Characterizing Zero Divisors of Group Rings

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(ABSTRACT)

The Atiyah Conjecture originates from a paper written 40 years ago by Sir Michael Atiyah, a famous mathematician and Fields medalist. Since publication of the paper, mathematicians have been working to solve many questions related to the conjecture, but it is still open.

The conjecture is about certain topological invariants attached to a group G. There are examples showing that the conjecture does not hold in general. These examples involve something like the lamplighter group (the wreath product $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$). We are interested in looking at examples where this is not the case. We are interested in the specific case where G is a finitely generated group in which the Prüfer group can be embedded as the center. The Prüfer group is a p-group for some prime p and its finite subgroups have unbounded order, in particular the finite subgroups of G will have unbounded order.

To understand whether any form of the Atiyah conjecture is true for G, it will first help to determine whether the group ring kG of the group G has a classical ring of quotients for some field k. To determine this we will need to know the zero divisors for the group ring kG. Our investigations will be divided into two cases, namely when the characteristic of the field k is the same as the prime p for the Prüfer group and when it is different.

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Chapter 1

Group Rings

1.1 Definitions

Definition 1.1. Let G be a group and R a ring. Set RG to be the set of all linear combinations of the form $\alpha = \sum_{g \in G} a_g g$ where $a_g \in R, g \in G$ and there are only finitely many $a_g \neq 0$. Given two elements of RG, $\alpha = \sum a_g g, \beta = \sum b_g g$, we define the sum $\alpha + \beta = \sum (a_g + b_g) g$ and the product to be $\alpha\beta = \sum_{g,h \in G} a_g g b_h h$. With these two operations, we can see that RG is a ring. RG is a ring with unity by $1 = \sum u_g g$ where $u_g = 1$ when g is the unit of G and $u_g = 0$ otherwise.

The set RG with the operations defined above is called the group ring of G over R. In the case where R is commutative, RG is also called the group algebra of G over R. [7, Definition 3.2.6]

We will work over the group ring kG where k is some field and G is a special group containing a countable abelian p-group H as its center.

Definition 1.2. A group G is said to be totally ordered if there is a translation invariant order $\leq$ such that for any two $x, y \in G$ either $x \leq y$ or $y \leq x$. If $x \leq y$ and $y \leq x$, then $x = y$.

Definition 1.3. A group G is said to be left ordered, given that G is totally ordered and for all $g \in G$, if $x < y$, then $gx < gy$.

Proposition 1.4. *A group G is left ordered if and only if it is right ordered.*

Proof. Let G be a left ordered group. Then there is an order $<$ on G such that G is left ordered under $<$. Define the order $<'$ by $x <' y$ means $y^{-1} < x^{-1}$. Let $x <' y$. We wish to show for any $g \in G$, $xg <' yg$. G is left ordered and $y^{-1} < x^{-1}$ so $g^{-1}y^{-1} < g^{-1}x^{-1}$. By definition of $<'$, this means that $xg <' yg$. Similarly, we could show if G is right ordered, then G is left ordered. □

Theorem 1.5. *Let k be a field and G a right ordered group. Then kG is a domain. [4, Theorem 4.1]*

We will assume that G/H is left ordered (and so right ordered by Proposition 1.4), so $k[G/H]$ is a domain.

Definition 1.6. The set $\{g \in G \mid a_g \neq 0\}$ is called the support of α and is denoted $\operatorname{Supp}\alpha$.

Using this definition, we have $kH = \{\alpha \in kG \mid \operatorname{Supp}\alpha \subset H\}$. Note that for any $\alpha \in kH$, $\operatorname{Supp}\alpha$ is a finite collection of elements of H, so we can use these finite elements to generate a group $A = \langle \operatorname{Supp}\alpha \rangle$ such that $\alpha \in kA$. We will do this many times in our proofs. It will also help in our proofs to have a standard way of representing an element $\alpha \in kG$. The following definition and lemma will help us with this.

Definition 1.7. Let G be a group and H a normal subgroup of G. Let X be the set consisting of exactly one representative for each coset of H in G. Then X is called a transversal for H is G.

Lemma 1.8. *Let Y be a transversal for H in G. Then every element $\alpha \in kG$ can be written uniquely as a finite sum of the form $\alpha = \sum_{y \in Y} y\alpha_y$ with $\alpha_y \in kH$. Thus kG is a free right kH module with Y as a free basis. [5, Lemma 1.1.3]*

We will write $\alpha = \sum \alpha_i x_i$ where $\alpha_i \in kH$ and $x_i \in X$ where X is a transversal for H in G.

1.2 Augmentation Ideal

Lemma 1.9. *Let E be an algebra over K and let $\tau : G \to U(E)$ be a group homomorphism of G into the group of units of E. Then τ extends to a K-algebra homomorphism $\tau : kG \to E$ be defining $\tau(\sum a_x x) = \sum a_x \tau(x)$. [5, Lemma 1.1.7]*

Let H be a normal subgroup of G. There is a natural homomorphism$^-$: $kG \to k[G/H]$ which maps $\alpha = \sum \alpha_g g \in kG$ to $\bar{\alpha} = \sum \alpha_g \bar{g} \in k[G/H]$ where $\bar{g} = gH$ is the image of g in G/H .

Definition 1.10. Consider the case when $H = G$. Then $G/H = \langle 1 \rangle$ and $\bar{g} = 1$ for all $g \in G$. Hence $\sum \alpha_g \bar{g} = (\sum \alpha_g)1$ and the kernel of this map is $\omega(kG) = \{\sum a_g g \mid \sum a_g = 0\}$. This is called the augmentation ideal of kG, and it has codimension 1 in kG. [5, p. 10]

The set $\{g - 1 : g \in G, g \neq 1\}$ is a basis of $\omega(kG)$ over k. In general, the kernel of the bar mapping is $\omega(kH)kG$ and by the first isomorphism theorem we have $\frac{kG}{\omega(kH)kG} \cong k(G/H)$.

The augmentation ideal will turn out to be very important in our classification of zero divisors. We will discuss some of its properties that will be useful in later proofs.

1.3 Nilpotent Elements

Definition 1.11. Let R be a ring and let $\mathbb{Z}^+$ represent the positive integers. An element $x \in R$ is said to be nilpotent if there is an $n \in \mathbb{Z}^+$ such that $x^n = 0$. An ideal I is said to be nilpotent if there exists an $n \in \mathbb{Z}^+$ such that $I^n = 0$.

Definition 1.12. An ideal I of a ring R is called a nil ideal if every element of I is a nilpotent element.

In a commutative ring, R, the nilpotent elements form an ideal called the nilradical. The nilradical is the intersection of all the prime ideals of R.

