Nonparallel stability of heated two-dimensional boundary layers

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The method of multiple scales is used to determine three partial differential equations describing the modulation of the amplitude and complex wavenumbers of three-dimensonal (3-D) waves propagating in two-dimensional (2-D) heated liquid layers. These equations are solved numerically along the characteristics subject to the condition that the ratio of the complex group velocities in the streamwise and transverse directions be real. A new criterion for the most dangerous frequency is proposed. For an n factor of 9, $F = 25 \times 10^{-6}$ is found to be the most dangerous frequency for the Blasius flow. Three-dimensional waves yield lower n factors than 2-D waves, irrespective of the heating distribution. For a power-law heating distribution of the form $T = T_e + Ax^N$, one cannot make a general statement on the effect of N on the stability. Numerical results are presented that show the n factor to increase with an increase or a decrease in N.

I. INTRODUCTION

The effects of heating/cooling on the stability of boundary layers were investigated by Linke[1] and Liepmann and Fila.[2] DiPrima and Dunn[3] quote unpublished results of McIntosh indicating large increases in the minimum critical Reynolds number for heated liquid boundary layers. Hauptmann[4] also predicted strong stabilization in water for small heating.

Wazzan, Okamura, and Smith[5,6] conducted extensive analytical studies of the stability of uniformly heated and cooled water boundary layers. They formulated the 2-D stability problem by taking into account the viscosity and temperature variations of the basic flow but neglecting the temperature and viscosity disturbances. The result is a fourth-order modified Orr–Sommerfeld equation. They showed that cooling destabilizes and heating stabilizes water boundary layers. Potter and Graber[7] also obtained similar results for plane Poiseuille flow. Lowell and Reshotko (e.g., Ref. 8) reformulated the 2-D stability problem and included the temperature and viscosity disturbances, thereby obtaining a sixth-order system. Lowell and Reshotko found that their sixth-order system yields neutral stability curves and growth rates that are sufficiently close to those obtained using the fourth-order system of Wazzan *et al.* The stabilizing effect of small uniform increases in the wall temperature of water boundary layers was confirmed experimentally by Strazisar, Reshotko, and Prahl[9] and Barker.[10] Barker found that the transition Reynolds number for water flowing in a tube can be increased from 10×10^6 to 42×10^6 by increasing the wall temperature by 7 °C. The results of Strazisar *et al.*[9] show that, as the wall temperature increases, the critical Reynolds number increases, the growth rates decrease, and the range of frequencies undergoing amplification decreases. These results qualitatively agree with all the parallel results.[5–7] Including the nonparallel terms in the stability analysis[11] yields results that quantitatively agree with the experimental data.

Since the flow over the portion of the body upstream of the critical Reynolds number is stable and does not need heating, one needs to heat only the portions downstream of the critical Reynolds number. This suggests the use of nonuniform wall heating. This led Strazisar and Reshotko[12] to examine experimentally the effects of two types of nonuniform wall heating: step changes in the wall temperature of magnitude ΔT occurring at a location x_s, and power-law temperature distributions of the form $T_w - T_e = Ax^N$ for positive and negative values of N, where T is the temperature and the subscripts w and e refer to the wall and boundary-layer edge, respectively. In the power law case, Strazisar and Reshotko kept the wall temperature difference $T_w(x_r) - T_e$ at some reference location x_r fixed, while varying the exponent N. They made all their measurements at x_r, which corresponds to a displacement-thickness Reynolds number of 800. They found that decreasing N is stabilizing because for all frequencies the case $N < 0$ results in growth rates that are lower than the case $N = 0$, which in turn results in growth rates that are lower than the case $N > 0$. These results could not be explained by the parallel stability analyses that calculate the mean flow using self-similar boundary-layer codes. Nayfeh and El-Hady[13] showed that the stabilizing influence of decreasing N at x_r can be explained only if the mean flow is calculated using nonsimilar boundary-layer codes. Although the growth rates and range of frequencies undergoing amplification decrease at x_r by decreasing N, we show in the present paper that the influence of decreasing N on the stability is not universal because the integrated growth rates (n factors) may increase or decrease with increasing N, depending on the position of the reference location x_r relative to branches I and II of the neutral stability curve. Moreover, we propose a method for calculating the most dangerous frequency; that is, the frequency that might be responsible for triggering large growth rates and eventually transition.

II. PROBLEM FORMULATION

The fluid density ρ^*, viscosity coefficient μ^*, and thermal conductivity κ^* are assumed to be functions of tempera-

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ture, which depends on the spatial coordinates. Nondimensional variables are introduced using $L^* = (2\nu_e^* x^*/U_e^*)^{1/2}$ as the length scale, where x^* is the surface distance and ν^* is the local edge kinematic viscosity, the local edge velocity U_e^* as the velocity scale, and the free stream values of viscosity, specific heat, and thermal conductivity as the reference values for the fluid properties. The quantities with an asterisk are dimensional.

A. Basic flow

We restrict our attention to time-independent 2-D boundary layer flows that are slightly nonparallel; that is, the transverse velocity is small compared with the streamwise component and all mean flow quantities are slowly varying functions of the streamwise position x. Then, the basic flow is governed by the following nondimensional set of equations:

$$\rho\left(U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y}\right) = \rho_e U_e \frac{dU_e}{dx} + \frac{\partial}{\partial y}\left(\mu \frac{\partial U}{\partial y}\right), \tag{1}$$

$$\frac{\partial P}{\partial y} = 0, \tag{2}$$

$$\frac{(\partial \rho U)}{\partial x} + \frac{\partial(\rho V)}{\partial y} = 0, \tag{3}$$

$$\rho\left(U\frac{\partial H}{\partial x} + V\frac{\partial H}{\partial y}\right) = \frac{\partial}{\partial y}\left\{\left[\left(\frac{\kappa}{c_p}\right)\frac{\partial H}{\partial y} + \mu\left(1 - \frac{1}{\mathrm{Pr}}\right)U\frac{\partial U}{\partial y}\right]\right\}, \tag{4}$$

where U and V are the velocity components in the x and y directions, P is the pressure, H is the total enthalpy, c_p is the specific heat at constant pressure, Pr is the Prandtl number, and the subscript e refers to conditions at the edge of the boundary layer.

The solution of Eqs. (1)–(4) subject to the appropriate boundary conditions yields the basic flow which can be expressed as

$$U_s = U_s(x_1,y), \quad V_s = \epsilon \hat{V}_s(x_1,y), \quad W_s = 0,$$
$$P_s = P_s(x_1), \quad T_s = T_s(x_1,y), \tag{5}$$

where

$$x_1 = \epsilon x, \quad \epsilon = 1/R \quad \text{and} \quad R = L^* U_e^*/\nu_e^*. \tag{6}$$

B. Disturbance equations

Since the effect of temperature perturbations on the stability results is negligible,[8] we consider the temperature profile obtained by solving the energy equation for the basic flow and do not consider the energy equation in deriving the disturbance equations. Small disturbances are introduced and superimposed on the basic flow so that the total-flow quantities can be expressed as

$$Q = Q_s(x_1,y) + q(x_1,y,z_1,t), \tag{7}$$

where Q represents U, V, W, and P, the subscript s refers to the basic state, and the lower cases u, v, w, p refer to the disturbance quantities.

