon_3}\}$	$\left\{\sqrt[3]{\frac{u_1^2u_2^2\epsilon_3}{u_3}}\right\}$	{9, 27}	{1, 2}
{6, 69, 414, 92}	{1, 1, 6, 3}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1\epsilon_2}, \sqrt[3]{\epsilon_1^2\epsilon_3}, \sqrt[3]{\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1^2u_3}{u_2^2\epsilon_2^2}}\right\}$	{18, 108}	{2, 3}
{6, 70, 420, 315}	{1, 9, 9, 3}	I	I	I	I	$\{\epsilon_1, \epsilon_3, \sqrt[3]{\epsilon_2}, \sqrt[3]{\epsilon_3^2\epsilon_4}\}$	{ }	{81, 2187}	{4, 7}

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Table B.1: *continued*

$\{m_1, m_2, m_3, m_4\}$	$\{h_1, h_2, h_3, h_4\}$	Type				$\hat{e}$	$\hat{E}$	$\{h, H\}$	ranks
{6, 74, 444, 333}	{1, 3, 3, 3}	I	I	I	IV	$\{\epsilon_1, \epsilon_4, \sqrt[3]{\epsilon_2}, \sqrt[3]{\epsilon_3}\}$	$\{\sqrt[3]{u_2 u_3}\}$	{9, 81}	{2,4}
{6, 82, 492, 369}	{1, 1, 6, 4}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1\epsilon_2}, \sqrt[3]{\epsilon_1^2\epsilon_3}, \sqrt[3]{\epsilon_1\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1^2 u_3}{u_2^2\epsilon_2^2}}\right\}$	{24, 192}	{1, 1}
{7, 10, 70, 490}	{3, 1, 9, 9}	III	I	I	I	$\{\epsilon_1, \epsilon_2, \epsilon_3, \sqrt[3]{\epsilon_2^2\epsilon_3^2\epsilon_4}\}$	{ }	{27, 243}	$\{3, 5^{(1)}\}$
{7, 12, 84, 126}	{3, 1, 3, 9}	III	I	I	III	$\{\epsilon_1, \epsilon_2, \epsilon_4, \sqrt[3]{\epsilon_2\epsilon_3}\}$	$\left\{\sqrt[3]{\frac{\epsilon_1^2\epsilon_1'\epsilon_4'}{\epsilon_4}}\right\}$	{9, 27}	$\{2, 3^{(1)}\}$
{7, 20, 140, 350}	{3, 3, 9, 3}	III	I	I	I	$\{\epsilon_1, \epsilon_2, \epsilon_3, \sqrt[3]{\epsilon_4}\}$	{ }	{27, 243}	{ }
{10, 12, 15, 150}	{1, 1, 2, 3}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1^2\epsilon_2}, \sqrt[3]{\epsilon_1^2\epsilon_3}, \sqrt[3]{\epsilon_1\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1^2 u_3^2\epsilon_4}{u_4}}\right\}$	{6, 12}	{1, 1}
{10, 14, 140, 175}	{1, 3, 9, 3}	I	I	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_1\epsilon_2^2\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_2\epsilon_4}\}$	{ }	{27, 243}	$\{3, 5^{(1)}\}$
{10, 22, 220, 275}	{1, 3, 9, 3}	I	I	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_1^2\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_2\epsilon_4}\}$	{ }	{27, 243}	{3, 4}
