Extending the Geometric Tools of Heterotic String Compactification and Dualities

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(ABSTRACT)

In this work, we extend the well-known spectral cover construction first developed by Friedman, Morgan, and Witten to describe more general vector bundles on elliptically fibered Calabi-Yau geometries. In particular, we consider the case in which the Calabi-Yau fibration is not in Weierstrass form but can rather contain fibral divisors or multiple sections (i.e., a higher rank Mordell-Weil group). In these cases, general vector bundles defined over such Calabi-Yau manifolds cannot be described by ordinary spectral data. To accomplish this, we employ well-established tools from the mathematics literature of Fourier-Mukai functors. We also generalize existing tools for explicitly computing Fourier-Mukai transforms of stable bundles on elliptic Calabi-Yau manifolds. As an example of these new tools, we produce novel examples of chirality changing small instanton transitions. Next, we provide a geometric formalism that can substantially increase the understood regimes of heterotic/F-theory duality. We consider heterotic target space dual (0,2) GLSMs on elliptically fibered Calabi-Yau manifolds. In this context, each half of the "dual" heterotic theories must, in turn, have an F-theory dual. Moreover, the apparent relationship between two heterotic compactifications seen in (0,2) heterotic target space dual pairs should, in principle, induce some putative correspondence between the dual F-theory geometries. It has previously been conjectured in the literature that (0,2) target space duality might manifest in F-theory as multiple $K3$-fibrations of the same elliptically fibered Calabi-Yau manifold. We investigate this conjecture in the context of both 6-dimensional and 4-dimensional effective theories and demonstrate that in general, (0,2) target space duality cannot be explained by such a simple phenomenon alone. In all cases, we provide evidence that non-geometric data in F-theory must play at least some role in the induced F-theory correspondence while leaving the full determination of the putative new F-theory duality to the future work. Finally, we consider F-theory over elliptically fibered manifolds, with a general conic base. Such manifolds are quite standard in F-theory sense, but our goal is to explore the extent of the heterotic/F-theory duality over such manifolds.

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(GENERAL AUDIENCE ABSTRACT)

String theory is the only physical theory that can lead to self-consistent, effective quantum gravity theories. However, quantum mechanics restricts the dimension of the effective spacetime to ten (and eleven) dimensions. Hence, to study the consequences of string theory in four dimensions, one needs to assume the extra six dimensions are curled into small compact dimensions. Upon this "compactification," it has been shown (mainly in the 1990s) that different classes of string theories can have equivalent four-dimensional physics. Such classes are called dual. The advantage of these dualities is that often they can map perturbative and non-perturbative limits of these theories. The goal of this dissertation is to explore and extend the geometric limitations of the duality between heterotic string theory and F-theory. One of the main tools in this particular duality is the Fourier-Mukai transformation. In particular, we consider Fourier-Mukai transformations over non-standard geometries. As an application, we study the F-theory dual of a heterotic/heterotic duality known as target space duality. As another side application, we derive new types of small instanton transitions in heterotic strings. In the end, we consider F-theory compactified over particular manifolds that if we consider them as a geometry dual to a heterotic string, can lead to unexpected consequences.

Dedication

I humbly dedicate this dissertation to Zoroaster.

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Chapter 1

Introduction

1.1 Motivation and overview

Since the 1980s, string theory has been considered as an area of fundamental physics that possibly could explain our four-dimensional universe. Even though it has been proven that string theory can provide i) a large family of phenomenological models for particle physics and cosmology [6, 56, 94, 127, 150], ii) it can be used to study field theories in various dimensions,[1] and iii) has led to important dialogs in mathematics,[2] there are still many open questions. During the 1980s and early 1990s, five anomaly free formulations of string theory in ten dimensions were known [1, 95], Type I $SO(32)$, Type IIA, Type IIB, Heterotic $E_8 \times E_8$ and Heterotic $SO(32)$. Some of these theories were known to be related through so-called "T-duality" [60, 64]. In 1995, E. Witten introduced M-theory in eleven dimensions [111, 115, 157], and it has been shown that all of the five theories are related to various limits of M-theory through a chain of dualities.

Almost a year later, C. Vafa introduced F-theory [151], which is a twelve-dimensional geometric theory. It has been one of the central topics of string theory studies in the past twelve years. One reason is that various limits of different string theory models are dual to F-theory compactified over certain complex manifolds.

[1] The literature on this is really huge, for one of the interesting aspects we refer the reader to [107].

[2] This is also a huge field itself, for some interesting aspects the reader can look at [101, 113] for topological strings and mirror symmetry, and [3] for moonshine.

In other words, M-theory compactified over a smooth Calabi-Yau manifold $\hat{X}_m$ can be realized as the Coulomb branch of F-theory compactification over the singular limit of the same Calabi-Yau X_m times a circle S^1. At the same time, it can be considered as the geometrization of the strong coupling limit of type IIB string theory compactified over B_{m-1} (the base of the Calabi-Yau manifold X_m). Also, if X_m is also a $K3$ fibration, this F-theory model also has a Heterotic dual model [131, 132].

Heterotic/F-theory duality has proven to be a robust and useful tool in the determination of F-theory effective physics as well a remarkable window into the structure of the string landscape. The seminal work on F-theory [89, 131, 132] appealed to Heterotic theories, and ever since, many new developments and tools have been built on or inspired by, the duality. Despite the important role that this duality has played, however, it has remained at some level limited by the geometric assumptions that have been frequently placed on the background geometries in both the Heterotic and F-theory compactifications.

We will relax these geometric assumptions and will try to study the new consequences and applications Heterotic/F-theory duality in slightly more general setups. We organized this dissertation as follows.

In chapter 2, we aim to broaden the consideration of background geometry of manifolds/bundles arising in Heterotic compactifications with an aim towards extending the validity and understanding of Heterotic/F-theory duality. In particular, we will focus on elliptically fibered Calabi-Yau geometries arising in Heterotic theories in the context of the so-called *Fourier Mukai* transforms of vector bundles on elliptically fibered manifolds (see e.g., [117] for a review).

In the context of Heterotic/F-theory duality, a range of geometries are possible in the elliptic and $K3$-fibered manifolds appearing in (1.71) and (1.72) (with many possible Hodge numbers, Picard groups, etc. appearing). However, thanks to the work of Nakayama [133], the existence of an *elliptic fibration* guarantees the existence of a particular minimal form for the dual CY geometries – the so-called *Weierstrass form* in which all reducible components of the fiber not intersecting the zero-section have been blown down.

It has been argued [151] that from the point of view of F-theory, Weierstrass models are the natural geometric point in which to consider/define the theory. In order to make sense of the axio-dilaton from a type IIB perspective, we require the existence of a section to the elliptic fibration, and for all reducible components of fibers not intersecting this zero section to be blown-down to zero size. This choice provides a unique value of the axio-dilaton for every point in the base geometry. Once it is further demanded that the torus fibration admits a section, it is guaranteed that the Weierstrass models are available and obtainable form the originally chosen geometry via birational morphisms [61].

If the F-theory geometry also admits a $K3$-fibration, then the choice of Weierstrass form described above also imposes the expected form of the Heterotic elliptically fibered geometry in the stable degeneration limit [30, 70, 90]. As a result, in much of the literature to date, it has simply been assumed that the essential procedure of Heterotic/F-theory duality must be to place both CY geometries, X_n and Y_{n+1} into Weierstrass form from the start.

However, this Weierstrass-centric procedure overlooks the fact that while the CY manifolds can be naturally transformed into Weierstrass form, the data of a vector bundle in a Heterotic theory may *crucially depend on the geometric features that are "washed out" in Weierstrass form.* In particular, due to a theorem of Shioda, Tate and Wazir [145, 146, 152], it is known that the space of divisors of an elliptically fibered CY threefold may be decomposed into the following groups:

1) Pull-backs, $\pi^*(D_\alpha)$ of divisors, D_α, in the base B_{n-1},
2) Rational sections to the elliptic fibration (i.e. elements of the Mordell-Weil group of X_n), and
3) So-called "fibral divisors" corresponding to reducible components of the fiber (i.e. vertical divisors not pulled back from the base).

As a result of the above decomposition, it is clear that the topology (i.e. Chern classes), cohomology (i.e. $H^*(X_3, V)$) and stability structure (i.e. stable regions within Kähler moduli space) of a stable, holomorphic bundle V on an elliptically fibered manifold can depend on these "extra" divisors (and elements of $h^{1,1}(X_3)$) which are not present in Weierstrass form. In addition, if X_n contains either a higher rank Mordell-Weil group or fibral divisors, the associated Weierstrass model is singular, leading to natural questions as to how to interpret the data of gauge fields/vector bundles over such spaces. As a result, in the processing of attempting to map the Heterotic CY manifold into Weierstrass form, important topological and quasi-topological information – and its ensuing physical consequences – could be lost.

It is the goal of this work to investigate Fourier-Mukai transforms of vector bundles over elliptically fibered manifolds *not in Weierstrass form* as a necessary first step in extending Heterotic/F-theory duality beyond the form considered in [89].

The key results of this work include:

- A generalization of the topological formulae for bundles described by smooth spectral covers to the case of Calabi-Yau threefolds involving fibral divisors and multiple sections (i.e., a higher rank Mordell-Weil group associated to the elliptic fibration).
- We generalize the available computational tools to *explicitly* construct the Fourier-Mukai transforms of vector bundles on elliptically fibered geometries. That is, given an explicit vector bundle constructed on an elliptic threefold (for example, built using the monad construction or as an extension bundle), we provide an algorithm to produce the spectral data (a key ingredient in determining an explicit F-theory dual of a chosen Heterotic background). This extends/generalizes important prior work in this area [36, 67, 122].
- We apply the generalized results for spectral cover bundles to the particular application of so-called "small instanton transitions" in Heterotic theories (i.e., M5-brane/fixed plane transitions in the language of Heterotic M-theory [111]). We find more general transitions possible than those previously cataloged in [135].

In chapter 3 work, we aim to explore the consequences of a conjectured duality between Heterotic string theories, i.e., the so-called target space duality (TSD) in the context of yet another duality – that between Heterotic string compactifications and F-theory. As has been observed since the first investigations into TSD [41, 52], this non-trivial duality of distinct Heterotic backgrounds could potentially also lead to an entirely new duality structure within F-theory. Since Heterotic and F-theory vacua consist of two of the most promising frameworks for string model building within 4-dimensional string compactifications, it makes sense to search for such novel and unexplored dualities to better understand redundancies within the space of such theories. In addition, if new dualities exist, they could also provide deep insights into the structure of the effective physics, or perhaps even new computational tools (as has manifestly proved to be the case with mirror symmetry in Type II compactifications of string theory, see e.g., [112]).

Heterotic target space duality was first observed in [65] and further explored in [5, 40, 41, 42, 140]. The basic premise is simple to state: two distinct $(0,2)$ GLSMs[3] sharing a non-geometric (i.e., Landau-Ginzburg or Hybrid) phase can be found to

[3] $(0,2)$ GLSMs are two dimensional linear (canonical kinetic term) gauge theories with $(0,2)$

have apparently identical 4-dimensional, $\mathcal{N} = 1$ target space theories. In these cases, the GLSMs are distinct, and the geometric phases of the two theories lead to manifestly different Calabi-Yau manifolds and vector bundles over them. However, the ensuing 4-dimensional theories, arising as large volume compactifications of the $E_8 \times E_8$ Heterotic string, contain at least the same gauge symmetry and 4-dimensional massless particle spectrum. Although not yet understood as a true string duality, this phenomenon has been referred to as *(0,2) Target Space Duality* (TSD) [65]. A more recent "landscape" survey of such theories [42, 140] showed that it is not just in special cases that such dualities can occur, but rather the vast majority of $(0, 2)$ GLSMs contain non-geometric phases that can be linked to other $(0, 2)$ GLSMs in this way. Moreover, recent work [5] demonstrated that in some cases, TSD also seems to preserve the form of non-trivial D- and F-term potentials of the 4-dimensional theory to a remarkable degree.

In chapter 4 we consider the geometry of $K3$-fibered Calabi-Yau manifolds in compactifications of F-theory [35, 89, 90, 131, 132]. As mentioned, and will be reviewed in some detail later, within the context of F-theory, the $(n+1)$-dimensional Calabi-Yau compactification geometry, Y_{n+1} is constrained to be elliptically fibered. That is, there exists a surjective map $\pi_f : Y_{n+1} \to \mathbb{B}_n$ with elliptic fiber, $\mathbb{E}$. Moreover, in the case that the geometry is also compatibly $K3$-fibered over a base B_{n-1}, the base to the elliptic fibration must be $\mathbb{P}^1$-fibered and the following relationships hold:

$$\begin{array}{ccc} Y_{n+1} & \xrightarrow{\mathbb{E}} & \mathcal{B}_n \\ \downarrow{\scriptstyle K3} & & \downarrow{\scriptstyle \mathbb{P}^1} \\ B_{n-1} & \longrightarrow & B_{n-1} \end{array} \tag{1.1}$$

Given a $K3$-fibered manifold as shown above, the effective physics of F-theory compactified on Y_{n+1} is known to be dual to that of the $E_8 \times E_8$ Heterotic string compactified on an elliptically fibered Calabi-Yau n-fold, $\pi_h : X_n \to B_{n-1}$, where the base to the Heterotic elliptic fibration is *the same* as the base to the $K3$ fibration shown above.

Heterotic/F-theory duality has long been a useful tool in the study of the resulting effective theories and has been extensively studied. Moreover, the structure of the base geometry to the F-theory elliptic fibration shown in (3.42) above – *namely that $\mathcal{B}_n$ is a $\mathbb{P}^1$-fibration* – has proven to be a tractable starting point in efforts to classify possible base manifolds for elliptically fibered Calabi-Yau 4-folds [7, 89, 103].

supersymmetry, which can have various phases. In some phases, it becomes a Landau-Ginzburg model, in other phases, its vacuum configuration becomes a Calabi-Yau manifold (geometric phase).

The goal of chapter 4 is to generalize the construction of elliptically fibered Calabi-Yau manifolds with a $\mathbb{P}^1$-fibered base considered in the literature to date and to study their consequences for Heterotic/F-theory duality. To begin, it is important to note that $\mathbb{P}^1$-fibrations can be simply divided into two classes:

- Those that are nowhere degenerate (i.e. $\mathbb{P}^1$-fibrations that are in fact $\mathbb{P}^1$-*bundles*) and
- Those that *do* degenerate over a higher co-dimensional discriminant locus.

1.2 Heterotic string compactification

In this section, we review the ten-dimensional Heterotic supergravity, derived as the effective theory of the ten-dimensional Heterotic string theory. Then we focus on the dimensional reduction of this theory over Calabi-Yau manifolds.

The ten-dimensional field theory we are considering is a $N = 1$ supergravity coupled to a Yang-Mills theory. The bosonic field content of this theory consists of the metric $g_{\mu\nu}$, the two form field $B_{\mu\nu}$, dilaton scalar Φ and the gauge fields A_μ with a gauge group G. The (minimal) Lagrangian of this theory, in string frame, up to the first order of α' is given by [126],

$$\begin{aligned} S_{10} \; &\simeq \; \frac{1}{2\kappa_{10}^2}\int_{M_{10}} \sqrt{-g}e^{-2\phi}\left[R + 4(\partial\phi)^2 - \frac{1}{2}H^2 + \frac{\alpha'}{4}trR^2 - \frac{\alpha'}{4}trF^2 + \dots\right] \\ &+ \; \text{(Fermionic Terms)} \end{aligned} \tag{1.2}$$

where the field H, which usually defined as $H = dB$, have to be modified when one couples the supergravity theory to the super Yang-Mill theory,

$$H = dB - \frac{\alpha'}{4}\left(\omega_3^{YM} - \omega_3^L\right), \tag{1.3}$$

where ω_3^{YM} is the Chern-Simons three form with the property $d\omega_3^{YM} \sim \mathrm{tr}(F \wedge F)$,

$$\omega_3^{YM} = tr\left(A \wedge F - \frac{1}{4}g_{YM}A \wedge A \wedge A\right). \tag{1.4}$$

Similarly, one can define a corresponding Chern-Simons term for the Lorentz spin connection.

Due to the existence of chiral fermions, this ten-dimensional field theory is generally anomalous. The source of the anomaly is coming from hexagonal loops of chiral fermions, where the six external legs can all be gravitons (gravitational anomaly), or all gauge fields (gauge anomaly), or a mixture of them (mixed anomaly). It turns out that if one adds a nonminimal term to the above Lagrangian of the form

$$\alpha'^2 \int_{M_{10}} B \wedge X_8, \tag{1.5}$$

where X_8 is a degree four polynomial in terms of F and R two-forms. Note that this term is the second order in α'. The addition of such terms can be justified if one considers the Heterotic supergravity as effective theory of Heterotic strings. After adding this term, it turns out that anomaly can be canceled only if the dimension of the gauge group G is 496. Also, the field strength must satisfy a certain group theoretical conditions [96]. The only solutions are $G = SO(32)$ and $G = E_8 \times E_8$.

One can also reach the same conclusion about G from the string worldsheet theory. From this view, in order to have an $N = 1$ superstring theory coupled with the Yang-Mills theory, one needs, in addition to the ten bosonic fields corresponding to the coordinates of M_{10}, to have sixteen right moving bosons over the worldsheet with values inside an even self-dual lattice. The number sixteen comes from the conformal anomaly on the worldsheet, and the properties of the sixteen-dimensional lattice are the result of the modular invariance of the torus partition functions [93].

Finally, to have an $N = 1$ theory, every correlation function $< \cdots >$ must be invariant under supersymmetry variations,

$$\delta_\epsilon < \cdots >= 0. \tag{1.6}$$

In particular, this requires the following equations of motions [96],

$$0 = \delta\psi_M = \frac{1}{\kappa} D_M \epsilon + \frac{1}{8\sqrt{3}\kappa} e^{-\phi} \left(\Gamma_M{}^{NPQ} - 9\delta^N{}_M \Gamma^{PQ} \right) \epsilon H_{NPQ}, \tag{1.7}$$

$$0 = \delta\chi^a = -\frac{1}{2\sqrt{2}g} e^{-\frac{\phi}{2}} \Gamma^{MN} F^a{}_{MN} \epsilon, \tag{1.8}$$

$$0 = \delta\lambda = -\frac{1}{\sqrt{2}} \left(\Gamma \cdot \partial\phi \right) \epsilon + \frac{1}{4\sqrt{6}\kappa} e^{-\phi} \Gamma^{MNP} \epsilon H_{MNP}, \tag{1.9}$$

where ϵ is 16-component, 10-dimensional (Majorana-Weyl) spinor parameterizing the supersymmetry transformations. For the moment, we assume $H = 0$, and let ϕ to be constant. So the equations of motion reduce to,

$$0 = \delta\psi_M = \frac{1}{\kappa} D_M \epsilon, \quad (1.10)$$

$$0 = \delta\chi^a = \Gamma^{MN} F^a{}_{MN} \epsilon, \quad (1.11)$$

$$0 = \delta\lambda. \quad (1.12)$$

Now, the purpose is to compactify the ten-dimensional space on a compact manifold M_d,

$$M_{10} = X_d \times M_{10-d}. \quad (1.13)$$

The equations of motion mentioned above imply M_d must be a complex and Kahler manifold. To see this, note that we can construct the Kahler and the almost complex structure as follows

$$J_{i\bar{j}} = \bar{\epsilon}\Gamma_{i\bar{j}}\epsilon, \quad (1.14)$$

$$J^i_j = g^{i\bar{i}} J_{\bar{i}j}. \quad (1.15)$$

Since ϵ is a covariantly constant spinor, $D_M \epsilon = 0$

$$dJ = D\bar{\epsilon}\Gamma\epsilon + \bar{\epsilon}\Gamma D\epsilon = 0. \quad (1.16)$$

In addition, one can show J^i_j satisfies the Nijenhuis tensor condition [96], so it must be a complex structure. Therefore X_d is a complex Kahler manifold, in particular the (real) dimension d must be even, $d = 2n$, and the structure group of the manifold reduces from $SO(d)$ to $U(n)$. Furthermore, since ϵ is covariantly constant, the holonomy group of the manifold must be contained in $SU(n)$ [48, 96]. This is satisfied if and only if the manifold M_{2n} has vanishing first Chern class i.e., it is Ricci flat. Such manifolds are called Calabi-Yau [159].