Substituting the total-flow quantities into the Navier–Stokes equations, subtracting the basic-flow terms, and linearizing the resulting equations, we obtain the following disturbance equations:

$$\rho_s\left(\frac{\partial u}{\partial t} + U_s\frac{\partial u}{\partial x} + u\frac{\partial U_s}{\partial x} + \epsilon V_s\frac{\partial u}{\partial y} + v\frac{\partial U_s}{\partial y} + W_s\frac{\partial u}{\partial z} + w\frac{\partial U_s}{\partial z}\right)$$
$$= -\frac{\partial p}{\partial x} + \frac{1}{R}\left\{\frac{\partial}{\partial x}\left[2\mu_s\frac{\partial u}{\partial x} + \lambda_s\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu_s\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu_s\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right]\right\}, \tag{8}$$

$$\rho_s\left(\frac{\partial v}{\partial t} + U_s\frac{\partial v}{\partial x} + \epsilon u\frac{\partial V_s}{\partial x} + \epsilon V_s\frac{\partial v}{\partial y} + \epsilon v\frac{\partial V_s}{\partial y} + W_s\frac{\partial v}{\partial z} + \epsilon w\frac{\partial V_s}{\partial z}\right)$$
$$= -\frac{\partial p}{\partial y} + \frac{1}{R}\left\{\frac{\partial}{\partial y}\left[2\mu_s\frac{\partial v}{\partial y} + \lambda_s\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] + \frac{\partial}{\partial x}\left[\mu_s\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[\mu_s\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right]\right\}, \tag{9}$$

$$\rho_s\left(\frac{\partial w}{\partial t} + U_s\frac{\partial w}{\partial x} + u\frac{\partial W_s}{\partial x} + \epsilon V_s\frac{\partial w}{\partial y} + v\frac{\partial W_s}{\partial y} + W_s\frac{\partial w}{\partial z} + w\frac{\partial W_s}{\partial z}\right)$$
$$= -\frac{\partial p}{\partial z} + \frac{1}{R}\left\{\frac{\partial}{\partial z}\left[2\mu_s\frac{\partial w}{\partial z} + \lambda_s\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] + \frac{\partial}{\partial x}\left[\mu_s\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu_s\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right]\right\}, \tag{10}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{11}$$

These equations need to be supplemented by boundary conditions. The no-slip and no-penetration boundary conditions demand the vanishing of the disturbance velocities at the wall; that is,

$$u = v = w = 0 \quad \text{at } y = 0. \tag{12}$$

Moreover, all disturbances must decay away from the wall; that is,

$$u, v, w, p \to 0 \quad \text{as } y \to \infty. \tag{13}$$

III. METHOD OF ANALYSIS

Using the method of multiple scales,[14,15] we seek a first-order uniform expansion for the disturbance quantities u, v, w, p in the traveling harmonic waveform,

$$q(x,y,z,t) = [q_0(x_1,y,z_1,t_1) + \epsilon q_1(x_1,y,z_1,t_1) + \cdots]\exp(i\theta), \tag{14}$$

where $z_1 = \epsilon z$, $t_1 = \epsilon t$, and

$$\frac{\partial\theta}{\partial x}=\alpha(x_1,z_1),\quad \frac{\partial\theta}{\partial z}=\beta(x_1,z_1),\quad \frac{\partial\theta}{\partial t}=-\omega. \qquad (15)$$

Here α is the wavenumber component in the x direction, β is the wavenumber component in the z direction, and ω is the nondimensional frequency. Using the chain rule, we have

$$\frac{\partial}{\partial x}=\alpha\frac{\partial}{\partial\theta}+\epsilon\frac{\partial}{\partial x_1}, \qquad (16a)$$

$$\frac{\partial}{\partial z}=\beta\frac{\partial}{\partial\theta}+\epsilon\frac{\partial}{\partial z_1}, \qquad (16b)$$

$$\frac{\partial}{\partial t}=-\omega\frac{\partial}{\partial\theta}+\epsilon\frac{\partial}{\partial t_1}. \qquad (16c)$$

For a spatial stability analysis, α and β are complex and ω is real; α_r and β_r then represent the wavenumber components, and $-\alpha_i$ and $-\beta_i$ represent the growth rates in the streamwise and spanwise directions, respectively. For a temporal stability analysis, α and β are real and represent the wavenumber components in the streamwise and spanwise directions; ω is complex, with a real part that represents the frequency and an imaginary part that represents the growth rate.

Substituting Eqs. (14)–(16) into Eqs. (8)–(13) and equating each of the coefficients of ϵ^0 and ϵ^1 on both sides, we have two sets of problems. These are called the zeroth-order and the first-order problems, respectively.

A. The zeroth-order problem

The zeroth-order problem is

$$L_1(u_0,v_0,w_0)=\rho_s(i\alpha u_0+Dv_0+i\beta w_0)+v_0D\rho=0, \qquad (17)$$

$$\begin{aligned}L_2(u_0,v_0,w_0,p_0)&\\=i\rho_s(-\omega+\alpha U_s+\beta W_s)u_0&\\+\rho_s v_0 DU_s+i\alpha p_0-(1/R)[i\alpha\mu_s(i\alpha r u_0+sDv_0&\\+i\beta s w_0)-\beta\mu_s(\alpha w_0+\beta u_0)+D\mu_s Du_0&\\+\mu_s D^2u_0+i\alpha(v_0D\mu_s+\mu_s Dv_0)]=0,&\qquad (18)\end{aligned}$$

$$\begin{aligned}L_3(u_0,v_0,w_0,p_0)&\\=i\rho_s(-\omega+\alpha U_s+\beta W_s)v_0+Dp_0&\\-(1/R)[i\alpha\mu_s(Du_0+i\alpha v_0)+i\alpha s u_0 D\mu_s&\\+i\alpha s\mu_s Du_0+rD\mu_s Dv_0+r\mu_s D^2v_0+i\beta s w_0 D\mu_s&\\+i\beta s\mu_s Dw_0+i\beta\mu_s(i\beta v_0+Dw_0)]=0,&\qquad (19)\end{aligned}$$

and

$$\begin{aligned}L_4(u_0,v_0,w_0,p_0)&\\=\rho_s[i(-\omega+\alpha U_s+\beta W_s)w_0+v_0Dw_s]&\\+i\beta p_0-(1/R)[i\alpha\mu_s(i\alpha w_0+i\beta u_0)+D\mu_s Dw_0&\\+\mu_s D^2w_0+iD\mu_s\beta v_0+i\beta\mu_s+i\beta\mu_s Dv_0&\\+i\beta\mu_s(i\alpha s u_0+sDv_0+i\beta r w_0)]=0,&\qquad (20)\end{aligned}$$

$$u_0=v_0=w_0=0\quad \text{at}\ y=0, \qquad (21)$$

$$u_0,v_0,w_0,p_0\to 0\quad \text{as}\ y\to\infty, \qquad (22)$$

where $s=\lambda_s/\mu_s$, $r=(2+s)$, and $D=d/dy$.

The homogeneous sixth-order system of Eqs. (17)–(22) represents an eigenvalue problem. For a given basic flow and four of the six parameters α_r, α_i, β_r, β_i, ω_r, and ω_i, we can calculate the other two as eigenvalues.