{10, 26, 260, 325}	{1, 3, 3, 3}	I	III	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_1\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1^2 u_4}{u_3^2\epsilon_3^2}}\right\}$	{9, 27}	{1, 2}
{10, 28, 35, 350}	{1, 3, 3, 3}	I	III	III	I	$\{\epsilon_1, \epsilon_2, \epsilon_3, \sqrt[3]{\epsilon_1\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{\epsilon_2^2\epsilon_2'}{\epsilon_3^2\epsilon_3'}}\right\}$	{3, 3}	{1, 2}
{10, 34, 340, 425}	{1, 3, 9, 6}	I	I	I	I	$\{\epsilon_1, \epsilon_3, \sqrt[3]{\epsilon_1^2\epsilon_2}, \sqrt[3]{\epsilon_1\epsilon_4}\}$	{ }	{54, 972}	{3, 4}
{10, 38, 380, 475}	{1, 3, 9, 3}	I	I	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_1^2\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_2\epsilon_4}\}$	{ }	{27, 243}	$\{2, 4^{(1)}\}$
{11, 12, 132, 198}	{2, 1, 3, 6}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1\epsilon_2}, \sqrt[3]{\epsilon_1\epsilon_3}, \sqrt[3]{\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1 u_3}{u_2\epsilon_2}}\right\}$	{36, 432}	{2, 3}
{12, 13, 156, 234}	{1, 3, 3, 6}	I	III	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_3}, \sqrt[3]{\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_3\epsilon_4}{\epsilon_2^2\epsilon_2' u_4}}\right\}$	{18, 108}	{2, 3}
{12, 14, 21, 252}	{1, 3, 3, 6}	I	I	III	I	$\{\epsilon_1, \epsilon_3, \sqrt[3]{\epsilon_1^2\epsilon_2}, \sqrt[3]{\epsilon_1\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1^2\epsilon_2^2 u_4}{u_2^2}}\right\}$	{18, 36}	$\{2, 3^{(1)}\}$
{12, 17, 204, 306}	{1, 1, 12, 3}	I	IV	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_3}, \sqrt[3]{\epsilon_4}\}$	$\{\sqrt[3]{u_3 u_4}\}$	{12, 144}	{1, 2}
{12, 19, 228, 342}	{1, 3, 9, 27}	I	III	I	III	$\{\epsilon_1, \epsilon_2, \epsilon_4, \sqrt[3]{\epsilon_3}\}$	$\left\{\sqrt[3]{\frac{\epsilon_2^2\epsilon_2'}{\epsilon_4^2\epsilon_4'}}\right\}$	{81, 2187}	{2, 5}
{12, 20, 30, 45}	{1, 3, 3, 1}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_2}, \sqrt[3]{\epsilon_1\epsilon_3}, \sqrt[3]{\epsilon_1\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1^2 u_3\epsilon_4}{u_4}}\right\}$	{9, 27}	{2, 3}
{12, 22, 33, 396}	{1, 3, 1, 3}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1\epsilon_2}, \sqrt[3]{\epsilon_1^2\epsilon_3}, \sqrt[3]{\epsilon_1\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1 u_4}{u_3^2\epsilon_3^2}}\right\}$	{9, 27}	{2, 3}