The equation $D_M\epsilon = 0$ also implies the effective non-compact spacetime M_{10-2n} must be flat. To see this choose M to be a direction tangent to M_{10-2n} (we denote them by greek letters μ, ν, etc.). Then

$$\Gamma^{\mu\nu}[D_\mu, D_\nu]\epsilon = 0 \Rightarrow R_{10-2n} = 0. \quad (1.17)$$

The only maximally symmetric flat space is the Minkowski space. To summarize, M_{2n} must be a Calabi-Yau manifold, and the effective spacetime must be Minkowski.

Finally, let us see what the second equation of motion implies about the gauge fields living over the Calabi-Yau n-fold X_n. One can use the definitions in (1.15) to show that the equation of motion $\Gamma^{MN}F_{MN}\epsilon = 0$ is equivalent to the following conditions (Hermitian Yang-Mills equation),

$$F_{ab} = F_{\bar{a}\bar{b}} = 0, \tag{1.18}$$

$$g^{a\bar{b}}F_{a\bar{b}} = 0. \tag{1.19}$$

These equations are tough to solve. However, we can identify F with the curvature of a vector bundle over X_n. The first equation means that the connection of the vector bundle is holomorphic, and the second equation one means the vector bundle is stable [85, 96, 149] (see Appendix A for definitions). These conditions (stability and holomorphicity) are of algebraic geometric nature rather than being differential equations, and they are easier to check.

The Bianchi identity of the field H (1.3), gives another (topological) constraint,

$$dH = \frac{\alpha'}{4}\left(trF \wedge F - trR \wedge R\right).$$

Taking the cohomology of both sides,

$$[trF \wedge F] - [trR \wedge R] = 0. \tag{1.20}$$

But $ch_2(V) = \frac{1}{4\pi^2}[trF \wedge F]$, so we can write the above equation as,

$$ch_2(V) - ch_2(X_n) = 0. \tag{1.21}$$

To summarize, to compactify the Heterotic string over Calabi-Yau n-fold X_n, we need to define a holomorphic stable vector bundle over X_n subject to the topological constraint (1.21) to cancel the anomaly. Note that in case of the $E_8 \times E_8$ Heterotic string the vector bundle V is a direct sum of two bundles $V_1 \oplus V_2$, one bundle for each E_8. The equation (1.21) becomes,

$$ch_2(V_1) + ch_2(V_2) - ch_2(X_n) = 0. \tag{1.22}$$

1.3 Heterotic M-theory

What has briefly explained in the last section was true only in the limit of the weak string coupling $g_s \to 0$. The strong coupling limit can be realized as an M-theory setup compactified on a manifold with a boundary [115]. More precisely, the idea is to consider M-theory on an eleven (real) dimensional manifold of the form $M_{11} = M_{10} \times S^1/\mathbb{Z}_2$. If S^1 is a circle of radius R, and parameterized with $x_{11} \in [0, 2\pi R)$, then $\mathbb{Z}_2$ acts as $x_{11} \to -x_{11}$. So there are two fixed points at $x_{11} = 0$ and $x_{11} = \pi R$.

The low energy limit of M-theory is an eleven-dimensional $N = 1$ supergravity theory. The supergravity multiplet consists of the graviton $g_{\mu\nu}$, three-form field $C_{\mu\nu\rho}$, and gravitino ψ_μ (which is a spin 3/2 Rarita-Schwinger field). There are also solitonic solutions to the field equations which correspond to a two-dimensional brane (M2-brane) and a five-dimensional brane (M5-brane), and the three form field couples to them electrically and magnetically respectively.

As usual, one needs to check whether the effective theory is anomaly free. Anomalies[4] can be traced back to loop diagrams with chiral fermions inside the loop, where the regularization methods can ruin the symmetries of the theory. So as long as M_{11} is smooth, the eleven-dimensional supergravity is anomaly free (there are not any chiral fermions in eleven dimensions). However for the case considered here, there are singularities over the fixed hyperplanes $x_{11} = 0$ and $x_{11} = \pi R$ (which are isomorphic to M_{10}), and Kaluza Klein reduction of ψ_μ induces massless chiral spin 3/2 fields in the ten-dimensional fixed hyperplanes (which are gravitinos in M_{10}). So, in principle, there are gravitational anomalies due to these reduced ten-dimensional degrees of freedom over $x_{11} = 0, \pi R$ [115].

To cancel the anomaly, one can use a generalized version of the Green-Schwarz mechanism [115]. In other words, note that the three-form field induces two-form B fields inside the fixed hyperplanes,[5]

$$\begin{aligned} B'_{\mu\nu} &:= C_{11\mu\nu}|_{x_{11}=0}, \\ B''_{\mu\nu} &:= C_{11\mu\nu}|_{x_{11}=\pi R}. \end{aligned} \tag{1.23}$$

The purpose is to use the nonminimal coupling of these two-form fields (as in ten-dimensional string theory), to cancel the gravitational anomaly. But, this is impossible unless we couple the induced ten-dimensional supergravity theories with super Yang-Mills theories i.e., we need to consider gauge degrees of freedom (vector multiplets) only over the fixed hyperplanes. One can think of these as something similar to the twisted sector states that show up in the ordinary orbifold compactification of string theory [115].

[4]We only consider local anomalies.

[5]Note that one cannot have a three-form inside M_{10} since a three-form with all legs inside M_{10} is odd under the orbifold action $x_{11} \to -x_{11}$ [115].

Again, similar to the ten-dimensional Heterotic string, the Green-Schwarz mechanism works only when the gauge group is 496 dimensional. Since the gauge degrees of freedom must be distributed evenly between the fixed hyperplanes, the gauge group in each hyperplane must be 248 dimensional. Therefore in each hyperplane, we must have a E_8 theory.

Furthermore, as in the perturbative theory, to have $N = 1$ supersymmetry in each hyperplane, the three form fields strengths of B', B'' must be modified. Consequently, the four-form field strength (G) of the three-form field in eleven dimensions must be modified,

$$\begin{aligned} G_{11\mu\nu\rho} = (dC)_{11\mu\nu\rho} \quad &+ \quad 4\sqrt{2}\pi(\frac{\kappa}{4\pi})^{2/3}\left[\delta(x_{11})\left(\omega^1_{3Y} - \frac{1}{2}\omega_{3L}\right)_{\mu\nu\rho}\right. \\ &+ \quad \left.\delta(x_{11} - \pi R)\left(\omega^2_{3Y} - \frac{1}{2}\omega_{3L}\right)_{\mu\nu\rho}\right], \end{aligned} \tag{1.24}$$

where ω_{3L} is the Lorentz Chern-Simons three-form of M_{10}, $\omega^{1,2}_{3Y}$ are the Chern-Simons three-forms which correspond to the gauge fields in the fixed hyperplanes, and κ is the eleven-dimensional Newton constant. The Bianchi identity becomes,

$$\begin{aligned} (dG)_{11\mu\nu\rho\gamma} \quad &= \quad 4\sqrt{2}\pi(\frac{\kappa}{4\pi})^{2/3}\left[-\delta(x_{11})\left(tr(F^1 \wedge F^1) - \frac{1}{2}tr(R \wedge R)\right)_{\mu\nu\rho\gamma}\right. \\ &- \quad \left.\delta(x_{11} - \pi R)\left(tr(F^2 \wedge F^2) - \frac{1}{2}tr(R \wedge R)\right)_{\mu\nu\rho\gamma}\right]. \end{aligned} \tag{1.25}$$

After integrating over x_{11} one get the same equation as (1.22). So it seems the M-theory set up considered above, has the same degrees of freedom as the Heterotic $E_8 \times E_8$ string theory. By comparing the effective theory of Heterotic $E_8 \times E_8$ string with the eleven-dimensional M-theory set up, one can identify R (the radius of S^1) with the Heterotic string coupling [157]. So when R is large, this M-theory set up, which we call Heterotic M-theory, can be identified with the strong coupling limit of the Heterotic $E_8 \times E_8$.

Before continuing to F-theory, since later we explore some new features of small instanton transitions, we should also review the effect of M5-branes in Heterotic M-theory. As mentioned before, M5-branes are magnetic sources of the three-form field.[6] This means if there are k M5-branes located at points $x_{11}^1, \dots, x_{11}^k$,[7] then the Bianchi identity must be modified to [126],

$$\begin{aligned}(dG)_{11\mu\nu\rho\gamma} &= 4\sqrt{2}\pi\left(\frac{\kappa}{4\pi}\right)^{2/3}\Bigg(-\delta\left(x_{11}\right)\left(tr\left(F^1\wedge F^1\right)-\frac{1}{2}tr\left(R\wedge R\right)\right)_{\mu\nu\rho\gamma}\\ &- \delta\left(x_{11}-\pi R\right)\left(tr\left(F^2\wedge F^2\right)-\frac{1}{2}tr\left(R\wedge R\right)\right)_{\mu\nu\rho\gamma}\Bigg)\\ &+ \frac{1}{2}\sum_{i=1}^{k}\left[J^i_{\mu\nu\rho\gamma}\left(\delta\left(x_{11}-x_{11}^i\right)+\delta\left(x_{11}+x_{11}^i\right)\right)\right], \end{aligned} \tag{1.26}$$

where J^i's are four-forms dual to (complex) codimension two cycles in M_{10} which we denote as $[W_i]$. This means the anomaly condition (1.22) becomes

$$c_2(V_1)+c_2(V_2)+\sum_{i=1}^{k}[W_i]-c_2(X)=0. \tag{1.27}$$

In addition, the M2-branes that are stretched between these M5-branes, or between M5-branes and fixed hyperplanes at $x_{11}=0,\pi R$, appear as strings in the worldvolume of M5-branes. Since the tensions of these strings are proportional to the distance of these M5-branes (or between M5-branes and the fixed hyperplanes), when these distances approach zero, the six-dimensional worldvolume theory will be a $N=(1,0)$ SCFT which contains non-perturbative tensionless strings [107].

1.4 F-theory compactification

There are many ways to "define" F-theory. As mentioned above, it can be defined as a limit of M-theory or as the strong coupling limit of type IIB string theory. Here we don't go through all the details; instead, we briefly mention the basics and then focus on the properties that are most relevant to the duality between Heterotic and F-theory.

[6] We should also mention in the effective Heterotic string theory, these M5-branes correspond to magnetic sources of the two-form field B. Such sources are called NS5 branes in the ten-dimensional string theory context.

[7] M5-branes cannot wrap around $S^1/\mathbb{Z}_2$, because it is inconsistent with the symmetry of the three-form filed under the orbifold symmetry.

1.4.1 F-theory as the strong limit of type IIB string theory

We begin by recalling the massless bosonic field content of type IIB:

$$\begin{array}{l}\text{Supergravity:} \quad B_{\mu\nu}, g_{\mu\nu}, \phi, \\ \text{RR fields:} \quad C_0, C_2, C_4^+,\end{array}$$

where B is the Kalb-Ramond 2-form, g is the metric, and ϕ is the dilaton. On the other hand, C_0, C_2 and C_4^+ are the Ramond-Ramond scalar, 2-form and the self dual four form respectively.[8] Due to supergravity, there are also fermionic superpartners of these fields, which include two gravitinos and two dilatinos, which we don't mention here. Generally, there are D1 and D3 branes which couple electrically to C_2 and C_4^+ respectively, and D5 and D7 branes which couple magnetically to C_2 and C_0 respectively.[9]

The action of these effective fields (in Einstein frame) is given by [138, 144],

$$\begin{aligned} S_{IIB} &= \frac{1}{2\kappa_{10}^2}\int d^{10}x(-g)^{1/2}\left(R - \frac{\partial_\mu\bar{\tau}\partial^\mu\tau}{2(Im\tau)^2} - \frac{\mathcal{M}_{ij}}{2}F_3^i\cdot F_3^j - \frac{1}{4}\tilde{F}_5\wedge *\tilde{F}_5\right) \\ &\quad -\frac{\epsilon_{ij}}{8\kappa_{10}^2}\int C_4\wedge F_3^i\wedge F_3^j, \end{aligned} \tag{1.28}$$

where

$$\tilde{F}_5 = F_5 - \frac{1}{2}C_2\wedge H_3 + \frac{1}{2}B_2\wedge F_3, \tag{1.29}$$

$$\mathcal{M} = \frac{1}{Im\tau}\begin{pmatrix} |\tau|^2 & -Re\tau \\ -Re\tau & 1\end{pmatrix}, \tag{1.30}$$

$$\tau = C_0 + ie^{-\Phi}, \quad F_3^1 = F_3, \quad F_3^1 = H_3. \tag{1.31}$$

$$\tag{1.32}$$

The important property of this action is the manifest $SL(2,\mathbb{Z})$ symmetry [10] defined

[8] C_4^+ has self dual field strength, i.e. $*dC_4^+ = dC_4^+$.

[9] In case that there are D9 branes, one needs a stack of $O9$-planes to cancel the charge of the D9 branes (i.e., tadpole cancellation), this leads to the type I string theory. We are not going to deal with such situations in the following.

[10] The symmetry of this action is $SL(2,\mathbb{R})$. However, the symmetry of the full non-perturbative type IIB string theory is $SL(2,\mathbb{Z})$. To extend this symmetry to the fermionic fields, we need to lift this to $MP(2,\mathbb{Z})$ [136].

in the following way,

$$\begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} C_2 \\ B_2 \end{pmatrix}, \tag{1.33}$$

$$\tau \to \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}). \tag{1.34}$$

The rest of the bosonic fields remain invariant.

We briefly list some of the important consequences of this symmetry.

1. The string coupling is defined as $g_s = e^{\Phi}$. This means under the transformation $\tau \to \frac{a\tau+b}{c\tau+d}$ the string coupling can change generally.

2. The two forms B_2 and C_2 couple electrically to the fundamental and D1-brane, respectively. Therefore the $SL(2, \mathbb{Z})$ symmetry mixes the two forms into a linear combination $pB_2 + qC_2$, and the corresponding string that couples electrically to this mixed two form is neither the fundamental string nor the D1-brane. Instead, it is a general (p, q)-string, which is a BPS string for coprime (p, q) [158].

3. The two form $pB_2 + qC_2$ and the scalar τ also magnetically couple to (p, q) 5-branes and 7-branes respectively. Such branes induce a $SL(2, \mathbb{Z})$ monodromy on strings, and it can be shown that a (p, q) string is invariant under the monodromy of (p, q)-branes [154].

Now we arrive at the main point, F-theory. F-theory is a way to compactify the type IIB theory on a sphere S^2 to derive an effective eight-dimensional $N = 1$ theory. This means we demand the ten-dimensional spacetime manifold for type IIB string must be a product,

$$M_{10} = M_2 \times \mathbb{R}^{1,7}, \qquad M_2 \simeq S^2. \tag{1.35}$$

Note that without turning on the two forms, self-dual four forms and/or scalars, i.e., without including the branes, the metric of the manifold M_2 has to be Ricci flat. But S^2 is not. Hence, one must include (p, q)-branes. In F-theory, the two forms fields B_2 and C_2 are frozen to zero on S^2, to get a $SL(2, \mathbb{Z})$ invariant solution. Then the only other option is to include the (p, q) 7-branes. Moreover, the Poincare symmetry in $\mathbb{R}^{1,7}$ should remain unbroken, so the 7-branes must fill the noncompact spacetime i.e., they are point-like sources from the view of the internal space S^2. It can be shown that τ satisfies the following equation of motion [97],

$$\bar{\partial}\partial\tau + \frac{\bar{\partial}\tau\partial\tau}{\bar{\tau}-\tau} = 0, \tag{1.36}$$

and the most general solution of this is of the form,

$$j(\tau(z)) = \frac{P(z)}{Q(z)}, \tag{1.37}$$

where z is the complex coordinate on S^2, P and Q are holomorphic functions of z, and j is the Jacobian function that maps the fundamental domain of the $SL(2,\mathbb{Z})$ to the sphere. For example in the case $P \sim z$ and $Q = 1$, one can see in the limit $|z| \to 0$,

$$\tau(z) \to \frac{lnz}{2\pi i}. \tag{1.38}$$

In particular, as $arg(z) \to arg(z) + 2\pi$, the axiodilaton field τ changes as $\tau \to \tau + 1$. By comparing with the general $SL(2,\mathbb{Z})$ monodromy of τ, we see that the source sitting at $z = 0$, must be a $(1,0)$ 7-brane, in another words a D7-brane.

It is also shown in [97] that, far away from $z = 0$, the metric takes the form $(\bar{z}z)^{-1/12}d\bar{z}dz$, and hence there will be a deficit angle [11] $-\frac{\pi}{6}$. To have a compact solution, the total deficit angle must be 4π. So one needs to include 24 separate 7-branes.

1.4.2 Elliptic fibration

Since the axio-dilation field τ has all of the properties of the complex structure of a torus, the idea of F-theory [151] to study all such solutions is to lift the internal space S^2 to an elliptic fibration over S^2. In other words, we consider a twelve-dimensional theory compactified over a four-dimensional (two complex dimension) space X_2 which admits an elliptic fibration, i.e. a map like

$$\pi : X_2 \longrightarrow S^2 \simeq \mathbb{CP}^1, \tag{1.39}$$

[11] To see this, [97] change the variable to $v = Z^{1-1/12}$, so the metric will be flat in v coordinate. But as $z \to e^{2\pi}z$, v transforms as $v \to e^{-\frac{\pi}{6}}v$. So there is a deficit angle $-\pi/6$ for each D7-brane.

where the fibers of π are elliptic curves (torus), and the complex structure of these curves are identified with the field τ.

There are many ways to define an elliptically fibered manifold, but one of them, the so-called Weierstrass model, is particularly important. Since any elliptically fibered manifold with a section is birationally equivalent to the Weierstrass model. So we briefly explain it here.

A Weierstrass model defined as a hypersurface in an ambient space as,

$$A := \mathbb{P}\left(\mathcal{L}^3 \oplus \mathcal{L}^2 \oplus \mathcal{O}_{P^1}\right), \tag{1.40}$$

where $\mathcal{L}$ is a line bundle over $\mathbb{P}^1 \simeq S^2$, and therefore A is a fibration of the weighted homogeneous projective space $\mathbb{WCP}^{3,2,1}$.[12] The coordinates of the fiber are given by (y, x, z), and the elliptic fibration is defined by,

$$y^2 + x^3 + f(u,v)xz^4 + g(u,v)z^6 = 0, \tag{1.41}$$

where (u, v) are coordinates of the base manifold $\mathbb{P}^1$, and the polynomials f and g are global sections of $H^0(\mathbb{P}^1, \mathcal{L}^4)$ and $H^0(\mathbb{P}^1, \mathcal{L}^6)$. There is a holomorphic one form Ω_1 associated to the elliptic curves defined in this way,

$$\Omega_1 = \frac{dx}{y}, \tag{1.42}$$

and the complex structure of the fibers is defined as,

$$\tau = \frac{\int_A \Omega_1}{\int_B \Omega_1}, \tag{1.43}$$

where A and B are the independent 1-cycles of the torus. Generically the elliptic fibers are smooth, and the singular locus can be found by the discriminant of the defining (degree three) equation,

$$\Delta := 4f^3 + 27g^2 = 0. \tag{1.44}$$

Finally one can show,

$$j(\tau(u,v)) \sim \frac{f^3}{\Delta}. \tag{1.45}$$

[12] We denote will this space as $\mathbb{P}^{321}$ in the rest of this dissertation.

