# General Boundary Value Problems of a Class of Fifth Order KdV Equations on a Bounded Interval

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**Abstract**

This paper investigates the initial boundary value problem (IBVP) associated with the fifth-order Korteweg-de Vries (KdV) equation on a finite interval:

$$\begin{cases} \partial_t u + u_x + \beta \partial_x^3 u + \partial_x^5 u = c_0 u \partial_x u + c_1 \partial_x u \partial_x^2 u + c_2 \partial_x (u \partial_x^2 u) + c_3 \partial_x (u^3),\ 0 < x < L\,, t > 0, \\ u(x,0) = \phi(x), \end{cases} \tag{0.1}$$

subject to non-homogeneous boundary conditions:

$$B_j u = h_j(t), \quad j = 1,2,3,4,5, \quad t > 0, \tag{0.2}$$

where

$$B_j u = \sum_{k=0}^{4} (a_{jk} \partial_x^k u(0,t) + b_{jk} \partial_x^k u(L,t)), \quad j = 1,2,3,4,5,$$

and $a_{jk}, b_{jk}$ are real constants for $k = 0,1,2,3,4$ and $j = 1,2,3,4,5$. Under certain general assumptions on these coefficients, the well-posedness of the IBVP $(0.1)-(0.2)$ is established, showing local **analytical** well-posedness in the space $H^s(0,L)$ for $s \geq 1$ (and for some specified cases, $s \geq 0$), with initial data $\phi \in H^s(0,L)$ and boundary values $h_j$ belonging to appropriate spaces with optimal regularity.

In contrast, for the pure initial value problem (IVP) of the fifth-order KdV equation posed on $\mathbb{R}$, it is known to be **continuous** ($C^0$) well-posed in the space $H^s(\mathbb{R})$ for $s \geq 2$, but it fails to be analytically well-posed in $H^s(\mathbb{R})$ for any $s \in \mathbb{R}$.

The remarkable strong Kato smoothing and double sharp Kato smoothing properties, attributed to Kenig, Ponce and Vega [38,41], of the pure initial value problem (IVP) of the linear inhomogeneous fifth order KdV equation posed on the whole line $\mathbb{R}$,

$$ptv + \beta \partial_x^3 v + \partial_x^5 v = g(x,t), \qquad v(x,0) = 0, \qquad x,\ t \in \mathbb{R},$$

have played a pivotal role in establishing the well-posedness of the IBVP $(0.1)-(0.2)$.



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Kawahara equation, Fifth order KdV equation, Non-homogeneous Initial-boundary value problem, General boundary conditions, Local well-posedness, Sharp Kato smoothing

# 1 Introduction

## 1.1 Problems to study

This paper is dedicated to the exploration of the initial boundary value problem (IBVP) associated with a versatile class of fifth-order Korteweg-de Vries (KdV) equations, defined within the interval $(0, L)$. The IBVP is structured as follows:

$$\partial_t u+\partial_x u+\beta\partial_x^3 u+\partial_x^5 u = c_0 u\partial_x u+c_1\partial_x u\partial_x^2 u+c_2\partial_x(u\partial_x^2 u)+c_3\partial_x(u^3),\ \ u(x,0)=\phi(x),\ \ x\in(0,L),\ t>0, \tag{1.1}$$

subject to non-homogeneous boundary conditions:

$$B_k u = h_k(t), \quad t>0, \quad \text{for} \quad k=1,2,3,4,5, \tag{1.2}$$

where the boundary operators $\mathcal{B}=(B_1,B_2,B_3,B_4,B_5)$ is defined as:

$$B_k u = \sum_{j=0}^{4}\left(a_{kj}\partial_x^j u(0,t)+b_{kj}\partial_x^j u(L,t)\right),$$

The real constants $\alpha, \beta, a_{kj}$, and $b_{kj}$ are essential parameters in the equation. The order of the boundary operator $B_k$ is denoted by $\eta_k$, which has the property that $a_{kj}^2+b_{kj}^2=0$ for any $j>\eta_k$, and $a_{kj}^2+b_{kj}^2\neq 0$ for $j=\eta_k$.

This paper's primary objective is to investigate the well-posedness of the IBVP (1.1)-(1.2) within the classical Sobolev space $H^s(0,L)$.

We commence by introducing the notion of solutions for the IBVP (1.1)-(1.2).

**Definition 1.1** (Classical solution)**.** *A function $u(x,t)$ is called a classical solution of* (1.1) *if*

$$u\in C([0,T];H^5(0,L))\cap C^1([0,T];L^2(0,L))$$

*and* (1.1) *is satisfied for any $t\in[0,T]$ and $x\in[0,L]$ a.e.*

**Definition 1.2.** *Let $s\in\mathbb{R}$ and $T>0$ be given. For any*

$$(\phi,\vec{h})\in X_{s,T,\mathcal{B}} := H^s(0,L)\times\mathcal{H}^s_{\mathcal{B}}(0,T)$$

*with $\vec{h}=(h_1,h_2,h_3,h_4,h_5)$ and*

$$\mathcal{H}^s_{\mathcal{B}}(0,T)=H^{\frac{s+2-\eta_1}{5}}(0,T)\times H^{\frac{s+2-\eta_2}{5}}(0,T)\times H^{\frac{s+2-\eta_3}{5}}(0,T)\times H^{\frac{s+2-\eta_4}{5}}(0,T)\times H^{\frac{s+2-\eta_5}{5}}(0,T).$$

*A function*

$$u\in C([0,T];H^s(0,L))$$

*is said to be a solution of the IBVP* (1.1)-(1.2) *on the time interval $[0,T]$ if there exists a sequence of classical solutions $\{u_n\}_{n=1}^{\infty}$ of the equation* (1.1) *with*

$$\phi_n(x)=u_n(x,0),$$

$$\vec{h}_n=(h_{1,n},h_{2,n},h_{3,n},h_{4,n},h_{5,n})=(B_1u_n,B_2u_n,B_{3u_n},B_4u_n,B_5u_n); \quad n=1,2,\cdots$$

*such that*

(i) $\lim_{n\longrightarrow\infty}\|u_n - u\|_{C(0,T;H^s(0,L))} = 0$;

(ii) $\lim_{n\longrightarrow\infty}\|\phi_n - \phi\|_{H^s(0,L)} = 0$ *and* $\lim_{n\longrightarrow\infty}\|\vec{h}_n - \vec{h}\|_{\mathcal{H}^s_{\mathcal{B}}(0,T)} = 0$.

**Remark 1.3.** The solutions of the IBVP (1.1)-(1.2) defined above solve the equation (1.1) in the sense of distribution.

**Definition 1.4** (Well-posedness)**.** Let $s \in \mathbb{R}$ and $T > 0$ be given. The IBVP (1.1)-(1.2) is said to be locally well-posed in $H^s(0, L)$ if for any $r > 0$, there exists a $T^* = T^*(r) \in (0, T]$ such that for any

$$(\phi, \vec{h}) \in X_{s,T,\mathcal{B}}$$

satisfying

$$\|(\phi, \vec{h})\|_{X_{s,T,\mathcal{B}}} \leq r$$

the IBVP (1.1)-(1.2) admits a unique solution

$$u \in C\left([0, T^*]; H^s(0, L)\right).$$

Moreover, the corresponding solution map $(\phi, \vec{h}) \mapsto u$ is continuous.

**Remark 1.5.**

(i) *If $T^*(r)$ can be chosen independently of $r$, the IBVP* (1.1)-(1.2) *is said to be globally well-posed in $H^s(0, L)$.*

(ii) *If the corresponding solution map $(\phi, \vec{h}) \mapsto u$ is real analytic, the IBVP* (1.1)-(1.2) *is said to be analytically well-posed in the space $H^s(0, L)$. On the other hand, if the corresponding solution map $(\phi, \vec{h}) \mapsto u$ is continuous, the IBVP* (1.1)-(1.2) *is said to be $C^0$ well-posed in the space $H^s(0, L)$.*

The main questions that will be addressed are:

**Question 1**: *Under what assumptions on the coefficients $a_{ij}$, $b_{ij}$ in* (1.2) *is the IBVP* (1.1)-(1.2) *well-posed in the space $H^s(0, L)$?*

**Question 2**: *For what values of $s$ is the IBVP* (1.1)-(1.2) *well-posed in space $H^s(0, L)$?*

By providing answers to these questions, we aim to establish the conditions under which the given fifth-order KdV equations with non-homogeneous boundary conditions are well-posed in the function space $H^s(0, L)$. This investigation will contribute to a deeper understanding of the behavior of the KdV equations and their solutions under various boundary conditions, which has implications for the stability and existence of solutions in related physical systems.