Equations (17)–(20) can be written as a set of six first-order equations that can be numerically integrated over the region of interest. To this end, we define

$$Z_{01}=u_0,\quad Z_{02}=Du_0,\quad Z_{03}=v_0, \qquad (23a)$$

$$Z_{04}=p_0,\quad Z_{05}=w_0,\quad Z_{06}=Dw_0. \qquad (23b)$$

This enables us to rewrite Eqs. (17)–(22) as

$$DZ_{0i}-\sum_{j=1}^{6}A_{ij}Z_{0j}=0\quad \text{for}\ i=1,2,3...,6, \qquad (24)$$

$$Z_{01}=Z_{03}=Z_{05}=0\quad \text{at}\ y=0, \qquad (25)$$

$$Z_{0n}\to 0\quad \text{as}\ y\to\infty, \qquad (26)$$

where (A_{ij}) is a 6×6 variable-coefficient matrix which is defined in Appendix A. The A_{ij} are also slowly varying functions of x_1 and z_1.

The solution of Eqs. (24)–(26) can be written as

$$Z_{0i}=A(x_1,z_1,t_1)\zeta_i(x_1,y,z_1)\quad \text{for}\ i=1,2,...,6, \qquad (27)$$

where A is an unknown function at this level of approximation. It is determined by imposing the solvability condition at the next level of approximation.[15]

B. The first-order problem

Substituting Eqs. (14)–(16) into Eqs. (8)–(13) and equating the coefficients of ϵ on both sides, we obtain

$$L_1(u_1,v_1,w_1)=I_1, \qquad (28a)$$

$$L_2(u_1,v_1,w_1,p_1)=I_2, \qquad (28b)$$

$$L_3(u_1,v_1,w_1,p_1)=I_3, \qquad (28c)$$

$$L_4(u_1,v_1,w_1,p_1)=I_4, \qquad (28d)$$

$$u_1,v_1,w_1=0\quad \text{at}\ y=0, \qquad (28e)$$

$$u_1,v_1,w_1,p_1\to 0\quad \text{as}\ y\to\infty, \qquad (28f)$$

where the operators L_1, L_2, L_3, and L_4 are defined in Eqs. (17)–(20) and the inhomogeneous terms I_1, I_2, I_3, and I_4 have contributions from the nonparallel effects of the basic flow and the variation of the eigenfunctions along the streamwise and spanwise directions. They are defined in Appendix B.

Equations (28) can be cast into a set of six first-order equations by defining

$$Z_{11}=u_1,\quad Z_{12}=Du_1,\quad Z_{13}=v_1, \qquad (29a)$$

$$Z_{14}=p_1,\quad Z_{15}=w_1,\quad Z_{16}=Dw_1. \qquad (29b)$$

The result is

$$DZ_{1i}-\sum_{j=1}^{6}A_{ij}Z_{1j}=D_i\frac{\partial A}{\partial t_1}+E_i\frac{\partial A}{\partial x_1}+F_i\frac{\partial A}{\partial z_1}+G_i, \qquad (30)$$

$$Z_{11}=Z_{13}=Z_{15}=0\quad \text{at}\ y=0, \qquad (31)$$

$$Z_{1n}\to 0\quad \text{as}\ y\to\infty, \qquad (32)$$

where D_i, E_i, F_i, and G_i are functions of the basic-flow quantities, the eigenfunctions of the zeroth-order problem, and the streamwise and spanwise derivatives of these quantities. They are defined in Appendix B.

Since the homogeneous part of the first-order problem has a nontrivial solution, the inhomogeneous first-order problem has a solution only if the inhomogeneous parts are orthogonal to every solution of the adjoint homogeneous problem.[15] This is the solvability or the consistency condition which must be satisfied by the first-order problem.

C. Solvability condition

We use the concept of adjoint to arrive at the solvability condition for the first-order problem. The adjoint problem can be defined as follows: the zeroth-order homogeneous system of equations (24) can be rewritten in matrix form as

$$\{DZ_0\} - [A]\{Z_0\} = 0, \tag{33}$$

where $\{Z_0\}$ is a column vector whose elements are the Z_{0n}. We multiply Eq. (33) from the left by $\{W\}^T$, where $\{W\}$ is the adjoint column vector, and obtain

$$\{W^T\}\{DZ_0\} - \{W^T\}[A]\{Z_0\} = 0. \tag{34a}$$

Integrating Eq. (34a) by parts from $y = 0$ to $y = \infty$, we obtain

$$\{W^T\}\{Z_0\}|_0^\infty - \int_0^\infty (\{DW^T\} + \{W^T\}[A])\{Z_0\}dy = 0. \tag{34b}$$

We now define the adjoint equation by setting the coefficient of $\{Z_0\}$ in the integrand equal to zero. The result is

$$\{DW^T\} + \{W^T\}[A] = 0,$$

or

$$\{DW\} + [A^T]\{W\} = 0. \tag{35}$$

The boundary conditions for the adjoint system can be obtained as follows. Using Eq. (35), we obtain from Eq. (34b) that

$$\{W^T\}\{Z_0\}|_0^\infty = 0,$$

or

$$[W_1Z_{01} + W_2Z_{02} + W_3Z_{02} + \cdots + W_6Z_{06}]|_0^\infty = 0. \tag{36}$$

Since $Z_{0n} \to 0$ as $y \to \infty$, the terms evaluated at infinity in Eq. (36) vanish if

$$W_n \to 0 \quad \text{as } y \to \infty. \tag{37}$$

Then using Eqs. (25), we find that Eq. (36) becomes

$$[W_2Z_{02} + W_4Z_{04} + W_6Z_{06}]_{y=0} = 0. \tag{38}$$

We define the adjoint boundary conditions at $y = 0$ such that each of the coefficients of Z_{02}, Z_{04}, and Z_{06} in Eq. (38) vanishes independently; that is,

$$W_2 = W_4 = W_6 = 0 \quad \text{at } y = 0. \tag{39}$$

Therefore the adjoint problem is defined by Eq. (35) subject to the boundary conditions (37) and (39).

Having defined the adjoint problem, we return to the inhomogeneous problem. We multiply the matrix form of Eq. (30) from the left by $\{W^T\}$, integrate the result by parts from $y = 0$ to $y = \infty$, use the definition of the adjoint and Eqs. (31) and (32), and obtain the solvability condition

$$\int_0^\infty \{W^T\}\{D\}\frac{\partial A}{\partial t_1}dy + \int_0^\infty \{W^T\}\{E\}\frac{\partial A}{\partial x_1}dy + \int_0^\infty \{W^T\}\{F\}\frac{\partial A}{\partial z_1}dy + \int_0^\infty \{W^T\}\{G\}dy = 0, \tag{40}$$

where $\{D\}$, $\{E\}$, $\{F\}$, and $\{G\}$ are column vectors whose components are the D_i, E_i, F_i, and G_i.