We see that over the singularity locus $\Delta = 0$, the complex structure τ goes to infinity. In addition, assuming the zeros of Δ are of order one, one can see that locally around one of the zeros, τ behaves in the same way as the axio-dilaton fields around a D7-brane mentioned before. Hence we must identify the zeros of the discriminant $\Delta = 0$ with the 7-branes, and since we must have 24 7-branes, the degree of Δ must be 24 too. So f must be degree 8, and g must be degree 12. With these properties, one can easily show that X_2 must be a K3 i.e., a complex, Kähler surface with holonomy inside $SU(2)$.

However, since the total monodromy due to the 7-branes are zero, we see even though every 7-brane locally behaves as a D7-brane, globally, not all of them can be D7-brane, rather they should be general (p, q) 7-branes. Therefore n reaches a condensate state of 7-branes [151].

Since every singular fiber of X_2 corresponds to a 7-brane, one may expect that the effective eight-dimensional gauge theory must be $U(1)^{24}$. But note that there are scalars on the world-volume of the 7-branes which are charged under the $U(1)$ in the world-volume, and their vev controls the fluctuations of the 7-brane in the normal directions. The global reparameterization symmetry of the sphere S^2 is $SL(2, \mathbb{C})$, and using this one can fix the position of three 7-branes i.e., fix the vev of six real scalar field, which Higgs, the gauge group down to $U(1)^{18}$. In addition, the Kaluza-Klein reduction of the metric (remember we turned off the two form fields inside the sphere S^2) gives another $U(1)^2$ which can be interpreted as the overall (rigid) shift of the 7-branes center of mass position inside the S^2. Consequently, the effective gauge group is [13] $U(1)^{20}$.

One may expect that, as in the perturbative D-brane setups, in the limit that several 7-brane coincide, the effective gauge group enhances to a non-abelian group. Indeed, due to the existence of (p, q) BPS strings, one can show it is possible to have setups of coincident 7-brane stacks, that their eight-dimensional world-volume theory corresponds to $N = 1$ super Yang-Mills theory with general ADE group[14] [92]. One can also compute the monodromy of strings around such stack of 7-branes and compare it with the monodromy of the vanishing 1-cycles of the elliptic fibers [63], and conclude that a stack of 7-branes with an ADE gauge group corresponds to a K3 manifold with ADE singularity.[15] This means by classifying the possible singularities

[13]The reduction of $U(1)^{24}$ to $U(1)^{18}$ is the sign of the fact that we have a condensate state of 7-branes, rather than 24 independent 7-branes [151].

[14]Note that in perturbative type IIB theory one can only get $SO(n)$, $Sp(n)$ and $SU(n)$ gauge groups, while in F-theory we can every ADE gauge group.

[15]By ADE singularity we mean a type of singularity such that after blowing up (replacing the

ord(f)	ord(g)	ord(Δ)	Singularity	Gauge Group
≥ 0	≥ 0	0	none	none
0	0	n	A_{n-1}	$SU(n)$
≥ 1	1	2	none	none
1	≥ 2	3	A_1	$SU(2)$
≥ 2	2	4	A_2	$SU(3)$
2	3	$n+6$	D_{n+4}	$SO(2n+8)$
≥ 2	≥ 3	6	D_4	$SO(8)$
≥ 3	4	8	E_6	E_6
≥ 3	5	9	E_7	E_7
≥ 4	5	10	E_8	E_8

Table 1.1: Kodaira classification of singularities of elliptically fibered $K3$.

of K3 surfaces we can classify all of the possible (from type IIB theories) eight-dimensional $N = 1$ supergravity theories with ADE gauge groups! The singularities of K3 were classified by Kodaira in the 1960s [119, 120].

1.4.3 Compactifying to lower dimensions.

It is possible to compactify F-theory to lower dimensions either by fibering the K3 over another manifold (this is the case that generally relates to Heterotic string theory) or by compactifying over the more general elliptically fibered manifold. In either case, the manifold on which we compactify F-theory, must be Calabi-Yau and elliptically fibered (or at least genus one fibration),[16]

$$\pi : X_{n+1} \longrightarrow B_n, \tag{1.46}$$

where n is the (complex) dimension of the base. The reason that X_{n+1} must be a Calabi-Yau manifold can be seen either by the supersymmetry requirement of the dual M-theory (which is explained later) or by studying the anomaly as before [141]. Again similar to the eight dimensional case, elliptic fibers becomes singular over a complex codimension one locus (divisors) in the base, and the singularity type

singularity with 2-cycles), the intersection form of the cycles becomes negative of the Cartan matrix of ADE Lie Groups.

[16]It is not necessary to require the elliptic fibration to have a section, we can rather work with genus one fibration. By section we mean an inclusion $\sigma : B_n \hookrightarrow X_{n+1}$, such that $\pi \circ \sigma = id_{B_n}$.

determines the gauge group in the effective theory. However, there are some novel features that can appear in lower dimensions.

- First of all, in this case the 7-branes wrap around the divisors in B_n, and gauge fields can live over divisors that can reduce the effective gauge group to something "smaller." As is explained later, information about these fields is necessary to "count" the zero modes [33, 37]. We will briefly explain this in the following subsection.

- Second, the divisors can intersect, and over the intersection locus (i.e., over the intersection locus of 7-branes) matter fields live as in the case of intersection D-branes [33, 35]. Again the information about gauge fields living over the divisors is necessary to count the number of various matter fields in the effective theory. Also, if $n \geq 3$, it is possible to have triple self-intersections in the discriminant, and such things correspond to Yukawa couplings in the effective theory.

- Third, it is possible to have non-trivial Mordel-Weil group, genus one fibration, or torsional section,[17] which respectively corresponds to extra $U(1)$ gauge groups, discrete groups, or dividing the gauge with a discrete group in the effective theory [28, 44, 98].

- Fourth, it is also possible to have monodromy around the singularity locus that can reduce the effective gauge group to some non-simply laced gauge group [37].

1.4.4 Higgs bundle

In this subsection, we briefly describe the fields living over the divisors following [33], since they are more relevant to the main topic of this dissertation. To make the connection between the geometrical features of the Calabi-Yau compactification of F-theory and the (lower-dimensional) effective field theories inside the world-volume of 7-branes, [33] considered a local Calabi-Yau compactification. In other words, it is possible to first consider a compact elliptically fibered Calabi-Yau X_{n+1} ($n \geq 2$) with discriminant locus $\Delta \subset B_n$ which the 7-branes wrap. The purpose here is to "zoom" on these 7-branes, such that we only consider a $\mathbb{C}^2$ patch of the normal directions. The Calabi-Yau condition and the ADE singularity, require the $\mathbb{C}^2$ fiber over Δ to be

[17] A torsional section is a divisor σ_t in X_{n+1} such that it intersects every fiber once. However, it is a torsion element of the Mordel-Weil group.

a non-compact K3 with ADE singularity [33] (note that Δ is still a compact, Kahler $n-1$ dimensional manifold).

A_n	$y^2 = x^2 + z^{n+1}$
D_n	$y^2 = x^2 z + z^{n-1}$
E_6	$y^2 = x^3 + z^4$
E_7	$y^2 = x^3 + xz^3$
E_8	$y^2 = x^3 + z^5$

Table 1.2: Algebraic equations of non-compact $K3$ with ADE singularities.

To get a non-compact Calabi-Yau manifold, one can fiber the non-compact $K3$ in table 1.2 over a complex Kahler manifold S. Demanding that the total space be Calabi-Yau, requires the affine coordinates (x, y, z) to be fibers of line bundles (K_S^a, K_S^b, K_S^c) for particular rational numbers (a, b, c). For more details, look at [33] and references there. Since X_{n+1} is non-compact (i.e., infinite volume), the gravity is decoupled in the effective theory. But Δ is still compact, and therefore there are still dynamical modes inside the 7-branes wrapping on Δ.

Let us begin by recalling the massless modes inside a 7-branes worldvolume ($N = 2$ super Yang-Mills theory on $\mathbb{R}^{1,7}$) first. The symmetries are $SO(1, 7)$ Lorentz group, $U(1)$ R-symmetry, and gauge symmetry. The super Yang-Mills multiplet consists of a gauge field A, two complex scalar fields ϕ, $\bar{\phi}$ with opposite $U(1)$ charge ± 1 (corresponding to the fluctuations of the 7-brane in the normal directions to the worldvolume), and finally two adjoint spinors $\psi_\pm$ with opposite chirality and opposite $U(1)$ charges $\pm\frac{1}{2}$. Similarly, the supersymmetry generators $Q_\pm$ transform in the same way as the spinors. Upon compactification of this eight-dimensional space over the complex Kahler surface $S := \Delta = 0$, the Lorentz group breaks as $SO(1, 7) \to SO(1, 3) \otimes SO(4)$, where $SO(4) \simeq SU(2) \otimes \mathrm{SU}(2)$ is the structure group of S viewed as an orientable, Riemannian manifold. The massless modes in the effective four-dimensional theory are shown in table 1.3.

Field/SUSY generator	$U(1)$ R-Charge	Type	Rep. under $SO(4) \times SO(1,3)$
A_μ	0	1-form	$[(2,2),(1,1)] \oplus [(1,1),(2,2)]$
ϕ	-1	scalar	$[(1,1),(1,1)]$
$\bar{\phi}$	1	scalar	$[(1,1),(1,1)]$
ψ_+	$\frac{1}{2}$	Spinor	$[(2,1),(2,1)] \oplus [(1,2),(1,2)]$
ψ_-	-$\frac{1}{2}$	Spinor	$[(2,1),(1,2)] \oplus [(1,2),(2,1)]$
Q_+	$\frac{1}{2}$	Spinor	$[(2,1),(2,1)] \oplus [(1,2),(1,2)]$
Q_-	$-\frac{1}{2}$	Spinor	$[(2,1),(1,2)] \oplus [(1,2),(2,1)]$

Table 1.3: Fields and the supercharges of the worldvolume theory of 7-branes.

The idea in [33] was to preserve the effective supersymmetry in four dimensions, and we must twist the structure group of S such that only one of the supersymmetry generators are scalars over S. This can be done by noting that since S is a Kähler manifold, the structure group reduces to $U(2)$. Hence one can embed the $U(1)_R$ inside the $U(2)$.

$$J_{top} = J \pm 2R, \tag{1.47}$$

where R is the generator of the R-symmetry and J is the center of $U(2)$. One can choose either plus or minus sign for twisting. Under the reduction of $SO(4) \simeq SU(2) \times SU(2)$ to $U(2)$ the representations $(2,1)$ and $(1,2)$ break as 2_0 and $1_+ \oplus 1_-$ respectively. Therefore after twisting, fields transform differently under the "new" Lorentz transformation. We summarize this in the following table (as in [33] we choose $J_{top} = J + 2R$),

Field/SUSY generator	Before twist	After Twist	Section
A_μ, $[(2,2),(1,1)]$	$2_{+1} \otimes (1,1) \oplus 2_{-1} \otimes (1,1)$	$2_{+1} \otimes (1,1) \oplus 2_{-1} \otimes (1,1)$	$\bar{\Omega}^1(S) \otimes ad(G) \oplus \Omega^1(S) \otimes ad(G)$, $A_{\bar{m}}$, A_m
A_μ, $[(1,1),(2,2)]$	$1 \otimes (2,2)$	$1 \otimes (2,2)$	$ad(G)$, A_μ
ϕ, $[(1,1),(1,1)]$	$1 \otimes (1,1)$	$1_{-2} \otimes (1,1)$	$\Omega^2(S) \otimes ad(G)$, ϕ_{mn}
$\bar{\phi}$, $[(1,1),(1,1)]$	$1 \otimes (1,1)$	$1_{+2} \otimes (1,1)$	$\bar{\Omega}^2(S) \otimes ad(G)$, $\bar{\phi}_{\bar{m}\bar{n}}$
ψ_+, $[(2,1),(2,1)]$	$2_0 \otimes (2,1)$	$2_{+1} \otimes (2,1)$	$\bar{\Omega}^1(S) \otimes ad(G)$, $\psi_{\alpha\bar{m}}$
ψ_+, $[(1,2),(1,2)]$	$(1_{+1} \oplus 1_{-1}) \otimes (1,2)$	$(1_{+2} \oplus 1_0) \otimes (1,2)$	$(\bar{\Omega}^2(S) \oplus 1) \otimes ad(G)$, $\psi_{\dot{\alpha}\bar{m}\bar{n}}$, $\psi_{\dot{\alpha}}$
ψ_-, $[(2,1),(1,2)]$	$2_0 \otimes (1,2)$	$2_{-1} \otimes (1,2)$	$\Omega^1(S) \otimes ad(G)$, $\psi_{\dot{\alpha}m}$
ψ_-, $[(1,2),(2,1)]$	$(1_+ \oplus 1_-) \otimes (2,1)$	$(1_0 \oplus 1_{-2}) \otimes (2,1)$	$(1 \oplus \Omega^2(S)) \otimes ad(G)$, ψ_α, $\psi_{\alpha mn}$

Table 1.4: Representation of the worldsvolume fields before and after the "topological" twist.

In particular, the fields ϕ and $\bar{\phi}$ transform as adjoint valued two forms i.e., they are sections of the bundles $K_S \otimes ad(G)$ and $\bar{K}_S \otimes ad(G)$ respectively (where

K_S is the canonical bundle of S, and G is the gauge group living on S). Moreover, by comparing the various Casimir invariants of ϕ, and the deformations of the ADE singularities, one confirms [33] that ϕ indeed corresponds to the fluctuations of the 7-brane inside B_n. This was, of course, a well-known feature of perturbative D7-branes.

Finally by considering the BPS equations of the eight-dimensional world-volume theory over $S \times \mathbb{R}^{1,3}$, or equivalently the F-term and the D-term equations of the effective four-dimensional theory, one finds the equation of motion for the bosons in the super Yang-Mills multiplet [33], the F-term equation is,

$$F^{(0,2)} = F^{(2,0)} = 0, \tag{1.48}$$

$$\bar{\partial}_A \phi = \partial_A \bar{\phi} = 0, \tag{1.49}$$

where ∂_A is the covariant derivative with connection A over S. The D-term equations are,[18]

$$\omega \wedge F^{(1,1)} + \frac{i}{2}[\phi, \bar{\phi}] = 0, \tag{1.50}$$

where ω is the Kahler class of S. In the mathematics literature, the doublet (E, ϕ), where E is a vector bundle with curvature given by the $(1,1)$ form F, is called the Higgs bundle, and the D-term and F-term equations above are analog of the Hitchin system equation for surfaces [109]. Indeed the F-term equations correspond to the fact that vector bundle E is a holomorphic bundle, and the D-term equation means E is a stable bundle [85, 149]. To study the solutions of these equations, [110] introduced the idea of "abelianization" or the spectral cover construction. Briefly, the idea is as follows. Since ϕ transforms as $ad(E)$, one can view the Higgs field as an element of $H^2(S, E^* \otimes E \otimes ad(E))$ i.e. a $rank(E) \times rank(E)$ matrix. Then one defines the spectral cover S_p as,

$$S_p = \{det\,(\lambda I - \phi) = 0\}, \tag{1.51}$$

where $\lambda = 0$ is the zero section of the total space $Tot(K_S)$. Note that $\Pi : S_p \to S$ is a finite morphism of degree $rank(E)$ i.e. it is a cover of S with $rank(E)$ number of sheets. Now consider a line bundle $\mathcal{L}$ on S_p, then it is shown that the pushforward $\Pi_* \mathcal{L}$ is a vector bundle of $rank(E)$ over the surface S, and it is also both holomorphic

[18] here it is assumed that the volume of S is large. Otherwise, these equations receive non-negligible quantum corrections. Also, in the case of intersection 7-branes, these equations have a source term on the right-hand side.

and stable, so it satisfies the Hitchin equations. Indeed, one can show $\Pi_*\mathcal{L} \simeq E \otimes K_S$. So it means there is a correspondence,

$$(E, \phi) \leftrightarrow (S_p, \mathcal{L}). \tag{1.52}$$

We summarize the main physical significance of these Higgs bundles,

- Given a solution (E, ϕ), one can find the particle spectrum of the effective theory. For example from the table above, one can see the (fermionic) spectrum of the effective theory is given by,

$$H^0(S, E) \oplus H^1(S, E) \oplus H^2(S, E), \tag{1.53}$$

  which correspond to $\phi_{\dot{\alpha}}$, $\psi_{\alpha\bar{m}}$ and $\psi_{\dot{\alpha}\bar{m}\bar{n}}$ respectively.

- In the case of intersecting brane models, the discussion above generalizes, and one needs explicit information about the flux induced over the intersection locus to find the matter spectrum of the effective theory, similarly, for Yukawa couplings (intersection of three branes).

- Suppose the singularity of the Calabi-Yau geometry corresponds to a gauge group of G_0. This means the effective gauge symmetry is G_0 *only if the background gauge field in the 7-brane stack is zero*, i.e., $F = 0$ in the Hitchin equations above. Otherwise, the unbroken gauge group H is the commutant of the group G inside G_0 [14].

One of the main points of this dissertation is to build the tools necessary to study more general examples (relative to the current examples) of $(S_p, \mathcal{L})$. To do that, we use the duality between Heterotic string theory and F-theory, and spectral cover construction of vector bundles.

1.4.5 F-theory as the singular limit of M-theory

Here we briefly review the M-theory "derivation" of F-theory. We don't use this picture in this dissertation, but we briefly review it here since it is somewhat more intuitive. First consider F-theory on Calabi-Yau $\pi : X_{n+1} \to B_n$ as before. This corresponds to the strong coupling limit of the type IIB on B_n. Then after compactifying over another circle S^1, so it is possible to apply T-duality over this circle [151],

$$\begin{aligned} \text{F-theory on } X_{n+1} \times S^1 \leftrightarrow \text{type IIB on } B_n \times S^1 \quad &\leftrightarrow \quad \text{type IIA on } B_n \times \tilde{S}^1 \\ R \quad &\leftrightarrow \quad \frac{\alpha'}{R'} \end{aligned} \tag{1.54}$$

where R and R' are the radii of the circles. On the other hand, M-theory on a circle corresponds to the strong coupling limit of the type IIA string theory, where the string coupling in type IIA is matched with the radius of the M-theory circle, therefore

$$\text{M-theory on } B_n \times \tilde{S}^1 \times S^1_M \sim B_n \times T^2 \leftrightarrow \text{F-theory on } X_{n+1} \times S^1.$$

In particular, one can generalize $B_n \times T^2$ to an elliptic fibration, and it is shown [144] that the complex structure of the elliptic fiber on M-theory equals to the complex structure of the elliptic fiber of X_{n+1}. However, note that in the decompactification limit $R \to \infty$, the volume of T^2 in M-theory must vanish. Therefore one can consider F-theory as the limit of the M-theory where the volume of the elliptic fibers vanish.[19]

Now, let us see where the gauge groups in the effective theory are coming from. Recall first that M-theory contains a three form C_3, which couples magnetically to M5-branes and electrically couples to M2-branes. Let us assume that M-theory is compactified over a smooth Calabi-Yau $\hat{X}_{n+1}$. On the other hand, by Poincare duality for every divisor (i.e., a complex codimension one cycle) D, there is a corresponding holomorphic two form ω. Then one can expand the three form C_3 relative to ω (see [154] and references there),

$$C_3 = A \wedge \omega, \tag{1.55}$$

where A is a one form in the effective theory.[20] On the other hand, since C_3 couples electrically with the M_2-brane, one gets the following coupling in the effective theory,

$$\int_{WV(M2)} C_3 \longrightarrow \int_C \omega \times \int_{WL(P)} A \sim q \int_{WL(P)} A, \tag{1.56}$$

[19]Indeed, the volume of the fiber is not a degree of freedom in F-theory.

