

# Two Aspects of Topology in Graph Configuration Spaces

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Thesis submitted to the faculty of  
the Virginia Polytechnic Institute and State University  
in partial fulfillment of the requirements for the degree of

Master of Science  
In  
Mathematics

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7 October 2005  
Blacksburg, Virginia

keywords: braid group, manifold, pseudomanifold with boundary, fundamental  
group, graph, configuration space

## Abstract

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A graph configuration space is generated by the movement of a finite number of robots on a graph. These configuration spaces of points in a graph are topologically interesting objects. By using local, combinatorial properties, we define a new classification of graphs whose configuration spaces are pseudomanifolds with boundary. In algebraic topology, graph configuration spaces are closely related to classical braid groups, which can be described as fundamental groups of configuration spaces of points in the plane. We examine this relationship by finding a presentation for the fundamental group of one graph configuration space.

# Acknowledgements

This paper would not have been possible without my advisor, Dr. Peter Haskell. He helped and encouraged me through every step.

I am also grateful to my husband, R. Matthew Gardner, for his love and support.

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# 1 Introduction

What are graph configuration spaces and why are we interested in studying them? A configuration space is the space arising from finite generators taking all possible positions that they may lawfully attain. For example, the configuration space of a single robot moving freely in ordinary Euclidean space is simply  $\mathbb{R}^3$ . In a graph configuration space, the movement of a finite number of robots is limited to the edges and vertices of a graph.

If graph configuration spaces were strictly mathematical, any number of robots might be on any given edge or vertex simultaneously. However, configuration spaces have practical applications in robotics, where a number of robots move along a network of rails. In a practical setting, it is necessary for these robots to avoid collisions with one another. Constraints must be set in place to prevent collisions.

One interesting aspect of these configuration spaces is their topological structure. Some configuration spaces are manifolds and some can be embedded in manifolds. In [1], Aaron Abrams lays a foundation for the theory behind graph configuration spaces as well as classifying which graphs give rise to 2-manifolds based upon the movement of two robots. Praphat Fernandes furthers this work in [2] by classifying the two-robot systems which give rise to 2-pseudomanifolds without boundary.

Another aspect of graph configuration spaces is their relationship to classical braid groups, which can be described as fundamental groups of configuration spaces of points in the plane. Graph configuration spaces are specialized configuration spaces of points contained within a graph.

In the following chapters, the previous work done by Abrams and Fernandes in graph classification will be continued and expanded. Furthermore, we will find a presentation of the fundamental group of one graph configuration space.

In Chapter 2, we will examine the two main motivations for our interest in graph configuration spaces, robotics and braid groups. The constraints upon robotic movement on a graph will be further described, as well as the cell structure composing the configuration space. Previous work with manifold and pseudo-manifold spaces will be outlined along with a more complete definition of these structures.

Chapter 3 is composed of original work and will classify which graphs give rise to a specific type of space, namely the 2-pseudomanifold with boundary. This work uses local, combinatorial principles to view the cell structure of the configuration space.

In Chapter 4, we take a closer look at the relationship between the graph configuration spaces and the classical braid groups. The techniques in algebraic topology and homotopy are described which will obtain a presentation of the fundamental group of the configuration space for the graph  $K_5$ , the complete graph on five vertices.

## 2 Background and Examples

This chapter includes an overview of the two main motivations for this work with graph configuration spaces, robotics and braid groups. It was mentioned briefly in the introduction that constraints must be set in order to avoid collisions between robots on a graph. These will be defined along with the notation for the cell structure of a configuration space. Lastly, we will consider the previous work done by Abrams and Fernandes in classifying graphs whose configuration spaces are either 2-manifolds or 2-pseudomanifolds without boundary.

### 2.1 Robotics

Imagine a number of robots moving mechanically along set paths on a factory floor. The movement of these robots, limited to the edges and vertices of a graph but otherwise freely moving, generates a graph configuration space. Some very common robots, called Automated Guided Vehicles (AGVs), are used to transport objects around a factory floor. AGVs are expensive and not designed to tolerate multiple collisions. However, if these robots are on a physical plane with no restrictions on their movement, collisions will be inevitable. Obviously some constraints are necessary in order to prevent the robots from being damaged.

There are a few ways that robots could be programmed to avoid collisions. If two robots were not constrained to a graph, it would be possible to design them such that one robot could sense the other and move around it if collision seemed imminent, making each collision avoidance a local phenomenon. These robots take a large amount of computing power and are therefore quite costly. Robots that are restricted to a graph of fixed paths are more practical and less expensive. “The resolution of a collision on a graph is a non-local phenomenon” [3], because at least one robot must make a change in its path before reaching another robot. Once a graph of paths has been determined, it is desirable to arrange each robot to start at a specific, predetermined location while a vector field controls its movement around the graph so that collisions are impossible. Ghrist, Koditschek and Rimon explain the theory and some solutions to this problem in [4] and [5].

In order to generate a graph configuration space from robotic movement with no collisions, we would like to create constraints so that every robot is at least some fixed distance from every other robot. The natural structure of a graph suggests the solution that every robot be at least one full edge apart at all times. When every robot is constrained in this way on a graph, the resulting generated space is a discretized configuration space. This discretized configuration space will be the type of graph configuration space used in the remainder of this paper.

To examine more thoroughly the applications of graph configuration spaces arising from robotics, see [3] or [6].

## 2.2 Braid Groups

The classical braid groups, which began as a way of studying knots, are an area of mathematics with applications in topology, group theory and mathematical physics. A braid group can be described as the fundamental group of a configuration space of points in the plane. The configuration space from which the braid group is derived is generated by a discrete number of particles moving in  $\mathbb{R}^2$  without collisions. As particles move in time, represented by height in  $\mathbb{R}^3$ , the particles' paths cross over and under one another, forming the "braid" pattern.

Some immediate comparisons can be made between a braid group configuration space and a discretized graph configuration space. The graph configuration space is also a configuration space of points, but its generators may only move upon points within the graph. Unlike the discretized graph configuration space which must remain at least one edge apart, the particles of the braid group configuration space are parameterized such that no two particles may occupy the same point in  $\mathbb{R}^2$ . Perhaps the largest difference between the two is that braid group configuration spaces are entirely dependent upon the number of generators employed. Because the generators are completely free to move anywhere within  $\mathbb{R}^2$ , every braid group configuration space with seven generators is isomorphic to any other braid group configuration space with seven generators. However, the discretized graph configuration spaces are dependent not only upon the number of generators employed, but also upon the structure of the graph. The discretized configuration space generated by three robots moving upon  $C_7$ , the cyclic graph on seven vertices, is a very different space than that generated by three robots moving upon  $K_7$ , the complete graph on seven vertices.

For more information on braid groups and graph braid groups, that is, the fundamental groups of configuration spaces of graphs, see [7].

## 2.3 Definitions and Notation

The following graph-related definitions have been taken from [1] and [8].

A *graph* is a triple consisting of a vertex set, an edge set, and a relation that associates with each edge two vertices called its endpoints. Another definition of a graph is as a 1-dimensional CW-complex. The 0-cells are vertices and the 1-cells are edges. The shortest path metric makes the vertex set of a graph a metric space, whose distances are denoted by  $d$ . The distance between two vertices,  $d(v_1, v_2)$  is the number of full edges on the shortest path between the two. Although the "distance" between two edges or a vertex and an edge can not be defined precisely under the metric, the distance between two objects can still be defined as the number of full edges on the shortest path between the two. Two edges with a common endpoint,  $E_1$  and  $E_2$ , or an edge  $E_1$  and its endpoint  $v_1$  would then have distances  $d(E_1, E_2) = 0$  and  $d(E_1, v_1) = 0$ , violating the definition of a metric space. Although this shortest path distance is not a true metric, we will use it as our distance measurement.

A *loop* is an edge whose endpoints are equal. We will use only loopless graphs in our classification of graphs generating configuration spaces. *Multiple edges* are edges having the same pair of endpoints. A graph is *simple* if it has no multiple edges. The *degree* of a vertex is the number of edges of which it is an endpoint. *Adjacent* vertices are endpoints of a common edge.

A *path* is a simple graph whose vertices can be listed so that vertices are adjacent if and only if they are consecutive in the list. A *connected graph* is one having a  $u, v$ -path for every pair of vertices  $u, v$ . A *cycle* is a simple graph whose vertices can be placed on a circle so that vertices are adjacent if and only if they appear consecutively on the circle. A *tree* is a connected graph with no cycles. The *complete graph*  $K_n$  has  $n$  vertices and  $\binom{n}{2}$  edges, one connecting each pair of vertices. The *complete bipartite graph*  $K_{m,n}$  has  $m + n$  vertices  $x_1, \dots, x_m, y_1, \dots, y_n$  and  $mn$  edges, one connecting  $x_i$  to  $y_j$  for each  $i$  and  $j$ .

Previously, we described the constraints upon robotic movement so that every robot must be at least one full edge apart from all other robots at all times. Now that we have our distance metric, we can define this parameterization for robots  $R_1$  and  $R_2$  so that  $d(R_1, R_2) \geq 1$ .

**Definition 2.1.** [2] Let  $R_1, \dots, R_n$  be a finite set of robots on a graph  $\Gamma$ . If some robot  $R_i$  is on a vertex  $v_i$  of  $\Gamma$  then let  $x_i = v_i$ . If  $R_i$  is on an edge  $E_i$  then let  $x_i = E_i$ . The *discretized configuration space*  $D^n(\Gamma)$  is the subset of all configurations of robots  $R_1, \dots, R_n$  such that  $d(x_i, x_j) \geq 1$  for all  $i, j$  with  $i \neq j$ .

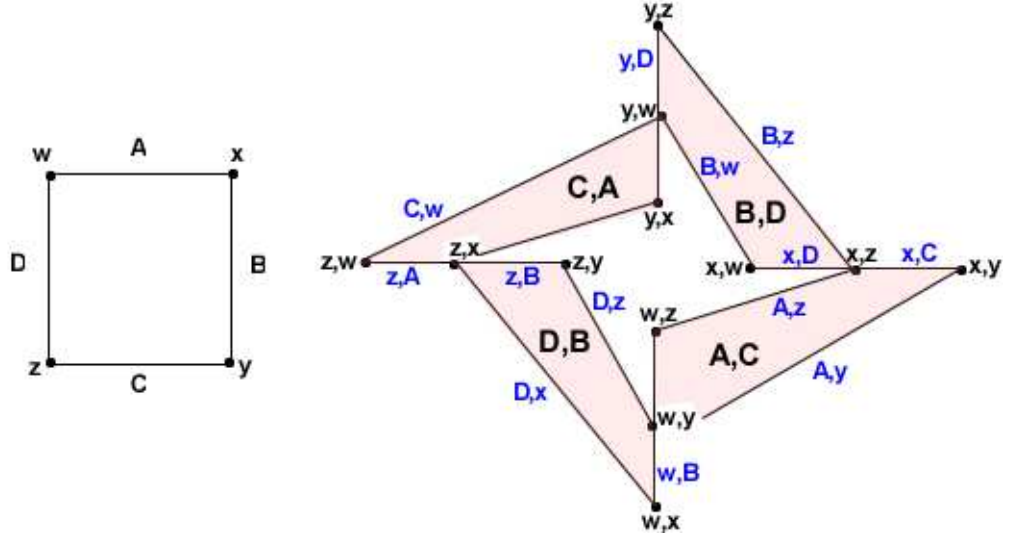
Note that the labeling of the robots is not trivial. The configuration of two robots where  $R_1$  is on vertex  $v_1$  and  $R_2$  is on vertex  $v_2$  is different than the configuration where  $R_1$  is on  $v_2$  and  $R_2$  on  $v_1$ .

For the remainder of this paper, when we refer to the configuration space, unless otherwise specified, we are speaking of the discretized graph configuration space.

The discretized configuration space is a cell complex. For any number of robots, a 0-cell is generated when each robot is positioned on a distinct vertex. Earlier classification has been done on configuration spaces generated by the movement of two robots on a graph. In the cell complex composing such configuration spaces, the 0-cells are generated by the stationary positions of two robots on two distinct vertices  $v_1$  and  $v_2$ . The 1-cells are generated by one stationary robot on a vertex  $v$  and one robot moving along an edge  $E$  such that  $d(v, E) \geq 1$ . The 2-cells are generated by two robots  $R_1$  and  $R_2$  moving along two distinct edges  $E_1$  and  $E_2$  such that  $d(E_1, E_2) \geq 1$ . A cell is labeled by its generating vertices and/or edges, so that the cells just described would be labeled as 0-cell  $(v_1, v_2)$ , 1-cell  $(v, E)$ , and 2-cell  $(E_1, E_2)$ . The tuples are ordered so that the first entry is the position of the  $R_1$  and the second entry is the position of  $R_2$ . In a cell complex generated by  $n$  robots, cells would be labeled by the  $n$ -tuple  $(p_1, p_2, \dots, p_n)$ , denoting the ordered positions of robots 1 through  $n$ .

Before moving on to more complex spaces, we will demonstrate the generation of a simple graph configuration space.

**Example 2.2.** Consider two robots moving on a square graph  $\Gamma$  with the constraint that  $d(R_1, R_2) \geq 1$ . When Robot 1 is on vertex  $w$ , Robot 2 is free to move on the vertices  $x, y$  and  $z$ , and the edges  $B$  and  $C$ . When Robot 1 is on edge  $A$ , Robot 2 is free to move on the vertices  $y$  and  $z$ , and the edge  $C$ . The space  $D^2(\Gamma)$  is connected, with twelve 0-cells, sixteen 1-cells, and four 2-cells. Every 1-cell is in the boundary of exactly one 2-cell.



Note that we have labeled the cells without parenthesis to save space. 1-cells have been labeled in blue, and 2-cells have been enlarged and highlighted.

In future visualizations, we will label only the 0-cells, using tuples without parenthesis.

## 2.4 Configuration Spaces: 2-manifolds and 2-pseudomanifolds Without Boundary

The structure of a graph configuration space is dependent upon both the structure of its graph and the number of robots on that graph. The configuration space of a graph with only one robot is trivial and has the same structure as the graph itself, so it makes sense to begin by studying configuration spaces generated by two robots. Working with the movement of two robots a graph led Abrams to discover that some of the resulting configuration spaces were special structures [1]. Abrams proved that two specific graphs give rise to 2-manifold configuration spaces. This classification was continued by Fernandes, who proved that two specific graphs give rise to 2-pseudomanifolds without boundary. See Fernandes' work [2] for an excellent background to special topological structures and graph classification.

Before stating the earlier results of graph classification, we should better understand the topology and properties of manifolds and pseudomanifolds.

**Definition 2.3.** The union of all  $k$ -dimensional and lower cells within a cell complex is called a *k-cell complex*. All cells of dimension lower than  $k$  are contained in the closure of some  $k$ -cell.

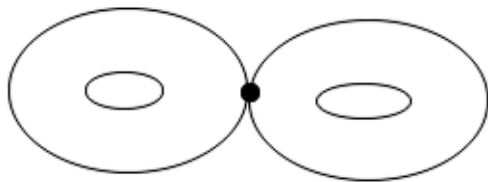
**Definition 2.4.** The union of all  $(k-1)$ -cells that do not lie in the intersection of at least two  $k$ -cells is called the *boundary* of a  $k$ -cell complex.

A 2-dimensional manifold, more commonly called the *2-manifold*, is a 2-cell complex with the same local properties as the plane in Euclidean space. Around every point of the 2-manifold, there is a neighborhood that is topologically the same as the open unit ball in  $\mathbb{R}^2$ . Using the properties of the cell structure, we define the 2-manifold to be a 2-cell complex in which every 1-cell lies in the intersection of exactly two 2-cells, permitting no singularities. A singularity in a graph configuration space would be a point in which the space was not locally Euclidean.

One property of the 2-manifold that promoted earlier research was its orientability or nonorientability. It is speculated that a discretized graph configuration space is always orientable, but this remains an open question.

**Definition 2.5.** Let a path be started at any point of a manifold with a definite choice of orientation and travel in that orientation around the surface of the manifold. If the path returns to its initial point with its original orientation reversed, the path is called *orientation-reversing*. If the path returns to its initial point with the same orientation with which it began, the path is called *orientation-preserving*. A 2-manifold is called *orientable* if every closed path is orientation-preserving [9].

A *2-pseudomanifold without boundary* is a 2-cell complex in which every 1-cell lies in the intersection of exactly two 2-cells. In a 2-pseudomanifold, singularities are permitted so that the space is not locally Euclidean at every point. A singularity is a 0-cell that does not have any Euclidean neighborhoods. No other singularities are possible in a 2-pseudomanifold except those occurring in 0-cells. A singularity occurring between two tori might have the following appearance:



A 2-pseudomanifold is also either orientable or non-orientable. For a 2-pseudomanifold to be orientable, all the 2-cells are able to be simultaneously oriented so any pair having a common 1-dimensional face are oriented coher-

ently. Pseudomanifolds are the most general structures that retain a meaningful concept of orientability. [9].