D. Amplitude-modulation equation

Substituting Eqs. (27) into Eq. (40) yields the following amplitude-modulation equation:

$$g_1\frac{\partial A}{\partial t_1} + g_2\frac{\partial A}{\partial x_1} + g_3\frac{\partial A}{\partial z_1} = \hat{h}_1 A,$$

or

$$\frac{\partial A}{\partial t_1} + \omega_\alpha\frac{\partial A}{\partial x_1} + \omega_\beta\frac{\partial A}{\partial z_1} = h_1 A, \tag{41}$$

where

$$\omega_\alpha = g_2/g_1, \quad \omega_\beta = g_3/g_1, \quad h_1 = \hat{h}_1/g_1, \tag{42}$$

and g_1, g_2, and g_3 are defined in Appendix C. Equation (41) describes the modulation of the amplitude function A with x_1, z_1, and t_1. Here, ω_α and ω_β are the components of the complex group velocity in the x and z directions, respectively. The functions g_1, g_2, and g_3 are given in quadratures in terms of the basic flow, the eigenvalues and the eigenfunctions of the zeroth-order problem, and the eigenfunctions of the adjoint problem. The function $\hat{h}_1$ is given in quadrature in terms of the basic flow, the adjoint eigenfunctions, the variation of the basic-flow quantities with the streamwise and spanwise directions, the nonparallel flow terms, and the variation of the eigenfunctions and eigenvalues of the zeroth-order problem in the streamwise and spanwise directions. It is defined in Appendix C.

E. Wavenumber modulation equations

The evaluation of h_1 demands the evaluation of the derivatives of the eigenfunctions of the zeroth-order problem. To this end we replace Z_{0i} by ζ_i, the zeroth-order eigenfunctions, in Eqs. (24)–(26), differentiate the resulting expressions with respect to x_1, and arrive at

$$D\left(\frac{\partial \zeta_i}{\partial x_1}\right) - \sum_{j=1}^{6} A_{ij}(\partial\zeta_j\partial x_1) = iE_i\frac{\partial \alpha}{\partial x_1} + iF_i\frac{\partial \beta}{\partial x_1} + H_{1i}, \tag{43}$$

subject to the boundary conditions

$$\frac{\partial \zeta_1}{\partial x_1} = \frac{\partial \zeta_3}{\partial x_1} = \frac{\partial \zeta_5}{\partial x_1} = 0 \quad \text{at } y = 0, \tag{44}$$

$$\frac{\partial \zeta_n}{\partial x_1} \to 0 \quad \text{as } y \to \infty. \tag{45}$$

Similarly, differentiating Eqs. (24)–(26) with respect to z_1 yields

$$D\left(\frac{\partial \zeta_i}{\partial z_1}\right) - \sum_{j=1}^{6} A_{ij}\left(\frac{\partial \zeta_j}{\partial z_1}\right) = iE_i\frac{\partial \alpha}{\partial z_1} + iF_i\frac{\partial \beta}{\partial z_1} + H_{2i}, \tag{46}$$

subject to the boundary conditions

$$\frac{\partial \zeta_1}{\partial z_1} = \frac{\partial \zeta_3}{\partial z_1} = \frac{\partial \zeta_5}{\partial z_1} = 0 \quad \text{at } y = 0, \tag{47}$$

$$\frac{\partial \zeta_n}{\partial z_1} \to 0 \quad \text{as } y \to \infty, \tag{48}$$

where the E_i and F_i are given in Appendix B and

$$H_{1i} = \sum_{m=1}^{6} \frac{\partial A_{im}}{\partial x_1}\bigg|_{\alpha,\beta} \zeta m, \quad H_{2i} = \sum_{m=1}^{6} \frac{\partial A_{im}}{\alpha z_1}\bigg|_{\alpha,\beta} \zeta m .$$

Since the homogeneous parts of Eqs. (43)–(48) have nontrivial solutions, the inhomogeneous systems (43)–(45) and (46)–(48) have solutions only if solvability conditions are satisfied. Application of these conditions yields

$$\omega_\alpha \frac{\partial \alpha}{\partial x_1} + \omega_\beta \frac{\partial \beta}{\partial x_1} = h_2, \tag{49}$$

and

$$\omega_\alpha \frac{\partial \alpha}{\partial z_1} + \omega_\beta \frac{\partial \beta}{\partial z_1} = h_3, \tag{50}$$

where h_2 and h_3 reflect the effects of nonparallelism in the streamwise and spanwise directions, respectively. They are given in Appendix C.

If the phase angle θ is assumed to be continuously differentiable, then it follows from Eqs. (15) that

$$\frac{\partial^2 \theta}{\partial z\, \partial x} = \epsilon \frac{\partial \alpha}{\partial z_1} = \frac{\partial^2 \theta}{\partial x\, \partial z} = \epsilon \frac{\partial \beta}{\partial x_1},$$

and hence

$$\frac{\partial \alpha}{\partial z_1} = \frac{\partial \beta}{\partial x_1}. \tag{51}$$

Therefore Eqs. (49) and (50) can be rewritten as

$$\omega_\alpha \frac{\partial \alpha}{\partial x_1} + \omega_\beta \frac{\partial \alpha}{\partial z_1} = h_2, \tag{52}$$

$$\omega_\alpha \frac{\partial \beta}{\partial x_1} + \omega_\beta \frac{\partial \beta}{\partial z_1} = h_3. \tag{53}$$

For this formulation to represent a physical problem, Nayfeh[16,17] showed that $\omega_\beta / \omega_\alpha$ must be a real quantity. Dividing Eqs. (52) and (53) by ω_α, we have

$$\frac{\partial \alpha}{\partial x_1} + \left(\frac{\omega_\beta}{\omega_\alpha}\right) \frac{\partial \alpha}{\partial z_1} = \frac{h_2}{\omega_\alpha}, \tag{54}$$

$$\frac{\partial \beta}{\partial x_1} + \left(\frac{\omega_\beta}{\omega_\alpha}\right) \frac{\partial \beta}{\partial z_1} = \frac{h_3}{\omega_\alpha}. \tag{55}$$

These two equations represent a Cauchy problem in x_1 and z_1.

A Cauchy problem needs initial data specified on an initial curve that is not a characteristic but intersects one of the characteristics. This initial data should satisfy the dispersion relationship and the condition $\omega_\beta / \omega_\alpha$ being real at a particular x and z location. The initial data are taken to be $\beta(x_1 = a, z_1) = \beta_0(z_1)$ on $z_1 = \Upsilon$.

IV. NUMERICAL PROCEDURE

The zeroth-order problem given by Eqs. (24)–(26) constitutes an eigenvalue problem. For a given mean flow the dispersion relationship is

$$\omega = \omega(\alpha, \beta, R). \tag{56}$$

The eigenvalues are determined numerically, in general, by integrating the system of Eqs. (24) in the transverse direction and imposing the boundary condition (25) and (26). The boundary conditions at $y = 0$ present no difficulties, but the boundary conditions (26) are reformulated in the numerical procedure that we use. Instead of applying the boundary conditions (26) numerically, which is very expensive and not very accurate, we determine an analytic solution to Eqs. (24) outside the boundary layer, apply these boundary conditions, and replace them with three other conditions at a finite value of y just outside the boundary layer. We reformulate the boundary conditions for both the zeroth-order problem and its adjoint following Ragab and Nayfeh.[18]

To solve the two-point boundary value problem given by Eqs. (24)–(26), we use the code SUPORT developed by Scott and Watts.[19] The integration procedure consists of the superposition of a set of linearly independent solutions coupled with an orthonormalization procedure that ensures the linear independence of the individual solution vectors. The boundary-value problem is converted into an initial value problem. The boundary conditions at $y = y_{\max}$ are known. These three conditions eliminate three of the six linearly independent solutions. The remaining three solutions are then integrated through the boundary layer. Linear combinations of these three solutions do not satisfy, in general, the boundary conditions at $y = 0$ unless the parameters ω, α, β, and R are chosen so that the dispersion relation (56) is satisfied. A simple Newton–Raphson procedure is used to iterate on the eigenvalues (two of the parameters $\omega_r, \omega_i, \alpha_r, \alpha_i, \beta_r, \beta_i$, and R given the other parameters) to satisfy the boundary conditions at $y = 0$.