Lemma 1.13. *If M is a prime ideal of kH where H is a locally finite abelian group, then M is a maximal ideal.*

This means that in the group ring kH, maximal ideals and prime ideals are equivalent, so the intersection of all the prime ideals will be the intersection of all the maximal ideals.

Definition 1.14. Let G be a group and p be a prime. G is called a p-group if every element of G has order a power of p. That is, for each $g \in G$, there exists $n \in \mathbb{Z}^+$ such that $g^{p^n} = 1$.

Lemma 1.15. *Let G be a nontrivial group. Then $\omega(kG)$ is nilpotent if and only if $char(k) = p$ for some prime p and G is a finite p-group. [5, Lemma 3.1.6]*

For a countable abelian p-group H, we cannot apply Lemma 1.14 because H is not necessarily finite. However, every finite collection of elements generate a finite group A (because a countable abelian p-group is locally finite). So in the case where $\text{char}(k) = p$, for any $\alpha \in kH$, there is a finite group A such that $\alpha \in kA$ where $\omega(kA)$ is nilpotent by the lemma above. Therefore, every element of $\omega(kH)$ is nilpotent.

However, in the case when $\text{char}(k)$ doesn't divide p, kH has no nonzero nilpotent elements. We see this through the following theorem.

Theorem 1.16. *Let kG be the group algebra of a finite group over a field k of characteristic $p \geq 0$. Then kG has a nonzero nilpotent ideal if and only if $p > 0$ and $p \mid |G|$ [7, Theorem 6.2.2]*

Assume $\beta \in kH$ is nilpotent. There is some $j \in \mathbb{Z}^+$ such that $\beta^j = 0$. Set $I = (\beta)$. Let $x \in I$. Then $x = \beta\gamma$ for some $\gamma \in kH$. $\beta^j = 0$, so $x^j = (\beta\gamma)^j = 0$ and x is nilpotent. Since every element of I is of the form $\beta\gamma$ for some γ, $I^j = 0$ so I is a nilpotent ideal. This contradicts Theorem 1.15, so there can be no nilpotent elements of kH. Hence, the intersection of all the maximal ideals of kH where k is a field of characteristic $\neq p$ will be 0.

1.4 Crossed Products

Definition 1.17. Let R be a ring with 1 and let G be a group. Then a crossed product $R * G$ of G over R is an associative ring which contains R and has an R-basis the set $\bar{G}$, a copy of G. Thus each element of $R * G$ is uniquely a finite sum $\sum_{x \in G} \bar{x} r_x$ with $r_x \in R$. Addition is as expected and multiplication is determined by two rules below. Specifically for $x, y \in G$ we have (twisting) $\bar{x}\bar{y} = \overline{xy}\tau(x, y)$ where $\tau : G \times G \to U = U(R)$, the group of units of R. Furthermore, for every $x \in G$ and $r \in R$ we have (action) $r\bar{x} = \bar{x} r^{\sigma(x)}$ where $\sigma : G \to Aut(R)$. [6]

Using this definition and setting $\sigma(x) = 1, \tau(x, y) = 1$, we have that $kG = k * G$. This allows us to consider group rings as crossed products.

Lemma 1.18. *Let $R * G$ be given and let $N \lhd G$. Then $R * G = (R * N) * (G/N)$ where the latter is some crossed product of the group G/N over the ring $R * N$. [6, Lemma 1.3]*

Set G equal to the special group containing a countable abelian p-group H as its center. Then $H \lhd G$ and $kH = k * H$, so applying Lemma 1.17 we see that $k * G = (kH) * (G/H)$.

Let $M \lhd kH$. Then $MkG = MG$ is an ideal of $k * G = kH * G/H$. Consider $\alpha_1 t_1 + ... + \alpha_n t_n$ where t_i are coset representatives of H in G and $\alpha_i \in kH$. In $kH * G/H$ this is $\sum \alpha_i \bar{t_i}$ where $\bar{t_i} = Ht_i$. Then an element of MkG is of the form $\sum m_i t_i$ where $m_i \in M$. In $kH * G/H$, MkG will be the ideal $M(kH) * G/H = M * G/H$ since M is an ideal of kH.

Lemma 1.19. *Let $R * G$ be given.*
i. If $J \lhd R * G$, then $J \cap R$ is a G-stable ideal of R and $(J \cap R) * G \subseteq J$.
ii. If I is a G-stable ideal of R, then $I * G \lhd R * G$ with $(I * G) \cap R = I$. Moreover $(R * G)/(I * G) \cong (R/I) * G$ where the latter is a suitable crossed product of G over R/I. [6, Lemma 1.4]

By Lemma 1.18, $\frac{kH*G/H}{M*G/H} \cong kH/M*G/H$. Combining these results we have $\frac{kG}{MkG} \cong \frac{k*G}{M*G/H} \cong \frac{kH*G/H}{M*G/H} \cong kH/M*G/H$. Therefore, $\frac{kG}{MkG} \cong kH/M*G/H$.

Theorem 1.20. *Let k be a domain, let G be a right ordered group, and let $k*G$ be a crossed product. Then $k*G$ is a domain. [4, Theorem 4.3]*

Let M be a maximal ideal. Above we showed that $\frac{kG}{MkG} \cong kH/M*G/H$. M being maximal means that kH/M is an integral domain. We assume G/H is left ordered (and so right ordered by Proposition 1.4), so by Theorem 1.19 $\frac{kG}{MkG}$ is a domain.

Other Ring Theory

Theorem 1.21. *Maschke's theorem: Let G be a finite group and let F be a field whose characteristic does not divide $|G|$. If V is any FG-module and U is any submodule of V, then V has a submodule W such that $V = U \oplus W$ (i.e., every submodule is a direct summand). [1]*

Let k be a field of characteristic p' such that p' does not divide p. Let A be a finite p-group. Then we can apply Maschke's theorem to kA. If I is a submodule of kA then there is a submodule J such that $kA = I \oplus J$.

Proposition 1.22. *Let R be a commutative ring. The map $^* : RG \to RG$ defined by $(\sum_{g\in G} a_g g)^* = \sum_{g\in G} a_g g^{-1}$ satisfies the properties*

i. $(\alpha + \beta)^* = \alpha^* + \beta^*$,
ii. $(\alpha\beta)^* = \beta^*\alpha^*$, *and*
iii. $\alpha^{**} = \alpha$

[7, Proposition 3.2.11]

Thus group rings over commutative rings are rings with involution. We will use this property later to help us understand the set of zero divisors.

Chapter 2

Motivation

The Atiyah Conjecture can be given an algebraic description using the group ring $\mathbb{C}G$. In the special case $\text{lcm}(G) = 1$ (i.e. G is torsion free) it implies the Kaplansky zero-divisor conjecture, namely that if G is torsion free, then $\mathbb{C}G$ is a domain. However the question we are interested in is the case where the finite subgroups of G have unbounded order. It was thought for a long time that the Conjecture would remain true in this case; however, this turned out not to be true.