## 1.2 Literature reviews

The fifth-order Korteweg-de Vries (KdV) equation,

$$\partial_t u + \partial_x u + \beta\partial_x^3 u - \alpha\partial_x^5 u = c_0 u\partial_x u + c_1\partial_x u\partial_x^2 u + c_2\partial_x(u\partial_x^2 u) + c_3\partial_x(u^3), \tag{1.3}$$

serves as a long-wave approximation to the water-wave equation and has been derived as a second-order asymptotic expansion for unidirectional wave propagation in what is known as the Boussinesq regime, as documented by Craig, Guyenne, and Kalisch [5], Olver [29], and other relevant references. The celebrated KdV equation,

$$\partial_t u + \partial_x u + \beta \partial_x^3 u = c_0 u \partial_x u, \tag{1.4}$$

represents the first-order expansion in this derivation process. In addition, included in the equation are the Kawahara equation:

$$\partial_t u + \partial_x u + u\partial_x u + \beta \partial_x^3 u + \partial_x^5 u = 0, \tag{1.5}$$

and the second equation from the KdV hierarchy:

$$\partial_t u - \partial_x^5 u + 10\partial_x u \partial_x^2 u + 10\partial_x(u\partial_x^2 u) - 10\partial_x^3(u^3) = 0, \tag{1.6}$$

which is completely integrable and possesses an infinite number of conservation laws.

Furthermore, in the special case where $c_1 = c_2$, equation (1.3) can be expressed as:

$$\partial_t u = \partial_x \mathrm{grad} H(u)$$

where

$$H(u) = \frac{1}{2}\int_{\mathbb{R}} \left(\alpha(\partial_x^2 u)^2 - c_1 u(\partial_x u)^2 + \frac{c_3}{2}u^4\right) dx.$$

As a consequence, both $H(u)$ and $M(u) = \int_{\mathbb{R}} u^2 dx$ are conserved by the flow of equation (1.3).

The well-posedness of the Cauchy problem for the fifth-order Korteweg-de Vries (KdV) equation (1.3) has been extensively investigated in the literature (cf [] and the references therein).

Regarding the KdV equation (1.4), the Cauchy problem has been a subject of study for over fifty years, engaging numerous mathematicians. It has been established to be analytically well-posed in the space $H^s(\mathbb{R})$ for any $s \geq -\frac{3}{4}$, and only $C_0$-well-posed in $H^s(\mathbb{R})$ for $-1 \leq s < -\frac{3}{4}$.

For the Kawahara equation (1.5), its Cauchy problem is known to be analytically well-posed in the space $H^s(\mathbb{R})$ for any $s \geq -2$, but it becomes ill-posed in $H^s(\mathbb{R})$ for any $s < -2$ (see [34, 35] and the references therein).

The Cauchy problem for the general fifth-order KdV equation (1.3) has been extensively investigated. Initially, Ponce [50] established its local well-posedness in the $C^0$ sense within the space $H^s(\mathbb{R})$, where $s$ is bounded below by 4. Kwon [47] later expanded the result to encompass any $s$ exceeding $\frac{5}{2}$. Subsequently, in separate endeavors, Kenig and Pilod [43], as well as Guo, Kwak, and Kwon [26], further extended the proof of $C^0$ well-posedness for the range $s \geq 2$.

**Remark 1.6.**

*(i) A natural inquiry arises regarding the analytical well-posedness of the Cauchy problem for equation* (1.3) *in the function space $H^s(\mathbb{R})$ for certain values of $s$. Remarkably, the answer is in the negative. Pilod [49] demonstrated that, if the solution map indeed exists, it cannot exhibit $C^2$ continuity in $H^s(\mathbb{R})$ for any $s \in \mathbb{R}$. Furthermore, Kwon [47] established that the solution map cannot even be uniformly continuous in $H^s(\mathbb{R})$ for any $s > \frac{5}{2}$.*

*(ii) This adverse outcome holds significant implications. Firstly, establishing the well-posedness of the Cauchy problem for equation* (1.3) *within the function space $H^s(\mathbb{R})$ cannot be accomplished through the application of the contraction mapping method. Secondly, it underscores that the nonlinear terms within equation* (1.3) *cannot be treated as mere small perturbations of the corresponding linear equation when addressing the well-posedness of the Cauchy problem in the space $H^s(\mathbb{R})$.*

As for the IBVPs of the KdV equation posed on a finite interval $(0, L)$, its study began with the work of Bubnov [9] in 1979, which studied the following IBVP of the KdV equation posed on $(0, L)$ with homogeneous boundary conditions,

$$\begin{cases} \partial_t u + u\partial_x u + \partial_x^3 u = f, \;\; u(x,0) = 0, \; x \in (0,L), \;\; t \in (0,T) \\ \alpha_1 \partial_x^2 u(0,t) + \alpha_2 \partial_x u(0,t) + \alpha_3 u(0,t) = 0, \\ \beta_1 \partial_x^2 u(L,t) + \beta_2 \partial_x u(L,t) + \beta_3 u(L,t) = 0, \\ \chi_1 \partial_x u(L,t) + \chi_2 u(L,t) = 0. \end{cases} \tag{1.7}$$

The following result was presented in [9].

**Theorem A** *Denoted by*

$$F_1 = \frac{\alpha_3}{\alpha_1} - \frac{\alpha_2^2}{2\alpha_1^2}, \; F_2 = \frac{\beta_2\chi_2}{\beta_1\chi_1} - \frac{\beta_3}{\beta_1} - \frac{\chi_2^2}{2\chi_1^2}$$

*and*

$$F_3 = \beta_2\chi_2 - \beta_3\chi_1.$$

*Assume that*

$$\begin{cases} F_1 > 0, \; F_2 > 0 & if \;\; \alpha_1\beta_1\chi_1 \neq 0, \\ \alpha_2 = 0, F_2 > 0, \alpha_3 \neq 0 & if \; \beta_1 \neq 0, \chi_1 \neq 0, \alpha_1 = 0, \\ F_1 > 0, F_3 \neq 0 & if \; \beta_1 = 0, \chi_1 \neq 0, \alpha_1 \neq 0, \\ \alpha_2 = 0, F_3 \neq 0, \alpha_3 \neq 0 & if \; \alpha_1 = \beta_1, \; \chi_1 \neq 0, \\ F_1 > 0, F_3 \neq 0 & if \;\; \beta_1 = 0, \alpha_1 \neq 0, \chi_1 = 0, \\ \alpha_2 = 0, \alpha \neq 0, F_3 \neq 0 & if \alpha_1 = \beta_1 = \chi_1 = 0. \end{cases} \tag{1.8}$$

*For any given*

$$f \in H^1_{loc}(0, \infty; L^2(0,1)) \; with \; f(x,0) = 0,$$

*there exists a* $T > 0$ *such that* (1.7) *admits a unique solution*

$$u \in L^2(0,T; H^3(0,1)) \; with \, u_t \in L^\infty(0,T; L^2(0,1)) \cap L^2(0,T; H^1(0,1)).$$

Since then, following the rapid advances of the study of the pure initial value problem (IVP) of the KdV equation posed either on $\mathbb{R}$ or on a periodic domain $\mathbb{T}$ (cf [1, 4, 7, 8, 15, 32, 39–42] and the references therein), various IBVPs of the KdV equation on $(0, L)$ such as

$$\begin{cases} \partial_t u + \partial_x u + u\partial_x u + \partial_x^3 u = 0, \quad u(x,0) = \phi(x), \quad x \in (0,L), \;\; t > 0 \\ \\ u(0,t) = h_1(t), \quad u(L,t) = h_2(t), \quad \partial_x u(L,t) = h_3(t) \end{cases} \tag{1.9}$$

and

$$\begin{cases} \partial_t u + \partial_x u + u\partial_x u + \partial_x^3 u = 0, \quad u(x,0) = \phi(x), \quad x \in (0,L), \;\; t > 0 \\ \\ u(0,t) = h_1(t), \quad u(L,t) = h_2(t), \quad \partial_x^2 u(L,t) = h_3(t), \end{cases} \tag{1.10}$$

for instance, or in more general cases,

$$\begin{cases} \partial_t u + \partial_x u + u\partial_x u + \partial_x^3 u = 0, , \quad x \in (0,L), \ t > 0 \\ u(x,0) = \phi(x), \\ D_1 u = h_1(t), \qquad D_2 u = h_2(t), \qquad D_3 u = h_3(t) \end{cases} \tag{1.11}$$

where

$$D_i u = \sum_{j=0}^{2} \left( a_{ij}\partial_x^j u(0,t) + b_{ij}\partial_x^j u(L,t) \right), \qquad i = 1,2,3,$$

have also been extensively investigated for their well-posedness in the space $H^s(0,L)$ [3, 6, 10, 11, 13–15, 22, 29, 31, 45, 46, 51]. It is now known that both IBVP (1.9) and IBVP (1.10) are locally well-posed in the space $H^s(0,L)$ for any $s > -1$. As for the IBVP of (1.11) of the KdV equation posed on the finite $(0,L)$ with general boundary conditions, the following theorem has been obtained by Capistrano-Filho, Sun and Zhang [10] recently.

