On Triangles and Quadrilaterals of Groups

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Abstract

This dissertation demonstrates the existence of a pair of algebraic and geometric structures on triangles of groups and on quadrilaterals of groups. These structures are an automatic and biautomatic structure. In addition, this paper also discusses the growth function for the quadrilaterals.

We show that these groups have these desired structures and discuss what they are. We also give an extraordinary example of a pair of quadrilaterals of groups that are defined nearly identically but do not behave alike.

This Thesis is Dedicated to my Parents

In Honor of my Mother

Terri Lynch

And in Honor of my Father

Michael Lynch

And to my Grandparents

In Honor of my Grandmother

Dascha Rabinowitz

In Honor of my Grandfather

Hersch Rabinowitz

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Chapter 1

Introduction

The goal of this thesis is to investigate certain structures on complexes of groups. In particular, we will look at automatic structures and growth functions for quadrilaterals of finite groups and for triangles of finite groups. We will address these questions through geometric group theory.

The history of geometric group theory goes back, at least, to the early 1900's with the work of Max Dehn. In the early 1980's people like William Thurston, Jim Cannon, and Mikhail Gromov took it in a couple of new directions. One direction gave rise to automatic groups (Cannon and Thurston), and the other gave rise to negatively curved groups (Gromov). This thesis gets most of its concepts and style from Thurston and Cannon (who together came up with *automatic groups*), but the subtext frequently reflects the ideas of Gromov. Other ideas in this thesis (particularly *growth functions*) are older.

The growth function of a finitely generated group G goes back to Milnor and Bourbaki. The *growth sequence* is defined as

$$a_n = \{g \in G| \, |g| = n\}$$

where $|g|$ is the word length of g in G associated to its presentation. The *growth function* is then defined to be

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

However, Milnor and Bourbaki were more interested in the asymptotic growth rate of $\{a_n\}$ than in the analytic function $f(z)$. In the early 80's, Cannon [3] addressed $f(z)$ more directly. He showed the growth function for cocompact hyperbolic groups is rational, no matter the choice of finite generators. Also Cannon developed some of the methods used in this thesis for calculating the growth function (in particular, the use of a *transition* matrix). This paper was also one of the pieces that led to the formulation of the idea of automatic groups. In 1978, Cannon and Thurston (at the International Congress of Mathematicians in Helsinki) discussed growth functions for hyperbolic groups, and made the conjecture that they were rational. In [3], Cannon continued these studies (using his idea of "cone types") and developed a relationship between hyperbolic phenomena and recursive processes. From here, Thurston introduced the notion of finite state automata, and Epstein, Holt, Patterson, and Marden completed the work. They came up with the final idea of automatic groups [5].

For the purposes of this thesis, there is one other notion that must be mentioned. This is the idea of an angle of a group. Gersten and Stallings in [8] introduced this idea (which we will define later) and we use it here extensively. All of these ideas lead us to be able to address questions that have been difficult to address in the past.

The methods of geometric group theory have been particularly successful in dealing with negatively curved groups. They have had much less success with non-positively curved groups. This thesis, as well as [6], deals with negatively curved groups and non-positively curved groups alike. The results of both of these papers show that for these groups we can use the same techniques on both negatively curved and non-positively curved groups to get desirable results. It is no coincidence, however, that the complexity of several of the arguments is increased when we take into account the possibility that the groups may not be negatively curved.

The automatic and biautomatic structures we develop in this thesis solve two of the major problems of geometric group theory for both of these types of groups. The automatic structure, it is easily shown, solves the word problem. This problem asks if G is a group with a given presentation $< S : R >$ and w is a word in S representing an element of G, then is it possible to find a (finite) algorithm to decide whether w represents the identity of G or not? This problem dates back to Dehn and has an interesting history. This problem is sometimes undecidable. But all automatic groups have a decidable word problem. The biautomatic structure solves a similar problem, known as the conjugacy problem. This question is similarly defined: if w_1 and w_2 are words representing elements of a group G with the same conditions as above, then is there a (finite) algorithm to decide whether the two group elements are conjugate? This problem is similarly troublesome, yet a biautomic structure allows one to easily find such an algorithm.

To define an automatic group, we first need the definition of a Finite State Automata (FSA). A FSA is essentially a simple machine which is capable of producing a certain type of language, a *regular language.* This type of language is one that can be generated by a simple set of rules from predicate calculus. We now give the formal definition.

Definition 1.1 (Finite State Automaton) *A finite state automaton over A is a* *quintuple* W=$(S$,A,μ,Y,$s_0)$, *where* S *is a finite set, called the* state set, A *is a finite set, called the* alphabet, $\mu : S \; X \; A \rightarrow S$ *is a function, called the* transition function, Y *is a (possibly empty) subset of* S *called the subset of* accept states, *and* $s_0 \in S$ *is called the* start state.

In our work, A will be the generators of a group G, and we imagine $\mu(s,a)$ as arrows from one state of S to another state. Frequently, the arrow itself will represent an element of A and the states will represent simple finite complexes (which are in some way representations of subgroups of G). We decide if an arrow takes us from one state to another by looking at an action of the group. This action will be defined on the complexes that the states represent. We calculate the new complex we get after acting on the original complex of the initial state by the generator the arrow represents. There will be some state of our FSA that represents this new complex as well. Then the arrow will go from the first state to the second. The set Y should be imagined as the set of all words of G (as interpreted from the arrows) that we want (what we "want" will be clear later). Finally, the significance of this is that since the machine is finite, the geometric actions can be described in finite way. This makes calculations relatively simple (and highly computable).

We also define the language $L(W)$ for W a FSA over A. It is the set of all words in A (as interpreted by the arrows) whose final state is an accept state. Now, we turn to the definition of an automatic group. We quote from [5].

Definition 1.2 (Automatic Group) *Let* G *be a group. An* automatic structure *on* G *consists of a set* A *of semigroup generators of* G, *a FSA* W *over* A, *and FSA* M_x *over* (A, A), *for* $x \in A \bigcup \{\epsilon\}$ *(where* ϵ *is the empty word), satisfying the following conditions:*

1. *The map* $\pi : L(W) \rightarrow G$ *is surjective.*

2. *For $x \in A \bigcup \{\epsilon\}$, we have $(w_1, w_2) \in L(M_x)$ if and only if $\pi(w_1 x) = \pi(w_2)$ (as elements of G) and both w_1 and w_2 are elements of $L(W)$.*

We call W the word acceptor. *An* automatic group *is one that admits an automatic structure.*

Intuitively, we imagine an automatic structure on a group G as being a language, which is easily calculated, for which we have at least one representative of every element of the group. Furthermore, we insist that the words of the language are "close by" in the Cayley graph of the group if they represent the same element of the group. More precisely, if we have two paths (which represent elements of G), starting from the initial point of the Cayley graph, and ending within distance one from each other which are both words in our language, then we insist that they stay within a bounded distance for the entire length of the paths. This may not be too clear from the definition, but it should be clear once we state the following theorem.

Theorem 1.3 *Let G be a group and let A be a finite set of semigroup generators for G. Let W be a FSA over A and suppose that $\pi : L(W) \to G$ is surjective. Then A and W are part of an automatic structure on G if and only if there is a number k with the property that whenever two strings w_1 and w_2 are accepted by W are such that $\pi(w_1 x) = \pi(w_2)$ for some $x \in A$, the corresponding paths of w_1 and w_2 are less than k apart in the Cayley graph of G.*

The set of paths in the Cayley graph of G generated by $L(W)$ will be called a *combing* if they satisfy this theorem. So, to prove we have an automatic structure all we have to do is find a language that can be produced by a FSA and which, as paths, is a combing.

Finally, we introduce the last structure we are interested in. The notion of *bicombing* and *biautomatic* come from Gersten and Short [7]. They have both a similar definition and a similar theorem to that of combing lines and automatic groups that we actually use in this thesis. The definition simply says that if G has an automatic structure $(A, L(W))$, then it is biautomatic if $(A, L(W)^{-1})$ is also an automatic structure for G, where $L(W)^{-1}$ is the set of formal inverses of $L(W)$. The theorem we use follows:

Theorem 1.4 *Let G be a group and let A be a finite set of generators closed under inversion. Let $L(W)$ be a regular language over A that maps onto G via π. Then L is biautomatic if and only if, for each $w \in L(W)$ and each pair of generators $x, y \in A$, the uniform distance between the path xwy and the path corresponding to any element of $L(W)$ representing $\pi(xwy)$ is bounded by a fixed constant $k > 1$.*

Again, if such a set of paths exists, we call it a *bicombing* of G. How is this different from an automatic structure? The difference lies in the two generators x, y. One, as before, insists that the two paths end up within one of each other, but the other generator allows the two paths to start from different points of the Cayley graph (which are distance one away).

In fact, in our work, we build (or in the case of triangles, have already built for us) a complex that the group G acts on. From this action, the dual complex of our complex is the Cayley graph of the group. We will not actually mention the Cayley graph in the body of the thesis, for we will do all our calculations in these complexes.

In general, our attack will be thus. We inductively build a complex that the group G acts on, and then we study the complex. We build a language that has only one representative for each element of G and so that any subset of an accepted path (in the Cayley graph) is again an element of our language (this is called a *prefix closed* language). These two properties are not necessary, but in showing that the paths are actually combings or bicombings, they simplify matters greatly. We will not need to worry whether the paths (when we check if they get far away from each other, as we shorten them in the Cayley graph) are in our language, and we will not mention this further. Once we show that the paths are bounded by a universal bound k, all that will remain for us is to show that the language can be generated by a FSA. (In fact, in the case of triangles [6] has already shown this and more. All we do here is show that the language defined there is a bicombing.)

Finally, the growth function (given our automatic language) will also be tackled with the complex we will build. By then, the complex will be sufficiently well understood so that the calculation of the growth function will be quick and easy.

We start with quadrilaterals of finite groups, because we have both more to do for them and less to do to them. This is because most of the structures we are discussing have already been shown to exist for triangles, but the triangles case is much more difficult than the quadrilaterals case.

Chapter 2

Quadrilaterals of Finite Groups

This chapter will deal with quadrilaterals of finite groups. After [6], we will investigate an automatic structure, a biautomatic structure, and the growth function for the amalgam group. A few assumptions will have to be made, though. In particular, we assume that we have trivial face group, non-trivial edge groups, and that no vertex group has angle greater than $\frac{\pi}{2}$. Given these assumptions, we can prove the main theorems.

Section 1 is just a few quick, necessary definitions.

In Section 2, we do the dirty work. To study the amalgam group, we use the geometry that is inherent in it. To this end, we build a complex which the amalgam group acts on and discover what we can from this complex. In fact, there already exists such a complex, called the amalgam complex. Unfortunately, though it is defined easily, its construction does not easily yield the necessary structures which we need to discover an automatic structure. With this in mind, we build a complex of our own, by induction, which the amalgam group acts on, and which is built so that we have control over the combinatorics of the boundaries. At the end of the section, we show that our complex is isomorphic (as a complex) to the amalgam complex. Therefore, we can use our complex to study the amalgam group.

Section 3 deals with the automatic and biautomatic structures. We find that the language we define is prefix closed, unique, and each member is a geodesic in the word norm. That is, we have a weakly-geodesic biautomatic structure. To prove our language is automatic and biautomatic, we show that the k-fellow traveler property holds for the combing lines. Then we give a full description of how to find the word acceptor automaton for the language. From [5] then, the proof is finished.

Section 4 grapples with the growth function. Following a method of Cannon's [3], we build a recurrence matrix for the "types" of vertices of the group, and this, together with a matrix for the initial types and a way of counting quadrilaterals given a type, gives the growth sequence. Calculating the growth function from these is simple. We note that the growth function turns out to be dependent solely on the order of the vertex groups and the edge groups, and not on the choice of edge injections.

In Section 5, we look at a pair of examples of particular interest. Both groups have the same vertex and edge groups, and the same edge injections, save one. As noted above, then, the growth function for both groups is the same. But one of the groups is negatively curved while the other is not. This follows from a theorem of Bridson [1].

The structure of this paper, and the work therein, is modeled after the work in [6], but in fact, it is much easier. The structure of the complex that we build turns out to be much "nicer" than the one for triangles of groups. We consider this a good thing.

2.1 Preliminary Definitions

Suppose we have a quadrilateral of finite groups: that is suppose we have four finite groups, called vertex groups, four other finite groups, called edge groups, and injections between them as in Figure 2.1.

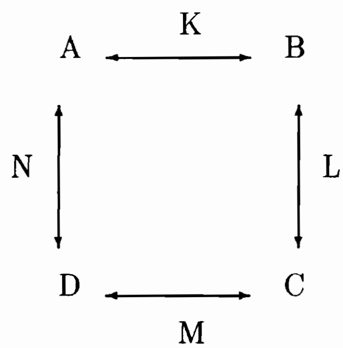


Figure 2.1: A quadrilateral of groups

We denote the injection from, for example, K into A by $\Psi_{K,A}$ and we write K_A for $\Psi_{K,A}(K)$. We also assume that K, L, M, and N are not trivial. The amalgam group G is the group with generators all the elements of the vertex groups, and relators all the relations of the vertex groups with, in addition, a few other relations coming from the injections. These are of the form $\Psi_{K,A}(k) = \Psi_{K,B}(k)$ for every $k \in K$. There are analogous relators for the edge groups L, M, and N. The amalgam complex X is a complex, with vertices the cosets of A_G (where A_G is interpreted as the image of A in G), B_G, C_G, and D_G in G. The edges are similarly made from the edge groups, e.g. K_G. Incidence is defined in the natural way, and there is a face for each element of G. There is a canonical action of G on X by left multiplication.

This is the complex we wish to understand, but instead of dealing with it directly, we give a few definitions that will be used to build a "better" complex. First, we define a *labeled* complex.

Definition 2.1 *A* labeled complex *is a complex X together with a label function*

$$l : \{vertices\ of\ X\} \bigcup \{edges\ of\ X\} \rightarrow \{A, B, C, D, K, L, M, N\}$$

such that the boundary of every closed cell is labeled as in Figure 2.2.

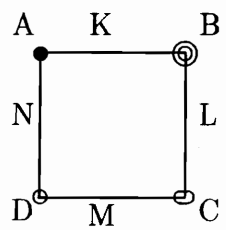


Figure 2.2: A labeled complex

We use the following convention when drawing figures: Vertices with label A are drawn as disks, vertices with label B are drawn as annuli, vertices with label D are drawn as circles, and vertices with label C are drawn as ovals. If a vertex appears as a point, the label is unknown, and most likely immaterial. In the complex we will define, all the cells will be constructed with labels like these.

Next, suppose E is an edge group. Then we have

Definition 2.2 (The Model Star of an Edge) *$st(E)$ is a labeled 2-complex which is the union of its closed 2-cells, with a label-preserving E-action, defined as follows: there is an edge $s(E)$ that is fixed pointwise by E with $l(s(E)) = E$;*

there is a 1-1 equivariant correspondence, $g \leftrightarrow s(g)$, between the elements of E and the cells of $st(E)$; and $s(E) \subset s(g)$ for every $g \in E$.

And similarly, we have the definition for vertex groups.

Definition 2.3 (The Model Star of a Vertex Group) *If V is a vertex group with associated edge groups E and F, then $st(V)$ is a labeled 2-complex which is the union of its closed 2-cells, with a label-preserving V-action, defined as follows: there is a vertex $s(V)$ that is fixed by V with $l(s(V)) = V$; for each coset $gE_V, gF_V, g \in V$, there is an edge $s(gE_V), s(gF_V)$ that contains $s(V)$ with $l(s(gE_V)) = E, l(s(gF_V)) = F$, and this gives a bijection between these cosets and the edges containing $s(V)$; there is a 1-1 equivariant correspondence, $g \leftrightarrow s(g)$, between the elements of V and the 2-cells of $st(V)$; and for every $g \in V$, $s(g)$ contains $s(gE_V)$ and $s(gF_V)$. We denote the link of $s(V)$ in $st(V)$ by $ln(V)$.*

Suppose V is a vertex group with an associated edge group E. Then there is a label-preserving embedding

$$\psi_{E,V} : st(E) \to st(V)$$

with $\psi_{E,V}(s(E)) = s(E_V)$ and $\psi_{E,V}(s(g)) = s(\Psi_{E,V}(g))$ for each $g \in E$.

Now, we recall a definition of Gersten and Stallings [8] that will give us the conditions we need on our amalgam groups to get the structures we are after.

Definition 2.4 (The angle between a pair of subgroups) *Let V be a group, and let E and F be subgroups of V. Define $st(V; E, F)$ and $ln(V; E, F)$ as above. The* angle *between E and F is 0 if $ln(V; E, F)$ is a forest and is $\frac{\pi}{n}$ if the shortest length of a circuit in $ln(V; E, F)$ is $4n$. We denote the angle between E and F by $\theta(V; E, F)$ or just $\theta(V)$ if E and F are obvious (which will usually be the case).*

These definitions are the major ones, but there are a few others that should be mentioned. Suppose X is a complex. If σ is a cell of X, then the *star* of σ is

$$star(\sigma) = star(\sigma, X) = \{\tau : \tau \text{ is a face of a 2-cell containing } \sigma\}$$

and the *link* of σ is

$$link(\sigma) = link(\sigma, X) = \{\tau : \tau \text{ is an element of } star(\sigma) \text{ and } \tau \cap \sigma = \phi\}.$$

If S is a subcomplex of X, then the *stellar neighborhood of S* is

$$nbhd(S) = nbhd(S, X) = \{\tau : \tau \text{ is a face of a 2-cell that contains a vertex of S}\}.$$

By construction, $star(\sigma)$, $link(\sigma)$, and $nbhd(S)$ are subcomplexes of X.

2.2 Building the Complex

We will construct a labeled complex inductively. We will build it level by level, such that $X(0)$ (the first level) is a quadrilateral as in Figure 2.2, and for each nonnegative integer n, $X(n+1) = nbhd(X(n), X(n+1))$ and there is a strong deformation retraction of $X(n+1)$ to $X(n)$. If σ is a face of $X(n)$ we will denote $star(\sigma, X(n))$ by $star(\sigma, n)$ and likewise for the *link*. For $n > 0$, we will denote by $\partial X(n)$ the subcomplex of $X(n)$ of 2-cells that are disjoint from $X(n-1)$. The *interior* of $X(n)$ is $X(n) \backslash \partial X(n)$. Given a vertex v in $X(n)$ we will say that its angle is $\theta(l(v))$ and will call it a $\theta(l(v))$ vertex.

2.2.1 The Axioms

We build the complex by induction on n. We suppose the following inductive hypotheses hold for $X(n)$.

D1) $X(n)$ is a labeled 2-complex. If $n > 0$, then $X(n-1) \subset X(n)$ and $X(n) = nbhd(X(n-1), X(n))$.

D2) For each edge $e \subset X(n)$, there is a label-preserving embedding $\varphi_{e,n} : star(e,n) \to st(l(e))$.

D3) For each edge $e \subset X(n)$ with $n > 0$ and $e \bigcap X(n-1) \neq \phi$, $\varphi_{e,n}$ is an isomorphism and if $e \subset X(n-1)$, then $\varphi_{e,n}|_{star(e,n-1)} = \varphi_{e,n-1}$.

D4) If e is an edge in $\partial X(n)$, then $star(e,n)$ is a face and $\varphi_{e,n}(star(e,n)) = s(1)$.

D5) For each vertex $v \in X(n)$, there is a label-preserving embedding $\varphi_{v,n} : star(v,n) \to st(l(v))$.

D6) If $n > 0$ and v is a vertex in $X(n-1)$, $\varphi_{v,n}$ is an isomorphism and $\varphi_{v,n}|_{star(v,n-1)} = \varphi_{v,n-1}$.

D7) If e is an edge in $X(n)$ with vertex v, then there is an element $g \in l(v)$ with

$$g \circ \psi_{l(e),l(v)} \circ \varphi_{e,n} = \varphi_{v,n}|_{star(e,n)}.$$

D8) If $n > 0$ and v is a vertex in $\partial X(n)$, then $\varphi_{v,n}(star(v,n))$ is one of the following:

1. equal to $s(1)$. That is, it is a unique face. We call this type of star a *corner star* and we call v a *corner star vertex.*

2. $nbhd(\varphi_{v,n}(w))$ for some vertex w in $link(v,n)$. We call w the *central vertex* of $star(v,n)$. Let e be the edge joining w and v in $X(n)$. We call e the *central edge* of $star(v,n)$ and we call v an *edge star vertex.*

The purpose of these axioms is to insure that the "attachings" of the model stars of vertices and edges to the boundary of the complex $X(n)$ behave the way we desire. The embeddings give a way of describing the part of the model star that is already in $X(n)$ and axioms 3 and 6 insure that the "attachings" agree with what is already there. Axiom 7's purpose is to insure that the "attachings" of the model stars of edges and of vertices agree. Axiom 8 gives the combinatorial description of the complex that we need for the desired structures.

We define $X(0)$ to be the complex of faces of a labeled quadrilateral as in Figure 2.2. For each $e \subset X(0)$, we define $\varphi_{e,0} : X(0) \to st(l(e))$ to be the label–preserving embedding with $\varphi_{e,0}(X(0)) = s(1)$ and for each vertex $v \in X(0)$, we define $\varphi_{v,0} : X(0) \to st(l(v))$ be the label-preserving embedding with $\varphi_{v,0} = s(1)$. (Since the embeddings are label-preserving they are unique). The axioms are trivially satisfied for $X(0)$. So let n be a nonnegative integer and suppose $X(n)$ has been constructed and satisfies D1)–D8). We will construct a labeled 2-complex $X(n+1)$ that satisfies D1)–D8).

2.2.2 The Equivalence Relation

We accomplish the "attachings" mentioned above using an equivalence relation. First let $\mathcal{E}$ be the set of edges in $\partial X(n)$ and let $\mathcal{V}$ be the set of vertices in $\partial X(n)$. For each $e \in \mathcal{E}$ (respectively $v \in \mathcal{V}$), let $str(e)$ (respectively $str(v)$) be a copy of $st(l(e))$ (respectively $st(l(v))$) and let $\imath_e : str(e) \to st(l(e))$ (respectively $\imath_v : str(v) \to st(l(v))$) be the label–preserving isomorphism induced by the identity.

Now let

$$Y(n+1) = X(n) \bigcup (\cup\{str(e) : e \in \mathcal{E}\}) \bigcup (\cup\{str(v) : v \in \mathcal{V}\})$$

and let $X(n+1) = Y(n+1)/\sim$, where $\sim$ is the equivalence relation generated by the following:

1. Suppose given $v \in \mathcal{V}$. We identify $star(v,n)$ with a subcomplex of $str(v)$. Let $\psi_v : star(v,n) \to str(v)$ be defined by $\psi_v = \imath_v^{-1} \circ \varphi_{v,n}$. Then $x \sim \psi_v(x)$ for every $x \in star(v,n)$.

2. Suppose given an edge $e \in \mathcal{E}$ which contains a vertex v. We identify $str(e)$ with a subcomplex of $str(v)$. For this we use D7), which gives us a $g \in l(v)$ with $g \circ \psi_{l(e),l(v)} \circ \varphi_{e,n} = \varphi_{v,n}|_{star(e,n)}$. We define $\psi_{e,v} : str(e) \to str(v)$ by $\psi_{e,v} = \imath_v^{-1} \circ g\psi_{l(e),l(v)} \circ \imath_e$. Then $x \sim \psi_{e,v}(x)$ for every $x \in str(e)$.

The first attaches a copy of $st(l(v))$ to $X(n)$ for each v in $\mathcal{V}$ and the second identifies the shared faces of two different model stars.

2.2.3 Showing D1)–D8) Hold for $X(n+1)$

Before we prove any of the axioms, we start with a few lemmas which capture the simple structure of this complex we have built.

Lemma 2.5 *Suppose $v \in \partial X(n)$. Then $str(v)$ is embedded in $X(n+1)$. Also $X(n)$ is embedded in $X(n+1)$.*

Proof: To show this we must investigate how the equivalence relation can relate to points that seem far away in $X(n)$. So we assume we have the following situation.

1. $e_1, e_2, \ldots, e_m \in \mathcal{E}$ and $x_i \in str(e_i)$ for $i = 1, \ldots, m$.

2. $e_i \cap e_{i+1} = v_i \in \mathcal{V}$ for $1 \leq i < m$.

3. $x_i \sim x_{i+1}$ by applying 2 of the equivalence relation to (e_i, v_i) and then to (e_{i+1}, v_i).

4. $v_i \neq v_{i+1}$ for $1 \leq i < m - 1$.

5. The image of x_i in X is not in $int(X(n))$.

What we wish to see is that m can't be too big. Then, we will be assured that the identifications don't match up points that are "far" apart, and the embeddings will follow from local arguments.

Let τ_i be a face in $str(e_i)$ such that $x_i \in \tau_i$. This face is unique unless $x_i \in \imath_{e_i}^{-1}(st(l(e_i)))$ in which case we choose $\tau_i = \imath^{-1}(st(1))$. Let $\hat{x}$ be the vertex in $\partial X(n+1)$ that is contained in all the τ_i's. This vertex exists since all the x_i's are equivalent and we are dealing with complexes and simplicial maps. Essentially, this means we never have to worry about identifications that are not of vertices. Now we choose v_i so that v_i and $\hat{x}$ are not adjacent in $X(n+1)$ (in other words, find an opposite vertex) and suppose that $\psi_{v_i}(v_i) \neq \psi_{e_i,v_i}(x_i)$. If either of these assumptions can not be realized, then $m = 2$ and we have what we want. (To see that the former implies $m = 2$, suppose that $\hat{x}$ is adjacent to v_i for all $1 \leq i \leq m-1$. Then, if $m > 2$ we have three vertices that are adjacent, and this is impossible, for all our faces have four vertices.)

Now, let us look at e_i and e_{i+1}. The images of τ_i and τ_{i+1} intersect in both v_i and $\hat{x}$, which by assumption are distinct and nonadjacent. This means that in $str(v_i)$ the images of τ_i and τ_{i+1} must intersect in at least two distinct vertices which are not adjacent. But $str(v_i)$ is a complex, so this means that the images of τ_i and τ_{i+1} must be the same. This can't be, for the inductive hypothesis D8) holds for $X(n)$.

We would have a circuit with length at most six in $st(l(v_i))$, and this would require $\theta(l(v_i)) > \frac{\pi}{2}$. We conclude, then, that $m = 2$.

Now, we can finish the proof. Because we have found that points can only be identified by a pair of adjacent edges and the vertex they intersect in (that is, $m = 2$), if we have two points $x, y \in X(n)$ or two points $x, y \in str(v)$ for some $v \in \mathcal{V}$ that are identified in $X(n+1)$, then the identification must occur within $St(l(v))$. Since both ψ_v and $\psi_{e,v}$ are embeddings, we are finished. Both $X(n)$ and $str(v)$ embed in $X(n+1)$ for all n and for all $v \in \mathcal{V}$. ■

Corollary 2.6 *Suppose σ and τ are faces in $X(n+1)$ with non-empty intersection, where at least one of the faces is not in $X(n)$. Then there exists $v \in \partial X(n)$ such that $v \in \sigma \cap \tau$.*

Now we have all the information needed, and we can prove that the axioms hold for our complex.

Note that Corollary 2.6 has proved D8).

Proof of D1):

We need to show that $X(n+1)$ is a labeled 2-complex. (All the other attributes are true by construction and the above lemma.) Since the equivalence relation respects the labeling, we need only show it is a 2-complex. To do this we need to prove three things:

1. If e is an edge in $Y(n+1)$ with vertices v_1 and v_2, then v_1 is not equivalent to v_2.
2. If e and e' are edges in $Y(n+1)$, with vertices v_1, v_2 and v_1', v_2' respectively and $v_1 \sim v_1'$ and $v_2 \sim v_2'$, then $e \sim e'$.