[20]Equation of motion of C_3 is (in the absence of $M2$-brane) $\Delta_{11} C_3 = (\Delta_{n+1} + \Delta_{9-2n}) C_3 = 0$. Since ω is a holomorphic form, this means $\partial^2 A = 0$. So A is a $U(1)$ gauge field.

where $WV(M2)$ is just the world-volume of the M2-brane, C is a curve dual to D in $\hat{X}_{n+1}$[21] where M2-brane wraps around it. In the effective theory, the wrapped M2-brane appears as a particle (P) moving one a world-line $WL(P)$. So the coupling above shows the corresponding particle in the effective theory is charged under $U(1)$ with charge $q = \int_C \omega$. In addition, the mass of this particle is proportional to the volume $Vol(C)$.

Now, note that not every gauge field in M-theory corresponds to a gauge field in F-theory. As mentioned, the effective theory of M-theory is related to the Kaluza-Klein reduction of the effective theory of F-theory. Therefore the anti-self-dual two forms in F-theory, that are due to the reduction of the self-dual four forms over the divisors in B_n can give $U(1)$ gauge fields In M-theory after Kaluza Klein reduction. Since the self-dual four forums in the IIB theory is related to C_3 in type IIA (and therefore in M-theory), we can see these gauge fields can also give rise by reducing the C_3 on the base divisors. On the other hand, the $U(1)$ gauge fields coming from the reducing the C_3 over the zero-section of $\hat{X}_{n+1}$, corresponds to Kaluza-Klein reduction of the metric over S^1 [154]. In conclusion, only the divisors that don't intersect the zero section, and are not the pullback of a divisor in the base, give rise to gauge fields in F-theory.

After resolving ADE singularities, the exceptional divisors satisfy the conditions of the last paragraphs, and the effective gauge fields correspond to the Cartan subalgebra of the ADE Lie algebra. But as mentioned in the type IIB picture, the effective gauge group must be a non-abelian group dictated by the type of the singularity. So, where are the non-Cartan generators of the gauge groups. The answer is since the manifold $\hat{X}_{n+1}$ is a Calabi-Yau, for every divisor that gives a $U(1)$ in the effective field theory, there is a dual rational curve (which is a $\mathbb{P}^1$) in the fiber, and therefore the M2-branes can wrap around them. One can check (see [154] and the references there) the charge of the corresponding states in the effective theory is the same as the root lattice of the ADE Lie algebras. However, since $\hat{X}_{n+1}$ is smooth, the volume of these rational curves are non-zero, so the states corresponding to the non-Cartan generators are massive in the M-theory limit. On the other hand, since F-theory is the limit where the volume of the fibers vanishes, these states become massless, and the gauge group enhances to the full non-abelian group dictated by the singularity.

[21] Since $\hat{X}_{n+1}$ is Calabi-Yau, the space of the 2-cycles and complex codimension one cycles are isomorphic.

1.5 Heterotic/F-theory duality

In this section we briefly review the duality between Heterotic $E_8 \times E_8$ string theory and F-theory.

1.5.1 Basic duality

In [151], the basic duality between Heterotic a torus T^2 and F-theory on an elliptically fibered K3 was conjectured. Later in [131, 132] for six-dimensional theory, and in [89] for general situations, the duality was confirmed. Here we review [151].

First, consider the Heterotic string on T^2. By turning on all of the Wilson lines over the T^2, the Lie algebra breaks into its Cartan subalgebra i.e., $U(1)^{16}$. On the other hand, recall, we also have the two forms and metrics. The Kaluza Klein reduction of these fields gives four other $U(1)$'s (two from metric and two from the two-form). So the eight-dimensional effective gauge group is $U(1)^{20}$. Now recall that this is precisely the effective gauge group of F-theory on K3. So this is the first sign of the duality.

Next, consider moduli space of a worldsheet theory (corresponding to Heterotic string) on a torus. In this case, one gets a four-dimensional momentum lattice $\Gamma_{2,2}$ [134, 138]. There is also a 16 dimensional negative definite, even, and self-dual lattice related to the root lattice of the gauge group. These two combine (by T-duality [39]) into a $\Gamma_{2,18}$ lattice. Hence the moduli space of this theory becomes,

$$O(\Gamma_{2,18}) \backslash O(2,18)/O(2) \times O(18). \tag{1.57}$$

On the other hand, it is well known [26] that the complex structure moduli of a smooth, elliptically fibered K3, is given by the same space. Since the Kahler structure and complex structure of a K3 are not independent, and the complex structure controls the position of 7-branes in F-theory (and hence the effective gauge theory), the moduli space of F-theory on a K3 is again the same space above. This is the second sign of duality.

Finally, there is one more (real) parameter left in Heterotic, and that one is the string coupling g_s. Note that on the F-theory side also only one parameter left, which is the volume of the base of the $K3$ (remember that the volume of the elliptic fibers vanishes in F-theory). So we must identify the volume of the base $\mathbb{P}^1$ of the K3 with the dual Heterotic string coupling constant.

1.5.2 Stable degeneration and Heterotic/F-theory duality in eight dimensions

In this section, we restrict ourselves to the $E_8 \times E_8$ gauge theory, and try to elaborate more on the basic duality of the last subsection by briefly reviewing the methods in [30, 68, 89]. The advantage of this method is that it makes the connection with Heterotic M-theory clear. The duality for general elliptically fibered Calabi-Yau's is left for the next subsection.

The purpose here is to consider a Weierstrass elliptically fibered $K3 \xrightarrow{\pi} \mathbb{P}^1$,

$$y^2 + x^3 + f(u,v)xz^4 + g(u,v)z^6 = 0,$$

and fiber this K3 over a disc (a complex affine line) parameterized by a complex number t to get family of K3 manifolds χ,

$$\begin{array}{ccc} K3 & \longrightarrow & \chi \\ & & \downarrow \Pi \\ & & D \end{array} \tag{1.58}$$

One can demand χ to be a semistable degeneration i.e., the fiber over generic t $\Pi^{-1}(t)$ be a smooth K3, but the central fiber can have normal crossing singularity. This means

$$\Pi^{-1}(0) = X_0 \cup X_1 \cup \cdots \cup X_N, \tag{1.59}$$

where the components X_i's are all reduced irreducible two-dimensional complex varieties. In addition, we demand that $K_\chi = 0$ (called a Kulikov model [121, 137]). Both of these conditions can be fulfilled after a series of birational transformations. So these assumptions are not too restrictive.

We can get a Kulikov model for our elliptically fibered K3 as follows. First, remember that f and g are degree 8 and 12 polynomials, respectively. So generally,

$$f = u^8 f_0 + u^7 v f_1 + \cdots + v^8 f_8, \tag{1.60}$$
$$g = u^{12} g_0 + u^{11} v g_1 + \cdots + v^{12} g_{12}. \tag{1.61}$$

We construct an initial fibration χ_0 as,

$$f_t^0 = t^4 u^8 f_0 + t^3 u^7 v f_1 + \cdots + u^4 v^4 f_4 + \cdots + v^8 f_8, \tag{1.62}$$
$$g_t^0 = t^6 u^{12} g_0 + t^5 u^{11} v g_1 + \cdots + u^6 v^6 g_6 + \cdots + v^{12} g_{12}. \tag{1.63}$$

Note the terms f_4 and higher in f_t^0, and g_6 and higher g_t^0 are independent of t. Obviously, χ_0 is not a semistable degeneration. Over a generic point t, we have a smooth $K3_t$, and over $t = 0$, we have a highly singular irreducible $K3$. To get a semistable degeneration one, apply the following birational transformation,

$$t \to et, \qquad v \to ev, \tag{1.64}$$
$$x \to e^2 x, \qquad y \to e^3 y. \tag{1.65}$$

After this birational transformation, the previous locus $t = 0$ is replaced by $t = 0$ and $e = 0$. Note that the coordinates (u, e) cannot be zero at the same time, similarly (v, t) have the same property. One gets the new family χ given by the equation,

$$0 = y^2 + x^3 + f^1(u, e; v, t)xz^4 + g^1(u, e; v, t)z^6, \tag{1.66}$$
$$f^1(u, e; v, t) = t^4u^8f_0 + t^3u^7vf_1 + \cdots + u^4v^4f_4 + eu^3v^5f_5 + \cdots + e^4v^8f_8, \tag{1.67}$$
$$g^1(u, e; v, t) = t^6u^{12}g_0 + t^5u^{11}vg_1 + \cdots + u^6v^6g_6 + eu^5v^7g_7 + \cdots + e^6v^{12}g_{12}. \tag{1.68}$$

Then we can see over generic point of D, where $e \cdot t \neq 0$, the fibers of χ and χ_0 are isomorphic, but the central fiber $e \cdot t = 0$, replaced by two two-dimensional varieties $e = 0$ and $t = 0$. Over $e = 0$, one gets,

$$f^1(1, 0; v, t) = t^4f_0 + t^3vf_1 + \cdots + v^4f_4, \tag{1.69}$$
$$g^1(1, 0; v, t) = t^6g_0 + t^5vg_1 + \cdots + v^6g_6. \tag{1.70}$$

This the equation of an elliptically fibered surface with $c_2 = 12$ (and of course $c_1 \neq 0$). Such surfaces are known as dP_9, and they correspond to blowing up $\mathbb{P}^2$ at nine points. In the elliptic fibration language, the ten generating divisors correspond to the anticanonical divisor (which is an elliptic curve), the section of the Weierstrass fibration, and eight other (-1)-curves. The intersection of the generators of the effective cone is the same as the (negative) Cartan matrix of E_8, and their intersections with the anticanonical divisor corresponds to non-trivial degree zero line bundles over that elliptic curve. In other words, they correspond to the Wilson lines E_8 gauge group over the elliptic curve. It is well known [89] that the complex structure of the dP_9 (keeping the anticanonical divisor fixed) parameterizes the E_8 vector bundle moduli over the corresponding elliptic curve.

Similarly, one can see the locus $t = 0$ is also a dP_9 surface, and the intersection $e = t = 0$ is an elliptic curve given by f_4 and g_6. Then the duality between Heterotic $E_8 \times E_8$ over a torus, and F-theory over $K3$ is just given by identifying the complex structure of the elliptic curve $e = t = 0$ with the complex structure of the torus in Heterotic string, and the complex structure of each dP_9's (leaving the elliptic curve fixed) are identified by the moduli space of either E_8's of Heterotic on the torus [89].

Note that this picture is very similar to that of Heterotic M-theory, and this is not an accident [131, 131]. By comparing the original of the $K3$ (1.60) with the central fiber of the family χ, one notes the $e = t = 0$ corresponds to the limit where f_i for $i \neq 4$ and g_i for $i \neq 6$ vanish. Intuitively this corresponds to stretching the base $\mathbb{P}^1$ of the original $K3$. In this limit, one gets E_8 singularities over each end of the $\mathbb{P}^1$. By identifying the volume of the base in this limit with the radius of $S^1/\mathbb{Z}_2$, the authors of [131, 131] get to the Heterotic M-theory picture.

1.5.3 Heterotic/F-theory duality in lower dimensions

In this subsection, we briefly review the generalization of the eight-dimensional duality mentioned before. The technical details are left for the next chapter. Consider the compactifications of the $E_8 \times E_8$ Heterotic theory on an elliptically fibered Calabi-Yau n-fold,

$$\pi_h : X_n \xrightarrow{\mathbb{E}} B_{n-1} \quad . \tag{1.71}$$

This will lead to the same effective physics as F-theory compactifications on a $K3$-fibered Calabi-Yau $n+1$-fold,

$$\pi_f : Y_{n+1} \xrightarrow{K3} B_{n-1} \quad . \tag{1.72}$$

Here the base manifold B_{n-1} appearing in (1.71) and (1.72) is the same Kähler manifold (thus inducing a duality fiber by fiber over the base from the 8-dimensional correspondence of [151]). Within the Heterotic theory, as discussed before, the geometry of the slope stable, holomorphic vector bundle, $\pi : V \to X_n$, must also be taken into account. In particular, to be understood in the context of the fiber-wise duality (induced from 8-dimensional correspondence), the data of the vector bundle must also be presented "fiber by fiber" in X_n over the base B_{n-1}.

To this end, the work of Friedman, Morgan, and Witten [89] introduced the tools of *Fourier-Mukai Transforms* into Heterotic theories. In this context, the data of a rank N, holomorphic, slope-stable vector bundle $\pi : V \to X$ is presented by its so-called "spectral data," loosely described as a pair

$$(S, \mathcal{L}_S) \tag{1.73}$$

consisting of an N-sheeted cover, S, of the base B_{n-1} (the "spectral cover") and a rank-1 sheaf $\mathcal{L}_S$ over it. Very loosely, this encapsulates the restriction of the bundle to each fiber (given by the N points on the elliptic curve over each point in the base) and the data of a line bundle, $\mathcal{L}_S$ encapsulating the "twisting" of this decomposition over the manifold.

Generally, one should "match" the spectral data in the Heterotic side to the spectral data of the Higgs bundle (1.52) in the F-theory side. This doesn't mean they are isomorphic. However, they are closely related. In [70], the Heterotic spectral cover was embedded into the F-theory geometry in the stable degeneration limit. There will be a $\mathbb{P}^1$ fiber over the spectral cover in each dP_9. This called the cylinder map in [70]. In six-dimensional compactification, one can pull back the non-trivial flat line bundles over the spectral cover using this cylinder mapping, and construct flat three-form fields in F-theory. In four-dimensional theories, there are not any non-trivial flat line bundles over the spectral cover (for generic smooth spectral cover). There are some "discrete" degrees of freedom in the connection of $\mathcal{L}_S$. These non-trivial connections can be pulled back using the cylinder mapping to get no trivial four-form fluxes in F-theory. Both of the flat three-form and four-form flux can be defined as the "remnants" of the M-theory three-form and its field strength in the F-theory limit. There are important attempts to compute such fluxes from the M-theory point of view [38]. The drawback of this approach is that all fluxes that one can get are abelian (direct sum of line bundles). However, starting from the Heterotic dual picture, one can get general no abelian fluxes.

The mathematical tool that enables us to do this is the Fourier-Mukai transform, and in the following, we review it. As usual, the details are left for the next chapter. A Fourier-Mukai transform is a relative integral functor acting on the bounded derived category of coherent sheaves $\Phi : D^b(X) \to D^b(\hat{X})$ (where $\hat{X}$ is the Altman-Kleinman compactification of the relative Jacobian of X). Let $\mathcal{E}^\bullet \in D^b(X)$ and define,

$$\begin{array}{ccccc} & & X \times_B \hat{X} & & \\ & \swarrow^{\pi_1} & \downarrow \rho & \searrow^{\pi_2} & \\ X & & B & & \hat{X} \end{array}$$

$$\mathcal{E} \to \Phi(\mathcal{E}^\bullet) := R\pi_{2*}(\pi_1^* \mathcal{E}^\bullet \otimes \mathcal{P}), \tag{1.74}$$

with $X \times_B \hat{X}$ is the fiber product and $\mathcal{P}$ is the "relative" Poincare sheaf and the so-called "kernel" of the Fourier-Mukai functor,

$$\mathcal{P} := \mathcal{I}_\Delta \otimes \pi_1^* \mathcal{O}_X(\sigma) \otimes \pi_2^* \mathcal{O}_{\hat{X}}(\sigma) \otimes \rho^* K_B^*, \tag{1.75}$$

and where $\mathcal{I}_\Delta$ is the ideal sheaf of the relative diagonal divisor,

$$0 \longrightarrow \mathcal{I}_\Delta \longrightarrow \mathcal{O}_{X \times_B \hat{X}} \longrightarrow \delta_* \mathcal{O}_X \longrightarrow 0,$$

$$\delta : X \hookrightarrow X \times_B \hat{X}, \tag{1.76}$$

and finally, K_B is the canonical bundle of the base B. This functorial/category-theoretic viewpoint proves to be a powerful tool as we examine and define the concepts above more carefully in the Sections to come and consider their generalizations.

Chapter 2

Heterotic Spectral Cover Constructions and Generalization

This chapter is based on the paper [21], written in collaboration with L.B. Anderson and X. Gao. The outline of the chapter is as follows. In Section 2.1 we review the basic tools and key results of Fourier-Mukai transforms and spectral cover bundles in the case of Weierstrass models. We then generalize these results to the case of elliptically fibered manifolds with fibral divisors in Section 2.2 and geometries with additional sections to the elliptic fibration in Sections 2.3 and 2.4. In Section 2.5 we provide explicit examples of Fourier-Mukai transforms by beginning with a bundle defined via some explicit construction (e.g. a monad or extension bundle) and then computing its spectral data directly. In Section 2.6 we apply our new results to the problem of chirality changing small instanton transitions. In Section 2.7 we illustrate the distinctions and obstructions that can arise between smooth and singular spectral covers. Finally in Section 5.1 we summarize this work and briefly discuss future directions. The appendices contain a set of well-known but useful mathematical results on the topics of derived categories and Fourier-Mukai functors. Although the material contained there is well-established in the mathematics literature, it is less commonly used by physicists and we provide a small overview in the hope that readers unfamiliar with these tools might find a brief and self-contained summary of these results useful.

2.1 A review of vector bundles over Weierstrass elliptic fibrations and Fourier-Mukai transforms

In this section we provide a brief review of some of the necessary existing tools and standard results of Fourier-Mukai transforms arising in elliptically fibered Calabi-Yau geometry. Since the literature on this topic is vast (see for example [89, 90]) and applications [24, 25, 58, 59, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 106], we make no attempt at a comprehensive review, but instead aim for a curated survey of some of the tools that will prove most useful in later Sections. Moreover, we hope that this review is of use in making the present paper somewhat self-contained. However, the reader familiar with this literature could skip straight on to Section 2.2. For more information about the applications of Fourier-Mukai functors in studying the moduli space of stable sheaves over elliptically fibered manifolds, the interested reader is referred to [32].

2.1.1 Irreducible smooth elliptic curve

To set notation and introduce the necessary tools let us begin by considering the case of $n = 1$ in (1.71), a one (complex) dimensional Calabi-Yau manifold – that is X is a smooth elliptic curve, E. In the case of a smooth elliptic curve, there is a classical result due to Atiyah [31] (which can generalized to abelian varieties [32]) which states that any (semi)stable coherent sheaf, V, of rank N and degree zero over E is S-equivalent[1] to a direct sum of general degree zero line bundles,

$$V \sim \bigoplus_i \mathcal{L}_i^{\oplus N_i}, \qquad \Sigma_i N_i = N, \quad deg(\mathcal{L}_i) = 0. \tag{2.1}$$

In the context of the moduli space of semi-stable sheaves on an elliptic curve, one can introduce an integral functor

$$\Phi^{\mathcal{P}}_{E\to E} : D^b(E) \longrightarrow D^b(E) \tag{2.2}$$

(note that here $\hat{E}$ the Jacobian of E is simply isomorphic to E and thus we do not make the distinction). This functor admits a canonical kernel, $\mathcal{P}$, the so-called

[1] For any semistable vector bundle (or torsion free) V with slope $\mu(V)$, there is a filtration – the Jordan-Holder filtration [88]) of the form $0 = F^0 \subset F^1 \subset \cdots \subset F^{k-1} \subset F^k = V$, where F^i/F^{i-1} is stable torsion free with $\mu(F^i/F^{i-1}) = \mu(V)$. Associated with this filtration there is a graded object $gr(V) = \oplus_{i=0}^k F^i/F^{i-1}$, and V and $gr(V)$ are said to be S-equivalent.

