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Schubert defects in Lagrangian Grassmannians

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Abstract: In this paper, we propose a construction of GLSM defects corresponding to Schubert cycles in Lagrangian Grassmannians, following recent work of Closset-Khlaif on Schubert cycles in ordinary Grassmannians. In the case of Lagrangian Grassmannians, there are superpotential terms in both the bulk GLSM as well as on the defect itself, enforcing isotropy constraints. We check our construction by comparing the locus on which the GLSM defect is supported to mathematical descriptions, checking dimensions, and perhaps most importantly, comparing defect indices to known and expected polynomial invariants of the Schubert cycles in quantum cohomology and quantum K theory.

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Contents

1 Introduction

Schubert cycles play an important role in mathematics in the study of Grassmannians. The paper [1] described a construction of defects in gauged linear sigma models (GLSMs) in two and three dimensions corresponding to Schubert cycles $\Omega_\lambda^{\mathrm{Gr}}$ in quantum cohomology and quantum K theory.

Briefly, the purpose of this paper is to propose an analogous construction for Schubert cycles $\Omega_\lambda^{\mathrm{LG}}$ in Lagrangian Grassmannians, realizing them in GLSMs as in [2, 3]. The construction is slightly subtle, as for example the F-term constraints imposed by the bulk and defect superpotentials have a redundancy, which does not have an analogue for the ordinary Grassmannians of [1], and which is accounted for in a subtle fashion. (In some sense, it is a bulk/boundary analogue of a non-complete-intersection construction; see e.g. [4] for a discussion of closed string non-complete-intersection GLSM constructions.)

Space	Ring	Polynomial
$Gr(k,n)$	H (H_T)	(factorial) Schur polynomials s_λ [7–11]
$Gr(k,n)$	QH (QH_T)	(factorial) Schur polynomials s_λ [10, 12]
$LG(n,2n)$	H (H_T)	(factorial) Schur Q-function Q_λ [13–15]
$LG(n,2n)$	QH (QH_T)	(factorial) Schur Q-function Q_λ [5]
$SG(k,2n)$, $k<n$	H, H_T	theta polynomials [16–18]
$SG(k,2n)$, $k<n$	QH, QH_T	not yet available
$OG(n,2n)$	H (H_T)	(factorial) Schur P-function P_λ [15–18]
$OG(n,2n)$	QH (QH_T)	(factorial) Schur P-function P_λ [5]
full flags	H (H_T)	(double) Schubert polynomials [19, 20]
full flags	QH (QH_T)	(double) quantum Schubert polynomials [21–23]

Table 1. Polynomials associated to Schubert cycles Ω_λ in (quantum, equivariant) cohomology.

We will confirm the interpretation much as in [1], by comparing partition functions to polynomials associated with the defects. Specifically we compute defect partition functions (as functions of the σ_a parametrizing the Coulomb branch), and compare to known and expected results for associated polynomials (see tables 1, 2), in quantum cohomology and quantum K theory. The idea is that the quantum cohomology and quantum K theory rings are, ultimately, rings of polynomials modulo some relations, and the Schubert cycles have polynomial representatives. In physics, these polynomials arise as defect partition functions. That relationship was used in [1] to establish that, in the case of ordinary Grassmannians, the proposed defects were related to Schubert cycles.

We will show that the partition functions of defects in both two-dimensional GLSMs (corresponding to quantum cohomology) and in three-dimensional GLSMs (corresponding to quantum K theory), for Lagrangian Grassmannians, are computed by Schur's Q-functions. For quantum cohomology, the mathematical relationship between Schubert cycles and Schur Q-functions has been proven rigorously, see e.g. [5]. For quantum K theory, the mathematical relationship between Schubert cycles and Schur Q-functions has not been proven, but matches expectations from e.g. [6].

For completeness, in tables 1, 2 we list functions associated to Schubert cycles in these and other contexts.

We begin in section 2 by reviewing the GLSM Schubert defect construction of [1] for ordinary Grassmannians. In particular, we elaborate on the mathematics implicit in the construction of [1], comparing to Schubert and Kempf-Laksov cycles to gain insight into the construction before working through the analogue for Lagrangian Grassmannians.

In section 3 we turn to Schubert defects in Lagrangian Grassmannians $LG(n,2n)$. After reviewing the construction of the bulk GLSM for $LG(n,2n)$ and some mathematics of Schubert cycles therein, we describe our proposal for a GLSM defect construction in $LG(n,2n)$, realizing

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Space	Ring	Polynomial
$Gr(k,n)$	K (K_T)	(factorial) Grothendieck polynomials G_λ [11, 24–27]
$Gr(k,n)$	QK (QK_T)	(factorial) Grothendieck polynomials G_λ [28]
$LG(n,2n)$ $(OG(n,2n))$	K, K_T	K-theoretic Schur Q-(P-)polynomials [29]
$LG(n,2n)$, $OG(n,2n)$	QK, QK_T	not yet available (but expected to match classical case)
$SG(k,2n)$, $k<n$	QK, QK_T	not yet available
full flags	K (K_T)	(double) Grothendieck polynomials [24, 25, 30, 31]
full flags	QK (QK_T)	(double) quantum Grothendieck polynomials [32, 33]

Table 2. Polynomials associated to Schubert cycles $\mathcal{O}_{\Omega_\lambda}$ in (quantum, equivariant) K theory.

Schubert cycles in $LG(n,2n)$ in GLSMs just as [1] gave GLSM realizations of Schubert cycles in ordinary Grassmannians $Gr(k,n)$. We also discuss the redundancy between bulk and defect superpotential F-term constraints, and its resolution via Higgsing by a field which otherwise does not enter either superpotential. As checks of our proposal, we walk through arguments for why the defects localize on the correct subvarieties, compare the resulting dimensions to mathematics results for the dimensions of Schubert cycles, and most importantly, compare index formulas/partition functions for the defects to corresponding characteristic polynomials. For the localization comparison and the characteristic polynomial computations, we provide both general arguments as well as independent computations in simple examples, to verify that the GLSM is behaving consistently with expectations for a Schubert cycle. We also outline a comparison of the GLSM description of the defect itself to the corresponding Kempf-Laksov cycle.

In appendix A we list some results for partition function computations in zero-dimensional defects that are utilized elsewhere. In appendix B we review the Schur Q-functions that arise as characteristic polynomials for Schubert cycles in Lagrangian Grassmannians. Finally, in appendix C we discuss an alternative toy model for dealing with redundant constraints in a (0,2) GLSM. We do not use this construction, but include it for completeness.

2 Review in ordinary Grassmannians

2.1 Mathematics of Kempf-Laksov and Schubert cycles

The Grassmannian $Gr(k,n)$ of k-planes in $\mathbb{C}^n$ can be described as the quotient $\mathbb{C}^{nk}//\mathrm{GL}(k)$. Physically, following [34], we represent this by a GLSM with gauge group $\mathrm{U}(k)$ and n fields ϕ in the fundamental of $\mathrm{U}(k)$.

The ϕ fields then form a $k \times n$ matrix. Every point in $Gr(k, n)$, every set of k-planes in $\mathbb{C}^n$, can be represented by a unique $k \times n$ matrix in reduced row echelon form. The columns of the leading 1's of the rows of the ϕ matrix, in reduced row echelon form, form a basis of the k-dimensional subspace of $\mathbb{C}^n$.

For a Young diagram $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$, define integers $\{\alpha_1, \cdots, \alpha_k\}$,

$$1 \leq \alpha_1 < \alpha_2 < \alpha_3 < \cdots < \alpha_k \leq n, \tag{2.1}$$

by

$$\lambda \; = \; [\alpha_k - k, \cdots, \alpha_1 - 1], \tag{2.2}$$

or in components,

$$\lambda_i \; = \; \alpha_{k+1-i} - (k + 1 - i). \tag{2.3}$$

We give a few examples below to illustrate this in $Gr(2, 4)$:

Young diagram	λ	α_1	α_2
$\emptyset$	$[0, 0]$	1	2
$\square$	$[1, 0]$	1	3
$\square\square$	$[2, 0]$	1	4
$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$[1, 1]$	2	3

A Schubert cell $\Omega_\lambda^{\mathrm{Gr},\circ} \subset Gr(k, n)$ is defined by points in $Gr(k, n)$ whose ϕ matrices have reduced row echelon form with rows with leading 1's in the columns $(\alpha_1, \cdots, \alpha_k)$, or in other words that on the nth row,

- the first $\alpha_n - 1$ entries vanish,
- the α_nth entry equals 1,
- the entries above the α_nth entry vanish.

(A Schubert variety $\Omega_\lambda^{\mathrm{Gr}}$ is the closure of the Schubert cell $\Omega_\lambda^{\mathrm{Gr},\circ}$. The closure contains Schubert cells $\Omega_{\widetilde{\lambda}}^{\mathrm{Gr},\circ}$ for Young tableau $\widetilde{\lambda} \supset \lambda$, graphically.)

The result has

$$\sum_{j=1}^{k-1} j\,(\alpha_{j+1} - \alpha_j - 1) \; + \; (n - \alpha_k)k \; = \; k(n-k) \; - \; \sum_j \lambda_j \; = \; k(n-k) \; - \; |\lambda| \tag{2.4}$$

undetermined entries, and hence describes a Schubert cycle of the same dimension. For example, in $Gr(2, 4)$,

Young diagram	α_1	α_2	dimension $\Omega_\lambda^{\mathrm{Gr}}$
$\emptyset$	1	2	4
$\square$	1	3	3
$\square\square$	1	4	2
$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	2	3	2

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In terms of ϕ matrices in reduced row echelon form, the Schubert cells $\Omega_\lambda^{\mathrm{Gr},\circ}$ and their closures, the Schubert varieties $\Omega_\lambda^{\mathrm{Gr}}$, respectively, are given by

$$\emptyset \leftrightarrow \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{bmatrix}, \quad \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \end{bmatrix} \tag{2.5}$$

$$\square \leftrightarrow \begin{bmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}, \quad \begin{bmatrix} * & * & * & * \\ 0 & 0 & * & * \end{bmatrix}, \tag{2.6}$$

$$\square\square \leftrightarrow \begin{bmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} * & * & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \tag{2.7}$$

$$\begin{smallmatrix}\square\\ \square\end{smallmatrix} \leftrightarrow \begin{bmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}. \quad \begin{bmatrix} 0 & * & * & * \\ 0 & 0 & * & * \end{bmatrix} \tag{2.8}$$

We make a few remarks concerning the matrix representations of the Schubert varieties in the table above:

- The matrices for the Schubert varieties are required to have maximal rank (two, in this case).
- The Schubert cells, given in row reduced echelon form, manifest the right dimension. The dimension of the Schubert varieties is not manifest in the description above.
- Notice for the Schubert cells that the zeroes to the left of the ones determine zeroes in the matrices for the Schubert varieties.

We can describe these more formally as follows. We will give a presentation that is tuned to match the GLSM description of [1], which we will also review shortly.

Fix a flag $\widetilde{F}_\bullet$ of subspaces

$$\widetilde{F}_1 \subset \widetilde{F}_2 \subset \cdots \subset \widetilde{F}_n = \mathbb{C}^n, \tag{2.9}$$

where $\dim \widetilde{F}_i = i$.

Fix a basis $\{e_i, 1 \leq i \leq n\}$ for $V = \mathbb{C}^n$ such that $\widetilde{F}_i = \langle e_1, \cdots, e_i \rangle$ for $1 \leq i$. Let λ_i denote the number of boxes in the ith row of λ, with $\lambda_0 = n - k$ and $\lambda_{\ell+1} = \ell - k + 1$, for ℓ the number of nonzero rows of the Young diagram λ. Define

$$V_i = \langle e_{\lambda_0 - \lambda_i + i + 1}, \ldots, e_{\lambda_0 - \lambda_{i+1} + i + 1} \rangle, \tag{2.10}$$

for $0 \leq i \leq \ell$, then

$$\widetilde{F}_{n-k+i-\lambda_i} = \bigoplus_{j=0}^{i-1} V_j. \tag{2.11}$$

Note that the V_i partition the vector space $V = \mathbb{C}^n$ into $\ell + 1$ distinct subspaces, where

$$\dim V_i = \begin{cases} \lambda_i - \lambda_{i+1} + 1 & 1 \leq i < \ell, \\ \lambda_\ell - \ell + k & i = \ell, \\ n - k + 1 - \lambda_1 & i = 0. \end{cases} \tag{2.12}$$

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As a consistency check, note that

$$\sum_{i=0}^{\ell} \dim V_i \;=\; n. \tag{2.13}$$

Define

$$F_i \;=\; V_0 \oplus \bigoplus_{j\neq i} V_j \;\subset\; V, \tag{2.14}$$

for $1 \leq i \leq \ell$, so that $\dim V/F_i = \dim V_i$.

Now, over the Grassmannian $Gr(k,n)$, we have the bundle of flags $Fl(\mathcal{S}) = F(1,2,\cdots,\ell;\mathcal{S})$ where $\mathcal{S}$ is the (rank k) universal subbundle on $Gr(k,n)$. Let $\mathcal{S}_i$ denote the universal subbundle on $Fl(\mathcal{S})$ of rank i, which has inclusions $\varphi_i^{i+1} : \mathcal{S}_i \hookrightarrow \mathcal{S}_{i+1}$. Let Z_i denote the vanishing locus in the total space of $Fl(\mathcal{S})$ of the map J_i defined to be the composition

$$\mathcal{S}_i \overset{\varphi_i^{i+1}}{\hookrightarrow} \mathcal{S}_{i+1} \overset{\varphi_{i+1}^{i+2}}{\hookrightarrow} \cdots \overset{\varphi_{\ell-1}^{\ell}}{\hookrightarrow} \mathcal{S}_\ell \overset{\varphi_\ell^{\ell+1}}{\hookrightarrow} \pi^*\mathcal{S} \hookrightarrow \mathbb{C}^n \to \mathbb{C}^n/F_i, \tag{2.15}$$

using the fact that the $\mathcal{S}_i$ inject into $\mathbb{C}^n$, which then projects onto $\mathbb{C}^n/F_i$.

Let

$$X_\lambda \subset \mathrm{Tot}\, F(1,2,\cdots,\ell;\mathcal{S}) \tag{2.16}$$

denote the intersection

$$Z_1 \cap Z_2 \cap \cdots \cap Z_\ell. \tag{2.17}$$

The space X_λ is sometimes called the Kempf-Laksov desingularization of the Schubert variety $\Omega_\lambda^{\mathrm{Gr}}$; cf. [35]. It is a smooth projective algebraic variety, and its fundamental cycle pushes forward to that of the corresponding Schubert variety:

$$\left[\Omega_\lambda^{\mathrm{Gr}}\right] \;=\; \pi_*\left[X_\lambda\right]. \tag{2.18}$$

We can see this (roughly) as follows. (We refer to [35] for further details.) Let $\Omega = \pi(Z_1 \cap Z_2 \cap \cdots \cap Z_\ell)$ and Σ be a k-dimensional vector subspace of $V = \mathbb{C}^n$. Then $\Sigma \in \Omega$ if and only if there exists a nested sequence of vector spaces $\Sigma_1 \subset \cdots \subset \Sigma_\ell \subseteq \Sigma$, where $\Sigma_i \subseteq F_i$, and in particular,

$$\Sigma_\ell \subseteq F_\ell \;=\; \widetilde{F}_{n-k+\ell-\lambda_\ell}, \tag{2.19}$$

$$\Sigma_{\ell-1} \subseteq F_{\ell-1} \cap F_\ell \;=\; \widetilde{F}_{n-k+\ell-1-\lambda_{\ell-1}}, \tag{2.20}$$

$$\cdots,$$

$$\Sigma_1 \subseteq \bigcap_{j=1}^{\ell} F_j \;=\; \widetilde{F}_{n-k+1-\lambda_1}. \tag{2.21}$$

This is if and only if $\dim(\Sigma \cap \widetilde{F}_{n-k+i-\lambda_i}) \geq i$ for $1 \leq i \leq \ell$, which defines the Schubert variety. Hence, Ω is the Schubert variety $\Omega_\lambda^{\mathrm{Gr}}$ corresponding to λ.

In fact, there is a commutative diagram

$$\begin{tikzcd} X_\lambda \arrow[r, hook] \arrow[d] & X_\emptyset \arrow[d] \\ \Omega_\lambda^{\mathrm{Gr}} \arrow[r, hook] & \Omega_\emptyset^{\mathrm{Gr}} = Gr(k,n), \end{tikzcd} \tag{2.22}$$

where the horizontal maps are inclusions, and the vertical maps are birational projections.

Briefly, the GLSM construction will implement these defects by enforcing constraints that force the zeroes to appear in the ϕ matrices above.

2.2 GLSM construction of Schubert defects

In this section we review the construction of [1] of GLSM defects corresponding to Schubert varieties in ordinary Grassmannians $G(k,n)$.