Since an element of the group ring has finite support, it will be contained in a finitely generated subgroup (the subgroup generated by the elements of its support) and then the Conjecture reduces to the case where G is finitely generated. We will focus on such groups. Examples showing that the Conjecture no longer holds when G has unbounded order involve something like the lamplighter group (the wreath product $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$). A group which is not the lamplighter group but has finite subgroups of unbounded order is the Prüfer group $\mathbb{Z}[1/p]/\mathbb{Z}$.

The Prüfer p-group may be denoted by $\mathbb{Z}(p^\infty)$ and one can express it as

$$\mathbb{Z}(p^\infty) = \left\{ e^{2\pi i m/p^n} \mid m \in \mathbb{Z}^+ \; n \in \mathbb{Z}^+ \right\}.$$

The Prüfer group is countably infinite abelian and locally cyclic (meaning every finite set of elements of the Prüfer group generates a cyclic group).

The Prüfer group is not finitely generated but can be embedded in a nice finitely generated group. In fact, Phillip Hall showed the Prüfer group can be embedded as the center of a finitely generated 3-step solvable group, which we will call G.

First we will define some notation. $[G'', G] = 1$ means that this group is 3-step solvable group. $\zeta(G) =$ center of G and $d(G)$ is the minimum number of elements which will suffice

to generate G.

Theorem 2.1. *Let C be a given countable abelian group $\neq 1$. Then there are uncountably many isomorphism types of groups G such that $d(G) = 2$, $\zeta(G) \cong C$ and $[G'', G] = 1$. [3, Theorem 6]*

We will not show a direct proof of this theorem. Instead we will show a sketch of how Hall uses Theorem 2.2 below to prove Theorem 2.1.

Theorem 2.2. *There exists a group G whose center Z is the free Abelian group with $\aleph_0$ generators such that $d(G) = 2$, $\zeta(G/Z) = Z/Z$ and $[G'', G] = 1$. [3, Theorem 7]*

We will create a mapping from Z to C by mapping half of the generators onto the countable set of generators for C. With the remaining half we will map an arbitrary subset, which we call K, to 1 and the remaining generators to some element $c \in C$. K will be the kernel of this homomorphism and by the first isomorphism theorem we have that Z/K is isomorphic to C. We could have selected any of the generators to be in K, so there will exist $2^{\aleph_0}$ different subgroups K of Z such that Z/K is isomorphic to C. Each of these K will be distinct, so we have an uncountable infinity of groups K.

Z is the center of G, so K is a normal subgroup of G and G/K makes sense. By assumption, $\zeta(G/Z) = Z/Z$, so $\zeta(G/K) = Z/K \cong C$ and $[G'', G] = 1$ implies that $[(G/K)'', G/K] = 1$. Lastly, $d(G/K)$ is less than or equal to $d(G)$ which equals 2. G/K is not abelian so not cyclic and must have $d(G/K)$ greater than 1. Hence $d(G/K)$ is exactly 2.

We see that the group G/K fits all the conditions needed for Theorem 2.1, so now all we need is that there are an uncountable infinity of different groups. Let Γ be one of the G/K. There can only be a countable number of homomorphisms from G to Γ because G is finitely generated and Γ is countable. So there are only countably many K such that G/K is isomorphic to Γ. Above we showed that the number of possible K is uncountable, so there are an uncountable infinity of non isomorphic G/K.

This shows that we can embed the Prüfer group as the center of a finitely generated group G. In fact, we can embed any countable abelian p-group H as the center of a finitely generated group G. Further, we can choose G so that $G/\zeta(G)$ is isomorphic to the wreath product $\mathbb{Z} \wr \mathbb{Z}$, which is bi-ordered. This allows us to assume G/H is left ordered.

Definition 2.3. A non-zero-divisor in R is an element x such that $rx \neq 0$ and $xr \neq 0$ for all nonzero $r \in R$. A classical right quotient ring for R is a ring Q which contains R

as a subring in such a way that every non-zero-divisor of R is invertible in Q and $Q = \{ab^{-1} \mid a, b \in R, b \text{ non-zero-divisor}\}$. [2, p. 96]

Proposition 2.4. *Let H be a countable abelian p-group and k some field of characteristic p. kH is its own classical quotient ring.*

Proof. Any finite collection of elements of H creates a finite subgroup A. A is a p-group and has order p^n for some n. Let $\alpha \in kH$. Then $\alpha \in kA$ for some A by generating a group from the finite number of elements in Suppα.

Let $\alpha = \sum a_i g_i$ such that $a_i \in k$ and $g_i \in A$. k has characteristic p, so $\alpha^{p^n} = \sum (a_i g_i)^{p^n} = \sum a_i g_i^{p^n} = \sum a_i$.

First suppose $\alpha \in \omega(kA)$. Then $\sum a_i = 0$. So $\alpha^{p^n} = 0$ and α is a zero divisor.

Next suppose $\sum a_i = 1$. Let $\beta = \sum b_i g_i$ where $b_i = a_i$ for $i \geq 1$ and $b_0 = a_0 - 1$. $\sum b_1 = 0$ so as above $\beta^{p^n} = 0$. $\alpha = 1 + \beta$ and has $1 - \beta + \beta^2 - \beta^3 ...$ as its inverse. Hence α is a unit.

Lastly suppose $\sum a_i = m$ for some $m \neq 0$. Let $\beta = \sum b_i g_i$ where $b_i = \frac{1}{m} a_i$ for $i \geq 1$ and $b_0 = \frac{1}{m}(a_0) - 1$. Then $\alpha = m(1 + \beta)$ with $\frac{1}{m}(1 - \beta + \beta^2 - \beta^3 ...)$ as its inverse. Hence α is a unit. Note this shows that if $\alpha \notin \omega(kA)$, then α is a unit.

So every element of kH is either a unit or a zero divisor. And $kH = \{ab^{-1} \mid a, b \in kH, b = 1\}$ so kH is its own classical quotient ring. □

Corollary 2.5. *If H is a countable abelian p-group then, $\alpha \in kH$ is a zero divisor if and only if $\alpha \in \omega(kH)$.*

We want to know if kG will have a classical ring of quotients when G is the special group containing the Prüfer group as its center.

From the Atiyah Conjecture, we have a ring $\mathcal{U}(G)$ containing $\mathbb{C}G$, which is constructed from unbounded operators acting on $l^2(G)$ (the Hilbert space with basis G). Every element of $\mathcal{U}(G)$ is either a zero divisor or unit. We would like to know if there is a smaller ring containing $\mathbb{C}G$ in which every element is a zero divisor or unit, in particular we would like this ring to be a classical quotient ring, (i.e., every element is of the form $\{ab^{-1} \mid a, b \in \mathbb{C}G, b \text{ non-zero-divisor}\}$).