3. If σ and τ are faces with vertices v_1, v_2, v_3, v_4, and v_1', v_2', v_3', v_4' respectively, and $v_1 \sim v_1', v_2 \sim v_2', v_3 \sim v_3'$, and $v_4 \sim v_4'$, then $\sigma \sim \tau$.

The first of these is easy, since the labels of v_1 and v_2 are different, and the equivalence relation respects the labeling. So suppose we have the situation of the second.

First, suppose that one of the vertices (wlog, we choose $v_1 \sim v_1'$) is in $X(n)$. If it is in the interior of $X(n)$, the result follows from Lemma 2.6. How? We will use this argument several times in the next few paragraphs, so we try to be as long-winded as possible here, so we can skip it later. There are images of e and e' in $X(n+1)$. We want to see what they can be. We notate the image of any face (in the classical sense of the word) o as $\hat{o}$. Now, we know that $v_1 \sim v_1'$, so this means that $\hat{v_1} = \hat{v_1'}$ of course. But this point is also equivalent to two points in $X(n)$, namely the vertices in the interior of $X(n)$ that correspond to v_1 and v_1'. If two points in $X(n)$ became identified in $X(n+1)$, we know that they must have been the same point to start with by Lemma 2.6. The same argument applies to the image of v_2, which may not be in the interior of $X(n)$ but is certainly in $X(n)$. That is, the second vertex of the image of e and of e' must also be the same in $X(n)$. Now, there is only one edge between these two vertices ($\hat{v_1}$ and $\hat{v_2}$ when thought of in $X(n)$), and this must be $\hat{e}$ and $\hat{e'}$. That is e and e' are equivalent to the same edge, and then the transitivity of the equivalence relation finishes the argument.

Now, suppose that $v = v_1 \sim v_1' \in \partial X(n)$. Then, we look at $str(v)$. There exists edges that e and e' are equivalent to in $str(v)$, so the results follows from the embedding of $str(v)$ in $X(n+1)$ with an argument analogous to the one above for $X(n)$.

Finally, we suppose that the images of e and e' are in $\partial X(n+1)$. Let v and v' be vertices in $\partial X(n)$ so that $e \in str(v)$ and $e' \in str(v')$. (Actually, one of the edges may be an element of $str(e_0)$ for some edge e_0 in $X(n)$ instead of an element of $str(v)$, but in this case we choose a vertex of e_0 and identify the edge with its image in $str(v)$.)

If $v = v'$ the embedding of $str(v)$ proves that the edges must be identified as above, so suppose not. If v and v' are adjacent, then the identification is a simple identification made by the chain v, e_1, v' where e_1 is the edge between v and v'. In this case, since $v_1 \sim v_1'$ and $v_2 \sim v_2'$, $e \sim e'$ because for the vertices to be identified, we must be identifying the image of a single face of $str(e_1)$ with a face in $str(v)$ and a face in $str(v')$, and this identification insists that e and e' be identified. Now, we also need to see what happens if we have a chain of edges and vertices v, e_1, v_0, e_2, v' so that we get the equivalence $v_1 \sim v_1'$ and another that leads to the equivalence $v_2 \sim v_2'$. We know that these chain can only have length two by Lemma 2.5, and this implies that $l(v_1) = l(v_2)$ since the middle vertices of each of the chains must have the same label. But e is supposed to be an edge between v_1 and v_2. So, this can not happen. We have shown that $v = v'$ and 2. follows.

To prove 3., we simply note that one of the vertices of σ and σ' is necessarily in $X(n)$ and, by the above argument, they must both be the same vertex. Then, the results follows from Lemma 2.5.

Proof of part of D2) and D3)

Suppose e is an edge in $X(n+1)$. We want to define $\varphi_{e,n+1}$. If e is in $X(n)$, then we want it to agree with $\varphi_{e,n}$. If e is in the interior of $X(n)$, we let $\varphi_{e,n+1} = \varphi_{e,n}$. If e is in $\partial X(n)$, then D4) gives that $star(e,n)$ is a single face and $\varphi_{e,n}(star(e,n)) = s(1)$. So we define $\varphi_{e,n+1}$ by the attaching. That is $\varphi_{e,n+1}|_{star(e,n)} = \varphi_{e,n}$, and any other face of $star(e, n+1)$ comes from a face of $str(e)$, so we identify those. This finishes D2) and D3) for $e \subset X(n)$. We need D4) and D5) to finish the arguments.

Proof of D6) and part of D5)

Suppose v is a vertex in $X(n+1)$. We want to define $\varphi_{v,n+1}$. As above, we do this for $v \in X(n)$. If $v \in X(n-1)$, then we define $\varphi_{v,n+1} = \varphi_{v,n}$. If $v \in \partial X(n)$, then as above, we define $\varphi_{v,n+1}$ by the attaching, i.e. we know where the $star(v, n)$ goes by the inductive hypothesis, so we map the others by the attaching relations.

Proof of D4)

Suppose $e \subset \partial X(n+1)$ and that $star(e, n+1)$ is not a single face. Then there are at least two faces and Corollary 2.6 says that they must share a vertex v in $X(n)$. But we know D1) holds, so the two faces must be the same.

Proof of D2) and D3)

All we have left is the case where $e \subset X(n+1) \setminus int(X(n))$. If $e \subset \partial X(n)$ then we have already defined $\varphi_{e,n+1}$ above, so suppose this is not the case. If $e \cap X(n) \neq \phi$, then let v be the intersection. We define $\varphi_{e,n+1}$ in terms of $\varphi_{v,n}$. Note that since $str(v)$ is embedded in $X(n+1)$, $star(e, n+1)$ is isomorphic to $str(e)$. Now given $\varphi_{v,n}(star(v, n))$, we can label all of the faces of $star(v, n+1)$ by the action. We put a well-ordering on all the elements of $l(v)$, which we use frequently and therefore make independent of e, and choose the minimal element of $star(e, n+1)$. This element is mapped to $s(1)$ and the others follow from the action.

Now, suppose $e \subset \partial X(n+1)$. Then by construction, $star(e, n+1)$ is a single face. We simply map this to $s(1)$ and we have proved all of D2) and D3).

Proof of D5)

The only case left is if $v \in \partial X(n+1)$. Here, there are two cases. If v is a corner type, then $star(v, n+1)$ is a single face and we map this to $s(1)$. If, however, v

is an edge star vertex, we define $\varphi_{v,n+1}$ in terms of the central edge of v. So for this purpose call that edge e. This is the case we dealt with above, so we have a definition for $\varphi_{e,n+1}$. But $star(v, n+1) = star(e, n+1)$ by D8), so we just define $\varphi_{v,n+1}$ to agree with $\varphi_{e,n+1}$. This finishes D5).

The proof of D7) is simply by definition of all the mappings. So all of the axioms hold, and we have our complex.

2.2.4 The Isomorphism to the Amalgam Complex

First, we need to see that there exists a strong deformation retraction of X to a point. To do this, we proceed by showing that there exists a strong deformation retraction of $X(n+1)$ to $X(n)$.

Suppose we have a face τ of $X(n+1) \setminus int(X(n))$. Then, there are exactly two edges of τ that have a vertex in $\partial X(n+1)$ and a vertex in $\partial X(n)$. We use a strong deformation retraction of the face onto these two edges if τ is a corner face, and we use a strong deformation retraction of the face onto these two edges and the edge which lies in $\partial X(n)$ if it is not a corner face. D4) guarantees that this is well-defined.

This leaves us with $X(n)$ and edges sticking out at some of the vertices. We simply retract these edges down to $X(n)$. This completes the retraction, and we are finished with this argument.

Now, we prove that X is isomorphic to the amalgam complex. We will proceed by defining a function that is a bijection between the faces of X and the faces of the amalgam complex. First, we define a function Φ from the free product F of A, B, C, and D to the set of faces in X. Suppose $g \in F$ and $g = g_1 g_2 g_3 ... g_k$ where g_i and g_{i+1} come from distinct vertex groups (this does not rule out any elements of F, but it insists on a particular form for them) for $i = 1, \ldots, k-1$. Then, we form the following chain of faces in X. We start at $X(0)$, and then we look at the vertex of $X(0)$ that has the label of the group that contains g_1. This allows us to define a face $\tau_1 = g_1 X(0)$ (this is actually an abuse of our language, for we really mean the image of $X(0)$ in the star of the vertex with the label of the group that contains g_1, but hopefully this is clear). Then, once we have τ_i defined, we define $\tau_{i+1} = g_{i+1}\tau_i$ with the multiplication taking place in the vertex group that contains g_{i+1} (which can be easily found by reading the label). We define $\Phi(g) = \tau_k$.

This is a map from F onto X. What we would like is a map from G, the amalgam group to X. So, we simply mod out by N, the kernel of the canonical homomorphism from F to G. We need to see that we still have a well-defined map. So, suppose that $h_1 \sim h_2$ under N. That is, $g = h_1 h_2^{-1} \in N$. If we can show that $\Phi(g) = X(0)$, then we have that h_1 and h_2 map to the same face, and Φ is well-defined.

Then $g = \Pi_{i=1}^{k} w_i r_i w_i^{-1}$, where the r_i are the basic relations given by the vertex groups or the edge identifications, and the w_i are elements of F. If $\Phi(w_i r_i w_i^{-1}) = X(0)$ for all i, then the same is true for g. Moreover, if $\Phi(r_i) = X(0)$, then $\Phi(w_i r_i w_i^{-1}) = X(0)$ by definition of Φ. Now, we have a simple question. Is $\Phi(r_i) = X(0)$ for any relator r_i of G? Certainly. This holds for any of the relators of the vertex groups since in the definition of Φ we would follow around a chain of τ_i's which never left the single vertex whose label contains r_i, and the equivariance of the action on $st(l(v))$ guarantees that this would have to lead to $X(0)$. The same is true for a relator which is an edge identification. So, Φ is well-defined.

Finally, we need to see that Φ is injective. Suppose, $h_1, h_2 \in G$ and $\Phi(h_1) = \Phi(h_2)$. Let h_1' and h_2' be lifts of h_1 and h_2, respectively, in F. Then $\Phi(h_1' h_2'^{-1}) = X(0)$. Hence, to show that Φ is injective, all we need to show is that if $\Phi(g) = X(0)$, then $g \in N$ (this would show that $h_1 \sim h_2$ under N).

For this purpose, let $\Phi(g) = X(0)$. Then, in the definition of Φ, we have a chain of faces $X(0), \tau_1, \ldots, \tau_k, X(0)$ such that each face intersects the previous one in a vertex. In fact, we know that successive vertices of this kind can be chosen to not have the same label since $g = g_1 \ldots g_k$ was made so that successive elements came from different vertex groups. This allows us to define a closed loop, α, starting and ending at a vertex of $X(0)$. Since X has a trivial fundamental group, this path must be null-homotopic.

We would like to have a triangulation of α and its interior. What we have right now is a set of quadrilaterals, rather than triangles, but this is easily fixed. For every face, choose a pair of opposite vertices. Then draw an edge between them. Now, we have a triangulation of the region. Then there exists a triangulation D_1 of a 2-disk in $\mathcal{R}^2$ and a simplicial map $f : D_1 \longrightarrow \bigcap_{n=0}^{\infty} X(n)$ such that α is gotten by restricting f to ∂D_1 oriented in the clockwise direction and starting at an appropriate base point. If β is a circuit in the 1-skeleton of D_1, denote by R_β the closed subset of D_1 bounded by β.

Before we can continue, we need to get rid of degenerate edges of D_1. For this, we choose a degenerate edge, e. This edge can not be in the boundary, so we form the set of all circuits that consist of an edge from a vertex $v_\delta \notin e$ to a vertex of e, e, and an edge from the other vertex of e to v_δ. We call the outermost two (there are precisely two since the edge is not in the boundary) δ_1 and δ_2. We foliate $R_{\delta_i} \setminus v_{\delta_i}$ by arcs such that e is one of the arcs and every other arc joins a point in one edge of $\delta_i \setminus e$ to the point on the other edge of $\delta_i \setminus e$ that has the same image under f. Then, we collapse δ_1 and δ_2 by identifying these arcs to a point. This makes the degenerate edge a point, and guarantees that any triangles that contain this edge are shrunk down to, at least, an edge. If we do this for all edges of D_1, then we have no degenerate edges. Call this new complex D_2, and observe that it is a set of triangles and edges, for which the map f induces a map f_1 which has the same properties on D_2 as f did on D_1.

Now, for each edge in the boundary of D_3, we attach the face τ_i (from the definition of $\Phi(g)$). We also wish to remove our added edges. That is, if the image of an edge under f_1 is one of the edges between opposite vertices in X, we remove it. Now, for each face in the complex we get, we find the barycenter, and we make a graph. We will have an edge between two barycenters if either they are the barycenters of τ_i and τ_{i+1} or if their faces share an edge. We will label these edges by the element of F that takes the image of one face to the other under the action on X.

Finally, we have the graph we want. Notice that the boundary of this graph, when read as a single word, is an element of gN. What we would like is to look at the components of R^2 minus this graph. The unbounded component has the boundary of the graph as its boundary, and as we said this can be interpreted as an element of gN. At the same time, any of the bounded components can be interpreted (in the same way) as an element of N (each one has an image in X that is a cycle around a single vertex). So, in X, the original word g, can be seen to be made up of conjugates of relators, so it itself must be an element of N. This proves that Φ is injective.

Once we have this bijection between the faces of X and the elements of G, it is easy to match up the labels of the faces of X with the cosets of A_G, B_G, C_G, and D_G correctly and make an isomorphism between X and the amalgam complex.

2.3 The Automatic and Biautomatic Structures

In this section, we define a language $\mathcal{L}$ and show that it is a part of a biautomatic structure for a quadrilateral of finite groups with no vertex angle greater than $\frac{\pi}{2}$ (and with trivial face group and non-trivial edge groups).

2.3.1 Preliminary Definitions

If V is a vertex group and E, F are the corresponding edge groups, let $S = \Psi_{E,V}(E) \bigcup \Psi_{F,V}(F)$ and let H be the subgroup of V generated by S. An element g of V has *length* k, denoted $|g| = k$, if $g \in H$ and $|g|_S = k$. Two elements $g, h \in V$ are at *distance* k, denoted $d(g,h) = k$, if $g^{-1}h \in H$ and $|g^{-1}h| = k$. If $g^{-1}h$ is not in H, then $d(g,h) = \infty$. This makes d a metric on V. We also write $d(s(g), s(h)) = k$ and say that $s(g)$ and $s(h)$ are at distance k, getting a metric on the set of faces in $st(V)$. We say that the faces σ and τ are *adjacent* if $d(\sigma, \tau) = 1$. We likewise have an edge path metric on the set of vertices in $ln(V)$ and a notion of adjacency of vertices in $ln(V)$. These metrics will all be denoted by d. Just as the metric on V induces a metric on the set of faces in $st(V)$, the ordering of V induces an ordering of the faces in $st(V)$. A sequence of faces in $st(V)$ is in a *circuit of length* k if the intersections of these faces with $ln(V)$ gives an edge path in a circuit of length k in $ln(V)$.

Let v be a vertex in $X(n)$ for some $n \in \mathcal{N}$. Then as above we have metrics $d_{v,n}$ and $d_{v,n+1}$ on the faces in $star(v,n)$ and $star(v, n+1)$, leading to $star(v,n)$–distances between faces in $star(v,n)$ and $star(v, n+1)$–distances between faces in $star(v, n+1)$. We also have analogous edge path metrics $d_{v,n}$ and $d_{v,n+1}$ on the vertices in $link(v,n)$ and $link(v, n+1)$. The circuit notion also lifts to $star(v, n+1)$ as does the ordering of the faces in $st(V)$. We call the ordering of the faces in

$star(v, n+1)$ the v–ordering of these faces.

2.3.2 Definition of $\mathcal{L}$

Let Σ be the alphabet $(A \bigcup B \bigcup C \bigcup D \bigcup K \bigcup L \bigcup M \bigcup N) \setminus \{1\}$, where if E is an edge group associated with vertex groups U and V, then for each $g \in E$, g is identified with $\Psi_{E,V}(g)$ and $\Psi_{E,U}(g)$. Let $\mathcal{L}$ be the language over Σ defined recursively as follows. The nullstring is in $\mathcal{L}$. Each element of Σ is in $\mathcal{L}$.

Now suppose $w \in \mathcal{L}$ and $|w| = n$. Let $\bar{w}$ denote the image of w in G, and let σ_0 represent $X(0)$. Using the action of G on the complex then, we let $\sigma = \bar{w}\sigma_0$.

We know that $\sigma \cap \partial X(n)$ is either one edge or two. In either case, suppose v is a vertex in this intersection. Let $g \in l(v)$. We want to know if $wg \in \mathcal{L}$. It will be if and only if given τ, a face in $star(v, n)$, the following holds:

1. $d(\sigma, \bar{w}g\sigma_0) \leq d(\tau, \bar{w}g\sigma_0)$,

2. if equality holds in 1., σ is less than τ in the v–ordering of the faces in $star(v, n+1)$.

2.3.3 That the Combing Lines Satisfy the k–fellow Traveler Property

This proof is short and sweet. Since we will prove the bicombing lines also satisfy the k–fellow traveler property, there is no real need to show this. But the proof is quick and enlightening (the geometric structure of the language is shown to be very nice), so we will share it.

Suppose given a face $\sigma \subset X$ whose interior is in $X(n+1) \setminus int(X(n))$ for some $n \in \mathcal{N}$. If $w \in \mathcal{L}$, $|\bar{w}|_\Sigma = n$, $\sigma \cap \bar{w}\sigma_0$ contains a vertex v, $g \in l(v)$, $wg \in \mathcal{L}$ and $\bar{w}g\sigma_0 = \sigma$, then we say that σ *combs back* to $\bar{w}\sigma_0$ and that it *combs through* v (which we will sometimes write as $\tau \xrightarrow{v}$.

Lemma 2.7 *Suppose σ is a face in $X(n+1) \setminus int(X(n))$ such that $v \in \sigma \cap \partial X(n)$ and v is a vertex. Then σ combs through v.*

Proof:

If σ intersects $\partial X(n)$ in more than just v, it must intersect $\partial X(n)$ in an edge. In this case, σ is distance one from a face in $star(v, n)$ and, therefore, σ combs to this face and through v. If σ only intersects in v, then it must comb back to a face in $star(v, n)$ for all other simplices are infinitely distant from σ by the definition of the metric. So, σ must comb through v. ■

This result, though clear, is central to all of the combing arguments to follow. It also fails for a certain case of the triangles, and leads to complications there.

Now we deal with the actual combing lines. For what follows, we assume that σ and τ are adjacent faces in X. What we would like to do is prove that they comb back to adjacent faces.

To this aim, we prove the following Lemma:

Lemma 2.8 *The combing lines of $\mathcal{L}$ have the k-fellow traveler property.*

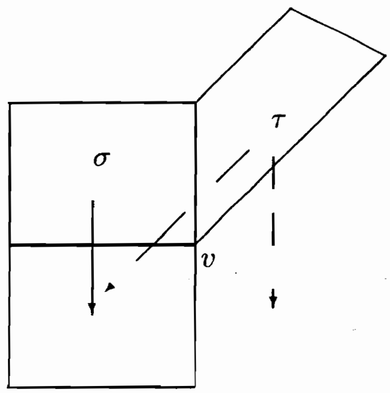


Figure 2.3: An example of Case 1

Case 1 $\sigma, \tau \in X(n+1) \setminus int(X(n))$. See Figure 2.3. Since the faces are adjacent, they must intersect in a vertex. And by Lemma 2.6 then, they must intersect in $\partial X(n)$. But then, Lemma 2.7 says they both comb through this vertex, and therefore, they comb back to adjacent faces.

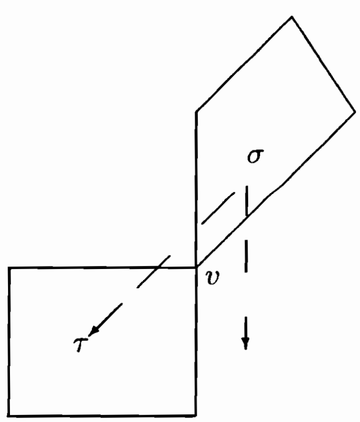


Figure 2.4: An example of Case 2

Case 2 $\sigma \in X(n+1) \setminus int(X(n))$, $\tau \in X(n)$. In this case, we find the two faces in different levels of the complex. See Figure 2.4. We note that if v is the vertex they intersect at, σ must comb through v, and now we have that $\sigma(n)$ and τ are in the configuration of Case 1.

This doesn't finish off the argument entirely. Though we see both cases have their faces combing back to other adjacent faces and therefore we seem to have a bound of 1, Case 2 has a little more to it than that. In fact, the bound is 2 here, because after one application of Case 2, the two combing lines are one step out of synch in terms of the word length. That is, though we are looking at $\sigma(n)$, we are still looking at τ. But the definition of the k–fellow traveler property insists that we look at $\sigma(n)$ and $\tau(n)$ together, i.e. we want to know $d_s(\sigma(n), \tau(n))$. This is where the bound of 2 comes in. (As a matter of fact, it's not hard to see that they are still bounded by 1, but we are trying to imitate the harder bicombing line argument, where the bound does get larger, where the same logic insists that the number is no bigger than 4.)

Note that since Case 1 only leads to Case 1, and Case 2 always leads to Case 1, we have no need to fear an infinite loop of cases. This proves that the combing lines have the k–fellow traveler property. ■

2.3.4 That the Bicombing Lines satisfy the k–fellow Traveler Property

The proof follows along the same line as the combing lines proof, but naturally it is more difficult. In addition to the above line of reasoning, we need to gain some understanding of what multiplication on the right can do to the combing lines, for this is the geometric interpretation of the bicombing lines. We multiply our original quadrilateral, σ_0, on the right, taking it to an adjacent face in X, which we call τ_0, and then we look at the same combing lines emanating out of τ_0. Our goal is to show that the combing lines emanating from both faces stay within a bound.

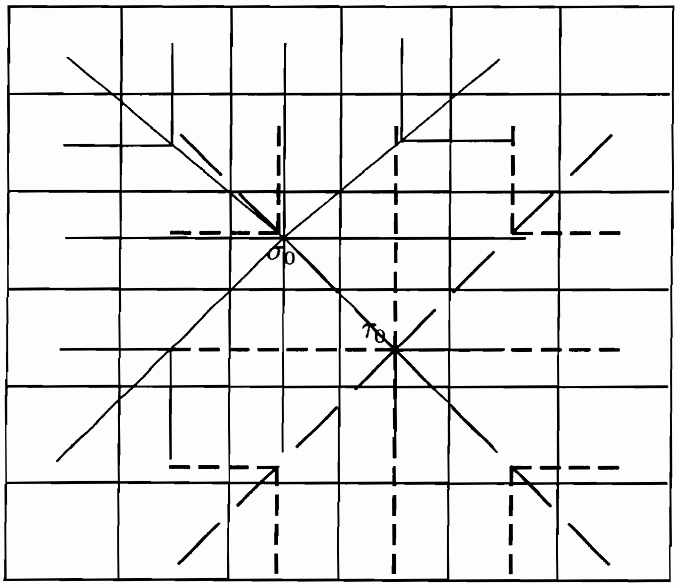


Figure 2.5: An example of a bicombing

As an example, see Figure 2.5. This is the case where all vertex groups are D_4 and the edge groups are all $\mathcal{Z}_2$. X turns out to be the complex pictured. The thin lines represent the combing lines of $\mathcal{L}$ when $\sigma_0 = X(0)$. The dashed lines are for the analogous situation with τ_0. Each set of lines is drawn out to length three in their respective views.

We suppose we have two adjacent faces in X and look at where they come from in terms of the combing lines, i.e., which faces they comb back to; but now we must deal with the possibility that the two combing lines come from two different original faces (σ_0 and τ_0). We should take a minute to be very clear as to what we mean by that. We suppose that we build the complex X inductively, starting with a face $X(0)$. Then we build the complex level by level. The regular language we have defined is dependent on the combinatorial structure of the n-balls of X. But, as the group acts on X we really needn't start our language with σ_0. Any face will do as a beginning–as the image of ϵ. What we do in this section is suppose that we have two distinct sets of combing lines: one set that emanates from σ_0, and another set of combing lines that emanate from a different face, τ_0, which is adjacent to σ_0. Actually, if they both come from the same face, of course, we have proved above that they stay within distance 2 of each other, since this is exactly what our automatic language guarantees. So, we are only interested in the case where the two combing lines are combing from the two different *views*, which is the term we will use to describe where the combing lines arise from. Either the σ_0 view, where the combing lines are drawn with σ_0 as the image of ϵ, or the τ_0 view, where the combing lines are drawn with τ_0 as the image of ϵ. Similarly, X_{σ_0} and X_{τ_0} will describe the two complexes as they are built with σ_0 and τ_0 as $X(0)$ respectively. And $star_{\sigma_0}(v, n)$, $star_{\tau_0}(v, n)$, $\partial X_{\sigma_0}, \partial X_{\tau_0}$ are defined analogously.

But before we can do this effectively, we need to get a handle on how the two views can differ. That is, if v is a vertex in X, how does it appear combinatorially from the viewpoint that $X(0)$ is σ_0, and how does it appear in terms of $X(0)$ being τ_0? For that is the effect of the σ_0 view versus the τ_0 view. For instance, in X_{σ_0} , v may appear as an edge star vertex in $\partial X_{\sigma_0}(n)$, but v may appear as a corner type vertex in $\partial X_{\tau_0}(n-1)$.

A series of quick lemmas follow, all used to prove the last lemma which gives us a powerful description of what possible differences there are in the two views. In fact, there is almost no possibility for change, i.e., if v is a vertex in X then the $star_{\sigma_0}(v, n)$ is either a superset or a subset of $star_{\tau_0}(v, m)$ except in one exceptional case. (In this case, the two *star*'s are not subsets of each other, but their differences are determined completely for our purposes. The case is so well-defined that we can just draw the necessary combing lines and observe.) Note that n is not necessarily equal to m, and which way the containment goes is dependent on the relationship of m and n.

Lemma 2.9 *If σ is a face and $\sigma \in X_{\sigma_0}(n)$, then $\sigma \in X_{\tau_0}(n+1)$. Moreover, if $\sigma \in X_{\sigma_0}(n) \backslash int(X_{\sigma_0}(n-1))$ then $\sigma \in X_{\tau_0}(m) \backslash int(X_{\tau_0}(m-1))$, where $m = n, n-1$, or $n+1$.*

Proof:

That $\sigma \in X_{\sigma_0}(n)$ implies $\sigma = g\sigma_0$ where $|g| \leq n$. Now let x be the generator of G such that $\sigma_0 = x\tau_0$ (note that there is such an x since σ_0 and τ_0 were chosen to be adjacent). Then $\sigma = gx\tau_0$. Since $|gx| \leq n+1$, $\sigma \in X_{\tau_0}(n+1)$. The rest of the lemma then follows simply by turning the letters around, i.e. swapping τ_0 and σ_0 in the first statement of the lemma. ■

The simplicity of this lemma does not obviate its use. In fact, it is central to many of the arguments, even beyond these following lemmas, which use it directly. The first of these is almost identical, but is a little less obvious.

Lemma 2.10 *If v is a vertex and $v \in \partial X_{\sigma_0}(n)$, then $v \in \partial X_{\tau_0}(m)$ where $m = n, n-1$, or $n+1$.*

Proof:

Since $v \in \partial X_{\sigma_0}(n)$, there exists a face $\sigma \in X_{\sigma_0}(n)$, such that $v \in \sigma$. From Lemma 2.9 then, $\sigma \in X_{\tau_0}(n+1)$. This implies $v \in X_{\tau_0}(n+1)$. But it doesn't show that v isn't in $\partial X_{\tau_0}(n-2)$, for though we know σ is not in $X_{\tau_0}(n-2)$, (by Lemma 2.9) some of its vertices may well be. We would just like to see that none of these vertices can possibly be v. To do this we assume the contrary.