V. METHOD OF DETERMINING THE MOST UNSTABLE DISTURBANCE

The most unstable disturbance is determined by solving the Cauchy problem defined by Eqs. (54) and (55) that govern the variation of the wavenumbers in the streamwise and spanwise directions. Their characteristics are given by

$$\frac{dx_1}{ds} = 1, \quad \frac{dz_1}{ds} = \frac{\omega_\beta}{\omega_\alpha}. \tag{57}$$

Along these characteristics, Eqs. (54) and (55) become

$$\frac{d\alpha}{ds} = \frac{h_2}{\omega_\alpha}, \quad \frac{d\beta}{ds} = \frac{h_3}{\omega_\alpha}. \tag{58}$$

Equations (58) show that if the wavenumber modulation equations were homogeneous, the wavenumbers would be constant along each characteristic obtained by integrating Eqs. (57). This would be the case for a "parallel boundary layer." Since the boundary layer is not "parallel" and all flow quantities are slowly varying functions of the streamwise and spanwise positions, the resulting wavenumber modulation equations are inhomogeneous, in general, and need to be numerically integrated with the dispersion relation (56) as a constraint subject to initial conditions on a given curve which intersects a characteristic curve. The initial conditions are given in the form

$$x_1(a,\tau) = a, \quad z_1(a,\tau) = \tau, \quad \beta(x_1 = a, \tau) = \beta_0(\tau). \tag{59}$$

The general solution of the characteristic Eqs. (57) can be written as

$$x_1 = s + c_1, \quad z_1 = \int_a^s \left(\frac{\omega_\beta}{\omega_\alpha}\right) dt + c_2, \tag{60}$$

where c_1 and c_2 are constants. Applying the initial conditions (59) and the choice $s = a$ at the initial curve, we obtain $x_1 = s$ and $c_2 = \tau$. Hence

$$x_1 = s, \quad z_1 = \int_a^s \left(\frac{\omega_\beta}{\omega_\alpha}\right) dt + \tau. \tag{61}$$

To proceed further, we need to determine the partial derivatives of α and β and hence the partial derivatives of the ζ_i with respect to x_1 and z_1 along the characteristics. It follows from Eq. (58b) and the initial condition (59c) that

$$\beta = \int_a^s \left(\frac{h_3}{\omega_\alpha}\right) dt + \beta_0(\tau). \tag{62}$$

Moreover, it follows from Eq. (61b) that

$$\tau = z_1 - \int_a^s \left(\frac{\omega_\beta}{\omega_\alpha}\right) dt. \tag{63}$$

Since

$$\frac{\partial\beta}{\partial x_1} = \left(\frac{\partial\beta}{\partial s}\right)\frac{\partial s}{\partial x_1} + \left(\frac{\partial\beta}{\partial\tau}\right)\frac{\partial\tau}{\partial x_1},$$

and

$$\frac{\partial s}{\partial x_1} = 1, \quad \frac{\partial\tau}{\partial x_1} = -\frac{\omega_\beta}{\omega_\alpha},$$

it follows from Eqs. (51) and (63) that

$$\frac{\partial\alpha}{\partial z_1} = \frac{\partial\beta}{\partial x_1} = \frac{h_3}{\omega_\alpha} - \left(\frac{\omega_\beta}{\omega_\alpha}\right)\beta_0'(\tau). \tag{64}$$

Then, it follows from Eq. (55) that

$$\frac{\partial\beta}{\partial z_1} = \beta_0'(\tau). \tag{65}$$

Substituting Eq. (64) into Eq. (54) yields

$$\frac{\partial\alpha}{\partial x_1} = \frac{h_2}{\omega_\alpha} - \left(\frac{\omega_\beta}{\omega_\alpha^2}\right) h_3 + \left(\frac{\omega_\beta}{\omega_\alpha}\right)^2 \beta_0'(\tau). \tag{66}$$

These partial derivatives can now be used to solve Eqs. (43)–(48) and hence determine $\partial\zeta_i/\partial x_1$ and $\partial\zeta_i/\partial z_1$, which in turn are used to determine h_1. Then, for a monochromatic frequency, Eq. (41) can be rewritten along the characteristic as

$$\frac{dA}{ds} = \left(\frac{h_1}{\omega_\alpha}\right) A. \tag{67}$$

To determine the most unstable disturbance, we determine the dimensional frequency ω^* and the real part of the dimensional spanwise wavenumber β_r^* such that the n factor is a maximum. To this end, for a given ω^*, we select a β_r^* and let $\beta_0(\tau) = \beta_r^*$ in Eq. (59c) and numerically integrate Eqs. (58), (61), and (67). The initial values for β_i, α_r, and α_i are determined by solving the zeroth-order eigenvalue problem and imposing the condition that $\omega_\beta/\omega_\alpha$ must be real. Then, s is incremented by Δs and new values for x_1, z_1, α and β are calculated from

$$x_1 = s + \Delta s, \quad z_1 = (\omega_\beta/\omega_\alpha)\Delta s + \tau, \tag{68}$$

$$\alpha = \alpha_{\text{old}} + (h_2/\omega_\alpha)\Delta s, \quad \beta = \beta_{\text{old}} + (h_3/\omega_\alpha)\Delta s. \tag{69}$$

The zeroth-order eigenvalue problem is solved at the new location to check whether the predicted values of α and β satisfy the eigenvalue problem and the condition that $\omega_\beta/\omega_\alpha$ must be real. If these conditions are not satisfied, the step size Δs is reduced until they are satisfied. Then, the n factor is calculated as

$$n = -\int_a^s \left[\alpha_i + \left(\frac{\omega_\beta}{\omega_\alpha}\right)\beta_i - \mathrm{Re}\left(\frac{h_1}{R\omega_\alpha}\right)\right] ds. \tag{70}$$

The process is repeated for other values of β_r^* and ω^*. Reed and Nayfeh[20] defined the most dangerous frequency and spanwise wavenumber (values that might be responsible for triggering large growth rates and eventually transition) to be the values that maximize n. Alternatively, we define the most dangerous frequency and spanwise wavenumber to be the values that make n exceed a critical value in the shortest possible distance.

VI. RESULTS AND DISCUSSION

Numerical results were obtained for the flow over a flat plate. The boundary layer equations were solved using the Transition Analysis Program System (TAPS).[21] In all calculations $U_e = 13.58$ m/sec, $T_e = 296.9$°K, $\rho_e = 0.9965$ g/cc, $\mu_e = 0.0091676$ cP, and the Reynolds number per meter (RPM) $= 14.76 \times 10^6$.

A. Unheated case

Table I shows the results for the unheated case. In each case ω^* is kept at a constant. To calculate the most dangerous frequency we proceed as follows. For flows over a flat plate we assume that transition corresponds to a given value of n, say 9. Keeping this in mind the most dangerous frequency should be defined as the one that gives $n = 9$ in the shortest possible distance along the plate. Based on this condition, Table I shows that the most dangerous frequency is $F \simeq 24.9 \times 10^{-6}$ corresponding to a dimensional frequency $\omega^* \simeq 5000$ Hz; R_x in all tables is the Reynolds number based on the surface distance x and is defined as $R_x = U_e^* x^*/\nu^*$. These results compare very well with those of Saric[22] who found that $F = 25 \times 10^{-6}$ is the most dangerous frequency for a flat plate. A sample plot of the variation of the n factor with distance is shown in Fig. 1.