We now turn our attention to the specific graphs whose configuration space is a special structure. Each of these graphs has a configuration space generated by two robots.

**Theorem 2.6.** [1] *Let  $\Gamma$  be a connected graph without loops. If  $D^2(\Gamma)$  is homeomorphic to a closed 2-dimensional manifold then  $\Gamma = K_5$ , the complete graph on five vertices, or  $K_{3,3}$ , the complete bipartite graph on six vertices.*

Specifically,  $D^2(K_5)$  is homeomorphic to the 6-holed torus and  $D^2(K_{3,3})$  is homeomorphic to the 4-holed torus.

**Theorem 2.7.** [2] *Let  $\Gamma$  be a connected graph without loops. If  $D^2(\Gamma)$  is homeomorphic to a closed 2-dimensional pseudomanifold without boundary then  $\Gamma =$  doubled  $K_4$ , the complete graph on four vertices with each edge doubled, or doubled  $C_4$ , the cyclic graph on four vertices with each edge doubled.*

The configuration space of the doubled  $C_4$  is composed of a cycle of four tori, each joined to its neighbors by a single 0-cell. The configuration space of the doubled  $K_4$  is best described as a cube built out of six tori, in which each torus shares a single 0-cell with four distinct tori.

We observe that in each case,  $\Gamma$  was required to be both loopless and connected. Every graph for the remainder of this paper will be assumed loopless unless otherwise specified. We will briefly discuss connectivity in the next section.

### 3 Configuration Spaces: 2-pseudomanifolds With Boundary

Now that we have defined 2-pseudomanifolds *without* boundary, we are ready to classify the graphs resulting in 2-pseudomanifolds *with* boundary. We recall that the boundary of a 2-pseudomanifold is the union of all 1-cells that do not lie in the intersection of at least two 2-cells. A 2-pseudomanifold with boundary is a 2-cell complex in which every 1-cell lies in the intersection of *either one or two* 2-cells. A cell complex having a 1-cell that lies in the intersection of no 2-cells is not a 2-pseudomanifold of any type, because singularities in 2-pseudomanifolds can only occur in 0-cells.

This remainder of this section is composed of original work. Each of the following graph configuration spaces is generated by two robots.

#### 3.1 Preliminaries

There are some properties common to all graphs resulting in 2-pseudomanifolds, both with and without boundary. Perhaps the most intuitive is the necessity of connectivity. Consider two robots on a disconnected graph. If the robots are on the same component of the graph, the configuration space will be generated only by that single component. If the robots are on two different components  $\Gamma_1$  and  $\Gamma_2$ , the configuration space will be  $\Gamma_1 \times \Gamma_2$ , and no interchange of positions will be possible. Thus it is necessary for configuration space generating graphs to be connected.

The following theorems deal primarily with vertex degrees in  $\Gamma$ . Corollaries 3.2 and 3.4 will be used repeatedly in later proofs to identify acceptable and unacceptable graph arrangements.

**Theorem 3.1.** *If  $\Gamma$  is a connected graph with a 2-pseudomanifold configuration space, with or without boundary, then every vertex in  $\Gamma$  is adjacent to at least two other vertices.*

*Proof.* Let  $\Gamma$  be a connected graph with a 2-pseudomanifold configuration space. By the definition of  $D^2(\Gamma)$ ,  $\Gamma$  cannot be a single vertex. If  $\Gamma$  consisted of two connected vertices, then  $D^2(\Gamma)$  would be a single 0-cell and not a 2-pseudomanifold. Thus  $\Gamma$  has at least three vertices.

Suppose that there exists some vertex  $v$  in  $\Gamma$  such that  $v$  is adjacent to exactly one other vertex  $w$ . Let  $\{E_1, \dots, E_m\}$ ,  $m \geq 1$ , denote the set of edges having both  $v$  and  $w$  as endpoints. Vertex  $w$  must be adjacent to at least one other vertex  $x$ , lest  $\Gamma$  have only two vertices, so that  $w$  and  $x$  are endpoints of at least one edge  $F$ . Now all edges with endpoint  $v$  are such that  $d(E_k, F) < 1$ ,  $1 \leq k \leq m$ . Therefore, the 1-cell  $(v, F)$  in  $D^2(\Gamma)$  is in the boundary of no 2-cells and  $D^2(\Gamma)$  is not a 2-pseudomanifold. Every vertex in  $\Gamma$  must be adjacent to at least two other vertices.  $\square$

**Corollary 3.2.** *If  $\Gamma$  is a simple, connected graph with a 2-pseudomanifold configuration space, with or without boundary, then every vertex in  $\Gamma$  has degree at least 2.*

**Theorem 3.3.** *If  $\Gamma$  is a connected graph with a 2-pseudomanifold configuration space, with or without boundary, then  $\Gamma$  contains no 3-cycle with a vertex adjacent to exactly two other vertices.*

*Proof.* Let  $\Gamma$  be a connected graph with a two-pseudomanifold configuration space. Assume that  $\Gamma$  contains a 3-cycle with vertices  $v$ ,  $x$  and  $y$ , where vertex  $v$  is adjacent to only the two vertices  $x$  and  $y$ . Vertices  $x$  and  $y$  are the mutual endpoints of edge(s)  $[E_1, \dots, E_m]$  with  $m \geq 1$ . Since  $d(v, E_k) = 1$  for  $1 \leq k \leq m$ ,  $D^2(\Gamma)$  contains the 1-cells  $(v, E_k)$ . For any edges  $F_i$  and  $G_j$  with respective endpoints  $x, v$  and  $y, v$ , then  $d(E_k, F_i) < 1$  and  $d(E_k, G_j) < 1$ . Thus, the 1-cells  $(v, E_k)$ ,  $1 \leq k \leq m$ , are in the boundary of no 2-cells in  $D^2(\Gamma)$  and  $D^2(\Gamma)$  is not a two-pseudomanifold. Then  $\Gamma$  contains no 3-cycle with a vertex adjacent to exactly two other vertices.  $\square$

**Corollary 3.4.** *If  $\Gamma$  is a simple, connected graph with a 2-pseudomanifold configuration space, with or without boundary, then  $\Gamma$  contains no 3-cycle with a vertex of degree 2.*

The next corollary allows us to begin to classify graphs systematically, as it puts a lower limit on the number of vertices that  $\Gamma$  is allowed.

**Corollary 3.5.** *If  $\Gamma$  is a connected graph with a 2-pseudomanifold configuration space, with or without boundary, then  $\Gamma$  has at least 4 vertices.*

*Proof.* Let  $\Gamma$  be a connected graph with a two-pseudomanifold configuration space. By Theorem 3.1, every vertex in  $\Gamma$  is adjacent to at least 2 other vertices. Thus,  $\Gamma$  contains at least 3 vertices.

In a connected graph with 3 vertices, either simple or non-simple, where every vertex is adjacent to at least 2 other vertices, every vertex in  $\Gamma$  is adjacent to exactly 2 other vertices. By the contrapositive of Theorem 3.3,  $D^2(\Gamma)$  is not a 2-pseudomanifold. Therefore,  $\Gamma$  has at least 4 vertices.  $\square$

In the next theorem and corollary, we begin to classify non-simple graphs having 2-pseudomanifold configuration spaces and find that they belong to a rather small set of possibilities for vertex arrangement. Further classification for non-simple graphs is found in Section 3.5 along with other graphs on four vertices.

**Theorem 3.6.** *Let  $\Gamma$  be a non-simple connected graph. Then if  $D^2(\Gamma)$  is a 2-pseudomanifold with or without boundary,  $\Gamma$  has at most 4 vertices.*

*Proof.* Let  $\Gamma$  be a connected graph with  $n \geq 5$  vertices and a pair of endpoints  $x$  and  $y$  with  $m$ -tuple edges  $[E_1, \dots, E_m]$  with  $m \geq 2$ . Either at least two of the remaining  $n - 2$  vertices are adjacent or no two are adjacent. If no two are adjacent, then each must be adjacent to at least one of  $x$  and  $y$  and not adjacent

to any other vertices besides  $x$  or  $y$ . By Theorem 3.1, each is adjacent to at least two other vertices in order for  $D^2(\Gamma)$  to be a 2-pseudomanifold, so then each is adjacent to both  $x$  and  $y$ . For every vertex  $v$  with  $v \neq x, y$ , there exists a 3-cycle  $v - x - y$  where  $v$  is adjacent to exactly 2 other vertices  $x$  and  $y$ . By Theorem 3.3,  $D^2(\Gamma)$  is not a 2-pseudomanifold. Thus at least two of the  $n - 2$  vertices are adjacent. Furthermore, each of the  $n - 2$  vertices not  $x$  or  $y$  must be adjacent to at least one other vertex not  $x$  or  $y$ , lest it be adjacent only to  $x$  and  $y$ .

Since each vertex is adjacent to at least two other vertices,  $x$  is adjacent to some vertex  $u$  creating edge  $G$ . If any two of the  $n - 3$  vertices not  $x, y$  or  $u$  are adjacent, these form edge  $H$  such that  $d(E_k, H) \geq 1$ ,  $2 \leq k \leq m$ , and  $d(G, H) \geq 1$ . Therefore in  $D^2(\Gamma)$ , the 1-cell  $(x, H)$  is in the boundary of at least three 2-cells, so that  $D^2(\Gamma)$  is not a 2-pseudomanifold. Thus no two of the  $n - 3$  vertices not  $x, y$  or  $u$  are adjacent.

Now we have seen that each of the  $n - 2$  vertices not  $x$  or  $y$  is adjacent to at least one other vertex not  $x$  or  $y$ , yet no two of the  $n - 3$  vertices not  $x, y$  or  $u$  are adjacent. Thus each of the  $n - 3$  vertices must be adjacent to  $u$  as well as to at least one of  $x$  and  $y$ , and since  $\Gamma$  has  $n \geq 5$  vertices, there exist at least two such vertices  $v$  and  $w$ , both adjacent to  $u$ , creating edges  $F$  and  $H$  respectively. Now vertex  $w$  must be adjacent to at least one of  $x$  and  $y$ . Already,  $d(E_k, F) = 1$  and  $d(E_k, H) = 1$ ,  $2 \leq k \leq m$ , and now if  $w$  is adjacent to  $x$  then  $d(wx, F) = 1$  or if  $w$  is adjacent to  $y$  then  $d(wy, F) = 1$ . In  $D^2(\Gamma)$ , either the 1-cell  $(x, F)$  or the 1-cell  $(y, F)$  is in the boundary of at least three 2-cells, and  $D^2(\Gamma)$  is not a 2-pseudomanifold. Therefore, if  $D^2(\Gamma)$  is a 2-pseudomanifold, then  $\Gamma$  has at most 4 vertices.  $\square$

**Corollary 3.7.** *Let  $\Gamma$  be a non-simple connected graph. Then if  $D^2(\Gamma)$  is a 2-pseudomanifold with or without boundary,  $\Gamma$  has exactly 4 vertices.*

*Proof.* Let  $\Gamma$  be a non-simple connected graph with a 2-pseudomanifold configuration space. By Corollary 3.5,  $\Gamma$  has at least 4 vertices. However, by Theorem 3.6,  $\Gamma$  has at most 4 vertices. Therefore,  $\Gamma$  must have exactly 4 vertices.  $\square$

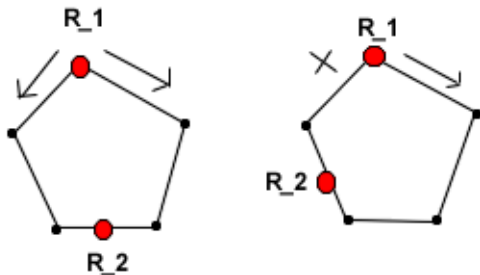
These theorems are the basis for all the others and will be used to rule out unusable cases from within a set of graph arrangements. They rely on local, combinatorial arguments, as do the other results in this chapter.

## 3.2 Cyclic Graphs

Knowing that  $\Gamma$  has at most 4 vertices, experimentation was begun with the most basic possible graph satisfying the theorems already established, the cyclic graph on four vertices,  $C_4$ . The configuration space for  $C_4$  has already been described in Example 2.2. We can see from the figure that every 1-cell is in the boundary of exactly one 2-cell, satisfying the requirements for a 2-pseudomanifold with boundary.

Consider a cyclic graph on more than 4 vertices. If  $R_1$  is on a vertex and  $R_2$  on an edge, forming a 1-cell in the configuration space, then  $R_1$  is free to

move onto either of two connecting edges to form a 2-cell, unless as in the  $C_4$  case, there is only one full edge in one direction between the initial positions of  $R_1$  and  $R_2$ . If that is the case, then  $R_1$  is free to move onto only one edge to form a 2-cell. We can see these possibilities clearly on  $C_5$ .



Now that we have an intuitive understanding of discretized movement on cyclic graphs, the next theorem proves that every cyclic graph with at least 4 vertices generates a 2-pseudomanifold with boundary.

**Theorem 3.8.** *If  $\Gamma$  is a cyclic graph with at least 4 vertices then  $D^2(\Gamma)$  is a 2-pseudomanifold with boundary.*

*Proof.* Let  $\Gamma$  be a cyclic graph with greater than 3 vertices, with some arbitrary vertex  $v$ . Every vertex has degree 2. For every edge  $E$  such that  $(v, E)$  is a 1-cell in  $D^2(\Gamma)$ ,  $d(v, E) \geq 1$ . If  $d(v, E) > 1$ , then  $v$  is the endpoint of two edges,  $F$  and  $G$ , such that  $d(F, E) \geq 1$  and  $d(G, E) \geq 1$ . Thus, in  $D^2(\Gamma)$ , the 1-cell  $(v, E)$  is in the boundary of exactly two 2-cells, namely  $(F, E)$  and  $(G, E)$ . If  $d(v, E) = 1$ , then  $v$  is the endpoint of one edge  $F$  where  $E$  and  $F$  have a common endpoint, with the result that  $d(E, F) < 1$ . Since  $\Gamma$  has  $> 3$  vertices,  $v$  is also the endpoint of one edge  $G$  where  $d(E, G) \geq 1$ . Thus, in  $D^2(\Gamma)$ , the 1-cell  $(v, E)$  is in the boundary of exactly one 2-cell, namely  $(E, G)$ . Therefore, every one cell in  $D^2(\Gamma)$  is in the boundary of either one or two 2-cells. Moreover, for every  $v$ , there exists at least one edge  $E$  such that  $d(v, E) = 1$ , so there exists some 1-cell in  $D^2(\Gamma)$  such that  $(v, E)$  is in the boundary of exactly one 2-cell, causing  $D^2(\Gamma)$  to be a two-pseudomanifold with boundary.  $\square$

Since cyclic graphs can have any number of vertices greater or equal to 4, there is no upper limit to how large a graph generating a 2-pseudomanifold with boundary can be. Consider the number of graph arrangements possible on 100 labeled vertices. If there were several possible graphs from this set, it might be impossible to identify them. Fortunately, only a select category of graphs can generate a 2-pseudomanifold with boundary configuration space.

One problem with graphs on a large number of vertices is immediate. Imagine a vertex and an edge with at least two full edges between them in a graph with 100 vertices. When  $R_1$  is situated on the edge and  $R_2$  on the vertex, the two robots form a 1-cell in their configuration space. If the vertex has degree

greater than 2, then  $R_2$  is free to move onto more than two edges, so that the original 1-cell is in the boundary of more than two 2-cells.

The following theorem and corollary prove that for a graph on 7 or more vertices, the only vertex arrangement that eliminates the above difficulty is that of the cyclic graph.

**Theorem 3.9.** *If  $\Gamma$  is a simple connected graph on at least 7 vertices with a configuration space that is a 2-pseudomanifold with boundary, then every vertex in  $\Gamma$  has degree at most two.*

*Proof.* Let  $\Gamma$  be a simple connected graph on  $n \geq 7$  vertices with a 2-pseudomanifold configuration space with boundary. Suppose that  $\Gamma$  has a vertex  $v$  of degree  $m \geq 3$ . We will consider the cases where  $m = 3$ ,  $4 \leq m \leq 5$ , and  $m \geq 6$ , and show that for each case,  $D^2(\Gamma)$  is not a 2-pseudomanifold with boundary.

Case 1: Vertex  $v$  has degree 3.

If  $v$  has degree 3, being adjacent to vertices  $x$ ,  $y$  and  $z$ , then there remain  $n - 4$  vertices that are not adjacent to  $v$ . Either at least two of these  $n - 4$  vertices are adjacent, or else no two are adjacent. If any two are adjacent, they form an edge  $E$  such that  $d(E, vx) \geq 1$ ,  $d(E, vy) \geq 1$ , and  $d(E, vz) \geq 1$ . Thus the 1-cell  $(v, E)$  in  $D^2(\Gamma)$  is in the boundary of three 2 cells, namely  $(E, vx)$ ,  $(E, vy)$ , and  $(E, vz)$  and  $D^2(\Gamma)$  is not a 2-pseudomanifold.