**Theorem B**: *Let $s \geq 0$ with $s \neq \frac{2j-1}{2}, j = 1,2,3...$, and $T > 0$ be given. If one of the assumptions below is satisfied,*

*(i)* $a_{12} = a_{11} = 0, \ a_{10} \neq 0, \ b_{12} = b_{11} = b_{10} = 0;$

$$a_{22} = a_{21} = a_{20} = 0, \ b_{20} \neq 0, b_{22} = b_{21} = 0;$$

$$a_{32} = a_{31} = 0, \ b_{31} \neq 0, \ b_{32} = 0.$$

*(ii)* $a_{12} = a_{11} = 0, \ a_{10} \neq 0, \ b_{12} = b_{11} = b_{10} = 0;$

$$b_{22} \neq 0, \ a_{22} = 0; \ a_{32} = a_{31} = 0, \ b_{31} \neq 0, \ b_{32} = 0.$$

*(iii)* $a_{12} \neq 0, \ b_{12} = 0; \ \ a_{22} = a_{21} = a_{20} = 0,$

$$b_{20} \neq 0, b_{22} = b_{21} = 0; \ a_{32} = a_{31} = 0, \ b_{31} \neq 0, \ b_{32} = 0.$$

*(iv)* $a_{12} \neq 0, \ b_{12} = 0; \ b_{22} \neq 0, \ a_{22} = 0; \ a_{32} = a_{31} = 0, \ b_{31} \neq 0, \ b_{32} = 0,$

*then, the IBVP (1.11) is locally analytically well-posed in the space $H^s(0,L)$ for any $s \geq 0$, with naturally compatible $\phi \in H^s(0,L)$ and boundary values $h_j$, $j = 1,2,3$, belonging to some appropriate spaces with optimal regularity.*

In contrast, there have been relatively fewer studies on the IBVPs of the fifth-order Korteweg-de Vries (KdV) equation posed on a finite interval $(0,L)$ (cf. [18, 19, 23, 48, 55, 56]). One particular investigation was conducted by Larkin and Simoes [48] concerning the following homogeneous boundary value problem:

$$\begin{cases} \partial_t u - \partial_x^5 u + \partial_x^3 u + u\partial_x u = f, \ u(x,0) = \phi(x), \ x \in (0,L), \\ \partial_x^2 u(0,t) = a_{32}\partial_x^2 u(0,t) + a_{31}\partial_x u(0,t) + a_{30}u(0,t) \\ \partial_x^4 u(0,t) = a_{42}\partial_x^2 u(0,t) + a_{41}\partial_x u(0,t) + a_{40}u(0,t), \\ \partial_x^2 u(L,t) = b_{21}\partial_x u(L,t) + b_{20}u(L,t), \\ \partial_x^3 u(L,t) = b_{31}\partial_x u(L,t) + b_{30}u(L,t), \\ \partial_x^4 u(L,t) = b_{41}\partial_x u(L,t) + b_{40}u(L,t), \ t > 0. \end{cases} \tag{1.12}$$

Their work resulted in the following theorem:

**Theorem C** *Assume*

$$\begin{cases} b_{20} - b_{40} - b_{20}^2 - \frac{1}{2}|b_{21}| - \frac{1}{2}b_{41} - \frac{1}{2}|b_{30}| > 0, \\ b_{31} - \frac{1}{2} - b_{21}^2 - \frac{1}{2}|b_{21}| - \frac{1}{2}b_{41} - \frac{1}{2}|b_{30}| > 0, \\ a_{40} - 1 - \frac{1}{2}|a_{41}| - \frac{1}{2}|a_{42}| - \frac{1}{2}|a_{30}| > 0, \\ \frac{1}{2} - a_{31} - \frac{1}{2}|a_{41}| - \frac{1}{2}|a_{30}| - \frac{1}{2}|a_{32}| > 0, \\ \frac{1}{4} - \frac{1}{2}|a_{42}| - \frac{1}{2}|a_{32}| > 0. \end{cases}$$

*Then for any naturally compatible* $u_0 \in H^5(0,L)$*, the IBVP (1.12) has a unique solution*

$$u \in L^\infty\big(0,T;H^5(0,L)\big) \cap L^2\big(0,T;H^7(0,L)\big).$$

Subsequently, Zhao and Zhang [55, 56] investigated the following sixteen different Initial-Boundary Value Problems (IBVPs) for the fifth-order Korteweg-de Vries (KdV) equation posed on the finite interval $(0,L)$:

$$\begin{cases} u_t + \partial_x^5 u + u\partial_x u = 0, \; x \in (0,L), \;\; t > 0 \\ u(x,0) = \phi(x), \qquad \mathcal{B}_{k,0}u = \vec{h}(t), \end{cases} \tag{1.13}$$

where $k = 1, 2, \cdots, 16$, and

$$\mathcal{B}_{k,0}u = (B_{1,k,0}, B_{2,k,0}, B_{3,k,0}, B_{4,k,0}, B_{5,k,0})u$$

with

$$\begin{aligned}
\mathcal{B}_{1,0}u &= \big(u(L,t), u(0,t), \partial_x u(L,t), \partial_x u(0,t), \partial_x^2 u(0,t)\big) \\
\mathcal{B}_{2,0}u &= \big(\partial_x u(L,t), \partial_x u(0,t), \partial_x^2 u(0,t), \partial_x^4 u(L,t), \partial_x^4 u(0,t)\big) \\
\mathcal{B}_{3,0}u &= \big(u(L,t), u(0,t), \partial_x^2 u(0,t), \partial_x^3 u(L,t), 3\partial_x^u(0,t)\big) \\
\mathcal{B}_{4,0}u &= \big(\partial_x^2 u(0,t), \partial_x^3 u(L,t), \partial_x^3 u(0,t), \partial_x^4 u(L,t), \partial_x^4 u(0,t)\big) \\
\mathcal{B}_{5,0}u &= \big(u(0,t), \partial_x u(L,t), \partial_x u(0,t), \partial_x^2 u(0,t), \partial_x^4 u(L,t)\big) \\
\mathcal{B}_{6,0}u &= \big(u(L,t), \partial_x u(L,t), \partial_x u(0,t), \partial_x^2 u(0,t), \partial_x^4 u(0,t)\big) \\
\mathcal{B}_{7,0}u &= \big(u(0,t), \partial_x^2 u(0,t), \partial_x^3 u(L,t), 3\partial_x^u(0,t), \partial_x^4 u(L,t)\big) \\
\mathcal{B}_{8,0}u &= \big(u(L,t), \partial_x u(0,t), \partial_x^3 u(L,t), \partial_x^3 u(0,t), \partial_x^4 u(0,t)\big) \\
\mathcal{B}_{9,0}u &= \big(u(L,t), u(0,t), \partial_x u(0,t), \partial_x^2 u(0,t), \partial_x^3 u(L,t)\big) \\
\mathcal{B}_{10,0}u &= \big(u(L,t), u(0,t), \partial_x u(L,t), \partial_x^2 u(0,t), \partial_x^3 u(0,t)\big) \\
\mathcal{B}_{11,0}u &= \big(\partial_x u(0,t), \partial_x^2 u(0,t), \partial_x^3 u(L,t), \partial_x^4 u(L,t), \partial_x^4 u(0,t)\big) \\
\mathcal{B}_{12,0}u &= \big(\partial_x u(L,t), \partial_x^2 u(0,t), \partial_x^3 u(0,t), \partial_x^4 u(L,t), \partial_x^4 u(0,t)\big)
\end{aligned}$$

$$\mathcal{B}_{13,0}u = \big(u(0,t), \partial_x u(0,t), \partial_x^2 u(0,t), \partial_x^3 u(L,t), \partial_x^4 u(L,t)\big)$$
$$\mathcal{B}_{14,0}u = \big(u(0,t), \partial_x u(L,t), \partial_x^2 u(0,t), \partial_x^3 u(0,t), \partial_x^4 u(L,t)\big)$$
$$\mathcal{B}_{15,0}u = \big(u(L,t), \partial_x u(0,t), \partial_x^2 u(0,t), \partial_x^3 u(L,t), \partial_x^4 u(0,t)\big)$$
$$\mathcal{B}_{16,0}u = \big(u(L,t), \partial_x u(L,t), \partial_x^2 u(0,t), \partial_x^3 u(0,t), \partial_x^4 u(0,t)\big)$$