Proposition 2.6. *We say that R satisfies the right Ore condition if given $a, x \in R$ with x a non-zero-divisor, there exist $b, y \in R$ with y a non-zero-divisor such that $ay = xb$. R has a*

classical right quotient ring if and only if it satisfies the right Ore condition. [2, p. 101 ex. 8]

This changes our question to: If S is the set of nonzero divisors of kG, does kG satisfy the Ore condition with respect to S? The first step is characterizing the zero divisors for kG. While characterizing the zero divisors for kG we realized that characterization remains the same for H any countable abelian p-group. We will generalize to this case.

For this paper we will set H to be a countable abelian p-group and G to be the group in which we embed H as the center. We will be assuming G/H is left ordered because this implies the Zero Divisor Conjecture for $k[G/H]$, that is, if $\alpha\beta = 0$ where $\alpha, \beta \in k[G/H]$ then either $\alpha = 0$ or $\beta = 0$.

Chapter 3

Characterizing the Zero Divisors

3.1 Case One: Same Characteristic

Unless otherwise noted, for this section we will use H to represent a countable abelian p-group, G to represent the special group containing H as its center, and k to represent a field of characteristic p. We will assume that G/H is left ordered.

Lemma 3.1. *Let A be a finite p-group of order p^n. Let $e = \sum_{a \in A} a$. Define $k_1 = ke$. Let $\beta \in kA$. If $a\beta = \beta$ for every $a \in A$, then $\beta \in k_1$.*

Proof. Write $\beta = \sum_{x \in A} \beta_x x$ where $\beta_x \in k$. We wish to show that β_x is the same for all x.

$\beta a = \beta$ for every $a \in A$, so $\sum_{x \in A} \beta_x (xa - x) = 0$. Consider the coefficient for y. When $xa = y$, $x = ya^{-1}$ so $\beta_x = \beta_{ya^{-1}}$ and we have $\sum \beta_{ya^{-1}}(y - ya^{-1}) = 0$. Rewriting this in terms of y, we have $\sum(\beta_{ya^{-1}} - \beta_y)y = 0$. The coefficient for one in this sum must be zero. Letting $y = 1$ we get $\beta_{a^{-1}} = \beta_1$ for every a in A, and we see that $\beta = \beta_1 \sum_{x \in A} x$. So $\beta \in k_1$. □

Lemma 3.2. *Let A be a finite p-group of order p^n. k_1 is the unique minimal ideal of kA.*

Proof. ke is an ideal of kA and $\dim_k ke = 1$ making ke minimal. We will prove uniqueness by way of contradiction.

Suppose k_1 is not the unique minimal ideal. Then there exists $I \lhd kA$ such that I is also minimal. Then $k_1 \cap I = 0$ (because I, k_1 are minimal). Let $J = \omega(kA)$, the augmentation ideal of kA. We may assume that $IJ = 0$. (If $IJ \neq 0$ then we can replace I with IJ. J is nilpotent, so eventually we will have that $IJ = 0$). $IJ = 0$ so for every a in A, $I(a - 1) = 0$. Choose $\beta \in I$. Then $\beta a = \beta$ for every $a \in A$, and $\beta \in k_1$. This shows that $I \subset k_1$, but this contradicts the fact that k_1 is a minimal ideal. Therefore, k_1 must be unique. □

Theorem 3.3. *Let $\bar{\alpha}$ be the image of $\alpha \in k[G/H]$. Then $\alpha \in kG$ is a zero divisor if and only if $\bar{\alpha} = 0$.*

Proof. First we will show if $\bar{\alpha} = 0$, then α is a zero divisor of kG

If $\bar{\alpha} = 0$, then $\alpha \in \omega(kH)kG$, the kernel of the bar mapping. Let X be a transversal for H in G. Then $\alpha = \sum \alpha_i x_i$ where $\alpha_i \in kH$ and $x_i \in X$. The support of each α_i is a finite set, so they generate some subgroup A such that $\alpha \in \omega(kA)kG \subset \omega(kH)kG$. We write $\alpha = \sum \alpha_i x_i$ such that $\alpha_i \in \omega(kA)$ and $x_i \in X$.

kA has a unique minimal ideal k_1 such that $k_1\omega(kA) = 0$ (because $k_1\omega(kA) \subset k_1$ and k_1 is minimal). Let $0 \neq \gamma \in k_1$. Then $\gamma\alpha_i = 0$ for every α_i and $\gamma\alpha = 0$. Since $\gamma \neq 0$ we have proven that α is a zero divisor of kG. □

Example 3.4. Before we continue, we will look at the specific case when H is a 2-group and $k \cong \mathbb{F}_2$ is the field with two elements. We will show in this case that all zero divisors of kG are contained in $\omega(kH)kG$.

Let $\alpha \in kG$ be a zero divisor. By way of contradiction, suppose that $\alpha \notin \omega(kH)kG$. Then there exists $\beta \in kG$ such that $\beta \neq 0$ and $\alpha\beta = 0$ or $\alpha\beta = 0$. Without loss of generality, we will assume $\alpha\beta = 0$.

$\alpha\beta = 0$ means that $\alpha\beta \in \omega(kH)kG$. Since $\alpha \notin \omega(kH)kG$, we have that $\beta \in \omega(kH)kG$. This is because $k[G/H]$ is a domain and $\omega(kH)kG$ is the kernel of the bar mapping from kG to $k[G/H]$.

Let $X = \{x_i\}$ be a transversal for H in G. We can write $\alpha = \sum \alpha_i x_i$ where $\alpha_i \in kH$ and $\beta = \sum \beta_i x_i$ where $\beta_i \in \omega(kH)$ (because $\beta \in \omega(kH)kG$).

α_i, β_i are in kA for some finitely generated subgroup A of H. $\alpha \notin \omega(kA)kG$ so there is at least one i such that $\alpha_i \notin \omega(kA)$.

We may assume that $\alpha_i, \beta_i \neq 0$. We multiply β repeatedly by elements of the form $a - 1$ for $1 \neq a \in A$ until $(a-1)\beta_i = 0$ for all i and for all $a \in A$ and not all β_i are zero. Since $a\beta_i = \beta_i$ for all $a \in A$, we see that $\beta_i \in k_1$ by Lemma 3.1.

Again we may assume that all β_i are nonzero and then furthermore, we may assume all $\alpha_i \in kA - \omega(kA)$ (because if $\alpha_i \in \omega(kA)$, then $\alpha_i k_1 = 0$ and hence $\alpha_i\beta_j = 0$ for all j).