If no two of the  $n - 4$  vertices not adjacent to  $v$  are adjacent to one another, then let the  $n - 4$  vertices along with vertex  $v$  be in a subset  $A$ , and let the 3 vertices adjacent to  $v$  be in a subset  $B$ , such that  $A \cup B = \Gamma$  and  $A \cap B = \phi$ .  $A$  contains at least 4 vertices, such that  $v$  has degree 3 and every other vertex has degree  $\geq 2$ . The sum of vertex degrees over  $A$ ,  $\sum_{v_i \in A} deg(v_i) \geq 3 + (3 * 2) = 9$ . No

two vertices in  $A$  are adjacent, thus there at at least 9 edges with one endpoint in  $A$  and the other endpoint in  $B$ . Since there are exactly 3 vertices in  $Y$  and  $9 \div 3 = 3$ , there exists a vertex in  $Y$  adjacent to at least 3 vertices in  $X$ , one of those being  $v$ . Without loss of generality, say that vertex  $x \in B$  is adjacent to  $v$ , and at least two additional vertices  $u$  and  $w$ . As  $A$  has at least 4 vertices, there exists at least one more vertex in  $A$  which may or may not be adjacent to  $x$ , label this vertex  $s$ . Since every vertex has degree  $\geq 2$ , vertex  $s$  must be adjacent to at least one vertex not  $x$  and since  $s \in A$ ,  $s$  is not adjacent to any other vertex in  $A$ . Thus  $s$  is adjacent to at least one of  $y$  or  $z$ . Now there exist at least 3 edges with endpoint  $x$  but not endpoint  $s$ ,  $y$ , or  $z$ , so without loss of generality, say that  $s$  is adjacent to  $z$ . Then in  $D^2(\Gamma)$ , the 1-cell  $(x, sz)$  is in the boundary of three 2-cells, namely  $(xv, sz)$ ,  $(xu, sz)$ , and  $(xw, sz)$  and  $D^2(\Gamma)$  is not a 2-pseudomanifold.

Case 2: Vertex  $v$  has degree  $4 \leq m \leq 5$ .

Vertex  $v$  is adjacent to either 4 or 5 vertices, and not every vertex in  $\Gamma$  is adjacent to  $v$ . Since  $\Gamma$  is connected, at least one vertex not adjacent to  $v$  must be adjacent to some vertex that is also adjacent to  $v$ . Say that vertex  $u$  is not adjacent to  $v$ , but is adjacent to  $x$ , where  $x$  is adjacent to  $v$ . Then there are at least 3 edges with endpoint  $v$ , but not endpoints  $u$  or  $x$ . Therefore in  $D^2(\Gamma)$ ,

the 1-cell  $(v, xu)$  is in the boundary of at least three 2-cells and  $D^2(\Gamma)$  is not a 2-pseudomanifold.

Case 3: Vertex  $v$  has degree  $\geq 6$ .

Either  $v$  is adjacent to every other vertex in  $\Gamma$  or else there exists at least one vertex not adjacent to  $v$ . If  $v$  is adjacent to every other vertex, then since every vertex has degree  $\geq 2$ , for vertex  $x \neq v$ , then  $x$  is adjacent to at least one other vertex  $y$ . There are at least 4 edges with endpoint  $v$ , but not endpoints  $x$  or  $y$ . Therefore in  $D^2(\Gamma)$ , the 1-cell  $(v, xu)$  is in the boundary of at least four 2-cells and  $D^2(\Gamma)$  is not a 2-pseudomanifold.

If there exists at least one vertex not adjacent  $v$ , then since  $\Gamma$  is connected, at least one vertex not adjacent to  $v$  must be adjacent to some vertex that is also adjacent to  $v$ . Say that vertex  $u$  is not adjacent to  $v$ , but is adjacent to  $x$ , where  $x$  is adjacent to  $v$ . Then there are at least 5 edges with endpoint  $v$ , but not endpoints  $u$  or  $x$ . Therefore in  $D^2(\Gamma)$ , the 1-cell  $(v, xu)$  is in the boundary of at least five 2-cells and  $D^2(\Gamma)$  is not a 2-pseudomanifold.

Having ruled out the cases where  $v$  has degree  $> 2$ , we conclude that if  $D^2(\Gamma)$  is a 2-pseudomanifold with boundary, then  $v$  has degree  $\leq 2$  for all  $v$  in  $\Gamma$ .  $\square$

**Corollary 3.10.** *If  $\Gamma$  is a simple, connected graph on  $n \geq 7$  vertices with a configuration space that is a two-pseudomanifold configuration space with boundary, then  $\Gamma$  is the cyclic graph on  $n$  vertices.*

*Proof.* Let  $\Gamma$  be a simple, connected graph on  $n \geq 7$  vertices with a two-pseudomanifold configuration space with boundary. By Theorem 3.9, every vertex in  $\Gamma$  has degree  $\leq 2$ , and by Theorem 3.1, every vertex has degree  $\geq 2$ . Therefore, every vertex must have exactly degree 2, making  $\Gamma$  the cyclic graph on  $n$  vertices.  $\square$

Now that we have a lower vertex limit for all graphs and an upper vertex limit for all graphs excluding cyclics, we can classify graphs according to their number of vertices. Graphs generating 2-pseudomanifolds with boundary must have 4, 5, or 6 vertices. Recall that graphs on 5 and 6 vertices must be simple. Beginning with graphs on 6 vertices, unworkable cases will be eliminated and desirable graphs identified.

### 3.3 Graphs on Six Vertices

Two graphs on 6 vertices whose configuration spaces are special structures have already been identified.  $D^2(K_{3,3})$  is a 2-manifold, the 4-holed torus, and  $D^2(C_6)$  is a 2-pseudomanifold with boundary.

As there are many vertex arrangements possible, a few preliminary lemmas greatly narrow down the possibilities. Note that they hold true for 2-pseudomanifolds both with and without boundary.

**Lemma 3.11.** *Let  $\Gamma$  be a simple, connected graph on 6 vertices. If  $D^2(\Gamma)$  is a 2-pseudomanifold configuration space, either with or without boundary, then for any vertex  $v$  in  $\Gamma$ , the degree of  $v$  is either 2 or 3.*

*Proof.* Let  $\Gamma$  be a simple, connected graph on 6 vertices. The maximum degree of any vertex is 5. Assume that vertex  $v$  has degree 5, so that every other vertex in  $\Gamma$  is adjacent to  $v$ . Since every vertex has degree  $\geq 2$ , some two vertices  $x$  and  $y$  adjacent to  $v$  must be the endpoints of some edge  $E$  such that  $d(v, E) = 1$ . Then there are exactly three edges with endpoint  $v$  but not endpoint  $x$  or  $y$ , labeled  $F$ ,  $G$ , and  $H$ , such that  $d(E, F) = 1$ ,  $d(E, G) = 1$ , and  $d(E, H) = 1$ . Therefore, in  $D^2(\Gamma)$ , the 1-cell  $(v, E)$  is in the boundary of three 2-cells, namely  $(E, F)$ ,  $(E, G)$ , and  $(E, H)$ , and  $D^2(\Gamma)$  is not a two-pseudomanifold.

Next assume that vertex  $v$  has degree 4, so that there is exactly one vertex  $w$  not adjacent to  $v$ . Since  $\Gamma$  is connected,  $w$  must be adjacent to some vertex  $x$  thus creating edge  $E$ . Since every vertex not  $w$  is adjacent to  $v$ ,  $x$  is adjacent to  $v$ , so  $d(v, E) = 1$ . There are exactly three edges with  $v$  but not  $x$  or  $w$  as an endpoint, labeled  $F$ ,  $G$ , and  $H$ . Therefore, in  $D^2(\Gamma)$ , the 1-cell  $(v, E)$  is in the boundary of three 2-cells, namely  $(E, F)$ ,  $(E, G)$ , and  $(E, H)$ , and  $D^2(\Gamma)$  is not a two-pseudomanifold. As  $v$  cannot have degree 4 or 5 if  $D^2(\Gamma)$  is to be a two-pseudomanifold, every vertex  $v$  in  $\Gamma$  has degree  $\leq 3$ .

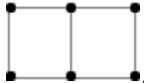
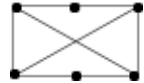
We have previously proven in Corollary 3.2 that if  $D^2(\Gamma)$  is a two-pseudomanifold with or without boundary, then every vertex in the simple, connected graph  $\Gamma$  has degree at least 2. Thus, every vertex in  $\Gamma$  has either degree 2 or degree 3.  $\square$

**Lemma 3.12.** *Let  $\Gamma$  be a simple, connected graph on 6 vertices. If every vertex in  $\Gamma$  has either degree 2 or 3, then there is an even number of vertices with degree 3 in  $\Gamma$ .*

*Proof.* Denote  $deg(\Gamma)$  to be the sum of vertex degrees over all vertices in  $\Gamma$ ,  $\sum_{i=1}^6 deg(v_i)$ , and  $E(\Gamma)$  to be the number of edges contained in  $\Gamma$ .  $deg(\Gamma) = 2 \times E(\Gamma)$ , since each edge has two vertices as endpoints. Thus  $deg(\Gamma)$  is even. Suppose that the number  $n$  of vertices in  $\Gamma$  having degree 3 is odd, and that all other vertices have degree 2.  $deg(\Gamma) = (3 \times n) + (2 \times (6 - n))$ . The product of two odd integers is odd, so  $deg(\Gamma)$  is the sum of an odd integer and an even integer, making  $deg(\Gamma)$  odd, a contradiction as  $deg(\Gamma)$  has been determined to be even. Therefore, there is an even number of vertices in  $\Gamma$  with degree 3, either 0, 2, 4, or 6.  $\square$

Using the results of the preceding lemma, we examine the graph on 6 vertices case by case. We learn that besides  $C_6$ , there are just two other graphs on 6 vertices which generate a 2-pseudomanifold with boundary.

**Theorem 3.13.** *Given that  $\Gamma$  is a simple, connected graph on 6 vertices, if  $D^2(\Gamma)$  is a 2-pseudomanifold configuration space with boundary, then  $\Gamma$  is either*

*the graph  $C_6$ , the graph , or the graph .*

*Proof.* We will demonstrate which specific graphs with 6 vertices have configuration spaces that are 2-pseudomanifolds. Suppose that  $D^2(\Gamma)$  is a 2-pseudomanifold configuration space with boundary. From the two previous lemmas, every vertex of  $\Gamma$  has either degree 2 or 3, and there is an even number of vertices in  $\Gamma$  with degree 3.

Case 1:  $\Gamma$  has no vertices of degree 3.

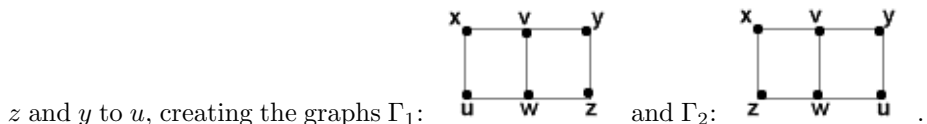
If  $\Gamma$  has no vertices with degree 3, every vertex must have degree 2. Then  $\Gamma$  is  $C_6$ , the cyclic graph with 6 vertices, which has been proven in Theorem 3.8 to be a 2-pseudomanifold with boundary.

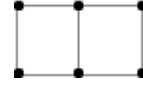
Case 2:  $\Gamma$  has two vertices of degree 3 and four of degree 2.

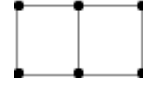
Let  $\Gamma$  have two vertices of degree 3, labeled  $v$  and  $w$ , and four of degree 2,  $u, x, y,$  and  $z$ . We will now show that  $v$  and  $w$  are adjacent. Suppose that  $v$  is not adjacent to  $w$ , so that  $v$  is adjacent to  $x, y$  and  $z$ . Then  $w$  cannot also be adjacent to  $x, y$  and  $z$ , giving  $w$  degree 3 and  $x, y$  and  $z$  each degree 2, because, with available degrees exhausted, there will remain one vertex  $u$  not adjacent to any other vertex in  $\Gamma$ . Thus,  $w$  must be adjacent to  $u$ , where  $u$  is not adjacent to  $v$ , forming edge  $E$ . As neither  $w$  nor  $u$  are adjacent to  $v$ ,  $d(v, E) \geq 1$ , corresponding to 1-cell  $(v, E)$  in  $D^2(\Gamma)$ . The three edges with endpoint  $v$  have endpoints  $x, y$  and  $z$  respectively. Since no common endpoints are shared with  $E$ ,  $d(vx, E) \geq 1$ ,  $d(vy, E) \geq 1$  and  $d(vz, E) \geq 1$ . In  $D^2(\Gamma)$ , the 1-cell  $(v, E)$  is in the boundary of three 2-cells, namely  $(vx, E)$ ,  $(vy, E)$ , and  $(vz, E)$ , so that  $D^2(\Gamma)$  is not a 2-pseudomanifold. Therefore,  $v$  is adjacent to  $w$ .

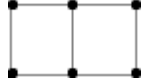
Letting  $v$  be adjacent to  $w$ , we will now show that no vertex in  $\Gamma$  is adjacent to both  $v$  and  $w$ . As  $v$  has degree 3, let  $v$  be adjacent to  $x$  and  $y$ .  $w$  must be adjacent to two additional vertices.  $w$  cannot be adjacent to both  $x$  and  $y$ , giving  $v$  and  $w$  degree 3 and  $x$  and  $y$  degree 2, because the two vertices  $u$  and  $z$  will then be unable to form edges with any other vertices. Suppose then that  $w$  is adjacent to one vertex  $x$  which is also adjacent to  $v$ . Immediately we see that vertex  $x$  has degree 2 and is in the 3-cycle  $x - v - w$ , resulting in a configuration space which is not a 2-pseudomanifold, according to Corollary 3.4. Thus, in  $D^2(\Gamma)$ , 1-cell  $(x, vw)$  is in the boundary of no 2-cells, so that  $D^2(\Gamma)$  is not a 2-pseudomanifold. Therefore, no vertex in  $\Gamma$  is adjacent to both  $v$  and  $w$ , the vertices of degree 3.

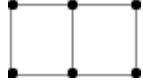
Finally we show that there is only one graph whose configuration space is a two-pseudomanifold with boundary, given that  $v$  and  $w$ , the vertices of degree 3, are adjacent and that no other vertex is adjacent to both  $v$  and  $w$ . Let  $v$  be adjacent to  $w, x$  and  $y$ . Then  $w$  is adjacent to  $v, u$  and  $z$ . Immediately we see that if  $x$  is adjacent to  $y$ , then  $v - x - y$  is a 3-cycle where both  $x$  and  $y$  have degree 2, creating 1-cells  $(x, vy)$  and  $(y, vx)$  in  $D^2(\Gamma)$  which are in the boundary of zero 2-cells. Therefore, either  $x$  is adjacent to  $u$  and  $y$  to  $z$ , or  $x$  is adjacent to





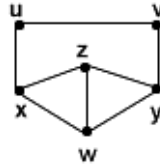
Since the labeling of the graphs is arbitrary,  $\Gamma_1 \cong \Gamma_2 \cong \Gamma =$  . It is easy to check that for every vertex  $v$  and edge  $E$  with  $d(v, E) \geq 1$ ,  $v$  is the endpoint of either one or two edges  $F, G$  with  $d(E, F) \geq 1$  and  $d(E, G) \geq 1$ , so that every 1-cell  $(v, E)$  in  $D^2(\Gamma)$  is in the boundary of either one or two 2-cells.

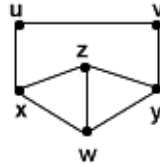


Thus,  $\Gamma =$   with two vertices of degree 3 is a 2-pseudomanifold with boundary.

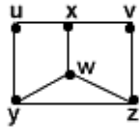
Case 3:  $\Gamma$  has four vertices of degree 3 and two vertices of degree 2.

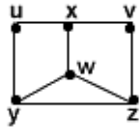
Let  $\Gamma$  have four vertices of degree 3, labeled  $w, x, y$ , and  $z$ , and two vertices of degree 2,  $u$  and  $v$ . We will first prove that  $u$  is not adjacent to  $v$ . Suppose that  $u$  is adjacent to  $v$ . No vertex  $x$  is adjacent to both  $u$  and  $v$ , since  $u-v-x$  would be a three cycle in which both vertices  $u$  and  $v$  would have degree 2, an impossibility according to Corollary 3.4. So  $u$  must be adjacent to some vertex  $x$  and  $v$  adjacent to a different vertex  $y$ . Consider the remaining vertices  $w$  and  $z$ . Since  $w$  has degree 3 and can no longer be adjacent to  $u$  or  $v$  as both currently have degree 2,  $w$  must be adjacent to  $x, y$  and  $z$ . Likewise,  $z$  must be



adjacent to  $x$  and  $y$ , resulting in the graph . In  $D^2(\Gamma)$ , the 1-cell  $(w, uw)$  is in the boundary of three 2-cells,  $(wx, uw)$ ,  $(wy, uw)$  and  $(wz, uw)$  and  $D^2(\Gamma)$  is not a 2-pseudomanifold. Therefore,  $u$  is not adjacent to  $v$ .