Suppose $v \in \partial X_{\tau_0}(n-2)$. Then there exists a face τ, such that $\tau \in X_{\tau_0}(n-2)$ and $v \in \tau$. Now, what happens to this face in X_{σ_0}? Well, $v \in \partial X_{\sigma_0}(n)$, so $\tau \in X_{\sigma_0}(n)$. But this cannot be by Lemma 2.9. So v behaves as we would like. ■

Next, we use these two lemmas to get what we want about the differences in the combinatorics of the two views. Namely, we pick a vertex and inspect what its respective stars can possibly look like in X_{σ_0} and in X_{τ_0}. Naturally, the cases vary depending on whether the vertex appears in different levels in each view or not. Surprisingly perhaps, if a vertex appears at a different level in each of the views, the two stars of the vertex are easily shown to have the property we desire, namely that one is the subset of the other. It is significantly more difficult if the vertex appears at the same level in both views (and, in fact, this is where the exceptional case can occur). So, we will prove it for the easy case.

But first, we make a note about the implications of the two above lemmas. If v is a vertex in X and σ is a face such that $\sigma \in star_{\sigma_0}(v, n)$, then the question arises as to whether σ is in $star_{\tau_0}(v, m)$ for some given m. (And even more if σ is not in $star_{\sigma_0}(v, n)$ can it be in $star_{\tau_0}(v, m)$? But this is really the same question.)

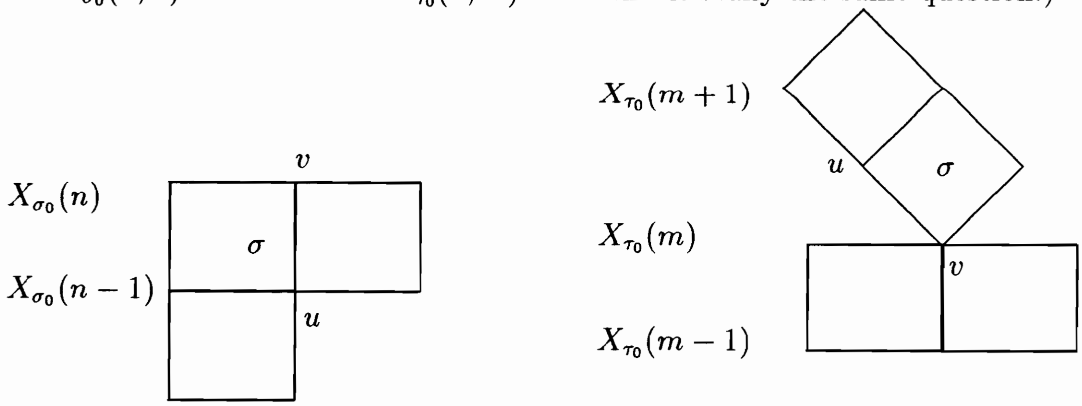


Figure 2.6: σ can't move too much (notice that the two figures are not necessarily at the same height)

The point of the above lemmas is one doesn't need to look very far. If σ doesn't intersect $star_{\tau_0}(v, m)$ other than in v, then it couldn't have been in $star_{\sigma_0}(v, n)$. (See Figure 2.6.) This is because there must be some vertex $u \in \partial X_{\sigma_0}(n-1)$ such that $u \in \sigma$, since $\sigma \in X_{\sigma_0}(n)$. Thus $u \in X_{\tau_0}(n)$. Now, there must be some face from which u came. This face is in $X_{\sigma_0}(n-1) \setminus int(X_{\sigma_0}(n-2))$ and $X_{\tau_0}(m+2) \setminus int(X_{\tau_0}(m+1))$ (the latter because σ only intersects $\partial X_{\tau_0}(m)$ in v). But we know from Lemma 2.9 that a face can not change levels by more than one from view to view. But this is exactly what we have here. So, to discover what type of changes a *star* can have from one view to the other, we need only look at faces which intersect the *star*. No others may be added (and, of course, this also means that no face that is in a particular *star* in a view can fail to intersect that analogous *star* in the other view).

Lemma 2.11 *Suppose v is a vertex and $v \in \partial X_{\sigma_0}(n)$ and $v \in \partial X_{\tau_0}(n+1)$. (In this case, we say the vertex "jumped up".) Then $star_{\sigma_0}(v, n) \subseteq star_{\tau_0}(v, n+1)$.*

Proof:

Let σ be a face such that $\sigma \in star_{\sigma_0}(v, n)$. Then $\sigma \in X_{\sigma_0}(n)$. So $\sigma \in X_{\tau_0}(n+1)$ by Lemma 2.9. Therefore, $\sigma \in star_{\tau_0}(v, n+1)$. ■

Of course the proof is similar if the vertex "jumps down" where this means the analogous thing. As promised, this case was simple. The next lemma is not so clear, and in fact the proof is split up into cases. Before we move on, we should make a note of what it takes for a face to be in $star(v, n)$ for a vertex $v \in \partial X(n)$ (notice the view we are talking about is irrelevant). For this to occur there must be another vertex of the face which appears, combinatorially, in $X(n-1)$. Conversely, if a face is not to be in the star of v, then none of its vertices can appear earlier than v. This is a necessary and sufficient condition for a face to be in the star of v and will be used for all the following without further mention. Each case corresponds to the possible combinatorial structure of $star(v,n)$. There are four possibilities depending on the combinatorics of the adjacent vertices: the corner type and three edge star types (the adjacent vertices are both edge star vertices, or one is an edge star vertex, while the other is a corner vertex, or both adjacent vertices are corner vertices).

Lemma 2.12 *Let v be a vertex such that $v \in \partial X_{\sigma_0}(n)$ and $v \in \partial X_{\tau_0}(n)$. Then either $star_{\sigma_0}(v,n) \subseteq star_{\tau_0}(v,n)$ or $star_{\tau_0}(v,n) \subseteq star_{\sigma_0}(v,n)$ unless two opposite angles are $\frac{\pi}{2}$, in which case a "slide" is possible.*

Proof:

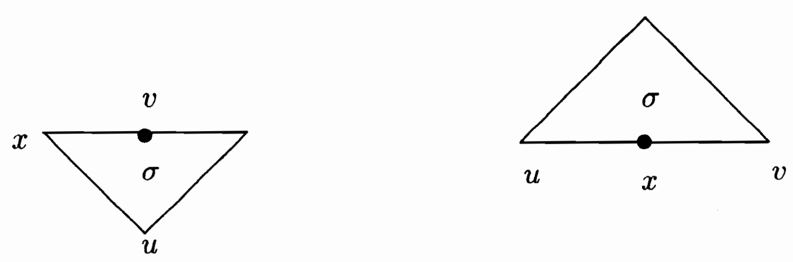


Figure 2.7: How things could go wrong for Case 1–but they can't

Case 1 v is a corner type vertex in the σ_0 view

See Figure 2.7. The only way this could fail to fulfill the lemma is if the unique face of $star_{\sigma_0}(v,n)$ were not in $star_{\tau_0}(v,n)$. But this can't occur. Let σ be this face and let u be the only vertex of σ that is in $\partial X_{\sigma_0}(n-1)$. Then, for σ to not be in $star_{\tau_0}(v,n)$, u must have "jumped up" and by Lemma 2.10 , u must be on the same level as v. But u and v are opposites, so since X is a complex, there must be a vertex that they are both adjacent to that is also in $\partial X_{\tau_0}(n)$. The fourth vertex of σ now violates D8).

Case 2 v is an edge star vertex

This case has various possibilities depending on the combinatorics of the vertices adjacent to v in $\partial X_{\sigma_0}(n)$.

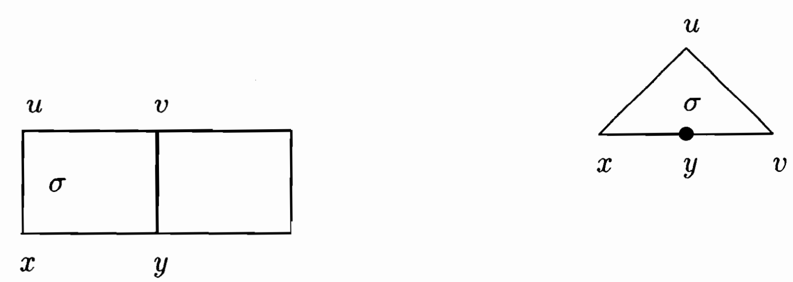


Figure 2.8: How things can go wrong for Case 2a–but they can't

Case 2a The adjacent vertices are all edge star vertices

See Figure 2.8. Again, we wonder if there is a face σ that is in $star_{\sigma_0}(v, n)$ but not in $star_{\tau_0}(v, n)$. But since σ has two vertices in $\partial X_{\sigma_0}(n)$ both of which are edge star vertices, then the other two vertices of σ must be in $X_{\sigma_0}(n-1)$ and therefore must be in $X_{\tau_0}(n)$. This leaves only one vertex to be in $\partial X_{\tau_0}(n+1)$, and again by D8) this can't happen.

Note that this argument says that a face can not jump if it is not the corner face of some vertex.

Case2b There is at least one adjacent corner type vertex

This case is more complex. In fact, we don't get the desired result in a very particular situation. What we show here is that the only way not to get containment one way or the other is if we get the picture of Figure 2.10. (This picture could be a little more complex, with fins poking out in all directions, depending on the orders of the edge groups, but the fundamental structure is as shown.) What that means is that the combing lines can be seen without worrying about the other possibilities.

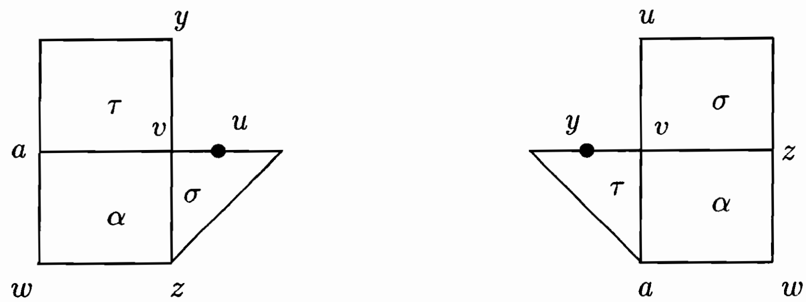


Figure 2.9: How things can go wrong for 2b–leads to sliding

So suppose that containment is held neither way. That is there exists a face σ such that $\sigma \in star_{\sigma_0}(v,n)$, but is not in $star_{\tau_0}(v,n)$ and there exists a face $\tau \in star_{\tau_0}(v,n)$ that is not in $star_{\sigma_0}(v,n)$. Then from the above arguments, σ must be the corner face of some vertex u in X_{σ_0} and τ must be the corner face of some vertex y in X_{τ_0} (see Figure 2.9). Let z be the central vertex of v in $X_{\sigma_0}(n)$ and let a be the central vertex of v in $X_{\tau_0}(n)$. Since all the other vertices adjacent to v in ∂X_{σ_0} are edge star vertices, we know what the faces containing z and a look like. Since σ jumps in X_{τ_0}, z does too, so a and z are the opposite vertices of some face (namely, the unique face that is in the star of v in both views), call it α. That α is unique is a direct consequence of D4). It is the other vertex of α that is interesting. This vertex, w, doesn't jump, but it's star changes just like v's.

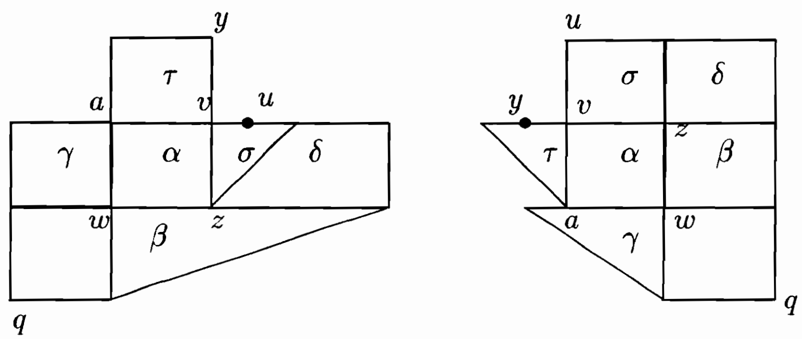


Figure 2.10: The hard case, where faces slide

See Figure 2.10. That is there exists a face that is in $star_{\sigma_0}(w, n-1)$, but is not in $star_{\tau_0}(w, n-1)$ (β) and there exists a face that is in $star_{\tau_0}(w, n-1)$, but is not in $star_{\sigma_0}(w, n-1)$ (γ). Now, this is exactly the situation we began with in this case. So all that we could say for v and its respective *stars* we can say for w. And then this leads to the vertex q, etc. This is the exceptional case and we need some terms here to simplify future arguments (rather than calling it the "exceptional case" repeatedly). We say that the vertices v, w, and q have faces (e.g., in v's case σ and τ) that *slide* (and will frequently say that the vertices themselves slide). It is not hard to see that for w to have faces that slide, the $star(w, n)$ must be as shown (in both views) in Figure 2.10. This guarantees that the face β has two vertices in $\partial X_{\tau_0}(n-1)$ and therefore, since it shares an edge with α, $\theta(l(w)) = \frac{\pi}{2}$ (we have demonstrated a circuit of length 4 no matter what other possible faces the figure is unable to show). The recursion of such vertices leads to the conclusion that this must also hold for $l(v)$. This is all we can say, but then again, this is all we need to say. ■

This finishes the argument, but not in an entirely satisfactory way. What we find is we get what we want about the stars of the vertices in the two different views (that is that one is contained in the other) in all but one case. Still, this isn't too hard to deal with, as this case is quite specific and we can deal with it separately. And that is what we will do.

The proof that the bicombing lines stay a bounded distance now follows. As with the combing lines, we take it case by case with the possible configurations that two faces at distance one from each other can have. We will save the special case of Case 2b above for last, and essentially draw on the Figure 2.10. The rest of the possibilities are fairly straightforward. In the following, we assume that σ and τ are faces in X that are adjacent. It is our goal to show that the bicombing lines have a bounded distance and we approach this problem as the lemmas above lead us to. Namely, we analyze the combing lines from each of the faces. But note, that to analyze the bicombing lines we need to look at the combing lines in the two different views. So, not only is it necessary to describe the situation of the faces, but to say which view the combing line of a face will be calculated in. To this end, we will say *σ combs in σ_0* when we mean that the face σ's combing lines will be calculated in the complex X_{σ_0} and likewise for X_{τ_0}.

Lemma 2.13 *The bicombing lines of $\mathcal{L}$ have the k-fellow traveler property.*

Proof:

We proceed inductively on n. For $X_{\sigma_0}(0)$, of course, the combing lines are trivially bounded (also for $X_{\tau_0}(0)$ but it is sufficient to do the work in the σ_0 view). So, we suppose that we have the combing lines bounded for a pair of adjacent faces in $X_{\sigma_0}(n)$ and look at the case where at least one of the faces is in $X_{\sigma_0}(n+1)$. We break up the proof into two cases depending on whether both faces are in $X_{\sigma_0}(n+1)$ or if only one is.

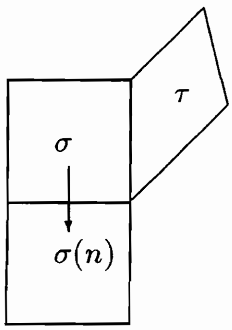


Figure 2.11: An example of Case 1

Case 1 σ **and** τ **are in** $X_{\sigma_0}(n+1) \setminus int(X_{\sigma_0}(n))$

See Figure 2.11. Since the faces are adjacent, Lemma 2.6 says that there exists a vertex v in $\partial X_{\sigma_0}(n)$ such that $v \in \sigma \cap \tau$. Then, wlog, we can say σ combs in σ_0, and by Lemma 2.7, σ combs through v. But then, we already have that $\sigma(n)$ and τ are adjacent, so this case leads to Case 2.

Case 2 $\sigma \in X_{\sigma_0}(n+1) \setminus int(X_{\sigma_0}(n))$ **and** $\tau \in X_{\sigma_0}(n)$

Now, if in X_{τ_0} we have the situation of Case 1 (that is the faces are in the same level of X_{τ_0}) then we are done. Otherwise, look at the combing lines of σ. If σ combs in σ_0, then following the same argument above, we find $\sigma(n)$ and τ are in the situation of Case 1 in X_{σ_0}, so we are done.

Figure 2.12: An example of Case 2 which leads to the sliding case

Finally, assume σ combs in X_{τ_0}. See Figure 2.12. Let v be a vertex in $\partial X_{\sigma_0}(n)$ in the intersection of σ and τ. If σ combs through v, then again, we have Case 1 for $\sigma(n)$ and τ in X_{σ_0}. So this just leaves the case where σ combs in τ_0 and does not comb through v. For this to be though, $\sigma \in star_{\tau_0}(v, m)$. (In this argument, like several others, if we do not know whether v has moved up, down, or stayed at the same level, we will just say that it is $X_{\tau_0}(m) \setminus int(X_{\tau_0}(m-1))$ and make no assumptions about m other than those the various lemmas have given us.) And since we have already taken care of the case where σ and τ are in $star_{\tau_0}(v, m)$ (that is the case where we have Case 1 in X_{τ_0}), we assume τ is not in $star_{\tau_0}(v, m)$. But that means $star(v, m)$ has slid as in Case 2b of Lemma 2.12 and that $n = m$. This is the one extraordinary case and the picture must then be as in Figure 2.13. (Actually, to be precise, the edge groups can be much larger, hence, the picture much more complex, but for the combing lines, these are irrelevant for we know that these faces share an edge with the respective *stars* of v, so the combing lines are simple to calculate.)

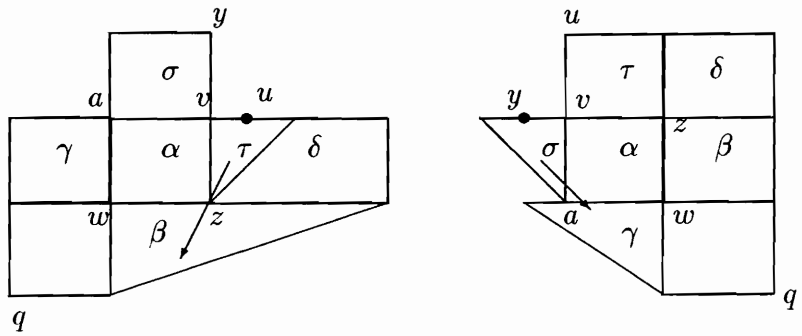


Figure 2.13: Drawing the bicombing lines for this particular case

Here, we see the situation as described in this paragraph. σ and τ must be the faces as shown or they could not do the sliding that is necessary. Then, it is easy to see that σ combs to γ (since γ is a corner star face and is the only one that could produce σ) in X_{τ_0} and τ combs to β (for the same reason) in X_{σ_0}. Finally, we notice that γ and β are now in the same configuration as σ and τ and this case (the difficult one) is recursive.

As with the combing line argument, this is an unsatisfactory ending. We have Case 1 leading to Case 2 which leads to Case 1, etc. This is very disturbing. In particular, we have a fear that with each successive Case 1 configuration, it is possible that we are comparing $\sigma(i)$ with $\tau(j)$ and there is no bound on the difference of i and j. But in fact, this is not a worry. For suppose we have two faces that are adjacent in a particular view (as they are in all of our cases). How far apart can their word lengths be? That is if $\sigma = g\sigma_0$ and $\tau = h\tau_0$ where $|g| = i$ and $|h| = j$, then how large can the difference of i and j be? This is exactly what Lemma 2.9 addresses. Since σ can only have word lengths between $i-1$ and $i+1$ and similarly for τ and j, then we see that if $|i-j| > 2$ then σ and τ can not be adjacent.

So this last worry is not actually a worry, and the lemma is proved. ■

2.3.5 The Word Acceptor Automaton

What we have proved above is not necessarily sufficient to give us an automatic or a biautomatic structure (though, as of yet, no one has proved that it is not). In this section, we show that our language is a regular language and thereby finish the proof of the automatic and biautomatic structures. To show $\mathcal{L}$ is regular we only need show that there is a finite state automaton which accepts it, and that is what we do here.

Theorem 2.14 *A quadrilateral of finite groups with trivial face group and non-trivial edge groups, and with no individual angle more than $\frac{\pi}{2}$ is both automatic and biautomatic.*

Proof:

We build the automaton. Our approach will be that for every face, σ, we will associate a type, $t(\sigma)$, and a state of the automaton, $s(t(\sigma))$. These types will depend on the geometry of the face, and our goal will be to retain just enough information "in" the state to decide what state a generator should lead to.

First, we define the type of the empty word to be $t(\epsilon_\Sigma)$. Now we define a type for each face of X. Suppose σ is a face of $X(n) \setminus int(X(n-1))$. Then σ either intersects $\partial X(n)$ in one edge or in two. Suppose the intersection is in one edge. Then let u and v be the vertices of the edge. We define

$$t(\sigma) = \{\varphi_{v,n}(star(v,n)), \varphi_{v,n}(\sigma), \varphi_{u,n}(star(u,n)), \varphi_{u,n}(\sigma)\}.$$

If σ intersects in two edges then we have three vertices, u,v, and w. Suppose w is the vertex that both edges share. Then define

$$t(\sigma) = \{\varphi_{v,n}(star(v,n)), \varphi_{v,n}(\sigma), \varphi_{u,n}(star(u,n)), \varphi_{u,n}(\sigma), l(w)\}.$$

This information is precisely the information needed to define where a state should lead. To each type, we will associate a state of our automaton. It is easy to see that for any given quadrilateral of groups the number of different possible types as defined above is finite; so then is our automaton. It is interesting to note, though, that the size of the automaton is dependent on the orders of the edge groups, so there is no universal bound on the size of the automaton for all quadrilaterals of groups. We need to see how this set of states can be made to produce our language $\mathcal{L}$. What remains is to define the transition function. This definition will reflect the definition of $\mathcal{L}$, not surprisingly.

Suppose s is a state of the automaton and $g \in \mathcal{L}$. To s is associated one of the two above types or it is the start state. If it is the start state, then we define g to lead to $s(t(g\sigma_0))$. All of our states will be accept states (with the exception of the single fail state that g will lead to if it does not fulfill one of the conditions below), so this reflects the fact that all the generators are in $\mathcal{L}$. Now, suppose s is the first of the above types. We wish to observe $g\sigma$. The action (and therefore the definition of type of $g\sigma$) only makes sense if $g \in l(u)$ or $g \in l(v)$, which, again, agrees with the definition of $\mathcal{L}$ (where u and v are the two vertices described in the description of this type of face). All other generators lead to the fail state. If $g \in l(u)$, then we use the criteria of $\mathcal{L}$. If σ is the closest of any face to $g\sigma$ in the $star(u, n+1)$–metric, then g takes s to an accept state or if σ is equally close to $g\sigma$ as some other face, τ, in $star(u,n)$, then we use the $l(u)$–ordering to decide whether $s(t(\sigma))$ leads to a fail state or to an accept state. That is, if $\sigma < \tau$, g leads to an accept state, otherwise g leads to the fail state. We do the same if $g \in l(v)$. If s is the second of the above types, then we treat u and v just the same, but w is easier. Since this vertex is a corner type vertex, σ combs to all of the faces w produces (that is all faces in $str(w)$) for it is the unique face of $star(w, n)$ by D8). So each element of $l(w)$ other than the identity leads from s to an accept state.

That this is a well-defined process will be clarified below. But it should be clear that if this can be done, then this automaton will produce $\mathcal{L}$ as this process is identical to the process in the definition of $\mathcal{L}$. So let us discuss how this can be done, and more precisely, define where a generator takes s, i.e., what type $s(t(g\sigma))$ is associated with.

Suppose we have a state $s(t(\sigma))$. And also, let us assume that this type is the first one described above. Let $g \in l(u)$. We look at $\varphi_{u,n}(star(u, n))$ and $\varphi_{u,n}(\sigma)$. This is some of the information that $s(t(\sigma))$ contains. We know that $g\varphi_{u,n}(\sigma) = \varphi_{u,n+1}(g\sigma)$ by definition, so whatever happens in $st(l(u))$ is geometrically the same as what happens in X. More precisely, if σ is at a minimal distance from $g\sigma$ in $star(u, n+1)$, than the respective thing is also true of $\varphi_{u,n}(\sigma)$ and $\varphi_{u,n}(g\sigma)$. So, the above description of deciding when g leads to an accept state is well-defined. But more than that, we can now define the type of $g\sigma$. This face must intersect the set $H_{\varphi_{u,n}(star(u,n))} = \{\phi : \phi \text{ is a face of } st(l(u)) \text{ that does not intersect } \varphi_{u,n}(star(u, n))\}$ (which will become part of $\partial X(n + 1)$) in one or two edges and we define its type as we did above for these two possibilities. All we have to do is decide what the maps look like for $g\sigma$. (Of course, it is sufficient to prove that one could decide without actually finding the maps, but since the proof can actually be constructive, we choose to do it that way.)

Suppose (for simplicity) that $g\varphi_{u,n}(\sigma)$ intersects $H_{\varphi_{u,n}(star(u,n))}$ in one edge, and call this edge's vertices x and y. Then we have all the information we need to define $\varphi_{x,n+1}(star(x,n+1))$ and likewise for y. To see this, we recall how $\varphi_{x,n+1}$ is defined. First, we need to find the vertex in $\varphi_{u,n}(star(u,n))$ that shares an edge with x. (That there is such a vertex is due to our assumption that $g\varphi_{u,n}(\sigma)$ intersects the boundary in one edge, but if this were not the case the definition would be simple, for a vertex for which there is no such vertex is a corner type vertex, and then all we need to know is the vertex's label to define the type.) It is how this vertex maps its neighborhood that tells us how to calculate $\varphi_{x,n+1}$. But what is this vertex? Either it is u or it is v. Just the two vertices we know everything (in terms of maps) about. See Figure 2.14.

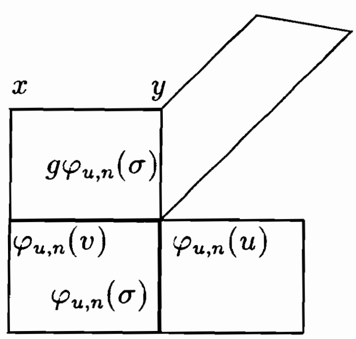


Figure 2.14: Calculating the map $\varphi_{x,n+1}$

Notice that in fact, we do not necessarily know what $star(x,n+1)$ looks like, for some of the faces of this subcomplex may not even be in $st(l(u))$ (or to be more precise, $star(x,n+1)$ may include faces that are not the image of $st(l(u))$). But we don't need to know what they are to know where they are mapped to, and the one face we do need to know about for the type definitions, $g\varphi_{u,n}(\sigma)$, is in $st(l(u))$. We figure out these things by using the injections we already know about. Notice that $g \in l(u)$ and $g \in l(v)$ (again this is by the assumption that $g\varphi_{u,n}(\sigma)$ hits the boundary in a single edge, but that is the only case that gives us this difficulty; for all the others only $\varphi_{u,n}$ is needed to determine the maps for x and y). To be more accurate, $g \in l(u)$ and the injection between $l(u)$ and $l(v)$ gives an element g' of $l(v)$ that is identified with g. Thus, we look at $\varphi_{v,n}(star(v,n))$ (which again, is exactly the information kept in the states), and using g' (and being careful to make any other necessary identification given by the injection), we do the same in $st(l(v))$. In this model star, we are now guaranteed to be able to treat x (again, this is not x but the vertex identified with x as given by the maps) as we can any vertex that is adjacent to the central vertex of the model star.

To find $\varphi_{y,n+1}(star_{y,n+1}(y,n+1))$, etc., we do similarly. For the second type, the one with the corner type vertex, as we noted above, there is little new to do, for the maps we need to understand are always defined easily, as there is only one face in the star of such a vertex and that is mapped to $s(1)$ in its model star.