The Grassmannian $Gr(k,n)$ itself is a $\mathrm{U}(k)$ gauge theory with n fundamentals denoted ϕ, as discussed in [34]. Fix a Young diagram λ, and let $\ell(\lambda)$ be the number of nonzero rows. Along a given defect, we construct a flag bundle, which is described as [36] a

$$\mathrm{U}(1)_\partial \times \mathrm{U}(2)_\partial \times \cdots \times \mathrm{U}(\ell(\lambda))_\partial \tag{2.23}$$

gauge theory with

- a bifundamental chiral φ_i^{i+1} (charged under $\mathrm{U}(i)_\partial \times \mathrm{U}(i+1)_\partial$) for $1 \leq i < \ell(\lambda)$,
- a bifundamental chiral of $\mathrm{U}(\ell(\lambda))_\partial \times \mathrm{U}(k)_{\mathrm{bulk}}$, in the fundamental of $\mathrm{U}(\ell(\lambda))_\partial$ and the antifundamental of $\mathrm{U}(k)_{\mathrm{bulk}}$, labelled $\varphi_\ell^{\ell+1}$,

We use ∂ subscripts to denote gauge factors lying solely along the defect. Furthermore, to implement the constraints, we add to the defect

- M_i^{Gr} Fermi superfields $\Lambda^{(i)}$, $1 \leq i \leq \ell(\lambda)$, in the antifundamental of $\mathrm{U}(i)_\partial$,

where the integers M_i^{Gr} $(0 \leq i \leq \ell(\lambda))$ partition n into $\ell+1$ blocks, where

$$M_i^{\mathrm{Gr}} = \begin{cases} \lambda_i - \lambda_{i+1} + 1 \; = \; \alpha_{k+1-i} - \alpha_{k-i} & 1 \leq i < \ell(\lambda), \\ \lambda_\ell - \ell + k \; = \; \alpha_{k+1-\ell} - 1 & i = \ell(\lambda), \\ n - \sum_{j=1}^{\ell} M_j^{\mathrm{Gr}} = n - k + 1 - \lambda_1 \; = \; n + 1 - \alpha_k & i = 0. \end{cases} \tag{2.24}$$

The defect is also given by a (0,2) superpotential

$$W_\partial = \sum_{i=1}^{\ell} \int d\theta \, \varphi_i^{i+1} \cdots \varphi_\ell^{\ell+1} \phi^{(i)} \Lambda^{(i)} = \sum_{i=1}^{\ell} \int d\theta \, J_i \Lambda^{(i)}, \tag{2.25}$$

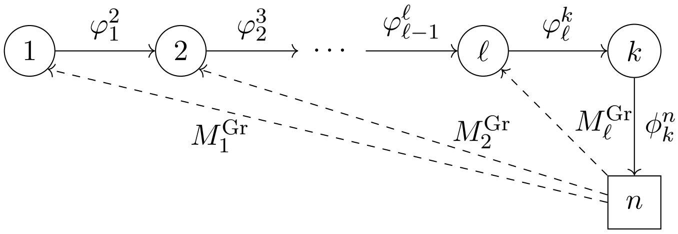
where $\phi^{(i)}$ is a $k \times M_i^{\mathrm{Gr}}$ submatrix of the ϕ fields defining the GLSM for $Gr(k,n)$, corresponding to $\mathbb{C}^n/F_i$, and each bulk fundamental ϕ field appearing couples to a different $\Lambda^{(i)}$, for $1 \leq i \leq \ell(\lambda)$. More explicitly, $\phi^{(i)}$ is a submatrix of the $k \times n$ matrix ϕ, with flavor indices running from $\widetilde{M}_i^{\mathrm{Gr}} + 1$, for

$$\widetilde{M}_r^{\mathrm{Gr}} = \sum_{j=0}^{r-1} M_j^{\mathrm{Gr}} \tag{2.26}$$

to $\widetilde{M}_i^{\mathrm{Gr}} + M_i^{\mathrm{Gr}}$. (This was described in [1] in terms of an index set.)

The maps $J_i = \varphi_i^{i+1} \cdots \varphi_\ell^{\ell+1} \phi^{(i)}$ are the same as the maps (2.15) appearing in the definition of the Kempf-Laksov desingularization. (The map $\pi^* \mathcal{S} \hookrightarrow \mathbb{C}^n \to \mathbb{C}^n/F_i$ is represented by $\phi^{(i)}$.) It should now be clear that this construction is precisely duplicating the Kempf-Laksov cycle X_λ.

A quiver describing this construction is



The effect of the quiver construction is to realize explicitly the Kempf-Laksov desingularization X_λ along the defect. Now, the defect itself is interpreted as living on the Grassmannian, so in this case, we interpret the defect as a pushforward of the Kempf-Laksov desingularization, $\pi_* X_\lambda$, which for ordinary Grassmannians matches the Schubert variety $\Omega_\lambda^{\rm Gr}$, from equation (2.18).

As a consistency check, let us check the dimensions of the GLSM defects against the dimensions of Schubert varieties. The bulk and boundary fields contribute as follows:

- The bulk chiral fields ϕ are of dimension nk, and so contribute nk.
- The boundary chiral fields φ_i^{i+1} of dimension $i(i+1)$ for $i<\ell$, with the last of dimension ℓk, so their total contribution is

$$\sum_{i=1}^{\ell-1} i(i+1) \;+\; k\ell. \tag{2.27}$$

- The bulk gauge fields contribute $-k^2$.
- The boundary gauge fields contribute

$$-\sum_{i=1}^{\ell} i^2. \tag{2.28}$$

- The boundary Fermi fields $\Lambda^{(i)}$ contribute constraints, hence

$$-\sum_{i=1}^{\ell} i M_i^{\rm Gr}. \tag{2.29}$$

The total contribution from the bulk fields is

$$nk - k^2 \;=\; k(n-k), \tag{2.30}$$

matching the dimension of the Grassmannian $Gr(k,n)$.

The total contribution from the remaining fields is

$$\sum_{i=1}^{\ell-1} i(i+1) \;+\; k\ell \;-\; \sum_{i=1}^{\ell} i^2 \;-\; \sum_{i=1}^{\ell} i M_i^{\rm Gr} \;=\; -\frac{1}{2}\ell(\ell+1) \;+\; k\ell \;-\; \sum_{i=1}^{\ell} i M_i^{\rm Gr}, \tag{2.31}$$

$$= -|\lambda|. \tag{2.32}$$

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Thus, the total dimension seen by the defect is

$$k(n-k) \; - \; |\lambda|, \tag{2.33}$$

matching the dimension of the Schubert cycle.

Finally, we add that for one-dimensional defects in three-dimensional GLSMs, in principle one also adds one-dimensional Chern-Simons terms (meaning, Wilson lines) to the boundary action. Doing so does not change the locus on which the defects localize, but does alter the index computation. Details are discussed in [1, section 3.3.4]. In the index computations reviewed here, following [1, section 3.3.4], such Chern-Simons terms for defect gauge fields are not added. (For bulk gauge fields, Chern-Simons terms are added, following [1].)

2.3 Schubert defects in two dimensions

We can consider the 2d GLSM coupled to a 0d $\mathcal{N}=2$ supersymmetric matrix model, which is defined exactly as in 3d/1d case. The 0d defect contributes to the 2d theory according to its supersymmetric partition function, which can be obtained by dimensional reduction of the 1d theory.

For a general 0d $\mathcal{N}=2$ supersymmetric matrix model, the partition function can be calculated exactly, as reviewed in appendix A, which is given by a Jeffrey-Kirwan residue of the form

$$\mathcal{I}^{0\mathrm{d}} = \frac{1}{|W|} \oint_{\mathrm{JK}} \prod_{r=1}^{\ell} \frac{d^r u^{(r)}}{(2\pi i)^r} \, Z^{0\mathrm{d}}_{\mathrm{vector}}(u) \, Z^{0\mathrm{d}}_{\mathrm{matter}}(u), \tag{2.34}$$

where W is the Weyl group of the gauge group, u is the complex scalar in the vector multiplet, Z_{vector} is the one-loop determinant of the vector multiplet, and Z_{matter} includes the one-loop determinant of both chiral multiplets and Fermi multiplets.

Now in our current case, we have a $\mathrm{U}(1) \times \mathrm{U}(2) \times \cdots \times \mathrm{U}(\ell)$ gauge group. The Coulomb branch is parametrized by ℓ groups of complex scalar fields $u^{(r)}_{i_r}$, $r = 1, 2, \cdots, \ell$, and i_r runs from 1 to r. The order of the Weyl group is then

$$|W| = \prod_{r=1}^{\ell} r!. \tag{2.35}$$

The vector one-loop determinant is given by

$$Z^{0\mathrm{d}}_{\mathrm{vector}} = \prod_{r=1}^{\ell} \left[\prod_{1 \le i_r \ne j_r \le r} \left(u^{(r)}_{i_r} - u^{(r)}_{j_r} \right) \right]. \tag{2.36}$$

The matter one-loop determinant is given by

$$Z^{0\mathrm{d}}_{\mathrm{matter}} = \frac{\prod_{i_\ell=1}^{\ell} \left(u^{(\ell)}_{i_\ell} \right)^{M^{\mathrm{Gr}}_\ell}}{\prod_{i_\ell=1}^{\ell} \prod_{a=1}^{k} \left(u^{(\ell)}_{i_\ell} - \sigma_a \right)} \prod_{r=1}^{\ell-1} \left[\frac{\prod_{i_r=1}^{r} \left(u^{(r)}_{i_r} \right)^{M^{\mathrm{Gr}}_r}}{\prod_{i_r=1}^{r} \prod_{j_{r+1}=1}^{r+1} \left(u^{(r)}_{i_r} - u^{(r+1)}_{j_{r+1}} \right)} \right], \tag{2.37}$$

where the numerators are from the Fermi multiplets, and the denominators are from the chiral multiplets. The first factor comes from the M^{Gr}_ℓ Fermi multiplets $\Lambda^{(\ell)}$, contributing

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to the numerator, and the chiral multiplet $\varphi_\ell^{\ell+1}$, contributing to the denominator. Notice that σ_a's are the complex scalar fields corresponding to the U(k) gauge group in the bulk, which are treated as complex masses. The product comes from the $M_r^{\rm Gr}$ Fermi multiplets $\Lambda^{(r)}$, for $r = 1, 2, \cdots, \ell-1$, contributing to the numerator, and the chiral multiplets φ_r^{r+1}, for $r = 1, 2, \cdots, \ell-1$, contributing to the denominator.

Since the Schubert variety is labeled by a Young diagram, we will also use λ to label the partition function. With all the ingredients, we can write

$$\begin{aligned}\mathcal{I}_\lambda^{\rm 0d} = &\left(\prod_{r=1}^{\ell}\frac{1}{r!}\right)\oint_{\rm JK}\left[\prod_{r=1}^{\ell}\frac{d^r u^{(r)}}{(2\pi i)^r}\prod_{1\le i_r\neq j_r\le r}\left(u_{i_r}^{(r)}-u_{j_r}^{(r)}\right)\right]\\ &\times\frac{\prod_{i_\ell=1}^{\ell}\left(u_{i_\ell}^{(\ell)}\right)^{M_\ell^{\rm Gr}}}{\prod_{i_\ell=1}^{\ell}\prod_{a=1}^{k}\left(u_{i_\ell}^{(\ell)}-\sigma_a\right)}\prod_{r=1}^{\ell-1}\left[\frac{\prod_{i_r=1}^{r}\left(u_{i_r}^{(r)}\right)^{M_r^{\rm Gr}}}{\prod_{i_r=1}^{r}\prod_{j_{r+1}=1}^{r+1}\left(u_{i_r}^{(r)}-u_{j_{r+1}}^{(r+1)}\right)}\right],\end{aligned} \tag{2.38}$$

where the λ dependence is encoded in M's. For reference,

$$\begin{aligned}M_r^{\rm Gr} &= \lambda_r - \lambda_{r+1} + 1, \quad r = 1, 2, \ldots, \ell-1,\\ M_\ell^{\rm Gr} &= \lambda_\ell - \ell + k.\end{aligned} \tag{2.39}$$

Also, we integrate over all the u's, and the index should be a function of σ's.

The Jeffrey-Kirwan residue above was computed and explained in [1, section 5]. As stated there, the contour integrals should be performed recursively, starting with the U(1) node. For every Young diagram λ, it can be shown [1, equ'n (5.14)] that

$$\mathcal{I}_\lambda^{\rm 0d} \;=\; s_\lambda(\sigma_a), \tag{2.40}$$

where s_λ is the ordinary Schur polynomial in k indeterminates, namely the σ_a parametrizing the Coulomb branch. If the theory has twisted masses (an equivariant structure), then instead of an ordinary Schur polynomial, $\mathcal{I}_\lambda^{\rm 0d}$ is a factorial Schur polynomial, in the σ_a and twisted masses.

In passing, the residue formula above is very similar to expressions in the mathematics literature, see for example [37, section 4.2, prop. 2], [38, theorem 4.1].

Below we work through a few examples to help make this clear.

For $Gr(2,4)$, if the Young diagram contains only a single row, i.e., $\lambda = (\lambda_1)$, the nonequivariant 0d partition function is given by

$$\mathcal{I}_{(\lambda_1)}^{\rm 0d} = \oint_{\rm JK}\frac{du}{2\pi i}\frac{u^{M_1^{\rm Gr}}}{(u-\sigma_1)(u-\sigma_2)} = \frac{\sigma_1^{M_1^{\rm Gr}}-\sigma_2^{M_1^{\rm Gr}}}{\sigma_1-\sigma_2} = \frac{\sigma_1^{\lambda_1+1}-\sigma_2^{\lambda_1+1}}{\sigma_1-\sigma_2}, \tag{2.41}$$

where we used the fact that $M_1^{\rm Gr} = \lambda_1 + 1$ according to (2.24). If the Young diagram contains two rows; that is $\lambda = (\lambda_1, \lambda_2)$, let $a = u^{(1)}, b = u^{(2)}$, the 0d partition function is given by

$$\begin{aligned}\mathcal{I}_{(\lambda_1,\lambda_2)}^{\rm 0d} &= \frac{1}{2}\oint_{\rm JK}\frac{da}{2\pi i}\frac{db_1}{2\pi i}\frac{db_2}{2\pi i}(b_1-b_2)(b_2-b_1)\frac{a^{M_1^{\rm Gr}}}{(a-b_1)(a-b_2)}\frac{(b_1 b_2)^{M_2^{\rm Gr}}}{\prod_{i,j=1}^{2}(b_i-\sigma_j)}\\ &= (\sigma_1\sigma_2)^{M_2^{\rm Gr}}\frac{\sigma_1^{M_1^{\rm Gr}}-\sigma_2^{M_1^{\rm Gr}}}{\sigma_1-\sigma_2} = (\sigma_1\sigma_2)^{\lambda_2}\frac{\sigma_1^{\lambda_1-\lambda_2+1}-\sigma_2^{\lambda_1-\lambda_2+1}}{\sigma_1-\sigma_2},\end{aligned} \tag{2.42}$$

where we have used the fact that $M_1^{\rm Gr} = \lambda_1 - \lambda_2 + 1$, and $M_2^{\rm Gr} = \lambda_2$.

Computing in simple examples we find

- $\mathcal{I}^{\text{0d}}_{\ydiagram{1}} = \sigma_1 + \sigma_2 = s_{\ydiagram{1}}$.
- $\mathcal{I}^{\text{0d}}_{\ydiagram{2}} = \sigma_1^2 + \sigma_1\sigma_2 + \sigma_2^2 = s_{\ydiagram{2}}$.
- $\mathcal{I}^{\text{0d}}_{\ydiagram{1,1}} = \sigma_1\sigma_2 = s_{\ydiagram{1,1}}$.
- $\mathcal{I}^{\text{0d}}_{\ydiagram{2,1}} = (\sigma_1\sigma_2)(\sigma_1 + \sigma_2) = s_{\ydiagram{2,1}}$.
- $\mathcal{I}^{\text{0d}}_{\ydiagram{2,2}} = (\sigma_1\sigma_2)^2 = s_{\ydiagram{2,2}}$.

2.4 Schubert defects in three dimensions

Similarly, following [1], we will review the 3d/1d case, which is a 3d GLSM coupled to a 1d $\mathcal{N} = (0,2)$ supersymmetric gauge theory, as relevant to Schubert cycles in quantum K theory. The Witten index of the 1d theory can be calculated as follows [39]

$$\mathcal{I}^{\text{1d}} = \frac{1}{|W|} \oint_{\text{JK}} \prod_{r=1}^{\ell} \frac{d^r u^{(r)}}{(2\pi i)^r} \, Z^{\text{1d}}_{\text{vector}}(u) \, Z^{\text{1d}}_{\text{matter}}(u), \tag{2.43}$$

where W is the Weyl group of the gauge group, u's are the Coulomb branch parameters, and Z_{vector} and Z_{matter} are one-loop determinant contributions from vector fields and matter fields respectively. Due to the large gauge symmetry $u \sim u + 1$, we will write the integrand as a function of the gauge invariant parameter

$$z = \exp(-2\pi i u). \tag{2.44}$$

In our case, we have $|W| = \prod_{r=1}^{\ell} r!$, and

$$Z^{\text{1d}}_{\text{vector}} = \prod_{r=1}^{\ell} \left[\prod_{1 \leq i_r \neq j_r \leq r} \left(1 - \frac{z^{(r)}_{i_r}}{z^{(r)}_{j_r}} \right) \right]. \tag{2.45}$$

The one-loop determinants from the matter fields are given by

$$Z^{\text{1d}}_{\text{matter}} = \frac{\prod_{i_\ell=1}^{\ell} \left(1 - z^{(\ell)}_{i_\ell}\right)^{M^{\text{Gr}}_\ell}}{\prod_{i_\ell=1}^{\ell} \prod_{a=1}^{k} \left(1 - \frac{z^{(\ell)}_{i_\ell}}{X_a}\right)} \prod_{r=1}^{\ell-1} \left[\frac{\prod_{i_r=1}^{r} \left(1 - z^{(r)}_{i_r}\right)^{M^{\text{Gr}}_r}}{\prod_{i_r=1}^{r} \prod_{i_{r+1}=1}^{r+1} \left(1 - \frac{z^{(r)}_{i_r}}{z^{(r+1)}_{i_{r+1}}}\right)} \right]. \tag{2.46}$$

Finally, we can write the 1d Witten index labeled by λ explicitly as

$$\begin{aligned} \mathcal{I}^{\text{1d}}_\lambda = {} & \left(\prod_{r=1}^{\ell} \frac{1}{r!} \right) \oint_{\text{JK}} \prod_{r=1}^{\ell} \left[\left(\prod_{i_r=1}^{r} \frac{-dz^{(r)}_{i_r}}{2\pi i z^{(r)}_{i_r}} \right) \prod_{1 \leq i_r \neq j_r \leq r} \left(1 - \frac{z^{(r)}_{i_r}}{z^{(r)}_{j_r}} \right) \right] \\ & \times \frac{\prod_{i_\ell=1}^{\ell} \left(1 - z^{(\ell)}_{i_\ell}\right)^{M^{\text{Gr}}_\ell}}{\prod_{i_\ell=1}^{\ell} \prod_{a=1}^{k} \left(1 - \frac{z^{(\ell)}_{i_\ell}}{X_a}\right)} \prod_{r=1}^{\ell-1} \left[\frac{\prod_{i_r=1}^{r} \left(1 - z^{(r)}_{i_r}\right)^{M^{\text{Gr}}_r}}{\prod_{i_r=1}^{r} \prod_{i_{r+1}=1}^{r+1} \left(1 - \frac{z^{(r)}_{i_r}}{z^{(r+1)}_{i_{r+1}}}\right)} \right], \end{aligned} \tag{2.47}$$

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which is evaluated to be a function of X's, which are the gauge invariant Coulomb branch parameters for the bulk $\mathrm{U}(k)$ gauge theory.