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Table B.1: *continued*

$\{m_1, m_2, m_3, m_4\}$	$\{h_1, h_2, h_3, h_4\}$	Type				$\hat{e}$	$\hat{E}$	$\{h, H\}$	ranks
{12, 23, 276, 414}	{1, 1, 3, 6}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1\epsilon_2}, \sqrt[3]{\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1^2\epsilon_4}{u_2u_4\epsilon_2}}\right\}$	{18, 108}	{2, 3}
{12, 26, 39, 468}	{1, 3, 6, 9}	I	III	III	III	$\{\epsilon_1, \epsilon_2, \epsilon_4, \sqrt[3]{\epsilon_2\epsilon_3\epsilon_4^2}\}$	$\left\{\sqrt[3]{\frac{\epsilon_2^2\epsilon_2'}{\epsilon_3^2\epsilon_3'}}, \sqrt[3]{\frac{\epsilon_2^2\epsilon_2'\epsilon_4'}{\epsilon_4}}\right\}$	{18, 108}	{2, 3}
{12, 28, 42, 63}	{1, 3, 3, 6}	I	III	I	III	$\{\epsilon_1, \epsilon_2, \epsilon_4, \sqrt[3]{\epsilon_1\epsilon_3}\}$	$\left\{\sqrt[3]{\frac{\epsilon_2^2\epsilon_2'\epsilon_4'}{\epsilon_4}}\right\}$	{6, 12}	{1, 2}
{12, 44, 66, 99}	{1, 1, 6, 1}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1\epsilon_2}, \sqrt[3]{\epsilon_1\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1u_3}{u_2\epsilon_2}}\right\}$	{6, 12}	{1, 1}
{12, 52, 78, 117}	{1, 3, 3, 3}	I	I	I	III	$\{\epsilon_1, \epsilon_4, \sqrt[3]{\epsilon_2}, \sqrt[3]{\epsilon_3}\}$	$\left\{\sqrt[3]{\frac{u_2^2\epsilon_3}{u_3}}\right\}$	{9, 27}	{2, 3}
{12, 68, 102, 153}	{1, 3, 3, 9}	I	I	I	IV	$\{\epsilon_1, \epsilon_4, \sqrt[3]{\epsilon_1^2\epsilon_2}, \sqrt[3]{\epsilon_1\epsilon_3}\}$	{ }	{27, 243}	{2, 3}
{12, 76, 114, 171}	{1, 6, 3, 6}	I	I	I	IV	$\{\epsilon_1, \epsilon_4, \sqrt[3]{\epsilon_2}, \sqrt[3]{\epsilon_3}\}$	$\left\{\sqrt[3]{\frac{u_2^2u_3}{\epsilon_2^2}}\right\}$	{36, 1296}	{2, 4}
{12, 92, 138, 207}	{1, 3, 3, 8}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1^2\epsilon_2}, \sqrt[3]{\epsilon_1^2\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1u_3\epsilon_4}{u_4}}\right\}$	{72, 1728}	{2, 3}
{12, 116, 174, 261}	{1, 1, 12, 1}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1\epsilon_2}, \sqrt[3]{\epsilon_1^2\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1^2u_3}{u_2^2\epsilon_2^2}}\right\}$	{12, 48}	{1, 1}
{12, 124, 186, 279}	{1, 9, 6, 3}	I	III	I	III	$\{\epsilon_1, \epsilon_2, \epsilon_4, \sqrt[3]{\epsilon_1^2\epsilon_3}\}$	$\left\{\sqrt[3]{\frac{\epsilon_2^2\epsilon_2'}{\epsilon_4^2\epsilon_4'}}\right\}$	{18, 108}	$\{2, 3^{(1)}\}$
{12, 140, 210, 315}	{1, 9, 18, 3}	I	I	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_1^2\epsilon_3}, \sqrt[3]{\epsilon_2^2\epsilon_4}\}$	{ }	{162, 8748}	$\{4, 6^{(1)}\}$
{12, 148, 222, 333}	{1, 6, 18, 3}	I	I	I	IV	$\{\epsilon_1, \epsilon_4, \sqrt[3]{\epsilon_1^2\epsilon_2}, \sqrt[3]{\epsilon_1^2\epsilon_3}\}$	{ }	{108, 3888}	$\{2, 3^{(1)}\}$
{12, 164, 246, 369}	{1, 6, 3, 4}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_2}, \sqrt[3]{\epsilon_1\epsilon_3}, \sqrt[3]{\epsilon_1\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1^2u_3\epsilon_4}{u_4}}\right\}$	{72, 1728}	{2, 3}
{12, 172, 258, 387}	{1, 3, 3, 3}	I	I	I	III	$\{\epsilon_1, \epsilon_4, \sqrt[3]{\epsilon_1\epsilon_2}, \sqrt[3]{\epsilon_1\epsilon_3}\}$	$\left\{\sqrt[3]{\frac{u_1^2u_2\epsilon_3}{u_3}}\right\}$	{9, 27}	{1, 2}
{12, 188, 282, 423}	{1, 1, 3, 7}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1\epsilon_2}, \sqrt[3]{\epsilon_1^2\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1u_3\epsilon_4}{u_4}}\right\}$	{21, 147}	{1, 1}
{12, 212, 318, 477}	{1, 6, 3, 9}	I	I	I	IV	$\{\epsilon_1, \epsilon_4, \sqrt[3]{\epsilon_1^2\epsilon_2}, \sqrt[3]{\epsilon_1^2\epsilon_3}\}$	{ }	{54, 972}	{2, 3}
{12, 220, 330, 495}	{1, 9, 9, 9}	I	I	I	I	$\{\epsilon_1, \epsilon_4, \sqrt[3]{\epsilon_1^2\epsilon_2}, \sqrt[3]{\epsilon_1^2\epsilon_3}\}$	{ }	{243, 19683}	{4, 6}
{14, 20, 35, 490}	{3, 3, 3, 9}	I	I	III	I	$\{\epsilon_1, \epsilon_2, \epsilon_3, \sqrt[3]{\epsilon_1^2\epsilon_4}\}$	{ }	{243, 19683}	{ }
{15, 21, 315, 175}	{2, 3, 3, 3}	I	III	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_3}, \sqrt[3]{\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_3^2\epsilon_4}{\epsilon_2^2\epsilon_2'u_4}}\right\}$	{18, 108}	{2, 3}