**Definition 1.7.** *For given $s \in \mathbb{R}$ and $T > 0$, define*

$$\mathcal{H}_k^s(0,T) \equiv H^{\frac{s+2-\eta_{1,k,0}}{5}}(0,T) \times H^{\frac{s+2-\eta_{2,k,0}}{5}}(0,T) \times H^{\frac{s+2-\eta_{3,k,0}}{5}}(0,T) \times H^{\frac{s+2-\eta_{4,k,0}}{5}}(0,T) \times H^{\frac{s+2-\eta_{5,k,0}}{5}}(0,T),$$

*where $\eta_{j,k,0}$ , $j = 1,2,\ldots,5$ are the orders of the boundary operators in $\mathcal{B}_{k,0}$, and*

$$X_{s,T}^k \equiv H^s(0,L) \times \mathcal{H}_k^s(0,T), \quad k = 0,1,2,\ldots,16.$$

The following result for the IBVP (1.13) is obtained.

**Theorem D**: *Let $T > 0$, $r > 0$ and $0 \le s \le 5$ with $s \ne \frac{2j-1}{2}, j = 1,2,3,4,5$ be given. There exists a $T^* \in (0,T]$ such that for any given naturally compatible $(\phi, \vec{h}) \in X_{s,T}^k$ with*

$$\|(\phi, \vec{h})\|_{X_{s,T}^k} \le r,$$

*the IBVP (1.13) admits a unique solution*

$$u \in C([0,T^*; H^s(0,L)) \cap L^2(0,T^*; H^{s+2}(0,L)).$$

*Moreover, the corresponding solution map is real analytical.*

**Remark 1.8.** *The choice of boundary operators $\mathcal{B}_{k,0}$, $k = 1,2,\cdots,16$ is designed such that the solution $u$ of the IBVP associated with the following linear equation linked to (1.13), subject to homogeneous boundary conditions,*

$$\begin{cases} u_t + \partial_x^5 u = 0, \ x \in (0,L), \ t > 0 \\ u(x,0) = \phi(x), \quad \mathcal{B}_{k,0}u = 0, \end{cases}$$

*satisfies the property*

$$\frac{d}{dt}\int_0^L u^2(x,t)\,dx + \frac{1}{2}\left(\partial_x^2 u\right)(L,t) = 0 \quad \textit{for any} \quad t \ge 0.$$

*This implicit introduction of a dissipative mechanism to the system (1.13) (i.e., the corresponding linearized problem with homogeneous boundary conditions) has played a crucial role in proving Theorem D.*

In [52], we continue to investigate the IBVP of the Kawahara equation:

$$\partial_t u + \partial_x u + u\partial_x u + \beta\partial_x^3 u + \partial_x^5 u = 0, \quad u(x,0) = \phi(x), \quad x \in (0,L), \ t > 0, \tag{1.14}$$

subject to non-homogeneous boundary conditions:

$$B_j u = h_j(t), \quad t > 0, \quad \text{for} \quad j = 1,2,3,4,5, \tag{1.15}$$

and we aim to identify more general boundary conditions under which the IBVP (1.14)-(1.15) is well-posed in the space $H^s(0,L)$.

**Definition 1.9.** *For $k=1,2,3,\dots,16$, we define the boundary operators $\mathcal{B}_k$ as follows:*

$$\mathcal{B}_k = \mathcal{B}_{k,0} + \mathcal{B}_{k,1}.$$

*with*

$$\mathcal{B}_{k,1} = (B_{1,k,1}, B_{2,k,1}, B_{3,k,1}, B_{4,k,1}, B_{5,k,1})$$

*and*

$$B_{j,k,1}u = \sum_{j=0}^{\eta_{j,k,0}-1} \left( a_{kj}\partial_x^j u(0,t) + b_{kj}\partial_x^j u(L,t) \right), \qquad j=1,2,3,4,5,$$

*where $\eta_{j,k,0}$ is the order of the boundary operator $B_{j,k,0}$, and $B_{j,k,1}=0$ if $\eta_{j,k,0}=0$.*

We consider the following sixteen different initial boundary value problems:

$$\begin{cases} \partial_t u + \partial_x u + u\partial_x u + \beta\partial_x^3 u + \partial_x^5 u = 0, \ \text{ for } x\in(0,L), t>0 \\ \\ u(x,0)=\phi(x), \quad \mathcal{B}_k u = \vec{h}, \end{cases} \tag{1.16}$$

where $k=1,2,3,4,\cdots,16$.

The following theorem represents the main result presented in [52]:

**Theorem E** *Let $T>0$, $r>0$ and $s\geq 0$ with $s\neq \frac{2j-1}{2}$, $j=1,2,\cdots$, be given. Then, there exists a $T^*=T^*(r)\in(0,T]$ such that for any naturally compatible $(\phi,\vec{h})\in X^k_{s,T}$ with*

$$\|(\phi,\vec{h})\|_{X^k_{0,T}} \leq r,$$

*the IBVP (1.16) admits a unique solution*

$$u\in C([0,T^*];H^s(0,L))\cap L^2(0,T^*;H^{s+2}(0,L)).$$

*Moreover,*

$$\partial_x^j u \in L_x^\infty(0,L;H_t^{\frac{s+2-j}{5}}(0,T)), \ j=1,2,3,4$$

*and the solution map $(\phi,\vec{h})\mapsto u$ from $X^k_{s,T}$ to $C([0,T^*];H^s(0,L))$ is real analytic.*

**Remark 1.10.**

*(i) The temporal regularity conditions imposed on the boundary values $\vec{h}$ are optimal.*

*(ii) The solutions of the IBVP of the linear equation associated to (1.16) with homogeneous boundary conditions (see (1.18)),*

$$\begin{cases} u_t + \beta u_{xxx} + u_{xxxxx} = 0, \ x\in(0,L), \ t>0 \\ \\ u(x,0)=\phi(x), \qquad \mathcal{B}_{k,0}u=0, \end{cases}$$

*does not satisfy the dissipative condition,*

$$\frac{d}{dt}\int_0^L u^2(x,t)dx \le 0 \qquad for\ any\ \ t \ge 0,$$

*anymore unless* $\beta = 0$.

## 1.3 Main Results

In this paper, we consider the following sixteen different initial boundary value problems.

$$\begin{cases} \partial_t u + \partial_x u + \beta\partial_x^3 u + \partial_x^5 u = c_0 u\partial_x u + c_1\partial_x u\partial_x^2 u + c_2\partial_x(u\partial_x^2 u) + c_3\partial_x(u^3) \text{ for } x \in (0,L), t > 0 \\ u(x,0) = \phi(x), \quad \mathcal{B}_k u = \vec{h}. \end{cases} \tag{1.17}$$

Here $k = 1, 2, 3, 4, \cdots, 16$ and the boundary operator $\mathcal{B}_k$ is as given in Definition 1.9.

The theorems below are the the main results of the paper.

**Theorem 1.11.** *For* $k = 1, 3, 9, 10$, *let* $T > 0$ *and* $r > 0$ *and* $s \ge 0$ *with* $s \ne \frac{2j-1}{2}$, $j = 1, 2, 3\cdots$, *be given there exists a* $T^* = T^*(r) \in (0, T]$ *such that for any naturally compatible* $(\phi, \vec{h}) \in X^k_{s,T}$ *with*

$$\|(\phi, \vec{h})\|_{X^k_{0,T}} \le r,$$

*the IBVP* (1.17) *admits a unique solution*

$$u \in C([0, T^*]; H^s(0, L)),$$

*and the solution map* $(\phi, \vec{h}) \mapsto u$ *from* $X^k_{s,T}$ *to* $C([0,T^*]; H^s(0,L))$ *is real analytic. Additionally, the solution* $u$ *also belongs to the space* $L^2(0, T^*; H^{s+2}(0, L))$ *and satisfies*

$$\partial_x^j u \in L_x^\infty(0, L; H_t^{\frac{s+2-j}{5}}(0, T)),\ j = 1, 2, 3, 4.$$

.