Following [1, section 3.3], and assuming that the FI parameters are all positive, the Jeffrey-Kirwan residue is evaluated by iteratively selecting poles from the denominators of the $Z^{\mathrm{1d}}_{\mathrm{matter}}$ above for the matter (chiral superfields), and then performing the integrations in order starting from the $\mathrm{U}(1)$ node and ending at the $\mathrm{U}(\ell)$ node.

Let us take $Gr(2,4)$ as an example. If the Young diagram contains only a single row, $\lambda = (\lambda_1)$, we have

$$\begin{aligned} \mathcal{I}^{\mathrm{1d}}_{(\lambda_1)} &= \oint_{\mathrm{JK}} \frac{-dz^{(1)}}{2\pi i z} \frac{(1-z)^{M_1^{\mathrm{Gr}}}}{\left(1-\frac{z}{X_1}\right)\left(1-\frac{z}{X_2}\right)} \\ &= \frac{X_1(1-X_2)^{M_1^{\mathrm{Gr}}} - X_2(1-X_1)^{M_1^{\mathrm{Gr}}}}{X_1 - X_2}, \end{aligned} \tag{2.48}$$

where $M_1^{\mathrm{Gr}} = \lambda_1 + 1$ according to (2.24).

If the Young diagram contains two rows, $\lambda = (\lambda_1, \lambda_2)$, we have

$$\begin{aligned} \mathcal{I}^{\mathrm{1d}}_{(\lambda_1,\lambda_2)} &= \frac{1}{2}\oint_{\mathrm{JK}} \frac{-dz^{(1)}}{2\pi i z^{(1)}} \frac{-dz_1^{(2)}}{2\pi i z_1^{(2)}} \frac{-dz_2^{(2)}}{2\pi i z_2^{(2)}} \left(1-\frac{z_1^{(2)}}{z_2^{(2)}}\right)\left(1-\frac{z_2^{(2)}}{z_1^{(2)}}\right) \\ &\quad \times \frac{\left(1-z^{(1)}\right)^{M_1^{\mathrm{Gr}}}}{\left(1-\frac{z^{(1)}}{z_1^{(2)}}\right)\left(1-\frac{z^{(1)}}{z_2^{(2)}}\right)} \frac{\left(1-z_1^{(2)}\right)^{M_2^{\mathrm{Gr}}}\left(1-z_2^{(2)}\right)^{M_2^{\mathrm{Gr}}}}{\left(1-\frac{z_1^{(2)}}{X_1}\right)\left(1-\frac{z_1^{(2)}}{X_2}\right)\left(1-\frac{z_2^{(2)}}{X_1}\right)\left(1-\frac{z_2^{(2)}}{X_2}\right)} \\ &= (1-X_1)^{M_2^{\mathrm{Gr}}}(1-X_2)^{M_2^{\mathrm{Gr}}} \cdot \frac{X_1(1-X_2)^{M_1^{\mathrm{Gr}}} - X_2(1-X_1)^{M_1^{\mathrm{Gr}}}}{X_1 - X_2}, \end{aligned} \tag{2.49}$$

where $M_1^{\mathrm{Gr}} = \lambda_1 - \lambda_2 + 1$, $M_2^{\mathrm{Gr}} = \lambda_2$ according to (2.24).

Computing in simple examples we find

- $\mathcal{I}^{\mathrm{1d}}_{\ydiagram{1}} = 1 - X_1 X_2$.
- $\mathcal{I}^{\mathrm{1d}}_{\ydiagram{2}} = 1 - 3X_1X_2 + X_1X_2(X_1 + X_2)$.
- $\mathcal{I}^{\mathrm{1d}}_{\ydiagram{1,1}} = (1 - X_1)(1 - X_2)$.
- $\mathcal{I}^{\mathrm{1d}}_{\ydiagram{2,1}} = (1 - X_1)(1 - X_2)(1 - X_1X_2)$.
- $\mathcal{I}^{\mathrm{1d}}_{\ydiagram{2,2}} = (1 - X_1)^2(1 - X_2)^2$.

As a consistency check, we observe that each of the indices above reduces to a corresponding index in the two-dimensional computation. To see this, write $X_i = \exp(2\pi R\sigma_i)$, where R is the radius of a S^1 we shall shrink. In the limit of small R, for example

$$\mathcal{I}^{\mathrm{1d}}_{\ydiagram{1}} = 1 - X_1X_2 \propto -R(\sigma_1 + \sigma_2) + \mathcal{O}(R^2), \tag{2.50}$$

and so in the limit, $\mathcal{I}^{\mathrm{1d}}_{\ydiagram{1}} \propto \mathcal{I}^{\mathrm{0d}}_{\ydiagram{1}}$, using the result we computed previously. More generally, it is similarly straightforward to verify that in the same limit of small R,

$$\mathcal{I}^{\mathrm{1d}}_{\lambda} \propto \mathcal{I}^{\mathrm{0d}}_{\lambda}. \tag{2.51}$$

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3 Schubert defects in Lagrangian Grassmannians

In this section, we will construct a proposal for GLSM defects which describe Schubert varieties in Lagrangian Grassmannians. We will argue that it correctly localizes onto Schubert varieties (both with a general argument and independently checking special cases), we will independently check that it has the correct dimension, and perhaps most importantly, we will check that its indices match the desired characteristic polynomials (Schur Q-functions) of Schubert varieties in both quantum cohomology and quantum K theory, both via general arguments and also by independent computations in simple examples.

3.1 Review of bulk GLSM for symplectic Grassmannians

Before we discuss Schubert varieties in Lagrangian Grassmannian, let us first review the GLSM construction of Lagrangian Grassmannians, and more generally, symplectic Grassmannians [2, 3].

Math definition. Given a $2n$-dimensional vector space $\mathbb{C}^{2n}$, equipped with a symplectic form ω. We will use a basis in which

$$\omega(e_i, e_{2n+1-i}) = -\omega(e_{2n+1-i}, e_i) = 1, \quad i = 1, 2, \cdots, n, \tag{3.1}$$

while other components vanish. (We will use this basis throughout this paper.) The symplectic Grassmannian $SG(k, 2n)$ ($k \leq n$) is defined as the collection of k-dimensional isotropic subspaces of the symplectic vector space $\mathbb{C}^{2n}$. When $k = n$, this is called a Lagrangian Grassmannian, denoted by $LG(n, 2n)$.

Given a k-dimensional isotropic subspace, we can find k linearly independent vectors in $\mathbb{C}^{2n}$,

$$\Phi^a = \sum_{i=1}^{2n} \Phi_i^a e_i, \quad a = 1, 2, \cdots, k. \tag{3.2}$$

Then the subspace being isotropic means that

$$\omega(\Phi^a, \Phi^b) = 0, \quad \forall\, 1 \leq a < b \leq k. \tag{3.3}$$

In our chosen basis, this means that

$$\sum_{i=1}^{n} \left(\Phi_i^a \Phi_{2n+1-i}^b - \Phi_i^b \Phi_{2n+1-i}^a \right) = 0. \tag{3.4}$$

Therefore, $SG(k, 2n)$ can be represented by $k \times 2n$ full rank matrices (Φ_i^a), subject to the above isotropy condition, which can be written in the matrix from

$$\Phi \omega \Phi^T = 0, \quad \text{with} \quad \omega = \begin{pmatrix} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ & & -1 & & & \\ & \iddots & & & & \\ -1 & & & & & \end{pmatrix}. \tag{3.5}$$

GLSM construction. For a symplectic Grassmannian $SG(k, 2n)$, the GLSM is a $\mathrm{U}(k)$ gauge theory with $2n$ chiral superfields Φ_i^a ($a = 1, 2, \cdots, k; i = 1, 2, \cdots, 2n$). There is another chiral superfield p_{ab} in the anti-symmetric tensor representation $\wedge^2 V^*$, with V^* being the anti-fundamental representation of $\mathrm{U}(k)$. We also have a superpotential

$$W = \sum_{a,b=1}^{k} \sum_{i=1}^{n} p_{ab} \Phi_i^a \Phi_{2n+1-i}^b. \tag{3.6}$$

Since p_{ab} is antisymmetric, the equation of motion with respect to p_{ab} gives

$$\sum_{i=1}^{n} \left(\Phi_i^a \Phi_{2n+1-i}^b - \Phi_i^b \Phi_{2n+1-i}^a \right) = 0, \tag{3.7}$$

which is exactly the isotropy condition in the chosen basis.

3.2 Mathematics of Schubert cycles

Math definition. We follow [40], which defines Schubert varieties in the symplectic Grassmannian $SG(n-k, 2n)$. We restrict ourselves to the Lagrangian Grassmannian case, where $k = 0$. First, let LG denote the Lagrangian Grassmannian $LG(n, 2n)$, which parametrizes the n-dimensional isotropic subspaces of $\mathbb{C}^{2n}$. Then, define an isotropic flag in $\mathbb{C}^{2n}$ to be a complete flag

$$F_\bullet : \quad 0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_{2n} = \mathbb{C}^{2n} \tag{3.8}$$

of subspaces such that

$$F_{n+i} = F_{n-i}^{\perp}, \quad 0 \leq i \leq n. \tag{3.9}$$

This choice of isotropic flags corresponds to our choice of symplectic forms.

Then the Schubert variety labeled by a strict partition λ, relative to this isotropic flag $F_\bullet$, is defined by

$$\Omega_\lambda^{\mathrm{LG}}(F_\bullet) = \{\Sigma \in LG \mid \dim\left(\Sigma \cap F_{n+1-\lambda_j}\right) \geq j, \quad \forall\, 1 \leq j \leq \ell\}. \tag{3.10}$$

Matrix description. Schubert cells in ordinary Grassmannian $Gr(k, n)$ can be represented by $k \times n$ full rank matrices in reduced row echelon form, with the position of leading 1's in each row fixed. We would like to give a similar matrix representation for Schubert cells in $LG(n, 2n)$. To this aim, we recall below the definition of the Schubert cells in $LG(n, 2n)$.

For a strict partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, define the rank sequence $r = (r_0 = 0 < r_1 < \ldots < r_n < r_{n+1} = 2n + 1)$ as follows:

- If $1 \leq i \leq \ell$, then $r_i = n + 1 - \lambda_i$;
- If $\ell < n$, then the elements $r_{\ell+1} < \ldots < r_n$ are determined by

  $$\{n+1, \ldots, 2n\} \setminus \{n + \lambda_1, \ldots, n + \lambda_\ell\} = \{r_{\ell+1}, \ldots, r_n\}.$$

  (One may think of this condition as the isotropy condition.)

Note in the case $\ell = n$, $\lambda = (n, n-1, \cdots, 1)$, and

$$\{n+1, \ldots, 2n\} \setminus \{n+\lambda_1, \ldots, n+\lambda_\ell\} = \emptyset.$$

Then the Schubert cell $\Omega_\lambda^{\mathrm{LG},\circ}$ is defined by

$$\Omega_\lambda^{\mathrm{LG},\circ} = \{\Sigma \in LG \mid \dim \Sigma \cap F_j = i \quad r_i \leq j < r_{i+1}; \quad 0 \leq i \leq n\}. \tag{3.11}$$

The Schubert varieties $\Omega_\lambda^{\mathrm{LG}}$ defined in (3.10) are the closures of the Schubert cells $\Omega_\lambda^{\mathrm{LG},\circ}$. The Schubert variety $\Omega_\lambda^{\mathrm{LG}}$ can be described by a matrix Φ of the form

$$\Phi = \begin{pmatrix} \star_{1,M_0^{\mathrm{LG}}} & 0_{1,M_1^{\mathrm{LG}}} & 0_{1,M_2^{\mathrm{LG}}} & \cdots & 0_{1,M_\ell^{\mathrm{LG}}} & 0_{1,n} \\ \star_{1,M_0^{\mathrm{LG}}} & \star_{1,M_1^{\mathrm{LG}}} & 0_{1,M_2^{\mathrm{LG}}} & \cdots & 0_{1,M_\ell^{\mathrm{LG}}} & 0_{1,n} \\ \star_{1,M_0^{\mathrm{LG}}} & \star_{1,M_1^{\mathrm{LG}}} & \star_{1,M_2^{\mathrm{LG}}} & \cdots & 0_{1,M_\ell^{\mathrm{LG}}} & 0_{1,n} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \star_{1,M_0^{\mathrm{LG}}} & \star_{1,M_1^{\mathrm{LG}}} & \star_{1,M_2^{\mathrm{LG}}} & \cdots & 0_{1,M_\ell^{\mathrm{LG}}} & 0_{1,n} \\ \star_{n-\ell,M_0^{\mathrm{LG}}} & \star_{n-\ell,M_1^{\mathrm{LG}}} & \star_{n-\ell,M_2^{\mathrm{LG}}} & \cdots & \star_{n-\ell,M_\ell^{\mathrm{LG}}} & \star_{n-\ell,n} \end{pmatrix}. \tag{3.12}$$

where

$$M_i^{\mathrm{LG}} = \begin{cases} \lambda_i - \lambda_{i+1} = r_{i+1} - r_i & 1 \leq i \leq \ell - 1, \\ \lambda_\ell - 1 = n - r_\ell & i = \ell, \\ n + 1 - \lambda_1 = r_1 & i = 0, \end{cases} \tag{3.13}$$

and where Φ is of full rank, and is also required to satisfy the isotropy condition. Phrased another way, the last potentially nonzero entry on row i is on column $n+1-\lambda_i$ for $1 \leq i \leq \ell$; remaining entries to the right must vanish.

We note that the Schubert varieties $\Omega_\lambda^{\mathrm{LG}}$ contain Schubert cells $\Omega_{\widetilde{\lambda}}^{\mathrm{LG},\circ}$ for Young tableau $\widetilde{\lambda} \supset \lambda$, in addition to the Schubert cell of which they are the closure.

In examples below, we also give explicit matrix representations of Schubert cells $\Omega_\lambda^{\mathrm{LG},\circ}$ and the Schubert varieties $\Omega_\lambda^{\mathrm{LG}}$.

Example: $\boldsymbol{LG(2,4)}$. We consider the matrix representation of Schubert cells in $LG(2,4)$.

- For $\lambda = \emptyset$, the Schubert cell has the form

$$\Phi = \begin{pmatrix} a & b & 1 & 0 \\ e & f & 0 & 1 \end{pmatrix}. \tag{3.14}$$

The isotropic condition gives a constraint that $a = f$. The corresponding Schubert variety is described by matrices

$$\begin{pmatrix} a & b & c & 0 \\ e & f & g & h \end{pmatrix} \tag{3.15}$$

subject to an isotropy condition $ah + bg - cf = 0$, and a constraint that the matrix have rank two. As a consistency check, this contains as special cases all of the matrices for Schubert varieties for all other λ, as expected.

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- For $\lambda = \ydiagram{1}$, the Schubert cell has the form

$$\Phi = \begin{pmatrix} a & 1 & 0 & 0 \\ b & 0 & c & 1 \end{pmatrix}, \quad a + c = 0. \tag{3.16}$$

  The corresponding Schubert variety is described by matrices

$$\begin{pmatrix} a & f & 0 & 0 \\ b & c & d & e \end{pmatrix}, \tag{3.17}$$

  subject to an isotropy condition $ae + fd = 0$, and a constraint that the matrix have rank two. As a consistency check, we note that this contains as special cases the matrices for both $\Omega^{\mathrm{LG}}_{\ydiagram{2}}$ and $\Omega^{\mathrm{LG}}_{\ydiagram{2,1}}$, as expected.

- For $\lambda = \ydiagram{2}$, the Schubert cell has the form

$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b & 1 & 0 \end{pmatrix}, \tag{3.18}$$

  and the Schubert variety has the form

$$\begin{pmatrix} e & 0 & 0 & 0 \\ a & b & c & 0 \end{pmatrix}, \tag{3.19}$$

  subject to the condition that the matrix has rank two. As a consistency check, we note that this contains as a special case the matrix for $\Omega^{\mathrm{LG}}_{\ydiagram{2,1}}$, as expected.

- For $\lambda = \ydiagram{2,1}$, the Schubert cell is given by a matrix of the form

$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{3.20}$$

  which automatically satisfies the isotropy condition. The Schubert variety is given by a matrix of the form

$$\begin{pmatrix} e & 0 & 0 & 0 \\ a & b & 0 & 0 \end{pmatrix}, \tag{3.21}$$

  subject to the condition that the matrix has rank two.

In summary, we have the Schubert cells

$$\emptyset \leftrightarrow \begin{pmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{pmatrix}, \quad a = d, \tag{3.22}$$

$$\ydiagram{1} \leftrightarrow \begin{pmatrix} a & 1 & 0 & 0 \\ b & 0 & c & 1 \end{pmatrix}, \quad a + c = 0, \tag{3.23}$$

$$\ydiagram{2} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 1 & 0 \end{pmatrix}, \tag{3.24}$$

$$\ydiagram{2,1} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{3.25}$$

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Example: $\boldsymbol{LG(3,6)}$. We directly list the results of Schubert cells in $LG(3,6)$. Notice that this is different from our above convention in the sense that the order of rows and the order of columns are both reversed, which is allowed due to the gauge symmetries.

$$\emptyset \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & a & b & c \\ 0 & 1 & 0 & d & e & f \\ 0 & 0 & 1 & g & h & i \end{pmatrix}, \quad a=i,\ b=f,\ d=h, \tag{3.26}$$

$$\ydiagram{1} \leftrightarrow \begin{pmatrix} 1 & 0 & a & 0 & b & c \\ 0 & 1 & d & 0 & e & f \\ 0 & 0 & 0 & 1 & g & h \end{pmatrix}, \quad a+h=0,\ b=f,\ d+g=0, \tag{3.27}$$

$$\ydiagram{2} \leftrightarrow \begin{pmatrix} 1 & a & 0 & b & 0 & c \\ 0 & 0 & 1 & d & 0 & e \\ 0 & 0 & 0 & 0 & 1 & f \end{pmatrix}, \quad a+f=0,\ b=e, \tag{3.28}$$

$$\ydiagram{2,1} \leftrightarrow \begin{pmatrix} 1 & a & b & 0 & 0 & c \\ 0 & 0 & 0 & 1 & 0 & d \\ 0 & 0 & 0 & 0 & 1 & e \end{pmatrix}, \quad d+b=0,\ a+e=0, \tag{3.29}$$

$$\ydiagram{3} \leftrightarrow \begin{pmatrix} 0 & 1 & 0 & a & b & 0 \\ 0 & 0 & 1 & c & d & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad a=d, \tag{3.30}$$

$$\ydiagram{3,1} \leftrightarrow \begin{pmatrix} 0 & 1 & a & 0 & b & 0 \\ 0 & 0 & 0 & 1 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad a+c=0, \tag{3.31}$$

$$\ydiagram{3,2} \leftrightarrow \begin{pmatrix} 0 & 0 & 1 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.32}$$

$$\ydiagram{3,2,1} \leftrightarrow \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.33}$$

For simplicity, we only list results for Schubert cells in $LG(3,6)$, not Schubert varieties.