This finishes the definition of the transition function and therefore completes the proof. ■

2.3.6 The Growth Function

In this section, we discuss the growth function of the amalgam group G of a quadrilateral of finite groups with trivial face group, and no particular angle more than $\frac{\pi}{2}$. Let the orders of the groups A, B, C, D, K, L, M, N be denoted by a, b, c, d, k, l, m, n, respectively.

The growth function of a group (G,Σ), where Σ is a finite generating set for G, is the analytic function $\sum_{n=0}^{n=\infty} a_n z^n$, where a_n is the number of elements of G with word length n in the generating set Σ.

Theorem 2.15 *Let G be the amalgam group of a quadrilateral of finite groups with the restrictions above. The growth function f is the rational function $f(z) = 1 + zV_0(I - zQ)^{-1}V_1$, where*

$$V_0 = \begin{bmatrix} \frac{k}{2} & \frac{n}{2} & \frac{k}{2} & \frac{l}{2} & \frac{n}{2} & \frac{m}{2} & \frac{l}{2} & \frac{m}{2} & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$V_1^t = \begin{bmatrix} \frac{a}{k} - 1 & 0 & 0 & \frac{b}{l} - 1 & \frac{d}{n} - 1 & \frac{d}{m} - 1 & 0 & 0 & 0 & 0 & b - k - l + 1 & a - n - k + 1 \end{bmatrix}$$
$$+$$
$$\begin{bmatrix} 0 & \frac{a}{n} - 1 & \frac{b}{k} - 1 & 0 & 0 & 0 & \frac{c}{l} - 1 & \frac{c}{m} - 1 & c - l - (m-1) & d - m - (n-1) & 0 & 0 \end{bmatrix},$$

and

$$Q = \begin{bmatrix}
0 & 0 & 0 & 0 & \frac{a}{k} - n & 0 & 0 & 0 & \frac{a}{k} - 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{a}{n} - 1 & 0 & 0 & 0 & \frac{a}{n} - 1 & 0 & 0 & 0 \\
\frac{b}{k} - 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{b}{l} - k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{d}{n} - 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{d}{m} - 1 & 0 \\
0 & 0 & 0 & \frac{c}{l} - 1 & 0 & \frac{c}{l} - m & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{c}{m} - l & 0 & \frac{c}{m} - 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c - lm & 0 & c - lm & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d - m - n + 1 & 0 \\
b - kl & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a - nk & 0 & 0 & 0 & a - n - k + 1 & 0 & 0 & 0
\end{bmatrix}$$
$$+$$
$$\begin{bmatrix}
0 & 0 & \frac{a}{k} - 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{a}{n} - k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{b}{k} - l & 0 & 0 & \frac{b}{k} - 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{b}{l} - 1 & 0 & 0 & \frac{b}{l} - 1 & 0 & 0 \\
0 & \frac{d}{n} - 1 & 0 & 0 & 0 & 0 & 0 & \frac{d}{n} - m & 0 & 0 & 0 & 0 \\
0 & \frac{d}{m} - n & 0 & 0 & 0 & 0 & 0 & \frac{d}{m} - 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{c}{l} - 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{c}{m} - 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c - l - (m-1) \\
0 & d - mn & 0 & 0 & 0 & 0 & 0 & d - mn & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & b - kl & 0 & 0 & b - k - (l-1) & 0 & 0 \\
0 & 0 & a - kn & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

Proof:

The proof is fairly straight-forward. We define types for each vertex of the amalgam complex X, and then use these types to count the faces. We find that there are basically only two types, corresponding to edge star vertices and corner vertices. The information we store in the types tells us what type each "produced" vertex is. Then, all we need is a method for each of the vertex types to count the number of faces in its neighborhood. That is what the above matrices give. V_1 is the initial state of types, Q is the transition matrix, that is it tells us how to go from one type of vertex to the next, and V_0 is the method of counting.

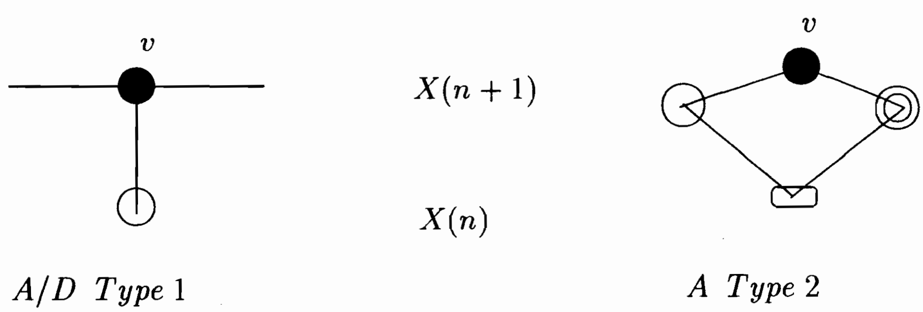


Figure 2.15: The types a vertex v can be

First, we need to define the types. Let v be a vertex in X. Then, we will define the type of v by looking at the combinatorics of X. Suppose $v \in \partial X(n+1)$. Then if v shares an edge with a vertex in $X(n)$, then we say it is of type 1 (it corresponds to an edge star vertex). If there is no such vertex (that is, v is a corner type vertex), then we say it is of type 2. But this is not quite enough. We also need to know the labels of both v and the vertex in $X(n)$ that is adjacent to v, if there exists such a vertex. We shall describe them as a B/A *Type* 1 *vertex* if $l(v) = B$ and the label of the vertex adjacent to v is A, e.g. This gives rise to twelve different types of vertices in X.

$t_1 : B/A$ Type 1 vertex

$t_2 : D/A$ Type 1 vertex

$t_3 : A/B$ Type 1 vertex

$t_4 : C/B$ Type 1 vertex

$t_5 : A/D$ Type 1 vertex

$t_6 : C/D$ Type 1 vertex

$t_7 : B/C$ Type 1 vertex

$t_8 : D/C$ Type 1 vertex

$t_9 : A$ Type 2 vertex

$t_{10} : B$ Type 2 vertex

$t_{11} : C$ Type 2 vertex

$t_{12} : D$ Type 2 vertex

See Figure 2.15.

Note that there is no need to label type 2 vertices with anything other than the label of v for we have uniquely determined the opposite vertex just by labels, and since v is a corner vertex, we know all the combinatorics just from this.

Calculating V_1, the initial types array, is quick. Suppose we have $X(0)$. Let us take the vertex labeled A as an example. The only types this vertex could "produce" are t_1, t_2, and t_{12}. How many t_1's does this vertex "produce"? First, the question is, how many B vertices are there in $st(A)$? The answer, as is seen by the definition, is $\frac{a}{k}$, the number of cosets of the image of K in A (recall that K is the edge group shared by A and B). But we don't want to count all of these. That is, we get into the question of what it means for a vertex to "produce" another vertex.

What we would like is to count each face once and only once. We can't quite accomplish that, but we will successfully count each face exactly twice. That is why V_0, the counting array, has the division by twos in it. So, we need to choose which vertices a vertex produces. The idea is that we shouldn't be redundant. Naturally, then, if v is in $\partial X(n)$, we don't want to count any other vertices in $X(n)$. We also don't want to count a vertex that another vertex in $X(n)$ may be producing.

To this aim, let us return to the question of the initial state array. There is a vertex labeled B in $X(0)$, and we don't want to count this vertex. This is our way of making sure we don't count vertices that appear at the same level (or one below even) as being "produced". We will say that our A vertex "produces" all the other B vertices, for a total of $\frac{a}{k} - 1$ vertices of type t_1 at the initial level. The same argument is applied to the t_2 vertices.

And finally, we need to look at the type 2 vertices. How many of them are produced by our A-vertex? First, we again start with how many possible vertices of this type there are in $st(A)$. We can see that any vertex with label C has no edge in common with the A vertex, so any of them is a viable possibility. How many vertices labeled thusly are there? There is one for each face, so there are exactly A such vertices. But again, we need to rule out some of these. First, we notice that one such vertex is already in X(0). We certainly don't want to count that one. But this time, there are more we don't want to "produce". In particular, if there is a vertex labeled C that is adjacent to a vertex in $X(0)$ (either the B vertex or the D vertex), we don't want to count it either. This is because the vertex that it is adjacent to will "produce" it, and they will get more information than the A vertex can.

So we need to rule out all such vertices. These are determined by the edge groups. That is, if a C vertex is adjacent to a vertex in $X(0)$, it must come from a face that is identified with an element of an edge group (otherwise it couldn't be adjacent to a vertex in $X(0)$). So, for each element of the edge group, we must rule out a vertex as being produced by the A vertex. The total, then, of all t_{12} vertices is $a - k - (l - 1)$. (We don't want to count the vertex in $X(0)$ twice, which is why the -1 is in the equation.)

Doing the same for each of the four vertices in $X(0)$, we find V_1. But in fact, we find more. We note that a vertex in $X(0)$ is really a representative of Type 2 vertices. We have actually calculated how many type 1 and type 2 vertices (and of which variety) any type 2 vertex produces. So all we have left to do is to describe the case of a type 1 vertex.

As above, the particular type 1 vertex is not important. Any of the first eight types act the same, with changes in the labels made carefully. The only difference now, is that we lose a few more vertices.

Suppose we have a type 1 vertex. Then it is an edge-star vertex. Let us call the vertex v and its central vertex u. Let's say, for simplicity's sake, that $l(v) = B$ and $l(u) = A$. Then we need to see how many vertices v produces and what type they are.

First, let us look at the type 2 vertices. These are the vertices labeled D. Again, there are a total of b vertices with this label in $st(B)$. But this time, we have quite a few that we don't want to count. First, there are all the one's in u's level (let's say $X(n-1)$). There is one for each element of the edge group K (the label that the edge from u to v has). In addition, as before, we don't want to count the opposite vertices that are adjacent to a vertex in v's level ($X(n)$). And, again, these vertices come from elements of the edge group, this time the edge group L (this is the edge from a B vertex (v) to a D vertex). Thus, the total of all t_{11} vertices that v produces is $b - kl$.

Now, we look at the type 1 vertices. This time, there is a different case for each of the other two labels. The label A is associated with the central vertex of v, so u is the only A-vertex to appear in $X(n)$. This is then, the only A-vertex that v does not produce. Therefore there are a total of $\frac{b}{k} - 1$ t_3-vertices. The label C, on the other hand appears many times in $X(n)$ adjacent to v. We need to rule out each of these. So there are a total of $\frac{b}{l} - k$ t_4—vertices produced by v.

Again, we use a similar argument for each of the type 1 vertex types. When we are all finished we find the matrix Q as given in the statement of the theorem.

All that is left is the method of counting faces. To do this, we rely on the types. For type 1 vertices, we count $\frac{1}{2}$ of each face. For type 2 vertices, we count none at all. The reason for this is that each face intersects the boundary of $X(n)$ either in two vertices or three. If it is three, then one vertex is a type 2 vertex. This one does not count the face. The other two each count the face $\frac{1}{2}$, so the face gets counted exactly once.

Now the argument is finished. V_0V_1 gives the number of faces in $X(1)\setminus int(X(0))$, or in other terms, a_1. QV_1 gives the number of vertices, with respect to type, in $X(1)$. So multiplying V_0QV_1 we get the number of faces in $X(2)$ that are not in $X(1)$. This is simply a_2. The analogous argument proves that $a_n = V_0Q^nV_1$.

We now also have the growth function as given in the statement of the theorem. ■

There are a few notes we would like to make about this theorem. First, we remark that the theorem is independent of any property of the groups except their orders. That is, given a quadrilateral of groups as above, the order of the groups are sufficient to determine the growth function. This seems a little surprising, for one might expect that the particular relations of the individual vertex groups or, even more, the injections of the edge groups into the vertex groups might be needed to calculate the growth function. As we show in the next section, the geometry of two such quadrilaterals of groups need not be alike, even though we now see their growth must be.

Another point we would like to make is of the simplification of the calculations if there is more symmetry involved. Suppose that all the vertex groups are the same and all the edge groups are the same. (Again we need not look at the injections.) Then, we have no need for 12 types. Since all of the vertex groups are the same, Type 1 and Type 2 are the only types needed. And as the edge groups are all the same, we will find the calculations much easier. Namely, Q will be a 2x2 matrix, and V_0 and V_1 will be 2x1 vectors. Obviously, finding the growth functions for these groups requires a minimum of work.

Both of these points are relevant to our final section.

2.3.7 An Interesting Pair of Examples

In this section, we discuss a pair of quadrilaterals of finite groups, and explore some curious geometric differences in their amalgam complexes. For this purpose, we will build these two amalgam groups with the same vertex groups and the same edge groups. The only difference will be the respective injections. As noted above, then, these groups both have the same growth function (and, of course, both are biautomatic).

We let $A = B = C = D = A_4$, the alternating group on four letters, and we let $K = L = M = N = \mathcal{Z}_3$. This is not enough to uniquely define the amalgam group (or complex). To differentiate, we need to define the injections. To this end, we need names for the elements of the groups.

Let κ, λ, μ, and η be the generators of K, L, M, and N, respectively. For A_4 we use the presentation

$$A_4 = \{a, b : a^3 = b^3 = abab = 1\}.$$

Then, let a_1, b_1 be the generators for A, a_2, b_2 be the generators for B, a_3, b_3 be the generators for D, and a_4, b_4 be the generators for C. For future reference, see Figure 2.16 which is a drawing of the link of $st(A_4)$. Note, also, that from this figure we can see that the angle of each of the vertex groups is $\frac{\pi}{2}$.

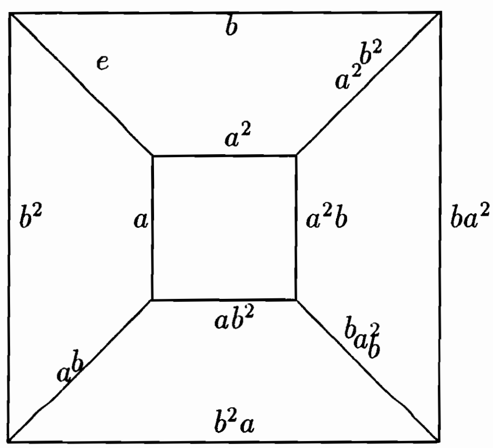


Figure 2.16: The link of A_4

Now we define our two amalgam groups. Let G be the amalgam group of the quadrilateral of groups defined with the above vertex groups and edge groups, and with

1. $\Psi_{K,A}(\kappa) = b_1, \Psi_{K,B}(\kappa) = a_2,$

2. $\Psi_{L,B}(\lambda) = b_2, \Psi_{L,C}(\lambda) = a_4$,

3. $\Psi_{M,C}(\mu) = b_4, \Psi_{M,D}(\mu) = a_3$, and

4. $\Psi_{N,D}(\eta) = b_3, \Psi_{N,A}(\eta) = a_1$.

The amalgam group G' will be defined by the same quadrilateral of groups, except that one of the monomorphisms in equation (4) will be changed, namely, $\Psi_{N,A}(\eta) = a_1^2$. This is the only change that will be made, but it is a dramatic one.

Our goal is to show that G is not negatively curved, while G' is. To show that G' is negatively curved, we use a theorem of Bridson. To wit, we show that G' does not have a flat plane.

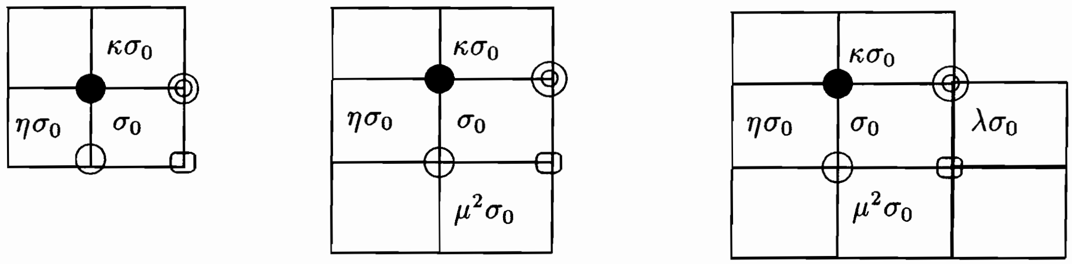


Figure 2.17: Successive steps in attempting to build a flat plane for G'

First, since G' acts transitively on the faces of X, if there is a flat plane, there is one that contains σ_0. In addition, it must contain either $\kappa\sigma_0$ or $\kappa^2\sigma_0$ since these are the only faces that intersect σ_0 in the edge labeled K. Wlog, let us say that the flat plane contains $\kappa\sigma_0$. But then, we notice that there is only one circuit of length four in $st(A)$ that contains both $s(\sigma_0)$ and $s(\kappa\sigma_0)$. This is the circuit which contains $\eta\sigma_0$. Notice that in the injections for G this circuit would not contain $\eta\sigma_0$ but $\eta^2\sigma_0$ (for in G, $\eta = a_1$ and in G', $\eta = a_1^2$). At the same time, we are forced to choose the faces as in Figure 2.17, for the same reasons (i.e., $\eta\sigma_0$ and σ_0 have a unique circuit of length 4 in $st(C)$). But as we approach the final side, we find that, because of our definition of the injection, there is no way to complete the plane (from A_4, we see that κ and λ^2 are in a circuit, but we also need λ to be in the circuit with μ^2). In fact, there can be no flat plane, and G' is negatively curved by [1]

In the definition of G, we see that all we have done is successfully avoided the above problem, and we do get a flat plane. We find that the elements $\lambda^2\eta$ and $\mu^{-1}\kappa$ (remember that $\mu^{-1} = \mu^2$) generate a subgroup isomorphic to $\mathcal{Z}$x$\mathcal{Z}$. See Figure 2.18

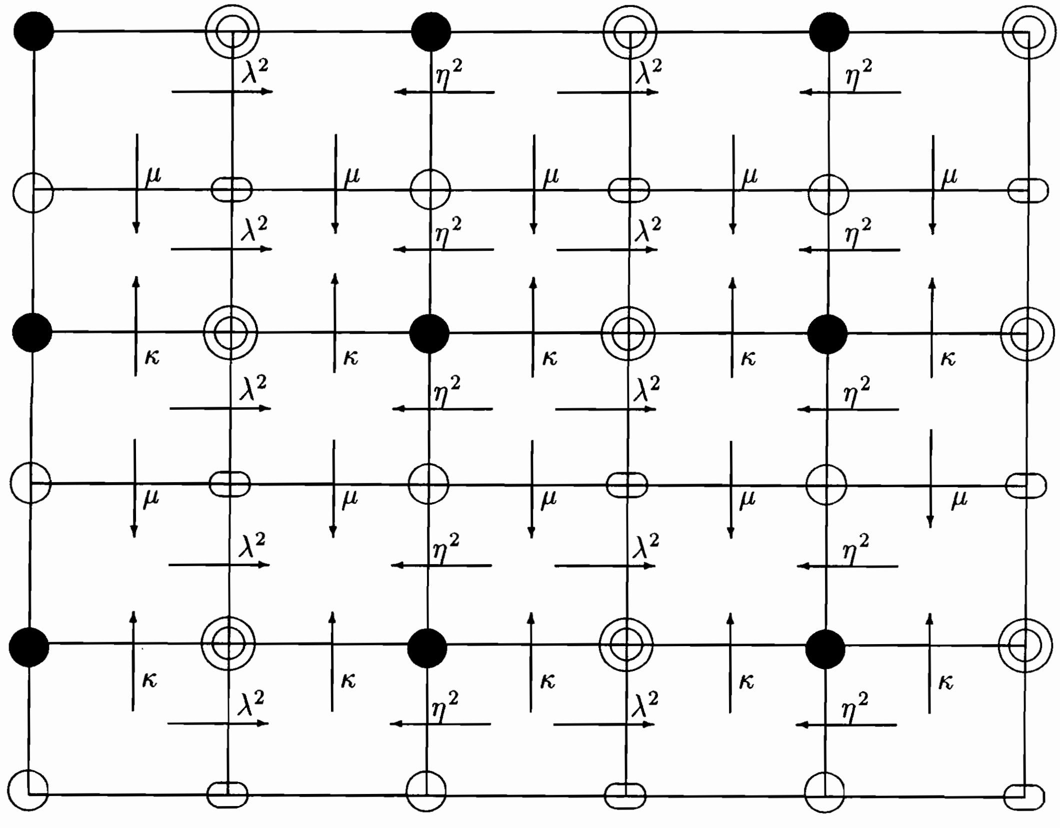


Figure 2.18: A flat plane in X for the group G

This last argument can be seen another way. It turns out that the group G is the commutator subgroup of the Picard group=$PSL(2, \mathcal{Z}[i])$, where $\mathcal{Z}[i]$ is the ring of Gaussian integers. As described by [2], we work with the elements

$$\eta = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \mu = \begin{bmatrix} 0 & i \\ i & 1 \end{bmatrix}, \lambda = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \kappa = \begin{bmatrix} 1 & i \\ i & 0 \end{bmatrix}.$$

Now, it is a straightforward calculation to show that we get the desired $\mathcal{Z}$x$\mathcal{Z}$.

This finishes the argument. These are two significantly different groups which are defined nearly identically and grow at the same rate. It turns out that the word acceptor automatons for these two groups, though the same size, are not the same. Using minimization arguments of finite state automata [4], we find that the two automata (each with 41 states), are not isomorphic.

2.3.8 Conclusion

We have successfully proved all of our theorems for quadrilaterals of groups with trivial face group, non-trivial edge groups, angle sum less than or equal to 2π, and no individual angle greater than $\frac{\pi}{2}$. We have defined a unique, prefix closed, regular language which obeys the k-fellow traveler property proving that these groups are automatic. In addition, we have shown that this same language and its inverse language together satisfy the k-fellow traveler property, thereby showing that the these groups are biautomatic. Moreover, we have shown how to construct the word acceptor automaton for this language. In addition, we have calculated the growth function for these groups, and have given an easily computable formula for any such group. Finally, we gave two examples of these groups that seem similar by definition, but geometrically are vastly different.

Next, we try to show that the automatic structure defined in [6] for triangles of groups is actually a biautomatic structure.

Chapter 3

Triangles of Finite Groups

Two theorems will be proved in this chapter: the first for the significantly easier problem of a triangle of groups with no angle $\frac{\pi}{2}$. The second is much longer and far more technical, but it is modeled on the first proof. For each proof, a few Lemmas will be proved to handle what possibilities a single vertex has in the two different "views" (e.g. an edge star vertex in σ_0 and a double edge star vertex in τ_0, where the boundary level the vertex appears in is the same, say). (The term will be used frequently and it means what is implied above. In investigating the bicombing lines, we need to observe combing lines when X is generated from two distinct simplices (in the sense that each one could be thought of as $X(0)$ and then the rest of the complex is built from there). The two views, then, are the way of describing the two different ways of creating the complex X and, more importantly, of identifying the combing lines built in each of these complexes. They are, of course, not necessarily related and it is the goal of this proof to show that they are, in fact, not too different.) Then, with a few technical Lemmas about what restrictions there are on combing lines, in the second proof, the proof of the biautomatic structure in each case is achieved with a case by case examination of the respective distances of the combing lines. The first three Lemmas here will apply to both proofs and are simple, but are the central facts used in the proofs. Lemma 3.12 is specific to the simpler proof and has its analogue in Lemma 3.14, where the more difficult proof truly begins.

Before we begin, a few comments should be made. The figures in the following proof are drawn with certain conventions which should be made clear. First, the page is split in two parts down the middle, where figures in the left half are drawn from the σ_0 view and figures in the right half are drawn from the τ_0 view (these terms will be defined precisely later–of course, I should just put this comment then, I know). Though, in many cases, the τ_0 view is not shown at all. The σ_0 view is drawn, with certain hypotheses that determine the figure (or determine the figure up to a point, and then the figure is drawn in the worst case scenario from there), and the τ_0 view is not drawn at all. In these case, this is done because the argument is entirely about what can happen in the τ_0 view. The argument will go through many possibilities (too many to bother drawing) that the τ_0 view can look like, and deal with them individually.

Secondly, unless mentioned, the heights of the simplices and vertices in the figures should not be assumed to be in scale. That is, even if the vertex v appears to be at the same height in both views, it is not necessarily. A convention that will go throughout this paper is that a simplex's height when seen from the σ_0 view will be denoted with n and a simplex's height when seen from the τ_0 view will be denoted with m. Frequently, these letters will be used without any other mention, and this means that there is no known relationship between n and m.

Another comment should be made about the simplicity of the figures. In most of the figures, there are no fins (that is simplices in the complex that are not in a plane with all the other simplices) drawn. But in the arguments, we do actually deal with them. They are left out of the figures most of the time, because most of the time they don't actually matter. So, enter the figures with caution. Hopefully, they will be a help, but one does need to be careful not to assume too much.

3.1 Preliminary Lemmas and Definitions

Lemma 3.1 *If $\sigma \in X_{\sigma_0}(n) \setminus int(X_{\sigma_0}(n-1))$, then $\sigma \in X_{\tau_0}(m) \setminus int(X_{\tau_0}(m-1))$, where m=n-1, n, or n+1.*

Proof:

$\sigma = g\sigma_0$ for some $g \in \mathcal{L}$ (the automatic structure defined by Floyd and Parry) where $|g| = n$. Now $\sigma_0 = x\tau_0$ for some generator x of the amalgam complex (so $|x| = 1$). This means $\sigma = gx\tau_0$, and $\sigma \in X_{\tau_0}(n+1)$. Switching the roles of σ_0 and τ_0 in this argument shows $\sigma \notin X_{\tau_0}(n-2)$. ■

Lemma 3.2 *If $v \in \partial X_{\sigma_0}(n)$, then $v \in \partial X_{\tau_0}(m)$, where m=n-1, n, or n+1.*

Proof:

Since $v \in \partial X_{\sigma_0}(n)$, there exists σ such that $v \in \sigma$ and $\sigma \in X_{\sigma_0}(n) \setminus int(X_{\sigma_0}(n-1))$. By Lemma 3.1 then, $\sigma \in X_{\tau_0}(n+1)$, so $v \in X_{\tau_0}(n+1)$. That $v \notin X_{\tau_0}(n-2)$ follows by switching the roles of σ_0 and τ_0 in the above argument. ■

Lemma 3.3 *If $v \in \partial X_{\sigma_0}(n)$ and $v \in \partial x_{\tau_0}(n+1)$, then $star_{\sigma_0}(v,n) \subseteq star_{\tau_0}(v,n+1)$. (And, of course, if v appears later in σ_0 the inclusion reverses.)*

Proof:

Suppose $\sigma \in star_{\sigma_0}(v,n)$. Then $\sigma \in X_{\sigma_0}(n)$, so by Lemma 3.1, $\sigma \in X_{\tau_0}(n+1)$. Since $v \in \partial X_{\tau_0}(n+1), \sigma \in star_{\tau_0}(v,n+1)$. ■

As is probably obvious by now, we will try to identify the "level" a vertex appears in, and in which view. Why? As the previous lemma shows, if a vertex appears in two different levels, then the *star* of that vertex can not change much. This is exactly what we want. For then, the combing lines can be calculated in both views simultaneously by simple arguments. Unfortunately, all is not so rosy if a vertex appears at the same level in both views, as we shall see below. Perhaps, it would be best to more precisely explain what "appears at a level" means. We mean the first n for which a vertex or a simplex is a member of $X(n)$. If we mean a particular view, we will make it clear. So we make a few definitions to clarify these points.

Definition 3.4 (A vertex v moves up, down, or stays in τ_0) *Suppose v is a vertex in $\partial X_{\sigma_0}(n)$. Then we say that v* goes up *(respectively* goes down *or* stays*) if v is an element of $\partial X_{\tau_0}(n+1)$ (respectively $\partial X_{\tau_0}(n-1)$ or $\partial X_{\tau_0}(n)$).*

Note that Lemma 3.2 says we have exhausted the possibilities for v in τ_0. Of course, we can say the same for v moving up or down or staying in σ_0.