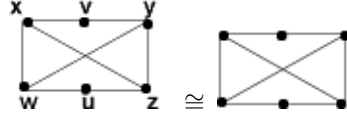
Next, we see that when  $u$  is not adjacent to  $v$ , no other vertex is adjacent to both  $u$  and  $v$ , where  $u, v$  are the vertices of degree 2. Suppose that  $x$  and  $y$  are adjacent to both  $u$  and  $v$ , so that both  $u$  and  $v$  have degree 2. Since  $w$  has degree 3, it must be adjacent to  $z, x$  and  $y$ . Now  $x, y$  and  $w$  all have degree 3, leaving  $z$  unable to form edges with any further vertices, so that  $z$  is unable to have degree 3. Therefore, both  $x$  and  $y$  cannot be adjacent to both  $u$  and  $v$ , so suppose that only  $x$  is adjacent to both  $u$  and  $v$ . Then  $v$  must be adjacent some vertex  $y$  and  $u$  adjacent some vertex  $z$ . Considering the sole remaining vertex  $w$  with degree 3,  $w$  must be adjacent to  $x, y$ , and  $z$ , giving  $x$  degree 3, forcing  $y$  and  $z$  to be adjacent, giving both degree 3 and forming graph



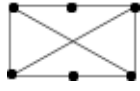
. In  $D^2(\Gamma)$ , the 1-cell  $(x, yz)$  is in the boundary of three 2-cells,  $(xu, yz)$ ,  $(xv, yz)$  and  $(xw, yz)$  and  $D^2(\Gamma)$  is not a 2-pseudomanifold. Therefore, no vertex is adjacent to both  $u$  and  $v$ .

Thus, in  $\Gamma$  with four vertices of degree 3 and two of degree 2, letting  $u$  be not adjacent to  $v$  and no vertex be adjacent to both  $u$  and  $v$ , we will show that

only one graph satisfying these conditions has a configuration space that is a 2-pseudomanifold. So  $v$  is adjacent to two vertices  $x$  and  $y$ , and  $u$  is adjacent to two different vertices  $w$  and  $z$ . It is immediately evident that  $w$  cannot be adjacent to  $z$ , as  $u-w-z$  would then be a 3-cycle with  $u$  having degree 2, and not satisfy the conditions of Corollary 3.4. For the same reason,  $x$  cannot be adjacent to  $y$ . Thus,  $w$  and  $z$  must be adjacent to both  $x$  and  $y$ , forming the



graph  $\cong$  as labeling is arbitrary. Again, it is easy to check that for every vertex  $v$  and edge  $E$  with  $d(v, E) \geq 1$ ,  $v$  is the endpoint of either one or two edges  $F, G$  with  $d(E, F) \geq 1$  and  $d(E, G) \geq 1$ , so that every 1-cell  $(v, E)$  in  $D^2(\Gamma)$  is in the boundary of either one or two 2-cells. Thus,  $\Gamma =$



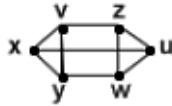
with four vertices of degree 3 is a 2-pseudomanifold with boundary.

Case 4:  $\Gamma$  has all six vertices of degree 3.

Last, we examine  $\Gamma$  with all six vertices of degree 3 and prove that none has a configuration spaces that is a two-pseudomanifolds with boundary. We have already seen that  $\Gamma = K_{3,3}$  has the 2-manifold without boundary homeomorphic to the 4-holed torus as its configuration space. Let  $\Gamma$  have every vertex of degree 3, where  $\Gamma$  is not  $K_{3,3}$ . Let some vertex  $v$  be adjacent to three vertices  $x, y$ , and  $z$ . Since  $\Gamma$  is not  $K_{3,3}$ , at least two of  $x, y$  and  $z$  must be adjacent to one another. Without loss of generality, assume that  $x$  is adjacent to  $y$ .

$z$  cannot be adjacent to both  $x$  and  $y$ , else each of  $v, x, y$ , and  $z$  have degree 3 and no other edges can be connected to those vertices, leaving the remaining two vertices  $u$  and  $w$  disconnected. Thus, suppose that  $z$  is adjacent to one of  $x$  or  $y$ , say  $x$  without loss of generality.  $z$  must be also adjacent to one other vertex  $u$ .  $u$  must be adjacent to  $y$  and the remaining vertex  $w$  in order to satisfy the condition that each vertex has degree 3. Now, every vertex except  $w$  has degree 3, so that no more edges can be adjoined to any vertex except  $w$ , and leaving the condition that every vertex be of degree 3 unsatisfied.

Next suppose that  $z$  is adjacent to neither  $x$  nor  $y$ , so that  $z$  is adjacent to  $u$  and  $w$ . Now  $u$  cannot be adjacent to both  $x$  and  $y$ , lest vertices  $[v, x, y, z, u]$  all have degree 3, excluding  $w$ . So  $u$  must be adjacent to one of  $x$  or  $y$ , say  $x$  without loss of generality, as well as  $w$ . In order to fulfill the condition that every vertex has degree 3, the two remaining vertices of degree 2,  $y$  and  $w$ , are made adjacent,



creating the graph . Thus in  $D^2(\Gamma)$ , the 1-cell  $(v, uw)$  is in the boundary of three 2-cells,  $(vx, uw)$ ,  $(vy, uw)$ , and  $(vz, uw)$ , so that  $D^2(\Gamma)$  is not a 2-pseudomanifold. Therefore, there are no graphs  $\Gamma_6$  with every vertex of degree 3 such that  $D^2(\Gamma)$  is a 2-pseudomanifold with boundary.  $\square$

This concludes the classification for graphs on 6 vertices. We find that there

might not be a great number of configuration space generating graphs on less than 7 vertices that result in 2-pseudomanifolds with boundary. The remaining classifications have shorter proofs and depend more upon already developed theorems.

### 3.4 Graphs on Five Vertices

We mentioned earlier that Corollaries 3.2 and 3.4 would be crucial in later proofs. It turns out that these two necessary conditions are also sufficient for defining 2-pseudomanifold generating graphs on 5 vertices. The following theorem proves their sufficiency, but does not outline the vertex arrangement of every graph that meets the criteria for a 2-pseudomanifold configuration space. We recall that the configuration space  $D^2(K_5)$  is a 2-manifold, specifically the 6-holed torus.

**Theorem 3.14.** *Let  $\Gamma$  be a simple connected graph on 5 vertices, with every vertex having degree at least 2. Then  $\Gamma$  has a 2-pseudomanifold configuration space with possible boundary if and only if no vertex in a 3-cycle has degree 2.*

*Proof.* If  $D^2(\Gamma)$  is a two-pseudomanifold, then no vertex in a 3-cycle has degree 2 in  $\Gamma$ , a simple connected graph on 5 vertices. This has been proven for a connected graph on any number of vertices in Theorem 3.3.

Conversely, because  $\Gamma$  has 5 vertices, any vertex  $v$  must have degree 2, 3, or 4. We will show that for each permissible degree of an arbitrary vertex  $v$ , if no vertex in any 3-cycle of  $\Gamma$  has degree 2, then where edge  $E$  is such that  $d(v, E) \geq 1$ , the 1-cell  $(v, E)$  is in the boundary of either one or two 2-cells.

Suppose that no vertex in any 3-cycle of  $\Gamma$  has degree 2.

Case 1: Vertex  $v$  has degree 2.

Because  $\Gamma$  is connected,  $\Gamma$  contains edges that do not have  $v$  as an endpoint. Suppose that  $E$  is one such edge in  $\Gamma$  such that  $(v, E)$  is a 1-cell in  $D^2(\Gamma)$ . If there is no edge  $F$  with endpoint  $v$  such that  $d(E, F) \geq 1$ , then  $v$  is in a 3-cycle with  $v$  adjacent to the endpoints of  $E$ . This is not permissible by the supposition. Thus  $v$  must be the endpoint of at least one edge  $F$  such that  $d(E, F) \geq 1$ . Specifically, if  $d(v, E) = 1$ , then  $v$  is adjacent to one endpoint of  $E$ , and is the endpoint of exactly one edge  $F$  with  $d(E, F) \geq 1$ . Thus, in  $D^2(\Gamma)$ , the 1-cell  $(v, E)$  is in the boundary of one 2-cell  $(E, F)$ . If  $d(v, E) > 1$ , then  $v$  is the endpoint of exactly two edges  $F$  and  $G$  with  $d(E, F) \geq 1$  and  $d(E, G) \geq 1$ . Similarly, in  $D^2(\Gamma)$ ,  $(v, E)$  is in the boundary of two 2-cells,  $(E, F)$  and  $(E, G)$ .  $v$  is never the endpoint of three or more such edges since the degree of  $v$  is 2. Therefore,  $(v, E)$  is in the boundary of either one or two 2-cells.

Case 2: Vertex  $v$  has degree 3.

Since  $\Gamma$  has 5 vertices,  $v$  is adjacent to three vertices  $x$ ,  $y$ , and  $z$ , but not to one vertex  $w$ . As in Case 1,  $\Gamma$  has edges that do not have  $v$  as an endpoint. If  $E$  does not have endpoint  $w$ , then both endpoints of  $E$  are adjacent to  $v$ . Without loss of generality, denote  $x$  and  $y$  the endpoints of  $E$  with the consequence that  $d(E, vx) < 1$  and  $d(E, vy) < 1$ . Since  $v$  and  $z$  are not endpoints of  $E$ ,  $d(E, vz) = 1$ . Therefore, in  $D^2(\Gamma)$ , the 1-cell  $(v, E)$  is in the boundary of the

2-cell  $(E, vz)$  and in the boundary of no other 2-cell. If  $w$  is an endpoint of  $E$ , then  $E$  has another endpoint that is adjacent to  $v$ . Without loss of generality, denote  $x$  an endpoint of  $E$  with the consequence that  $d(E, vx) < 1$ . Now neither  $v$ ,  $y$ , nor  $z$  is an endpoint of  $E$ , so  $d(E, vy) \geq 1$  and  $d(E, vz) \geq 1$ . Accordingly, in  $D^2(\Gamma)$ , the 1-cell  $(v, E)$  is in the boundary of exactly two 2-cells,  $(E, vy)$  and  $(E, vz)$ . Therefore  $(v, E)$  is in the boundary of either one or two 2-cells.

Case 3: Vertex  $v$  has degree 4.

$v$  is adjacent to every other vertex in  $\Gamma$ , namely  $w$ ,  $x$ ,  $y$ , and  $z$ . Every edge  $E$  such that  $d(v, E) \geq 1$  has endpoints that are both adjacent to  $v$ . Such an edge exists because the degree of every vertex is  $\geq 2$ . Without loss of generality, denote  $x$  and  $y$  the endpoints of  $E$ . Then  $d(E, vw) = 1$  and  $d(E, vz) = 1$ , but  $d(E, vx) < 1$  and  $d(E, vy) < 1$ . Therefore, in  $D^2(\Gamma)$ , the 1-cell  $(v, E)$  is in the boundary of exactly two 2-cells,  $(E, vw)$  and  $(E, vz)$ .

In conclusion, if no vertex in  $\Gamma$  which is in a 3-cycle has degree 2, then any vertex  $v$  and edge  $E$  such that  $d(v, E) \geq 1$  corresponds to a 1-cell  $(v, E)$  in  $D^2(\Gamma)$  that is in the boundary of either one or two 2-cells, so that  $D^2(\Gamma)$  is a 2-pseudomanifold.  $\square$

Note that the proof has been written for graphs generating 2-pseudomanifold configuration spaces with *possible* boundary. Under the suppositions of the proof, it is impossible to rule out that a graph configuration space may be a 2-pseudomanifold without boundary. However, we know from earlier work that  $K_5$  is the only graph on five vertices giving rise to a 2-manifold or 2-pseudomanifold without boundary. We must then conclude that any other graph on 5 vertices meeting the criteria given in the theorem generates a 2-pseudomanifold *with* boundary.

Although the theorem does not identify by vertex arrangement which graphs meet its criteria, these graphs are easy to identify. Checking to see that no vertex has degree 0 or 1 is immediate. Verifying that no vertex in a 3-cycle has degree 2 is best done by identifying all vertices with degree 2, then confirming for each that its two adjacent vertices are not mutual endpoints of a common edge.

### 3.5 Graphs on Four Vertices

The set of graphs on 4 vertices differ from the previous classifications by containing both simple and non-simple graphs. We begin with the simple graphs, which also use Corollaries 3.2 and 3.4 as sufficient conditions for generating a 2-pseudomanifold with boundary configuration space. Because there are fewer cases, we specify which two specific graphs fall under this classification.

**Theorem 3.15.** *If  $\Gamma$  is a simple connected graph on 4 vertices such that every vertex has degree at least two and such that no vertex in a 3-cycle has degree 2, then the configuration space  $D^2(\Gamma)$  is a 2-pseudomanifold with boundary. The only two such graphs are the cyclic graph  $C_4$  and the complete graph  $K_4$ .*

*Proof.* Let  $\Gamma$  be a simple connected graph on 4 vertices such that every vertex has degree  $\geq 2$  and such that no vertex in a 3-cycle has degree 2. Label the

vertices of  $\Gamma$  as  $v, x, y$  and  $z$ . Vertex  $v$  is adjacent to either 2 or 3 of the other vertices. Suppose that  $v$  is adjacent to three vertices  $x, y$  and  $z$ . Each of  $x, y$  and  $z$  must be made adjacent to at least one other vertex. Without loss of generality, let  $x$  and  $y$  be adjacent. Now  $v - x - y$  is a 3-cycle in which both  $x$  and  $y$  have degree 2. Thus both  $x$  and  $y$  must be adjacent to the remaining vertex  $z$ . The resulting graph is the complete graph on 4 vertices,  $K_4$  and no more edges may be added.

In the complete graph, every vertex is adjacent to every other vertex, so for any three vertices  $v, x$  and  $y$  in  $K_4$ ,  $d(v, x) = 1$  and  $d(v, y) = 1$  so that  $(v, xy)$  is a 1-cell in  $D^2(K_4)$ . Now  $v$  is adjacent to exactly one more vertex  $z$ , so that  $(v, xy)$  is in the boundary of exactly the one 2-cell  $(vz, xy)$  in  $D^2(\Gamma)$ . Therefore  $D^2(\Gamma)$  is a 2-pseudomanifold with boundary.

Next suppose that  $v$  is adjacent to two vertices  $x$  and  $y$ . Vertex  $x$  can not be adjacent to  $y$ , else  $v - x - y$  is a 3-cycle in which  $v$  has degree 2. So for each vertex to have degree  $\geq 2$ , both  $x$  and  $y$  must be adjacent to the remaining vertex  $z$ . Vertex  $z$  cannot be adjacent to any more vertices, as it is not adjacent to  $v$ . The resulting graph is the cyclic graph on 4 vertices,  $C_4$ , which is a 2-pseudomanifold with boundary, as proven in Theorem 3.8.  $\square$

**Corollary 3.16.** *Let  $\Gamma$  be a simple connected graph on 4 vertices. If  $D^2(\Gamma)$  is a 2-pseudomanifold with boundary, i.e.  $\Gamma$  is  $C_4$  or  $K_4$ , then every 1-cell in  $D^2(\Gamma)$  is in the boundary of exactly one 2-cell.*

*Proof.* It has been proven in Theorem 3.15 that for the four vertices arbitrarily labeled  $v, x, y$  and  $z$  in  $K_4$ ,  $(v, xy)$  is a 1-cell in  $D^2(K_4)$  which is in the boundary of exactly the 2-cell  $(vz, xy)$ . Let the 4-cycle  $C_4$  have arbitrarily ordered vertices  $v - x - y - z$ . Thus,  $(v, xy)$  is a 1-cell in  $D^2(C_4)$  in the boundary of exactly the 2-cell  $(vz, xy)$ . Likewise,  $(v, yz)$  is a 1-cell in the boundary of exactly the 2-cell  $(vx, yz)$ . Because of the symmetries of  $K_4$  and  $C_4$ , these relationships hold for every arbitrary labeling.  $\square$

Moving into the non-simple graph cases, the next lemma addresses multiple edges. Although a pair of vertices in a non-simple graph may normally be mutual endpoints for any number of common edges, a 2-pseudomanifold generating graph is restricted to two common edges per pair of vertices.