3.3 GLSM for Schubert defects in Lagrangian Grassmannians

In this section we will propose a GLSM construction of defects corresponding to Schubert classes in Lagrangian Grassmannians. In the next section, we will outline an analysis of the GLSM physics to argue that the defect localizes on the matrix Φ given in the mathematical description of the corresponding Schubert variety in the previous section. Later we will independently check that the resulting defects localize on a subvariety of $LG(n,2n)$ of the correct dimension, and also check that their indices match the Schur Q-functions expected to describe Schubert cycles in Lagrangian Grassmannians in both quantum cohomology and quantum K theory.

A Schubert class in Lagrangian Grassmannian $LG(n,2n)$ is labeled by a strict Young diagram $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_\ell, 0, \cdots, 0)$, where $n \geq \lambda_1 > \lambda_2 > \cdots > \lambda_\ell > 0$, ℓ is the number of nonzero rows in the Young diagram, and $1 \leq \ell \leq n$. The GLSM construction of this Schubert class is given as follows.

Matter content. In the bulk, the field description is the same as what we discussed above in the previous subsection. Repeated here, we have a $\mathrm{U}(n)_{\mathrm{bulk}}$ gauge theory with $2n$ fundamental chiral multiplets Φ_i^a, and a chiral multiplet p_{ab} transforming in $\wedge^2 V^*$. There is also a superpotential

$$W = \sum_{a,b=1}^{n} \sum_{i=1}^{n} p_{ab} \Phi_i^a \Phi_{2n+1-i}^b. \tag{3.34}$$

On the boundary, we have a

$$\mathrm{U}(1)_\partial \times \mathrm{U}(2)_\partial \times \cdots \mathrm{U}(\ell)_\partial \tag{3.35}$$

gauge theory, with

- chiral muliplets $\left(\varphi^{(r)}\right)^{\alpha_r}_{\alpha_{r+1}}$ transforming as bifundamentals of $\mathrm{U}(r)_\partial \times \mathrm{U}(r+1)_\partial$ for $1 \leq r \leq \ell - 1$,
- a chiral multiplet $\left(\varphi^{(\ell)}\right)^{\alpha_\ell}_{a}$ transforming as the bifundamental of $\mathrm{U}(\ell)_\partial \times \mathrm{U}(n)_{\mathrm{bulk}}$,
- M_r^{LG} Fermi multiplets $\left(\Lambda^{(r)}\right)^{i}_{\alpha_r}$ in the antifundamental of $\mathrm{U}(r)_\partial$ for $1 \leq r \leq \ell$, where[1] M_r^{LG} is given in equation (3.13)
- a chiral multiplet $q^{\alpha_\ell \beta_\ell}$ in $\wedge^2 V_\ell$, where V_ℓ is the fundamental representation of $\mathrm{U}(\ell)_\partial$,
- a Fermi multiplet $\Gamma_{\alpha_\ell a}$ in $V_\ell^* \times V^*$, where V is the fundamental representation of $\mathrm{U}(n)_{\mathrm{bulk}}$.

Superpotential. The boundary superpotential can be written as two parts,

$$W_\partial = W_{\partial,0} + W_{\partial,1}, \tag{3.36}$$

and

$$W_{\partial,0} = \sum_{r=1}^{\ell} \sum_{i=1}^{M_r^{\mathrm{LG}}} \left(\varphi^{(r)}\right)^{\alpha_r}_{\alpha_{r+1}} \left(\varphi^{(r+1)}\right)^{\alpha_{r+1}}_{\alpha_{r+2}} \cdots \left(\varphi^{(\ell)}\right)^{\alpha_\ell}_{a} \Phi^a_{\widetilde{M}^{\mathrm{LG}}_{r-1}+i} \left(\Lambda^{(r)}\right)^{i}_{\alpha_r}, \tag{3.37}$$

where

$$\widetilde{M}_r^{\mathrm{LG}} = \sum_{j=0}^{r} M_j^{\mathrm{LG}}, \tag{3.38}$$

(In passing, for later use, note that

$$M_0^{\mathrm{LG}} = n - \sum_{j=1}^{\ell} M_j^{\mathrm{LG}}, \tag{3.39}$$

[1]The M_r^{LG} Fermi multiplets transform in a $\mathrm{U}(M_r^{\mathrm{LG}})$ subgroup of the global $\mathrm{U}(n) \subset \mathrm{Sp}(2n)$ symmetry acting on the bulk fields.

for the $M_i^{\rm LG}$ defined in equation (3.13).) This superpotential is of the same form as studied by [1]. The other superpotential is given by

$$W_{\partial,1} = \sum_{a,b} \sum_{\alpha_\ell} \sum_{i=1}^{n} \left(\varphi^{(\ell)}\right)_a^{\alpha_\ell} \Phi_i^a \Phi_{2n+1-i}^b \Gamma_{\alpha_\ell b}. \tag{3.40}$$

Redundancy. Now, the constraints implied by the boundary superpotential are at least partially redundant with respect to the bulk isotropy condition enforced by the bulk superpotential. This will play an important role in the analysis in the next section, but is somewhat subtle, so we shall describe this carefully below.

First, define an $n \times n$ matrix A by

$$A^{ab} \;=\; \sum_{i=1}^{n} \Phi_i^a \Phi_{2n+1-i}^b. \tag{3.41}$$

- F-terms from the bulk superpotential constrain A to be symmetric: $A^{ab} = A^{ba}$, or more simply $A^{[ab]} = 0$.
- F-terms from the defect superpotential $W_{\partial,1}$ give the constraint

  $$\left(\varphi^{(\ell)}\right)_a^{\alpha_\ell} A^{ab} \;=\; 0. \tag{3.42}$$

  If we take $\varphi^{(\ell)}$ to generically have full rank, consistent with D-terms, then the constraint above implies that the maximum rank of A is $n - \ell$.

Now, in principle, the space of symmetric $n \times n$ matrices of rank at most $n - \ell$ has dimension (see e.g. [41, section 1(a)])

$$\frac{1}{2} n(n+1) \;-\; \binom{\ell+1}{2} \;=\; \frac{1}{2} n(n+1) \;-\; \frac{1}{2}\ell(\ell+1). \tag{3.43}$$

However, if we try to count the dimension by applying the bulk and defect F-term constraints separately, we find a mismatch: the symmetry condition on $n \times n$ matrices provides $(1/2)n(n-1)$ constraints, and the rank condition on $n \times n$ matrices provides[2] ℓ^2 constraints. Treating bulk and defect F-terms separately suggests, incorrectly, that the dimension of the space of symmetric $n \times n$ matrices of rank at most $n - \ell$ has dimension

$$n^2 \;-\; \ell^2 \;-\; \frac{n(n-1)}{2}. \tag{3.44}$$

The difference between the correct dimension (3.43) and the naive counting is

$$\frac{n(n+1)}{2} - \frac{\ell(\ell+1)}{2} - \left(n^2 - \ell^2 - \frac{n(n-1)}{2}\right) = \binom{\ell}{2}. \tag{3.45}$$

[2]An $m \times n$ matrix of rank at most r has $mn - (m-r)(n-r)$ parameters (see e.g. [42, theorem 14.4]), as can be seen by e.g. writing the matrix in terms of a set of r linearly independent rows. In the case of an $n \times n$ matrix of rank $r = n - \ell$, there are $n^2 - \ell^2$ parameters, hence the space of such matrices is codimension ℓ^2 in the space of $n \times n$ matrices.

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This tells us that the bulk and defect F-term constraints are redundant (that they define an analogue of a non-complete-intersection), with redundancy given by (3.45).

It may also be helpful to see this more explicitly in an example. Suppose $n = \ell = 2$. Write

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{3.46}$$

and let φ be an arbitrary full-rank 2×2 matrix. A priori, the bulk symmetry constraint implies $b = c$, and the condition $\varphi A = 0$ gives four more constraints, for a total of five constraints. However, if φ has rank two, then it is invertible, and the condition $\varphi A = 0$ implies $A = 0$, for which the bulk symmetry constraint is redundant. In this case, one of the constraints is redundant, consistent with the counting above as

$$\binom{2}{2} = 1. \tag{3.47}$$

This redundancy will play an important role in the physical analysis of the defects. Schematically, because of this redundancy between the bulk and defect superpotential F-term constraints, along the location of the defect, the bulk p fields will not be completely constrained by F-terms, and part of the bulk gauge symmetry will not be Higgsed by the bulk Φ fields. The defect $\varphi^{(\ell)}$ fields can be used to Higgs the residual bulk gauge symmetry, and the defect q fields (which do not appear in either the bulk or defect superpotentials) will Higgs the remaining residual $\mathrm{U}(\ell)_\partial$ gauge symmetry. In particular, the q fields are necessary in order to get the right defect indices to match mathematics results, but as they do not appear in either the bulk or defect superpotentials, their role may appear obscure. We claim that, in effect, they act to Higgs defect gauge symmetries which remain due to bulk/defect F-term constraint redundancies. We will outline the details in the next section.

Finally, we mention that in principle, for one-dimensional defects in three-dimensional GLSMs, one can add one-dimensional Chern-Simons terms. We do not do so in this paper.

3.4 Localization of the GLSM construction on Schubert cycles

In this section we will outline an argument that the defects localize on the correct locus in $LG(n, 2n)$ to describe Schubert cells. Later, we will independently verify that they have the correct dimension, and that their indices are Schur Q-functions in both quantum cohomology and quantum K theory, as expected for Schubert cycles in $LG(n, 2n)$.

We first study the F-term equations arising from the superpotentials.

F-term equations. The superpotential $W_{\partial,0}$ was previously studied in [1]. From $W_{\partial,0}$, the equations of motion with respect to $\Lambda^{(i)}$ are

$$\left(\varphi^{(r)}\right)^{\alpha_r}_{\alpha_{r+1}} \left(\varphi^{(r+1)}\right)^{\alpha_{r+1}}_{\alpha_{r+2}} \cdots \left(\varphi^{(\ell)}\right)^{\alpha_\ell}_{a} \Phi^a_{\widetilde{M}^{\mathrm{LG}}_{r-1}+i} = 0, \quad i = 1, 2, \cdots, M^{\mathrm{LG}}_r, \quad r = 1, 2, \cdots, \ell. \tag{3.48}$$

This means that the rank of each $n \times M_r^{\rm LG}$ submatrix of Φ, for $1 \le r \le \ell$, is reduced by r. Therefore, the matrix Φ is of the form

$$\Phi = \begin{pmatrix} *\cdots* & 0\cdots0 & 0\cdots0 & \cdots & 0\cdots0 & *\cdots* \\ *\cdots* & *\cdots* & 0\cdots0 & \cdots & 0\cdots0 & *\cdots* \\ *\cdots* & *\cdots* & *\cdots* & \cdots & 0\cdots0 & *\cdots* \\ \vdots\quad\vdots & \vdots\quad\vdots & \vdots\quad\vdots & \cdots & \vdots\quad\vdots & \vdots\quad\vdots \\ *\cdots* & *\cdots* & *\cdots* & \cdots & 0\cdots0 & *\cdots* \\ *\cdots* & *\cdots* & *\cdots* & \cdots & *\cdots* & *\cdots* \\ \vdots\quad\vdots & \vdots\quad\vdots & \vdots\quad\vdots & \cdots & \vdots\quad\vdots & \vdots\quad\vdots \\ *\cdots* & *\cdots* & *\cdots* & \cdots & *\cdots* & *\cdots* \end{pmatrix}. \tag{3.49}$$

To make the size of each submatrix more explicit, we can write

$$\Phi = \begin{pmatrix} \star_{1,M_0^{\rm LG}} & 0_{1,M_1^{\rm LG}} & 0_{1,M_2^{\rm LG}} & \cdots & 0_{1,M_\ell^{\rm LG}} & \star_{1,n} \\ \star_{1,M_0^{\rm LG}} & \star_{1,M_1^{\rm LG}} & 0_{1,M_2^{\rm LG}} & \cdots & 0_{1,M_\ell^{\rm LG}} & \star_{1,n} \\ \star_{1,M_0^{\rm LG}} & \star_{1,M_1^{\rm LG}} & \star_{1,M_2^{\rm LG}} & \cdots & 0_{1,M_\ell^{\rm LG}} & \star_{1,n} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \star_{1,M_0^{\rm LG}} & \star_{1,M_1^{\rm LG}} & \star_{1,M_2^{\rm LG}} & \cdots & 0_{1,M_\ell^{\rm LG}} & \star_{1,n} \\ \star_{n-\ell,M_0^{\rm LG}} & \star_{n-\ell,M_1^{\rm LG}} & \star_{n-\ell,M_2^{\rm LG}} & \cdots & \star_{n-\ell,M_\ell^{\rm LG}} & \star_{n-\ell,n} \end{pmatrix}. \tag{3.50}$$

In the matrix above, for example the left-most column has ℓ copies of $\star_{1,M_0^{\rm LG}}$. Note also that the set of $M_i^{\rm LG}$ for $0 \le i \le \ell$ partitions n, so that altogether, Φ is an $n \times 2n$ matrix.

For the other superpotential $W_{\partial,1}$, the equation of motion with respect to Γ gives

$$\sum_{i=1}^{n} \left(\varphi^{(\ell)}\right)_a^{\alpha_\ell} \Phi_i^a \Phi_{2n+1-i}^b = 0. \tag{3.51}$$

If we define a matrix

$$A^{ab} = \sum_{i=1}^{n} \Phi_i^a \Phi_{2n+1-i}^b, \tag{3.52}$$

then the constraint (3.51) can be written in the form

$$\varphi^{(\ell)} A = 0. \tag{3.53}$$

For generic $\varphi^{(\ell)}$ (as dictated by D-terms), since $\varphi^{(\ell)}$ is an $\ell \times n$ matrix, we see that the rank of the $n \times n$ matrix A must be reduced by ℓ.

In terms of the matrix Φ, the left-hand $n \times n$ matrix is of full rank, so we restrict the rank of the right-hand $n \times n$ matrix, by setting most of the last column to zero. The matrix Φ then takes the form

$$\Phi = \begin{pmatrix} \star_{1,M_0^{\rm LG}} & 0_{1,M_1^{\rm LG}} & 0_{1,M_2^{\rm LG}} & \cdots & 0_{1,M_\ell^{\rm LG}} & 0_{1,n} \\ \star_{1,M_0^{\rm LG}} & \star_{1,M_1^{\rm LG}} & 0_{1,M_2^{\rm LG}} & \cdots & 0_{1,M_\ell^{\rm LG}} & 0_{1,n} \\ \star_{1,M_0^{\rm LG}} & \star_{1,M_1^{\rm LG}} & \star_{1,M_2^{\rm LG}} & \cdots & 0_{1,M_\ell^{\rm LG}} & 0_{1,n} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \star_{1,M_0^{\rm LG}} & \star_{1,M_1^{\rm LG}} & \star_{1,M_2^{\rm LG}} & \cdots & 0_{1,M_\ell^{\rm LG}} & 0_{1,n} \\ \star_{n-\ell,M_0^{\rm LG}} & \star_{n-\ell,M_1^{\rm LG}} & \star_{n-\ell,M_2^{\rm LG}} & \cdots & \star_{n-\ell,M_\ell^{\rm LG}} & \star_{n-\ell,n} \end{pmatrix}. \tag{3.54}$$

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Also taking into account the bulk isotropy condition $\Phi\omega\Phi^T = 0$, arising from the bulk superpotential, we observe that this matrix matches the matrix given as equation (3.12), which establishes that this is the desired Schubert cycle.

D-term equations. So far we have focused on the F-term equations. Now let us turn to the D-term equations, whose analysis we will outline. We will also provide a consistency check that the defect q fields provide needed degrees of freedom to Higgs the gauge symmetry.

The bulk D-term equations take the form

$$\sum_i (\Phi_i^a)^\dagger \Phi_i^b \,-\, 2\sum_c (p_{bc})^\dagger p_{ca} \,=\, \xi\delta_a^b, \tag{3.55}$$

where ξ denotes the FI parameter. Away from the defect, smoothness and the F-term constraint from the bulk superpotential imply that $p = 0$, hence this reduces to the same D-term equation as an ordinary Grassmannian without a defect. In that case, the D-term just provides an orthonormality condition on the bulk Φ fields, as discussed in [34].

Here, the boundary isotropy condition is partially redundant with respect to the bulk isotropy condition. As a result, the boundary F-term condition forces the bulk Φ to lie along the critical locus, so that the bulk superpotential is redundant, and the bulk p field has some degrees of freedom that are not completely removed by F terms along the location of the defect.

From the matrix (3.54), the matrix A of (3.53) vanishes for $b > \ell$, so

$$\binom{\ell}{2} \tag{3.56}$$

components of p_{ab} are not completely constrained by F terms. The bulk D terms then provide

$$n^2 \,-\, \binom{\ell}{2} \tag{3.57}$$

constraints. As a result, the bulk gauge field has

$$n^2 - \left(n^2 - \binom{\ell}{2}\right) \,=\, \binom{\ell}{2} \tag{3.58}$$

remaining degrees of freedom.

Next, we will check that, in the presence of the q fields, there are enough degrees of freedom to Higgs the bulk and boundary gauge symmetries.