Poincare sheaf,

$$\mathcal{P} := \mathcal{I}_{\Delta} \otimes \pi_1^* \mathcal{O}_E(p_0) \otimes \pi_2^* \mathcal{O}_E(p_0) \tag{2.3}$$

where π_1, π_2 are the projection of $E \times E$ to the first and second factor respectively, p_0 is the divisor corresponding to the zero element of the abelian group on the elliptic curves, and Δ is the diagonal divisor in $E \times E$ (and also δ is the diagonal morphism). It is not hard to prove that $\mathcal{P}$ satisfies the conditions due to Orlov and Bondal ([32], see Appendix B) that guarantee that $\Phi^{\mathcal{P}}_{E \to E}$ is indeed a Fourier-Mukai transform (i.e. it is an equivalence of derived categories).

To illustrate how this specific Fourier-Mukai functor acts on coherent sheaves of degree zero, it is useful to highlight its specific behavior in several explicit cases. To begin, consider the simplest possible case of $V = \mathcal{O}_E(p - p_0)$, i.e. a generic degree zero line bundle over E. Here,

$$\Phi^{\mathcal{P}}_E \left(\mathcal{O}_E(p - p_0) \right) = R\pi_{2*} \left(\pi_1^* \mathcal{O}_E(p - p_0) \otimes \mathcal{P} \right).$$

To compute this explicitly, consider the following short exact sequence induced by the morphism $\delta : E \longrightarrow E \times E$,

$$0 \longrightarrow \mathcal{P} \longrightarrow \pi_1^* \mathcal{O}_E(p_0) \otimes \pi_2^* \mathcal{O}_E(p_0) \longrightarrow \delta_* \mathcal{O}_E(2p_0) \longrightarrow 0. \tag{2.4}$$

Twist in the sequence above with $\mathcal{O}_E(p - p_0)$, and then applying the (left exact) functor $R\pi_*$ to that yields the following long exact sequence (to see the properties of derived functors refer to Appendix B),

$$\begin{array}{l} 0 \longrightarrow \Phi^0 \left(\mathcal{O}_E(p - p_0) \right) \longrightarrow \left(R^0 \pi_{2*} \pi_1^* \mathcal{O}_E(p) \right) \otimes \mathcal{O}_E(p_0) \longrightarrow \mathcal{O}_E(p_0) \otimes \mathcal{O}_E(p) \\ \hookrightarrow \Phi^1 \left(\mathcal{O}_E(p - p_0) \right) \longrightarrow \left(R^1 \pi_{2*} \pi_1^* \mathcal{O}_E(p) \right) \otimes \mathcal{O}_E(p_0) \longrightarrow 0. \end{array} \tag{2.5}$$

To determine the the FM transform, it is necessary to understand the sheaves appearing in the middle column, and to that end, it is possible to apply the base change formula for flat morphisms,

$$\begin{array}{ccc} E \times E & \xrightarrow{\pi_2} & E \\ {\scriptstyle \pi_1} \downarrow & & \downarrow {\scriptstyle P} \\ E & \xrightarrow{P} & p \end{array}$$

$$R\pi_{2*} \pi_1^* \simeq P^* R P_*, \tag{2.6}$$

where P is just a projection to a point. Therefore,

$$R\pi_{2*}\pi_1^*\mathcal{O}_E(p) = P^*R\Gamma(E, \mathcal{O}_E(p)) = \mathcal{O}_E. \tag{2.7}$$

Consequently, it follows that $\mathcal{O}_E(p-p_0)$ must be a WIT_1, and it is supported[2] on p,

$$\Phi^{\mathcal{P}}(\mathcal{O}_E(p-p_0)) = \mathcal{O}_p[-1]. \tag{2.8}$$

In summary, the Fourier-Mukai transform of any direct sum degree zero line bundles on an elliptic curve, is a direct sum of torsion sheaves supported on the corresponding points of the Jacobian.

As another simple example, consider the non-trivial extension of two trivial line bundles,

$$0 \longrightarrow \mathcal{O}_E \longrightarrow V_2 \longrightarrow \mathcal{O}_E \longrightarrow 0. \tag{2.9}$$

Applying Φ on this short exact sequence yields

$$\begin{aligned} 0 \longrightarrow \Phi^0(\mathcal{O}_E) \longrightarrow \Phi^0(V_2) \longrightarrow \Phi^0(\mathcal{O}_E) \\ \hookrightarrow \Phi^1(\mathcal{O}_E) \longrightarrow \Phi^1(V_2) \longrightarrow \Phi^1(\mathcal{O}_E) \longrightarrow 0. \end{aligned} \tag{2.10}$$

From the previous discussion we have reviewed that $\Phi^{\mathcal{P}}(\mathcal{O}_E) = \mathcal{O}_{p_0}[-1]$, so the first row must be zero (i.e. $\Phi^0(V_2) = 0$), and

$$0 \longrightarrow \mathcal{O}_{p_0} \longrightarrow \Phi^1(V_2) \longrightarrow \mathcal{O}_{p_0} \longrightarrow 0, \tag{2.11}$$

but this cannot be a non-trivial extension of the torsion sheaves, and one concludes,

$$\Phi^{\mathcal{P}}(V_2) = (\mathcal{O}_{p_0} \oplus \mathcal{O}_{p_0})[-1]. \tag{2.12}$$

Note that V_2 is S-equivalent to $\mathcal{O}_E^{\oplus 2}$ but not equal, however, Fourier-Mukai of both of them is the same. These results can be generalized to general semistable vector bundles of degree zero over elliptic curves.

[2] Note that there is a more intuitive way of getting the same result. The presheaf of the Fourier-Mukai transform of $\mathcal{O}_E(p-p_0)$ over any point q is related to $H^i(E, \mathcal{O}_E(p-q))$, and for $i = 0, 1$ it is zero unless $p = q$, so naively, both $\Phi^0(\mathcal{O}_E(p-p_0))$ and $\Phi^1(\mathcal{O}_E(p-p_0))$ are some torsion sheaves supported over the point p. However, note that since $\mathcal{O}_E(p-p_0)$ is a locally free sheaf, and the projections are flat morphisms, $\Phi^0(\mathcal{O}_E(p-p_0))$ cannot be a torsion sheaf, so only $\Phi^1(\mathcal{O}_E(p-p_0))$ is non-zero, and the only possibility is the skyscraper sheaf $\mathcal{O}_p$.

2.1.2 Weierstrass elliptic fibration

With the results above in hand for a single elliptic curve, they can now be extended fiber-by-fiber for a smooth elliptic fibration. We begin with the simplest case, that of a smooth Weierstrass elliptic fibration $\pi : X \longrightarrow B$. This fibration admits a holomorphic section $\sigma : B \to X$ and every fiber $X_b = \pi^{-1}(b)$ is integral, and generically smooth for $b \in B$. Note that from here onward we will mainly work with smooth Calabi-Yau threefolds and since there exists an isomorphism, $\hat{X} \simeq X$, we will ignore the distinction between X and its relative Jacobian.

In general, on a fibered space, it is possible to define a relative integral functor Φ in almost the same way it was defined for a trivial fibration (i.e. $E \times B$, see Appendix C for more information on integral functors). So for any $V^\bullet \in D^b(X)$ there exists the following:

$$\begin{array}{ccccc} & & X \times_B X & & \\ & \swarrow^{\pi_1} & \downarrow{\scriptstyle \rho} & \searrow^{\pi_2} & \\ X & & B & & X \end{array}$$

$$\Phi(V^\bullet) := R\pi_{2*}(\pi_1^* V^\bullet \otimes^L \mathcal{K}^\bullet), \tag{2.13}$$

with $X \times_B X$ is the fiber product and the kernel is chosen as $\mathcal{K}^\bullet \in D^b(X \times_B X)$. In the case at hand, the kernel is required to be the "relative" Poincare sheaf,

$$\mathcal{P} := \mathcal{I}_\Delta \otimes \pi_1^* \mathcal{O}_X(\sigma) \otimes \pi_2^* \mathcal{O}_X(\sigma) \otimes \rho^* K_B^*, \tag{2.14}$$

where $\mathcal{I}_\Delta$ is the ideal sheaf of the relative diagonal divisor,

$$0 \longrightarrow \mathcal{I}_\Delta \longrightarrow \mathcal{O}_{X \times_B X} \longrightarrow \delta_* \mathcal{O}_X \longrightarrow 0,$$

$$\delta : X \hookrightarrow X \times_B X, \tag{2.15}$$

and K_B is the canonical bundle of the base B (which is chosen to make the restriction $\mathcal{P}|_{\pi_1^* \sigma_1} \simeq \mathcal{O}_X$, and similarly for σ_2).

From this relative integral functor, it is possible to define "absolute" integral functor with kernel $j_* \mathcal{P}$, where $j : X \times_B X \hookrightarrow X \times X$ is a closed immersion. Note that $\Phi(V^\bullet) \simeq \Phi^{j_* \mathcal{P}}_{X \to X}(V^\bullet)$ for any $V^\bullet$. It can be proved [32] that this kernel is indeed strongly simple, so the corresponding integral functor is fully faithful. Moreover, since X is a smooth Calabi-Yau manifold, it follows that this integral functor is indeed an equivalence, i.e. a Fourier-Mukai functor. Look at Appendix B, references there.

It should also be noted that there exist simple formulas for base change compatibility (see Appendix C), and it can be readily verified that the restriction of this Fourier Mukai functor over a generic smooth elliptic fiber is the same as the absolute integral functor that was reviewed briefly in the last Subsection with p_0 being the point chosen by the section.

2.1.3 Spectral cover

It is proved in [128] that the restriction of a stable coherent sheaf on a generic fiber is (semi)stable. As we have seen, the relative Fourier-Mukai transform defined in the last subsection, is compatible with base change, and hence its restriction on generic fibers, is the same as the Fourier-Mukai transform on elliptic curves defined in Section 2.1.1. On the other hand, the Fourier-Mukai transform of a (semi)stable degree zero sheaves of rank N over the elliptic curves is a torsion sheaf of length N (roughly speaking, the support of a torsion sheaf is a set of N points, these points can be infinitesimally close).

These set of N points over generic fibers define a surface $S \subset X$ and a finite morphism, $\pi_S : S \longrightarrow B$, of degree N. This surface S is called a *spectral cover*,[3] and is the support[4] of $\Phi^1(V)$.

On the other hand, the restriction of the torsion sheaf $\Phi^1(V)$ over its support (which is S), is a rank one coherent sheaf. This can be seen from the fiberwise treatment (note that $ch_0(\Phi^1(V)) = 0$, and $ch_1(\Phi^1(V)) = N = \text{Rank }(V)$ when restricted over a generic fiber, since S is actually an N-sheeted cover of the base). As a result, the rank of the torsion sheaf over its support must be one (for the cases the support is a non-reduced scheme this argument should be modified a little, and

[3]Depending on the choice of gauge group, there are constraints on the position of the points. For example for $SU(n)$ bundles (to which we will restrict our focus in this chapter) the sum of these points under the group law of the elliptic curve must be zero. This implies that the spectral cover must be given by a holomorphic function on that torus. For other gauge groups refer to [89], and [84].

[4]Note that spectral cover can wrap around some elliptic fibers. This is a symptom of the fact that the restriction of the vector bundle over those elliptic fibers is unstable. The restricted Fourier-Mukai transform on these fibers returns non-WIT objects (see Appendix B for definitions), and yet, if V is a vector bundle, the global Fourier-Mukai still returns a WIT_1 object. This is due the flatness of the morphisms and the kernel involved in defining the integral functor.

it is possible to show that the numerical rank of the spectral sheaf is one, see [32]). The rank one sheaf $\mathcal{L} := \Phi^1(V)|_S$ is referred to as the *spectral sheaf*, and the doublet $(\mathcal{L}, S)$ is called the *spectral data*.

If in addition, if the spectral cover is smooth, the spectral sheaf $\mathcal{L}$ is in fact, a line bundle. In the seminal paper [89] some restrictions on the topology of $\mathcal{L}$ are derived, with the assumption that spectral cover S is an integral scheme (reduced and irreducible). We turn to these now, before generalizing them in later sections.

2.1.4 Topological data

A goal of this work is to generalize the results of [89] and [57] for the topology of a vector bundle associated to a smooth spectral cover in the following sections. As a result, it is useful to briefly review the derivation of constraints on the topological data (i.e. the relations between the topology of $\mathcal{L}$ and $ch(V)$). In the following we will assume that the spectral cover is an integral scheme, V is a WIT_1, locally free sheaf (vector bundle) of rank N with vanishing first Chern class, $c_1(V) = 0$, and that the Chern character of V can be written generally as,

$$
\begin{aligned}
ch(V) &= N - c_2(V) + \frac{1}{2}c_3(V), \\
c_2(V) &= \sigma\eta + \omega[f],
\end{aligned}
$$

where η is the pullback of a base divisor, $[f]$ is the fiber class (ω is an integer).

We will derive the form of the Chern classes of a smooth spectral cover bundle using a slightly different method than that employed in [57, 89], using tools that are well known in mathematics literature (see for example, [91]) and generalize more readily to the geometries studied in later sections.

Recall that $\Phi(V) = R\pi_{2*}(\pi_1^*V \otimes \mathcal{P})$. Thus, we can begin by computing the Chern characters of $\Phi(V)$, using the (singular[5]) Grothendieck-Riemann-Roch theorem [91] for π_2:

$$
ch\left(\Phi(V)\right) = \pi_{2*}\left(\pi_1^* ch(V) ch(\mathcal{P}) td(T_{X/B})\right), \tag{2.16}
$$

where $td(T_{X/B})$ is the Todd class of the virtual relative tangent bundle of $\pi : X \longrightarrow B$. In addition, it is also necessary to compute the Chern character of the relative

[5]Note that $X \times_B X$ is singular over the discriminant of X, even though X is smooth.

Poincare sheaf, and for that, one needs to compute $ch(\mathcal{I}_\Delta)$. This latter is straightforward to find by applying GRR to the diagonal morphism δ,

$$0 \longrightarrow \mathcal{I}_\Delta \longrightarrow \mathcal{O}_{X\times_B X} \longrightarrow \delta_*\mathcal{O}_X \longrightarrow 0,$$

$$ch(\mathcal{I}_\Delta) = 1 - \delta_*\left(\tfrac{1}{td(T_{X/B})}\right). \tag{2.17}$$

With these results in place, it remains simply to compute the pullback and push forward of cycles by using the following identities:

$$\pi_{2*}\pi_1^* D = 0, \quad D \in Div(B), \tag{2.18}$$

$$\pi_{2*}\pi_1^* f = 0, \quad f \quad \text{fiber class}, \tag{2.19}$$

$$\pi_{2*}(\pi_1^* c \cdot \delta_* d) = c \cdot d, \quad c, d \in A_\bullet(X), \tag{2.20}$$

$$\pi_{2*}(\pi_1^*(\sigma) \cdot b) = b, \quad b \in A_\bullet(B). \tag{2.21}$$

The first two identities are the result of the fact that if the dimension of the image of a cycle has a lower dimension the corresponding push forward will be zero as a homomorphism between the cycles in the Chow group. The last two follow from the definition of the diagonal morphism and the section (together with projection formula for cycles).

After putting all of these together, the result is as follows,

$$ch_0(\Phi(V)) = 0, \tag{2.22}$$

$$ch_1(\Phi(V)) = -(N\sigma + \eta), \tag{2.23}$$

$$ch_2(\Phi(V)) = (N\sigma + \eta)\left(\frac{c_1(B)}{2}\right) + \frac{1}{2}c_3(V)f, \tag{2.24}$$

$$ch_3(\Phi(V)) = -\frac{1}{6}Nc_1(B)^2 + \omega. \tag{2.25}$$

On the other hand, it should be recalled that V is WIT_1, i.e. $\Phi(V) = i_{S*}\mathcal{L}[-1]$, where $i_S : S \hookrightarrow X$, is the closed immersion of S into X, and $\mathcal{L}$ is the spectral sheaf (or spectral line bundle in this case). Therefore one can write,

$$\begin{aligned} ch(\Phi(V)) &= -ch(i_{S*}\mathcal{L}), \qquad (2.26)\\ ch(i_{S*}\mathcal{L}) &= i_{S*}\left(e^{c_1(\mathcal{L})}\frac{1}{T_{X/S}}\right)\\ &= [S]+[S]\cdot\left(c_1(L)-\frac{1}{2}[S]\right)+[S]\cdot\left(\frac{c_1(\mathcal{L})^2}{2}-\frac{1}{2}c_1(\mathcal{L})\cdot[S]+\frac{1}{6}[S]^2\right), \qquad (2.27)\end{aligned}$$

where in the second line the GRR theorem can be applied for the morphism i_{S*}, and $T_{X/S}$ is the virtual relative tangent bundle. Importantly, in the third line it is assumed $c_1(\mathcal{L})$ can be written in terms of the divisors of X, restricted to S, by writing $[S]\cdot c_1(\mathcal{L})$ instead of $i_{S*}\mathcal{L}$ (we'll return to this point in Section 2.2).

In summary then, by comparing these two ways of calculating the Chern character of the Fourier-Mukai transform, it is possible to obtain the constraints originally calculated in [57, 89]. The first equation (2.22) yields simply that $Rank(\Phi^0(V))-Rank(\Phi^1(V))=0$, and since we have restricted ourselves to WIT_1 sheaves, $\Phi^0(V)=0$ (see Appendix C for definitions), so this means that $Rank(\Phi^1(V))=0$ i.e. $\Phi^1(V))$ is a torsion sheaf (which is not surprising). From the first Chern character, the divisor class of the spectral cover can be read (noting the relative minus sign),

$$[S]=N\sigma+\eta. \qquad (2.28)$$

The next comparison puts non-trivial constraints on $c_1(\mathcal{L})$,

$$-[S]\cdot\left(c_1(\mathcal{L})-\frac{1}{2}[S]\right)=(N\sigma+\eta)\left(\frac{c_1(B)}{2}\right)+\frac{1}{2}c_3(V)f. \qquad (2.29)$$

Therefore the general form of the first Chern class must be of the form,

$$\begin{aligned} c_1(\mathcal{L}) &= \frac{1}{2}\left(-c_1(B)+[S]\right)+\gamma, \qquad (2.30)\\ [S]\cdot\gamma &= -\frac{1}{2}c_3(V)f. \qquad (2.31)\end{aligned}$$

The only solution for the second equation above is

$$\gamma=\lambda\left(N\sigma-\eta+Nc_1(B)\right), \qquad (2.32)$$

where λ is a constant which can be half integer or integer. So the general solutions for the $c_1(\mathcal{L})$ and $c_3(V)$ are,

$$c_1(\mathcal{L}) = \frac{1}{2}\left(-c_1(B) + [S]\right) + \lambda\left(N\sigma - \eta + nc_1(B)\right), \quad (2.33)$$

$$c_3(V) = 2\lambda\eta\left(\eta - Nc_1(B)\right), \quad (2.34)$$

where in general, λ must satisfy constraints (i.e. be either integer or half integer) in order for $c_1(\mathcal{L})$ to be integral [89]. Note that there is sign difference between (2.33), and the similar formula in [89]. This arises because either $\mathcal{P}^\vee$ or $\mathcal{P}$ may be used as the kernel of the Fourier-Mukai functor. Finally it is possible to obtain ω from (2.25),

$$-\frac{1}{6}Nc_1(B)^2 + \omega = -[S]\cdot\left(\frac{c_1(\mathcal{L})}{2} - \frac{1}{2}c_1(\mathcal{L})\cdot[S] + \frac{1}{6}[S]^2\right). \quad (2.35)$$

By plugging (2.33) and (2.28) into this one gets,

$$\omega = -\frac{c_1(B)^2N^3}{24} + \frac{c_1(B)^2N}{24} + \frac{1}{8}c_1(B)\eta N^2 - \frac{\eta^2 N}{8} - \frac{1}{2}c_1(B)\eta\lambda^2N^2 + \frac{1}{2}\eta^2\lambda^2N. \quad (2.36)$$

As a result, we arrive finally at the following well-known formulas for the Chern classes of a bundle corresponding to a smooth spectral cover within a Weierstrass CY 3-fold:

$$c_1(V) = 0, \quad (2.37)$$

$$c_2(V) = \eta\sigma - \frac{N^3 - N}{24}c_1(B_2)^2 + \frac{N}{2}\left(\lambda^2 - \frac{1}{4}\right)\eta\cdot\left(\eta - Nc_1(B_2)\right), \quad (2.38)$$

$$c_3(V) = 2\lambda\sigma\eta\cdot\left(\eta - Nc_1(B_2)\right). \quad (2.39)$$

This is identical with the result of [89]. Having reproduced this classic result, we turn in the next section to our first generalization: Fourier-Mukai transforms and spectral cover bundles for elliptically fibered CY 3-folds exhibiting reducible fibers over co-dimension 1 loci in the base (i.e. the 3-folds contain so-called “fibral” divisors).