$e = \sum_{x \in A} x$ is uniquely nonzero in k_1 since k is a field of only two elements. $\alpha\beta = \sum \alpha_i x_i \beta_j x_j = \sum(\alpha_i \beta_j) x_i x_j$ and $\alpha_i \beta_j \neq 0$ for every i, j because α_i is not in $\omega(kA)$ for some i and so α_i is not a zero divisor. $\alpha_i \beta_j \in k_1$ because k_1 is an ideal of kA. Because $\alpha_i \beta_j \neq 0$ for every i, j, we can ignore those instances where the product does equal zero and consider only those where $\alpha_i \beta_j \neq 0$ and therefore must equal e. We now have $\alpha_i \beta_i = e$ for all i, j and therefore $\alpha\beta = e \sum_i x_i \sum_j x_j$.

We have $0 = \bar{\alpha}\bar{\beta} = e \sum \bar{x}_i \sum \bar{x}_j$. By the zero divisor conjecture for $k[G/H]$, we have that $\sum_{i,j} x_i \sum x_j \notin \omega(kH)kG$. This means that if we write $\sum_{i,j} x_i \sum x_j$ in the form $\sum_l \gamma_l x_l$where $\gamma_l \in kH$, then at least one $\gamma_l \notin \omega(kH)$. It follows that $e\gamma_l \neq 0$ (because $\gamma_l \notin \omega(kH)$ implies γ_l not a zero divisor) and hence $e \sum_{i,j} x_i \sum x_j \neq 0$.

Proof. Now we will show if $\alpha \in kG$ is a zero divisor, then $\bar{\alpha} = 0$.

Suppose by way of contradiction that α is a zero divisor and $\bar{\alpha} \neq 0$. There exists $\beta \neq 0$ such that $\beta \in kG$ and $\alpha\beta = 0$ or $\beta\alpha = 0$. Without loss of generality, assume $\alpha\beta = 0$. Consider the natural epimorphism that maps kG onto $k[G/H]$. Let $\bar{\alpha}$ and $\bar{\beta}$ represent the image of α and β under this mapping respectively. $\alpha\beta = 0$ so $\bar{\alpha}\,\bar{\beta} = 0$. We are assuming the zero divisor conjecture for $k[G/H]$, so $\bar{\beta} = 0$ and $\beta \in \omega(kH)kG$, the kernel.

Let X be a transversal for H in G. We can write $\alpha = \sum \alpha_i x_i$ and $\beta = \sum \beta_i x_i$ where $x_i \in X$ and $\alpha_i, \beta_i \in kH - 0$. Let A be the subgroup of H generated by the support of the α_i, β_i. A is a finite abelian p-group and has a unique minimal ideal $k_1 = k \sum_{a \in A} a$.

We may assume that $\alpha_i, \beta_i \neq 0$. We multiply β repeatedly by elements of the form $a - 1$ for $1 \neq a \in A$ until $(a-1)\beta_i = 0$ for all $a \in A$ and for all i and not all β_i are zero. Since $a\beta_i = \beta_i$ for all $a \in A$ and for all i, we see that $\beta_i \in k_1$ by Lemma 3.1. So $\beta_i = ek_i$.There is some i such that $\alpha_i \notin \omega(kA)$ because if $\alpha_i \in \omega(kA)$ for all i, then $\alpha \in \omega(kA)kG$.

Again we may assume that all β_i are nonzero and then furthermore, we may assume all $\alpha_i \in kA - \omega(kA)$ (because if $\alpha_i \in \omega(kA)$, then $\alpha_i k_1 = 0$ and hence $\alpha_i \beta_j = 0$ for all j).

$\alpha_i \beta_j \neq 0$ for every i, j because α_i is not in $\omega(kA)$. $\alpha_i \beta_j \in k_1$ because k_1 is an ideal of kA. Because $\alpha_i \beta_j \neq 0$ for every i, j, we can ignore those instances where the product does equal zero and consider only those where $\alpha_i \beta_j \neq 0$ and therefore must equal $ek_{i,j}$ for some $k_{i,j} \in k$. We now have $\alpha\beta = e \sum_{i,j} x_i \sum k_{i,j} x_j$.

We have $0 = \bar{\alpha}\bar{\beta} = e \sum \bar{x}_i \sum k_{i,j} \bar{x}_j$. By the zero divisor conjecture for $k[G/H]$, we have

that $\sum_{i,j} x_i \sum x_j \notin \omega(kH)kG$. This means that if we write $\sum_{i,j} x_i \sum k_{i,j} x_j$ in the form $\sum_l \gamma_l x_l$where $\gamma_l \in kH$, then at least one $\gamma_l \notin \omega(kH)$. It follows that $e\gamma_l \neq 0$ and hence $e \sum_{i,j} x_i \sum k_{i,j} x_j \neq 0$. □

Corollary 3.5. *Let H be the Prüfer p-group and G the special group containing H as its center. Then $\alpha \in kG$ is a zero divisor if and only if $\bar{\alpha} = 0$.*

3.2 Case Two: Mixed Characteristic

Previously we have been considering kG where H is a countable abelian p-group with prime p and k is a field of characteristic p. We were able to show that α is a zero divisor in kG if and only if $\alpha \in \omega(kH)kG$. However, in the case of mixed characteristic this is not true.

Unless otherwise noted, for this section we will use H to represent a countable abelian p-group, G to represent the special group containing H in its center, and k to represent a field of characteristic p' such that p' does not divide p. We will assume that G/H is left ordered.

Example 3.6. Consider the case where H is a 2-group and $k = \mathbb{F}_3$, a finite field with characteristic 3. Let $a \in H$ be of order 2, and let $e = 2(1 + a) \in kH$. $e^2 = 4(1 + 2a + a^2) = 4(2 + 2a) = 8 + 8a = 2 + 2a = 2(1 + a) = e$. So we see that $e^2 = e$ and thus e is idempotent. e is an example of a zero divisor of kG that is not in $\omega(kH)kG$. $e(1 - e) = e - e^2 = 0$ and $e \notin \omega(kH)kG$.

In the case of mixed characteristics, it is true that $\omega(kH)kG \subseteq$ the set of zero divisors of kG, but it is not true that all zero divisors of kG are contained in $\omega(kH)kG$ (as we saw in the above example).