Combining the remaining bulk gauge degrees of freedom with the $\mathrm{U}(\ell)_\partial$ along the defect, we have a total of

$$\ell^2 \,+\, \binom{\ell}{2} \tag{3.59}$$

gauge boson degrees of freedom, between the bulk and the defect.

Now, also, from the matrix (3.54), we see that the F-term equation $\varphi^\ell\Phi = 0$ provides $\ell(n-\ell)$ constraints on the $n \times \ell$ matrix φ^ℓ, hence there are

$$n\ell \,-\, \ell(n-\ell) \,=\, \ell^2 \tag{3.60}$$

degrees of freedom. We also have the boundary q fields, which have a total of

$$\binom{\ell}{2} \tag{3.61}$$

degrees of freedom.

As a result, taking into account the contribution from the q fields, we see that the number of gauge boson and ordinary boson degrees of freedom match, which is consistent with the claim that we can Higgs both the bulk and the defect gauge groups.

For the gauge groups $\mathrm{U}(i)_\partial$ for $i < \ell$, the D-terms are the same as for defects in ordinary Grassmannians, with only couplings to defect fields and defect gauge groups, no bulk couplings, so their analysis proceeds largely as in [1]. Note that here, they have (defect-only) F-term couplings not present in [1], so the solution space will differ, but their analysis is similar.

So, we see that the boundary q fields are needed to completely Higgs the defect gauge field, and are ultimately required due to the fact that the bulk and defect superpotentials are redundant. A different way of resolving such redundancies is outlined in appendix C. (We speculate that appendix C may be more nearly relevant in mirror constructions.)

The defects themselves. In the case of the ordinary Grassmannian, the defect could be identified very closely with the Kempf-Laksov cycle X_λ^{Gr} whose pushforward is the Schubert cycle.

Mathematically, an analogue X_λ^{LG} for $LG(n,2n)$ of the Kempf-Laksov cycle X_λ^{Gr} was constructed in [43, section 1]. We briefly review the main definition. Fix a strict partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ and recall the standard symplectic flag $\mathbb{C} \subset \mathbb{C}^2 \subset \ldots \subset \mathbb{C}^{2n}$. Then X_λ^{LG} is the subvariety of the symplectic flag manifold $SF(1, \ldots, \ell, n; \mathbb{C}^{2n})$ defined by sequences $(A_1 \subset A_2 \subset \ldots \subset A_\ell \subset A_n \subset \mathbb{C}^{2n})$ such that $\dim A_i = i$, for any i, A_n is isotropic, and for $1 \leq i \leq \ell$, $A_i \subset \mathbb{C}^{n+1-\lambda_i}$. This variety has a natural projection $X_\lambda^{\mathrm{LG}} \to LG(n,2n)$ given by $(A_1 \subset A_2 \subset \ldots \subset A_\ell \subset A_n) \to A_n$. As before, one can show that this is a birational map onto the corresponding Schubert variety in $LG(n,2n)$. Again similar to the Grassmannian case, one may also realize X_λ^{LG} as tower of bundles over a point. Namely, choosing an isotropic one-dimensional subspace $A_1 \subset \mathbb{C}^{n+1-\lambda_1}$ is a projective space $\mathbb{P}(\mathbb{C}^{n+1-\lambda_1})$. Since $A_1 \subset \mathbb{C}^{n+1-\lambda_1}$, it follows that $\mathbb{C}^{n+1-\lambda_2} \subset \mathbb{C}^{n-1+\lambda_1} \subset A_1^\perp$; then choosing an isotropic subspace $A_1 \subset A_2 \subset \mathbb{C}^{n+1-\lambda_2}$ is the same as choosing a dimension one subspace $A_2/A_1 \subset \mathbb{C}^{n+1-\lambda_2}/A_1$, which globalizes to the projective bundle $\mathbb{P}(\mathbb{C}^{n+1-\lambda_2}/\mathcal{O}(-1))$ over $\mathbb{P}(\mathbb{C}^{n+1-\lambda_1})$. One continues this process to build the sequence $A_1 \subset \ldots \subset A_\ell$ as a tower of projective bundles. Finally, choosing the Lagrangian space $A_n \supset A_\ell$ leads to picking a point in the Lagrangian Grassmannian $LG(n-\ell, A_\ell^\perp/A_\ell)$, which globalizes to an appropriate Lagrangian Grassmann bundle over the previously constructed space.

We conjecture that the Kempf-Laksov cycle X_λ^{LG} of [43, section 1], described above, is realized by the defect GLSM, which we outline here. Certainly the ingredients above, the flag of subspaces, matches the structure of the GLSM defect construction and GLSMs for symplectic flag manifolds discussed in [3, section 2.6]. The mathematical description above does not involve any representations related to the q fields — but in our analysis, the q fields are all Higgsed away, and so would not participate. Finally, for completeness, another possibility is that the GLSM construction is merely giving a cycle in the same class as $X_\lambda^{\rm LG}$, if not necessarily the same variety. (See for example [43, cor. 2.6] for the class.) We leave a detailed analysis for future work.

3.5 Consistency check: dimensions

Let $\Omega_\lambda^{\rm LG}$ be the Schubert defect in $LG(n,2n)$, labeled by λ. In this section we will check that our physics construction describes a space of the same dimension as $\Omega_\lambda^{\rm LG}$, as an independent consistency check on our claim that the proposed GLSM construction realizes $\Omega_\lambda^{\rm LG}$.

Schematically,

$$\dim(\Omega_\lambda^{\rm LG}) \;=\; (\text{bulk d.o.f.}) + (\text{dim. } \varphi) + (\text{dim. } q) - (\text{Fermi constraints}) - (\text{gauge d.o.f.}).$$

In this accounting,

$$(\text{bulk d.o.f.}) = \dim LG(n,2n) \;=\; \frac{n(n+1)}{2}, \tag{3.62}$$

$$(\text{dim. } \varphi) = \sum_{r=1}^{\ell-1} r(r+1) \;+\; \ell n, \tag{3.63}$$

$$(\text{dim. } q) = \frac{\ell(\ell-1)}{2}, \tag{3.64}$$

$$(\text{Fermi constraints}) = \sum_{r=1}^{\ell} r M_r^{\rm LG} \;+\; n\ell, \tag{3.65}$$

$$(\text{gauge d.o.f.}) = \sum_{r=1}^{\ell} r^2. \tag{3.66}$$

In the expression above, the Fermi constraints include both the $\Lambda^{(r)}$ as well as Γ. Putting these together, we have

$$\begin{aligned} \dim(\Omega_\lambda^{\rm LG}) &= \frac{n(n+1)}{2} - \sum_{r=1}^{\ell-1} r(M_r^{\rm LG}-1) - \ell(M_\ell^{\rm LG}+\ell-n) - n\ell + \frac{\ell(\ell-1)}{2} \\ &= \frac{n(n+1)}{2} - \sum_{r=1}^{\ell-1} r(M_r^{\rm LG}-1) - \ell M_\ell^{\rm LG} - \frac{\ell(\ell+1)}{2}. \end{aligned} \tag{3.67}$$

Given $M_i^{\rm LG}$ defined as in equation (3.13), then we have

$$\dim(\Omega_\lambda^{\rm LG}) = \frac{n(n+1)}{2} - \sum_{r=1}^{\ell} \lambda_\ell. \tag{3.68}$$

This matches the known dimension of the Schubert cell in mathematics (see e.g. [16, section 0]).

3.6 Examples of GLSM constructions in $LG(2,4)$

In this section, we outline computations of Schubert varieties in examples of GLSM constructions in $LG(2,4)$, and demonstrate in each case that the defect localizes onto the correct subvariety. This section is intended to complement the general analysis presented earlier, to confirm in special cases via independent arguments that the GLSM really is correctly localizing on Schubert varieties.

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- $\lambda = \emptyset$

  In this case, $\ell = 0$, so the theory along the defect is trivial (no matter, no gauge factors). The only constraint we have is $\Phi\omega\Phi^T = 0$, arising from the bulk superpotential. Therefore, we can write

$$\Phi = \begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix}, \quad ah + bg - cf - de = 0. \tag{3.69}$$

  There is also the D-term constraint implying that Φ have rank two. We can then perform a bulk U(2) rotation to write this in the form

$$\Phi = \begin{pmatrix} a & b & c & 0 \\ e & f & g & h \end{pmatrix}, \quad ah + bg - cf = 0. \tag{3.70}$$

  which matches the mathematical description of the Schubert variety $\Omega^{\mathrm{LG}}_{\emptyset}$ in (3.15), as expected.

- $\lambda = \square$

  In this case, $\ell = 1$. The defect GLSM is a $\mathrm{U}(1)_\partial$ gauge theory with

  - a chiral $\varphi^{(1)}$ of charge $+1$ under $\mathrm{U}(1)_\partial$, and transforming in the antifundamental of $\mathrm{U}(n)_{\mathrm{bulk}}$,
  - since $M_1^{\mathrm{LG}} = 0$, there is no Fermi multiplet $\Lambda^{(1)}$,
  - since $\wedge^2 V_\ell = 0$, there is no chiral q,
  - a Fermi Γ in $V_1^* \times V^*$, meaning it is of charge -1 under $\mathrm{U}(1)_\partial$ and in the antifundamental of $\mathrm{U}(n)_{\mathrm{bulk}}$.

  In this case, $W_{\partial,0} = 0$, so the only contribution to the defect superpotential is

$$W_{\partial,1} = \sum_{i=1}^{n} \left(\varphi^{(1)}\right)_a \Phi_i^a \Phi_{2n+1-i}^b \Gamma_b \tag{3.71}$$

  (where we have suppressed the α_ℓ index, as it only takes one value).

  Write

$$\Phi = \begin{pmatrix} a & f & g & h \\ b & c & d & e \end{pmatrix}, \tag{3.72}$$

  then the constraint from $W_{\partial,1}$ can be expressed as

$$\varphi(\Phi_1\Phi_2)(\Phi_4\Phi_3)^T = \varphi \begin{pmatrix} a & f \\ b & c \end{pmatrix} \begin{pmatrix} h & e \\ g & d \end{pmatrix} = 0. \tag{3.73}$$

  Here, φ is a two-component vector. Each component is charge 1, and since it is the only scalar charged under the $\mathrm{U}(1)_\partial$, from the D-terms the two components behave like homogeneous coordinates on a $\mathbb{P}^1$: they cannot both vanish. (We can also utilize the bulk U(2) gauge symmetry to rotate φ to any convenient point on $\mathbb{P}^1$.) This means that the product of the pair of 2×2 matrices must have a zero eigenvalue, that its rank must be less than 2. Utilizing global symmetries, without loss of generality, we can take $g = h = 0$.

Furthermore, if we use the bulk U(2) gauge symmetry to rotate φ to the form $(*, 0)$, then (3.73) implies $ae + fd = 0$.

Putting this together, the boundary F-term equation tells us that the first row of the matrix can be fixed as follows (together with the constraint from the isotropy condition)

$$\Phi = \begin{pmatrix} a & f & 0 & 0 \\ b & c & d & e \end{pmatrix}, \quad ae + fd = 0. \tag{3.74}$$

Combined with the D-term constraint that Φ have rank two, this matches the mathematical description of the Schubert variety $\Omega^{\mathrm{LG}}_{\square}$ in (3.17), as expected.

- $\lambda = \square\square$

  In this case, $\ell = 1$. The defect GLSM is a $\mathrm{U}(1)_\partial$ gauge theory with

  - a chiral $\varphi^{(1)}$ of charge $+1$ under $\mathrm{U}(1)_\partial$, and transforming in the antifundamental of $\mathrm{U}(n)_{\mathrm{bulk}}$,
  - since $M_1^{\mathrm{LG}} = 1$, one set of Fermi multiplets $\Lambda^{(1)}$ of charge -1 under $\mathrm{U}(1)_\partial$,
  - since $\wedge^2 V_1 = 0$, there is no chiral q,
  - a Fermi Γ in $V_1^* \times V^*$, meaning it is of charge -1 under $\mathrm{U}(1)_\partial$ and in the antifundamental of $\mathrm{U}(n)_{\mathrm{bulk}}$,

  together with superpotential terms as described earlier.

  Write

$$\Phi \;=\; \begin{pmatrix} e & f & g & h \\ a & b & c & d \end{pmatrix}, \tag{3.75}$$

  Proceeding as before, φ is a two-component vector, with components that behave like the homogeneous coordinates of a $\mathbb{P}^1$. We can apply the bulk $\mathrm{U}(2)_{\mathrm{bulk}}$ gauge symmetry to rotate to any convenient point on $\mathbb{P}^1$, and without loss of generality we choose $\varphi = (*, 0)$.

  The superpotential terms $W_{\partial,0}$ imply the constraint

$$\varphi_a \Phi^a_2 \;=\; 0, \tag{3.76}$$

  which for our choice of φ implies that $f = 0$.

  The superpotential terms $W_{\partial,1}$ imply the constraint

$$\varphi \begin{pmatrix} e & 0 \\ a & b \end{pmatrix} \begin{pmatrix} h & d \\ g & c \end{pmatrix} \;=\; 0, \tag{3.77}$$

  which implies

$$eh \;=\; 0, \quad de \;=\; 0. \tag{3.78}$$

From the bulk D-term constraints, Φ must have rank two. We also have a global Sp(2) rotation that we can use to interchange columns. Without loss of generality, we assume that the leftmost two columns of Φ are of rank two, hence

$$eb - af \;\neq\; 0. \tag{3.79}$$

We have already set $f = 0$, hence $e \neq 0$. From the constraint implied by $W_{\partial,1}$, we see that $h = d = 0$.

This means that we can write

$$\Phi \;=\; \begin{pmatrix} e & 0 & g & 0 \\ a & b & c & 0 \end{pmatrix}. \tag{3.80}$$

The bulk isotropy condition implies that $bg = 0$. In order for the rank of the left-most 2×2 submatrix to be two, we must have $b \neq 0$, hence $g = 0$.

This means we can write

$$\Phi \;=\; \begin{pmatrix} e & 0 & 0 & 0 \\ a & b & c & 0 \end{pmatrix}. \tag{3.81}$$

Combined with the D-term constraint that Φ have rank two, this matches the mathematical description of the Schubert variety $\Omega^{\mathrm{LG}}_{\square\square}$ in (3.19), as expected.

- $\lambda = \begin{smallmatrix}\square & \square \\ \square & \end{smallmatrix}$

  In this case, $\ell = 2$. The defect GLSM is a $\mathrm{U}(1)_\partial \times \mathrm{U}(2)_\partial$ gauge theory with

  – chiral multiplets $\varphi^{(1)}$ in the bifundamental of $\mathrm{U}(1)_\partial \times \mathrm{U}(2)_\partial$,

  – chiral multiplets $\varphi^{(2)}$ transforming in the bifundamental of $\mathrm{U}(2)_\partial \times \mathrm{U}(2)_{\mathrm{bulk}}$,

  – $M_1^{\mathrm{LG}} = 1$ Fermi multiplet $\Lambda^{(1)}$ of charge -1 under $\mathrm{U}(1)_\partial$,

  – a chiral q transforming in $\wedge^2 V_\ell$, where V_ℓ is the fundamental representation of $\mathrm{U}(2)_\partial$,

  – a Fermi multiplet Γ in $V_\ell^* \times V^*$, where V is the fundamental representation of $\mathrm{U}(n)_{\mathrm{bulk}}$,

  together with superpotential terms as discussed elsewhere.

  As before, $\varphi^{(1)}$ is a two-component vector with components of charges $+1$ under $\mathrm{U}(1)_\partial$. The D-term conditions imply that the components behave like homogeneous coordinates on $\mathbb{P}^1$, and we can use $\mathrm{U}(2)_\partial$ symmetry to rotate $\varphi^{(1)}$ to a convenient element of $\mathbb{P}^1$, which we will take to be $(*, 0)$.

  From D-terms, $\varphi^{(2)}$ is diagonalizable, and using $\mathrm{U}(2)_{\mathrm{bulk}}$ and remaining $\mathrm{U}(2)_\partial$ symmetries, we diagonalize it.

  Write

$$\Phi \;=\; \begin{pmatrix} e & f & g & h \\ a & b & c & d \end{pmatrix}. \tag{3.82}$$

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The constraint arising from $W_{\partial,0}$ is of the form

$$\left(\varphi^{(1)}\varphi^{(2)}\right)_a \Phi_2^a \;=\; 0, \tag{3.83}$$

which implies that $f=0$.

The constraint arising from $W_{\partial,1}$ can be shown to imply that $g=h=c=d=0$.

The resulting matrix

$$\Phi \;=\; \begin{pmatrix} e & 0 & 0 & 0 \\ a & b & 0 & 0 \end{pmatrix}. \tag{3.84}$$

still needs to be of rank two, from bulk D-term conditions, hence the determinant of the remaining left-most 2×2 matrix must be nonzero. This matches the mathematical description of the Schubert variety $\Omega^{\mathrm{LG}}_{\yng(2,1)}$ in (3.21), as expected.

The remaining residual gauge symmetry can be applied to generate row reduction operations. The resulting Φ has the form

$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{3.85}$$

and it automatically satisfies the isotropy condition. This matches the mathematical description of the Schubert cell $\Omega^{\mathrm{LG},\circ}_{\yng(2,1)}$, as expected.

3.7 Schubert defects in two dimensions

In the previous section, we studied the geometric constraints implicit in the GLSM construction, to argue that the GLSMs were correctly describing Schubert varieties. In this section and the next, we shall compute indices, and compare to expected polynomials, as a strong consistency check on the underlying idea. In particular, in this section we focus on zero-dimensional defects in two-dimensional GLSMs, corresponding to describing Schubert cycles in terms of quantum cohomology. We will find in general, and also independently in computations in specific examples, that the defect indices are given by Schur Q-functions, appropriate for the given Schubert cycle.