Definition 3.5 (v gains or loses in τ_0) *We say that v* gains one *in τ_0 if there exists a simplex, σ, such that $d_g(\sigma, star_{\sigma_0}(v,n)) = 1$ and $\sigma \in star_{\tau_0}(v,m)$. We will also say that v gains σ if we want to talk about a particular simplex. v* gains two *in τ_0 if there exists a simplex, τ, such that $d_g(\tau, star_{\sigma_0}(v,n)) = 2$ and $\tau \in star_{\tau_0}(v,m)$. The definition for* lose one or two *and the definitions for gaining or losing in σ_0 are analogous.*

We will prove that these are the only possibilities for change for the star of a vertex in one view with respect to the other.

Definition 3.6 (v gains on the z side in τ_0) *Suppose v and z are adjacent vertices in $\partial X_{\sigma_0}(n)$. We say that v* gains on the z side *if there exists a simplex that v gains in τ_0 and this simplex is no more than 2 away from $star_{\sigma_0}(\vec{vz}, n)$.*

Definition 3.7 (v is at most a *blank* star vertex in σ_0)

Suppose $v \in \partial X_{\sigma_0}(n)$. Then we say v is at most *an edge star vertex (or any of the other possibilities) if v can not be any of the other possibilities where the diam[link$_{\sigma_0}$ (v, n)] is greater than 2 (and analogously for the others).*

Definition 3.8 (σ combs to τ) *Suppose σ is a simplex in $X_{\sigma_0}(n)$. Then there is a unique simplex, τ, that produced σ in terms of the combing lines. This is because the automatic language defined in [6] is unique. We say that σ* combs to *[probably should say "combs back to"] τ.*

Definition 3.9 (σ combs through v) *Suppose σ is a simplex containing v in $X_{\sigma_0}(n)$. Let τ be the simplex that σ combs to. If $v \in \tau$, then we say that σ* combs through *v.*

Lemma 3.10 (The non-cyclic reasoning Lemma) *If $\sigma \in X_{\sigma_0}(n)$ and $\tau \in X_{\sigma_0}(n+1)$, where $\sigma \hookrightarrow^{\sigma_0}$ and $\tau \hookrightarrow^{\tau_0}$ and $d_s(\sigma, \tau) \leq K$ (a positive constant), then $d_s(\sigma, \tau(n)) \leq K + 2$.*

Proof:

Since $|\tau|_{\sigma_0} = n + 1$, $n \leq |\tau|_{\tau_0} \leq n + 2$ by Lemma 3.1. Now we know (at worst) $d_s(\sigma, \tau(n+2)) \leq K$, so $d_s(\sigma, \tau(n)) \leq K + 2$. ∎

Note: If $\sigma \in X_{\sigma_0}(n+1)$ and $\tau \in X_{\sigma_0}(n)$ we have the same condition, i.e., $d_s(\sigma, \tau(n)) \leq K+2$. Now in the configuration arguments all the configurations have one of these three conditions, i.e. $\sigma \in X_{\sigma_0}(n \text{ } or \text{ } n+1)$ and $\tau \in X_{\sigma_0}(n \text{ } or \text{ } n+1)$, and we have that $K \leq 2$. Therefore the configurations give for $\sigma, \tau \in L$, $d_s(\sigma(i), \tau(i)) \leq 4$. (Here, we must be careful, for $\tau \in L$, while in all other above statements τ may not have been in L, for it had a generator multiplied on the left. But in having $\tau \hookrightarrow^{\tau_0}$ we are approaching the question equivalently.)

The following lemma guarantees that the stars in each of the views can't be "too" different. That is, if one star contains a simplex then that simplex can't be more than two away from the star in the other view. Furthermore, the simplex can only be one away if it is not a phantom edge simplex. This allows succeeding lemmas to ignore simplices that are "far" away.

Lemma 3.11 *If $v \in \partial X_{\sigma_0}(n)$ and $v \in \partial X_{\tau_0}(m)$, and if $\tau \in star_{\tau_0}(v,n)$, then $d_g(\tau, star_{\sigma_0}(v,m)) \leq 2$. Moreover, if $n = m$, then the distance equals one unless τ is a phantom edge simplex in σ_0 (either inner or outer).*

Proof:

Suppose $n \neq m$ and $d_g(\tau, star_{\sigma_0}(v,n)) \geq 3$. Then look at a simplex adjacent to τ that is distance two from $star_{\sigma_0}(v,n)$. This simplex has two vertices in $\partial X_{\sigma_0}(n+1)$, and for v to gain τ these two vertices must have gone down one in τ_0. But what about a simplex (there is at least one) in the model star of this edge? It must have gone down two. This is impossible by Lemma 3.1.

Now suppose $n = m$ and suppose τ is not a phantom edge simplex in σ_0 and $\tau \in star_{\tau_0}(v,n)$ but $\tau \notin star_{\sigma_0}(v,n)$ and $d_g(\tau, star_{\sigma_0}(v,n)) > 1$. τ does not share an edge with any simplex in $star_{\sigma_0}(v,n)$, so there exists $u, w \in \tau$ such that $u, w \in \partial X_{\sigma_0}(n+1)$. Since $\tau \in star_{\tau_0}(v,n)$ and $v \in \partial X_{\tau_0}(n)$, either u or w is an element of $\partial X_{\tau_0}(n-1)$. But this can't be by Lemma 3.2. If τ is a phantom edge simplex, then the result follows by definition. ■

The following Lemma and Theorem prove that a triangle of groups where no angle is greater than $\frac{\pi}{3}$ is biautomatic.

3.2 The $\frac{\pi}{3}$ Case

3.2.1 Star Possibilities

Lemma 3.12 *If $v \in \partial X_{\sigma_0}(n)$ and $v \in \partial X_{\tau_0}(n)$, then $star_{\sigma_0}(v,n) \subseteq star_{\tau_0}(v,n)$ or $star_{\tau_0}(v,n) \subseteq star_{\sigma_0}(v,n)$, unless v is a 2-simplex star vertex in both, and v slides one.*

Proof:

For this proof, we simply break it up into the possible vertex types that v may be in σ_0 and investigate the possibilities in τ_0. But before we do this, recall that since there are no $\frac{\pi}{2}$vertices, there can be no phantom edges, no double edge star vertices, and no triple edge star vertices. This makes the above lemma even simpler. Of course, there are no phantom edges to worry about, but even more, it is impossible to gain two, since even an edge star vertex becomes a double edge star vertex that way.

Figure 3.1: v is an edge star vertex in σ_0

Case 1 Suppose v is an edge star vertex in σ_0

See Figure 3.1. Suppose there exists $\alpha \in star_{\sigma_0}(v,n)$ but $\alpha \notin star_{\tau_0}(v,n)$. By Lemma 3.2, the central vertex of v (call it y) in σ_0, which is in α, must be an element of $\partial X_{\tau_0}(n)$ (it has gone up at least one since α has and at most one by Lemma 3.2). Therefore, there exists a 2-simplex, β, that is in $star_{\sigma_0}(v,n) \cap star_{\tau_0}(v,n)$. Let w be the vertex of β that is in $X_{\tau_0}(n-1)$. Now look at the two Stars of w: $\beta \in star_{\sigma_0}(w,n)$ and $\beta \notin star_{\tau_0}(w,n-1)$. But $d_g(\beta, star_{\tau_0}(w,n-1)) > 1$, since $y \in \partial X_{\tau_0}(n)$ and $y \in \beta$. But then Lemma 3.11 (actually the comment at the beginning of this proof about Lemma 3.11) says this can not occur. So this case is finished.

Figure 3.2: v is a 2-simplex star vertex which slides

Case 2 Suppose v is a 2-simplex star vertex in σ_0

The conclusion of the theorem is fulfilled unless v is a 2-simplex star vertex in τ_0 as well and v slides (or the star would be the same in both views). See Figure 3.2. Let β be the central 2-simplex of v in σ_0 and let γ be the central 2-simplex of v in τ_0. Let y be the vertex of β that is in $\partial X_{\sigma_0}(n-1)$ and also is in $\partial x_{\tau_0}(n)$ (i.e., the vertex which moves up a level). Now, y has gained β in τ_0, so it must be a 2-simplex star vertex in τ_0, by Lemma 3.3. If z is the third vertex of β then, in τ_0, z must also be a 2-simplex star vertex or else $l(z)$ has an odd cycle. This implies that $\theta(l(z)) = \frac{\pi}{3}$ (it has a cycle of length six). Repeating this argument with γ (and with y's counterpart w) we find that z must be a 2-simplex star vertex in σ_0 as well. In addition, we find z is the same type of vertex as v (in that it slides), only one level lower. That $l(v) \neq l(z)$ implies that this recursion insists that for this case to occur, all the angles must be $\frac{\pi}{3}$'s (the next vertex in this recursion is a and it is the third type, different from both $l(v)$ and $l(z)$). Now this case does occur in this very special case, but this structure we have discovered is sufficient to control the combing lines. Note the Lemma is now proved and with additional information. ■

Now with all of these Lemmas finally in place, we can actually prove the simpler theorem.

3.2.2 The Biautomatic Structure for the $\frac{\pi}{3}$ Case

Theorem 3.13 *If G is the amalgam complex for a triangle of groups with non-empty edge groups, trivial face group, angle sum less than or equal to π, and no angle greater than $\frac{\pi}{3}$, then $(G,\mathcal{L})$ is biautomatic structure for G (where $\mathcal{L}$ is the regular language defined in [6].*

Proof:

The proof follows from [6] which says that to prove that an automatic structure is biautomatic, one only needs to look at the bicombing lines. Now Floyd and Parry have already shown that the structure we are looking at is automatic, so we need only look at the bicombing lines.

Goal: Show that for $\sigma_0 = x\tau_0$, where x is a generator of G, and for $g, h \in \mathcal{L}$ that if $d_s(g\sigma_0, h\tau_0) = 1$, then $d_s(g(i)\sigma_0, h(i)\tau_0)) < K$, for some constant K, and for all

$i \leq max(|g|, |h|)$.

To accomplish this goal, we look at the possible configurations where two simplices, σ and τ intersect in a vertex (that is in the simplicial metric where their distance is one).

Configuration 1 $\sigma, \tau \in X_{\sigma_0}(n+1) \setminus int(X_{\sigma_0}(n))$ and there exists $v \in \sigma \cap \tau$ with $v \in \partial X_{\sigma_0}(n)$

One of σ or τ combs in σ_0, so say σ does. Then $d_s(\sigma(n-1), \tau) = 1$ since σ combs through v and $v \in \tau$. So this leads to Configuration 3. (The non-cyclic reasoning Lemma exists for the purpose of insuring that this incestuous logic doesn't harm us.)

Configuration 2 $\sigma, \tau \in X_{\sigma_0}(n+1) \setminus int(X_{\sigma_0}(n))$ and $v = \sigma \cap \tau \in \partial X_{\sigma_0}(n+1)$

Now, in this case, $\sigma, \tau \in star_{\sigma_0}(v, n+1)$, and they do not share an edge, so v must be a 2-simplex star vertex in σ_0, and neither σ nor τ is this central 2-simplex of v. Once again, wlog, let σ comb in σ_0. See Figure 3.3. Let y and x be the other two vertices of the central 2-simplex of v in σ_0. Let $\alpha = star_{\sigma_0}(\vec{xy}, n)$. Then by Lemma 3.2 of Floyd and Parry, $\sigma \overset{\sigma_0}{\rightarrow} \alpha$ (and if this is the case, we are done) unless y is a $\frac{\pi}{3}$ 2-simplex star vertex and σ is in a 6-cycle of y's in σ_0. Also, for this to occur, $l(x) < l(y)$ must be true so that $l(x) > Q$, where Q is the vertex label different from $l(x)$ and $l(y)$. In this case, we must look at where τ combs. If it goes to α then all is well: the distance between $\sigma(n)$ and $\tau(n)$ is 1.

Now the only way $\tau \overset{\tau_0}{\not\rightarrow} \alpha$ is if x loses α in τ_0 (if not $d_g(\tau, \alpha) = 2$ and $l(x) < l(y)$), so x must be a 2-simplex star vertex in τ_0 for this to occur. But then, y must move up one and gain α in τ_0, but this is impossible, for the

Star of y is as large as it can be in σ_0 (and Lemma 3.3 says y must keep of all its σ_0 star in τ_0.) So, we have shown that $\tau \stackrel{\tau_0}{\rightarrow} \alpha$, so the combing lines stay within one. This configuration is recursive.

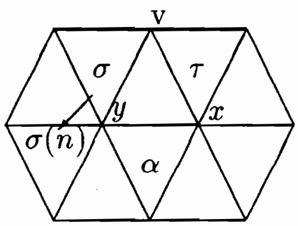


Figure 3.3: Configuration 2

Configuration 3 $\sigma \in X_{\sigma_0}(n+1) \setminus int(X_{\sigma_0}(n))$ and $\tau \in X_{\sigma_0}(n) \setminus int(X_{\sigma_0}(n-1))$

Now, in these cases (which follow) the only occurrence that is not taken care of by the first two configurations (where σ_0 and τ_0 could be interchanged), is if σ and τ are in different levels in both views. If both σ and τ are in their same respective levels in both views (i.e., they don't slide), then, wlog, say σ combs in σ_0. Then σ combs through v and we have Configuration 1 or 2. So suppose σ and τ switch levels. That is, suppose v is a 2-simplex star vertex in both views and that it slides. Then we have the very determined conditions in Case 2 of Lemma 3.12. Now note that if $\sigma \stackrel{\sigma_0}{\rightarrow}$ then again $\sigma(n)$ and τ are in Configuration 1 or 2. So suppose $\sigma \stackrel{\tau_0}{\rightarrow}$. See Figure 3.2 (with $\sigma = \delta$ and $\tau = \alpha$), the figure for Case 2 of Lemma 3.12. Now for τ and σ to be in different levels in both views, τ must be a simplex which shares an edge with β and σ must be a simplex which shares an edge with γ. But then Lemma 3.2 of Floyd and Parry, says that $\tau \stackrel{\sigma_0}{\to} \phi_1$ and $\sigma \stackrel{\tau_0}{\to} \phi_4$. And $d_s(\sigma(n), \tau(n-1)) = 1$. In fact, this is precisely the case. So this configuration is also recursive.

So, we have shown that all the configurations of distance one are recursive, and in fact, except for word length differences (the non-cyclic reasoning lemma only guarantees the word length difference is less than 2), the constant K described in the theorem is 1. So the proof is finished and the furthest two combing lines can get apart is 3. ■

3.3 The $\frac{\pi}{2}$ Case

3.3.1 Star Possibilities

As before, we start with a lemma about what possible changes there are from one view to the other.

Lemma 3.14 *If $v \in \partial X_{\tau_0}(n)$ and $v \in \partial X_{\sigma_0}(n)$, then $star_{\tau_0}(v,n) \subseteq star_{\sigma_0}(v,n)$ or the inclusion is reversed, unless v is a $\frac{\pi}{2}$-edge star vertex in both views. In this case, v can slide one.*

Proof: The proof is simply a case by case study of the possible vertex-star types: edge star vertices, 2-simplex star vertices, double edge star vertices, and the two types of triple edge star vertices. In each case, we suppose v is a certain type in X_{σ_0} and observe what possibilities v has in X_{τ_0}. The possibilities listed at the beginning of each case (e.g. v is an edge star vertex), are all the possibilities remaining after applying the various lemmas. In particular, notice that v can only lose or gain simplices in τ_0 that are at most distance two away from the boundary in σ_0 and if v loses or gains a simplex with its distance to the boundary equal to two, then the simplex must be an outer phantom edge simplex (this is the point of Lemma 3.11).

Case 1 Suppose v is an edge star vertex in σ_0

Possibilities: v can slide one or two or gain two and lose one

(a) Suppose v slides one

Say w is the central vertex of v in σ_0. Then since w gains two in τ_0, w is either a double edge star or a triple edge star in τ_0. This breaks up into three cases (one double edge star case and two different types of triple edge stars).

i. Say w is not an edge star in σ_0.

Then w is a triple edge star in τ_0.

A. Suppose that $diam[link_{\tau_0}(w, n+1)] = 6$

Then w is a double edge star in σ_0. See Figure 3.4. Let x be the central vertex of v in τ_0. Let a be the central vertex of w in τ_0. We know that $\Theta(l(a)) = \frac{\pi}{3}$, so v must be a $\frac{\pi}{2}$vertex. Then, since x gains two in σ_0, $\Theta(x) = \frac{\pi}{3}$ and $\Theta(w) \leq \frac{\pi}{6}$. The figure is now determined by repeated applications of C9) and C10). Not much else can be said, but that the link of e in σ_0 and the link of c in τ_0 have diameter at most 4. And most remarkably, that v slides implies that z, d, and g also do. So we have a recursive structure. This can happen.

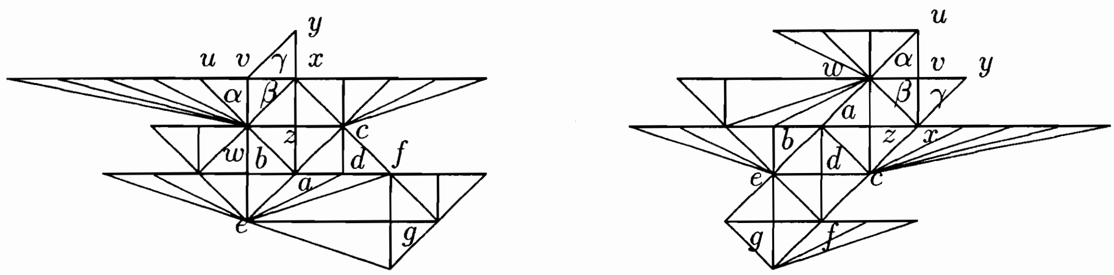


Figure 3.4: $diam[link_{\tau_0}(w, n+1)] = 6$

B. Suppose $diam[link_{\tau_0}(w, n+1)] = 5$

Then w is a 2-simplex star in σ_0 and the angles are as above. The diagram is similar. This also happens. See 3.5.

Figure 3.5: $diam[link_{\tau_0}(w, n+1)] = 5$

ii. Say w is a edge star vertex in σ_0

Again, the situation is determined (see 3.6.) w and x both gain two (w in τ_0 and x in σ_0), so $\theta(v) = \theta(z) = \frac{\pi}{2}$. If either w or x has angle $\frac{\pi}{3}$ then we have Case 1A.i, so suppose there are no $\frac{\pi}{3}$'s. This can happen.

Figure 3.6: w is a edge star vertex in σ_0

These three cases actually happen, but just like in the earlier proof where we have a case of sliding, the structure is so determined that proving that the combing lines are bounded is just a matter of actually drawing them. There are no options in any of these arguments.

(b) Suppose v slides two

For a vertex to gain two (at the same level), one of the adjacent vertices must be a $\frac{\pi}{2}$-2 simplex star vertex, by Lemma 3.11. Let z be such a vertex (since v is an edge star z must be a 2 simplex star vertex in both views). Let y be the central vertex of v in σ_0 and let ϕ_1 be the central simplex of z in σ_0. Then $\phi_1 \in star_{\tau_0}(y, n)$ but is not in $star_{\sigma_0}(y, n-1)$, so y gains in τ_0, and can't be an edge star. But then C9) says that $\theta(y) \leq \frac{\pi}{6}$ and that $\theta(v) = \frac{\pi}{3}$. Now let a be the other vertex of ϕ_1. There are now two cases depending on whether or not $\vec{ya}$ is a phantom edge in σ_0.

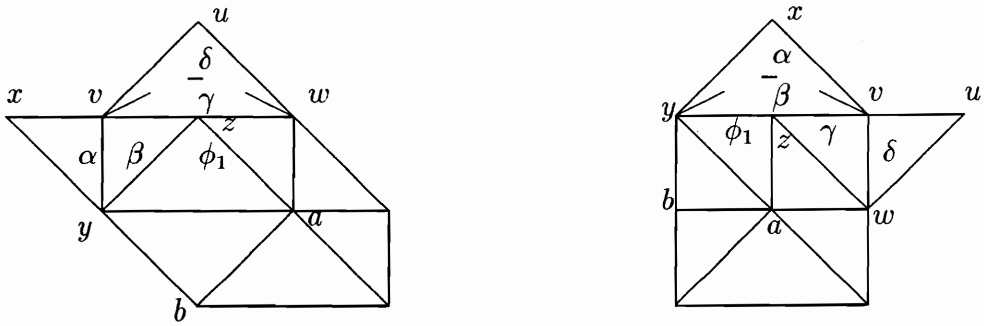


Figure 3.7: $\vec{ya}$ is not a phantom edge

i. $\vec{ya}$ is not a phantom edge in σ_0.

See Figure 3.7. Let w be the third vertex (a and z being the first two) of the central simplex of z in τ_0. As y did in τ_0, w gains in σ_0 so a must be a 2-simplex star in both views (for the diameter of its links in both views must be three or five, and, since a is a $\frac{\pi}{3}$vertex, by C8) it can't be five in either). Let b be a $\frac{\pi}{2}$ vertex adjacent to a in τ_0. Notice that b must go down in σ_0 since w goes up. Therefore, b gains $St(\vec{ba}, m-1)$ in τ_0, so it must be a 2-simplex star vertex there. But C9) then says that a must be an edge star in τ_0. This is a contradiction, so this case does not occur.

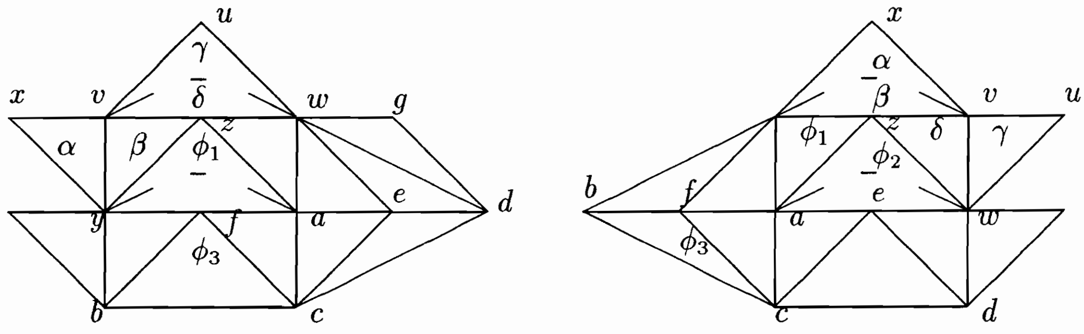


Figure 3.8: $\vec{ya}$ is a phantom edge

ii. Suppose $\vec{ya}$ is a phantom edge

See Figure 3.8. Let ϕ_2 be the central simplex of z in τ_0 and let w be the vertex other than z and a of ϕ_2. Now suppose $\vec{aw}$ is not a phantom edge in τ_0. Then again, a is a 2-simplex star in τ_0 and f is also, so C9) says this can not occur. So suppose $\vec{aw}$ is a phantom edge. Notice $\phi_3 \in star_{\tau_0}(b, n-1)$ but is not it $star_{\sigma_0}(b, n-2)$, so b is not an edge star in τ_0. But now $\vec{cd}$ must be a phantom edge in τ_0 and c must be a triple edge star in τ_0 or else the diameter of the link of $st(l(c))$ is less than 12. But C9) says this can not occur (triple edge stars cannot be adjacent to phantom edges). So this case can not occur.

(c) Suppose v gains two and loses one

As before, for v to gain two it must have a $\frac{\pi}{2}$-2 simplex vertex adjacent to it. Call this vertex u. Let y be the central vertex of v in σ_0 and let β be the simplex that contains v, u, and y. Then $\beta \in star_{\sigma_0}(u, n)$ but is not in $star_{\tau_0}(u, n-1)$. But β intersects $\partial X_\sigma(n-1)$ only in

u, so $d_g(\beta, star_{\tau_0}(u, n-1)) > 1$. That is, u lost two in τ_0, but it was only a 2-simplex star vertex in σ_0, so this is a contradiction and this case does not occur.

This finishes Case 1. The only possible slides are when $\theta(v) = \frac{\pi}{2}$ and in this case v only slides one.

Case 2 v is a 2-simplex star in σ_0

Possibilities: v slides one or two or gains two and loses one (the other possibilities have already been covered in Case 1

(a) v slides one

Let u and w be vertices adjacent to v in σ_0 with different labels, let β be the central simplex of v in σ_0, let γ be the central simplex of v in τ_0 and let x and y be the other vertices of β and w the other vertex of γ. This case breaks up into four cases depending on the angle of $l(v)$.

Figure 3.9: $\theta(l(v)) = \frac{\pi}{2}$

i. $\theta(l(v)) = \frac{\pi}{2}$

See Figure 3.9. Since x gains at least one (β, definitely) in τ_0, C9) says $\theta(l(x)) \leq \frac{\pi}{6}$ and $\theta(l(u)) = \frac{\pi}{3}$ and that u is an edge star vertex in both views. So u shifts and Case 1 insists that

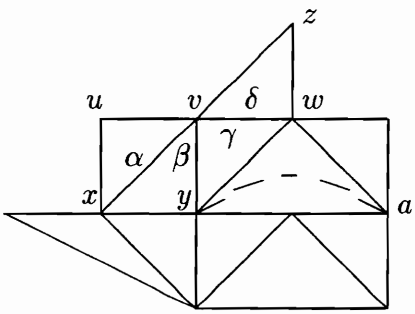
$\theta(l(u)) = \frac{\pi}{2}$. This is not the case by assumption. So this doesn't occur.

Figure 3.10: $\theta(l(v)) = \frac{\pi}{3}$

ii. $\theta(l(v)) = \frac{\pi}{3}$

See Figure 3.10. x gains in τ_0, so again C9) says that $\theta(l(x)) \neq \frac{\pi}{2}$ for then v could not be a 2-simplex star vertex in τ_0. Therefore $\theta(l(y)) = \frac{\pi}{2}$. So $\vec{yw}$ is not a phantom edge in τ_0. Let $\phi = star_{\tau_0}(\vec{yw}, n)$ and let a be the other vertex of ϕ. Since $\vec{yw}$ is not a phantom edge, $a \in \partial X_{\tau_0}(n-1)$ and therefore is in $\partial X_{\sigma_0}(n)$ by Lemma 3.1. Now for the link of y to make sense, then, y must be a 2-simplex star vertex in σ_0. But then y can't be an edge star vertex in τ_0 by Case 1(c), and can't be a 2-simplex star vertex by Case 2(a)(i). So this can not occur either.

iii. $\theta(l(v)) \leq \frac{\pi}{6}$ and $\theta(B) = \frac{\pi}{3}$



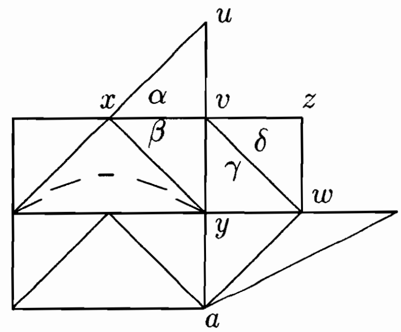


Figure 3.11: $\theta(l(v)) \leq \frac{\pi}{6}, \theta(B) = \frac{\pi}{3}$, and $\theta(l(x)) = \frac{\pi}{2}$

A. $\theta(l(x)) = \frac{\pi}{2}$

x gains one in τ_0, so it is an edge star vertex in σ_0 and a 2-simplex star vertex in τ_0. This is true of w also, mutatis mutandi. Let a be the third vertex of the central simplex of w in σ_0. If $\vec{ya}$ is not a phantom edge in σ_0, then (again by the diameter of the link) y must be a 2-simplex star vertex in σ_0. Now x goes up in τ_0, so either y slides or loses one and gains two. But the former can't occur by Case 2(a)(ii) and the latter can't by Case 1(c). So suppose $\vec{ya}$ is a phantom edge in σ_0. See Figure 3.10. By C9) y must be an edge star in σ_0. If y is a 2-simplex star vertex in τ_0, we again have Case 1(c) and if y is an edge star vertex, we have Case 1(b). So this case does not occur.