**Theorem 1.12.** *Let* $T > 0$ *and* $r > 0$ *and* $s \ge 1$ *with* $s \ne \frac{2j-1}{2}$, $j = 2, 3\cdots$, *be given there exists a* $T^* = T^*(r) \in (0, T]$ *such that for any naturally compatible* $(\phi, \vec{h}) \in X^k_{s,T}$ *with*

$$\|(\phi, \vec{h})\|_{X^k_{1,T}} \le r,$$

*the IBVP* (1.17) *admits a unique solution*

$$u \in C([0, T^*]; H^s(0, L)),$$

*and the solution map* $(\phi, \vec{h}) \mapsto u$ *from* $X^k_{s,T}$ *to* $C([0,T^*]; H^s(0,L))$ *is real analytic. Additionally, the solution* $u$ *also belongs to the space* $L^2(0, T^*; H^{s+2}(0, L))$ *and satisfies*

$$\partial_x^j u \in L_x^\infty(0, L; H_t^{\frac{s+2-j}{5}}(0, T)),\ j = 1, 2, 3, 4.$$

.

The theorem will be proved using the contraction mapping principle in the space

$$Z^k_{s,T} := \left\{ u \in C(0,T; H^s(0,L)) \cap L^2(0,T; H^{s+2}(0,L));\ \mathcal{B}_{k,0} u \in \mathcal{H}^s_k(0,T) \right\}.$$

with

$$\|u\|_{Z^k_{s,T}} := \left( \|u\|^2_{C(0,T;H^s(0,L))} + \|u\|^2_{L^2(0,T;H^{s+2}(0,L))} + \|\mathcal{B}_{k,0}u\|^2_{\mathcal{H}^s_k(0,T)} \right)^{\frac{1}{2}}.$$

For a pair of given initial and boundary value $(\phi, \vec{h})$, consider the following linear IBVP for a given $v \in Z_{s,T}$:

$$\begin{cases} \partial_t u + \beta \partial_x^3 u + \partial_x^5 u = -\partial_x v + c_0 v \partial_x v + c_1 \partial_x v \partial_x^2 v + c_2 \partial_x (v \partial_x^2 v) + c_3 \partial_x (v^3) \ \text{ for } x \in (0,L), t > 0 \\ u(x,0) = \phi(x), \quad \mathcal{B}_{k,0} u = \vec{h} - \mathcal{B}_{k,1} v. \end{cases} \tag{1.18}$$

Its solution $u$ defines a nonlinear map $\Gamma_k$ on $Z_{s,T}$:

$$\Gamma_k(v) := u.$$

The goal is to show that the nonlinear map $\Gamma_k$ is a contraction, and its fixed point in $Z_{s,T}$ is then the desired solution of the IBVP (1.16). In this process, the following smoothing properties for solutions of the inhomogeneous linear fifth-order KdV equation due to Kenig, Ponce, and Vega [38, 41] will play an indispensable role. [1]

$$\partial_t w + \beta \partial_x^3 w + \partial_x^5 w = g(x,t),\ w(x,0) = 0, \quad x,\ t \in \mathbb{R} \tag{1.19}$$

**Theorem F** *There exists a constant $C > 0$ such that the following estimates hold for the solution $w$ of the IVP (1.19).*

*(i) If $g \in L^1_t L^2_x$, then for any $T > 0$*

$$\|w\|_{L^2(0,T;H^2_{loc}(\mathbb{R}))} \leq C \|g\|_{L^1_T L^2_x} \tag{1.20}$$

*and*

$$\|\partial_x^2 w\|_{L^\infty_x L^2_t} \leq C \|g\|_{L^1_t L^2_x} \tag{1.21}$$

*(ii) If $g \in L^1_x L^2_t$, then any $T > 0$,*

$$\|w\|_{L^\infty_T H^2_x} \leq C \|g\|_{L^1_x L^2_t}, \tag{1.22}$$

*and*

$$\|\partial_x^4 w\|_{L^\infty_x L^2_t} \leq C \|g\|_{L^1_x L^2_t}. \tag{1.23}$$

Here

$$\|g\|_{L^p_x L^q_T} = \left( \int_{-\infty}^{\infty} \left( \int_{-T}^{T} |g(x,t)|^q dt \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}$$

---

[1] Kenig, Ponce and Vega studied in [41] the IVP of the linear KdV equation

$$w_t + w_{xxx} = g(x,t),\ u(x,0) = \psi(x), \quad x,\ t \in \mathbb{R}$$

for its Kato and sharp Kato smoothing properties. But their arguments there can be applied to the IVP (1.19) with minor modifications.

and

$$\|g\|_{L^q_T L^p_x} = \left( \int\limits_{-T}^{T} \left( \int\limits_{-\infty}^{\infty} |g(x,t)|^p dx \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}}$$

and the interval $[-T, T] = \mathbb{R}$ if $T = t$.

**Remark 1.13.** *The estimate (1.20) is commonly referred to as local Kato smoothing, denoting a localized smoothing effect. On the other hand, the estimate (1.21) is known as sharp Kato smoothing, emphasizing its precise and refined smoothing properties. As for the estimate (1.22), it represents a strong global Kato smoothing effect, extending the smoothing behavior over the entire domain. Lastly, the estimate (1.23) is referred to as sharp double Kato smoothing, indicating its doubly enhanced and accurate smoothing characteristics.*

The rest of the paper is organized as follows.

— In Section 2 we will study various linear IBVPs associated to the nonlinear IBVP (1.17) and present the linear estimates needed to study the nonlinear IBVP (1.17).

— Section 3 is devoted to study the well-posedness of the nonlinear IBVP (1.17) and present the proof of Theorem 1.12 and Theorem 1.11, the main results of the paper.

# 2 Linear problems

Consideration is first given to the pure initial value problem (IVP)

$$w_t + \beta w_{xxx} + w_{xxxxx} = g(x,t), \; u(x,0) = \psi(x), \quad x, \; t \in \mathbb{R}. \tag{2.1}$$

The solution to this problem can be expressed as:

$$w(x,t) = W_{\mathbb{R}}(t)\psi + \int_0^t W_{\mathbb{R}}(t-\tau) g(\tau) d\tau := z_1(x,t) + z_2(x,t)$$

where

$$z_1(x,t) = W_{\mathbb{R}}(t)\psi, \qquad z_2(x,t) = \int_0^t W_{\mathbb{R}}(t-\tau) g(\tau) d\tau$$

Here, $W_{\mathbb{R}}(t)$ represents the $C^0$ group associated with the IVP (2.1). The solutions of this problem exhibit the following smoothing properties.

**Proposition 2.1.** *Let $T > 0$ and $s \geq 0$ be given, there exists a constant $C > 0$ such that the following estimates hold for the solution $w$ of the IVP (2.1).*

*(i) (local Kato smoothing)*

$$\|z_1\|_{L^\infty_T H^s_x} + \|z_1\|_{L^2_T H^{s+2}_{x,loc}} \leq C\|\psi\|_{H^s(\mathbb{R})}, \quad \|z_2\|_{L^\infty_T H^s_x} + \|z_2\|_{L^2_T H^{s+2}_{x,loc}} \leq C\|g\|_{L^1_T H^s_x}, \tag{2.2}$$

*(ii) (Sharp Kato smoothing)*

$$\|\partial^j_x z_1\|_{L^\infty_x H_t^{\frac{s+2-j}{5}}} \leq C\|\psi\|_{H^s(\mathbb{R})}, \; j = 1, 2, 3, 4, \tag{2.3}$$

$$\|\partial^2_x z_2\|_{W^{s,\infty}_x L^2_t} \leq C\|g\|_{L^1_t H^s_x} \tag{2.4}$$

(ii) (Strong global Kato smoothing)

$$\|z_2\|_{L^\infty_T H^{s+2}_x} \le C\|g\|_{W^{s,1}_x L^2_T} \tag{2.5}$$

(iv) (Sharp double Kato smoothing)

$$\|\partial_x^4 w\|_{W^{s,\infty}_x L^2_t} \le C\|g\|_{W^{s,1}_x L^2_t}. \tag{2.6}$$

The trace estimates given below follow from Proposition 2.1 through interpolation.