3.7.1 General claim

For a Schubert variety in Lagrangian Grassmannian $LG(n,2n)$ labeled by a strict Young diagram λ, with the number of nonzero rows in λ denoted by ℓ, consider

$$\mathcal{I}_\lambda = \left(\prod_{r=1}^{\ell}\frac{1}{r!}\right)\oint_{\mathrm{JK}}\prod_{r=1}^{\ell}\left[\left(\prod_{i_r=1}^{r}\frac{ds^{(r)}_{i_r}}{2\pi i}\right)\prod_{1\le i_r\neq j_r\le r}\left(s^{(r)}_{i_r}-s^{(r)}_{j_r}\right)\right]\cdot\widetilde{Z}_m, \tag{3.86}$$

where

$$\widetilde{Z}_m = \left(\prod_{1\le i_\ell<j_\ell\le\ell}\frac{1}{s^{(\ell)}_{i_\ell}+s^{(\ell)}_{j_\ell}}\right)\left(\prod_{i_\ell=1}^{\ell}\frac{\left(s^{(\ell)}_{i_\ell}\right)^{M_\ell^{\mathrm{LG}}}\prod_{a=1}^{n}\left(s^{(\ell)}_{i_\ell}+\sigma_a\right)}{\prod_{a=1}^{n}\left(s^{(\ell)}_{i_\ell}-\sigma_a\right)}\right)\cdot\prod_{r=1}^{\ell-1}\prod_{i_r=1}^{r}\left(\frac{\left(s^{(r)}_{i_r}\right)^{M_r^{\mathrm{LG}}}}{\prod_{j_{r+1}=1}^{r+1}\left(s^{(r)}_{i_r}-s^{(r+1)}_{j_{r+1}}\right)}\right). \tag{3.87}$$

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In the above, the first factor comes from the one-loop determinant of $q^{\alpha_\ell \beta_\ell}$. In the second factor, the numerator is due to M_ℓ^{LG} Fermi multiplets $\Lambda^{(\ell)}$ and the Fermi multiplet $\Gamma_{\alpha_\ell a}$, while the denominator is due to $\varphi^{(r)}$. The last factor in the equation above is the contribution from the M_r^{LG} Fermi multiplets $\Lambda^{(r)}$ and the bifundamentals $\varphi^{(r)}$ for $r = 1, \dots, \ell - 1$.

For these defects in two dimensions, we can evaluate the Jeffrey-Kirwan residue in the same form as for Schubert defects in ordinary Grassmannians as discussed previously in section 2.3. Then it can be shown that $\mathcal{I}_\lambda$ equals the Schur Q-function Q_λ, for M_r^{LG} defined by equation (3.13).

We will check in detail in examples that $\mathcal{I}_\lambda = Q_\lambda$, then give a general argument for why it is always the case.

3.7.2 Example: $LG(2, 4)$

For $\lambda = (\lambda_1, 0)$, $\ell = 1$, writing $a \equiv s^{(1)}$, we have

$$\mathcal{I}_\lambda = \oint \frac{da}{2\pi i} a^{\lambda_1 - 1} \frac{(a + \sigma_1)(a + \sigma_2)}{(a - \sigma_1)(a - \sigma_2)} = 2 \frac{\sigma_1 + \sigma_2}{\sigma_1 - \sigma_2} (\sigma_1^{\lambda_1} - \sigma_2^{\lambda_1}). \tag{3.88}$$

Hence we have

$$\mathcal{I}_{\yng(1)} = 2(\sigma_1 + \sigma_2) = Q_{\yng(1)}(\sigma), \quad \mathcal{I}_{\yng(2)} = 2(\sigma_1 + \sigma_2)^2 = Q_{\yng(2)}(\sigma), \tag{3.89}$$

where $Q_\lambda(\sigma)$ denotes the Q Schur polynomial in the σ's, as listed in appendix B in equations (B.6), (B.7).

For $\lambda = (\lambda_1, \lambda_2)$, $\ell = 2$, writing $a \equiv s^{(1)}, b \equiv s^{(2)}$, we have

$$\begin{aligned} \mathcal{I}_\lambda &= \frac{1}{2} \oint \frac{da}{2\pi i} \frac{db_1}{2\pi i} \frac{db_2}{2\pi i} (b_1 - b_2)(b_2 - b_1) \frac{1}{b_1 + b_2} \frac{a^{\lambda_1 - \lambda_2}}{(a - b_1)(a - b_2)} \\ &\quad \cdot \frac{(b_1 b_2)^{\lambda_2 - 1} (b_1 + \sigma_1)(b_1 + \sigma_2)(b_2 + \sigma_1)(b_2 + \sigma_2)}{(b_1 - \sigma_1)(b_1 - \sigma_2)(b_2 - \sigma_1)(b_2 - \sigma_2)}, \qquad (3.90) \\ &= 4 \frac{\sigma_1 + \sigma_2}{\sigma_1 - \sigma_2} (\sigma_1 \sigma_2)^{\lambda_2} (\sigma_1^{\lambda_1 - \lambda_2} - \sigma_2^{\lambda_1 - \lambda_2}). \qquad (3.91) \end{aligned}$$

So we have

$$\mathcal{I}_{\yng(2,1)} = 4(\sigma_1 \sigma_2)(\sigma_1 + \sigma_2) = Q_{\yng(2,1)}(\sigma), \tag{3.92}$$

matching the Schur Q-function in equation (B.8).

3.7.3 Example: $LG(3, 6)$

For $\lambda = (\lambda_1, 0, 0)$, $\ell = 1$, writing $a \equiv s^{(1)}$, we have

$$\begin{aligned} \mathcal{I}_\lambda &= \oint \frac{da}{2\pi i} a^{\lambda_1 - 1} \frac{(a + \sigma_1)(a + \sigma_2)(a + \sigma_3)}{(a - \sigma_1)(a - \sigma_2)(a - \sigma_3)}, \qquad (3.93) \\ &= 2 \frac{(\sigma_1 + \sigma_2)(\sigma_1 + \sigma_3)(\sigma_2 + \sigma_3)}{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)(\sigma_2 - \sigma_3)} \left(\sigma_1^{\lambda_1} \frac{\sigma_2 - \sigma_3}{\sigma_2 + \sigma_3} - \sigma_2^{\lambda_1} \frac{\sigma_1 - \sigma_3}{\sigma_1 + \sigma_3} + \sigma_3^{\lambda_1} \frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2} \right). \end{aligned}$$

It is easy to see that

$$\mathcal{I}_{\ydiagram{1}} = 2(\sigma_1+\sigma_2+\sigma_3) = Q_{\ydiagram{1}}(\sigma), \quad \mathcal{I}_{\ydiagram{2}} = 2(\sigma_1+\sigma_2+\sigma_3)^2 = Q_{\ydiagram{2}}(\sigma), \tag{3.94}$$

$$\begin{aligned}\mathcal{I}_{\ydiagram{3}} &= 2(\sigma_1^3+\sigma_2^3+\sigma_3^3+2\sigma_1\sigma_2(\sigma_1+\sigma_2)+2\sigma_1\sigma_3(\sigma_1+\sigma_3)+2\sigma_2\sigma_3(\sigma_2+\sigma_3)+4\sigma_1\sigma_2\sigma_3)\\ &= Q_{\ydiagram{3}}(\sigma).\end{aligned} \tag{3.95}$$

For $\lambda=(\lambda_1,\lambda_2,0)$, $\ell=2$, writing $a\equiv s^{(1)}, b\equiv s^{(2)}$, we have

$$\mathcal{I}_\lambda = \frac{1}{2}\oint \frac{da}{2\pi i}\frac{db_1}{2\pi i}\frac{db_2}{2\pi i}\frac{(b_1-b_2)(b_2-b_1)}{b_1+b_2}\frac{a^{\lambda_1-\lambda_2}}{(a-b_1)(a-b_2)} \tag{3.96}$$
$$\cdot\frac{(b_1b_2)^{\lambda_2-1}(b_1+\sigma_1)(b_1+\sigma_2)(b_1+\sigma_3)(b_2+\sigma_1)(b_2+\sigma_2)(b_2+\sigma_3)}{(b_1-\sigma_1)(b_1-\sigma_2)(b_1-\sigma_3)(b_2-\sigma_1)(b_2-\sigma_2)(b_2-\sigma_3)},$$
$$= 4\frac{(\sigma_1+\sigma_2)(\sigma_1+\sigma_3)(\sigma_2+\sigma_3)}{(\sigma_1-\sigma_2)(\sigma_1-\sigma_3)(\sigma_2-\sigma_3)} \tag{3.97}$$
$$\cdot\Big((\sigma_1\sigma_2)^{\lambda_2}(\sigma_1^{\lambda_1-\lambda_2}-\sigma_2^{\lambda_1-\lambda_2})-(\sigma_1\sigma_3)^{\lambda_2}(\sigma_1^{\lambda_1-\lambda_2}-\sigma_3^{\lambda_1-\lambda_2})+(\sigma_2\sigma_3)^{\lambda_2}(\sigma_2^{\lambda_1-\lambda_2}-\sigma_3^{\lambda_1-\lambda_2})\Big).$$

Then we have

$$\begin{aligned}\mathcal{I}_{\ydiagram{2,1}} &= 4(\sigma_1+\sigma_2)(\sigma_1+\sigma_3)(\sigma_2+\sigma_3)\\ &= Q_{\ydiagram{2,1}}(\sigma),\end{aligned} \tag{3.98}$$

$$\begin{aligned}\mathcal{I}_{\ydiagram{3,1}} &= 4(\sigma_1+\sigma_2)(\sigma_1+\sigma_3)(\sigma_2+\sigma_3)(\sigma_1+\sigma_2+\sigma_3)\\ &= Q_{\ydiagram{3,1}}(\sigma),\end{aligned} \tag{3.99}$$

$$\begin{aligned}\mathcal{I}_{\ydiagram{3,2}} &= 4(\sigma_1+\sigma_2)(\sigma_1+\sigma_3)(\sigma_2+\sigma_3)(\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_2\sigma_3)\\ &= Q_{\ydiagram{3,2}}(\sigma).\end{aligned} \tag{3.100}$$

For $\lambda=(\lambda_1,\lambda_2,\lambda_3)$, $\ell=3$, we wil only write down the result,

$$\mathcal{I}_{\ydiagram{3,2,1}} = 8\sigma_1\sigma_2\sigma_3(\sigma_1+\sigma_2)(\sigma_1+\sigma_3)(\sigma_2+\sigma_3)(\sigma_1+\sigma_2+\sigma_3) = Q_{\ydiagram{3,2,1}}(\sigma). \tag{3.101}$$

3.7.4 General argument

In this section we will give a general argument for why index computations match Schur Q-functions.

These will be exercises in computing Jeffrey-Kirwan residues; see for example [46, section 3.1, equ'n (3.7)], [47, equ'n (2.34)] and in particular [1, section 3.3] for further information. We will perform the nested contour integrals in (3.86) and (3.111) similarly as in [1] by iteratively calculating $\mathrm{U}(k)$-integral from $k=1$ to $k=\ell$. In the nested integrals, we should choose the poles from the chiral multiplets in the denominators of (3.87) and (3.112), respectively. This choice makes sure that the indices we are computing count the contributions from Higgs branch vacua, corresponding to FI parameters of the defect quiver gauge theories all being positive. The only difference with [1] in the computation is that here we start with a fixed pole configuration and do the computation, then the final result is obtained by applying Weyl permutations to include all possible pole configurations.

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Given a strict partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ and the dictionary between λ and M_i^{LG}'s is given as in equation (3.13), which we repeat here:

$$M_i^{\mathrm{LG}} = \begin{cases} \lambda_i - \lambda_{i+1} & 1 \le i \le \ell - 1, \\ \lambda_\ell - 1 & i = \ell. \end{cases} \tag{3.102}$$

The index for the $0d$ defect quiver gauge theory is given as follows

$$\begin{aligned} \mathcal{I}_\lambda = \left(\prod_{r=1}^{\ell} \frac{1}{r!}\right) \oint_{\mathrm{JK}} \left[\prod_{r=1}^{\ell} \frac{d^r s^{(r)}}{(2\pi i)^r} \prod_{1\le i\neq j\le r} (s_i^{(r)} - s_j^{(r)})\right] \left[\prod_{1\le i<j\le \ell} \frac{1}{s_i^{(\ell)} + s_j^{(\ell)}}\right] \\ \times \left[\prod_{i=1}^{\ell} (s_i^{(\ell)})^{M_\ell^{\mathrm{LG}}} \prod_{a=1}^{n} \frac{(s_i^{(\ell)} + \sigma_a)}{(s_i^{(\ell)} - \sigma_a)}\right] \left[\prod_{r=1}^{\ell-1} \prod_{i=1}^{r} \frac{(s_i^{(r)})^{M_r^{\mathrm{LG}}}}{\prod_{j=1}^{r+1} (s_i^{(r)} - s_j^{(r+1)})}\right]. \end{aligned} \tag{3.103}$$

We compute the above JK residue as follows.

First, the factor of

$$\prod_{r=1}^{\ell} \frac{1}{r!}$$

accounts for the multiplicity due to the Weyl group symmetry $\mathfrak{S}_r$ of $\mathrm{U}(r)_\partial$ for each r. We can drop that factor by fixing an ordering of $s_a^{(r)}$'s for each r.

Next, consider the contribution from the pole configuration

$$\left\{ s_a^{(r)} = \sigma_a, \text{ for } a = 1, \ldots, r \text{ and } r = 1, \ldots, \ell. \right\}. \tag{3.104}$$

(As mentioned above, the only poles that contribute to the JK residue are of this form, as in e.g. [47, section 4.6].) All other possible contributing poles are obtained by applying permutations $\omega \in \mathfrak{S}_n$ to $\{\sigma_1, \ldots, \sigma_n\}$. That said, those permutations overcount by $(n - \ell)!$, because the residues are invariant under $\mathfrak{S}_{n-\ell}$, which will be important later.

The residue of this one pole contribution is

$$\begin{aligned} &\left(\prod_{r=1}^{\ell} \prod_{i\neq j=1}^{r} (\sigma_i - \sigma_j)\right) \left(\prod_{1\le i<j\le \ell} \frac{1}{\sigma_i + \sigma_j}\right) \left(\prod_{i=1}^{\ell} \frac{(\sigma_i)^{M_\ell^{\mathrm{LG}}} \prod_{k=1}^{n} (\sigma_i + \sigma_k)}{\prod_{k\neq i} (\sigma_i - \sigma_k)}\right) \\ &\qquad \times \left(\prod_{r=1}^{\ell-1} \prod_{i=1}^{r} \frac{(\sigma_i)^{M_r^{\mathrm{LG}}}}{\prod_{j\neq i}^{r+1} (\sigma_i - \sigma_j)}\right), \end{aligned} \tag{3.105}$$

$$\begin{aligned} &= \left(2^\ell \prod_{i=1}^{\ell} (\sigma_i)^{M_\ell^{\mathrm{LG}}+1} \prod_{r=1}^{\ell-1} \prod_{i=1}^{r} (\sigma_i)^{M_r^{\mathrm{LG}}}\right) \left(\frac{\prod_{i=1}^{\ell} \prod_{k=1}^{i-1} (\sigma_i + \sigma_k) \prod_{k=i+1}^{n} (\sigma_i + \sigma_k)}{\prod_{i=1}^{\ell-1} \prod_{j=i+1}^{\ell} (\sigma_i + \sigma_j)}\right) \\ &\times \frac{\left(\prod_{r=1}^{\ell} \prod_{i=1}^{r} \prod_{j=1}^{i-1} (\sigma_i - \sigma_j) \prod_{j=i+1}^{r} (\sigma_i - \sigma_j)\right)}{\left(\prod_{i=1}^{\ell} \prod_{k=1}^{i-1} (\sigma_i - \sigma_k) \prod_{k=i+1}^{n} (\sigma_i - \sigma_k)\right) \left(\prod_{r=1}^{\ell-1} \prod_{i=1}^{r} \prod_{j=1}^{i-1} (\sigma_i - \sigma_j) \prod_{j=i+1}^{r+1} (\sigma_i - \sigma_j)\right)}, \\ &= \left(2^\ell \prod_{j=1}^{\ell} (\sigma_j)^{\lambda_j}\right) \left(\prod_{i=1}^{\ell} \prod_{j=i+1}^{n} \frac{\sigma_i + \sigma_j}{\sigma_i - \sigma_j}\right), \end{aligned} \tag{3.106}$$

where we have used the fact that

$$\lambda_i = 1 + \sum_{j=i}^{\ell} M_j^{\mathrm{LG}}, \tag{3.107}$$

as well as

$$\prod_{r=1}^{\ell}\prod_{i=1}^{r}\prod_{j=i+1}^{r}(\sigma_i-\sigma_j) \;=\; \prod_{r=1}^{\ell-1}\prod_{i=1}^{r}\prod_{j=i+1}^{r+1}(\sigma_i-\sigma_j)\,. \tag{3.108}$$

One sees immediately that (3.106) is the $\omega = \text{id}$ term in the expression for the Schur Q-function with $\beta = 0$, (B.3), using the fact that

$$[\![x|0]\!]^{\lambda} \;=\; \prod_{j=1}^{\ell}\left((2\sigma_j)\sigma_j^{\lambda_j-1}\right) \;=\; 2^{\ell}\prod_{j=1}^{\ell}\sigma_j^{\lambda_j}. \tag{3.109}$$

Summing the contributions from various poles, which corresponds to acting by permutations, and taking into account the $|\mathfrak{S}_{n-\ell}|$ invariance mentioned earlier, we have

$$\mathcal{I}_{\lambda}(\sigma) \;=\; \frac{1}{(n-\ell)!}\sum_{\omega\in\mathfrak{S}_n}\omega\left\{\left(2^{\ell}\prod_{j=1}^{\ell}(\sigma_j)^{\lambda_j}\right)\left(\prod_{i=1}^{\ell}\prod_{j=i+1}^{n}\frac{\sigma_i+\sigma_j}{\sigma_i-\sigma_j}\right)\right\}, \tag{3.110}$$

where the factor of $1/(n-\ell)!$ compensates for the fact that we are summing over additional permutations. This matches the expression (B.3) for the Schur Q-function at $\beta = 0$, and so we see $\mathcal{I}_{\lambda}(\sigma) \;=\; Q_{\lambda}(\sigma)$.

3.8 Schubert defects in three dimensions

In this section, we turn to one-dimensional defects in three-dimensional GLSMs, in which Schubert cycles are described in terms of quantum K theory. We will verify, both through general arguments and via independent computations in simple examples, that the physical defect indices match K-theoretic Schur Q-functions, as expected to describe Schubert cycles in Lagrangian Grassmannians in quantum K theory.