2.2 Elliptically fibered manifolds with fibral divisors

In this section we extend the classic results of Section 2.1.4 and consider the Fourier-Mukai transform of a vector bundle over a smooth elliptically fibered Calabi-Yau threefold $\pi : X \longrightarrow B$ with a (holomorphic) section σ and so-called *fibral divisors* – divisors D_I, $I = 1, \ldots m$, which project to a curve in the base B_2. In the absence of any additional sections to the elliptic fibration, we have a simple decomposition of the Picard group of X into a) a holomorphic section b) Divisors pulled back from the base, B, and c) fibral divisors. Hence, $h^{1,1}(X_3) = 1 + h^{1,1}(B_2) + m$. Moreover, as a result of the fibral divisors, it is clear that there will be new contributions to the Picard group of S, $Pic(S)$, compared to a Weierstrass model. These new geometric integers clearly affect the Heterotic theory (and could potentially change the G_4 flux present in an F-theory dual geometry).

Our first effort will be to derive topological formulas for the topology of a bundle over an X_3 of the form described above and compare these to the standard case (i.e. (2.1.4) in Section 2.1). We will demonstrate that although the new divisors in X_3 do in general effect the topology of possible smooth spectral cover bundles defined over X_3, they do not contribute to the chiral index.

In general, the form of the fibral divisors (at codimension 1 in B_2) will be of the form expected by Kodaira-Tate [118, 147] and a rich array of possibilities is possible. For simplicity, here we will consider the case of I_n-type reducible fibers only. It should be noted that even in this simple case, it is clear that the intersection numbers of divisors in X_3 and the topology of a spectral cover bundle $\pi : V \to X_3$ will be more complicated than in the simple case of Weierstrass models considered in Section 2.1. For instance, although some triple intersection numbers of X_3 can be simply parameterized in terms of the intersection structure of B_2, not all can (see e.g. [99] for a list of the triple intersection numbers of an elliptic manifold which are currently known in general). For instance, it is not currently known how to generally parameterize triple intersection numbers involving only fibral divisors in a base-independent way.

Since generic fibers in X_3 are still irreducible smooth elliptic curves, we will begin by briefly considering what happens over fibers with "exceptional curves," taking the case of I_2 fibers for simplicity. For more details the interested reader is referred to [46, 47, 108].

2.2.1 (Semi) stable vector bundles over I_2 elliptic curves

The I_2 degeneration of an elliptic fiber is a union of two rational curves $C_1 \cup C_2$ with two intersection points. We assume the section of the elliptic fibration intersects transversely with C_1 at a point p_0. In general any locally free sheaf V of rank N over such a reducible fiber can be characterized by its restriction over the components [124],

$$0 \longrightarrow V \longrightarrow V_{C_1} \oplus V_{C_2} \longrightarrow T \longrightarrow 0, \tag{2.40}$$

where T is a torsion sheaf supported over the intersection points of I_2. Now consider a torsion free rank one sheaf $\mathcal{L}$ of degree zero (it is useful to recall that here the notions of degree and rank are defined by the Hilbert polynomial). If $\mathcal{L}$ is strictly semistable, the restrictions $\mathcal{L}_{C_1}$ and $\mathcal{L}_{C_2}$ are $\mathcal{O}_{C_1}(-1)$ and $\mathcal{O}_{C_2}(+1)$ or the other way around. In any case the graded object (defined by the Jordan-Holder filtration) is [124],

$$Gr(\mathcal{L}) = \mathcal{O}_{C_1}(-1) \oplus \mathcal{O}_{C_2}(-1). \tag{2.41}$$

On the other hand the graded object of the stable ones are,

$$Gr(\mathcal{L}) = \mathcal{O}_{C_1}(p - p_0) \oplus \mathcal{O}_{C_2}. \tag{2.42}$$

Therefore the graded object of any semi stable bundle over I_2 is a direct sum of the cases mentioned above. One can also note that the compactified Jacobian of I_2 is a nodal elliptic curve in which all of the semistable line bundles (2.41), map to the singular node, and the line bundles map uniquely to the smooth points as in the smooth elliptic curve [108, 124].

It is proved in [46, 47] that the integral functor $\Phi^{\mathcal{P}_0}_{I_2 \to I_2}$ defined by the usual Poincare sheaf $\mathcal{P}_0 = \mathcal{I}_\Delta \otimes \pi_1^* \mathcal{O}_{I_2}(p_0)$, satisfies the criteria mentioned in Appendix C, and therefore it is a Fourier-Mukai functor. The action of this functor over the stable line bundles (2.42) is the same as that defined in Section 2.1,

$$\Phi^{\mathcal{P}_0}_{I_2 \to I_2}(\mathcal{L}) = \mathcal{O}_p[-1]. \tag{2.43}$$

It remains, then, to compute the other case. Assume $\mathcal{L} = \mathcal{O}_{C_1}(-1)$. As before, by using the exact sequence for $\mathcal{I}_\Delta$ and base change formula, one can compute,

$$\begin{aligned} 0 \longrightarrow \quad & \Phi^{\mathcal{P}_0 0}_{I_2 \to I_2}\left(\mathcal{O}_{C_1}(-1)\right) \quad \longrightarrow \pi^* \pi_* \mathcal{O}_{C_1} \longrightarrow \mathcal{O}_{C_1} \to \\ \to \quad & \Phi^{\mathcal{P}_0 1}_{I_2 \to I_2}\left(\mathcal{O}_{C_1}(-1)\right) \quad \longrightarrow \pi^* R^1 \pi_* \mathcal{O}_{C_1} \longrightarrow 0, \end{aligned} \tag{2.44}$$

since $\pi^* R\pi_* \mathcal{O}_{C_1} = \mathcal{O}_{I_2}$, and the third map in the first row is surjective, we conclude,

$$\Phi^{\mathcal{P}_0}_{I_2 \to I_2}(\mathcal{O}_{C_1}(-1)) = \mathcal{I}_{C_1}. \tag{2.45}$$

In the same way one finds,

$$\Phi^{\mathcal{P}_0}_{I_2\to I_2}(\mathcal{O}_{C_2}(-1)) = \mathcal{O}_{C_2}(-1)[-1]. \tag{2.46}$$

Therefore, the Fourier-Mukai transform of a strictly semistable rank one torsion free sheaf (2.41) is,

$$\Phi^{\mathcal{P}_0}_{I_2\to I_2}(\mathcal{L}) = \mathcal{I}_{C_1} \oplus \mathcal{O}_{C_2}(-1)[-1]. \tag{2.47}$$

In contrast to the stable line bundles, we see the Fourier-Mukai of (2.41) is non-WIT. However as mentioned before, in the case of elliptic fibration, the Fourier-Mukai transform of a vector bundle can be WIT_1 as long as it is stable (and of course flat over the base).

Note that contrary to the case in Section 2.1, the "Fourier transform" of stable degree zero sheaves over an elliptic fibration X with fibral divisors cannot live in the Jacobian $J(X)$ of X. This is because $J(X)$ is indeed a singular variety, and as reviewed in Appendix C, Fourier-Mukai functors are sensitive to singularities, i.e. a singular and a smooth variety cannot be Fourier-Mukai partners. This means if someone tries to "parameterize" the stable degree zero vector bundles over X by some "spectral data" in $J(X)$ some important information will be lost. We will return to this in Section 2.2.3. However, as we will see, it is possible to uniquely "parameterize" the stable degree zero vector bundle moduli in terms of the resolution of $J(X)$, i.e. X itself.

2.2.2 Topological data

The results of the previous section give us the tools to extend the Fourier-Mukai transform discussed in previous Sections to the singular/reducible fibers present in the case of an elliptic threefold with I_n reducible fibers. In this subsection, the same tools used for Weierstrass models are employed to determine the topology (i.e. Chern classes) of smooth spectral cover bundles on elliptic Calabi-Yau manifolds with fibral divisors. As in Section 2.1 we define the an integral functor with Poincare sheaf as the kernel, and as discussed above, it will be Fourier-Mukai again. So it is still possible to use (2.16) to derive some topological constraints.

The only geometric difference within the CY 3-fold is the existence of new fibral divisors $D_I \in Div(X)$ $(I = 1, \ldots r)$ which in general will not intersect the holomorphic zero section, and in every "slice" π^*D (with D a divisor pulled back from the base) in the intersection $D_I \cdot \pi^*D$ is a (-2)-curve.[6]

With these information, the essential non-zero intersections of divisors are,

$$\sigma^2 = -c_1 \cdot \sigma, \tag{2.48}$$
$$\sigma \cdot D_I = 0, \quad for \quad I = 1, \ldots, r, \tag{2.49}$$
$$h_{\alpha\beta} := \sigma \cdot D_\alpha \cdot D_\beta \quad h_{\alpha\beta} \textit{ is a symmetric, invertible, integral matrix}, \tag{2.50}$$
$$D_\alpha \cdot D_I \cdot D_J = -\mathcal{C}_{IJ}\mathcal{S} \cdot D_\alpha, \tag{2.51}$$
$$\textit{For } I_n\textit{:} \quad \mathcal{C}_{I,I} = 2, \quad \mathcal{C}_{I,I+1} = -1. \tag{2.52}$$

With the above constraints we can write the second Chern class of the tangent bundle as,

$$c_2(X) = 12\sigma \cdot c_1 + c_2 + 11c_1^2 + \sum \xi_I D_I. \tag{2.53}$$

Let us turn now to the computation of the topology of a smooth spectral cover bundle. The general form of the Chern character of a bundle $\pi : \mathcal{E} \to X$ can be expanded as

$$ch(V) = N - (\sigma\eta + \omega f + \sum \zeta_I D_I) + \frac{1}{2}c_3(V), \tag{2.54}$$

where ζ and η are $\mathbb{Q}$-Cartier divisors pulled back from the base B. Similar to the Weierstrass case, we can compute the Chern character of $\Phi^{\mathcal{P}}_{X\to X}(V)$,

$$\begin{aligned}
ch_0(\Phi(V)) &= 0, && (2.55)\\
ch_1(\Phi(V)) &= -\left(N\sigma + \eta\right), && (2.56)\\
ch_2(\Phi(V)) &= \left(N\sigma + \eta\right)\frac{c_1(B)}{2} + \frac{1}{2}c_3(V)f + \sum \zeta_I D_I, && (2.57)\\
ch_3(\Phi(V)) &= \omega - \frac{1}{6}nc_1(B)^2. && (2.58)
\end{aligned}$$

As explained before, since V is locally free, $\Phi(V)$ must be WIT_1. If, as in [89], we assume the support of $\Phi^1(V)$, which is the spectral cover S , is a generic integral scheme, then

$$\Phi(V) = i_{S*}\mathcal{L}[-1], \tag{2.59}$$
$$i_{S*} : S \hookrightarrow X, \tag{2.60}$$

[6] From now on, in this section, we define the base divisor D as $D := \frac{1}{\mathcal{S}\cdot\mathcal{S}}\mathcal{S}$, where $\mathcal{S}$ is the "image" of the fibral divisors in the base.

where $\mathcal{L}$ must be a line bundle over S as long as V is given by a smooth spectral cover. After using GRR for the surface S, the following results obtained,

$$\begin{aligned} [S] &= n\sigma + \eta, && (2.61) \\ c_1(\mathcal{L}) &= \frac{1}{2}\left(-c_1 + [S]\right) + \gamma + \sum \beta_{iI} e_{iI}, && (2.62) \\ [S]\cdot\gamma &= -\frac{1}{2} c_3(V) f, && (2.63) \end{aligned}$$

where e_{iI}'s are the fibral (-2)-curves intersecting the spectral cover. I labels the generator of the algebra, i labels the number of the isolated curves (determined by η). Note that the number of such curves with the spectral cover can be determined by computing the intersection number $[S] \cdot D_I^2$ and dividing by -2. Furthermore, these (-2)-curves intersect as,

$$e_{iI} \cdot e_{jJ} = -\delta_{ij} \mathcal{C}_{IJ}. \tag{2.64}$$

After proceeding as before, we obtain the following solutions,

$$\begin{aligned} \gamma &= \lambda\left(n\sigma - \eta + nc_1(B)\right), && (2.65) \\ c_3(V) &= 2\lambda\eta\left(\eta - nc_1(B)\right), && (2.66) \\ \omega &= \omega_{std} - \left(-\sum_{i,I} \beta_{iI}^2 + \sum_{i,I} \beta_{iI}\beta_{i,I+1}\right), && (2.67) \end{aligned}$$

where ω_{std} is the same as (2.36). However not all parameters β_{iI} are free, instead they should satisfy the following equations,

$$\sum_i^k \beta_{iI} D \cdot D_I = -\zeta_I \cdot D_I, \quad \text{for each I}, \tag{2.68}$$

where k is the number of the "sets" of (-2)-curves inside the spectral cover,

$$k = \eta \cdot \mathcal{S}. \tag{2.69}$$

Therefore the only contribution of the (-2)-curves will appear in $c_2(V)$ via the correction to (2.36).[7]

Unlike the case of Weierstrass models explored in the previous subsection, here it is difficult to write a fully general expression for the Chern classes of V due

[7] Note that similar results were derived in [77, 80].

to the incomplete knowledge of triple intersection numbers within the CY geometry. In order to make this explicit, we turn to the case of a single fibral divisor here – that is a CY 3-fold with resolved $SU(2)$ singular fibers.

In this case I=1 and the correction to the second Chern class is of the form,

$$\omega = \omega_{std} + \sum_{i}^{k} \beta_i^2. \tag{2.70}$$

The condition on β_i is,

$$\left(\sum_{i=1}^{k} \beta_i \right) \frac{\mathcal{S}}{\mathcal{S} \cdot \mathcal{S}} \cdot D_1 = -\zeta_1 D_1. \tag{2.71}$$

This is equivalent to (by multiplying with D_1),

$$\sum_{i=1}^{k} \beta_i = -\zeta_1 \cdot \mathcal{S}. \tag{2.72}$$

Therefore the correction would be,

$$\omega = \omega_{std} + \left(\zeta_1 \cdot \mathcal{S} + \sum_{i=2}^{k} \beta_i \right)^2 + \sum_{i=2}^{k} \beta_i^2. \tag{2.73}$$

It should be noted that this correction term will contribute to anomaly cancellation in the Heterotic theory and to the G-flux in the dual F-theory geometry. We'll return to this point in later sections. In summary then,

$$c_2(V) = \sigma \cdot \eta + \omega_{std} + \left(\zeta_1 \cdot \mathcal{S} + \sum_{i=2}^{k} \beta_i \right)^2 + \sum_{i=2}^{k} \beta_i^2 + \zeta_1 \cdot D_1, \tag{2.74}$$

$$c_3(V) = 2\lambda\eta \left(\eta - n c_1(B) \right), \tag{2.75}$$

and λ is subject to the same integrality conditions as [89].

2.2.3 What is missing in the singular limit

There is a common belief in the literature that if one need to find the F-theory dual of a perturbative Heterotic model on a non-Weierstrass elliptically Calabi-Yau with fibral divisors, then one should shrink the exceptional divisors first, and try to find the F-theory dual by working with spectral data in the singular Weierstrass limit. Here we will comment on this from the Heterotic string point of view, and explain what will be missed if one uses the naive spectral data in the singular limit.

As it should be clear by now, the naive spectral data in the singular limit are not in a one to one correspondence with the bundles in the smooth limit where the exceptional divisors have non-zero size i.e. the integral functor is not going to be an equivalence. Hence, if one use the "singular spectral data" to find the F-theory dual, some information will be lost.

More concretely, as mentioned before, the actual spectral cover in the smooth elliptic fibration will generically wrap around a finite number of (-2)-curves, and the spectral sheaf may or may not be dependent on them. So in the blow down limit, the (-2)-curves shrink into double point singularities. These singularities are located at the points where the double points of the branch curve intersect with the singularity locus of the Weierstrass model. In other words, if we look at their image on the base, Fig (2.1), they correspond to the points where the double point singularity of the branch curve hits the singularity locus of the elliptic fibration on the base. On the other hand, locally near these singularities, two sheets of the spectral cover meet each other, and one can use a local model in $\mathbb{C}^3$ as,

$$S = z^2 - xy = 0, \tag{2.76}$$

where x, y, z are the coordinates of the $\mathbb{C}^3$. Here S is a cone, and can be viewed as the double cover of the $x - y$ plane with branch locus on the lines $x = 0$ and $y = 0$. The double point singularity is located on the vertex of the coin i.e. $x = y = 0$. Now, as it is well known (see for example [104] example 6.5.2), the generator of the curve will be a Weil divisor. So instead of the original Cartier (-2)-curves on the smooth spectral cover, one gets Weil divisors in the singular limit, and any line bundles on the singular spectral cover will be independent of them. Now lets look at the situation the other way around. Suppose we naively choose a generic n-sheeted cover of B_2 in the singular Weierstrass limit, and a line bundle over that, and use these to find the F-theory dual or study the moduli space of the Heterotic string. First of all, for any choice of complex structure of this generic spectral cover, it contains a finite number of double point singularities. To see this, restrict the elliptic fibration over a singular locus where the Weierstrass equation factors as (in the patch $Z = 1$),

$$Y^2 = (X - b_0)^2(X - b_1), \tag{2.77}$$

where b_0 and b_1 are suitable polynomials. In addition a generic n-sheeted cover can

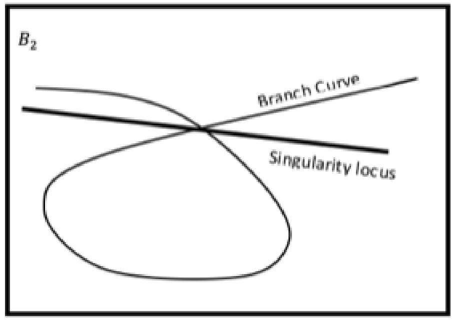


Figure 2.1: Branch locus and singularity locus near the singularity of the spectral cover.

be written as,

$$S = g_{n-4}(Y)X^2 + g_{n-2}(Y)X + g_n(Y), \tag{2.78}$$

where g_{n-4}, g_{n-2} and g_n are polynomial in terms of Y and appropriate local coordinates on base, and the subscripts determine the degree in tems of Y .[8] After eliminating X in these to equation we get the following interesting degree n polynomial in terms of Y,

$$\left(b_0^2 g_{n-4} + b_0 g_{n-2} + g_n\right)^2 \left(b_1^2 g_{n-4} + b_1 g_{n-2} + g_n\right) + Y^2 G_{n-2}(Y), \tag{2.79}$$

where G_{n-2} is polynomial in terms Y (of degree $n-2$) and base coordinates which we don't need to know the details. Zeros of this polynomial (with multiplicity) are the points where the n-sheeted cover hits the (singular) elliptic curve. Now, note that if

$$b_1^2 g_{n-4}(Y=0) + b_1 g_{n-2}(Y=0) + g_n(Y=0) = 0, \tag{2.80}$$

or

$$b_0^2 g_{n-4}(Y=0) + b_0 g_{n-2}(Y=0) + g_n(Y=0) = 0. \tag{2.81}$$

However from the above equation it is clear that the zeros of (2.81) are order two, this means the over these points the n-sheeted cover is locally like (2.76) (for suitable

[8]For example Y itself is of degree 3.

x, y, z) i.e. a double point singularity. The conclusion from the above calculations that we want to emphasize, is that the ubiquitous double point singularities of the n-sheeted covers in the singular Weierstrass limit, signals the necessity of working in the blown up limit.