**Proposition 2.2.** *For any given $N > 0$ and $T > 0$, there exists a constant $C$ such that for any function $g$ with its support contained in $(-N, N) \times (-T, T)$, the following estimates hold:*

$$\|\partial_x^j z_2\|_{L^\infty_x H_T^{\frac{4-j}{5}}} \le C\|g\|_{L^2_T L^2_x}, \quad j = 0, 1, 2, 3, 4.$$

*Proof.* Notice that if $g$ has compact support, then $g \in L^2_t L^2_x$ implies that $g \in L^1_x L^2_t$ and we have

$$\|g\|_{L^1_x L^2_t} \le C\|g\|_{L^2_t L^2_x}.$$

Thus, it follows from (2.6) (with $s = 0$) that

$$\|\partial_x^4 z_2\|_{L^\infty_x L^2_T} \le C\|g\|_{L^2_T L^2_x}, \tag{2.7}$$

which implies further that

$$\|z_2\|_{L^\infty_x L^2_T} \le C\|g\|_{L^2_T H^{-4}_x}. \tag{2.8}$$

We also have

$$\|\partial_x^5 z_2\|_{L^\infty_x L^2_T} \le C\|g\|_{L^2_T H^1_x}$$

and

$$\|\partial_x^3 z_2\|_{L^\infty_x L^2_T} \le C\|g\|_{L^2_T H^{-1}_x} \le C\|g\|_{L^2_T H^1_x}.$$

As

$$\partial_t z_2 = g - \beta \partial_x^2 z_2 - \partial_x^5 z_2,$$

we have

$$\|\partial_t z_2\|_{L^\infty_x L^2_t} \le \|\partial_x^5 z_2\|_{L^\infty_x L^2_t} + C\|\partial_x^3 z_2\|_{L^\infty_x L^2_t} + C\|g\|_{L^\infty_x L^2_t} \le C\|g\|_{L^2_t H^1_x}.$$

Consequently,

$$\|z_2\|_{L^\infty_x H^1_t} \le C\|g\|_{L^2_t H^1_x}. \tag{2.9}$$

It then follows from (2.8) and (2.9) by interpolation that

$$\|z_2\|_{L^\infty_x H_T^{\frac{5}{4}}} \le C\|g\|_{L^2_t L^2_x} \tag{2.10}$$

The proof is completed by interpolation between (2.8) and (2.10). □

The following smoothing estimates are need to prove Theorem 1.11

**Corollary 2.3.** *For any given $N > 0$ and $T > 0$, there exists a constant $C$ such that for any function $g$ with its support contained in $(-N,N) \times (-T,T)$, the following estimates hold:*

$$\|\partial_x^{j-1} z_2\|_{L_x^\infty H_T^{\frac{4-j}{5}}} \le C\|g\|_{L_T^2 H_x^{-1}}, \quad j = 1,2,3,4.$$

**Remark 2.4.** *It seems reasonable to guess that*

$$\|\partial_x^4 z_2\|_{L_x^\infty H_T^{-\frac{1}{5}}} \le C\|g\|_{L_T^2 H_x^{-1}}.$$

*But we cannot prove it. If it were true, then Theorem 1.11 would hold for $j = 1,2,3,\cdots,16$.*

**Corollary 2.5.** *For any given $N > 0$ and $T > 0$, there exists a constant $C$ such that for any function $g$ with its support contained in $(-N,N) \times (-T,T)$, the following estimates hold:*

$$\|\mathcal{B}_{k,0} z_2\|_{\mathcal{H}_k^2(0,T)} \le C\|g\|_{L_T^2 L_x^2}, \quad k = 1,2,\cdots,16$$

*and*

$$\|\mathcal{B}_{k,0} z_2\|_{\mathcal{H}_k^1(0,T)} \le C\|g\|_{L_T^2 H_x^{-1}}, \quad k = 1,3,9,10.$$

**Proposition 2.6.** *For any given $N > 0$ and $T > 0$, there exists a constant $C$ such that for any function $g$ with its support contained in $(-N,N) \times (-T,T)$, the following estimates hold:*

$$\|z_2\|_{L_T^\infty L_x^2} + \|z_2\|_{L_T^2 H_{x,loc}^2} \le C\|f\|_{L_T^1 L_x^2}, \tag{2.11}$$

$$\|z_2\|_{L_T^\infty H_x^2} + \|z_2\|_{L_T^2 H_{x,loc}^4} \le C\|f\|_{L_T^2 L_x^2}, \tag{2.12}$$

$$\|z_3\|_{L_T^\infty H_x^1} + \|z_2\|_{L_T^2 H_{x,loc}^3} \le C\|f\|_{L_T^{\frac{4}{3}} L_x^2} \tag{2.13}$$

*and*

$$\|z_2\|_{L_T^\infty L_x^2} + \|z_2\|_{L_T^2 H_{x,loc}^2} \le C\|f\|_{L_T^{\frac{4}{3}} H_x^{-1}} \tag{2.14}$$

*Proof.* (2.11) and (2.12) follow directly from Proposition 2.1. By interpolation between (2.11) and (2.12), we obtain (2.13). Additionally, (2.14) is a different version of (2.13) which is needed in the proof of Theorem 1.11. □

Next we turn to consider the IBVP of the linear fifth order equation associated to the nonlinear IBVP (1.1)-(1.2):

$$\begin{cases} u_t + \beta u_{xxx} + u_{xxxxx} = f, \quad x \in (0,L), t > 0 \\ u(x,0) = \phi(x), \qquad \mathcal{B}_{k,0} u = \vec{h} \end{cases} \tag{2.15}$$

In the case of $f \equiv 0$, $\phi \equiv 0$, the corresponding solution map

$$\vec{h} \in \mathcal{H}_k^0(0,T) \mapsto u \in C([0,T]; L^2(0,L))$$

for the IBVP

$$\begin{cases} u_t + \beta u_{xxx} + u_{xxxxx} = 0, \quad \text{for } x \in (0,L), t > 0 \\ u(x,0) = 0, \quad \mathcal{B}_{k,0} u = \vec{h}, \end{cases} \tag{2.16}$$

is called the boundary integral operator denoted by $W_{bdr}^{(k)}$:

$$u(x,t) = W_{bdr}^{(k)}[\vec{h}](x,t).$$

Following the approach originally developed by Bona-Sun-Zhang in dealing with the non-homogeneous boundary value problems of the KdV equation in [2, 3, 5, 6] we have obtained an explicit integral representation formula for the boundary integral operator $W_{bdr}^{(k)}[\vec{h}]$, which enable us to establish the following Kato smoothing and sharp Kato smoothing estimates for solutions of the IBVP (2.16) [].

**Proposition 2.7.** *Let $T > 0$, $s \geq 0$ with $s \neq \frac{2l-1}{2}, l = 1, 2, \cdots,$ and $k \in \{1, 2, \cdots, 16\}$ be given. There exists a constant $C > 0$ such that for any naturally compatible $\vec{h} \in \mathcal{H}_k^s(0,T)$, the corresponding solution*

$$u(x,t) = W_{bdr}^{(k)}[\vec{h}](x,t)$$

*of the IBVP (2.16) satisfies*

$$\|u\|_{Z_{s,T}^k} \leq C\|\vec{h}\|_{\mathcal{H}_k^s(0,T)}$$

*and*

$$\sup_{0<x<L} \|\partial_x^j u(x,\cdot)\|_{H_t^{\frac{s+2-j}{5}}(0,T)} \leq C\|\vec{h}\|_{\mathcal{H}_k^s(0,T)}, j = 0, 1, 2, 3, 4.$$

As for the IBVP (2.15) with $\phi \neq 0$ and $f \neq 0$, let $f$ and $\phi$ be extensions of $f$ and $\phi$ from the interval $(0, L)$ to $\mathbb{R}$ with compact support, and consider the pure IVP:

$$v_t + \beta v_{xxx} + v_{5x} = f^*, \quad v(x,0) = \phi^*, \quad x, \ t \in \mathbb{R}.$$

The solution $v(x,t)$ can be written as

$$v(x,t) = W_{\mathbb{R}}(t)\phi^* + \int_0^t W_{\mathbb{R}}(t-\tau) f^*(\tau) d\tau,$$

Let

$$\vec{q}(t) := \mathcal{B}_{k,0} v.$$

The solution u of the IBVP (2.15) can then be written as

$$u(x,t) = W_{\mathbb{R}}(t)\phi^* + \int_0^t W_{\mathbb{R}}(t-\tau) f^*(\tau) d\tau + W_{bdr}^{(k)}[\vec{h} - \vec{q}](x,t).$$

The estimates presented in Proposition 2.8 below for solutions of the IBVP (2.15) follows from Proposition 2.1 and Proposition 2.7.