Proposition 3.7. *Let H be a locally finite group (meaning every finitely generated subgroup is finite) and let k be a field of characteristic p' where p' does not divide p. If $\alpha \in \omega(kH)kG$, then α is a zero divisor of kG.*

Proof. $\alpha \in \omega(kA)kG$ where A is some finitely generated subgroup of H. A is a group of order $q = p^n$ for some n. Since characteristic of k does not divide p, we can divide by q and consider the element $\beta = \sum_{a \in A} \frac{1}{q} a$. $a\beta = \beta$ for any $a \in A$ because multiplication by a only permutes the elements. So $(1 - a)\beta = 0 \ \forall a \in A$. And since α is a k linear sum of $(1 - a)$ where $a \in A$, we see that $\alpha\beta = 0$ and α is a zero divisor. □

Lemma 3.8. *Let k be a field of characteristic p' such that p' does not divide p. Let M_i be the maximal ideals of kH. Then the intersection of $M_i kG$ is 0.*

Proof. kH has no nilpotent elements (see Theorem 1.15), so $\cap M_i = 0$. We will use this to show that $\cap M_i kG = 0$. Let $t_1, t_2, ...$ be coset representatives for H in G. G is the disjoint union of Ht_i. If $\beta \in kG$ we can write $\beta = \beta_1 t_1 + \beta_2 t_2 + ...$ where $\beta_i \in kH$ for every i. If $\beta \in M_i kG$ for every i, then for any $j, \beta_j \in M_i$ for every i, so $\beta_j \in \cap M_i$ and $\beta_j = 0$. Therefore $\beta = 0$. □

Theorem 3.9. *Let k be a field of characteristic p'. Then $\alpha \in kG$ is a zero divisor if and only if $\alpha \in MkG$ where M is some maximal ideal of kH.*

Proof. First we will show if α is a zero divisor of kG then there is some maximal ideal M of kH such that $\alpha \in MkG$.

We showed in the introduction that $kG/MkG \cong kH/M * [G/H]$ where M is an ideal of kH. We will use this in our proof. Suppose α is a zero divisor in kG such that $\alpha \notin MkG$ for every maximal ideal of kH. Then there is an element $\beta \neq 0$ such that $\alpha\beta = 0$ or $\beta\alpha = 0$. Without loss of generality, assume $\alpha\beta = 0$. Consider the homomorphism $\theta_i : kG \to kH/M_i * [G/H]$. $\alpha\beta = 0$ so $\theta_i(\alpha\beta) = 0$ for all i. $\alpha \notin M_i kG$ for all i and $kH/M * [G/H]$ is a domain, so $\beta \in M_i kG$ for all i. This means that β is in the intersection of the $M_i kG$, so $\beta = 0$ and we have reached a contradiction.

Now we will show that if $\alpha \in MkG$ where M is some maximal ideal of kH, then α is a zero divisor of kG.

Let X be a transversal for H in G. $G = \cup Hx_i$ where $x_i \in X$ and $Hx_i \cap Hx_j = \emptyset$ if $i \neq j$. Consider $\alpha \in MkG$ where M is a maximal ideal in kH. Then $\alpha = m_1 x_1 \alpha_1 + ... + m_n x_n \alpha_n$ for some n where $\alpha_i \in kH$ and $m_i \in M$.

Set $M' = M \cap kA$. M' is an ideal of kA, not equal to kA (because if it were equal then we would have that $M \supset kA$ which contradicts M being maximal). M' is nonempty because $m_i \alpha_i \in M \cap kA$. By Maschke's Thm, $kA = M' \oplus J$ where $J \lhd kA$ is nonempty. Take $0 \neq j \in J$. $jm_i\alpha_i \in M' \cap J$, so $jm_i\alpha_i = 0$ and α is a zero divisor. □

Corollary 3.10. *Let k be a field of characteristic p', let H be the Prüfer p-group, and let G be the group containing H as its center. Then $\alpha \in kG$ is a zero divisor if and only if $\alpha \in MkG$ where M is some maximal ideal of kA.*

Now that we have classified the zero divisors for kG, the next step is to determine whether kG has a classical quotient ring. We predict that if $k[G/H]$ has a classical quotient ring then kG will as well.

Chapter 4

Additional Results

4.1 Showing $\beta' \in kA$

When considering polynomials in $R[x]$, where R is a ring, one can show that if f is a zero divisor, then there is some $r \in R$ such that $rf = 0$ or $fr = 0$. We wish to adapt this result to our group ring. First we looked at the case where G/H is bi-ordered. Then we extended to the case where G/H is left-ordered.

4.1.1 G/H Bi-ordered

Proposition 4.1. *Let H be a subgroup of G such that H is central in G and G/H is bi-ordered. If $\alpha \in kG$ is a zero divisor, then there is some finitely generated subgroup $A \leq H$ and some $\beta' \in kA$ such that $\alpha\beta' = 0$.*

Proof. Let α be a zero divisor of kG. Let X be a transversal for H in G. G/H is bi-ordered, so the cosets of H in G have an order and we can use the same order on their coset representatives. Meaning, if $x_iH < x_jH$ we can say that $x_i < x_j$. Relabel the elements so that $x_1 < x_2 < ....$ We can write $\alpha = \sum \alpha_i x_i$ where $\alpha_i \in kH$ and $x_i < x_j$ for $i < j$. Since α is a zero divisor there exist β such that $\alpha\beta = 0$ or $\beta\alpha = 0$. Without loss of generality assume $\alpha\beta = 0$. Select β to be the element with fewest nonzero terms when written as $\beta = \sum \beta_j y_j$ where $y_j \in X$ and $\beta_j \in kH$.

$\alpha\beta = 0$ so $\sum \alpha_i\beta_j x_i y_j = 0$. These sums are finite so we can choose x_n to be the largest x_i and y_m to be the largest y_j. $x_n y_m > x_i y_j$ for $i \neq n, j \neq m$. This is the largest product of the $x_i y_j$, so its coefficient must be zero (we can write $0 = \sum 0 z_i$ where z_i are the distinct coset representatives). $\alpha_n\beta_m = 0$, which gives that $\alpha_n\beta = 0$ because $\alpha\alpha_n\beta = 0$ and $\alpha_n\beta$ has fewer nonzero terms than β. $\alpha_n\beta = 0$ and since the y_j are distinct coset representatives, their

coefficients must be zero and we have $\alpha_n\beta_j = 0$ for $j = 1, 2, ..., m$.

Now consider the next largest product. $x_{n-1}y_m > x_iy_j$ for $i < n, j < m$ and $i < n-1, j \leq m$. Similarly, $x_ny_{m-1} > x_iy_j$ for $i < n, j < m$ and $i \leq n, j < m-1$. So we can consider two cases: either $x_{n-1}y_m = x_ny_{m-1}$ or one is larger than the other. In both cases we must have that $\alpha_{n-1}\beta_m x_{n-1}y_m + \alpha_n\beta_{m-1}x_ny_{m-1} = 0$. We have already shown that $\alpha_n\beta_{m-1} = 0$. Therefore $\alpha_{n-1}\beta_m$ must be 0, and as before this gives that $\alpha_{n-1}\beta = 0$.