The second problem with "parameterizing" the vector bundle moduli with the singular data is that since the line bundle in the singular limit doesn't depend on the (-2)-curves, the vector bundle that is constructed will not land on some specific components of the moduli space. In particular, physically, at least one consequence of this is missing some new possibilities for the small instanton transitions through exchanging 5 branes in the Heterotic M-Theory picture. In the context of Heterotic/F-theory duality, we expect that (-2)-curves inside the spectral cover correspond to new G_4-fluxes in the F-theory dual, consistent with the Fourier-Mukai calculations above, and if one considers only the singular spectral cover such possibilities could be missed.

2.3 Non-trivial Mordell-Weil group with a holomorphic zero section

In this section we continue our generalization away from Weierstrass elliptic fibrations by considering a Fourier-Mukai transform of vector bundles on elliptically fibered geometries in which the fibration admits more than one section – that is a higher rank Mordell-Weil group (the group of rational sections to the elliptic fibration [125, 155]). In the case that the zero section is strictly holomorphic (rather than rational) the definition of the Fourier-Mukai transform introduced in [89, 90] can actually be applied directly. In this case there are also isolated reducible fibers, but as we saw before one can still define a Poincare sheaf, and the corresponding integral functor will be a Fourier-Mukai transform.[9] Therefore, the new Fourier-Mukai functor required for this case is the same as that introduced for fibral divisors in Section 2.1. We defer to later the more generic case of geometries with higher-rank Mordell-Weil group and *only* rational sections (see Section 2.4).

In the case of a holomorphic section and additional (possibly rational) sec-

[9]Note that if there exists more than one holomorphic section, there is a redundancy in the choice of the "zero section." The Fourier Mukai functors defined by different choices will be equivalent to each other, and can be written in terms of each other, so we fix the zero section throughout the calculations in this section.

tions, it is clear that the CY 3-fold X_3 contains new elements in its Picard group and as a result, their restriction to the spectral cover and $Pic(S)$ will lead to generalizations of the formulas, (2.1.4), derived in Weierstrass form. We will compute these generalized Chern character formulas directly in the following subsections and independently compare these results to those found in explicit examples in Section 2.1 (the latter will be obtained by direct computation of the Fourier-Mukai transforms of a set of simple bundles).

To set notation, note that we will consider the case of multiple sections to the elliptic fibration and consider the case where the zero section (denoted σ) is holomorphic. In addition, there are σ_m with $m = 1, \dots rk(MW)$ (in general rational) sections present. Here we take $Pic(X)$ of the CY 3-fold to be generated by generated by,

$$\sigma \quad \text{the zero section,} \tag{2.82}$$
$$S_m = \sigma_m - \sigma - \pi^*\pi_*\sigma_m\sigma - c_1(B), \tag{2.83}$$
$$D_\alpha, \quad \alpha = 1, \dots h^{1,1}(B), \tag{2.84}$$

where S_m is the Shioda map of the rational section. Since σ, there exists a general relation of the form,

$$\sigma \cdot S_p = \sum_{m=1}^{r} D_{m,p} S_m, \tag{2.85}$$

where $D_{m,p}$ are specific divisors in Pic(B). This is because,

$$\sigma^2 \cdot S_m = -c_1(B) \cdot \sigma S_m = 0, \tag{2.86}$$
$$\sigma \cdot D_b \cdot S_m = 0. \tag{2.87}$$

2.3.1 Topological data

As in the case of Weierstrass models considered in Section 2.1, we begin by asking what topological formulas can be derived (in as much generality as possible) for a bundle, V on the manifold above, defined by a *smooth spectral cover*.

On an elliptic CY 3-fold as described above, the general form of the Chern character of a degree zero vector bundle can be written as

$$ch(V) \quad = N - (\sigma\eta + \textstyle\sum_{i=1}^{r} S_i\eta_i + \omega f) + \frac{1}{2}c_3(V), \tag{2.88}$$

where N is the rank of the bundle, S_i are the image of the Shioda map [145, 146] of the generators of the Mordell-Weil group, r is the rank of the Mordell Weil group, and σ is the zero section we chose. With the help of GRR theorem, one gets the topology of the Fourier-Mukai transform of this bundle.

$$ch(\Phi(V)) = -(N\sigma+\eta) + (N\sigma+\eta)\frac{c_1(B)}{2} + \sum_{i=1}^{r} S_i\eta_i + \frac{1}{2}c_3(V)f + \left(\omega - \frac{1}{6}Nc_1(B)^2\right). \tag{2.89}$$

Since V is locally free, it must be WIT_1 and $\Phi^1(V)$ will be a torsion sheaf. If the support of this torsion sheaf is a generic smooth surface, then,

$$\Phi^1(V) = i_{S*}\mathcal{L},$$

where $\mathcal{L}$ is line bundle.[10] So by applying GRR to i_S, topological constraints we are looking for can be obtained,

$$[S] = N\sigma + \eta, \tag{2.90}$$

$$c_1(\mathcal{L}) = \frac{1}{2}\left(-c_1(B) + [S]\right) + \sum_{i}^{r} \beta_i S_i + \lambda\left(N\sigma - \eta + Nc_1(B)\right), \tag{2.91}$$

$$\sum_{i,j=1}^{r} S_j\left(\beta_i\left(\eta\delta_{i,j} + ND_{j,i}\right) + \eta_j\delta_{i,j}\right) = 0, \tag{2.92}$$

$$c_3(V) = 2\lambda\eta\left(\eta - Nc_1(B)\right), \tag{2.93}$$

$$\omega = \omega_{std} - \frac{1}{2}\sum_{m,n,p} \beta_m\beta_n\left(\eta\delta_{p,m} + ND_{p,n}\right) S_k S_j, \tag{2.94}$$

where the third equation is a constraint on the β_m's, and clearly they contribute in Chern characters of V only through the corrections in ω, and there is not any correction in $c_3(V)$, i.e. the chirality of the effective theory is unchanged.

[10] Recall that smoothness of V implies the smoothness of $\mathcal{L}$ on S.

2.3.2 Rank one Mordell-Weil group

In this section, we derive explicit correction to the formulas in Section 2.1.4 in the case $rk(MW) = 1$. The formulas above can be rewritten as,

$$c_1(\mathcal{L}) = \frac{1}{2}(-c_1(B) + [S]) + \beta_1 S_1 + \lambda(N\sigma - \eta + Nc_1(B)), \tag{2.95}$$

$$\sigma \cdot S_1 = D_{11} \cdot S_1, \quad D_{11} \text{ is a specific base divisor}, \tag{2.96}$$

$$\omega = \omega_{std} - \frac{1}{2}\beta^2(\eta + ND_{11})S_1^2, \tag{2.97}$$

$$\beta_1(\eta + ND_{11})S_1 + \eta_1 \cdot S_1 = 0. \tag{2.98}$$

Note that σ_1 induces an integral divisor in S, so the coefficient of σ_1 in $c_1(\mathcal{L})$, i.e. β_1 must be integer,

$$\beta \in \mathbb{Z}. \tag{2.99}$$

This condition fixes η_1 in terms of η. More precisely, if one expand η and η_1 in terms of the base divisor,

$$\eta = \eta^\alpha D_\alpha, \tag{2.100}$$

$$\eta_1 = \eta_1^\alpha D_\alpha, \tag{2.101}$$

$$\tag{2.102}$$

then we get the following,

$$\eta_1^\alpha = -\beta_1(\eta^\alpha + ND_{11}^\alpha), \tag{2.103}$$

where β_1 is an integer. Therefore the Chern classes of V in this case is given by,

$$\begin{aligned} c_2(V) = \sigma \cdot \eta \quad &- \quad \beta_1 (\eta + ND_{11}) \cdot S_1 \\ &+ \quad \left(\omega_{std} - \frac{1}{2}\beta_1^2(\eta + ND_{11})S_1^2\right) f, \end{aligned} \tag{2.104}$$

$$c_3(V) \quad = \quad 2\lambda\eta\,(\eta - Nc_1(B))\,. \tag{2.105}$$

2.4 Non-trivial Mordell-Weil group with rational generators

In this section we consider the last piece that will allow us to compute the Fourier-Mukai transform of vector bundles (or even any coherent sheaf) over any smooth elliptically fibered Calabi Yau variety $\pi : X \longrightarrow B$. In the previous Section we considered the case in which the elliptic threefold with a non-trivial Mordell-Weil Group and (importantly) the zero section was holomorphic. But this is far from the general case, in which all sections to the fibration are birational (i.e. the locus $\sigma = 0$ for such a section is birational to B_2 rather than equal to it).

Here we will consider the moduli space of vector bundles over these more general elliptic fibrations. We emphasize again that such information is potentially very important to the study of both the Heterotic theory and its F-theory dual. Below, we demonstrate that it is possible in principle for the chirality of the effective theory to change compared to the computation in Weierstrass form. So this case is distinct from those studied in previous Sections.

What makes this situation a little more complicated is that to define a Poincare sheaf one needs a "true" section (i.e. an inclusion $i_B : B \hookrightarrow X$ such that $\pi \circ i_B = id_B$). In that case the section is holomorphic. The key property is that a holomorphic section intersects *every* fiber at exactly one point. However if the section is rational, this is not satisfied for finitely many fibers containing reducible curves. As a result, the Poincare sheaf will not be a good kernel for the Fourier-Mukai functor. It is not clear at this moment how to deal with this in general, but there are cases which after a flop transition, the zero rational section becomes holomorphic. We restrict ourselves to this in the following, and general case will be studied in a future work.

The key point is that one can see that derived categories stay "invariant" under flop transitions.[11] So if after a finite number of flop transition one of the sections becomes holomorphic, then it is possible to reduce the problem to one of the cases described before. The disadvantage to this approach is that it is not guaranteed that such flops exists generally.

2.4.1 Flop transitions

Suppose $C \subset X$ be a rational curve in the Calabi-Yau threefold X, and $N_C X$ is the corresponding normal bundle (obviously, with rank 2) over C. In general one can always blow up X around this curve $p : \tilde{X} \longrightarrow X$, and the corresponding exceptional divisor $E \in Div(\tilde{X})$ will be isomorphic to $\mathbb{P}(N_C X)$, which is therefore a $\mathbb{P}^1$ bundle over $C \simeq \mathbb{P}^1$. If $N_C X \simeq \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$, then one can show that the

[11] This is the theorem of Bondal and Orlov (see [117]Theorem 11.23, and the references therein).

exceptional divisor is just a trivial $\mathbb{P}^1$ bundle over another $\mathbb{P}^1$, i.e. $e \simeq \mathbb{P}^1 \times \mathbb{P}^1$ (see e.g. [117]). In any case, after blowing X up, one can decide to blow the rational curve C down to get another threefold variety $q : \tilde{X} \longrightarrow X'$. Such geometric birational transformations are called standard flip transitions, and depending on the normal bundle $N_C X$, they can change the canonical bundle of the variety. So in general X' is not a Calabi-Yau variety. However in the special case which is described above, $N_C X \simeq \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$, the canonical bundle will remain unchanged (X' will be Calabi-Yau), this is called the standard flop transition.

For a general flip transition, the functor $Rq_* Lp^* : D^b(X) \longrightarrow D^b(X')$, is a fully faithful functor, and its image can be characterized by using the semi-orthogonal decomposition [117]. But here we restrict ourselves to the standard flop transitions, and in this case $Rq_* Lp^*$ will be an equivalence. To be more clear, consider the following diagram,

$$\begin{array}{ccccc} & & \tilde{X} & & \\ & \swarrow^{p} & & \searrow^{q} & \\ X & & & & X' \end{array} \tag{2.106}$$

To compute the topological data, we start with a bundle with most general Chern character as before,

$$ch(V) = N - \left(\sigma\eta + \sum_i S_i \eta_i + \omega f\right) + \frac{1}{2} c_3(V),$$

where σ is the rational zero section of X, and the Chern character of the object $\mathcal{F}^\bullet := Rp_* q^* V$ is needed,

$$ch(Rp_* q^* V) = p_* \left(ch(q^* V) \frac{Td(\tilde{X})}{Td(X')} \right), \tag{2.107}$$

then, since the zero section is holomorphic in X', we will be able to compute the Chern characters of $\mathcal{F}^\bullet$ in X' as in the last section. To compute (2.107), we can find the relations between the Chern characters of $T\tilde{X}$ and TX. To see this, consider the following diagram,

$$\begin{array}{ccc} E := \mathbb{P}(N_c X) & \overset{j}{\hookrightarrow} & \tilde{X} \\ \downarrow g & & \downarrow p \\ c & \overset{i}{\hookrightarrow} & X \end{array} \tag{2.108}$$

One can prove [91] the following short exact functors,

$$\begin{aligned}
&I. \quad 0 \longrightarrow \mathcal{O}_g(-1) \longrightarrow g^* N_C X \longrightarrow \mathcal{G} \longrightarrow 0, \\
&II. \quad 0 \longrightarrow TE \longrightarrow j^* T\tilde{X} \longrightarrow \mathcal{O}_g(-1) \longrightarrow 0, \\
&III. \quad 0 \longrightarrow T\tilde{X} \longrightarrow p^* TX \longrightarrow j_* \mathcal{G} \longrightarrow 0,
\end{aligned} \tag{2.109}$$

where the first one is the relative version of the famous Euler sequence,[12] the second one is the adjunction, and the third sequence is proved by noting that $T\tilde{X}$ and TX are isomorphic almost everywhere (for details see [91], Chapter 15). In addition if the divisor in the fiber of g is t, and we denote the hyperplane in C as d then one can show,

$$t^2 = 2, \quad t \cdot d = 1. \tag{2.110}$$

By using this information, and the GRR theorem, one can compute the Chern classes of $\tilde{X}$. The result is the following,

$$c_1\left(\tilde{X}\right) = -E, \tag{2.111}$$

$$c_2\left(\tilde{X}\right) = p^* c_2\left(X'\right) + j_*\left(t - g^* c_1\left(\mathbb{P}^1\right)\right). \tag{2.112}$$

Using these data we can get the Chern characters in X',

$$ch\left(Rp_* q^* V\right) = N - \left(\sigma' \eta + \sum_i S_i' \eta_i + \omega f\right) + \frac{1}{2} c_3(V). \tag{2.113}$$

The next part of the calculations will be the same as the previous section, but with intersection numbers in X' not X. So is is possible to employ the same formulas in Section 2.3.1, but the intersection formulas are in X' rather than X.

Carrying out the flops explicitly The discussion above is somewhat abstract in nature, and as a result, it's helpful to illustrate these geometric transitions in an explicit Calabi-Yau geometry.

[12] Therefore, $\mathcal{G}$ is the relative tangent bundle times $\mathcal{O}(-1)$, i.e. $T_g \otimes \mathcal{O}(-1)$

We can illustrate the results stated above with the following simple rank 2 bundle defined by extension:

$$0 \longrightarrow \mathcal{O}_X(-\sigma_1 + D_b) \longrightarrow V_2 \longrightarrow \mathcal{O}_X(\sigma_1 - D_b) \longrightarrow 0\ . \tag{2.114}$$

For the Calabi-Yau threefold, we will take the anti-canonical hypersurface of the following toric variety,

x_1	x_2	x_3	e	u_1	v_1	u_2	v_2	$-K$
0	0	0	0	1	0	1	0	2
0	0	0	0	0	1	1	1	3
1	1	1	0	0	2	3	0	8
1	1	0	1	0	1	2	0	6

(2.115)

In this manifold, the flop transition described above (which converts a rational section to a holomorphic one) corresponds simply to a different triangulation of the toric polytope. Each triangulation corresponds to a specific Stanley Reisner ideal,

$$\mathcal{I}_{SR1} = \{u_1u_2, x_3v_1, v_1v_2, ev_2, x_1x_2x_3, x_1x_2e\}\,, \tag{2.116}$$

$$\mathcal{I}_{SR2} = \{eu_1, u_1u_2, v_1v_2, ev_2, x_1x_2x_3, x_1x_2e, x_3v_1u_2\}\,. \tag{2.117}$$

In both cases the sections are,

$$\sigma_1 = (1, 0, 0, 0), \tag{2.118}$$

$$\sigma_2 = (-1, 1, 2, 2). \tag{2.119}$$

However, in the first triangulation, both section are rational, and in particular, σ_1 wraps around two (-1)-curves. After the flop transition, in the second triangulation, the section σ_1 becomes holomorphic, and the section σ_2 remains rational, but it wraps around two more (-1)-curves (which are the flop transition of the initial ones).

To fix notation, we denote the sections in the initial geometry as σ_1, σ_2 and the sections in the second geometry as σ_1', σ_2' respectively.[13] To find out the corresponding cycles the that σ_1 wraps around them, we should compute the intersection formulas. So for the first geometry,

$$\sigma_1^2 = -c_1(B)\cdot\sigma_1 + \sigma_1\cdot E, \tag{2.120}$$

$$\sigma_1\cdot E = D\cdot E - \frac{1}{4}D\cdot S - 2f, \tag{2.121}$$

$$\sigma_2^2 = -c_1(B)\cdot\sigma_2 + \frac{19}{4}D\cdot S + D\cdot e + 38f. \tag{2.122}$$

[13] Also note that both geometries contain an exceptional divisor E, and D as the hyperplane in the base $\mathbb{P}^2$, which are common to both geometries.

The corresponding intersection formulas after the flop transition are,

$$\sigma_1'^2 = -c_1(B) \cdot \sigma_1', \tag{2.123}$$
$$\sigma_1' \cdot E = 0, \tag{2.124}$$
$$\sigma_2'^2 = -c_1(B) \cdot \sigma_2' + 5D \cdot S' + 40f. \tag{2.125}$$

It is clear that the codimension two cycle that is disappearing from the first geometry in the flop transition is,

$$[C] = D \cdot e - \frac{1}{4} D \cdot S - 2f, \tag{2.126}$$

and the codimension two cycle appearing in the new geometry is,

$$[C'] = -D \cdot e + \frac{1}{4} D \cdot S' + 2f. \tag{2.127}$$

In particular, note that $\sigma_1' \cdot [C'] = +2$.