**Proposition 2.8.** *Let $T > 0$ and $k \in \{1, 2, \cdots, 16\}$ be given. There exists a constant $C > 0$ such that for any $(\phi, \vec{h}) \in X_{0,T}^k$, the following estimates hold f for solutions of the IBVP (2.15):*

$$\|u\|_{L^\infty(0,T;L^2(0,L))} \leq C \left( \|(\phi, \vec{h})\|_{X_{0,T}^k} + \|f\|_{L^1(0,T;L^2(0,L))} \right), \tag{2.17}$$

$$\|u\|_{L^2(0,T;H^2(0,L))} \leq C \left( \|(\phi, \vec{h})\|_{X_{0,T}^k} + \|f\|_{L^1(0,T;L^2(0,L))} \right), \tag{2.18}$$

$$\|\partial_x^2 u\|_{L_x^\infty(0,L;L^2(0,T))} \leq C \left( \|(\phi, \vec{h})\|_{X_{0,T}^k} + \|f\|_{L^1(0,T;L^2(0,L))} \right) . \tag{2.19}$$

*Also*

$$\|\partial_x^j u\|_{L_x^\infty(0,L;H^{\frac{2-j}{5}}(0,T))} \le C\|(\phi,\vec{h})\|_{X_{0,T}^k}, j=0,1,2,3,4 \tag{2.20}$$

*assuming* $f\equiv 0$*, and*

$$\|\partial_x^4 u\|_{L_x^\infty(0,L;L^2(0,T))} \le C\|f\|_{L^2(0,T;L^2(0,L))} \tag{2.21}$$

*assuming* $(\phi,\vec{h})\equiv 0$.

Next we present estimates for solutions of the IBVP (2.15) with $(\phi,\vec{h})\equiv 0$.

**Proposition 2.9.** *Let* $T>0$ *and* $k\in\{1,2,\cdots,16\}$ *be given and assume* $(\phi,\vec{h})\equiv 0$*. There exists constants* $\theta>0$ *and* $C>0$ *such that the following estimates hold for the solution* $u$ *of the IBVP (2.15):*

$$\|u\|_{Z_{1,T}^k} \le CT^\theta\|f\|_{L^2(0,T;L^2(0,L))} \tag{2.22}$$

*and*

$$\|u\|_{Z_{0,T}^k} \le CT^\theta\|f\|_{L^2(0,T;H^{-1}(0,L))} \tag{2.23}$$

*Proof.* We only need to prove (2.22). The solution $u$ can be written as

$$u=u_1-u_2$$

with

$$u_1=\int_0^t W_{\mathbb{R}}(t-\tau)f^*(\tau)d\tau$$

and

$$u_2=W_{bdr}^{(k)}\vec{q}, \qquad \vec{q}=\mathcal{B}_{k,0}u_1.$$

First of all, by Propositions 2.2,

$$\vec{q}\in\mathcal{H}_k^2(0,T).$$

Thus, using Proposition 2.7,

$$\|u_2\|_{Z_{1,T}^k} \le C\|\vec{q}\|_{\mathcal{H}_k^1(0,T)} \le CT^\theta\|\vec{q}\|_{\mathcal{H}_k^2(0,T)} \le CT^\theta\|f\|_{L^2(0,T;L^2(0,2))}$$

Then applying Proposition 2.6 we arrive at

$$\|u_1\|_{Z_{1,T}^k} \le C\left(\|f\|_{L^{\frac{4}{3}}(0,T;L^2(0,L))} + \|\vec{q}\|_{\mathcal{H}_k^1(0,T)}\right) \le CT^\theta\|f\|_{L^2(0,T;L^2(0,2))}$$

for some $\theta>0$.

□

# 3 Nonlinear problems

Armed with the linear estimates from Section 2, we are now ready to study the well-posedness of the nonlinear IBVPs In this paper, we consider the following sixteen different initial boundary value problems.

$$\begin{cases} \partial_t u+\partial_x u+\beta\partial_x^3 u+\partial_x^5 u = c_0 u\partial_x u + c_1\partial_x u\partial_x^2 u + c_2\partial_x(u\partial_x^2 u)+c_3\partial_x(u^3) \ \text{ for } x\in(0,L), t>0 \\ u(x,0)=\phi(x), \quad \mathcal{B}_k u=\vec{h}. \end{cases} \tag{3.1}$$

Here $k = 1, 2, 3, 4, \cdots, 16$ and the boundary operator $\mathcal{B}_k$ is as given in Definition 1.9.

**Lemma 3.1.** *Let*

$$N(u, v, w) := c_0 v\partial_x u + c_1 \partial_x v \partial_x^2 u + c_2 \partial_x (v\partial_x^2 u) + c_3 \partial_x (uvw)$$

*Then for any* $u, v, w \in Z^k_{1,T}$,

$$\|N(u, v, w)\|_{L^2(0,T;L^2(0,L))} \leq C \left( \|u\|_{Z^k_{1,T}} \|v\|_{Z^k_{1,T}} + \|u\|_{Z^k_{1,T}} \|v\|_{Z^k_{1,T}} \|w\|_{Z^k_{1,T}} \right)$$

*and* $u, v, w \in Z^k_{0,T}$,

$$\|N(u, v, w)\|_{L^{\frac{4}{3}}(0,T;H^{-1}(0,L))} \leq CT^{\theta} \left( \|u\|_{Z^k_{0,T}} \|v\|_{Z^k_{0,T}} + \|u\|_{Z^k_{0,T}} \|v\|_{Z^k_{0,T}} \|w\|_{Z^k_{0,T}} \right)$$

*Proof.* We will only prove the estimate for $\|\partial_x(v\partial_x^2 u)\|_{L^2(0,T;L^2(0,L))}$ and $\|\partial_x(v\partial_x^2 u)\|_{L^{\frac{4}{3}}(0,T;H^{-1}(0,L))}$, as the proof for other terms in $N(u)$ is similar and simpler. To this end, note that

$$\|\partial_x(v\partial_x^2 u)\|_{L^2_x} = \|\partial_x v \partial_x^2 u + v\partial_x^3 u)\|_{L^2_x} \leq \|v\|_{H^1_x} (\|u\|_{H^2_x} + \|u\|_{H^3_x}) \leq 2\|v\|_{H^1_x} \|u\|_{H^3_x},$$

$$\int_0^T \|\partial_x(v\partial_x^2 u)\|^2_{L^2_x} dt \leq 2 \sup_{0<t<T} \|v(\cdot, t)\|^2_{H^1_x} \int_0^T \|u(\cdot, t)\|^2_{H^3_x} dt \leq C\|v\|^2_{Z^k_{1,T}} \|u\|^2_{Z^k_{1,T}}$$

Thus

$$\|\partial_x(v\partial_x^2 u)\|_{L^2(0,T;L^2(0,L))} \leq C\|v\|_{Z^k_{1,T}} \|u\|_{Z^k_{1,T}}.$$

As for $\|\partial_x(v\partial_x^2 u)\|_{L^{\frac{4}{3}}(0,T;H^{-1}(0,L))}$,

$$\begin{aligned}
\|\partial_x(v\partial_x^2 u)\|_{L^{\frac{4}{3}}(0,T;H^{-1}(0,L))} &\leq C\|v\partial_x^2 u\|_{L^{\frac{4}{3}}(0,T;L^2(0,L))} \\
&\leq \|\|v\|_{L^\infty_x} |\partial_x^2 u|\|\|_{L^{\frac{4}{3}}(0,T;L^2(0,L))} \\
&\leq C\|v\|_{L_t^{\frac{10}{3}}(0,T;H_x^{\frac{1}{2}+\epsilon}(0,L))} \|u\|_{L^2(0,T;H^2_x(0,L))} \\
&\leq CT^{\frac{1}{6}} \|v\|_{L^4(0,T;H_x^{\frac{1}{2}+\epsilon}(0,L))} \|u\|_{L^2(0,T;H^2_x(0,L))} \\
&\leq CT^{\frac{1}{6}} \|u\|_{Z^k_{0,T}} \|v\|_{Z^k_{0,T}}.
\end{aligned}$$

□

**Proposition 3.2.** *Let* $T > 0$ *and* $r > 0$ *be given.*

*(i) There exists a* $T^* = T^*(r) \in (0, T]$ *such that for any* $(\phi, \vec{h}) \in X^k_{1,T}$ *with*

$$\|(\phi, \vec{h})\|_{X^k_{1,T}} \leq r,$$

*the IBVP* (3.1) *admits a unique solution*

$$u \in Z^k_{1,T^*}.$$

*Moreover, the corresponding solution map* $(\phi, \vec{h}) \mapsto u$ *from* $X^k_{1,T}$ *to* $Z^k_{1,T^*}$ *is real analytic.*

*(ii) For $k = 1, 3, 9, 10$, there exists a $T^* = T^*(r) \in (0, T]$ such that for any $(\phi, \vec{h}) \in X^k_{0,T}$ with*

$$\|(\phi, \vec{h})\|_{X^k_{0,T}} \leq r,$$

*the IBVP (3.1) admits a unique solution*

$$u \in Z^k_{0,T^*}.$$

*Moreover, the corresponding solution map $(\phi, \vec{h}) \mapsto u$ from $X^k_{0,T}$ to $Z^k_{0,T^*}$ is real analytic.*

*Proof.* We only provide the proof of (ii). The proof of (i) is similar. Let $\delta > 0$ and $0 < \eta \leq \min\{1, T\}$ be constants to be determine and

$$S^k_{\eta,\delta} = \left\{ u \in Z^k_{0,\eta} : \|u\|_{Z^k_{0,\eta}} \leq \delta \right\},$$

which is a bounded closed convex subset of $Z^k_{0,\eta}$.