It is interesting to note that $F = 12.46 \times 10^{-6}$ gives n factors greater than 20 if the integration is carried out along a much larger distance on the plate. Had we based the definition of the most dangerous frequency on the maximum value of n, irrespective of how far the disturbance travels to reach this value,[20] the results would have been erroneous. Since transition on a flat plate is assumed to occur when n reaches

TABLE I. Variation of n factor with frequency for unheated case.

ω^*	$F \times 10^{-6}$	n	$R_x \times 10^{-6}$
2500	12.46	18.50	9.28
2500	12.46	9.00	5.25
4500	22.42	9.00	3.23
4900	24.44	9.00	3.14
4950	24.68	9.00	3.13
5000	24.93	9.00	3.13
5100	25.43	9.00	3.15
5200	25.93	9.00	3.15

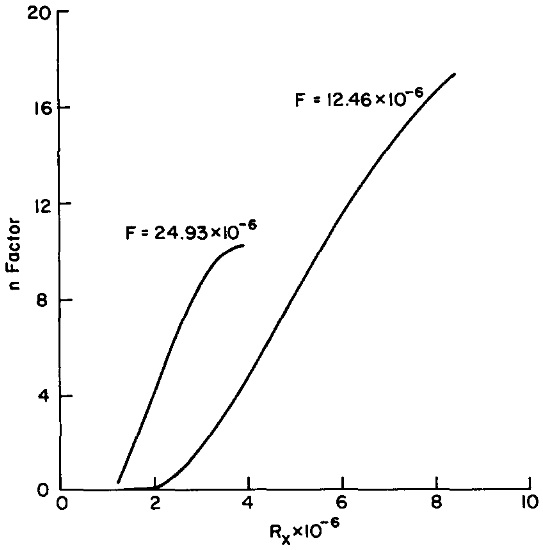

FIG. 1. Variation of the n factor with distance for the Blasius flow.

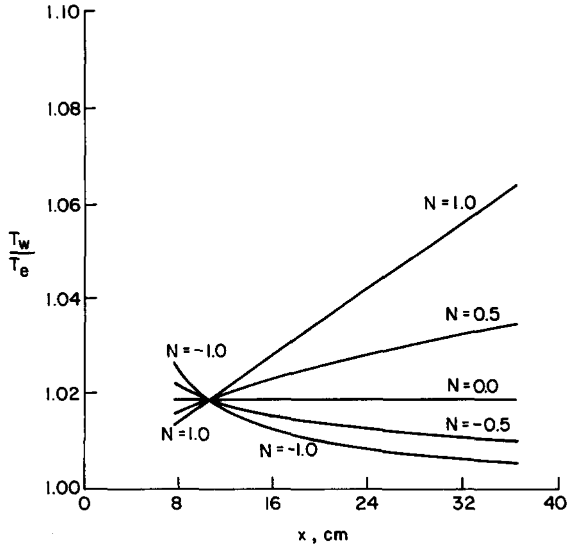

FIG. 2. Temperature distributions.

9, there is no point in performing any calculations once this critical value is reached for any frequency. Consequently the modified definition of the most critical frequency should be used.

Next, an investigation is considered into the influence of 3-D disturbances with different dimensional wavelengths in the spanwise direction. In all the cases considered, the resulting n factors are equal to or less than those obtained for 2-D disturbances as shown in Table II. Introduction of 3-D disturbances reduces the wavenumber as well as the growth rates. The stronger the three dimensionality is, the lower the growth rates.

B. Heated case

Next, we evaluate the influence of power-law wall heating on the stability of the flow past a flat plate. The wall temperature is varied as $T_w = T_e + Ax^N$, keeping $\Delta T = T_w - T_e = 5.56\,^\circ\mathrm{C}$ at $x = 10.71$ cm for all N. The temperature distribution along the length of the plate is shown in Fig. 2. The exponent N varies between -1.0 and 1.0. Since the wall overheat is a function of the streamwise position, we use nonsimilar boundary-layer codes to solve for the mean-flow quantities. This was shown to be a requirement by Nayfeh and El-Hady.[13] The mean flow is obtained using the TAPS code. The variation of the growth rates at $x_r = 10.71$ cm with N is shown in Table III for $F = 19.95 \times 10^{-6}$. Inspection of Table III shows that decreasing N produces smaller growth rates and hence is stabilizing, in agreement with the experimental results of Strazisar and Reshotko[12] and the analytical results of Nayfeh and El-Hady.[13] The stabilizing effect (decrease of growth rates and range of frequencies undergoing amplification) produced by decreasing the exponent N at the reference location can be explained as follows: As N decreases, Fig. 2 shows that ΔT increases at all locations upstream of x_r, resulting in full velocity profiles and hence a more stable flow. Therefore, the stabilizing effect at x_r is the culmination of all upstream stabilizing effects. However, as N decreases, ΔT decreases at all locations downstream of x_r, resulting in less full velocity profiles and hence a less stable flow. Consequently, there exists a location x_c downstream of x_r such that for $x > x_c$ a distribution with a larger exponent will produce smaller growth rates and range of frequencies undergoing amplification. Therefore, the effect of the exponent N on the stability should be based on the integration of the growth rates (n factor) rather than on the values of the growth rates at a given location. In the present case, Fig. 3 shows the variation of the n factors with distance for $F = 24.44 \times 10^{-6}$. It shows that the n factor for the case $N = -1.0$ is less than that for the case $N = 1.0$ only at locations corresponding to $R_x \approx 2.4 \times 10^6$. Therefore, if the criterion for comparing the

TABLE II. Effect of three-dimensional disturbances for unheated case.

ω^*	$F \times 10^{-6}$	β_r^*	n	$R_x = 10^{-6}$
4900	24.44	1.0e − 04	9.35	3.26
4900	24.44	3.0e + 00	9.16	3.26
4900	24.44	5.0e + 00	8.76	3.26
5000	24.93	1.0e − 04	9.35	3.26
5000	24.93	1.0e − 01	9.35	3.26
5000	24.93	3.0e − 01	9.35	3.26
5000	24.93	5.0e − 01	9.35	3.26
5000	24.93	1.0e + 01	6.16	3.12
5000	24.93	1.5e + 01	3.09	2.46

TABLE III. Effect of exponent N on growth rates at $x_r = 10.71$ cm.

N	α_i
$N = -1.0$	− 2.51e − 04
$N = -0.5$	− 0.54e − 03
$N = 0.5$	− 0.15e − 02
$N = 1.0$	− 0.17e − 02

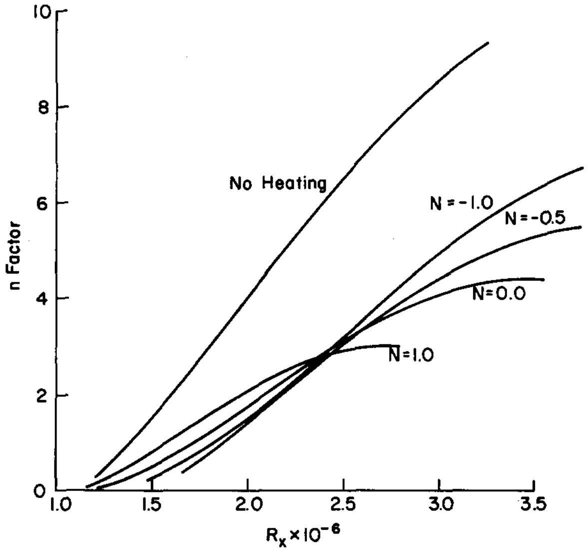

FIG. 3. Effect of exponent N on n factors.