We will show by induction on r that $\alpha_{n-r}\beta = 0$ for $0 \leq r \leq n$.

We have shown above that $\alpha_n\beta_j = 0$ for $j = 1, 2, ..., m$. Assume $\alpha_{n-r}\beta = 0$ for $0 \leq r < k$. We will show that $\alpha_{n-k}\beta = 0$. Let $E = \{x_iy_j \mid i = 1, 2, ..., n-k,\ j = 1, 2, ..., m\}$ and consider $\max\{E\}$.

$x_{n-k}y_m > x_iy_j$ for $i < n-k, j \leq m$ and for $i \leq n-k, j < m$, so $x_{n-k}y_m$ is the maximal element of E. $x_{n-k}y_m < x_iy_k$ for $i > n-k, j = m$, so the only products that might equal $x_{n-k}y_m$ are of the form x_iy_j with $i > n-k, j < m$. We assumed $\alpha_i\beta = 0$ for $i > n-k$, so the coefficients for these products must be zero. Therefore, the coefficient for $\alpha_{n-k}\beta_m$ must be zero as well. Then, $\alpha_{n-k}\beta = 0$ because $\alpha\alpha_{n-k}\beta = 0$ and $\alpha_{n-k}\beta$ has fewer nonzero terms than β.

$\alpha_{n-k}\beta = 0$ for $0 < k \leq n$, so $\alpha_{n-k}\beta_j = 0$ for $0 < k \leq n, j = 1, 2, ...m$ and we see that $\alpha\beta_j = 0$ for $j = 1, 2, ..., m$. β_j is nonzero by assumption, and an element of kH, so we can use $\operatorname{Supp}\beta_j$ to generate a subgroup A of H. Set $\beta' = \beta_j$ and we have that $\alpha\beta' = 0$ where $\beta' \in kA$. □

4.1.2 G/H Left Ordered

Proposition 4.2. *Let H be a subgroup of G such that H is central in G and G/H is left-ordered. Let $\alpha \in kG$ be a zero divisor. Then there is some finitely generated subgroup $A \leq H$ and some $\beta' \in kA$ such that $\beta'\alpha = 0$ or $\alpha\beta' = 0$.*

Proof. We have shown the result holds for when G/H is bi-ordered. We will prove for left-ordered in the same fashion, but our choice of maximal element will be different. When choosing $\max\{x_iy_j \mid i = 1, ..., n, j = 1, ..., m\}$ we can still say that $x_iy_j < x_iy_m$ for all $i, j \neq m$, but we can no longer say that x_ny_m will be maximal.

Let α be a zero divisor of kG. Let X be a transversal for H in G. G/H is left ordered

means that the cosets of H in G have an order, and we can use the same order on their coset representatives. Write $\alpha = \sum \alpha_i x_i$ where $x_i \in X$ such that $x_1 < x_2 < ... < x_n$ and $0 \neq \alpha_i \in kH$.

Since α is a zero divisor there exist β such that $\beta\alpha = 0$ or $\alpha\beta = 0$. Without loss of generality assume $\beta\alpha = 0$. G/H is left ordered, so G/H is right ordered as well, and the proof for $\alpha\beta = 0$ will follow the same way but with using the right order for G/H. Select β to be the element with fewest nonzero terms when written as $\beta = \sum \beta_j y_j$ where $y_j \in X$ and $\beta_j \in kH$.

We will prove by induction on the number of terms $\beta_j x_j$ in the sum for β. First assume there is only one. Then $\beta = \beta_j y_j$. $yx_i < yx_j$ if $i < j$, so in particular $yx_i \neq yx_j$ if $i \neq j$. $\beta\alpha = \sum \beta_j y \alpha_i x_i = 0$ so $\beta_j \alpha_i = 0$ for $i = 1, 2, ..., m$ because yx_i are all distinct coset representatives. We see that $\beta_j \alpha = 0$ where $\beta_j \in kH$. Support of β_j is finite, so it generates some subgroup A of H and we have $\beta_j \alpha = 0$ where $\beta_j \in kA$.

Next assume that for all $k < m$, this holds for the case where β has k nonzero terms in its sum. We will show that it holds for the case where the number of nonzero terms of β is m. We can write $\beta = \sum \beta_j y_j$ where $0 \neq \beta_j \in kH$ and $y_i < y_j$ for $i < j$ and $\max\{y_j \mid j = 1, 2, ..., m\} = y_m$.

$\beta\alpha = 0$ so $\sum \beta_j \alpha_i y_j x_i = 0$. These sums are finite and totally ordered, so we can find maximal elements in $\{y_j x_i \mid i = 1, 2, ..., n, j = 1, 2, ..., m\}$. We will show that in fact there is only one maximal element. As shown above, $y_j x_i < y_j x_n$ for all j and for all $i \neq n$, so we are only considering the set $\{y_j x_n \mid j = 1, 2, ..., m\}$. Assume $y_i x_n = y_j x_n$. Multiplying on the right by x_n^{-1} we have that $y_i = y_j$. So every element of $\{y_j x_n \mid j = 1, 2, ..., m\}$ is distinct and we can find one maximal element. Without loss of generality, say $y_s x_n$. So $y_s x_n > y_j x_i$ such that $j \neq s$. Then, as in the proof for bi-ordered we see that $\beta_s \alpha_n = 0$ implies that $\beta\alpha_n = 0$ because $\alpha\beta\alpha_n = 0$ and $\beta\alpha_n$ has fewer nonzero terms than β. Therefore $\beta_j \alpha_n = 0$ for $j = 1, ..., n$.

We will show by induction on k that $\beta\alpha_{n-k} = 0$ for $0 \leq k \leq n$.

We have shown above that when $k = 0$, $\beta\alpha_n = 0$ and $\beta_j \alpha_n = 0$ for $j = 1, ..., m$. Assume $\beta\alpha_{n-r} = 0$ for $0 \leq r < k$. We will show that $\beta\alpha_{n-k} = 0$. We need to find all the elements of $\{y_j x_i \mid i = 1, 2, ..., n, j = 1, 2, ..., m\}$ that might equal $y_k x_{n-k}$. We have seen if $y_j x_{n-k} = y_j x_{n-k}$, then $y_i = y_j$. Similarly, if $y_j x_i = y_j x_{n-k}$, then $x_i = x_{n-k}$. Because G/H is left-ordered, $y_j x_i < y_j x_{n-k}$ for $i = 1, ..., n-k-1$. For $y_j x_i$ where $i > n-k$ we have seen that $\beta_j \alpha_i = 0$ for $j = 1, 2, ..., m$ so in the sum these terms will be zero and can be ignored.