**Lemma 3.17.** *Let  $D^2(\Gamma)$  be a 2-pseudomanifold configuration space, with or without boundary. If  $\Gamma$  is a non-simple, connected graph, then no pair of endpoints in  $\Gamma$  serves as the endpoint set of more than two edges.*

*Proof.* Suppose that a non-simple, connected graph  $\Gamma$  has a pair of vertices  $x$  and  $y$  with  $m$ -tuple edges  $\{E_1, E_2, \dots, E_m\}$ , such that  $m \geq 3$ . If there exists edge  $F$  such that neither  $x$  nor  $y$  is an endpoint of  $F$ , then  $d(E_k, F) \geq 1$  for all  $1 \leq k \leq m$ . Thus the 1-cell  $(x, F)$  in  $D^2(\Gamma)$  is in the boundary of  $m$  2-cells,  $m \geq 3$ , so that  $D^2(\Gamma)$  is not a 2-pseudomanifold. Suppose that no such edge  $F$  exists, so that every vertex  $v$  in  $\Gamma$  must be adjacent to  $x$  or  $y$ . In fact, since every vertex must be adjacent to at least 2 other vertices, then every vertex  $v$  must be adjacent to both  $x$  and  $y$ . For all such edges  $F_i$  and  $G_j$  with respective

endpoints  $x, v$  and  $y, v$ , then  $d(E_k, F_i) < 1$  and  $d(E_k, G_j) < 1$  for all  $1 \geq k \geq m$ . Thus the 1-cells  $(v, E_k)$ ,  $1 \geq k \geq m$ , are in the boundary of no 2-cells in  $D^2(\Gamma)$  and  $D^2(\Gamma)$  is not a 2-pseudomanifold. Therefore, no pair of endpoints in  $\Gamma$  have  $m$ -tuple edges such that  $m \geq 3$ .  $\square$

Finally we come to our last classification. Non-simple graphs  $\Gamma$  on 4 vertices are exactly the graphs that can be formed from  $C_4$  or  $K_4$  by doubling any subset of the  $C_4$  or  $K_4$  edges.

**Theorem 3.18.** *Let  $\Gamma$  be a non-simple connected graph on 4 vertices, where any pair of vertices in  $\Gamma$  are the mutual endpoints of at most double edges.  $\Gamma$  has a 2-pseudomanifold configuration space, with or without boundary, if and only if the non-simple graph  $\Gamma$  has the same vertex adjacencies as the simple graph  $C_4$  or the simple graph  $K_4$ .*

*Proof.* Let  $\Lambda$  be a simple connected graph on 4 vertices. Suppose that  $\Lambda$  is neither  $C_4$  nor  $K_4$ . From the conditions of Theorem 3.15, we know that  $\Lambda$  either contains a vertex with degree less than 2 or that  $\Lambda$  has a vertex of degree 2 within a 3-cycle. First suppose that  $\Lambda$  contains a vertex  $v_\Lambda$  with degree  $< 2$ . In a simple connected graph, this implies that  $v_\Lambda$  is adjacent to only one other vertex. For any non-simple connected graph  $\Gamma$  on 4 vertices, where  $\Gamma$  and  $\Lambda$  have the same vertex adjacencies, since  $v_\Lambda$  is adjacent to only one other vertex, then  $v_\Gamma$  is also adjacent to one other vertex. Then  $D^2(\Gamma)$  is not a 2-pseudomanifold, with or without boundary, since  $\Gamma$  contains a vertex which is adjacent to only one other vertex.

Next suppose that  $\Lambda$  contains a vertex  $v_\Lambda$  of degree 2 within a 3-cycle  $v_\Lambda - x_\Lambda - y_\Lambda$ . This implies that  $v$  is adjacent to exactly two other vertices,  $x_\Lambda$  and  $y_\Lambda$ . For any non-simple connected graph  $\Gamma$  on 4 vertices, where  $\Gamma$  and  $\Lambda$  have the same vertex adjacencies, since  $v_\Lambda$  is adjacent to exactly  $x_\Lambda$  and  $y_\Lambda$ , then  $v_\Gamma$  is also adjacent to exactly  $x_\Gamma$  and  $y_\Gamma$ . Thus  $\Gamma$  contains at least one 3-cycle on vertices  $v_\Gamma, x_\Gamma$ , and  $y_\Gamma$  where  $v_\Gamma$  is adjacent to exactly 2 vertices. So  $D^2(\Gamma)$  is not a 2-pseudomanifold. This proves that if  $\Lambda$  is a graph such that  $D^2(\Lambda)$  is not a 2-pseudomanifold then any non-simple connected graph  $\Gamma$  having the same vertex adjacencies as  $\Lambda$  also does not have a 2-pseudomanifold configuration space. Therefore, if a non-simple connected graph on 4 vertices,  $\Gamma$ , has a 2-pseudomanifold configuration space, then there exists a simple graph  $\Lambda$  on 4 vertices, having a 2-pseudomanifold configuration space with boundary, such that  $\Lambda$  and  $\Gamma$  have the same vertex adjacencies.

Conversely, any graph  $\Gamma$  on 4 vertices having the same vertex adjacencies as  $C_4$  or  $K_4$  has the properties that every vertex is adjacent to at least two other vertices, and that no vertex in a 3-cycle is adjacent to exactly two other vertices, since the same must hold true for  $\Lambda$  as seen in Theorem 3.15.

Suppose that  $\Gamma$  is some non-simple graph with at most double edges where  $\Gamma$  has the same vertex adjacencies as  $C_4$ . Arbitrarily order the vertex adjacencies  $v - x - y - z$ . There exists at least one edge  $E$ , and possibly a second edge  $F$  with endpoints  $x$  and  $y$ , and at least one edge  $G$  and possibly a second  $H$  with endpoints  $v$  and  $z$ . Thus, in  $D^2(\Gamma)$ , the 1-cell  $(v, E)$  is in the boundary

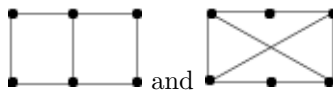
of either one or two 2-cells  $(G, E)$  and  $(H, E)$ . If edge  $F$  exists, then the 1-cell  $(v, F)$  is also in the boundary of one or two 2-cells  $(G, F)$  and  $(H, F)$ . A similar proof holds where there exists at least one edge  $K$ , and possibly a second  $L$  with endpoints  $z$  and  $y$ , and at least one edge  $M$  and possibly a second  $N$  with endpoints  $v$  and  $x$ . Then, in  $D^2(\Gamma)$ , the 1-cell  $(v, K)$  is in the boundary of either one or two 2-cells  $(M, K)$  and  $(N, K)$ , and if  $L$  exists, the one cell  $(v, L)$  is also in the boundary of one or two 2-cells  $(M, L)$  and  $(N, L)$ . Due to the symmetry and arbitrary ordering of  $C_4 = \Lambda$ , similar relationships hold for any chosen vertex. Therefore, every 1-cell is in the boundary of at least one and at most two 2-cells in  $D^2(\Gamma)$ , where  $\Gamma$  and  $C_4$  have the same vertex adjacencies and  $\Gamma$  has at most double edges.

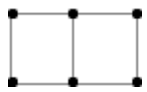
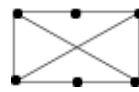
Suppose that  $\Gamma$  is some nonsimple graph with at most double edges where  $\Gamma$  has the same vertex adjacencies as  $K_4$ . Every vertex in  $\Gamma$  is adjacent to every other vertex by at least one edge. For any three vertices  $v, x$  and  $y$ , the distance between each is exactly one. There exists at least one edge  $E$  and possibly a second edge  $F$  with endpoints  $x$  and  $y$ , creating possible 1-cells  $(v, E)$  and  $(v, F)$  in  $D^2(\Gamma)$ . Then  $v$  is adjacent to exactly one other vertex  $z$ , with either one or two edges  $G$  and  $H$  having endpoints  $v$  and  $z$ . Therefore, the 1-cell  $(v, E)$  is in the boundary of either one or two 2-cells in  $D^2(\Gamma)$ ,  $(G, E)$  and  $(H, E)$ , and if edge  $F$  exists, then the 1-cell  $(v, F)$  is in the boundary of either one or two 2-cells,  $(G, F)$  and  $(H, F)$ . Since  $K_4$  is completely symmetrical at every vertex, similar relationships hold for any chosen vertex. Therefore, every 1-cell is in the boundary of at least one and at most two 2-cells in  $D^2(\Gamma)$ , where  $\Gamma$  and  $K_4$  have the same vertex adjacencies and  $\Gamma$  has at most double edges.  $\square$

Again, we note that the theorem includes 2-pseudomanifold configuration spaces, both with and without boundary. We know that the graphs generating 2-pseudomanifolds without boundary are doubled  $C_4$  and doubled  $K_4$  and none other, so we can conclude that every other graph in the set besides these two generates a 2-pseudomanifold configuration space with boundary.

This concludes a complete classification of connected graphs  $\Gamma$  such that  $D^2(\Gamma)$  is a 2-pseudomanifold configuration space possibly with boundary. These graphs are:

- the cyclic graphs  $C_n, n \geq 4$



- the simple graphs on 6 vertices:  and 
- the simple graphs on 5 vertices such that every vertex has degree at least 2 and no 3-cycle contains a vertex of degree 2
- the simple graph on 4 vertices:  $K_4$
- the nonsimple graphs on 4 vertices formed from  $C_4$  or  $K_4$  by doubling any subset of the  $C_4$  or  $K_4$  edges.

## 4 The Fundamental Group of a Configuration Space

As we have seen, the question of whether a graph configuration space is a pseudomanifold can be answered by local combinatorial reasoning. When the configuration space is an orientable 2-dimensional manifold, the space can be identified by similar reasoning because the Euler characteristic provides a complete classification of such manifolds. In this way, the space  $D^2(K_5)$  is known to be the 6-holed torus and the space  $D^2(K_{3,3})$  to be the 4-holed torus. In general, classification depends on more sophisticated algebraic invariants, such as the fundamental group.

A catalogue of 2-dimensional pseudomanifolds with boundary can never be as straightforward and useful as the catalogue of 2-dimensional manifolds. From any 2-dimensional pseudomanifold one can make a new one by identifying vertices. However, independent of any application to classification, the fundamental group of a graph configuration space remains interesting. Such fundamental groups are an analogue of the better known braid groups, which are fundamental groups of configuration spaces of points in the plane. The braid group associated with configuration of a pair of points in the plane can quickly be seen to be isomorphic to  $\mathbb{Z}$ . In contrast, the fundamental group of the configuration space of two points on a graph can be much more interesting due to the natural topology of the graph.

In what follows, we illustrate the technique needed to calculate these fundamental groups by calculating the fundamental group of  $D^2(K_5)$ .

### 4.1 Fibrations

Over each distinct vertex or edge in a graph  $\Gamma$ , a fibration in  $D^2\Gamma$  may be created by fixing one robot on the given point or edge, while allowing the second robot to move freely upon other permissible paths in the graph, namely those vertices and edges which are at least of distance one away from the given vertex or edge. In the specific graph  $K_5$ , when one robot is fixed on any given vertex, the second robot is free to move around the remaining four points and the edges connecting them to one another. Likewise, when one robot is fixed on any given edge, the second robot is permitted to move on the triangular path connecting the three vertices that are not endpoints of the given edge.

Visualizations of a fibration over a point and an edge are given in Figures 1 and 2 in the Appendix of Figures. The vertices in the resulting space  $D^2\Gamma$  are labeled with the position of Robot 1 then the position of Robot 2, separated by a comma.

The entire configuration space can be viewed as a union of these individual fibrations. The local topology of each fibration gives a clearer picture of the global topology of the manifold. In  $D^2(K_5)$ , each edge fibration is a triangular tube, and each vertex fibration has the structure of a triangular pyramid composed of the end cross-sections of four distinct triangular tubes. The end

cross-section of each triangular tube, or edge fibration, shares exactly one edge with each of the three other triangular tubes sharing a common vertex fibration. There are five such distinct vertex fibrations, one for each vertex, with each being the end cross-section of four triangular tubes. As there are ten edges in  $K_5$ , there are ten distinct edge fibrations. See [1] for more information on the algebra of graph fibrations.

Each edge in  $K_5$  corresponds to a tubular edge fibration in  $D^2(K_5)$ . The edge's two endpoints correspond to two vertex fibers. To calculate the fundamental group of  $D^2(K_5)$ , we will work with a space homotopically equivalent to  $D^2(K_5)$ . This space is found by contracting longitudinally the tubes corresponding to the edges in a maximal tree in  $K_5$ . The contraction of each tube "removes" the tube and identifies the tube's end cross-section in the fiber over one endpoint with its end cross-section in the fiber. Before giving a more detailed explanation and proof of the properties of one such contraction, let us examine some definitions and results relevant to the fundamental group.

## 4.2 Algebraic Topology

These definitions and theorems explain what is meant by the fundamental group and homotopy equivalences, two essential concepts to our calculations. These preliminaries are taken from [9].

Let  $X$  be a topological space and  $x_0$  a point in  $X$ . The fundamental group  $\pi_1(X, x_0)$  is generated by continuous functions  $f : [0, 1] \rightarrow X$  satisfying  $f(0) = x_0 = f(1)$ . Such functions, which are continuous maps of a closed interval into a topological space, are called *paths*, and a path with the same value for both the initial and the terminal points is called a *loop*.

Next, we wish to define the operation of multiplication on paths and loops. Two paths  $f$  and  $g$  may be multiplied together provided that the terminal point of  $f$  is the initial point of  $g$ . "It is the path that first traverses  $f$ , then  $g$ , but it must do so at double speed to complete the trip in the same unit time [10]." Where  $f, g : [0, 1] \rightarrow X$  and the terminal point of  $f$  is the initial point of  $g$ , the product  $f \cdot g$  is defined by:

$$(f \cdot g)t = \begin{cases} f(2t) & 0 \leq t \leq 1/2 \\ (2t - 1) & 1/2 \leq t \leq 1 \end{cases}$$

It is evident from the definition of multiplication that loops sharing the same basepoint can always be multiplied. Loops sharing a common basepoint can also be partitioned into equivalency classes, as described by the next definition.

**Definition 4.1.** Consider two loops  $f, g : [0, 1] \rightarrow X$  where  $f(0) = x_0 = f(1)$  and  $g(0) = x_0 = g(1)$ . There is an equivalence relation  $f \sim g$ , provided that there is a continuous function called *homotopy* under which:

$$h : [0, 1] \times [0, 1] \rightarrow X$$

such that

$$\left. \begin{array}{l} h(0,t) = f(t) \\ h(1,t) = g(t) \end{array} \right\} \quad t \in [0,1]$$

$$h(s,0) = x_0 = h(s,1) \quad s \in [0,1].$$

**Lemma 4.2.** For loops  $f_1, g_1, f_2,$  and  $g_2,$  all with basepoint at  $x_0,$  if  $f_1 \sim f_2$  and  $g_1 \sim g_2,$  then  $f_1 \cdot g_1 \sim f_2 \cdot g_2.$

This multiplication on homotopy classes of loops based at  $x_0$  satisfies the properties of group multiplication and allows us to define the fundamental group.

**Definition 4.3.** The group of equivalence classes of loops in  $X$  with basepoint  $x_0$  is called the *fundamental group*  $\pi_1(X, x_0).$

Our next definition describes the effect of a continuous mapping on the fundamental group.

**Definition 4.4.** Let  $F : X \rightarrow Y$  be a continuous function satisfying  $F(x_0) = y_0.$  The mapping  $F_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$  is defined by  $F_*([f]) = [F \circ f].$  The mapping  $F_*$  is well-defined on equivalence classes and has the following properties (Massey 63):

1. If  $f$  and  $g$  are paths in  $X$  such that  $f \cdot g$  is defined, then  $F_*([f] \cdot [g]) = F_*([f]) \cdot F_*([g]).$
2.  $F_*([f]^{-1}) = (F_*([f]))^{-1}.$
3. If  $G : Y \rightarrow Z$  is also a continuous map, then  $(G \circ F)_* = G_* \circ F_*.$
4. If  $F : X \rightarrow X$  is the identity map, then  $F_*([f]) = [f].$

Thus  $F_*$  is a homomorphism induced by the map  $F.$

The continuous map  $F$  induces the homomorphism  $F_*,$  and if  $F$  is a homomorphism, then  $F_*$  is an isomorphism. In order to study this induced homomorphism, we begin with defining the homotopy of continuous maps.