TABLE V. Effect of heating on three dimensional disturbances.

β_r^*	T_w		n	$R_x \times 10^{-6}$
1.0e − 04	296.6		9.36	3.26
3.0e + 00	296.6		9.16	3.26
5.0e + 00	296.6		9.01	3.40
1.0e − 04	$A = 10.0,$	$N = 0.0$	4.40	3.48
1.0e + 00	$A = 10.0,$	$N = 0.0$	4.37	3.40
1.0e − 04	$A = 16.87,$	$N = 0.5$	3.63	3.04
1.0e + 00	$A = 16.87,$	$N = 0.5$	3.61	3.12
3.0e + 00	$A = 16.87,$	$N = 0.5$	3.45	2.98
5.0e + 00	$A = 16.87,$	$N = 0.5$	3.12	2.98
1.0e − 04	$A = 5.93,$	$N = -0.5$	5.52	3.83
1.0e + 00	$A = 5.93,$	$N = -0.5$	5.47	3.83
1.0e − 04	$A = 3.52,$	$N = -1.0$	6.89	4.18
1.0e + 00	$A = 3.52,$	$N = -1.0$	6.84	4.18
1.0e − 04	$A = 28.46,$	$N = 1.0$	3.02	2.73
1.0e + 00	$A = 28.46,$	$N = 1.0$	3.01	2.73

effect of the exponent N on the stability is based on the maximum value of the n factor, or the value of n that exceeds a given value in the shortest distance, then decreasing N is destabilizing, which appears to contradict the conclusions of Strazisar and Reshotko[12] and Nayfeh and El-Hady.[13]

It should be emphasized that, if x_r is greater than or equal to or even slightly less than the streamwise position of branch II of the neutral stability curve, decreasing N will produce smaller values of n and hence is stabilizing. This is the reason why Nayfeh and El-Hady,[13] after calculating the n factors for distributions corresponding to different values of the exponent N, have arrived at the conclusion that decreasing N is stabilizing. As a test we changed x_r to 25.0 cm for the case $\omega^* = 4900$, for which the branch II point occurs at $x = 27.5$ cm, and calculated the effect of the exponent N on the n factors. For this value of x_r, Table IV shows that decreasing N is stabilizing, in agreement with the conclusions of Strazisar and Reshotko[12] and Nayfeh and El-Hady.[13] Therefore the conclusion that decreasing or increasing N is stabilizing actually depends on the location of the reference point x_r and is thus not universal.

Table V shows the calculated n factors for 3-D disturbances for $F = 24.44 \times 10^{-6}$. Also shown are the results of the unheated case. It is clear from the data that 3-D disturbances result in lower growth rates for both unheated and heated boundary layers over a flat plate, irrespective of the wall overheat. The introduction of three dimensionality actually lowers the streamwise wavenumbers α_r and the streamwise growth rates $-\alpha_i$.

TABLE IV. Effect of exponent N on n factors for $x_r = 25$ cm.

T_w		n	$R_x \times 10^{-6}$
296.9 °K		9.36	3.26
$A = 8.21,$	$N = -1.0$	completely stable	
$A = 9.06,$	$N = -0.5$	3.04	3.69
$A = 10.00,$	$N = 0.0$	4.40	3.48
$A = 11.04,$	$N = 0.5$	5.56	3.26
$A = 12.19,$	$N = 1.0$	6.48	3.26

VII. CONCLUDING REMARKS

We have analyzed the 3–D stability of 2-D boundary-layer flows. Numerical results are presented for the case of heated water boundary layers.

A modified definition for determining the critical frequency has been suggested. The critical frequency is the one that gives $n = n_{tr}$, where n_{tr} corresponds to transition, in the shortest possible distance on the body. This definition has been used to obtain the most dangerous frequency for the Blasius boundary layer.

Three dimensional disturbances have been studied. The results show that 2-D waves are more critical and yield higher n factors than 3-D waves for heated and unheated boundary layers.

The effect of power-law heating distributions in a water boundary layer on the n factors has been evaluated. The results show that the stability strongly depends on the actual heat distribution.

ACKNOWLEDGMENT

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APPENDIX A: COMPONENTS OF MATRIX A

All $A_{ij} = 0$ except the following:

$A_{12} = 1.0,$

$A_{21} = (R/\mu_s)(-i\omega\rho_s + i\alpha U_s\rho_s) + (\alpha^2 + \beta^2),$

$A_{22} = -D\mu_s/\mu_s,$

$A_{23} = (R/\mu_s)[\rho_s DU_s - i\alpha D\mu_s/R + i\alpha\mu_s(s+1)D\rho_s/R\rho_s]$

$A_{24} = i\alpha R/\mu_s,$

$$A_{31} = -i\alpha,$$
$$A_{33} = -D\rho_s/\rho_s,$$
$$A_{35} = -i\beta,$$
$$A_{41} = [-2i\alpha D\mu_s + ir\alpha\mu_s(D\rho_s/\rho_s)]/R,$$
$$A_{42} = -i\alpha\mu_s/R,$$
$$A_{43} = -i\rho_s(-\omega + \alpha U_s) - \{\mu_s[\alpha^2 + \beta^2 + rD(D\rho_s/\rho_s)] + r[D\mu_s D\rho_s/\rho_s - \mu_s(D\rho_s/\rho_s)^2]\}/R,$$
$$A_{45} = i\beta(r\mu_s D\rho_s/\rho_s - 2D\mu_s)/R,$$
$$A_{46} = -i\beta\mu_s/R,$$
$$A_{56} = 1.0,$$
$$A_{63} = -i\beta[D\mu_s/\mu_s - (s+1)D\rho_s/\rho_s],$$
$$A_{64} = i\beta R/\mu_s,$$
$$A_{65} = i\rho_s R(-\omega + \alpha U_s)/\mu_s + \alpha^2 + \beta^2,$$
$$A_{66} = -D\mu_s/\mu_s.$$