3.8.1 General claim

For a K-theoretic Schubert class in $LG(n, 2n)$, labeled by a strict Young diagram λ, with the number of nonzero rows in λ denoted by ℓ, consider

$$\mathcal{I}_{\lambda}^{\text{1d}} = \left(\prod_{r=1}^{\ell}\frac{1}{r!}\right)\oint_{\text{JK}}\prod_{r=1}^{\ell}\left[\left(\prod_{i_r=1}^{r}\frac{-dz_{i_r}^{(r)}}{2\pi i z_{i_r}^{(r)}}\right)\prod_{1\le i_r\neq j_r\le r}\left(1-\frac{z_{i_r}^{(r)}}{z_{j_r}^{(r)}}\right)\right]\cdot Z_m, \tag{3.111}$$

where

$$\begin{aligned} Z_m =& \left(\prod_{1\le i_\ell<j_\ell\le \ell}\frac{1}{1-z_{i_\ell}^{(\ell)}z_{j_\ell}^{(\ell)}}\right)\left(\prod_{i_\ell=1}^{\ell}\frac{\left(1-z_{i_\ell}^{(\ell)}\right)^{M_\ell^{\text{LG}}}\prod_{a=1}^{n}\left(1-z_{i_\ell}^{(\ell)}X_a\right)}{\prod_{a=1}^{n}\left(1-z_{i_\ell}^{(\ell)}/X_a\right)}\right) \\ &\cdot\prod_{r=1}^{\ell-1}\prod_{i_r=1}^{r}\left(\frac{\left(1-z_{i_r}^{(r)}\right)^{M_r^{\text{LG}}}}{\prod_{j_{r+1}=1}^{r+1}\left(1-z_{i_r}^{(r)}/z_{j_{r+1}}^{(r+1)}\right)}\right), \end{aligned} \tag{3.112}$$

for M_r^{LG} defined by equation (3.13), following the conventions of [1, section 3.3.4]. We will first check in examples, and then in subsection 3.8.4 we give a general argument that $\mathcal{I}_{\lambda}^{\text{1d}}$ gives the K-theoretic Schur Q-function that is reviewed in appendix B (with $\beta = -1$).

3.8.2 Example: $LG(2,4)$

For $\lambda = (\lambda_1, 0)$, the corresponding Schubert defect is realized by a 1d U(1) gauge theory with $M_1^{\rm LG} = \lambda_1 - 1$. The supersymmetric localization formula gives

$$
\begin{aligned}
\mathcal{I}^{\rm 1d}_{\lambda}(X) &= \oint \left(\frac{-dz}{2\pi i z}\right)(1-z)^{M_1^{\rm LG}} \prod_{i=1}^{2} \frac{1 - zX_i}{1 - z/X_i} \\
&= (1-X_1)^{M_1^{\rm LG}} \frac{(1-X_1^2)(1-X_1X_2)}{1 - X_1X_2^{-1}} + (1-X_2)^{M_1^{\rm LG}} \frac{(1-X_2^2)(1-X_1X_2)}{1-X_2X_1^{-1}}\,,
\end{aligned}
\tag{3.113}
$$

Consider the cases when $\lambda_1 = 1$ and $\lambda_1 = 2$, then

$$
\begin{aligned}
\mathcal{I}^{\rm 1d}_{\square}(X) &= \frac{(1-X_1^2)(1-X_1X_2)}{1-X_1X_2^{-1}} + \frac{(1-X_2^2)(1-X_1X_2)}{1-X_2X_1^{-1}} = (1-X_1X_2)(1+X_1X_2)\,, \\
\mathcal{I}^{\rm 1d}_{\square\square}(X) &= (1-X_1)\frac{(1-X_1^2)(1-X_1X_2)}{1-X_1X_2^{-1}} + (1-X_2)\frac{(1-X_2^2)(1-X_1X_2)}{1-X_2X_1^{-1}} \\
&= (1-X_1X_2)(1+X_1X_2 - X_1^2X_2 - X_1X_2^2)\,,
\end{aligned}
\tag{3.114}
$$

which match equations (B.10), (B.12) for the corresponding Schur Q-functions.

For $\lambda = (\lambda_1, \lambda_2)$, the Schubert defect is realized by a 1d U(1) $\times$ U(2) gauge theory with $M_1^{\rm LG} = \lambda_1 - \lambda_2$ and $M_2^{\rm LG} = \lambda_2 - \lambda_1$. Then, we have

$$
\begin{aligned}
\mathcal{I}^{\rm 1d}_{\lambda}(X) &= \frac{1}{2!}\oint \left(\frac{-dz^{(1)}}{2\pi i z^{(1)}}\frac{-dz_1^{(2)}}{2\pi i z_1^{(2)}}\frac{-dz_2^{(2)}}{2\pi i z_2^{(2)}}\right)\left(1 - \frac{z_1^{(2)}}{z_2^{(2)}}\right)\left(1-\frac{z_2^{(2)}}{z_1^{(2)}}\right)\frac{1}{1-z_1^{(2)}z_2^{(2)}} \\
&\quad \times \left(\prod_{a=1}^{2}(1-z_a^{(2)})^{M_2^{\rm LG}}\prod_{i=1}^{2}\frac{1-z_a^{(2)}X_i}{1-z_a^{(2)}/X_i}\right)\frac{\left(1-z^{(1)}\right)^{M_1^{\rm LG}}}{\prod_{a=1}^{2}\left(1-z^{(1)}/z_a^{(2)}\right)} \\
&= (1-X_1^2)(1-X_2^2)(1-X_1)^{M_1^{\rm LG}+M_2^{\rm LG}}(1-X_2)^{M_2^{\rm LG}}\frac{1-X_1X_2}{1-X_1/X_2} \\
&\quad + (1-X_1^2)(1-X_2^2)(1-X_2)^{M_1^{\rm LG}+M_2^{\rm LG}}(1-X_1)^{M_2^{\rm LG}}\frac{1-X_1X_2}{1-X_2/X_1}\,.
\end{aligned}
\tag{3.115}
$$

In the special case when $\lambda = (2,1)$, namely, $M_1^{\rm LG} = 1$ and $M_2^{\rm LG} = 0$, one can derive

$$
\begin{aligned}
\mathcal{I}^{\rm 1d}_{\substack{\square\square\\ \square}}(X) &= (1-X_1^2)(1-X_2^2)(1-X_1)\frac{1-X_1X_2}{1-X_1/X_2} + (1-X_1^2)(1-X_2^2)(1-X_2)\frac{1-X_1X_2}{1-X_2/X_1} \\
&= (1-X_1^2)(1-X_2^2)(1-X_1X_2)\,,
\end{aligned}
\tag{3.116}
$$

which matches equation (B.14) for the corresponding Schur Q-function.

By direct comparison, one can conclude that the Schubert defects $\mathcal{I}^{\rm 1d}_{\square}$, $\mathcal{I}^{\rm 1d}_{\square\square}$ and $\mathcal{I}^{\rm 1d}_{\substack{\square\square\\ \square}}$ give exactly the K-theoretic Schur Q-functions. This is in accord with (not yet proven) expectations from e.g. [6].

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3.8.3 Example: $LG(3,6)$

For $\lambda=(\lambda_1,0,0)$, the supersymmetric localization formula of the $1d$ defect quiver gives

$$\begin{aligned}\mathcal{I}^{1d}_{\lambda}(X)&=\oint\frac{-dz}{2\pi i z}(1-z)^{M_1^{\mathrm{LG}}}\prod_{i=1}^{3}\frac{1-zX_i}{1-z/X_i}\\&=(1-X_1)^{M_1^{\mathrm{LG}}}(1-X_1^2)\left(\frac{1-X_1X_2}{1-X_1/X_2}\right)\left(\frac{1-X_1X_3}{1-X_1/X_3}\right)\\&\quad+(1-X_2)^{M_1^{\mathrm{LG}}}(1-X_2^2)\left(\frac{1-X_1X_2}{1-X_2/X_1}\right)\left(\frac{1-X_2X_3}{1-X_2/X_3}\right)\\&\quad+(1-X_3)^{M_1^{\mathrm{LG}}}(1-X_3^2)\left(\frac{1-X_1X_3}{1-X_3/X_1}\right)\left(\frac{1-X_2X_3}{1-X_3/X_2}\right).\end{aligned}\tag{3.117}$$

In the special cases, $\lambda_1=1$, 2 or 3, one can obtain

$$\begin{aligned}\mathcal{I}^{1\mathrm{d}}_{\square}(X)&=(1-X_1^2)\left(\frac{1-X_1X_2}{1-X_1/X_2}\right)\left(\frac{1-X_1X_3}{1-X_1/X_3}\right)+(1-X_2^2)\left(\frac{1-X_1X_2}{1-X_2/X_1}\right)\left(\frac{1-X_2X_3}{1-X_2/X_3}\right)\\&\quad+(1-X_3^2)\left(\frac{1-X_1X_3}{1-X_3/X_1}\right)\left(\frac{1-X_2X_3}{1-X_3/X_2}\right)\\&=(1-X_1X_2X_3)(1+X_1X_2X_3)\,,\end{aligned}\tag{3.118}$$

$$\begin{aligned}\mathcal{I}^{1\mathrm{d}}_{\square\square}&=1-X_1^2X_2^2X_3-X_1^2X_2X_3^2-X_1X_2^2X_3^2-X_1^2X_2^2X_3^2\\&\quad+X_1^3X_2^2X_3^2+X_1^2X_2^3X_3^2+X_1^2X_2^2X_3^3\,,\end{aligned}\tag{3.119}$$

$$\begin{aligned}\mathcal{I}^{1\mathrm{d}}_{\square\square\square}&=1-X_1^2X_2X_3-X_1X_2^2X_3-X_1X_2X_3^2-2X_1^2X_2^2X_3-2X_1^2X_2X_3^2-2X_1X_2^2X_3^2\\&\quad+X_1^3X_2^2X_3+X_1^2X_2^3X_3+X_1^3X_2X_3^2+X_1X_2^3X_3^2+X_1X_2^2X_3^3+X_1^2X_2X_3^3\\&\quad+2X_1^2X_2^2X_3^3+2X_1^3X_2^2X_3^2+2X_1^2X_2^3X_3^2+2X_1^2X_2^2X_3^3\\&\quad-X_1^4X_2^2X_3^2-X_1^2X_2^4X_3^2-X_1^2X_2^2X_3^4-X_1^3X_2^3X_3^2-X_1^3X_2^2X_3^3-X_1^2X_2^3X_3^3\,,\end{aligned}\tag{3.120}$$

and it can be shown that these match the corresponding Schur Q-functions.

When $\lambda=(\lambda_1,\lambda_2,0)$, one can obtain the Schubert classes by the following localization formula

$$\begin{aligned}\mathcal{I}^{1d}_{\lambda}(X)&=\frac{1}{2!}\oint\left(\frac{-dz^{(1)}}{2\pi i z^{(1)}}\frac{-dz^{(2)}_1}{2\pi i z^{(2)}_1}\frac{-dz^{(2)}_2}{2\pi i z^{(2)}_2}\right)\left(1-\frac{z^{(2)}_1}{z^{(2)}_2}\right)\left(1-\frac{z^{(2)}_2}{z^{(2)}_1}\right)\frac{1}{1-z^{(2)}_1z^{(2)}_2}\\&\quad\times\left(\prod_{a=1}^{2}(1-z^{(2)}_a)^{M_2^{\mathrm{LG}}}\prod_{i=1}^{3}\frac{1-z^{(2)}_aX_i}{1-z^{(2)}_a/X_i}\right)\frac{\left(1-z^{(1)}\right)^{M_1^{\mathrm{LG}}}}{\prod_{a=1}^{2}\left(1-z^{(1)}/z^{(2)}_a\right)}.\end{aligned}\tag{3.121}$$

For example,

$$\mathcal{I}^{1d}_{\yng(2,1)}(X)=(1-X_1X_2)(1-X_1X_3)(1-X_2X_3)(1+X_1X_2+X_1X_3+X_2X_3)\,,\tag{3.122}$$

$$\begin{aligned}\mathcal{I}^{1d}_{\yng(3,1)}(X)&=(1-X_1X_2)(1-X_1X_3)(1-X_2X_3)\\&\quad\times(1+X_1X_2+X_1X_3+X_2X_3-X_1^2X_2X_3-X_1X_2^2X_3-X_1X_2X_3^2-X_1^2X_2^2X_3^2)\end{aligned}\tag{3.123}$$

$$\begin{aligned}\mathcal{I}^{1d}_{\yng(3,2)}(X)&=(1-X_1X_2)(1-X_1X_3)(1-X_2X_3)\\&\quad\times(1+X_1X_2+X_1X_3+X_2X_3-X_1^2X_2-X_1X_2^2-X_1^2X_3-X_1X_3^2-X_2^2X_3-X_2X_3^2\\&\qquad-2X_1X_2X_3+X_1^2X_2X_3+X_1X_2^2X_3+X_1X_2X_3^2+X_1^2X_2^2X_3^2)\,.\end{aligned}\tag{3.124}$$

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When $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, according to the localization formula

$$
\begin{aligned}
\mathcal{I}^{1d}_{\lambda}(X) = \frac{1}{2!3!} \oint & \left[\frac{-dz^{(1)}}{2\pi i z^{(1)}} \prod_{\alpha=1}^{2} \frac{-dz^{(2)}_{\alpha}}{2\pi i z^{(2)}_{\alpha}} \prod_{\beta=1}^{3} \frac{-dz^{(3)}_{\beta}}{2\pi i z^{(3)}_{\beta}} \right] \left[\prod_{1\le\alpha_1\neq\alpha_2\le 2} 1 - \frac{z^{(2)}_{\alpha_1}}{z^{(2)}_{\alpha_2}} \right] \left[\prod_{1\le\beta_1\neq\beta_2\le 3} 1 - \frac{z^{(3)}_{\beta_1}}{z^{(2)}_{\beta_2}} \right] \\
& \times \frac{1}{1 - z^{(2)}_1 z^{(2)}_2} \left[\prod_{1\le\beta_1<\beta_2\le 3} \frac{1}{1 - z^{(3)}_{\beta_1} z^{(3)}_{\beta_2}} \right] \left[\prod_{\beta=1}^{3} (1 - z^{(3)}_{\beta})^{M^{\mathrm{LG}}_3} \prod_{i=1}^{3} \frac{1 - z^{(3)}_{\beta=1} X_i}{1 - z^{(3)}_{\beta=1} X_i^{-1}} \right] \\
& \times \left[\prod_{\alpha=1}^{2} (1 - z^{(2)}_{\alpha})^{M^{\mathrm{LG}}_2} \prod_{\beta=1}^{3} \frac{1 - z^{(2)}_{\alpha} z^{(3)}_{\beta}}{1 - z^{(2)}_{\alpha} / z^{(3)}_{\beta}} \right] \frac{\left(1 - z^{(1)}\right)^{M^{\mathrm{LG}}_1}}{\prod_{\alpha=1}^{2} \left(1 - z^{(1)} / z^{(2)}_{\alpha}\right)} .
\end{aligned}
\tag{3.125}
$$

one can obtain

$$
\mathcal{I}^{1d}_{\ydiagram{3,2,1}}(X) = (1 - X_1^2)(1 - X_2^2)(1 - X_3^3)(1 - X_1 X_2)(1 - X_1 X_3)(1 - X_2 X_3) . \tag{3.126}
$$

In each case above, it can be shown that the indices $\mathcal{I}_{\lambda}$ match K-theoretic Schur Q-functions $Q_{\lambda}(X)$, in agreement with expectations from e.g. [6].

3.8.4 General argument

In this section, we will give a general argument that indices of the $1d$ defect quiver gauge theories match K-theoretic Schur Q-functions. Our argument will be closely analogous to that in section 3.7.4, to which we refer for technicalities.

The index of the $1d$ defect quiver gauge theory is given as

$$
\begin{aligned}
\mathcal{I}_{\lambda} = \left(\prod_{r=1}^{\ell} \frac{1}{r!} \right) \oint_{\mathrm{JK}} & \left[\prod_{r=1}^{\ell} \frac{d^r z^{(r)}}{(2\pi i)^r} \prod_{i=1}^{r} \left(\frac{-1}{z^{(r)}_i} \right) \prod_{1\le i\neq j\le r} \left(1 - \frac{z^{(r)}_i}{z^{(r)}_j} \right) \right] \left[\prod_{1\le i<j\le \ell} \frac{1}{1 - z^{(\ell)}_i z^{(\ell)}_j} \right] \\
& \times \left[\prod_{i=1}^{\ell} (1 - z^{(\ell)}_i)^{M^{\mathrm{LG}}_{\ell}} \prod_{k=1}^{n} \frac{1 - z^{(\ell)}_i X_k}{1 - z^{(\ell)}_i {X_k}^{-1}} \right] \left[\prod_{r=1}^{\ell-1} \prod_{i=1}^{r} \frac{(1 - z^{(r)}_i)^{M^{\mathrm{LG}}_r}}{\prod_{j=1}^{r+1} (1 - z^{(r)}_i {z^{(r+1)}_j}^{-1})} \right] .
\end{aligned}
\tag{3.127}
$$

We will compute this residue integral following the same procedure as in the $0d$ case. Let us start with the following pole configuration:

$$
z^{(r)}_a = X_a, \quad \text{for } r = 1, \ldots, \ell \text{ and } a = 1, \ldots, r. \tag{3.128}
$$

The residue due to this pole configuration is

$$\left[\prod_{r=1}^{\ell}\prod_{i=1}^{r}\left(-\frac{1}{X_i}\right)\prod_{j=1}^{i-1}\left(1-X_iX_j{}^{-1}\right)\prod_{j=i+1}^{r}\left(1-X_iX_j{}^{-1}\right)\right]\left[\prod_{i=1}^{\ell}\prod_{j=1}^{i-1}\frac{1}{1-X_iX_j}\prod_{j=i+1}^{\ell}\frac{1}{1-X_iX_j}\right]$$
$$\times\left[\prod_{i=1}^{\ell}(1-X_i)^{M_\ell^{\mathrm{LG}}}(-X_i)\frac{\prod_{k=1}^{n}(1-X_iX_k)}{\prod_{k=1,k\neq i}^{n}(1-X_iX_k{}^{-1})}\right]\left[\prod_{r=1}^{\ell-1}\prod_{i=1}^{r}\frac{(1-X_i)^{M_r^{\mathrm{LG}}}}{\prod_{j=1,j\neq i}^{r+1}(1-X_iX_j{}^{-1})}(-X_i)\right],$$
$$=\left[\prod_{j=1}^{\ell}(1-X_j{}^2)\prod_{r=1}^{\ell}\prod_{i=1}^{r}(1-X_i)^{M_r^{\mathrm{LG}}}\right]$$
$$\times\frac{\prod_{i=1}^{\ell}\prod_{j=i+1}^{\ell}\left(1-X_iX_j{}^{-1}\right)}{\prod_{i=1}^{\ell}\prod_{k=i+1}^{n}(1-X_iX_k{}^{-1})}\frac{\prod_{r=1}^{\ell-1}\prod_{i=1}^{r}\prod_{j=i+1}^{r}\left(1-X_iX_j{}^{-1}\right)}{\prod_{r=1}^{\ell-1}\prod_{i=1}^{r}\prod_{j=i+1}^{r+1}(1-X_iX_j{}^{-1})}$$
$$\times\frac{\prod_{i=1}^{\ell}\prod_{k=i+1}^{n}(1-X_iX_k)}{\prod_{i=1}^{\ell}\prod_{j=i+1}^{\ell}(1-X_iX_j)}, \tag{3.129}$$
$$=\left[\prod_{j=1}^{\ell}(1-X_j{}^2)\prod_{r=1}^{\ell}\prod_{i=1}^{r}(1-X_i)^{M_r^{\mathrm{LG}}}\right]\prod_{i=1}^{\ell}\prod_{k=i+1}^{n}\frac{1-X_iX_k}{1-X_iX_k{}^{-1}}, \tag{3.130}$$

which is the $\omega=$ identity term appearing in the expression (B.5) for the Schur Q-function with $\beta=-1$ (the K-theoretic Schur Q-functions).