B. $\theta(l(x)) = \frac{\pi}{3}$

As above y must slide, and, as above, this can not occur.

iv. $\theta(l(v)) \neq \frac{\pi}{2}$ and $\theta(B) \neq \frac{\pi}{3}$

Once more, x gains in σ_0 so C9) says it can't be a $\frac{\pi}{2}$ vertex since v is a 2-simplex star, so $\theta(l(y)) = \frac{\pi}{2}$. But then y slides and this is Case 2(a)(i), which we know can not occur. So this possibility does not occur.

(b) v slides two

By Lemma 3.11, a vertex can only slide two (while remaining on the same level) if an adjacent vertex is a $\frac{\pi}{2}$ 2-simplex star vertex in σ_0 and in τ_0. But since v is a 2-simplex star vertex, these two vertices are distinct and are adjacent. This is absurd. So this case does not occur.

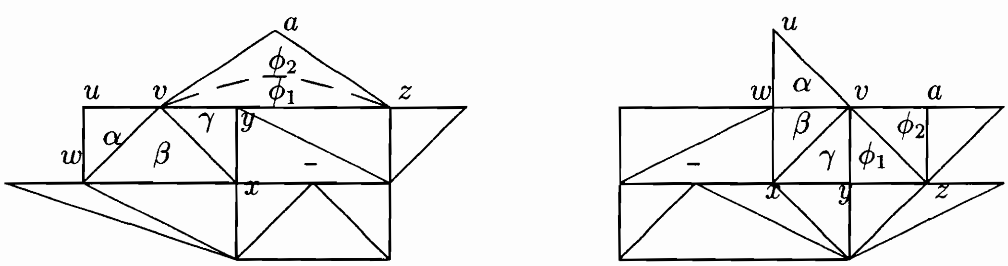


Figure 3.12: v loses one and gains two

(c) v loses one and gains two

See Figure 3.12. Again by Lemma 3.11, to gain two a vertex must have a $\frac{\pi}{2}$ 2-simplex star vertex adjacent to it, so let y be such a vertex. Then by C9), $\theta(l(v)) \leq \frac{\pi}{6}$ and $\theta(B) = \frac{\pi}{3}$. Let β be the central simplex of v in σ_0, w be the $\frac{\pi}{2}$ vertex of β and x be the other one. Now w gains (at least) β in τ_0, so it must be a 2-simplex star vertex in τ_0 and an edge star vertex in σ_0. x definitely loses one in τ_0, so it either slides or loses one and gains two. But it can not lose one and gain two, because by this very case, it would have to have angle $\leq \frac{\pi}{6}$. And $\theta(l(x)) \neq \frac{\pi}{2}$ so if x is an edge star vertex, it can't slide and by Cases 2(a) and 2(b) x can not slide if it is a 2-simplex star vertex. So then the only other possibility is that x is a double edge star vertex in τ_0, but this would mean w would have to be in $X_{\tau_0}(n-1)$. So this case does not occur.

So this completes Case 2, and we find that there are no aberrations when v is a 2-simplex star vertex.

Case 3 Suppose v is a double edge star vertex in σ_0

Possibilities are v slides one or two or gains two and loses one (again the other cases are already done).

Figure 3.13: v slides one

(a) Suppose v slides one

See Figure 3.13. Let x be the central vertex of v in σ_0 and let y be the central vertex of v in τ_0. Then by C8) iii, $\theta(l(x)) = \frac{\pi}{2}$ or $\frac{\pi}{3}$ and the same is true of y. So, $\theta(B) = \frac{\pi}{3}$, and $\theta(l(v)) \leq \frac{\pi}{6}$. Wlog, say x is the $\frac{\pi}{2}$ vertex. Let w be a vertex adjacent to x (distinct from y) in $X_{\sigma_0}(n-1)$. Let α be $\Delta(v, x, w)$, and let $\beta = star_{\sigma_0}(\vec{wx}, n-1)$. Now $\alpha \in star_{\tau_0}(w, n)$ and $\beta \in star_{\tau_0}(x, n-1)$, but this is ridiculous, for now $\vec{wx}$ must be in two places at once. So this case does not occur.

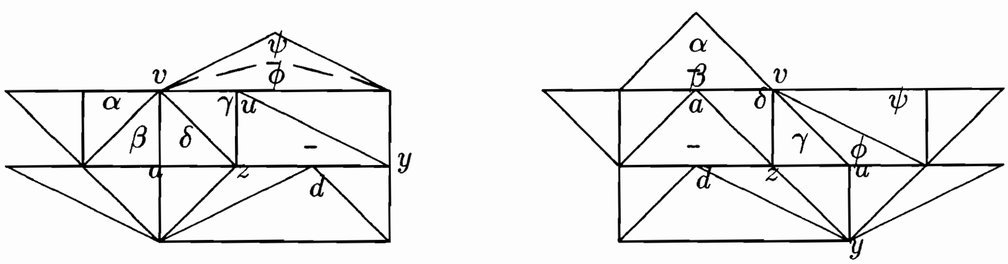


Figure 3.14: v slides two

(b) Suppose v slides two

See Figure 3.14. For v to gain two while remaining at the same level, it must have a $\frac{\pi}{2}$-2 simplex star vertex adjacent to it in σ_0, and to lose two it must have a $\frac{\pi}{2}$-2 simplex star vertex adjacent to it in τ_0. Let the former be called u and the latter be called a. C9) says that $\theta(v) \leq \frac{\pi}{6}$ since it is a double edge star vertex and is adjacent to a $\frac{\pi}{2}$-2 simplex star vertex. Let z be the third vertex of $star_{\sigma_0}(\vec{vu})$. Now $\theta(z) = \frac{\pi}{3}$. Now, z can not be a double edge star vertex since it will have slid two, and this very case says that to slide two a vertex must have angle $\leq \frac{\pi}{6}$. This means that z can be either a 2 simplex star vertex in both views or an edge star in σ_0. But if z is a 2 simplex star vertex in both, it must shift and Case 2 ruled that out. So suppose it is an edge star vertex in σ_0 (wlog). And let y be the other vertex of the phantom edge and let d be the $\frac{\pi}{2}$-2 simplex star vertex that produced $\vec{zy}$. If z is a 2-simplex star vertex in τ_0, z must gain two and lose one (in τ_0) and this can not occur by Case 2(c). So suppose that z is an edge star vertex in τ_0. Then z slides one and since $\theta(l(z)) \neq \frac{\pi}{2}$, Case 1(a) says no. So this case can not occur.

(c) Suppose v gains two and loses one

As above to gain two v must have a $\frac{\pi}{2}$-2 simplex star vertex adjacent to it in σ_0. Let this vertex be z. Let x be the central vertex of v in σ_0 and let y be the central vertex of v in τ_0. $\theta(l(x)) = \frac{\pi}{2}$ and $\theta(l(y)) = \frac{\pi}{3}$. As in (a), x must be an edge star in both views (in σ_0 since y must be a 2-simplex star vertex), but for v to lose one, a vertex in $star_{\sigma_0}(x, n-1)$ must move up to $X_{\tau_0}(n)$. And at the same time, another vertex adjacent to x in $X_{\sigma_0}(n-1)$, namely y, stays at the same level. So x does not slide, remains an edge star vertex, and it loses a simplex. This is an impossibility.

So, once again, none of these possibilities can occur. Case 3 is finished.

Case 4 Suppose v is a triple edge star vertex.

(a) The link has diameter 6

If v slides one, C8) iv says that two vertices (the central vertices in their respective views) have angle $\frac{\pi}{3}$ and the third angle is a $\frac{\pi}{2}$. This is of course, not possible. For v to slide two, it would have to have a $\frac{\pi}{2}$-2 simplex star vertex adjacent to it, and by C9) this can not occur. So this case gives no possibilities.

(b) The link has diameter 5

As above v can neither slide two or gain two, so the only possibility is that v slides one. So let z be the central vertex of v in σ_0. In τ_0, if z is not the central vertex then by C10) (with respect to the central vertex in τ_0), z must be an edge star. But this means it lost two and remained at the same level, so z must be adjacent to a $\frac{\pi}{2}$-2 simplex star vertex. But this vertex "produces" another $\frac{\pi}{2}$-2 simplex star vertex, and this one is adjacent to v. This contradicts C9). If , on the other hand, z is the central vertex in both views, then a vertex adjacent to z, has gained one and stayed at the same level, which says that the central vertex of v has a $\frac{\pi}{2}$-2 simplex star vertex adjacent to it, which contradicts C9). So this case fails to be realized.

■

3.3.2 Technical Lemmas

The following two lemmas are technical lemmas about the combing lines. They are proved due to the frequency of their respective situations in the proof of the big theorem.

Lemma 3.15 *If v is a triple edge star vertex in $\partial X_{\sigma_0}(n)$ with diameter of $link(v,n)$ equal to five, then v can not gain in τ_0.*

Proof: Lemma 3.3 rules out the possibility that v can gain two by going up and C9) does the same if v stays, while Lemma 3.14 implies that v can't slide so all that remains is showing that v can not gain one. That is, that v can not be a triple edge star vertex with diameter six in τ_0. Suppose this is the case. Let α be a simplex that v gains in τ_0, let $u \in \alpha$ be the vertex adjacent to v that is in $X_{\sigma_0}(n)$ and let $\beta = star_{\sigma_0}(\vec{vu}, n)$. Then, let z be the third vertex of β. Now, observe that z is a $\frac{\pi}{2}$-edge star vertex in both views (simply because v is a triple edge star in both). Now u has lost β and gained one in τ_0. This says that u has slid. But $l(u) = \frac{\pi}{3}$. This is a contradiction. ■

Lemma 3.16 *If α and α' are the inner and outer phantom edge simplices of a vertex v in σ_0, respectively, then they both comb to the same simplex as long as they both comb in the same view.*

Proof: If they both comb in σ_0, then this is true by definition of the automatic structure. So suppose they both comb in τ_0. By Lemma 3.14, α and α' can not be in $star_{\tau_0}(v,n)$, since v is a $\frac{\pi}{2}$- 2 simplex star vertex. Again, the two simplices comb to the same one if they are still phantom edges in σ_0, by definition, so the only other case is if v lost one in τ_0. But then $d_s(\alpha, star_{\tau_0}(v,n)) = 1$ and $d_s(\alpha', star_{\tau_0}(v,n)) = 2$, so by a simple distance argument, the theorem is proved (it is worth noting that $l(v) = A$, so even if the vertex adjacent to v is a $\frac{\pi}{3}$-2 simplex vertex, $A > B$ implies that α and α' comb together). ■

Lemma 3.17 *Suppose x, z, and w are adjacent vertices in $\partial X_{\sigma_0}(n)$ and that z is a $\frac{\pi}{2}$-2 simplex star vertex. If x is an edge star vertex in σ_0 and $l(x) \neq C$ or $\theta(B) \neq \frac{\pi}{3}$,* *or if* $d_{x,n}(z,w) \neq 2$*, then* x *can not gain on the* w*-side.*

Proof: By [FP, Lemma 3.8], which basically says that any vertex that fulfills the hypothesis of this lemma can not have two adjacent $\frac{\pi}{2}$-2 simplex star vertices, w is not a $\frac{\pi}{2}$-2 simplex star vertex in σ_0. This means x can't gain two on the w side while staying at the n-level, by Lemma 3.11. By C9, x can't gain if it moves up, since z also moves up (and can't lose by Lemma 3.14). The only other possibility is x gains one and stays at n-level. But then, w loses one in τ_0 and it therefore was a 2-simplex star vertex in σ_0, and we know this isn't the case. ■

The following are two extremely specific lemmas about the possibilities of combing lines and of stars of vertices. They are no fun and should only be referred to when necessary. They are included only because they come up several times, and it is not worth proving them over several times when one time will do.

Lemma 3.18 *Suppose* $\tau \in star_{\sigma_0}(v, n+1)$ *and* $v \in \partial X_{\sigma_0}(n)$*. Say* $\Theta(v) \leq \frac{\pi}{6}$ *and* $d(\tau, star_{\sigma_0}(v,n)) = 2$*. If* τ *combs in* τ_0 *through* v*, then* τ *must comb to one of the labeled simplices in Figure 3.15*

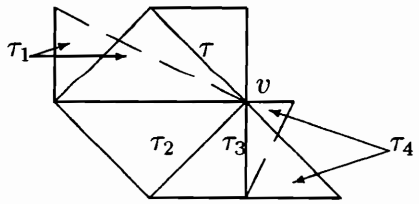


Figure 3.15: Possibilities for τ to comb to

Note that in this figure, there are more fins possible (e.g. where τ_2 and τ_3 intersect). These, too, are possibly the simplices that produce τ in terms of the the combing lines.

Proof:

The only other possibility is if v loses τ_2 and τ_3, by [FP, Lemma 3.6]. In this case $d_s(\tau_4, \tau) = 4$ and [FP, Lemma 3.6] does not guarantee which simplex τ combs to. But since v lost two in this case, $diam[link_{\tau_0}(v)] \leq 4$, so $d_s(\tau, \phi) \geq 5$ where ϕ is any other simplex in the star of v. This is because $\Theta(v) \leq \frac{\pi}{6}$ and no cycle can be less than 12. So $\tau \rightarrow \tau_4$. ■

Lemma 3.19 *If $d_g(\tau, \partial X_{\sigma_0}(n)) = 2$ and $\tau \bigcap \partial X_{\sigma_0}(n) = y$ and u is the vertex (other than y) in τ that is also contained in a simplex of distance one from the boundary, then $\tau \xrightarrow{y}$ or $\tau \xrightarrow{u}$ in τ_0 and u is a double edge star vertex or a triple edge star vertex in τ_0.*

Proof:

The lemma is obvious if τ combs in σ_0 (the hypotheses insure that τ is not a phantom edge simplex). So suppose τ combs in τ_0 and that $\tau \in star_{\tau_0}(y)$ (this is the only way for τ not to comb through y). Let v be the other vertex of τ. For τ to comb through v, it must not be in $star_{\tau_0}(v)$. But this is not possible, since $\tau \in star_{\tau_0}(y)$. So τ must comb through u, and u must gain at least two in τ_0. So the lemma is proved. ■

3.3.3 The Biautomatic Structure for the $\frac{\pi}{2}$ Case

Now, we have finished with all of the lemmas and are ready to prove the theorem for the $\frac{\pi}{2}$ case. As in the previous theorem, the proof will follow from a case by case analysis of the possible configurations of two simplices which are one away from each other in the simplicial metric, where one simplex's combing line is calculated in X_{σ_0} and the others combing line is calculated in X_{τ_0}. This is how we introduce the left multiplication by a generator in the definition of biautomatic.

The major added complexity (beyond that Lemma 3.14 has a more complex aberration) compared to the $\frac{\pi}{3}$ theorem is the existence of phantom edge simplices. In all but the easiest cases, these phantom edge possibilities will be dealt with in their own cases. The theorem still follows as before. Only now, the constant K is 3, even before the application of the Lemma 3.10 which gives a final bound of 6 for the biautomatic structure.

Theorem 3.20 *If G is the amalgam complex for a triangle of groups with non-trivial edge groups and a trivial face group, and with angle sum less than or equal to π and no angle greater than $\frac{\pi}{2}$, then G is biautomatic.*

Proof:

As before, this proof is done by induction in X on n. We will assume that the combing lines are bounded for any two adjacent simplices in $X_{\sigma_0}(n)$, and then prove that they remain in a configuration we can deal with when we look at what they comb to in their respective views. These resulting simplices will not necessarily be adjacent but they will be in configurations which we will prove remain bounded. As mentioned, the proof follows from a case by case study. We will start with the simplest case. In all the cases σ and τ are simplices in X.

Configuration 1 $\sigma, \tau \in X_{\sigma_0}(n+1) \setminus int(X_{\sigma_0}(n))$ and $\sigma \bigcap \tau \bigcap \partial X_{\sigma_0}(n) \neq \emptyset$

Let $v \in \sigma \bigcap \tau \bigcap \partial X_{\sigma_0}(n)$. Wlog, say $\sigma \xrightarrow{\sigma_0}$. Then, if σ is not a phantom edge simplex in σ_0, [FP, Lemma 3.7] says $v \in \sigma(n)$. So $(\sigma(n), \tau)$ are in a basic configuration. So, suppose σ is a phantom edge simplex in σ_0. If v is the vertex which produced σ or if $l(v) = C$, then $\sigma \xrightarrow{v}$, and as before, $(\sigma(n), \tau)$ are in a basic configuration.

Therefore, let $l(v) = B$ and let u be the vertex which produces σ. Then by C9), v is an edge star vertex in σ_0. By [FP Lemma 3.8], z is also an edge star vertex in σ_0, where z is any vertex adjacent to v in $\partial X_{\sigma_0}(n)$ other than u. The only remaining case of interest is when τ combs to a simplex that doesn't intersect u (if it does then $(\sigma, \tau(n))$ are in a basic configuration). But even in this case, τ must comb through v, for the only way not to, is for $\tau \in star_{\tau_0}(v, m)$ (where m is not necessarily n). But for v to have gained in τ_0, some vertex z (as defined above) must have lost a simplex (β in the figure), and since it was an edge star vertex, this isn't possible. So τ must comb through v (to a simplex such as β). We split up the cases now depending on the angle of v.

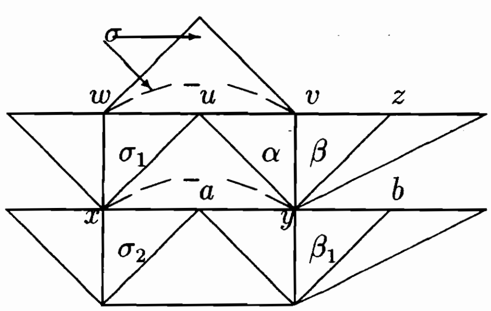


Figure 3.16: σ is a phantom edge simplex

(a) $\Theta(l(v)) \neq \frac{\pi}{3}$

In this case, since $l(v) = B \neq \frac{\pi}{3}$, all vertices adjacent to u are edge star vertices by C9). We know, by definition, that σ combs to σ_1. Then, if $\vec{xy}$ is not a phantom edge, σ_1 combs through y, and $(\sigma_1(n-1), \beta)$ are in a basic configuration, and we are done.

So, we suppose that $\vec{xy}$ is a phantom edge as drawn in Figure 3.16. Let a be the vertex that produces the phantom edge simplex. Let $\sigma_2 = star_{\sigma_0}(\vec{xa}, n-1)$. Then σ_1 combs to σ_2 by distance. That is, σ_1 must comb through x, so it is just a question of which simplex of $star_{\sigma_0}(\vec{xa}, n-1)$. But, we know that x is an edge star vertex, since a is a 2-simplex star vertex (C9), and then, just because $St(l(x))$ must have an even number of simplices in a cycle, σ_1 must comb to σ_2 (by a simple distance argument). Now σ_2 and β are in the same situation that σ_1 and τ were above. That is, we know that the worst possible case is when β combs to β_1 (as in the figure), and this case is identical to the case just dealt with (σ_1, β). So, in this case the combing lines will stay at distance two (until they get to $X(0)$). (Actually, we must be careful in saying these cases are identical, since $l(y) \neq l(v)$, but since $l(v) \neq \frac{\pi}{3}$, we can get away with it here.)

(b) $\Theta(l(v)) = \frac{\pi}{3}$

This time the argument gets a little too complex to handle in one paragraph, so we break this case up into two subcases, depending on the edge $\vec{xy}$.

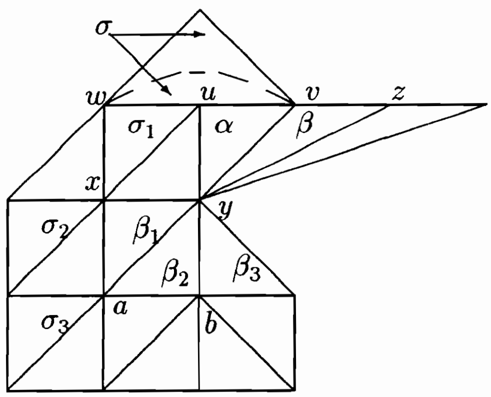


Figure 3.17: $\vec{xy}$ is not a phantom edge

i. $\vec{xy}$ is not a phantom edge

Here we define $\beta_1 = star_{\sigma_0}(\vec{xy}, n-1)$ and a to be its third vertex. If σ_1 combs to β_1, then we are done, since (β_1, β) are in a basic configuration. So suppose that σ_1 does not comb to β_1. How is this possible? Well, since $\Theta(x) = \frac{\pi}{3}$, [FP, Lemma 3.4] says that x must be a 2-simplex star vertex in σ_0, and there must be a simplex like σ_2 in Figure 3.17 that σ_1 combs to. In this situation, if β then combs to β_1, we are done. So again, how could β comb to another simplex? A distance argument says that the only other possible simplices are β_2 and β_3 as in the figure (if they exist at all), unless y is a triple edge star vertex in σ_0. But in this case, y can't lose one by Lemma 3.15 and it can't lose two for then a gains two and this is impossible since it is a $\frac{\pi}{2}$vertex. So, even here, β combs to β_2 or β_3 is the only troublesome case.

Now, in this situation, a must be a 2-simplex star vertex, and then σ_2 must comb to σ_3 (as in the figure). Now we see that if β combs to β_2 we are done. This just leaves us with the β_3 case. To comb to β_3, y will have to have lost two in τ_0. If y moves down, then a gains or slides and neither is a possibility. If y stays, then it must be adjacent to a $\frac{\pi}{2}$-2 simplex star vertex in τ_0, which would have to be a. But then y went up (or a went down) in σ_0 and this is impossible. So this case is finished.

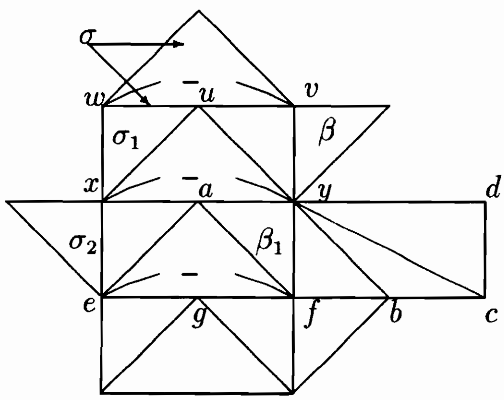


Figure 3.18: $\vec{xy}$ is a phantom edge

ii. $\vec{xy}$ is a phantom edge

This situation is as in Figure 3.18 (where $star_{\sigma_0}(y, n-1)$ is seen in the worst case scenario). y can not lose β_1 in τ_0, since a is already a $\frac{\pi}{2}$ 2-simplex star vertex. So, either y gains in τ_0 and is a double edge star vertex in σ_0 or β combs to β_1 by a simple distance argument. But in fact, we have already shown in

Lemma 3.17 that y can not gain except on the a-side, and this would lead to no problem. So, suppose $\beta \longrightarrow \beta_1$. This finishes the argument, for then, either σ_1 combs to a simplex distance 1 from β_1, or it combs to a simplex like σ_2 (some fin of edge $\vec{xe}$), and this is the situation we started with in this case for (σ_1, β) where β_1 plays the role of σ_1 and σ_2 plays the role of β.

This ends the argument of Configuration 1.

Configuration 2 $\sigma \bigcap \tau = v \in \partial X_{\sigma_0}(n+1)$ and both σ and τ intersect $\partial X_{\sigma_0}(n)$ in an edge.

There are two different ways for this case to occur, depending on $star_{\sigma_0}(v, n+1)$. We are guaranteed that v is a triple edge star vertex in σ_0 just by the hypotheses. The two cases then come from the two types of triple edge star vertices.

(a) Say $diam[link_{\sigma_0}(v, n+1)] = 5$.

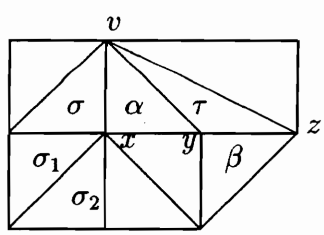


Figure 3.19: Configuration 2(a)

See Figure 3.19. Let x be the central vertex of v in σ_0, and let us suppose that, wlog, σ is the simplex that contains x. Then there exists a $\frac{\pi}{2}$ vertex adjacent to x in $link_{\sigma_0}(v, n+1)$ which τ contains. Let us call this vertex y.

If τ combs in σ_0, then we know τ combs to β. We would like to show that σ must comb through x. How could it not? Only if $\sigma \in star_{\tau_0}(x, m)$. But x can't gain since it is a $\frac{\pi}{3}$ double edge star vertex, nor can it slide. So σ must comb through x.

Now, we look at σ_1 and β (or possibly σ_2 and β). This situation where we have a $\frac{\pi}{3}$ double edge star vertex adjacent to a $\frac{\pi}{2}$ vertex and simplices distance two apart as shown comes up frequently in these configurations. That these configurations do not lead to unbounded combing lines is true and not too hard to show, but because we need the argument several times and it isn't a basic configuration (meaning the simplices are more than distance one apart), we delay this question until later. This is called Complex Configuration #1, and will be dealt with after all of the other basic configurations have been shown to either lead to other basic configurations or to one of the Complex Configurations. There will be a total of 7 of them.

But we still have more to do here. We still need to know what happens if σ combs in σ_0. Then, we are certain of where σ combs to (σ_1 in the figure.) And now, like above, we would like to see that τ must comb through y. Then we will have the above situation of Complex Configuration #1. So how could τ fail to comb through y? Since y is a $\frac{\pi}{2}$ vertex, there is no worry that if τ is a phantom edge simplex, it fails to comb through y. So, as before, the only worry is if y gains τ in τ_0. But for it to, y would become a 2-simplex star vertex, and so x can no longer be adjacent to it, by C9). This suggests that y has actually also gained α, and this is of course ridiculous. So these cases just lead to Complex Configuration #1.

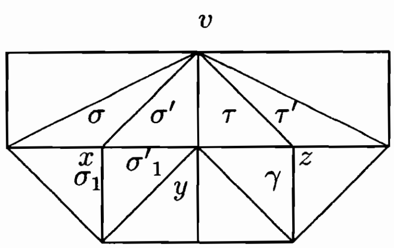


Figure 3.20: Configuration 2(b)

(b) Say $diam[link_{\sigma_0}(v, n+1)] = 6$

See Figure 3.20. This time we can say that σ (or σ') comb in σ_0 wlog. Let y be the central vertex of v in σ_0 and let x and z be the vertices adjacent to y in $link_{\sigma_0}(v, n+1)$ that contain σ (σ') and τ (τ'), respectively. Now, we know where $\sigma(\sigma')$ combs, so once again it is just a matter of making sure that $\tau(\tau')$ combs through z. For τ this argument is immediate since y can't gain. For $(\sigma'(n), \tau(m))$ we then have a basic configuration, and for $(\sigma(n), \tau(m))$ we have complex configuration #1. For τ', the argument that it combs through z is identical to the argument of the last case for τ. In this case, $(\sigma'(n), \tau'(m))$ are in complex configuration #1, and $(\sigma(n), \tau'(m))$ are in complex configuration #3 (to be defined precisely later).

We are done with this configuration then.

Configuration 3 $\sigma \bigcap \tau = v \in \partial X_{\sigma_0}(n+1)$ and σ intersects $X_{\sigma_0}(n)$ in an edge but

τ only intersects $X_{\sigma_0}(n)$ in a vertex (call it y).

We have two cases depending on in which view the simplices comb.

(a) σ combs in σ_0.

It is easy to see where σ combs to, since it shares an edge with the boundary of X_{σ_0} (n). This configuration guarantees that v is at least a double edge star vertex in σ_0. But we can make no further conclusions from the hypothesis so we must break up this configuration even further into cases based on the *star* of v.

i. v is a double edge star vertex in σ_0.