APPENDIX B: INHOMOGENEITIES FOR FIRST-ORDER PROBLEM

$$I_1 = -\rho_s\left(\frac{\partial u_0}{\partial x_1} + \frac{\partial w_0}{\partial z_1}\right) - u_0\frac{\partial \rho_s}{\partial x_1}$$

$$\begin{aligned} I_2 = &-\rho_s\left(\frac{\partial u_0}{\partial t_1} + U_s\frac{\partial u_0}{\partial x_1} + V_s\frac{\partial u_0}{\partial y} + u_0\frac{\partial U_s}{\partial x_1}\right) + W_s\frac{\partial u_0}{\partial z_1} - \frac{\partial p_0}{\partial x_1} + \frac{i}{R}\left(2\mu_s\, i\alpha\frac{\partial u_0}{\partial x_1} + 2\mu_s i u_0\frac{\partial \alpha}{\partial x_1} + 2i\alpha u_0\frac{\partial \mu_s}{\partial x_1} + 2i\alpha\mu_s\frac{\partial u_0}{\partial x_1}\right. \\ &+ i\alpha\lambda_s\frac{\partial u_0}{\partial x_1} + i\alpha u_0\frac{\partial \lambda_s}{\partial x_1} + i\alpha\lambda_s\frac{\partial u_0}{\partial x_1} + i\lambda_s u_0\frac{\partial \alpha}{\partial x_1} + \lambda_s\frac{\partial^2 v_0}{\partial x_1\,\partial y} + \frac{\partial v_0}{\partial y}\frac{\partial \lambda_s}{\partial x_1} + i\beta w_0\frac{\partial \lambda_s}{\partial x_1} + i\lambda_s\beta\frac{\partial w_0}{\partial x_1} + i\alpha\lambda_s\frac{\partial w_0}{\partial z_1} \\ &\left.+ iw_0\lambda_s\frac{\partial \beta}{\partial x_1} + \frac{\partial \mu_s}{\partial y}\frac{\partial v_0}{\partial x_1} + \mu_s\frac{\partial^2 v_0}{\partial y\,\partial x_1} + i\alpha\mu_s\frac{\partial w_0}{\partial z_1} + i\beta\mu_s\frac{\partial w_0}{\partial x_1} + i\mu_s w_0\frac{\partial \alpha}{\partial z_1} + 2i\beta\mu_s\frac{\partial u_0}{\partial z_1} + i\mu_s u_0\frac{\partial \beta}{\partial z_1}\right), \end{aligned}$$

$$\begin{aligned} I_3 = &-\rho_s\left(\frac{\partial v_0}{\partial t_1} + U_s\frac{\partial v_0}{\partial x_1} + V_s\frac{\partial v_0}{\partial y} + v_0\frac{\partial V_s}{\partial y}\right) + \frac{1}{R}\left(i\alpha v_0\frac{\partial \mu_s}{\partial x_1} + i\alpha\mu_s\frac{\partial v_0}{\partial x_1} + i\mu_s v_0\frac{\partial \alpha}{\partial x_1} + i\alpha\frac{\partial v_0}{\partial x_1} + \mu_s\frac{\partial^2 u_0}{\partial x_1\,\partial y}\right. \\ &\left.+ \lambda_s\frac{\partial^2 u_0}{\partial x_1\,\partial y} + \frac{\partial \lambda_s}{\partial y}\frac{\partial u_0}{\partial x_1} + \frac{\partial \lambda_s}{\partial y}\frac{\partial w_0}{\partial z_1} + \lambda_s\frac{\partial^2 w_0}{\partial y\,\partial z_1} + i\mu_s v_0\frac{\partial \beta}{\partial z_1} + i\beta\mu_s\frac{\partial v_0}{\partial z_1} + i\mu_s\beta\frac{\partial v_0}{\partial z_1} + \mu_s\frac{\partial^2 w_0}{\partial y\,\partial z_1}\right), \end{aligned}$$

$$\begin{aligned} I_4 = &-\rho_s\left(\frac{\partial w_0}{\partial t_1} + U_s\frac{\partial w_0}{\partial x_1} + V_s\frac{\partial w_0}{\partial y}\right) - \frac{\partial p_0}{\partial z_1} - \frac{1}{R}\left[-w_0\left(i\alpha\frac{\partial \mu_s}{\partial x_1} + i\mu_s\frac{\partial \alpha}{\partial x_1} + 2i\mu_s\frac{\partial \beta}{\partial z_1} + i\lambda_s\frac{\partial \beta}{\partial z_1}\right) - 2i\alpha\mu_s\frac{\partial w_0}{\partial x_1}\right. \\ &+ u_0\left(-i\mu_s\frac{\partial \beta}{\partial x_1} - i\beta\frac{\partial \mu_s}{\partial x_1} - i\lambda_s\frac{\partial \alpha}{\partial z_1}\right) + \frac{\partial u_0}{\partial x_1}(-i\beta\mu_s - i\beta\lambda_s) + \frac{\partial u_0}{\partial z_1}(-i\alpha\mu_s - i\alpha\lambda_s) \\ &\left.+ \frac{\partial^2 v_0}{\partial y\,\partial z_1}(-\mu_s - \lambda_s) + \frac{\partial v_0}{\partial z_1}\left(-\frac{\partial \mu_s}{\partial y}\right) + \frac{\partial w_0}{\partial z_1}(-2i\mu_s\beta - 2i\mu_s - 2i\beta\lambda_s)\right], \end{aligned}$$

$$D_n = i\sum_{m=1}^{6}\frac{\partial A_{nm}}{\partial \omega}\zeta_m,$$

$$E_n = -i\sum_{m=1}^{6}\frac{\partial A_{nm}}{\partial \alpha}\zeta_m,$$

$$F_n = -i\sum_{m=1}^{6}\frac{\partial A_{nm}}{\partial \beta}\zeta_m,$$

$$G_1 = G_5 = 0,$$

$$G_2 = \rho_s\left(\frac{\partial U_s}{\partial x_1}\zeta_1 + U_s\frac{\partial \zeta_1}{\partial x_1} + V_s\frac{\partial \zeta_1}{\partial y} + W_s\frac{\partial \zeta_1}{\partial z_1} + \frac{1}{\rho_s}\frac{\partial \zeta_4}{\partial x_1}\right)(\mu_s)^{-1},$$

$$G_3 = -\left(\frac{\partial \zeta_1}{\partial x_1} + \frac{\zeta_1}{\rho_s}\frac{\partial \rho_s}{\partial x_1} + \frac{\partial \zeta_5}{\partial z_1}\right),$$

$$G_4 = -\rho_s\left(U_s\frac{\partial \zeta_3}{\partial x_1} + V_s\frac{\partial \zeta_3}{\partial y} + \zeta_3\frac{\partial V_s}{\partial y}\right),$$

$$G_6 = \frac{\rho_s R}{\mu_s}\left(U_s\frac{\partial \zeta_5}{\partial x_1} + V_s\frac{\partial \zeta_5}{\partial y} + \frac{1}{\rho_s}\frac{\partial \zeta_4}{\partial z_1}\right).$$

APPENDIX C: COEFFICIENTS OF MODULATION EQUATIONS

$$g_1 = \int_0^\infty\left(\frac{\rho_s R}{\mu_s}(W_2\zeta_1 + W_6\zeta_5) - \rho_s\zeta_3 W_4\right)dy,$$

$$g_2 = \int_0^\infty\left\{\frac{\rho_s R U_s}{\mu_s}\left[W_2\left(\zeta_1 + \frac{\zeta_4}{U_s}\right) + W_{16}\zeta_5\right] - W_3\zeta_1 - \rho_s U_s\zeta_3 W_4\right\}dy,$$

$$g_3 = \int_0^\infty\left(-W_3\zeta_5 + \frac{RW_6\zeta_4}{\mu_s}\right)dy,$$

$$\hat{h}_1 = \sum_{m=1}^{6}\int_0^\infty G_m W_m\, dy,$$

$$h_2 = \sum_{n,m=1}^{6}\int_0^\infty \left.\frac{\partial A_{nm}}{\partial x_1}\right|_{\alpha,\beta}\zeta_m W_n\, dy,$$

$$h_3 = \sum_{n,m=1}^{6}\int_0^\infty \left.\frac{\partial A_{nm}}{\partial z_1}\right|_{\alpha,\beta}\zeta_m W_n\, dy.$$

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