Proceeding as before, and taking into accoun the $|\mathfrak{S}_{n-\ell}|$ overcounting, the full index is then

$$\mathcal{I}_\lambda(X) \;=\; \frac{1}{(n-\ell)!}\sum_{\omega\in\mathfrak{S}_n}\omega\left\{\left[\prod_{j=1}^{\ell}(1-X_j{}^2)\prod_{r=1}^{\ell}\prod_{i=1}^{r}(1-X_i)^{M_r^{\mathrm{LG}}}\right]\prod_{i=1}^{\ell}\prod_{k=i+1}^{n}\frac{1-X_iX_k}{1-X_iX_k{}^{-1}}\right\}, \tag{3.131}$$

Next, we use the fact that

$$[\![1-z\,|\,0]\!]^\lambda = \prod_{j=1}^{\ell}\left[(1-X_j^2)(1-X_j)^{\lambda_j-1}\right], \tag{3.132}$$
$$= \prod_{j=1}^{\ell}\left[(1-X_j^2)\prod_{r=1}^{j}(1-X_r)^{M_j^{\mathrm{LG}}}\right], \tag{3.133}$$

which is a consequence of the fact that the power of $(1-X_i)$ appearing is

$$\sum_{r=i}^{\ell}M_r^{\mathrm{LG}} \;=\; \lambda_i-1. \tag{3.134}$$

As a result, $\mathcal{I}_\lambda(X)$ exactly matches (B.5), hence

$$\mathcal{I}_\lambda(X) \;=\; Q_\lambda(X). \tag{3.135}$$

4 Conclusions

In this paper we have described a proposal for a GLSM construction of Schubert cycles in Lagrangian Grassmannians. We have argued that the GLSM localizes onto the correct locus (both by general arguments and by independent computations in simple examples), that the dimension matches that of the Schubert cycles, and that the physical defect indices match the expected characteristic polynomials (Schur Q-functions) in both quantum cohomology and quantum K theory, both by general arguments and by independent computations in specific examples.

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A Partition functions of 0d defects

In this section we briefly outline the computation of partition functions of (zero-dimensional) defects in two-dimensional GLSMs. Briefly, our strategy will be to dimensionally-reduce the one-dimensional results of [39, section 4].

We begin with the index computations of [39, section 4]. Results there are given for the index of a (one-dimensional) quantum mechanical theory, not a supersymmetric matrix model. However, since all the fields are periodic in the time direction in the computation of the one-dimensional index, we can take the zero-radius limit, in which the theory reduces to a SUSY gauged matrix model of interest to us.

Naively, the Lagrangian of this gauged matrix model descends from the quantum mechanics by omitting the kinetic terms and the Chern-Simons terms. However, there are some subtleties here. One can define the theta-vacua in even dimensional theory; furthermore, the vector field variables are real in the one-dimensional gauged theory, but there is no vector field in the zero dimensional theory, and the scalar of the vector multiplet is a complex one. So a more precise reduction should come from the 2d $\mathcal{N} = (0,2)$ gauged linear sigma model. So unlike the one-dimensional gauged supersymmetric quantum mechanics, the non-compact Coulomb branch can be lifted by the theta angle in the gauged matrix model.

The resulting partition functions in the zero-dimensional case can be understood as follows. Following [39, section 4], the desired partition function can be expressed in terms of a (higher dimensional) contour integral, computed as the Jeffrey-Kirwan (JK) residue:

$$S \;=\; \frac{1}{\mid W \mid} \oint_{\mathrm{JK}} \prod_{r=1}^{\ell} \frac{d^r u^{(r)}}{(2\pi i)^r} \, \mathrm{Z}^{\mathrm{0d}}_{\mathrm{vector}}(u) \, \mathrm{Z}^{\mathrm{0d}}_{\mathrm{matter}}(u,m), \tag{A.1}$$

where r is the rank of the gauge group, W is the Weyl group of the gauge group, u is the complex scalar in the vector multiplet of the gauged matrix model, and m is its complex mass. The one-loop determinants can be computed straightforwardly:

- The one-loop determinant of the vector mulitplet can be computed by restricting to the zero Fourier mode of [39, equation (4.17)], which gives

$$\mathrm{Z}^{\mathrm{0d}}_{\mathrm{vector}}(u) = \alpha(u), \tag{A.2}$$

where α is the roots of the gauge group.

- The one-loop determinant of the chiral multiplet can be obtained from the zero mode in [39, equation (4.23)], which tells

$$\mathrm{Z}^{\mathrm{0d}}_{\mathrm{chiral}}(u,m) = \prod_{i\in\mathcal{R}} \frac{1}{\rho_i(u) + \rho^F_i(m)}, \tag{A.3}$$

where $\mathcal{R}$ denotes the representation of bosonic matter fields, ρ are weights of the gauge group, and ρ^F are the weights of the flavor symmetry.

- The one-loop determinant of the Fermi multiplet can be gained from the [39, equation (4.26)]

$$\mathrm{Z}^{\mathrm{0d}}_{\mathrm{fermi}}(u,m) = \prod_{j\in\tilde{\mathcal{R}}} \left(\tilde{\rho}_j(u) + \tilde{\rho}^F_j(m)\right), \tag{A.4}$$

where $\tilde{\mathcal{R}}$ stands for the representation of Fermi multiplets, $\tilde{\rho}$ are the associated weights of the gauge group, and $\tilde{\rho}^F$ are the weights of its flavor group.

B Schur Q-function

The Schur Q-function can be found in, for example, [13, section 3.1], [29]. For completeness, we briefly recall its definition and basic properties in this appendix.

B.1 Definition

Given a strict partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, the Schur Q-function of n variables $\{x_1, \ldots, x_n\}$ is defined as [29, def'n (2.1)]

$$Q_\lambda(x_1,\ldots,x_n) := \frac{1}{(n-\ell)!} \sum_{\omega\in\mathfrak{S}_n} \omega \left[[\![x|b]\!]^\lambda \prod_{i=1}^{\ell} \prod_{j=i+1}^{n} \frac{x_i \oplus x_j}{x_i \ominus x_j} \right], \tag{B.1}$$

where

$$\begin{aligned} x\oplus y &:= x + y + \beta x y\,, \\ x\ominus y &:= \frac{x-y}{1+\beta y}\,, \\ [\![x|b]\!]^k &:= (x\oplus x)(x\oplus b_1)\cdots(x\oplus b_{k-1})\,, \\ [\![x|b]\!]^\lambda &:= \prod_{j=1}^{\ell} [\![x_j|b]\!]^{\lambda_j}\,. \end{aligned} \tag{B.2}$$

The parameter β can be taken to be 0 or -1. Both of the allowed values are relevant for this paper:

- $\beta = 0$ is relevant to Schubert cycles in quantum cohomology (two-dimensional physical models),

- $\beta = -1$ is relevant to Schubert cycles in quantum K theory (three-dimensional physical models). The resulting polynomials are also known as the K-theoretic Schur Q-functions.

The b_i are determined by equivariant parameters; for example, in the case of K theory, $b_i = 1 - \exp(t_i)$, see for example [29, section 8]. In the non-equivariant case, which we focus on in this paper, we should set the b_i's to zero.

When $\beta = 0$, $x \oplus y = x + y$ and $x \ominus y = x - y$, then (B.1) can be simplified as

$$Q_\lambda(x_1, \ldots, x_n) = \frac{1}{(n-\ell)!} \sum_{\omega \in \mathfrak{S}_n} \omega \left[[\![x|b]\!]^\lambda \prod_{i=1}^{\ell} \prod_{j=i+1}^{n} \frac{x_i + x_j}{x_i - x_j} \right] \tag{B.3}$$

When $\beta = -1$, to compare with the physics results, it is more convenient to do the following change[3] of variables, $x_i = 1 - X_i$. Then one can obtain

$$\begin{aligned} x_1 \oplus x_2|_{\beta=-1} &= x_1 + x_2 - x_1 x_2 = 1 - X_1 X_2\,, \\ x_1 \ominus x_2|_{\beta=-1} &= \frac{x_1 - x_2}{1 - x_2} = 1 - \frac{X_1}{X_2}\,, \end{aligned} \tag{B.4}$$

and in this case

$$Q_\lambda(X_1, \ldots, X_n) = \frac{1}{(n-\ell)!} \sum_{\omega \in \mathfrak{S}_n} \omega \left[[\![1 - X|1 - y]\!]^\lambda \prod_{i=1}^{\ell} \prod_{j=i+1}^{n} \frac{1 - X_i X_j}{1 - X_i/X_j} \right], \tag{B.5}$$

where we have defined $y_i := 1 - b_i$ for $i = 1, 2, \ldots$.

Remark. The factor $1/(n-\ell)!$ appearing in front of (B.1) reflects the fact that (B.1) is invariant under the group $\mathfrak{S}_{n-\ell}$ of permutations of $\{x_{\ell+1}, \ldots, x_n\}$, of order $(n-\ell)!$. In other words, the factor $1/(n-\ell)!$ is there to compensate for a redundancy of multiplicity $(n-\ell)!$ in the summation over elements of $\mathfrak{S}_n$.

B.2 Examples: $n = 2$

In this subsection we provide some examples of Schur Q-functions, in the case that $n = 2$, meaning $LG(2, 4)$.

B.2.1 Quantum cohomology ($\beta = 0$)

- $\lambda = (1)$

$$Q_{\ydiagram{1}}(x) = 2x_1 \frac{x_1 + x_2}{x_1 - x_2} + 2x_2 \frac{x_2 + x_1}{x_2 - x_1} = 2(x_1 + x_2)\,. \tag{B.6}$$

- $\lambda = (2)$

$$Q_{\ydiagram{2}}(x) = 2x_1^2 \frac{x_1 + x_2}{x_1 - x_2} + 2x_2^2 \frac{x_2 + x_1}{x_2 - x_1} = 2(x_1 + x_2)^2\,. \tag{B.7}$$

- $\lambda = (2, 1)$

$$Q_{\ydiagram{2,1}}(x) = 4x_1^2 x_2 \frac{x_1 + x_2}{x_1 - x_2} + 4x_2^2 x_1 \frac{x_2 + x_1}{x_2 - x_1} = 4x_1 x_2 (x_1 + x_2)\,. \tag{B.8}$$

[3] In passing, the x_i's here coincide with the shifted variables used in [44, 45].

B.2.2 Quantum K theory ($\beta = -1$)

- $\lambda = (1)$

$$
\begin{aligned}
Q_{\square}(x) &= (2x_1 - x_1^2)\frac{(x_1+x_2-x_1x_2)(1-x_2)}{(x_1-x_2)} + (2x_2-x_2^2)\frac{(x_1+x_2-x_1x_2)(1-x_1)}{(x_2-x_1)}, \\
&= (x_1+x_2-x_1x_2)(2-x_1-x_2+x_1x_2), \\
&= 2(x_1+x_2) - (x_1+x_2)^2 - 2x_1x_2 + 2x_1x_2(x_1+x_2) - x_1^2x_2^2, \\
&= (1-X_1X_2)(1+X_1X_2)\,.
\end{aligned} \tag{B.9}
$$

Or, we can directly compute in terms of shifted variables as

$$
\begin{aligned}
Q_{\square}(X) &= (1-X_1^2)\frac{1-X_1X_2}{1-X_1{X_2}^{-1}} + (1-X_2^2)\frac{1-X_1X_2}{1-X_2{X_1}^{-1}}, \\
&= (1-X_1X_2)(1+X_1X_2)
\end{aligned} \tag{B.10}
$$

- $\lambda = (2)$

$$
\begin{aligned}
Q_{\square\square}(x) &= (2x_1-x_1^2)x_1\frac{(x_1+x_2-x_1x_2)(1-x_2)}{(x_1-x_2)} \\
&\quad + (2x_2-x_2^2)x_2\frac{(x_1+x_2-x_1x_2)(1-x_1)}{(x_2-x_1)}, \\
&= (x_1+x_2-x_1x_2)(2x_1+2x_2-x_1^2-x_2^2-3x_1x_2+x_1^2x_2+x_1x_2^2), \\
&= 2(x_1+x_2)^2 - x_1^3 - x_2^3 - 6x_1x_2(x_1+x_2) + 2x_1x_2(x_1^2+x_2^2) + 5x_1^2x_2^2 \\
&\quad - x_1^2x_2^2(x_1+x_2), \\
&= (1-X_1X_2)(1+X_1X_2-X_1^2X_2-X_1X_2^2)\,.
\end{aligned} \tag{B.11}
$$

Alternatively, one can compute in terms shifted variables directly as

$$
\begin{aligned}
Q_{\square\square} &= (1-X_1^2)(1-X_1)\frac{1-X_1X_2}{1-X_1{X_2}^{-1}} + (1-X_2^2)(1-X_2)\frac{1-X_1X_2}{1-X_2{X_1}^{-1}}, \\
&= (1-X_1X_2)(1+X_1X_2-X_1^2X_2-X_1X_2^2).
\end{aligned} \tag{B.12}
$$

- $\lambda = (2,1)$

$$
\begin{aligned}
Q_{(2,1)}(x) &= (2x_1-x_1^2)x_1(2x_2-x_2^2)\frac{(x_1+x_2-x_1x_2)(1-x_2)}{(x_1-x_2)} \\
&\quad + (2x_2-x_2^2)x_2(2x_1-x_1^2)\frac{(x_1+x_2-x_1x_2)(1-x_1)}{(x_2-x_1)}, \\
&= x_1x_2(2-x_1)(2-x_2)(x_1+x_2-x_1x_2), \\
&= 4x_1x_2(x_1+x_2) - 2x_1x_2(x_1^2+x_2^2) - 8x_1^2x_2^2 + 3x_1^2x_2^2(x_1+x_2) - x_1^3x_2^3, \\
&= (1-{X_1}^2)(1-{X_2}^2)(1-X_1X_2)\,,
\end{aligned} \tag{B.13}
$$

and, similarly, one can directly compute using X-variables

$$
\begin{aligned}
Q_{(2,1)} &= (1-X_1^2)(1-X_1)(1-X_2^2)\frac{1-X_1X_2}{1-X_1{X_2}^{-1}} \\
&\quad + (1-X_2^2)(1-X_2)(1-X_1^2)\frac{1-X_1X_2}{1-X_2{X_1}^{-1}}, \\
&= (1-{X_1}^2)(1-{X_2}^2)(1-X_1X_2)\,.
\end{aligned} \tag{B.14}
$$

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C Toy model of eliminating redundant constraints

In this section we outline a simple toy model of how one can add fields to remove redundant constraints. We do not use this method in our construction, but include it here as a counterpoint.

Consider a two-dimensional (0,2) supersymmetric theory. For simplicity, we will assume not include a gauge factor, though it is trivial to add. Suppose we have two Fermi superfields Λ_1, Λ_2, and a function $f(\Phi)$ of chiral superfields. Then, consider the (0,2) superpotential

$$W \;=\; \Lambda_1 f(\Phi) \;+\; \Lambda_2 f(\Phi). \tag{C.1}$$

Now, ordinarily we would interpret a (0,2) superpotential term of the form $\Lambda f(\Phi)$ as enforcing a constraint on the low-energy space of vacua, of the form $\{f = 0\}$. Here, however, we have two identical constraints. In fact, in this toy model, we can trivially write the superpotential as

$$W \;=\; (\Lambda_1 + \Lambda_2) f(\Phi), \tag{C.2}$$

which makes it clear that the combination $\Lambda_1 - \Lambda_2$ decouples.

Now, we can remove that degeneracy by adding a new chiral superfield q and adding a superpotential term $mq(\Lambda_1 - \Lambda_2)$. This terms gives a mass m to both q and to the difference $\Lambda_1 - \Lambda_2$, effectively removing them from the low-energy theory.

As a result, this theory with the extra chiral superifield q and superpotential

$$W \;=\; (\Lambda_1 + \Lambda_2) f(\Phi) \;+\; mq(\Lambda_1 - \Lambda_2) \tag{C.3}$$

is equivalent at low energies to a theory containing only a single Fermi superfield Λ and the superpotential

$$W \;=\; \Lambda f(\Phi). \tag{C.4}$$

Data Availability Statement. This article has no associated data or the data will not be deposited.

Code Availability Statement. This article has no associated code or the code will not be deposited.

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