Using left orderability, we set up the following set of inequalities:

$$\begin{gathered} y_1x_1 < y_1x_2 < y_1x_3 < \ldots < y_1x_{n-k-1} < y_1x_{n-k} \\ y_2x_1 < y_2x_2 < y_2x_3 < \ldots < y_2x_{n-k-1} < y_2x_{n-k} \\ \vdots \\ y_{m-1}x_1 < y_{m-1}x_2 < y_{m-1}x_3 < \ldots < y_{m-1}x_{n-k-1} < y_{m-1}x_{n-k} \\ y_mx_1 < y_mx_2 < y_mx_3 < \ldots < y_mx_{n-k-1} < y_mx_{n-k} \end{gathered}$$

$y_ix_{n-k} \neq y_jx_{n-k}$ if $i \neq j$, so we can place the products of the form y_ix_{n-k} in strictly increasing order. There are a finite number of these, so there must be some maximal element. Without loss of generality, let y_tx_{n-k} be the maximal element. The coefficient for this product must be zero, so $\beta_t\alpha_{n-k} = 0$ and we have $\beta\alpha_{n-k} = 0$.

If $\beta\alpha_{n-k} = 0$ for $0 \leq k \leq n$, then $\beta_j\alpha_i = 0$ for $i = 1, 2, ..., n$ $j = 1, 2, ..., m$. So we have $\beta_j\alpha = 0$ for $j = 1, 2, ..., m$ where $\beta_j \in kH$. The support of β_j is finite, so we can generate a group $A \leq H$ such that $\beta_j \in kA$. Set $\beta' = \beta_j$ and we are done. □

Note that $\beta_j\alpha_i = 0$ for all j, i. This gives the following proposition:

Proposition 4.3. *If $\alpha = \sum \alpha_i x_i$ is a zero divisor, then there exists $\gamma \in kA$, where A is a finitely generated subgroup of H, such that $\gamma\alpha_i = 0$ or $\alpha_i\gamma = 0$ for all i.*

4.2 The Left Annihilator of a Zero Divisor

Lastly we wish to classify the left annihilator for an element $\alpha \in \mathbb{C}G$.

Definition 4.4. The left annihilator of an element α in $\mathbb{C}G$ is the set of all $\beta \in \mathbb{C}G$ such that $\beta\alpha = 0$. We denote the left annihilator of α by $l(\alpha)$.

Proposition 4.5. *Let A be a finite p-group. Let I be an ideal of $\mathbb{C}A$. Using Maschke's theorem, there exists a unique projection $e \in \mathbb{C}A$ such that $I = e\mathbb{C}A$.*

Proof. Let I be an ideal of $\mathbb{C}A$. Then by Maschke's theorem there is an ideal K of $\mathbb{C}A$ such that $\mathbb{C}A = I \oplus K$. Then $1 = e + f$ for some $e \in I, f \in K$. So $e = e1 = e^2 + ef$, but $ef \in I, K$ so $ef \in I \cap K$ and $ef = 0$. So $e = e^2$ and this shows that e is idempotent. We also want that e preserves involution (i.e., $e^* = e$).

On $\mathbb{C}G$ we have involution, $\sum \alpha_g g \to \sum \bar{\alpha} g^{-1}$ with the properties: $(\alpha+\beta)^* = \alpha^*+\beta^*, (\alpha\beta)^* = \beta^*\alpha^*, (\alpha^*)^* = \alpha$, and if $\alpha\alpha^* = 0$, then $\alpha = 0$.

We have an inner product on $\mathbb{C}A$ defined by $\langle g, h\rangle = \delta_{g,h}$ for $g, h \in A$, where δ is the Kronecker delta. More generally, $\langle \sum a_g g, \sum b_h h\rangle = \sum a_g \bar{b_h}$ where bar denotes complex conjugation.

Now $\langle ge, f\rangle = 0$ for all $g \in G$. Therefore, $\langle g, e^* - e^*e\rangle = \langle g, e^*(1-e)\rangle = \langle g, e^*f\rangle = \langle ge, f\rangle = 0$ for all $g \in G$ and hence $e^* = e^*e$. Using $(xy)^* = y^*x^*$, we see that $e = (e^*)^* = (e^*e)^* = e^*$ and hence e is a projection. Let $i \in I$. Then $i = 1i = (e + f)i = ei + fi = ei$ because $fi \in I \cap K$. Therefore $I = e\mathbb{C}A$. □

Proposition 4.6. *Assume that G is left ordered and let $\alpha \in \mathbb{C}G$. Then the left annihilator of $\alpha \in \mathbb{C}G$ is equal to $\mathbb{C}Ge$ for a unique projection e in $\mathbb{C}H$.*

Proof. We assume that G/H is left orderable. Set $\alpha = \sum \alpha_i g_i \in \mathbb{C}G$ where $\alpha_i \in \mathbb{C}H$ and g_i transversal for H in G. We may assume the α_i are in $\mathbb{C}A$ where A is a finite subgroup of H.

Let I be the ideal of $\mathbb{C}A$ generated by α_i. Using Maschke's Theorem, there exists a unique projection $e \in \mathbb{C}A$ such that $I = e\mathbb{C}A$. $(1 - e)e\mathbb{C}A = 0$ and $\alpha_i \in e\mathbb{C}A$, so $(1 - e)\alpha_i = 0$ for all i and $(1 - e)\alpha = 0$. Then $\beta(1 - e) \in l(\alpha) = \{\beta \in kG | \beta\alpha = 0\}$ for all $\beta \in \mathbb{C}G$. Therefore, $\mathbb{C}G(1 - e) \subset l(\alpha)$.

Conversely, suppose $\beta\alpha = 0$ and $\beta \notin \mathbb{C}G(1 - e)$. $\beta = \beta(1 - e) + \beta e$ so $\beta e \neq 0$. $(1 - e)\alpha = 0$ so $\alpha = e\alpha$. Set $\alpha' = \alpha + (1 - e)$. $\beta e\alpha' = \beta e\alpha + \beta e(1 - e) = 0$ and α' is a zero divisor. So there is $\gamma \neq 0$ such that $\gamma\alpha_i = 0$ for all i and $\gamma(1 - e) = 0$. But this is not possible because the α_i and $1 - e$ generate the whole ideal $\mathbb{C}A$ (because $\mathbb{C}A = (1 - e)\mathbb{C}A + e\mathbb{C}A = (1 - e)\mathbb{C}A + I$ and I is generated by α_i). So if $\beta\alpha = 0$ then $\beta \in \mathbb{C}G(1 - e)$ and $l(\alpha) \subset \mathbb{C}G(1 - e)$. Therefore, $l(\alpha) = \mathbb{C}G(1 - e)$. □

Chapter 5

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