Let x be the central vertex of v in σ_0. C8) then guarantees that this vertex is either a $\frac{\pi}{2}$ vertex or it is a $\frac{\pi}{3}-$ double edge star vertex. We have to further break this up into cases to make the argument tractable.

A. $\Theta(l(x)) = \frac{\pi}{2}$.

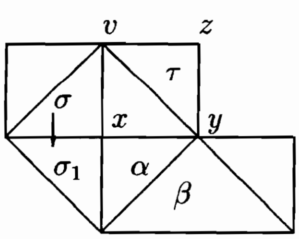


Figure 3.21: Configuration 3(a)iA

See Figure 3.21 (where $star_{\sigma_0}(y, n)$ is drawn in the hardest case). If τ combs to a simplex intersecting σ or σ_1, then we are done. So, what other possibilities are there? y must be adjacent to x for v is a double edge star vertex in σ_0. If y were an edge star vertex in σ_0, then it can't slide (it's not a $\frac{\pi}{2}$ vertex) or lose. So suppose y gains in τ_0. If x is a $\frac{\pi}{2}$-2 simplex star vertex in σ_0 and y is a $\frac{\pi}{3}$ vertex, then we have complex configuration #6 for (σ_1, τ) (in the worst case). If y is not a $\frac{\pi}{3}$ (and x is as in last sentence), then either Lemma 3.18 gives complex configuration #2 or we have a basic configuration for $(\sigma_1, \tau(m-1))$.

This takes care of the possibility that x may be a 2-simplex star vertex (the figure does not insist that x is an edge star vertex, incidentally, for x may have more simplices that are not drawn). Now, let us suppose x is an edge star vertex. Then τ must be distance 2 from $X_{\sigma_0}(n)$. If $\Theta(l(y)) \leq \frac{\pi}{6}$, Lemma 3.18 gives that either we have a basic configuration or complex configuration #2 for $(\sigma_1, \tau(m-1))$. If $\Theta(l(y)) > \frac{\pi}{6}$ vertex, then a simple distance argument says that we have a basic configuration for $(\sigma_1, \tau(m-1))$. So this case is finished.

B. $\Theta(l(x)) = \frac{\pi}{3}$

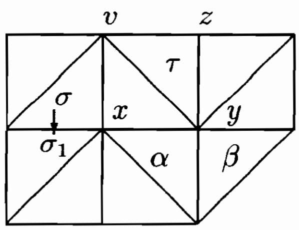


Figure 3.22: Configuration 3(a)iB

See Figure 3.22. In this case, we know that $\Theta(l(y)) = \frac{\pi}{2}$ and $\Theta(l(v)) \leq \frac{\pi}{6}$. We also know, by C8) that x is a double edge star vertex and by C9) that y is an edge star vertex in σ_0.

If τ combs to α, then (σ_1, α) are in a basic configuration, and if τ combs to β then (σ_1, β) are in complex configuration #1. So what other simplices can τ comb to? Let's look at the possible changes in τ_0 for y. It can't lose (it's an edge star) or slide since x would have to gain and x is already as big as it can be. So let's suppose y gains one in τ_0. But this would mean that x would have to be an edge star vertex in τ_0 (by C9) and then gain two in σ_0 without moving up. This can only happen if there exists a w that is a $\frac{\pi}{2}$-2 simplex star vertex distinct from y adjacent to x in τ_0. But this violates [FP, Lemma 3.8] since y is already such a vertex in τ_0. So, this case is finished.

ii. v is a triple edge star vertex with link 5 in σ_0

We have to break this case up into a pair of cases, depending on the relationship between the central vertex of v in σ_0 and σ.

A. $\Theta(l(x)) = \frac{\pi}{2}$

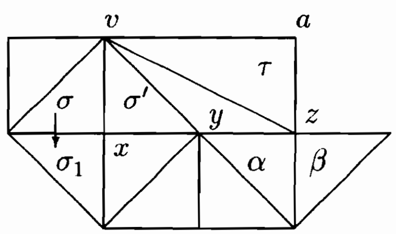


Figure 3.23: Configuration 3(a)iiA

First, we assume we have the situation of Figure 3.23. Let y be the central vertex of v in σ_0. Let x be the $\frac{\pi}{2}$ - edge star vertex in σ_0 that σ (σ') contains. Since σ (σ') combs in σ_0, we can easily see where it combs to. For (σ'(n),τ) we now have the same situation as with Configuration 3(a)iB. So let us deal with (σ_1, τ). If τ combs through v we are done, and we know τ can't comb through a (for v would have to slide or gain and neither are possible), so we need only worry about z. If τ combs to α, then we have complex configuration #1 and if τ combs to β then we have complex configuration #3. So what other simplices could τ comb to (of course, the drawing may be misleading if we pretend there is no other simplex that share the edge that α and β do, but any such simplex is identical to β in terms of these arguments–and pictures). Well, maybe z gains one. But then z would be a 2-simplex star vertex, and y would have to lose two (by C8) and this isn't possible, since then x would then also have to be a 2-simplex star vertex (in violation of [FP], Lemma 3.8). Also, z can't slide for y would have to gain, and it is already at it's largest. So, this case is taken care of.

B. $\Theta(l(x)) = \frac{\pi}{3}$.

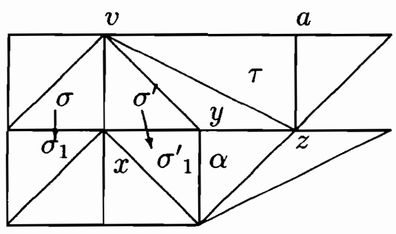


Figure 3.24: Configuration 3(a)iiB

See Figure 3.24. Here we assume that x is the central vertex of v in σ_0. And now, y is the $\frac{\pi}{2}$-edge star vertex. Again, it is easy to see where σ (σ') combs to. Again, if τ combs through v we are done and τ can not comb through a. So, we look at the possibilities with z. z can't lose one in τ_0 nor can it slide (it isn't a $\frac{\pi}{2}$ vertex), so all it can do is gain one or two. If z gains one, τ combs to α by [FP, Lemma 3.3] –essentially a distance argument combined with the tie-breaking A is less than C. So suppose z gains two. τ is still distance 2 from α in τ_0, so unless τ is closer to some other simplex in $star_{\tau_0}(v, m)$, the tie-breaking rule will again say τ combs to α. But for τ to be distance 1 from $star_{\tau_0}(v, m)$, v will have to have gained in τ_0 and this is not possible by Lemma 3.15. That means that τ will comb to α, at worst. Then $({\sigma'}_1, \alpha)$ are in a basic configuration and (σ_1, α) are in complex configuration #1. So we are finished with this possibility.

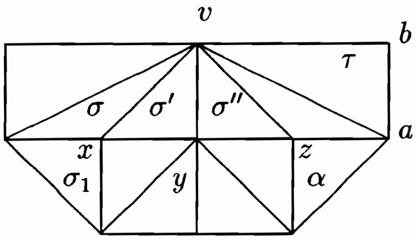


Figure 3.25: Configuration 3(a)iii

iii. v is a triple edge vertex with $diam[link_{\sigma_0}(v, n)] = 6$.

See Figure 3.25

For σ (or σ'or σ''), we can see the simplex it combs to. So, once more we just look into where τ combs. If τ combs through v then we are finished, and as above, τ can not comb through b, so let's suppose τ combs through a. We need to see what a can do in τ_0. a is a $\frac{\pi}{3}$-edge star (by C10), so a can not lose in τ_0 nor can it slide. And as in the last argument, if a gains one, then τ combs to α. And if a gains two and τ is distance 1 from $star_{\tau_0}(a, m)$ (the only case where τ doesn't comb to α), then v would have to gain in τ_0, but of course this can't happen.

So, in all cases of interest, τ combs to α. Then, $(\sigma''(n), \alpha)$ are in a basic configuration, $(\sigma'(n), \alpha)$ are in complex configuration #1, and $(\sigma(n), \alpha)$ are in complex configuration #4. And this case is finished.

We have now finished the argument for Configuration 3, in the case where σ combs in σ_0.

(b) τ combs in σ_0.

As before, we need to break this up into three cases depending on the possibilities for v in σ_0. (This time, though, the cases are easier for determining where σ combs is easier than determining where τ combed before since $\sigma \bigcap \partial X_{\sigma_0}(n)$ is an edge).

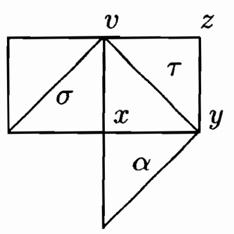


Figure 3.26: Configuration 3(b)i

i. v is a double edge star in σ_0.

See Figure 3.26. (This figure is similar to the one for Configuration 3(a)iA, and the letters are defined as for that figure.) If τ combs to α we are done, for then (σ, α) are in a basic configuration. So, we need to explore how τ could fail to comb to α. There are only two ways by [FP, Lemma 3.3]. First, y could be a $\frac{\pi}{2}$-2 simplex star vertex in σ_0, but then x would have to be a $\frac{\pi}{3}$-double edge star vertex by C8), and C9) says this can't happen. Second, y could be a $\frac{\pi}{3}$-2 simplex star vertex with τ

distance 2 from $star_{\sigma_0}(y,n)$. But then in this case $l(x) = A$, so τ still combs to α.

We have finished this case.

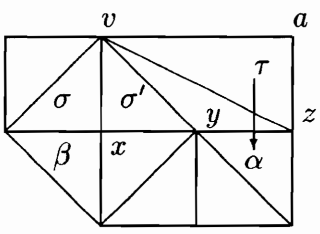


Figure 3.27: Configuration 3(b)iiA

ii. v is a triple edge star vertex with $diam[link_{\sigma_0}(v,n)] = 5$.

As before, we break this case up into two, depending on the relationship between the central vertex of v in σ_0 and the intersection of τ with $\partial X_{\sigma_0}(n)$.

A. y is the central vertex of v (where y is defined as in Figure 3.27 where the definitions come from Configuration 3(a)ii).

As in the above argument, τ combs to α. So, for σ'we are done. All that is left is to look at where σ combs. If σ combs through x we are done for either we have a basic configuration or we have complex configuration #1 (in the case where σ combs to β). To not comb through x, σ would have to be in $star_{\tau_0}(x,m)$. This could only occur if x gained or slid in τ_0. x can't gain for if it gained, y would have to be an edge star by C9) in τ_0, and this can't be done, by [FP, Lemma 3.8]. On the other hand, if x slides, v has to lose three, which isn't possible. So we are finished.

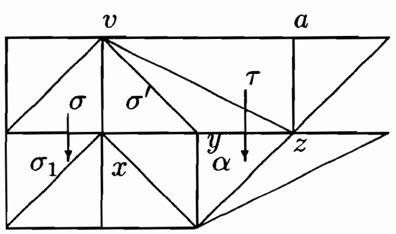


Figure 3.28: Configuration 3(b)iiB

B. x is the central vertex of v.

See Figure 3.28. This case is simple. As above τ combs to α. And x can not pick up σ (or σ') so, σ (σ') must comb through x. In this case then, the worst possible situation we have then is complex configuration #1.

This finishes off this case.

iii. v is a triple edge star vertex $diam[link_{\sigma_0}\ (v,n)] = 6$.

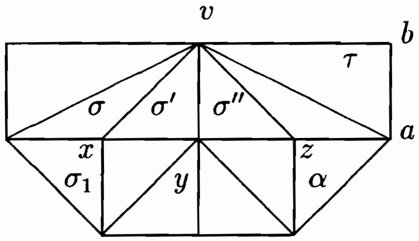


Figure 3.29: Configuration 3(b)iii

See Figure 3.29. Once more, τ combs to α. So, we are finished for σ''. Again, σ and σ'must comb through x, so we either have a basic configuration, complex configuration #1, or complex configuration #3. In any case, though, we are finished with this case.

And now, we have (finally) finished the argument for Configuration 3.

Configuration 4 $\sigma \bigcap \tau = w \in \partial X_{\sigma_0}(n+1)$, $\sigma \bigcap \partial X_{\sigma_0}(n) = v$, and $\tau \bigcap \partial X_{\sigma_0}(n) = x$.

First, we note that in this configuration, there is no difference between σ and τ combinatorically speaking, so wlog, we say σ combs in σ_0. Now, this makes for a shorter argument than Configuration 3, but on the other hand, we now have the added case where w may be a 2-simplex star vertex in σ_0. And in fact, we now have to break up this case into 3 cases.

(a) w is a 2-simplex star vertex in σ_0.

Unfortunately, we now have to break this case up into two cases depending on whether $\vec{vx}$ is a phantom edge or not. ($\vec{vx}$ exists by the assumptions.)

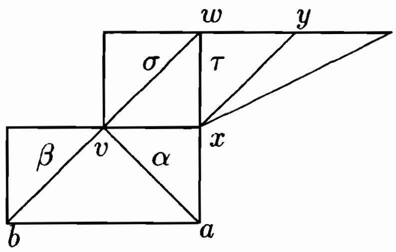


Figure 3.30: Configuration 4(a)i

i. $\vec{vx}$ is not a phantom edge.

See Figure 3.30. Let $\alpha = star_{\sigma_0}(\vec{vx})$. If σ combs to α, then we are done, for (α, τ) are in a basic configuration. So, we need only work on the cases where σ does not comb to α. By [FP, Lemma 3.8] these are when v is a $\frac{\pi}{2}$ or $\frac{\pi}{3}$ 2-simplex star vertex in σ_0. So let us start with the former case. We still haven't guaranteed that σ doesn't comb to α. In fact, we have a tie, so for this to occur, we assume $l(x) = B$. But then, C9) says x is an edge star vertex in σ_0. Now, we calculate where τ can comb. τ combs to α (or through w) unless x gains but by Lemma 3.17 this can't occur. So, we know that τ combs to α, and we have a basic configuration for (β, α).

It is also possible that v is a $\frac{\pi}{3}$-2 simplex star vertex in σ_0. This time assume $l(x) = C$ so that σ combs to a simplex other than α. So $l(w) = A$ and C9) says y must be an edge star vertex in σ_0. This guarantees that τ combs through x (or w but this is trivial). By a simple distance argument (recalling that $\Theta(x) \leq \frac{\pi}{6}$), unless x loses in τ_0, τ combs to α and we are done. But if x loses, a gains in τ_0, and this is not possible, for it would have to be the central vertex of v in τ_0, which would be a double edge star vertex, and it would have to have gained, which isn't possible by C8).

So this case is done.

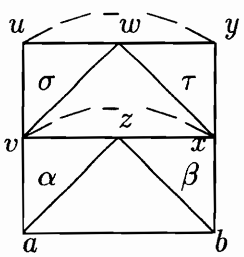


Figure 3.31: Configuration 4(a)ii

ii. $\vec{vx}$ is a phantom edge.

See Figure 3.31. First, we suppose $\Theta(B) = \frac{\pi}{3}$. If $\Theta(v) \le \frac{\pi}{6}$, then σ combs to α and (α, τ) are in complex configuration #6. If $\Theta(v) = \frac{\pi}{3}$, then x is at most a double edge star vertex in σ_0, and since x can not lose in τ_0 (z would have to gain), τ combs to β. If σ combed to α we would be done, so suppose σ combs to some other simplex (remembering that v is an edge star vertex). Now, if $\vec{ab}$ is not a phantom edge, $(\sigma(n-2), \beta)$ are in a basic configuration, since $\Theta(a) \le \frac{\pi}{6}$. On the other hand, if $\vec{ab}$ is a phantom edge, then (by a distance argument), $(\sigma(n-2), \beta)$ are in complex configuration #6.

Finally, we assume that $\Theta(B) \ne \frac{\pi}{3}$. In this case u, y, v, and x must all be edge star vertices by C9). It's easy to see that σ must comb to α. If τ combs to β, then, we are done. The only way this could fail to occur (for x can not lose or slide) is for x

to gain in τ_0. But for x to gain, a vertex adjacent to it must go down and lose (at least) one in τ_0. All vertices adjacent to x (in $\partial X_{\sigma_0}(n)$) are $\frac{\pi}{2}$ vertices, and in fact, by [FP, Lemma 3.8] must be edge star vertices. These vertices can not lose, so x can not gain.

We are done with the case of w being a 2-simplex star vertex.

(b) w is a double edge star vertex in σ_0

From C8), we see that there are two possibilities for the central vertex of w in σ_0 (let's call it x), and we take those two cases separately.

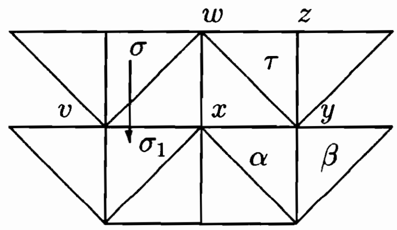


Figure 3.32: Configuration 4(b)i

i. x is a $\frac{\pi}{3}$—double edge star vertex in σ_0.

See Figure 3.32. By C9), then, we see that all $\frac{\pi}{2}$ vertices adjacent to x in $\partial X_{\sigma_0}(n)$ are edge star vertices, and this gives us Figure 3.32 (note that $\Theta(w) \neq \frac{\pi}{2}$ since it is a double edge star vertex). Now, a simple distance argument gives that σ combs to σ_1. If τ combs through w then we are done, and Lemma 3.19 says that τ can not comb through z, so we suppose τ combs through y. If τ combs to α, we are done. How can τ fail to comb to α? y

can neither lose nor slide (for x would have to gain), so the only way is for y to gain. This would make y a 2-simplex star vertex in τ_0, and for this to be, x would have to lose two (by C9). But this says, in τ_0, that y is a 2-simplex star vertex and x gains two. Lemma 3.17 rules this out.

So in this case we have a basic configuration.

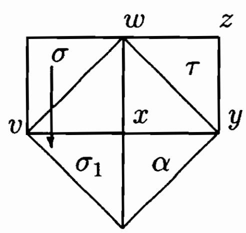


Figure 3.33: Configuration 4(b)ii

ii. $\Theta(x) = \frac{\pi}{2}$.

See Figure 3.33. σ combs to σ_1 by [FP, Lemma 3.3] (even if v is a $\frac{\pi}{3}$-2 simplex star vertex in σ_0, σ_1 wins since $l(x) = A$). Again, if τ combs through w, we are done, and τ can't comb through z by Lemma 3.19, so we suppose τ combs through y. If $\Theta(y) \leq \frac{\pi}{6}$, then Lemma 3.18 says that we either have a basic configuration, or we have complex configuration #2 (in the case where y loses two and $\Theta(B) = \frac{\pi}{3}$).

So, suppose $\Theta(y) > \frac{\pi}{6}$. If $\Theta(y) \neq \frac{\pi}{3}$, τ combs to a simplex that is distance one from σ_1, so we have a basic configuration (this just by a simple distance argument) or we may have complex configuration #5 if y is a double edge star vertex in σ_0 and it loses one or two in τ_0. All that is left is to deal with the case where $\Theta(y) = \frac{\pi}{3}$. If y is a double edge star in σ_0, we again have a complex configuration if y loses, but this time it is complex configuration #1. For the cases where y is a 2-simplex star vertex or an edge star vertex, we again have a basic configuration, unless y gains one. In this case, we have complex configuration #1, but not in the usual manner, and we want to be careful here. This time, the configuration is as viewed from X_{τ_0} , and that is why the proof of complex configuration #1 needs to be done from both views. It and complex configuration #7 are the only two which are done from both viewpoints.

This finishes off the possibility that w is a double edge star vertex in σ_0.

(c) w is a triple edge star vertex in σ_0.

As always, we divide this into two cases.

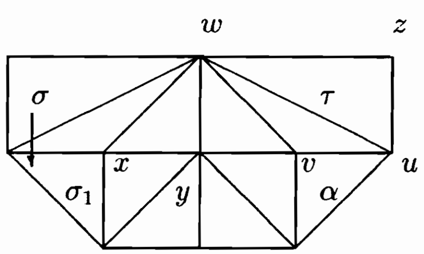


Figure 3.34: Configuration 4(c)i

i. w is a triple edge star vertex $diam[link_{\sigma_0}(v,n)] = 6$.

See Figure 3.34 σ combs to σ_1 by distance. If τ combs through w, we are finished, and it can't comb through z, so suppose τ combs through u. If τ combs to α, then we have complex configuration #5. Since u can't lose or slide in τ_0, the only other possibility is that u gains in τ_0. But if u gains one, then τ still combs to α since $A < C$. Suppose u gains two in τ_0. But for this to insure that τ combs to a simplex other than α, w would have to gain, and this isn't possible. So this case can not occur.

ii. w is a triple edge star vertex $diam[link_{\sigma_0}(v,n)] = 5$.

Again, as usual, we divide this case up into two cases depending on the location of the central vertex of w in σ_0.

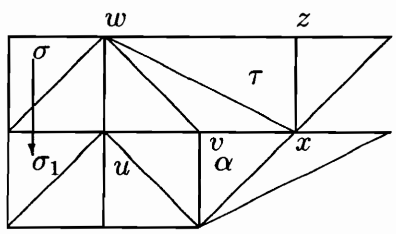


Figure 3.35: Configuration 4(c)iiA

A. u is the central vertex of w in σ_0

(See Figure 3.35). C9) says that all $\frac{\pi}{2}$vertices adjacent to u must be edge star vertices, so σ combs to σ_1. As in all the above arguments, we need only look at the case where τ combs through x. This argument is identical to the above one, for x is an edge star vertex by C10) and can't slide or lose. We find again that τ must comb to α, and this leaves us with complex configuration #1 from the τ_0 view.

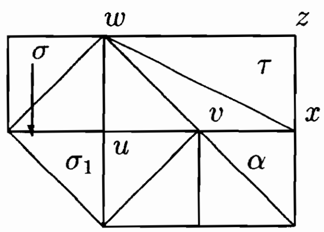


Figure 3.36: Configuration 4(c)iiB

B. v is the central vertex of w in σ_0

(See Figure 3.36). σ combs to σ_1 by C10). As always, we only deal with the case where τ combs through x. If τ combs to α we have complex configuration #1. The only other worry is if x gains in τ_0, but again this violates Lemma 3.17 from the τ_0 viewpoint. So this case is finished.

Indeed, this finishes off Configuration 4, and we are now done with the longest (and pickiest) of the configurations. Whew.

Configuration 5 $|\sigma|_{\sigma_0} \neq |\tau|_{\sigma_0}$ and $|\sigma|_{\tau_0} \neq |\tau|_{\tau_0}$ and $\sigma \bigcap \tau = v \in \partial X_{\sigma_0}\ (n)$.

The reason we insist that the two simplices be at different levels in both views is that if they were at the same level in one view (and not the other), the arguments for Configuration #1-4 would do. That is, in which ever view the two simplices appeared in the same level, they would be in one of those configurations, and the argument will have already been made. (In fact, in quite a few of the cases above, the τ_0 view did have the simplices at different levels.) So the only remaining cases of interest are when the simplices are at different levels in both views.

Let us suppose that, wlog, $|\sigma|_{\sigma_0} = n$ and $|\tau|_{\sigma_0} = n + 1$. In all cases, if τ combs through v, then $(\sigma, \tau(n))$ are in one of Configuration #1-4 and we are done. So, the only cases of interest are those where τ does not comb through v. By [FP, Lemma 3.2], this is when τ is a phantom edge simplex (which will be dealt with later) or its combing lines are calculated in τ_0 and it is a member of $star_{\tau_0}(v, n)$ (note that we know $\tau \in X_{\tau_0}(n)$).

But let us look at what we are requiring in the latter statement. v must lose σ in τ_0, and at the same time v must gain τ in τ_0. There is only one time a vertex can gain one simplex and lose another. It is the very special situation of Case 1 of Lemma 3.14.

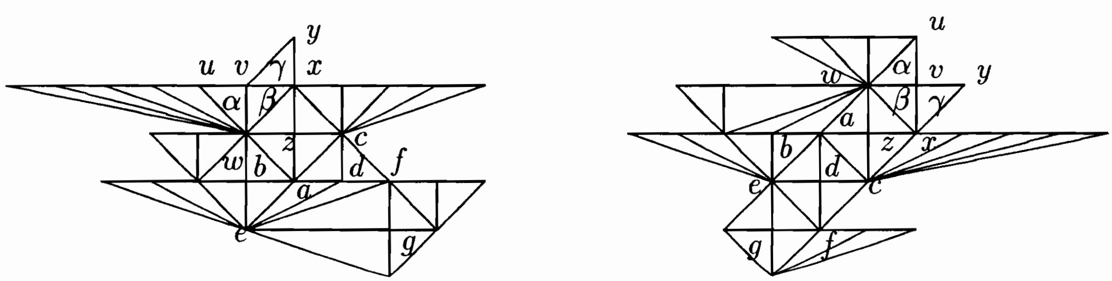


Figure 3.37: Configuration 5

This figure is only one of the three possibilities, but in each the argument is essentially the same. The figures are forced. The combing lines have little room for argument. In each, it is clear where σ combs by [FP, Lemma 3.4]. τ has few choices but all of them lead either to complex configuration #1 or complex configuration #5. So, the strange case, though very strange, is not hard to deal with.

Now, we deal with the possibility that τ is a phantom edge simplex in σ_0. This is indeed a real possibility (unlike with most of the other configurations).

For this, we need to break up the cases, depending on σ.

(a) σ intersects ∂X_{σ_0} $(n-1)$ in a vertex y.

If v is the vertex which produced τ as a phantom edge simplex, then no matter where τ combs, we have a basic configuration since τ must comb through v. And if v didn't produce τ, then σ can not be a phantom edge simplex in τ_0, for the $\frac{\pi}{2}$ vertex of σ would have to gain exactly one in τ_0, and not contain σ. This is not possible, so σ is not a phantom edge simplex in either view. This implies that the only interesting case is when τ is a phantom edge simplex in both views (or one view would have no phantom edge simplices at all, and this case is already argued).

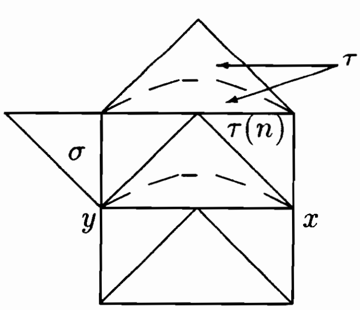


Figure 3.38: Configuration 5(a)

See Figure 3.38.

In this situation, there is basically no possibility of change in the two views. In fact, for τ to be a phantom edge simplex in both views, it must be produced by the same vertex in both views. So if $l(v) = C$, τ will comb to a simplex distance one from σ and we are done. So, we need to assume that $l(v) = B$ (having already ruled out v as the vertex which produced τ). This assumption guarantees that v is an edge star vertex in both views, by C9).

Let u be the vertex which produced τ and let x and y be the other vertices of the central simplex of u. If $\Theta(y) \leq \frac{\pi}{6}$, $(\sigma(n-1), \tau(n))$ are in a basic configuration by [FP, Lemma 3.6], so we suppose this isn't the case. But then $\Theta(y) \neq \frac{\pi}{3}$ and in all remaining cases, $(\sigma(n-1), \tau(n-2))$ are in a basic configuration. And this case is concluded.

(b) σ intersects ∂X_{σ_0} $(n-1)$ in an edge.

For this to be the case, $l(v) = C$, by C9), for it can not be an edge star vertex in σ_0. So if τ combs in σ_0, or if τ is a phantom edge simplex in both views, as in the previous case, τ combs to a simplex distance one from σ and we are done. So, suppose that this is not the case. Now, we only need to worry if $\tau \in$ $star_{\tau_0}$ (v, n). But unless v loses σ in τ_0, we have already finished this case (in Configuration 2 or 3). And this implies that v slides, and that is ridiculous since $l(v) = C$. So this case is finished.

This finishes off Configuration 5.