**Definition 4.5.** Let  $X$  and  $Y$  be topological spaces with respective basepoints  $x_0$  and  $y_0.$  Two continuous functions  $F : X \rightarrow Y$  and  $G : X \rightarrow Y$  such that  $F(x_0) = y_0$  and  $G(x_0) = y_0$  are called *basepoint-preserving homotopic* if there exists a continuous function

$$H : [0,1] \times X \rightarrow Y$$

such that

$$\left. \begin{array}{l} H(0,x) = F(x) \\ H(1,x) = G(x) \end{array} \right\} \quad x \in X$$

$$H(s, x_0) = y_0 \quad s \in [0,1].$$

The next two theorems describe some relationships between induced homomorphisms and the fundamental group.

**Theorem 4.6.** *Let  $f, g : X \rightarrow Y$  be maps that are homotopic relative to the basepoints  $x_0 \in X, y_0 \in Y$ , then*

$$f_* = g_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0).$$

**Theorem 4.7.**  $(id_X)_* = id_{\pi_1(X, x_0)}$

The second important concept used in calculating the fundamental group of a graph configuration space is the concept of homotopy equivalences.

**Definition 4.8.** (Massey 82) Let  $X$  and  $Y$  be two spaces. The continuous maps  $F : X \rightarrow Y$  and  $G : Y \rightarrow X$  are called homotopy equivalences if  $G \circ F \sim id_X$  and  $F \circ G \sim id_Y$ . The spaces  $X$  and  $Y$  are then homotopically equivalent.

Finally we come to the theorem that allows our calculations. Earlier we mentioned that in order to calculate the fundamental group of  $D^2(K_5)$ , we would work with a space homotopically equivalent to  $D^2(K_5)$ . What the following theorem states is that if two spaces are homotopically equivalent, then their fundamental groups are isomorphic.

**Theorem 4.9.** *If  $F : X \rightarrow Y$  is a homotopy equivalence, then  $F_* : \pi_1(X, x) \rightarrow \pi_1(Y, f(x))$  is an isomorphism for any  $x \in X$ .*

*Proof.* Let  $X \begin{matrix} \xrightarrow{F} \\ \xleftarrow{G} \end{matrix} Y$  such that  $F \circ G \sim id_Y$  and  $G \circ F \sim id_X$ . By Theorem 4.6, since  $F \circ G \sim id_Y$ , then  $(F \circ G)_* = (id_Y)_*$ . By Definition 4.4, property 4,  $(F \circ G)_* = F_* \circ G_*$ . By Theorem 4.7,  $(id_Y)_* = id_{\pi_1(Y, f(x))}$ .

Thus,  $F_* \circ G_* = id_{\pi_1(Y, f(x))}$  and similarly,  $G_* \circ F_* = id_{\pi_1(X, x)}$ .  $G_*$  is an inverse of  $F_*$  and  $F_*$  is bijective. From the properties of the mapping  $F_*$ , we know that  $F_*$  is a homomorphism. Therefore  $F_* : \pi_1(X, x) \rightarrow \pi_1(Y, f(x))$  is an isomorphism for any  $x \in X$ .  $\square$

With these definitions and theorems, we are prepared to find a space homotopically equivalent to  $D^2(K_5)$ .

### 4.3 Homotopy Equivalence

These preliminaries lead to the main statement and proof of the procedure used to contract the space  $D^2(K_5)$  into a homotopically equivalent space. A presentation of the fundamental group for  $D^2(K_5)$  could be obtained by using the standard process for finding the fundamental group of a cell complex. The advantage of having a contracted space, however, is that the process for finding the fundamental group gives a simpler presentation when applied to the contracted space than when applied to  $D^2(K_5)$ . Theorem 4.9 ensures that the contracted space will have a fundamental group isomorphic to the fundamental group of  $D^2(K_5)$ .

**Theorem 4.10.** *Let  $T_5$  be a maximum tree in  $K_5$ . If each triangular tube in  $D^2(K_5)$  that is the edge fibration over an edge in  $T_5$  is contracted longitudinally to its cross-section, the resulting space is homotopically equivalent to  $D^2(K_5)$ .*

*Proof.* In order to prove that  $D^2(K_5)$  and its contracted space  $X$  are homotopically equivalent, two continuous functions  $F : D^2(K_5) \rightarrow X$  and  $G : X \rightarrow D^2(K_5)$  are needed such that  $G \circ F \sim id_{D^2(K_5)}$  and  $F \circ G \sim id_X$ .

Every edge fibration is the same for all edges and every vertex fibration is the same for all vertices in  $K_5$ . It suffices to prove that the edge fibration and its connected endpoint fibrations corresponding to one edge of a maximum tree in  $K_5$  is homotopically equivalent to the space where that edge fibration has been contracted, since every other edge fibration will contract in an identical manner and the arguments establishing homotopy equivalence can be pieced together consecutively. Therefore, letting the space  $T$  be an edge fibration and its connected endpoint fibrations and letting  $X$  be the contracted space, two continuous functions  $p : T \rightarrow X$  and  $i : X \rightarrow T$  are needed such that  $i \circ p \sim id_T$  and  $p \circ i \sim id_X$ .

In  $D^2(K_5)$  where a triangular tube is the fibration over an edge and two triangular pyramids are fibrations over the endpoints in  $K_5$ , we desire the function  $p : T \rightarrow X$  to be a vertical projection that reduces a triangular tube to a flat triangle while preserving the end pyramids. Denote the triangular tube as  $K_T \times [1, 2]$  where  $K_T$  corresponds to the triangular structure and  $[1, 2]$  represents the tube length. The contracted space  $X$  may be defined as  $X = T / \sim$  where  $\sim : (x, \alpha) \sim (y, \beta)$  if and only if  $x = y$  for  $(x, \alpha), (y, \beta) \in K \times [1, 2]$ , allowing a mapping that will preserve the triangular structure independent of position on  $[1, 2]$ .

Then the function

$$p : T \rightarrow X = T / \sim$$

is given by

$$p(w) = [w].$$

Each points not on the triangular tube, namely those points in the two triangular pyramids but not in the triangular tube ends, has an equivalency class made up of that point itself and no other. Adjacencies that occurred in the original  $K_T$  are preserved in the contracted  $K_X$ .

The mapping  $p$  has been pictured in Figure 3. Edge  $ab$  along with its endpoints is part of a maximum tree in  $K_5$  and has its fibration in  $D^2(K_5)$  above it. As the edge  $ab$  is contracted, the three edges  $ac - bc$ ,  $ad - bd$ , and  $ae - be$ , which are generated by one robot's motion restricted to edge  $ab$ , also contract and become three points, denoted  $(ba), c$ ,  $(ba), d$  and  $(ba), e$ . All other points that were previously adjacent to the endpoint of a contracted edge remain adjacent to the new combined point, both in  $K_5$  and  $D^2(K_5)$ .

In the remainder of this proof, we will regard  $r = 0$  as the tip of the top

pyramid, both in  $X$  and in  $T$ . At every horizontal cross-section of the top pyramid, the depth variable is associated with three points, making up the edges of the ‘faceless’ pyramid. In  $T$ , the depth variable of the triangular tube is associated with a triangle of points. All contractions leave the variable which controls either the triangle of points or the triple of points constant, so the following notation will suppress all but the  $r$  or depth variable. Thus, the interval  $[a, b]$  refers to all points between depths  $a$  and  $b$ .

The map  $i : X \rightarrow T$  is more difficult to describe. The top pyramid in  $X$  must be stretched along both the edges of the tube and the top pyramid in  $T$ . The difficulty in finding such a function lies with the impossibility of separating the two triangular pyramids which have been ‘glued’ onto one another.

To get a function which will fix the bottom pyramid while stretching the top pyramid along the tube, the bottom pyramid in  $X$  will be identified with the bottom pyramid in  $T$ . The remainder of the spaces  $T$  and  $X$  will be expressed purely by the depth variable,  $r \in [0, 2]$  in  $T$  and  $r \in [0, 1]$  in  $X$ . Beginning with the tip of the top pyramid as  $r = 0$  in  $X$ , the top half of the top pyramid,  $[0, \frac{1}{2}]$ , will be identified with the top half of the top pyramid in  $T$ , the next fourth,  $[\frac{1}{2}, \frac{3}{4}]$  in  $X$ , will stretch to re-create the remainder of the pyramid,  $[\frac{1}{2}, 1]$  in  $T$ , and the final fourth of each edge,  $[\frac{3}{4}, 1]$  in  $X$ , will reconstruct its corresponding tube edge,  $[1, 2]$  in  $T$ .

This function  $i : X \rightarrow T$  is given by

$$i(r) = \begin{cases} r & r \in [0, \frac{1}{2}] \\ \frac{1}{2} + 2(r - \frac{1}{2}) & r \in [\frac{1}{2}, \frac{3}{4}] \\ 1 + \frac{r - \frac{3}{4}}{\frac{1}{4}} & r \in [\frac{3}{4}, 1] \end{cases}$$

The mapping  $i$  is pictured in Figure 4.

Now that we have the functions  $p : T \rightarrow X$  and  $i : X \rightarrow T$ , it remains to show that  $p \circ i \sim id_X$  and  $i \circ p \sim id_T$ .

First we will show that  $i \circ p \sim id_T$ . Since  $id_T : T \rightarrow T$  is the identity function, then  $id_T(r) = r$  for  $0 \leq r \leq \frac{1}{2}$ , where the depth interval  $[0, \frac{1}{2}]$  represents the top half of the top pyramid in  $T$ . Recall that the mapping  $p$  did not alter the top pyramid, and the mapping  $i$  was an identity mapping in the domain interval  $[0, \frac{1}{2}]$ . Thus  $i \circ p : T \rightarrow T$  also defines  $(i \circ p)(r) = r$  for  $0 \leq r \leq \frac{1}{2}$ . The bottom pyramid is identified with itself in the natural way in both cases.

Recalling definition 4.5, a function

$$H : [0, 1] \times T \rightarrow T$$

is needed such that

$$\left. \begin{array}{l} H(0, r) = id_T(r) \\ H(1, r) = (i \circ p)(r) \end{array} \right\} r \in T$$

$$H(s, r_0) = r_0 \quad s \in [0, 1], 0 \leq r_0 \leq \frac{1}{2}.$$

Partition the domain  $[0, 1] \times T$  into subdomains  $[0, 1] \times R_n$ , where  $R_0 = [0, \frac{1}{2}]$ ,  $R_1 = [\frac{1}{2}, \frac{3}{4}]$ ,  $R_2 = [\frac{3}{4}, 1]$ , and  $R_3 = [1, 2]$ .

The homotopy variable  $s \in [0, 1]$  slides the value of  $H$  between the identity mapping  $id_T$ , for which every depth variable  $r$  goes to itself, and  $i \circ p$ , for which  $r = \frac{1}{2} \mapsto \frac{1}{2}$ ,  $\frac{3}{4} \mapsto 1$ ,  $1 \mapsto 2$ , and  $2 \mapsto 2$ . The image of  $r = \frac{3}{4}$  is given by  $b_1 = \frac{3}{4} + \frac{1}{4}s$ , so that  $b_1(0) = \frac{3}{4} = id_T(\frac{3}{4})$  and  $b_1(1) = 1 = (i \circ p)(\frac{3}{4})$ . Likewise, the image of  $r = 1$  is given by  $b_2 = 1 + s$ . At every fixed value of  $s$ , the map arises from convex combinations in  $r$ , of the form  $(1 - q)a + q \cdot b$ . These convex combinations determine the values between  $\frac{1}{2}$  and  $\frac{3}{4} + \frac{1}{4}s$  for  $r \in R_1$ ,  $\frac{3}{4} + \frac{1}{4}s$  and  $1 + s$  for  $r \in R_2$ , and  $1 + s$  and  $2$  for  $r \in R_3$ .

The function

$$H : [0, 1] \times T \rightarrow T$$

is given by

$$H(s, r) = \begin{cases} r & r \in R_0, s \in [0, 1] \\ \frac{1}{2} + (\frac{r - \frac{1}{2}}{\frac{1}{4}})(\frac{3}{4} + \frac{1}{4}s - \frac{1}{2}) & r \in R_1, s \in [0, 1] \\ \frac{3}{4} + \frac{1}{4}s + (\frac{r - \frac{3}{4}}{\frac{1}{4}})(1 + s - (\frac{3}{4} + \frac{1}{4}s)) & r \in R_2, s \in [0, 1] \\ 1 + s + (r - 1)(2 - (1 + s)) & r \in R_3, s \in [0, 1] \end{cases}$$

The functions  $i \circ p$  and  $H$  are continuous since they are both piecewise polynomial, sharing the same value at interval boundaries.

When  $s = 0$ ,  $H(0, r) = id_T(r)$  and when  $s = 1$ ,  $H(1, r) = (i \circ p)(r)$ . Since  $H(s, r_0)$  is an identity where  $0 \leq r_0 \leq 1/2$ ,  $H(s, r_0) = r_0$ . Therefore,  $i \circ p \sim id_T$ .

To prove that  $p \circ i \sim id_X$ , again we note that since  $id_X : X \rightarrow X$  is the identity function, then  $id_X(r) = r$  for  $0 \leq r \leq \frac{1}{2}$ , where the depth interval  $[0, \frac{1}{2}]$  represents the top half of the top pyramid in  $X$ . Neither the mapping  $i$  nor  $p$  altered the composition of the top half of the top pyramid, so  $(p \circ i)(r) = r$  for  $0 \leq r \leq \frac{1}{2}$ . The bottom pyramid is identified with itself in the natural way.

A function

$$H : [0, 1] \times X \rightarrow X$$

is needed such that

$$\left. \begin{array}{l} H(0, r) = id_X(r) \\ H(1, r) = (p \circ i)(r) \end{array} \right\} r \in X$$

$$H(s, r_0) = r_0 \quad s \in [0, 1], 0 \leq r_0 \leq \frac{1}{2}.$$

Partition the domain  $[0, 1] \times X$  into subdomains  $[0, 1] \times R_n$ , where  $R_0 = [0, \frac{1}{2}]$ ,  $R_1 = [\frac{1}{2}, \frac{3}{4}]$ , and  $R_2 = [\frac{3}{4}, 1]$ .

In this case, the homotopy variable  $s \in [0, 1]$  slides the value of  $H$  between

the identity mapping  $id_x$  and  $p \circ i$ , for which  $r = \frac{1}{2} \mapsto \frac{1}{2}$ ,  $\frac{3}{4} \mapsto 1$ , and  $1 \mapsto 1$ . The image of  $r = \frac{3}{4}$  is given by  $b_1 = \frac{3}{4} + s(1 + \frac{3}{4})$ , so that  $b_1(0) = \frac{3}{4} = id_X(\frac{3}{4})$  and  $b_1(1) = 1 = (p \circ i)(\frac{3}{4})$ . Convex combinations determine the values between  $\frac{1}{2}$  and  $\frac{3}{4} + s(1 + \frac{3}{4})$  for  $r \in R_1$  and  $\frac{3}{4} + s(1 + \frac{3}{4})$  and 1 for  $r \in R_2$ .

The function

$$H : [0, 1] \times X \rightarrow X$$

is given by

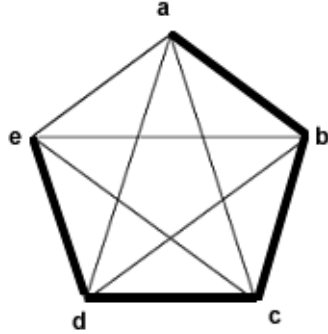
$$H(s, r) = \begin{cases} r & r \in R_0, s \in [0, 1] \\ \frac{1}{2} + (\frac{r - \frac{1}{2}}{\frac{1}{4}})(\frac{3}{4} + s(1 - \frac{3}{4}) - \frac{1}{2}) & r \in R_1, s \in [0, 1] \\ \frac{3}{4} + s(1 - \frac{3}{4} + (\frac{r - \frac{3}{4}}{\frac{1}{4}})(1 - (\frac{3}{4} + s(1 - \frac{3}{4})))) & r \in R_2, s \in [0, 1] \end{cases}$$

The functions  $p \circ i$  and  $H$  are continuous since they are both piecewise polynomial, sharing the same value at interval boundaries.

When  $s = 0$ ,  $H(0, r) = id_X(r)$  and when  $s = 1$ ,  $H(1, r) = (p \circ i)(r)$ . Since  $H(s, r_0)$  is an identity where  $0 \leq r_0 \leq 1/2$ ,  $H(s, r_0) = r_0$ .

Therefore,  $p \circ i \sim id_X$  and  $i \circ p \sim id_T$  so that  $D^2(K_5)$  and its contracted space  $X$  are homotopically equivalent.  $\square$

Now that we have proven that  $p : T \rightarrow X$  is a homotopy equivalence, we may apply an analogous argument contracting each tube corresponding to the edges of a maximum tree in  $K_5$ . Although any maximum tree may be chosen, the following specific tree  $T_5$  will be used for demonstration:



Longitudinally contracting each tube to its cross-section using the methods described in Theorem 4.10 results in a space  $\tilde{X}$ . One such a contraction has already proven to result in a space homotopically equivalent to  $D^2(K_5)$ , and applying identical contractions over the remaining tubes four times in succession likewise results in the space  $\tilde{X}$  being homotopically equivalent to  $D^2(K_5)$ .