For given $(\phi, \vec{h}) \in X^k_{0,\theta}$, we define a map $\Gamma_k$ on $S^k_{\eta,\delta}$ by

$$u = \Gamma_k(v)$$

for any $v \in S^k_{\eta,\delta}$, where $u$ is the unique solution of

$$\begin{cases} \partial_t u + \partial_x^5 u + \beta \partial_x^3 u = N(v, v, v), \quad x \in (0, L),\ t \in (0, T) \\ \\ u(x, 0) = \phi(x), \quad \mathcal{B}_{k,0} u = \vec{h}_k - \mathcal{B}_{k,1} v. \end{cases} \tag{3.2}$$

Applying Proposition 3.4 and Proposition 3.5 yields that there exist constants $C > 0$ and $\theta > 0$ such that

$$\|\Gamma_k(v)\|_{Z^k_{0,\eta}} \leq C\|(\phi, \vec{h})\|_{X^k_{0,\eta}} + C\eta^\theta \left(1 + \|v\|_{Z^k_{0,\eta}}\right) \|v\|_{Z^k_{0,\eta}}$$

Choose $\delta > 0$ and $\eta > 0$ such that

$$\delta = 2C\|(\phi, \vec{h})\|_{X^k_{0,\eta}}, \quad C\eta^\theta(1 + \delta) < \frac{1}{2}.$$

Then

$$\|\Gamma_k(v)\|_{Z^k_{0,\eta}} < \frac{\delta}{2} + \frac{\delta}{2} < \delta \quad \text{for any} \quad v \in S^k_{\eta,\delta}.$$

Furthermore, for any $v, w \in S^k_{\eta,\delta}$,

$$\|\Gamma_k(v) - \Gamma_k(w)\|_{Z^k_{0,\eta}} \leq C\eta^\theta \left(1 + \|v\|_{Z^k_{0,\eta}} + \|w\|_{Z^k_{0,\eta}}\right) \|v - w\|_{Z^k_{0,\eta}} \leq \frac{1}{2}\|v - w\|_{Z^k_{0,\eta}}.$$

The map $\Gamma_k$ is a contraction mapping on $S^k_{\theta,\delta}$. Moreover, its fixed point

$$u = \Gamma_k(u)$$

is desired solution of the nonlinear IBVP (3.1). □

Next, we consider the following linearized IBVPs associated to the nonlinear problems,

$$\begin{cases} \partial_t w + \beta\partial_x^3 w + \partial_x^5 w + \partial_x \left(a_1 w + \partial_x a_2 \partial_x w + a_3 \partial_x^2 w + a_4^2 w + \partial_x^2 a_5 w\right) = 0, \quad x \in (0,L),\ t > 0, \\ w(x,0) = \phi(x), \quad \mathcal{B}_k w = \vec{h} \end{cases} \tag{3.3}$$

for $k = 1, 2, 3. \dots, 16$ where $a_j(x,t)$, $j = 12, \cdots, 5$ are given functions.

**Proposition 3.3.** *Let $T > 0$ be given.*

*(i) Assume $K = 1, 3, 9, 10$ and $a_j \in Z^k_{0,T}, j = 1, 2, \cdots, 5$. Then for any $(\phi, \vec{h}) \in X^k_{0,T}$, the IBVP* (3.3) *admits a unique solution*

$$u \in Z^k_{0,T}$$

*satisfying*

$$\|u\|_{Z^k_{0,T}} \le C\|(\phi, \vec{h})\|_{X^k_{0,T}}$$

*where the constant $C > 0$ depending only on $T$ and $\|a_j\|_{Z^k_{0,T}}$, $j = 1, 2 \cdots, 6$.*

*(ii) Assume $K = 1, 2, \cdots, 16$ and $a_j \in Z^k_{1,T}, j = 1, 2, \cdots, 5$. Then for any $(\phi, \vec{h}) \in X^k_{1,T}$, the IBVP* (3.3) *admits a unique solution*

$$u \in Z^k_{1,T}$$

*satisfying*

$$\|u\|_{Z^k_{1,T}} \le C\|(\phi, \vec{h})\|_{X^k_{1,T}}$$

*where the constant $C > 0$ depending only on $T$ and $\|a_j\|_{Z^k_{1,T}}$, $j = 1, 2, \cdots, 6$.*

*Proof.* We only present the proof of (i). The proof of (ii) is similar. Let

$$S^k_{\eta,\delta} = \left\{u \in Z^k_{0,\eta} : \|u\|_{Z^k_{0,\eta}} \le \delta\right\}$$

and for given $(\phi, \vec{h}) \in X^k_{0,\theta}$, define a map $\Gamma$ on $S_{\eta,\delta}$ by

$$u = \Gamma_k(v)$$

for any $v \in S^k_{\eta,\delta}$, where $u$ is the solution of

$$\begin{cases} \partial_t w + \partial_x^5 w + \beta\partial_x^3 w = -\partial_x \left(a_1 v + \partial_x a_2 \partial_x v + a_3 \partial_x^2 v + a_4^2 v + \partial_x^2 a_5 v\right), \quad x \in (0,L),\, t \in (0,T) \\ w(x,0) = \phi(x), \quad \mathcal{B}_{k,0} w = \vec{h}_k - \mathcal{B}_{k,1} v. \end{cases}$$

The same argument as that in the proof of Proposition 4.1 shows that $\Gamma_k$ is a contraction in the ball $S^k_{\eta,\delta}$ as long as $\eta$ and $\delta$ are chosen appropriately. Its fixed point $w = \Gamma_k(w)$ is the desired solution of the IBVP (3.3) in the temporal interval $[0, \eta]$. Note that $\eta$ only depends on $\|a_j\|_{Z^k_{0,T}}, j = 1, 2, \cdot, 5$ and, especially, not on $\phi$. Thus by a standard extension argument, $\eta$ can be extended to any large given number $T$. □

***Proof of Theorem 1.11.***
The case $s = 0$ is Proposition 3.2 proved earlier. For the case of $s = 5$, let $w = \partial_t u$. Then $v$ solves the

following IBVP of the linear fifth order KdV equation with variable coefficients

$$\begin{cases} \partial_t w + \beta\partial_x^3 w + \partial_x^5 w + \partial_x \left(a_1 w + \partial_x a_2 \partial_x w + a_3 \partial_x^2 w + a_4^2 w + \partial_x^2 a_5 w + a_6 \partial_x^2 w\right) = 0, \quad x \in (0, L),\ t > 0, \\ w(x, 0) = \phi^*(x), \quad \mathcal{B}_k w = \vec{h}' \end{cases}$$

where

$$a_1 = -c_0 u, \quad a_2 - c_1 u, \quad a_3 = a_5 = -c_2 u, \quad a_4 = -3c_3 u,$$

$$\phi^*(x) = -\phi'(x) - \beta\phi'''(x) - \phi'''''(x) + N(\phi, \phi, \phi).$$

Applying Proposition 3.3 yields that $\partial_t u = w \in Z^k_{0,T^*}$. Then it follows from the relation

$$\partial_x^5 u = -\partial_t u - \partial_x u - \beta\partial_x^3 u + N(u, u, u)$$

together with $u \in Z^k_{0,T^*}$ that $\partial_x^5 u \in Z^k_{0,T^*}$. Consequently $u \in Z^k_{5,T^*}$. The case of $0 < s < 5$ follows using Tartar's nonlinear interpolation theory [53]. Similarly, the case $s > 5$ follows from the same standard approach developed in [3]. The proof is complete. □

***Proof of Theorem 1.12***. It is similar to the proof of Theorem 1.11 and is therefore omitted. □

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