It is worth stopping to quell the rumors at this point. It seems that we may have run into some circular reasoning at this point. In Configurations #1-4 we often stopped when we were led to Configuration 5, and now in Configuration

5, we find that we stop when we get back to something in one of Configurations #1-4. Highly suspicious we would agree, but that we have already taken care of this suspicion. Lemma 3.10 was proved just for the sake of this worry. We see that this kind of trading from one configuration to the other can not lead to two words that are far apart, not geometrically as simplices, but simply in spelling length. The lemma says that they can't possibly be in any of the basic configurations, unless they are of similar word lengths. So, this is not really the trouble it seems to be.

But after all that, there is a big problem we have ducked all along. That is what is left for the final basic configuration.

Phantom Edge cases In all of the possible previous configurations, suppose that at least one of σ or τ is a phantom edge simplex.

The argument for Configuration 1 has actually already dealt with this possibility, so we move on to the next three which all have the property that σ and τ intersect in a vertex in ∂X_{σ_0} $(n+1)$. If either were a phantom edge simplex, it would have to be outer phantom edge simplex (for an inner phantom edge simplex does not have a vertex in the boundary of the level it appears in). But an outer phantom edge simplex does not have an edge in ∂X_{σ_0} (n), so Configuration 2 makes no sense here.

For Configuration 3, this makes sense if τ is the outer phantom edge simplex (σ can not be one). (Actually, this case is not Configuration 3, since here we have τ intersecting the boundary in two vertices instead of one. But since this can only occur when we have an outer phantom edge, we did not put this in the definition of Configuration 3. Nonetheless, we take care of it now, as if it fulfilled the hypothesis of Configuration 3 because, in spirit, this is where the case belong.) Now, σ and τ do not share an edge, and neither of them have a vertex in $\partial X_{\sigma_0}(n+1)$, so $diam[link_{\sigma_0}(v, n+1)]$ is at least 5. But this means that $\Theta(v) \leq \frac{\pi}{6}$. But this is ridiculous, for v is the third vertex of τ, where the other two are adjacent to the vertex that produced τ (as a phantom edge simplex). That means $l(v) = A$. So Configuration 3 is taken care of.

For Configuration 4, both σ and τ could be outer phantom edge simplices, but the above argument about the *link* of v applies again, and this case can not occur.

This is the end of all basic configuration cases. We have shown that for any two simplices that are distance one away from each other that the combing lines either stay in a basic configuration (with the caveat that the word lengths of the two simplices may not be the same) or lead to the so-called "complex configurations". It is time to get to these configurations, then. We, again, want to show that the combing lines lead to simplices that are in a basic configuration or that lead back to these complex configurations and are therefore at a bounded distance from one another.

So, why the need for these complex configurations? Well, we find that it is best to put these possibilities in a different place than the cases we have already taken care of for a couple of reasons. First, they are not basic configurations, that is the simplices involved are more than distance one away from each other, and we found that it was best to keep these apart from the basic configurations. And second, we have these cases here (rather than just keep them within the arguments from which they arise) because they tend to appear frequently in the previous arguments. This way we can just make the argument once, hopefully clearly, and move on.

In the following we will have the situation that σ and τ are the two simplices that we will want to calculate combing lines for, and they will be in a configuration other than the basic configurations. With two exceptions, we will assume that σ combs in σ_0, as this is all that is required from the above cases. We will be careful to mention the two for which this is not true when we get to them.

So, with no further ado...

Complex Configuration #1 u is a $\frac{\pi}{2}$-edge star vertex in $\partial X_{\sigma_0}(n)$ and v is a $\frac{\pi}{3}$-double edge star vertex in $\partial X_{\sigma_0}(n)$ and u and v are adjacent. σ is a simplex in $star_{\sigma_0}(u,n)$ which doesn't contain v and τ is a simplex in $star_{\sigma_0}(v,n)$ which is distance 2 away from σ in the gallery metric.

This is one of the cases where we need to look in both views. That is, we do not assume that σ combs in σ_0. So, we break up this case into the two different views.

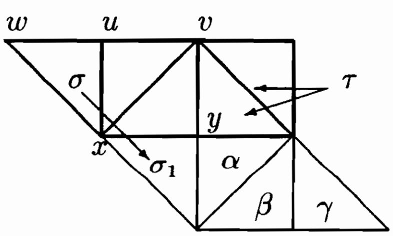


Figure 3.39: Complex Configuration #1(a)

(a) σ combs in σ_0.

See Figure 3.39. Let y be the central vertex of v in σ_0 and let x be the vertex which is adjacent to y, u and v. The situation implies that $\Theta(x) \leq \frac{\pi}{6}$ so [FP, Lemma 3.6] says that σ combs to σ_1 (as in the figure). If τ contains y then we are done $((\sigma_1, \tau)$ would be in a basic configuration), so we assume τ does not contain y. If τ combs through v we are done, and by Lemma 3.19 it must comb through z (the vertex that is in τ and is adjacent to both v and y) or v. Lemma 3.18 then says that we have a basic configuration unless τ combs to γ. In this case we have complex configuration #2.

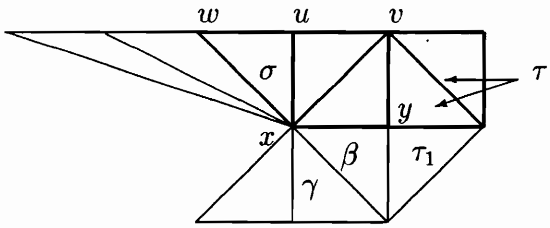


Figure 3.40: Complex Configuration #1(b)

(b) τ combs in σ_0.

See Figure 3.40. It is clear that τ combs to τ_1 no matter which of the two simplices it is. We only need see where σ combs. But as before, we know that $\Theta(x) \leq \frac{\pi}{6}$, so this will prove easy to do. We know that σ must be at least distance 3 from $X_{\sigma_0}(n-1)$ (or $l(x)$ would have a circuit of less than length 12), so x can not gain σ in τ_0, and σ must comb through x. If σ combs to a simplex that is distance one from τ_1 we are, of course, done. The only way for this not to occur is for x to lose in τ_0. Then σ will comb to a simplex more than distance one away from τ_1 and this can occur. But this is precisely complex configuration #7 (when looked at with τ_1 as σ and $\sigma(n)$ as τ).

Complex Configuration #2 u is a $\frac{\pi}{2}$-edge star vertex in ∂X_{σ_0} $(n+1)$ and x is a vertex adjacent to u in ∂X_{σ_0} $(n+1)$ which is a triple edge star vertex. $\sigma = star_{\sigma_0}$ $(\vec{ux}, n+1)$ and τ is a simplex in $star_{\sigma_0}$ $(x, n+1)$ that is distance two from σ in σ_0 with the gallery metric. We assume that σ combs in σ_0.

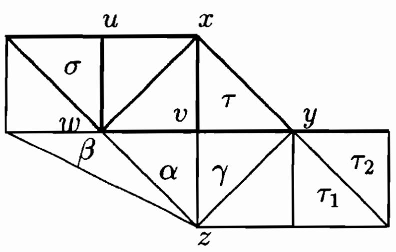


Figure 3.41: Complex Configuration #2

See Figure 3.41. Suppose τ combs through v. Then unless τ combs to γ we are done (distance one from σ). If τ does comb to γ we still have distance one if w is an edge star vertex in σ_0 (obviously) or if w is a 2-simplex star vertex in σ_0 (by a distance argument and noting that $A < C$ in case of a tie). Finally, if w is a double edge star vertex in σ_0, we have complex configuration #1 (where τ combs in σ_0).

So suppose τ does not comb through v. This implies that v must have gained τ in τ_0. So either v gained one or it slid. If v gained one in τ_0, (τ, σ) are in complex configuration #6, where τ plays the role of σ and vice-versa.

(Note that in that case we only do the case that σ combs in σ_0, and in fact what we are doing is τ combs in τ_0, which is equivalent. Another thing to watch for here, is that we haven't actually done anything in this argument.

We are just delaying it until later. We need to be careful that we don't do the same thing in the argument of complex configuration #6, but we don't.)

So, all that is left is the case where v slides. In this case, w gains two in τ_0, so it must have been an edge star vertex in σ_0 (since it is a $\frac{\pi}{3}$vertex). (This is why this case is drawn in the figure.) This means σ either combs to a simplex distance one from τ (that is, α) or one adjacent to α. But notice that y had to lose two in τ_0 for v to slide. This means (again since y is a $\frac{\pi}{3}$vertex) that y is an edge star vertex in τ_0 and a double edge star vertex in σ_0 (naturally this is all as must be in the case where a vertex slides). That means that τ must comb to τ_1 or τ_2. In either case, we have complex configuration #4 for (β, τ_i) for $i = 1$ or 2.

This then completes the argument for complex configuration #2 (though it once again delays some of the argument).

Complex Configuration #3 u is a $\frac{\pi}{2}$-edge star vertex and v is a $\frac{\pi}{3}$-double edge star vertex which is adjacent to u and w is another $\frac{\pi}{2}$-edge star vertex adjacent to v in ∂X_{σ_0} $(n+1)$. Suppose $\sigma \in star_{\sigma_0}$ $(u, n+1)$ but $\sigma \notin star_{\sigma_0}$ $(v, n+1)$ and $\tau \in star_{\sigma_0}$ $(w, n+1)$ but $\tau \notin star_{\sigma_0}$ $(v, n+1)$. Finally, suppose σ combs in σ_0.

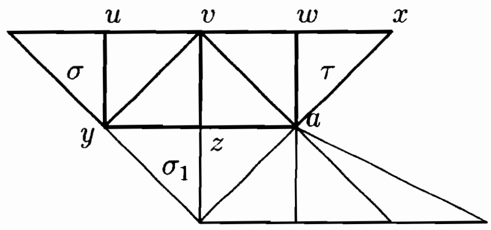


Figure 3.42: Complex Configuration #3

See Figure 3.42. Let z be the central vertex of v in σ_0. Let y and a, respectively, be the vertices in $link_{\sigma_0}(v, n+1)$ that are adjacent to z and to u and w, respectively. Then $\Theta(y) = \Theta(a) \leq \frac{\pi}{6}$ by assumption. [FP, Lemma 3.6] then gives that σ combs to σ_1 (the simplex $star(\vec{yz})$).

If τ combs through w, we get complex configuration #1 for $(\sigma, \tau(n))$ (the only way this could occur is for w to slide). And τ does not comb through x, so we suppose τ combs through a. Now, if τ combs to a simplex that is distance one from σ_1, we are done, of course, so suppose it does not. How could this occur? [FP, Lemma 3.6] again gives that if a stays the same or gains or even loses one, τ will comb to a simplex that is distance one from σ_1. So, all that leaves is for a to lose two in τ_0. For a to lose two in τ_0, a must have been at least a double edge star vertex in σ_0. If a is a double edge star vertex in σ_0, no matter which simplex in $star_{\tau_0}(a, m)$ τ combs to, we have complex configuration #1 in σ_0 for $(\sigma_1, \tau(m))$. If a is a triple edge star vertex in σ_0, then we either get complex configuration #2 or complex configuration #7 for $(\sigma_1, \tau(n))$.

This finishes of complex configuration #3.

Complex Configuration #4 Suppose w,v, and u are vertices that are adjacent in $\partial X_{\sigma_0}\ (n+1)$ and that w is a $\frac{\pi}{3}$-edge star vertex, v is a $\frac{\pi}{2}$-edge star vertex and u is a $\frac{\pi}{3}$-double edge star vertex. Let σ be a simplex in $star_{\sigma_0}\ (w, n+1)$ that is not in $star_{\sigma_0}\ (v, n+1)$ and let τ be a simplex in $star_{\sigma_0}\ (u, n+1)$ that is not int $star_{\sigma_0}\ (v, n+1)$. Assume that σ combs in σ_0.

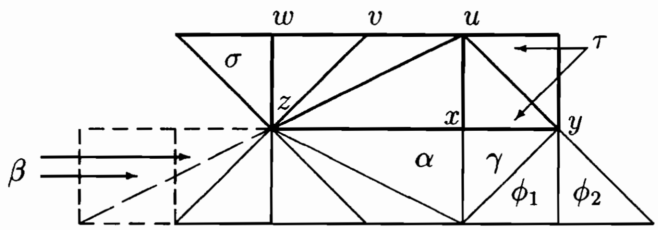


Figure 3.43: Complex Configuration #4

See Figure 3.43. Let z be the central vertex of w (and v) and x be the central vertex of u in σ_0. Let y be the vertex, other than x, that τ intersects $\partial X_{\sigma_0}\ (n)$ in. Now we know, by assumption, that $\Theta(y) = \Theta(z) \leq \frac{\pi}{6}$, so by Lemma 3.18, τ combs to γ, ϕ_1, or ϕ_2 (or it combs through u and is distance one from α) and if z is not a triple edge star vertex in σ_0, σ combs to α (as in the figure). In the case that τ combs to γ or ϕ_1 we have a basic configuration. If τ combs to ϕ_2, we have complex configuration #2 for (α, ϕ_2).

So, assume z is a triple edge star vertex in σ_0. Now, σ could possibly comb to β, so say it does. We will show that τ must comb to γ and then (β, γ) are in complex configuration #7. So, we need to rule out the possibilities that τ combs to ϕ_1 or ϕ_2. To comb to either one, y would have to lose (one or two) in τ_0. This implies that x gains. But, this again, requires that x be a 2-simplex star vertex in τ_0 and we must violate Lemma 3.17. So, τ must comb to γ (or through u) and we are done.

Complex Configuration #5 Let u be a $\frac{\pi}{2}$-edge star vertex and let v be a $\frac{\pi}{4}$ or $\frac{\pi}{5}$ double edge star vertex in σ_0, and say they are adjacent in ∂X_{σ_0} $(n+1)$. Suppose $\sigma \in star_{\sigma_0}$ $(u, n+1)$ but $\sigma \notin star_{\sigma_0}$ $(v, n+1)$ and $\tau \in Star_{\sigma_0}$ $(v, n+1)$ but $\tau \notin star_{\sigma_0}$ $(u, n+1)$ and that the distance between τ and σ is greater than 1. Finally, assume σ combs in σ_0.

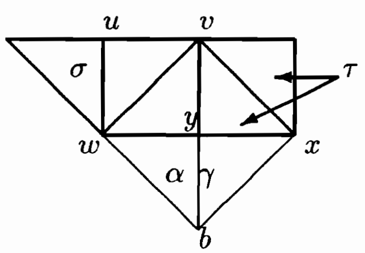


Figure 3.44: Complex Configuration #5

See Figure 3.44. Let y be the central vertex of v in σ_0 and let w be the central vertex of u in σ_0. Let $\alpha = star_{\sigma_0}$ $(\vec{wy}, n)$. If σ combs to α, the argument is fairly quick. The only way α and τ or $\tau(n)$ are not distance one away is if x (the vertex other than y that τ intersects ∂X_{σ_0} (n) in) is a double edge star vertex (this is just a distance argument). And in that case, $(\alpha, \tau(n))$ are in this very complex configuration in σ_0 (y must be an edge star vertex in σ_0, by C9)).

So suppose σ does not comb to α. Let us look at where τ combs. This occurs in τ_0. If τ combs through v, we are done, and there is no possibility left besides

τ combing through x, so let us explore that one. How could τ comb to any simplex other than γ? Well, a simple distance argument says that that can only happen if x loses γ in τ_0. But if x loses γ, it must have been at least a 2-simplex star vertex in σ_0, and this implies that b must have been a double edge star vertex in σ_0 (or it would have to be a $\frac{\pi}{3}$vertex). But wait. If x loses in τ_0, b gains in τ_0, and that isn't possible. So, x can't lose (nor can it gain), so τ combs to γ.

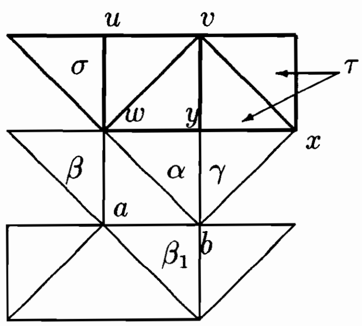


Figure 3.45: Complex Configuration #5–w is a 2-simplex star vertex in σ_0

Now, we must go back and look at w in σ_0. For σ to fail to comb to α, w must be either a 2-simplex star vertex or a double edge star vertex in σ_0. If it is a 2-simplex star vertex, $l(w) = C$ (otherwise, since C is greater than A, ties go to α). See Figure 3.45. Now, σ combs to β. But β combs to β_1 and we have a basic configuration for (β_1, γ) unless a is a 2-simplex star vertex in σ_0. But in this case, C9) gives that b is an edge star vertex in σ_0 and can't gain except on the a side. This implies that γ must comb to β_1 and we are done.

Finally, suppose w is a double edge star vertex in σ_0. See Figure 3.46.

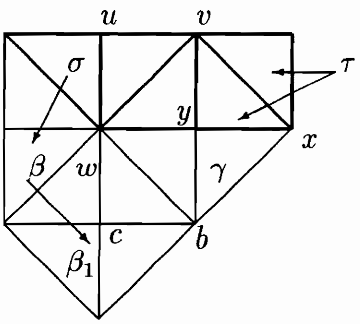


Figure 3.46: Complex Configuration #5–w is a double edge star vertex in σ_0

First, it is easy to see, by a distance argument, noting the angles of the vertices, that β combs to β_1. Now, we must figure out where γ can comb in τ_0. Let b be the third vertex of γ. This vertex is what we need to investigate. If b is a double edge star vertex in σ_0, either γ combs to a simplex adjacent to β_1 and we are done, or we have this very complex configuration for $(\beta_1, \gamma(m-1))$ (actually b could also be a 2-simplex star vertex in τ_0, but this case will be dealt with below).

But we still need to deal with the other two possibilities. They are complex enough that they warrant separate investigation.

(a) Suppose b is an edge star vertex in σ_0.

See Figure 3.47.

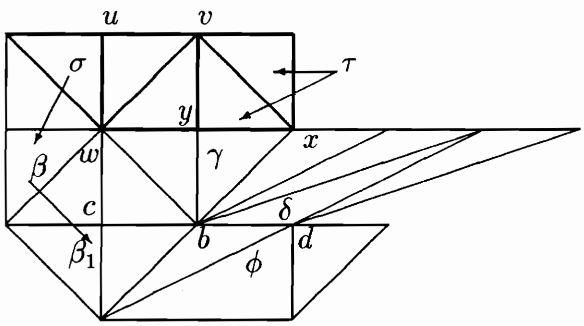


Figure 3.47: Complex Configuration #5–w is an edge star vertex in σ_0

If γ combs to a simplex adjacent to β_1, we are done, of course. So let us examine how that could fail to occur. One way, is for b to gain two on the d side in τ_0. Then it is possible that γ will comb to a simplex that is not adjacent to β_1. But in this case, we have the situation of (β, γ) above (note that it is not the situation of the beginning of this case, since β_1 combs in σ_0, etc.). So, that case gets no further apart. The other possibility is that b gains one in τ_0 (on the d side). In that case, we have, in τ_0, the situation of Figure 3.48.

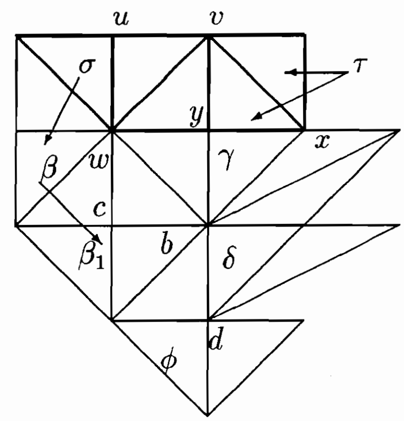


Figure 3.48: Complex Configuration #5–b gains one in τ_0. τ_0 view

In this case, a distance argument gives that δ combs to ϕ, and we have a basic configuration for (β_1, ϕ).

This finishes off this case.

(b) b is a 2-simplex star vertex in σ_0.

This time the case is as in Figure 3.48 above, though this time the view is in σ_0. But figuring out where δ comes is not as easy. We have to take into account the possibility that b changes in τ_0. Actually we have done this already. If b gains, it is a double edge star vertex, and we have the case of (β, γ) in the above arguments for (β_1, δ). b can't gain two, and if it loses or stays the same, then (β_1, δ) are in the situation of Figure 3.48.

Thus, Complex Configuration #5 is no problem. Whew.

Complex Configuration #6 Suppose u is a $\frac{\pi}{2}$-2 simplex star vertex and v is a $\frac{\pi}{3}$vertex adjacent to it in $\partial X_{\sigma_0}(n)$. Let σ be a simplex in $star_{\sigma_0}(u, n)$ that does not intersect the central vertex of v in X_{σ_0} (let's call it z), and let τ be a simplex that is in both $star_{\sigma_0}(w, n+1)$ and $star_{\sigma_0}(v, n+1)$ which does not intersect σ. Finally, assume σ combs in σ_0.

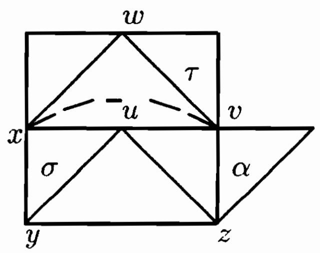


Figure 3.49: Complex Configuration #6

See Figure 3.49. By Lemma 3.17, v can't gain except on the u side, and it can't lose (since it is an edge star) in τ_0, so τ must comb either to a simplex which is distance one from σ or to α. From here, we break this case up into two cases depending on whether $\vec{yz}$ is a phantom edge or not.

(a) Suppose $\vec{yz}$ is not a phantom edge.

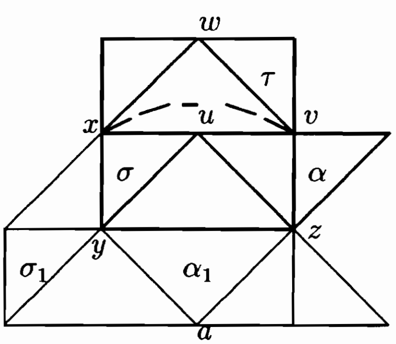


Figure 3.50: Complex Configuration #6–$\vec{yz}$ is not a phantom edge

[FP, Lemma 3.4] says that σ combs to a simplex adjacent to α unless y is a 2-simplex star vertex in σ_0. So we suppose it is; see Figure 3.50.

Now, σ combs to σ_1. If α combs to α_1, we are done, so we look at how that could fail to occur. The only way is for z to lose two in τ_0. But, to lose two, a would have to gain in τ_0, and it is already a $\frac{\pi}{2}$-2 simplex star vertex in σ_0 (to lose two z must be at least a double edge star vertex in σ_0). This is ridiculous, so α combs to α_1 and we are done.

(b) $\vec{yz}$ is a phantom edge.

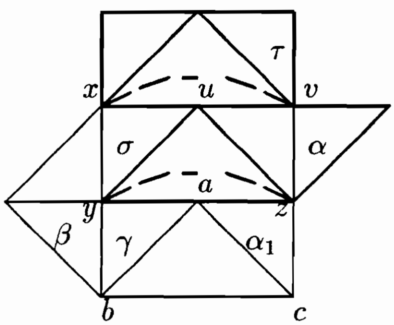


Figure 3.51: Complex Configuration #6–$\vec{yz}$ is a phantom edge

See Figure 3.51. z can not lose or a gains and this can't be. If z isn't an edge star vertex in σ_0, Lemma 3.17 says it can't gain, so in all cases, α combs to α_1 (just a distance argument with the help of C9)). Now, if σ combs to γ, we are done, so suppose σ combs to β (as in the figure). If $\vec{bc}$ is not a phantom edge [FP, Lemma 3.6] says that $(\beta(n-2), \alpha_1)$ are adjacent, and if $\vec{bc}$ is a phantom edge, $(\beta(n-2), \alpha_1)$ are in this very configuration. So, we are finished with complex configuration #6.

Complex Configuration #7 Suppose v is a triple edge star vertex and u is a $\frac{\pi}{2}$-edge star vertex which is adjacent to it in ∂X_{σ_0} $(n+1)$. Suppose σ is a simplex of $star_{\sigma_0}$ $(u, n+1)$ that is not in $star_{\sigma_0}$ $(v, n+1)$ and that τ is a simplex in $star_{\sigma_0}$ $(v, n+1)$ such that $d_g(\sigma, \tau) > 4$.

We have two cases for this configuration, depending on whether σ or τ combs in σ_0.

(a) σ combs in σ_0.

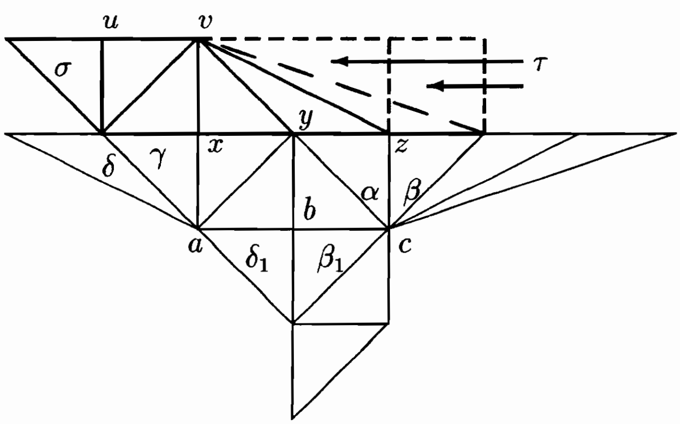


Figure 3.52: Complex Configuration #7(a)–σ combs in σ_0

See Figure 3.52. As neither y or z can gain in τ_0, τ combs to either α or to β (depending on the link of v). σ combs to either δ or to γ. (γ, α) are in complex configuration #1. (γ, β) are in complex configuration #3. (δ, α) are in complex configuration #4. So the only trouble is if σ combs to δ and τ combs to β. If β combs to β_1, we either get a basic configuration (if a is not a triple edge star vertex in σ_0), or we get this complex configuration (though case (b)). Hence, we are done if β combs to β_1. So, how could this fail to happen? By [FP, Lemma 3.6], c must have lost one or two in τ_0. In this case, b must slide or gain, and then a can be at most a double edge star vertex in σ_0 (by C9) and from sliding Case 1A). Therefore, in this case, δ combs to δ_1. Then, no matter where β combs, we have either complex configuration #2 or this complex configuration (and this case of it). In all cases we are done.

(b) τ combs in τ_0.

By the symmetry of x and z and w etc., this case is close to the previous one. That is, τ still combs to α or β and if σ combs to δ or γ the argument is the same, since anything that can happen for δ or γ could have happened for α or β in part (a) of this argument (taking into account that δ does not actually correspond to β–the argument does not change). If σ combs to a simplex in $star_{\sigma_0}(v, n+1)$, we are done since these simplices are distance one from τ. So, the only other possibility is if w gains on a side other than x's. But for w to gain, u must gain, and be a $\frac{\pi}{2}$-2 simplex star vertex in τ_0. Then, by C9), v must lose two, and z must be a 2-simplex star vertex in τ_0. But then, v has two adjacent $\frac{\pi}{2}$-2 simplex star vertices and [FP, Lemma 3.8] says no dice. So, this case is okay, and we are done.

This finishes off the complex configurations. Each of them led either to other complex configurations or to basic configurations. In either case, we stay within a bounded number at all times when we compare the σ and τ of any given case. But this can be misleading. To really finish the proof, we must insist that σ be compared to the simplex that has the same length as it, and many times in our arguments, we find that we are not comparing two simplices of the same length. But this is what Lemma 3.10 is for. It says, that if two simplices are distance one away from each other in a view, than in terms of their lengths (word lengths in their respective views), they can't be more than one letter apart. So, for all the basic configurations, this is enough to say we don't need to add more than one to get a bound that works when taking into account different word lengths. But in the complex configurations, the simplices we compare are sometimes three away when we start, so their word lengths can also be three away.

So, in conclusion, this means that 6 is a sufficient bound on the bicombing lines for a triangle of finite groups with empty face group, non- empty edge groups, total angle sum $\leq \pi$ and no single angle greater than $\frac{\pi}{2}$. These groups are biautomatic. ■

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Vita

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