Figures 5 through 9 will show the specific contraction in  $D^2(K_5)$ , contracting one tube at a time as a fiber over the given maximum tree  $T_5$ . In Figure 5, the tubes which are fibers over the maximum tree  $T_5$  and their end triangles have

been denoted by color. Points have been labeled with the position of Robot 1 then the position of Robot 2, separated by a comma. As each tube is contracted, the labeling of the first position is changed to the names of the contracted points in  $K_5$  separated by parenthesis. For example, in Figure 6, the tube which was the fiber of edge  $ab$  in  $K_5$  has been contracted, and the resulting cross-section is labeled by the points  $(ba), c, (ba), d$  and  $(ba), e$ .

#### 4.4 Application of the Seifert-Van Kampen Theorem

After the contraction on the four tubes being the fibers of edges  $ab, bc, cd,$  and  $de$ , the space  $\tilde{X}$  is composed of six remaining tubes. The end triangles of each tube are loops in the space  $\tilde{X}$  so that there are twelve such loops connected in pairs by six tubes. In Figure 10, these loops have been designated such that two loops with the same shading are the mutual end triangles of a tube. It can also be confirmed that each edge of an end triangle is a shared edge with exactly one other distinct end triangle.

Besides its two end triangles, each triangular tube contains three distinct tube edges, each connecting a vertex on an end triangle to its corresponding vertex on the second end triangle. In  $\tilde{X}$ , these are represented as two triple edges, a quadruple edge, and eight loops containing a single vertex each, totaling eighteen edges, three for each of six tubes. It can be confirmed that the number of these edges at any given vertex is equal to the number of end triangles containing that particular vertex.

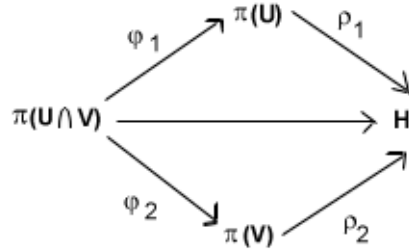
Our calculation of the fundamental group of  $D^2(K_5)$  will be based on the Seifert-Van Kampen theorem, whose statement characterizes the fundamental group of a space in terms of fundamental groups of subspaces. Before applying the theorem, we will have to do one more contraction to create a space homotopically equivalent to  $\tilde{X}$ . This space decomposes into subspaces in such a way that the Seifert-Van Kampen theorem implies that the fundamental group of the space is the free product of the fundamental group of the subspaces.

We wish to apply the Seifert-Van Kampen theorem to the contracted space  $\tilde{X}$  in order to generate a new homotopically equivalent space and its fundamental group. According to this theorem, after certain prerequisites are met, the fundamental group can be freely generated by the union of its components.

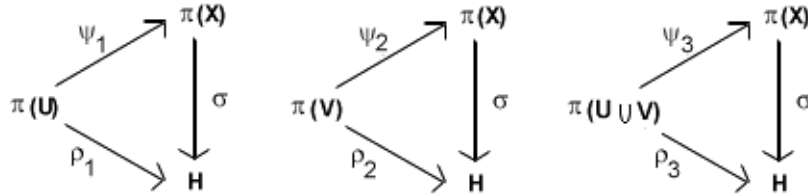
**Definition 4.11.** The space  $X$  is *arcwise-connected* if for any two points  $x$  and  $y$  in  $X$  there exists a continuous function  $f : [0, 1] \rightarrow X$  such that  $f(0) = x$  and  $f(1) = y$ .

**Theorem 4.12.** (*Seifert-Van Kampen*)

[9] Assume that  $U$  and  $V$  are arcwise-connected open subsets of  $X$  such that  $X = U \cup V$  and  $U \cap V$  is nonempty and arcwise connected. Choose a basepoint  $x_0 \in U \cap V$  for all fundamental groups under consideration. Let  $H$  be any group, and  $\rho_1, \rho_2,$  and  $\rho_3$  any three homomorphisms such that the following diagram is commutative:



Then, there exists a unique homomorphism  $\sigma : \pi_1(X) \rightarrow H$  such that the following three diagrams are commutative:



This theorem holds for arcwise-connected open subsets  $U_1, \dots, U_n$  of  $X$  such that  $X = \bigcup_{i=1}^n U_i$  and  $\bigcap_{i=1}^n U_i$  is nonempty and arcwise connected

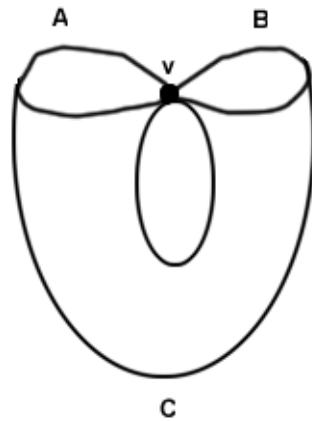
The space  $\tilde{X}$  has arcwise-connected subsets  $U_1, \dots, U_6$  such that  $\tilde{X} = \bigcup_{i=1}^6 U_i$ . These subsets are the six tubes that make up  $\tilde{X}$ . However,  $\bigcap_{i=1}^6 U_i$  is empty and one of the prerequisites for this theorem is that there must be a base point contained in the intersection of every subset  $U_i$ . In order to guarantee that each of the loops which are end triangles of a tube will intersect some base point, we will choose a maximum tree in  $\tilde{X}$  such that each of the twelve end triangles has at least one edge in the tree. When the tree is contracted to any chosen vertex  $v$  within the tree, every original end triangle intersects that single vertex, which is then designated as the basepoint. Contraction of a maximum tree results in a homotopically equivalent space. Then  $\bigcap_{i=1}^6 U_i = v$ . Although these subsets are closed, Seifert - Van Kampen can be applied to closed sets that are deformation retracts of open sets.

In Figure 11, a specific maximum tree which meets the requirements described in the previous paragraph has been chosen for demonstration purposes. The uncontracted edges have been numbered. This figure also shows the orientation of the remaining edges, a necessity which will be discussed in the following paragraphs.

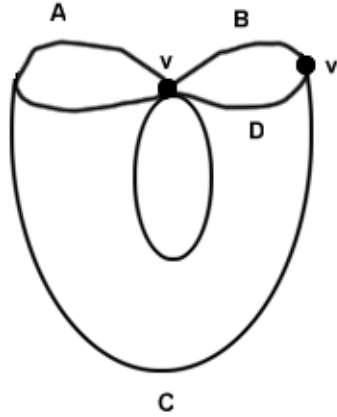
Because the intersection of the tubes after the tree is contracted is nonempty,

arcwise-connected, and simply connected, the Seifert - Van Kampen Theorem insures that the fundamental group of  $\tilde{X}$  will be freely generated by the fundamental group of each of its six tubes. Thus, in order to calculate  $\pi_1(\tilde{X})$ , we must first calculate  $\pi_1(\text{tube})$ , or more specifically, the fundamental group of a tube that has been modified by the contraction of a maximum tree in  $\tilde{X}$ . Each such tube is defined by related generators, each of the two end triangles being represented by its one or two edges remaining apart from the contracted tree, and the tube being represented by exactly one. Although each triangular tube has three tube edges, these edges are related to one another in such a way that only one is needed to represent the tube.

Since each tube end now intersects a single base point, a single tube has the following general appearance:



where  $A$  is the generator for one tube end,  $B$  is the generator for the second tube end,  $C$  is the tube generator, and  $v$  is the basepoint. However, if a tube end is composed of two edges, then that tube end will have two generators with the following appearance:

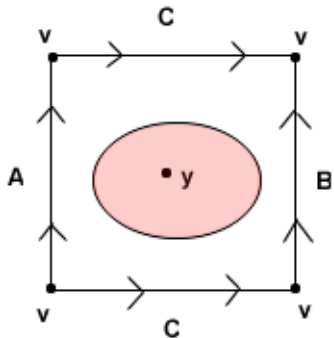


Here  $B$  and  $D$  are two generators for a single tube end. Since a maximal tree contains every vertex, after its contraction, both  $B$  and  $D$  are loops with endpoint  $v$ .

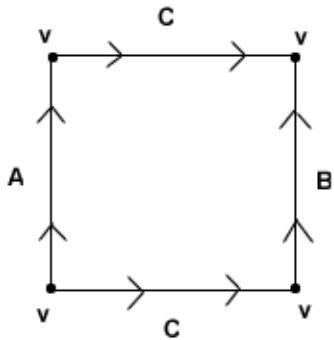
We recall that the fundamental group is a group made up of loops with the common basepoint  $v$ . Therefore, the path representing the tube must be directional, requiring orientation on the graph  $\tilde{X}$  after its maximum tree has been contracted. This orientation may be arbitrary and can be obtained by orienting the graph  $K_5$  so that orientation in the contracted  $\tilde{X}$  is dependent upon the movement of Robot 2. For example, if  $b \mapsto c$  in  $K_5$  then edge 11 with endpoints  $(edc), b$  and  $(ed), c$  has the orientation  $(edc), b \mapsto (ed), c$ . The orientation that we will use has been given in Figure 11. The sides  $C$  become the tube with end loops  $A$  and  $B$ .

Finally let us examine the fundamental group of any given tube  $T$  having been contracted to intersect the basepoint  $v$ . Represent the tube by identifying two opposite sides of a square, labeled in the same manner as the single contracted tube with each tube end having one generator pictured previously. Note that every labeled vertex of the square is the basepoint  $v$ .

We apply the Seifert-Van Kampen theorem again to calculate the fundamental group for this single tube. The theorem allows us to choose two subsets  $U$  and  $V$  such that  $U \cup V = T$  and  $U \cap V$  is nonempty and arcwise-connected. Let  $y$  be a point within the tube, and let  $U = T - \{y\}$ . Let  $V$  be a disk on the interior of the tube containing  $y$ .

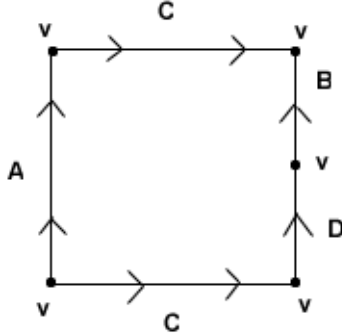


Since  $U$  is a disk, then  $\pi_1(U) = \{e\}$ , the identity.  
 The whole square minus a point radially retracts to the edge graph



Thus  $\pi_1(V) = \langle A, B, C \rangle$ , a group on three generators.  
 The subset  $U \cap V$  is the disk  $U$  minus the point  $y$ , which retracts to a circle, so that  $\pi_1(U \cap V) = \mathbb{Z}$ .  
 The resulting relationship according to the Seifert-Van Kampen theorem tells us that  $CBC^{-1}A^{-1} \sim e$  in  $\pi_1(U \cup V)$ . Therefore, the presentation of the fundamental group for this tube is the generators  $\langle A, B, C \rangle$  with the relation  $CBC^{-1}A^{-1} \sim e$ .

If a tube end is composed of two edges, the edge graph of the tube will be:



with the fundamental group  $\langle A, B, C, D \rangle$  having the relation  $C(DB)C^{-1}A^{-1} \sim e$ .

With any specific tube, the paths for specific end triangles are substituted for the values  $A$  and  $B$ , while a unique numbering variable is substituted for the tube generator  $C$ . Referring back to Figures 10 and 11, we see that the triangle defined by points  $a, b, (ba), c$  and  $(cba), d$  and the triangle defined by points  $e, d, (ed), c$  and  $(edc), b$  are the mutual endpoints of a tube. Every edge in the first triangle has been contracted in the maximum tree except for edge 1, and every edge in the second triangle has been contracted except for edge 11. The presentation for this particular tube is  $c_1 a_1 c_{11}^{-1} a_1^{-1}$ , where  $c_i$  represents edges in the triangular end of a tube and  $a_i$  uniquely numbers the tube. In an end triangle with two uncontracted edges, the substitution is defined by both edges with attention to orientation.

#### 4.5 Presentation of the Fundamental Group

Having satisfied the requirements of the Seifert-Van Kampen Theorem, the fundamental group of  $\tilde{X}$  with its maximal tree contracted will be freely generated by the fundamental group of each of its six tubes. The fundamental group of each tube is calculated in the same manner as described previously.

The space  $\tilde{X}$  and the space resulting in a maximal tree on  $\tilde{X}$  being contracted are both homotopically equivalent to  $D^2(K_5)$ , the 6-holed torus. Therefore, the following fundamental group is the fundamental group for  $D^2(K_5)$ . Since there are six tubes and eleven uncontracted edges, the presentation has seventeen generators with the following relations:

$$\begin{array}{llll}
 c_1 & a_1 & (c_{11})^{-1} & (a_1)^{-1} \\
 c_6 c_8 & a_2 & (c_8 c_9^{-1})^{-1} & (a_2)^{-1} \\
 c_4 c_5 & a_3 & (c_5 c_6)^{-1} & (a_3)^{-1} \\
 c_2 c_1 & a_4 & (c_9 c_{10})^{-1} & (a_4)^{-1} \\
 c_3 c_4^{-1} & a_5 & (c_{11} c_7)^{-1} & (a_5)^{-1} \\
 c_2 c_3 & a_6 & (c_{10} c_7)^{-1} & (a_6)^{-1}
 \end{array}$$

The  $c_i$ s represent uncontracted edges in the triangular end of a tube and the  $a_i$ s uniquely number each tube.

We recall that the braid group associated with the configuration of a pair of points in the plane is isomorphic to  $\mathbb{Z}$ . The fundamental group of a graph configuration space generated by two robots is a much more complex space. Some difficulties arise when trying to compare it to the standard presentation of the fundamental group of the 6-holed torus, although we know that both presentations must give isomorphic groups.

The standard presentation for the 6-holed torus, to which  $D^2(K_5)$  is isomorphic, consists of the set of generators  $\{\alpha_1, \beta_1, \dots, \alpha_n, \beta_n\}$  and the single relation  $\Pi_n^{i=1}[\alpha_i, \beta_i]$ , where  $\Pi_n^{i=1}[\alpha_i, \beta_i]$  denotes the commutator  $\alpha_i\beta_i\alpha_i^{-1}\beta_i^{-1}$ .

“Any group admits many different presentations, which may look quite different. Conversely, given two presentations... it is often nearly impossible to determine whether or not the two groups thus defined are isomorphic” [9]. So although we cannot determine whether the given presentation is isomorphic to the standard presentation, the standard presentation has twelve generators. Although abelianizing our presentation will not prove isomorphism, a total of twelve generators after abelianization will show that our presentation may be correct, while any other number will show a definite mistake in our calculations. Each of the tube generators  $a_1, a_2, \dots, a_6$  is independent, so the relations between the end triangle generators must cancel out some of these generators.

We find that every generator  $c_i$  is a product of the six generators  $c_1, c_5, c_7, c_8, c_9,$  and  $c_{10}$ .

There are five remaining generators, but each may be re-expressed in terms of the above six. The equations containing these five are all tautologies.

Thus the abelianized fundamental group is freely generated by twelve generators, six tube generators  $a_1, a_2, \dots, a_6$  and six edge generators  $c_1, c_5, c_7, c_8, c_9$  and  $c_{10}$ . This confirms that the abelianized fundamental group of  $D^2(K_5)$  has the same number of generators as the standard presentation for the fundamental group of the 6-holed torus.

The relationship to braid groups and the study of special structures are just a few aspects of topology in graph configuration spaces. There are numerous remaining questions about these spaces, in the topics we have covered as well other areas of topology. Graph configuration spaces may be used in a practical physical setting as with the robots in a factory, or in almost purely theoretical branches of mathematics. Hopefully work in understanding graph configuration spaces and their fundamental groups will continue until they are as well understood as the braid groups that they resemble.