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Table B.1: *continued*

$\{m_1, m_2, m_3, m_4\}$	$\{h_1, h_2, h_3, h_4\}$	Type				$\hat{e}$	$\hat{E}$	$\{h, H\}$	ranks
{15, 33, 495, 275}	{2, 1, 9, 3}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1^2\epsilon_2}, \sqrt[3]{\epsilon_3}, \sqrt[3]{\epsilon_1\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1\epsilon_4}{u_2\epsilon_2 u_4}}\right\}$	{54, 972}	{2, 3}
{20, 28, 70, 175}	{3, 3, 9, 3}	I	III	I	I	$\{\epsilon_1, \epsilon_2, \epsilon_3\sqrt[3]{\epsilon_3\epsilon_4}\}$	{ }	{36, 1296}	{ }
{20, 44, 110, 275}	{3, 1, 9, 3}	I	I	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_1^2\epsilon_2\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_2^2\epsilon_4}\}$	{ }	{27, 243}	{3, 5}
{20, 52, 130, 325}	{3, 3, 9, 3}	I	I	I	I	$\{\epsilon_1, \epsilon_3, \sqrt[3]{\epsilon_1^2\epsilon_2}, \sqrt[3]{\epsilon_3^2\epsilon_4}\}$	{ }	{27, 243}	{ }
{20, 68, 170, 425}	{3, 3, 3, 6}	I	I	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_2\epsilon_3}, \sqrt[3]{\epsilon_2^2\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_2^2 u_4}{\epsilon_2^2\epsilon_4}}\right\}$	{27, 243}	{ }
{20, 76, 190, 475}	{3, 6, 3, 3}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1^2\epsilon_2}, \sqrt[3]{\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_4}\}$	{ }	{27, 243}	{ }
{30, 84, 315, 350}	{3, 3, 3, 3}	I	I	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_1^2\epsilon_2^2\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_2\epsilon_4}\}$	{ }	{27, 243}	{ }
{42, 60, 315, 490}	{3, 3, 3, 9}	I	I	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_2\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_2^2\epsilon_4}\}$	{ }	{27, 243}	{ }
{45, 63, 105, 175}	{1, 6, 6, 3}	I	III	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_1\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1 u_3^2\epsilon_4}{u_2^2\epsilon_2 u_4}}\right\}$	{36, 432}	$\{2, 3^{(1)}\}$
{45, 99, 165, 275}	{1, 1, 9, 3}	I	I	I	I	$\{\epsilon_1, \sqrt[3]{\epsilon_1\epsilon_2}, \sqrt[3]{\epsilon_1^2\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1^2\epsilon_4}{u_2\epsilon_2 u_4}}\right\}$	{27, 243}	{2, 3}
{45, 117, 195, 325}	{1, 3, 6, 3}	I	III	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_1^2\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1 u_3\epsilon_4}{\epsilon_2^2\epsilon_2' u_4}}\right\}$	{18, 108}	{1, 1}
{45, 126, 210, 350}	{1, 9, 18, 3}	I	III	I	I	$\{\epsilon_1, \epsilon_2, \epsilon_3, \sqrt[3]{\epsilon_1\epsilon_3\epsilon_4}\}$	{ }	{54, 972}	$\{3, 5^{(1)}\}$
{45, 153, 255, 425}	{1, 9, 3, 6}	I	IV	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_1\epsilon_3}, \sqrt[3]{\epsilon_1\epsilon_4}\}$	{ }	{54, 972}	{2, 3}
{45, 171, 285, 475}	{1, 6, 15, 3}	I	IV	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_3}, \sqrt[3]{\epsilon_4}\}$	$\{\sqrt[3]{u_3 u_4}\}$	{90, 8100}	{2, 4}
{63, 90, 210, 490}	{6, 3, 18, 9}	III	I	I	I	$\{\epsilon_1, \epsilon_2, \epsilon_3, \sqrt[3]{\epsilon_4}\}$	{ }	{90, 8100}	{ }
{63, 150, 350, 420}	{6, 3, 3, 9}	III	I	I	I	$\{\epsilon_1, \epsilon_2, \epsilon_3, \sqrt[3]{\epsilon_2\epsilon_3^2\epsilon_4}\}$	{ }	{90, 8100}	{ }
{90, 105, 350, 252}	{3, 6, 3, 6}	I	I	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_2\epsilon_3}, \sqrt[3]{\epsilon_2^2\epsilon_4}\}$	{ }	{90, 8100}	{ }
{90, 126, 420, 175}	{3, 9, 9, 3}	I	III	I	I	$\{\epsilon_1, \epsilon_2, \epsilon_3, \sqrt[3]{\epsilon_3\epsilon_4}\}$	{ }	{90, 8100}	{ }
{105, 126, 490, 150}	{6, 9, 9, 3}	I	III	I	I	$\{\epsilon_1, \epsilon_2, \epsilon_3, \sqrt[3]{\epsilon_1^2\epsilon_3^2\epsilon_4}\}$	{ }	{90, 8100}	{ }
{150, 175, 210, 252}	{3, 3, 18, 6}	I	I	I	I	$\{\epsilon_1, \epsilon_2, \sqrt[3]{\epsilon_1\epsilon_3}, \sqrt[3]{\epsilon_1^2\epsilon_4}\}$	$\left\{\sqrt[3]{\frac{u_1 u_3^2\epsilon_4}{u_4}}\right\}$	{90, 8100}	{ }

Vita

Alberto Pablo Chalmeta was born in Lausanne, Switzerland in 1970. He graduated from Virgina Tech with a Bachelors degree in Mechanical Engineering in 1994. He entered the Masters program at Virginia Tech in 1996 and after earning his degree he entered the Ph.D. progam. In 2002 he went to work full time at New River Community College as a mathematics teacher while completing his Ph.D..