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## 5 Appendix of Figures

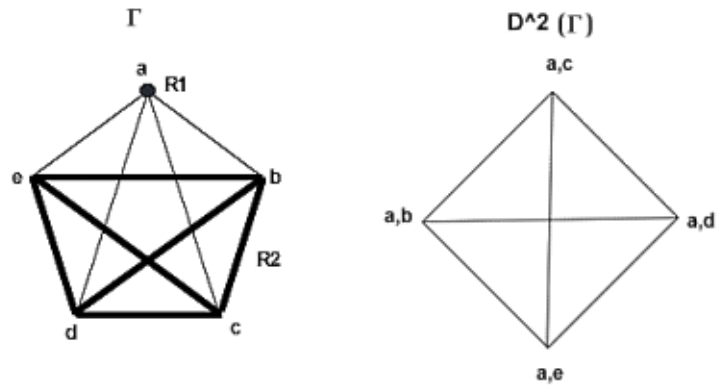


Figure 1: Fibration over the point  $a$  in  $K_5$

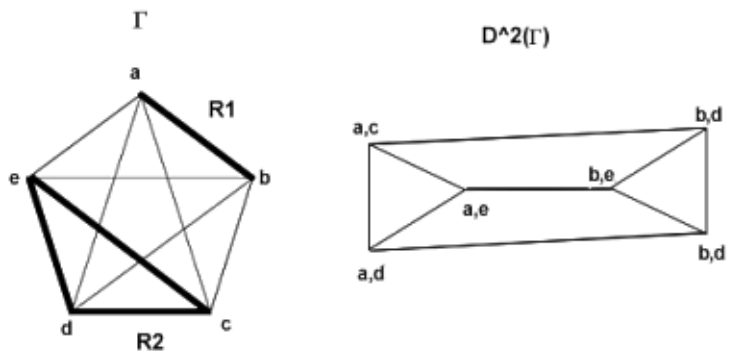


Figure 2: Fibration over the edge  $ab$  in  $K_5$

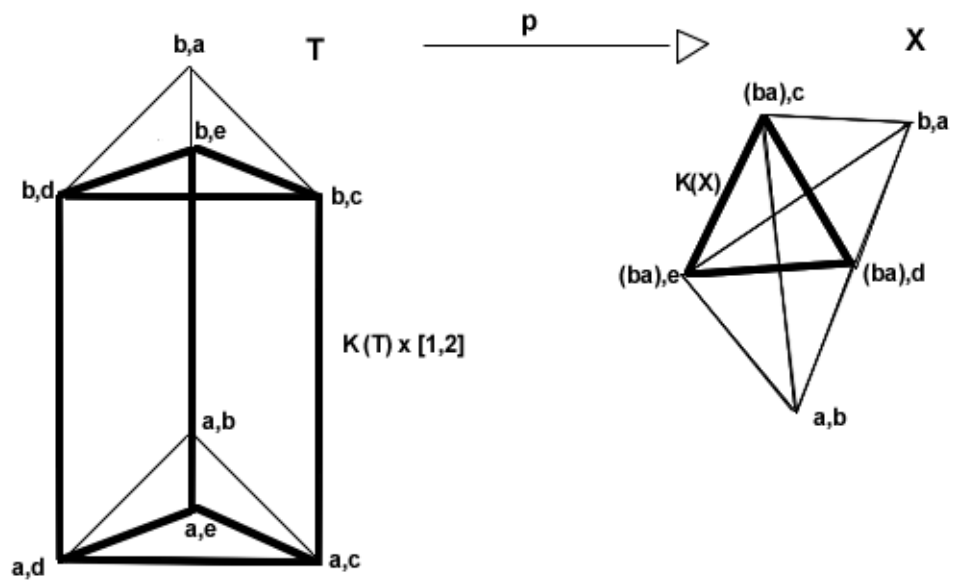


Figure 3: The mapping  $p : T \rightarrow X$

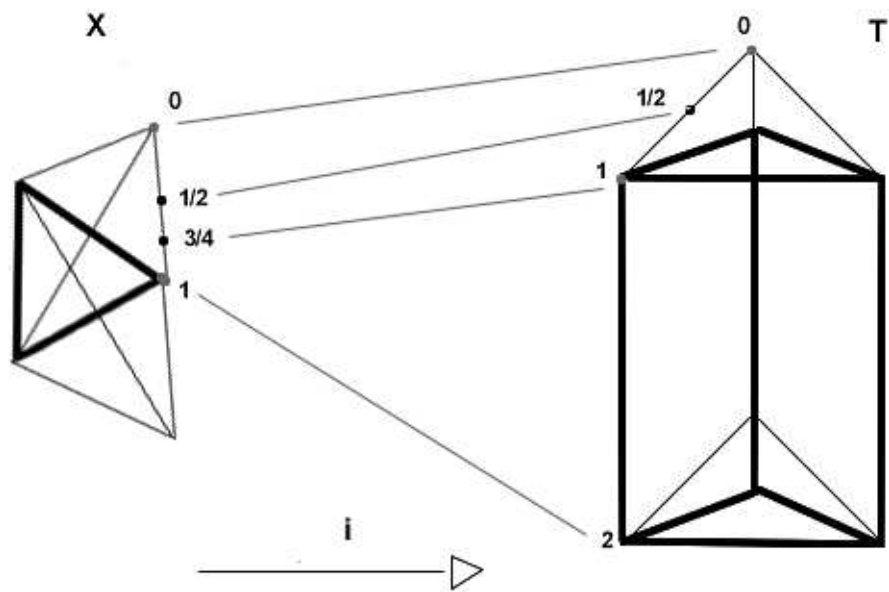


Figure 4: The mapping  $i : X \rightarrow T$

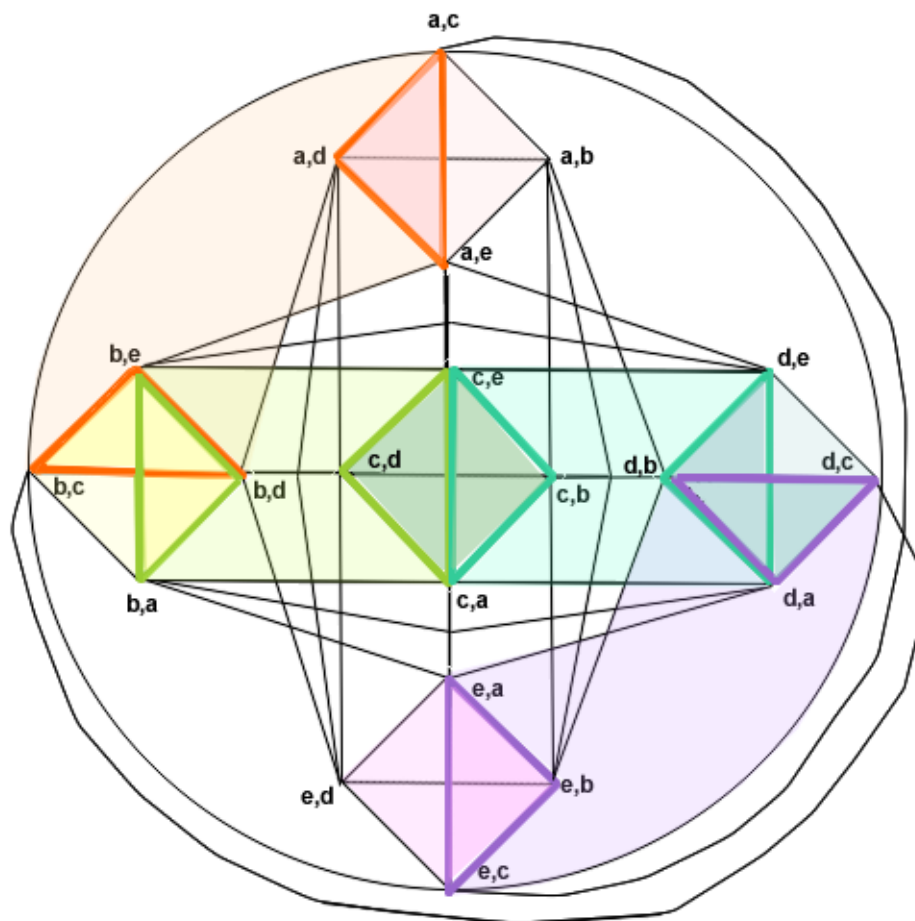


Figure 5:  $D^2(K_5)$

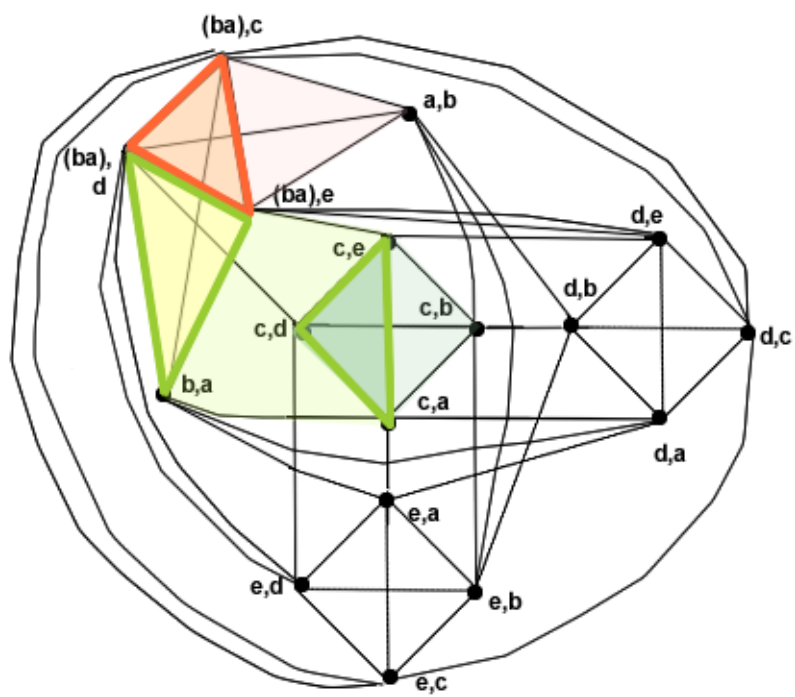


Figure 6: Contraction of the fibration over edge  $ab$  in  $K_5$

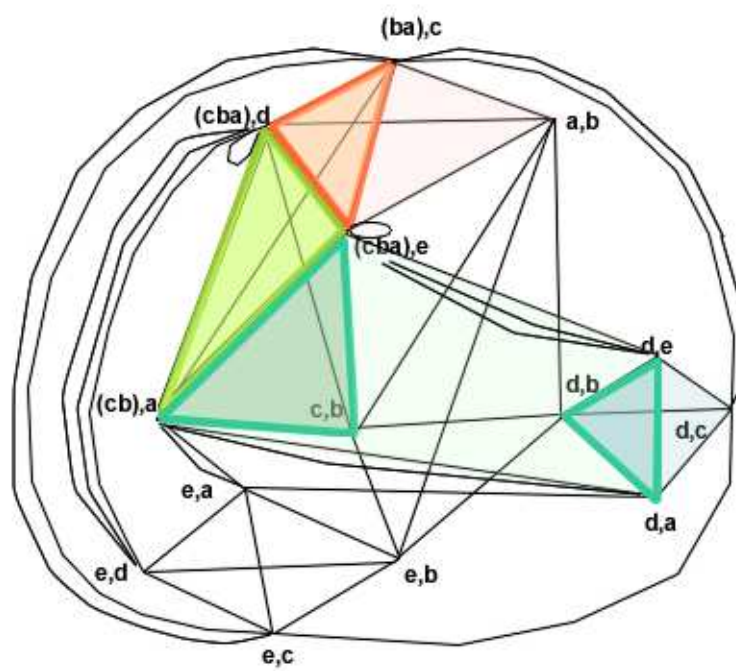


Figure 7: Contraction of the fibration over edge  $bc$  in  $K_5$

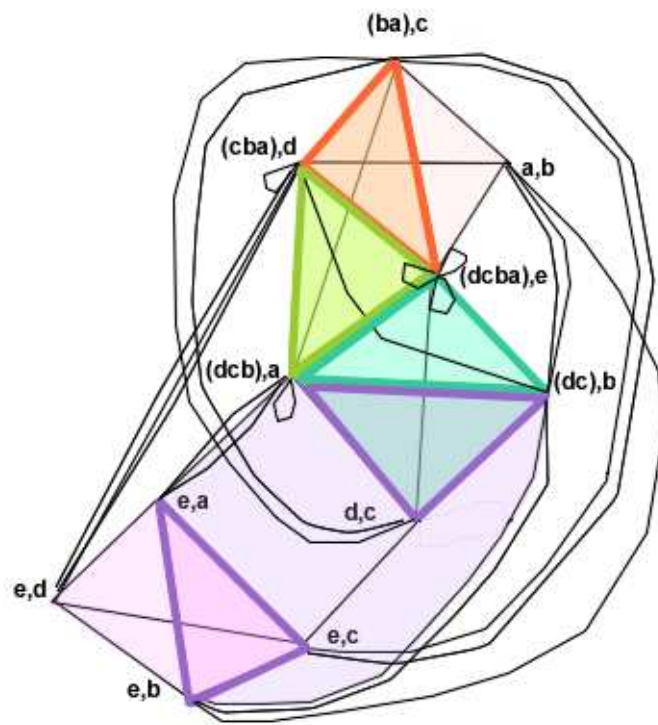


Figure 8: Contraction of the fibration over edge  $cd$  in  $K_5$



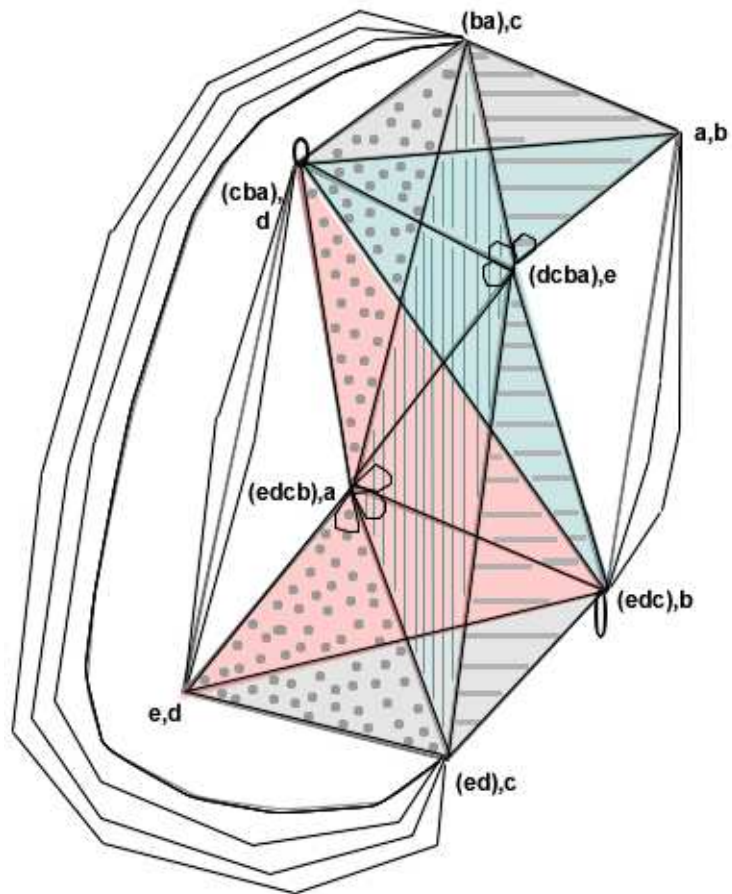


Figure 10: Six triangular tubes with tube ends specified by shading

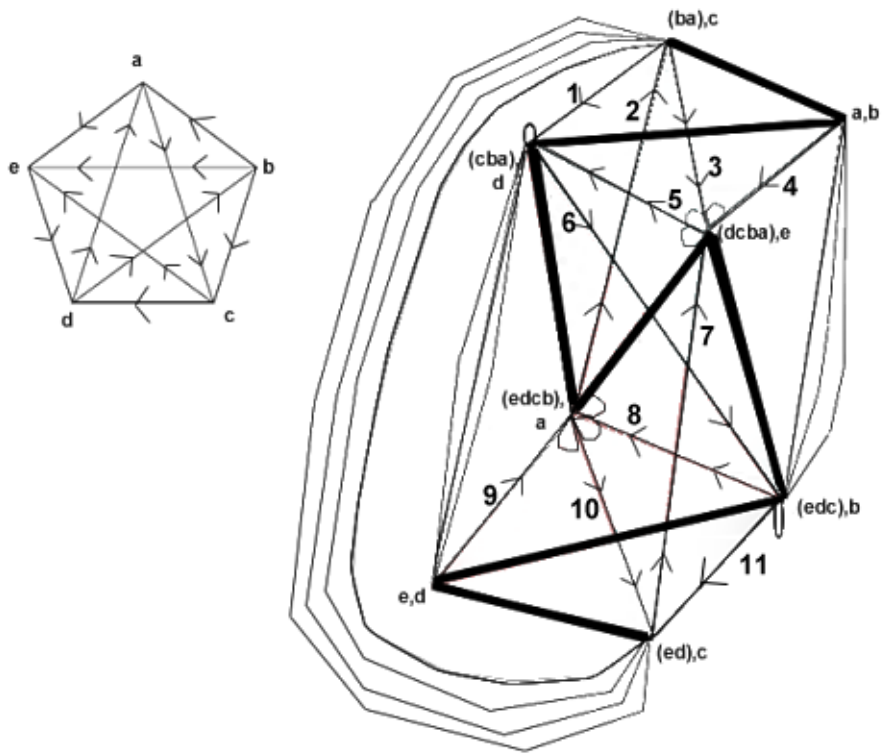


Figure 11: Maximum tree and orientation