It is also possible to compute the explicit Fourier-Mukai transform of the vector bundle given in (2.114). The details of such a computation are outlined in Section 2.5. Here we simply state the following result to illustrate the general arguments above.

The Chern characters before and after the flop transition are given by

$$Ch\,(V) = 2 - \left((2D_b + c_1\,(B))\,\sigma_1 + \frac{1}{4} D \cdot S - D \cdot e - \left(D_b^2 - 2f\right) \right), \tag{2.128}$$
$$Ch\,(Rp_* q^* V_2) = 2 - \left((2D_b + c_1\,(B))\,\sigma_1' - D_b^2 + [c'] \right). \tag{2.129}$$

By substituting the formula for the codimension two class $[C']$ we see V_2 and $p_* q^* V_2$ have the same Chern class in accordance with the general result of the previous subsection.

2.4.2 Comment on the chirality of the effective theory

Here we want to study the effect of the (-1)-curves in the rational zero section in the spectrum of the effective theory. We will fix notation as,

$$\mathcal{F}^\bullet := Rp_* q^* V,$$
$$\mathcal{L}^\bullet := \Phi^{\mathcal{P}}_{X' \to X'}\,(\mathcal{F}^\bullet)\,. \tag{2.130}$$

The goal then is to compute the zero-mode spectrum (i.e. bundle-valued cohomology groups) of V in X. Suppose the support of $\mathcal{L}^\bullet$ takes the most general form,[14] this task reduces to computation of $R^1\pi_*V$ by using Leray spectral sequence. To find this, first notice that inverse functor of Rq_*Lp^* is given by,[15]

$$V = Rp_* \left(Lq^* \mathcal{F}^\bullet \otimes \mathcal{O}_{\tilde{X}}(e) \right). \tag{2.131}$$

Therefore we get,

$$\begin{aligned} R\pi_* V &= R\pi'_* \left(\mathcal{F}^\bullet \otimes Rq_* \mathcal{O}_{\tilde{X}}(e) \right) \\ &= R\pi'_* \left(\mathcal{F}^\bullet \right), \end{aligned} \tag{2.132}$$

where we used $Rq_*\mathcal{O}_{\tilde{X}}(e) = \mathcal{O}_{X'}$. Next, one can use the same techniques as before to compute the $R\pi'_*\mathcal{F}^\bullet$ in terms of the "spectral data" in X',

$$R\pi_* V = R\pi'_* \mathcal{F}^\bullet = R\pi'_* \left(\mathcal{L}^\bullet \otimes \mathcal{O}_{\sigma'} \right). \tag{2.133}$$

Naively the above result is the same as in the standard cases. But notice that $\mathcal{L}^\bullet$ is the Fourier-Mukai transform of a (may be non-WIT or singular) object $\mathcal{F}^\bullet$ in $D^b(X')$, and it may receive new contributions from the original (-1)-curve in X. In the example computed before, the component $[C'_2]$ doesn't intersect with the zero section, so the only contribution to the spectrum of the effective theory is through the line bundle over the component S.

2.5 Examples of explicit Fourier-Mukai transforms

The power of a Fourier-Mukai transform (and its inverse) is that in principle we can move freely between descriptions of stable vector bundles on elliptically fibered manifolds and the spectral data that we have been studying in Sections 2.1, 2.2, and 2.3 . In this section we now utilize this potential to explicitly compute FM transforms of stable bundles defined by the monad construction or by extension (see e.g. [53]). Several explicit realizations of this type have been accomplished before in the literature [36] and we will provide some generalizations. In particular, we will develop general tools that are applicable away from Weierstrass 3-folds.

[14] The restriction of the support on the generic irreducible fiber is a set of points such that none of them are coincident.

[15] Remember that this is Fourier-Mukai functor so it has an inverse.

In these examples, we shall also observe that although we have derived general formulas for bundles defined via *smooth* spectral covers, this proves to be too limited to describe the explicit bundles we consider in the majority of cases. We will return to this point – namely that there remain important gaps in our description of general points in the moduli space of bundles – in Section 2.5.2.

Beginning with the simplest possible elliptic CY 3-fold geometry – i.e. Weierstrass form, we will illustrate the ideas that can be generalized to compute the Fourier-Mukai transform of sheaves which are defined by extension sequences or monads.

2.5.1 Bundles defined by extension on Weierstrass CY three-folds

To illustrate the techniques of taking explicit FM transforms, we begin with the simplest possible extension bundle – a rank two vector bundle defined by extension of two line bundles:

$$0 \longrightarrow \mathcal{L}_1 \longrightarrow V_2 \longrightarrow \mathcal{L}_1^\vee \longrightarrow 0. \tag{2.134}$$

We require V_2 to be stable, and $c_1(V_2) = 0$. Note that a necessary (though not sufficient) constraint on the line bundles appearing in this sequence is that $\mathcal{L}_1$ must not be effective (i.e. have global sections). For such a stable bundle the restriction of V_2 over $E_t = \pi^{-1}(t)$ for a generic $t \in B$ is one of the following cases [31],

$$\begin{aligned} V_2|_{E_t} &= \mathcal{O}_{E_t} \oplus \mathcal{O}_{E_t}, \\ V_2|_{E_t} &= V_2 \otimes \mathcal{F}, \quad deg(\mathcal{F}) = 0, \\ V_2|_{E_t} &= \mathcal{O}_{E_t}(-p-p_0) \oplus \mathcal{O}_{E_t}(p-p_0). \end{aligned} \tag{2.135}$$

In the first case, the support of the Fourier-Mukai sheaf (i.e. spectral cover), will be a non-reduced scheme (supported over the the section σ). In the second case V_2 is the unique non-trivial extension of trivial line bundles, and $\mathcal{F} = \mathcal{O}_{E_t}(p - p_0)$ for some p (here p_0 is the point on E_t chosen by the section), but for Weierstrass fibration, $p = p_0$ for generic fibers, and $V_2|_{E_t} = V_2$. So again the spectral cover will be non-reduced and supported over the zero section. In the final case, the spectral cover can be non-singular. So it is clear that in the majority of cases, we *do not* expect the FM transform of V_2 to be in the same component of moduli space as a *smooth* spectral cover of the form described in Section 2.1. We will illustrate this effect with two choices of $\mathcal{L}_1$ below.

Applying the Fourier-Mukai functor to (2.134) produces a long exact sequence involving the FM transform of the line bundles defining V_2. Thus, we can compute $\Phi(V_2)$ if we can compute $\Phi(\mathcal{L}_1)$. To begin, the definition of the Poincare sheaf, (2.1.2) and (2.1.2), allows us to write the following short exact sequence:

$$\begin{aligned} 0 \longrightarrow \pi_1^*\mathcal{L}_1 \otimes \mathcal{P} \quad &\longrightarrow \quad \pi_1^*\left(\mathcal{L}_1 \otimes \mathcal{O}_X(\sigma)\right) \otimes \pi_2^*\left(\mathcal{O}_X(\sigma) \otimes \pi^* K_B^*\right) \\ &\longrightarrow \quad \delta_*\left(\mathcal{L}_1 \otimes \mathcal{O}_X(2\sigma) \otimes \pi^* K_B^*\right) \longrightarrow 0. \end{aligned} \tag{2.136}$$

Now, by applying, $R\pi_{2*}$ to the above sequence, we can compute $\Phi(\mathcal{L}_1)$,

$$\begin{aligned} 0 \quad &\longrightarrow \quad \Phi^0\left(\mathcal{L}_1\right) \longrightarrow R^0\pi_{2*}\pi_1^*\left(\mathcal{L}_1 \otimes \mathcal{O}_X(\sigma)\right) \otimes \left(\mathcal{O}_X(\sigma) \otimes \pi^* K_B^*\right) \longrightarrow \left(\mathcal{L}_1 \otimes \mathcal{O}_X\left(2\sigma\right) \otimes \pi^* K_B^*\right) \to \\ &\longrightarrow \quad \Phi^1\left(\mathcal{L}_1\right) \longrightarrow R^1\pi_{2*}\pi_1^*\left(\mathcal{L}_1 \otimes \mathcal{O}_X(\sigma)\right) \otimes \left(\mathcal{O}_X(\sigma) \otimes \pi^* K_B^*\right) \longrightarrow 0. \end{aligned} \tag{2.137}$$

With these general observations in hand, we will first consider the case where $\mathcal{L}_1 = \mathcal{O}_X(D_b)$ with D_b a divisor pulled back from the base, B_2. To use (2.137), in this case, $R\pi_{2*}\pi_1^*(\mathcal{L}_1 \otimes \mathcal{O}_X(\sigma))$ must be computed. To accomplish this, we can use the base change formula (see Appendix C), which relates the following push-forwards,

$$\begin{array}{ccc} X \times_B X & \xrightarrow{\pi_1} & X \\ \Big\downarrow{\scriptstyle \pi_2} & & \Big\downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\pi} & B \end{array}$$

$$R\pi_{2*}\pi_1^* \simeq \pi^* R\pi_*. \tag{2.138}$$

Therefore $R\pi_{2*}\pi_1^*(\mathcal{L}_1 \otimes \mathcal{O}_X(\sigma)) = (\pi^* R\pi_*\mathcal{O}_X(\sigma)) \otimes \mathcal{O}_X(D_b)$. On the other hand, by Koszul sequence for the section (σ) we have,

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(\sigma) \longrightarrow \mathcal{O}_\sigma(K_B) \longrightarrow 0. \tag{2.139}$$

It is well-known for Weierstrass CY elliptic fibration $\pi : X \longrightarrow B$, $R^0\pi_*\mathcal{O}_X = \mathcal{O}_B, R^1\pi_*\mathcal{O}_X = K_B$ (see e.g. [87]). So the above sequence implies $R\pi_*\mathcal{O}_X(\sigma) = \mathcal{O}_B$ and hence $R\pi_{2*}\pi_1^*(\mathcal{L}_1 \otimes \mathcal{O}_X(\sigma)) = \mathcal{O}_X$. Plugging this into (2.137), we see that this sequence is just Koszul sequence again which is twisted $\mathcal{O}_X(\sigma) \otimes \pi^* K_B^*$,

$$\Phi(\mathcal{L}_1) = \mathcal{O}_\sigma(D_b)[-1]. \tag{2.140}$$

We can apply this result then to obtain the FM transform of V_2 for this chosen line bundle to find

$$0 \longrightarrow \mathcal{O}_\sigma(D_b) \longrightarrow \Phi^1(V_2) \longrightarrow \mathcal{O}_\sigma(-D_b) \longrightarrow 0\ . \tag{2.141}$$

In this case by the arguments given above, $\Phi^1(V_2)$ is supported over the section[16] and its rank (when restricted over the support) is two (the rank is one when restricted to the modified support). As a result, from the arguments above, we do not expect the topology of this bundle to match the formulas given in (2.16) (and indeed they do not though we will not yet make this comparison explicitly).

Let us not contrast this with another (non-generic) choice of line bundle,

$$\mathcal{L}_1 = \mathcal{O}_X(-\sigma + D_b). \tag{2.142}$$

In this case

$$\Phi\left(\mathcal{O}_X(\sigma + D_b)\right) = \mathcal{O}_X(-\sigma + K_B + D_b), \tag{2.143}$$
$$\Phi(\mathcal{O}_X(-\sigma + D_b)) = \mathcal{O}_X(\sigma + D_b)[-1]. \tag{2.144}$$

For the choice of line bundle in (2.142), the extension bundle V_2 is defined by a non-trivial element of the following space of extensions:

$$Ext^1(\mathcal{L}_1^\vee, \mathcal{L}_1) = H^1(X, \mathcal{L}_1^2) = H^0\left(B, \mathcal{O}_B\left(2D_b + c_1(B)\right) \oplus \mathcal{O}_B\left(2D_b - c_1(B)\right)\right), \tag{2.145}$$

(note that the last equality follows from a Leray spectral sequence on the elliptic threefold (see (B.42)), and $R\pi_*\mathcal{O}_X(-2\sigma) = K_b \oplus K_b^{-1}$. As a brief aside, we remark here that the form of this space of extensions gives us some information about the form of the possible FM dual spectral cover.

It is clear from the expression above that if $2D_b + c_1(B)$ is not effective, then there exists no non-trivial extension, and the vector bundle is simply a direct sum $\mathcal{L}_1 \oplus \mathcal{L}_1^\vee$ (and therefore not strictly stable). If $2D_b + c_1(B) = 0$ there is only one non-zero extension. On the other hand, if the degree of D_b is large enough to make $2D_b - c_1(B)$ effective then for any generic choice of extension there are $(2d_b + c_1(B)) \cdot (2D_b - c_1(B))$ isolated curves which the spectral cover must wrap.

Returning to our primary goal of computing the FM transform of V_2, it can be observed that there is enough information in (2.143) and (2.144) to compute $\Phi(V_2)$ explicitly.

$$0 \longrightarrow \Phi^0(V_2) \longrightarrow \mathcal{O}_X(-\sigma + K_B - D_b) \xrightarrow{\text{F}} \mathcal{O}_X(\sigma + D_b) \longrightarrow \Phi^1(V_2) \longrightarrow 0. \tag{2.146}$$

By fully faithfulness of Fourier-Mukai functor, one can show

$$F \in Ext^0(\mathcal{O}_X(-\sigma + K_B - D_b), \mathcal{O}_X(\sigma + D_b)) \simeq Ext^1(\mathcal{L}_1^\vee, \mathcal{L}_1).$$

[16] It is also possible to have vertical components, depending on the degree of the divisor D_b

Therefore it is necessary $2D_b - c_1(B)$ be effective to have a non-zero F, and $\Phi^0(V_2) = 0$ (and hence stability of V_2). Assuming that this is satisfied, we can find the Fourier-Mukai transform of V_2 as

$$\Phi(V_2) = \mathcal{O}_{2\sigma+2D_b-K_B}(\sigma + D_b). \tag{2.147}$$

At last we are in a position to compute the topological data, and directly compare the bundle constructed here with what would be expected from the formulas derived in [57, 89] and reviewed in Section 2.1. The Chern character of V_2 is,

$$ch(V_2) = 2 - \left(\sigma(2D_b + c_1(B)) + D_b^2\right). \tag{2.148}$$

Therefore from (2.28), the divisor class of spectral cover must be

$$[S] = 2\sigma + 2D_b + c_1(B). \tag{2.149}$$

This is the same as the divisor class of the support of the torsion sheaf in (2.147), In addition, since we require $[S]$ to be the divisor class of a algebraic surface it must be the case that $2D_b + c_1(B)$ is effective. This was exactly the requirement for the non-trivial extension discussed above.

For this example, the general algebraic formula for S takes the form

$$\begin{aligned} S &= f_1 x + f_2 z^2, \\ div(f_1) &= 2D_b - c_1(B), \\ div(f_2) &= 2D_b + c_1(B). \end{aligned} \tag{2.150}$$

So we see if $2D_b + c_1(B)$ is effective, but $2D_b - c_1(B)$ is not effective, then the coefficient f_1 vanishes, and the locus $f_2 = 0$ is the position of the vertical components mentioned above. Moreover, when $2D_b - c_1(B)$ is effective then the position of those vertical fibers is given by the points where $f_1 = f_2 = 0$, again as discussed before. Comparing this with the sequence before (2.147), we see the map F is indeed given by S, and therefore S uniquely determines an element in the extension group.

Now from equation (2.33), $c_1(\mathcal{L}) = \sigma + D_b + \lambda(2\sigma + 2D_b + c_1(B))$. This is compatible with (2.147) if we choose $\lambda = 0$. With $\lambda = 0$ and $N = 2$, the equation (2.36) produces

$$\omega = D_b^2, \tag{2.151}$$

and also from (2.34) it follows that $c_3(v_2) = 0$, in agreement with the Chern character computed directly above. Also note that the divisor class of the matter curve must be $\sigma \cdot [S] = 2D_b - c_1(B)$ [89]. So the FM transform of this vector bundle is indeed a smooth spectral cover and agrees with the topological formulas found in [57, 89] as expected.

2.5.2 FM transforms of monad bundles over 3-folds

In the following section we will provide an explicit construction of the spectral data a bundle defined via a monad. This construction is somewhat lengthy, but is useful to present in detail to demonstrate that FM transforms can be explicitly constructed for bundles that appear frequently in the Heterotic literature.

Over a Weierstrass CY 3-fold of the form studied in Section 2.1 consider a bundle defined as a so-called "monad" (i.e. as the kernel of a morphism between two sums of line bundles over X_3):

$$0 \longrightarrow V \longrightarrow \oplus_{i=1}^{l} \mathcal{O}_X(n_i\sigma + D_i) \xrightarrow{\mathrm{F}} \oplus_{j=1}^{k} \mathcal{O}_X(m_j\sigma + D_j) \longrightarrow 0, \qquad (2.152)$$

where $Rank(V) = N = l - k$, and the divisors D_i are pulled back from the base, B_2. To compute the Fourier-Mukai transform V we will see that it is necessary to begin with the transform of line bundles of the form $\mathcal{O}_X(n_i\sigma + D_i)$, as well as the morphism $\Phi(F)$. With that information, we can compute $\Phi(V)$. We should point out that for the geometry in question, none of the n_i's nor m_j's are allowed to be negative. This is necessary for stability of the bundle.[17] Upon applying the FM functor to (2.152), we get a sequence of the following form,

$$\begin{aligned} 0 \longrightarrow \Phi^0(V) \longrightarrow \oplus_{i=1}^{\prime l} \Phi^0(\mathcal{O}_X(n_i\sigma + D_i)) \overset{\Phi(F_0)}{\longrightarrow} \oplus_{j=1}^{\prime k} \Phi^0(\mathcal{O}_X(m_j\sigma + D_j)) \\ \hookrightarrow \Phi^1(V) \longrightarrow \oplus_{i=1}^{\prime\prime l} \Phi^1(\mathcal{O}_X(n_i\sigma + D_i)) \longrightarrow \oplus_{j=1}^{\prime\prime k} \Phi^1(\mathcal{O}_X(m_j\sigma + D_j)) \longrightarrow 0. \end{aligned} \qquad (2.153)$$

In the diagram above we employ the sign $\oplus^{'}$ to refer to the direct sum over the line bundles with positive definite relative degree, and use $\oplus^{''}$ to mean the direct sum over the line bundles with with relative degree zero (i.e. pull back of line bundles in the base). So to compute the Fourier-Mukai transform of V we need to compute the Fourier-Mukai transform of the line bundles in (2.152). To do this, one can simply use the defining sequence of the diagonal divisor in Section 2.1. Combining this with

[17] Actually if we naively compute the Fourier-Mukai of such sheaves (with some n_i's being negative), the result is either non-WIT_1 or $\Phi^1(V)$ is not a torsion sheaf. But we know V is stable if and only if it is WIT_1 respect to Φ, and $\Phi^1(V)$ is a torsion sheaf. In practice, this is a way to check the stability of a degree zero vector bundle over elliptically fibered manifolds.

the sequence above, give the following diagram,

$$\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & \cdots & \longrightarrow & \oplus'^{l}_{i=1}\Phi^0(\mathcal{O}_X(n_i\sigma+D_i)) & \xrightarrow{\Phi(F_0)} & \oplus'^{k}_{j=1}\Phi^0(\mathcal{O}_X(m_j\sigma+D_j)) & \longrightarrow & \cdots & & \\
 & & & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K_1 & \longrightarrow & \mathcal{A}\otimes\mathcal{O}_X(\sigma+c_1(B)) & \xrightarrow{F_0} & \mathcal{N}\otimes\mathcal{O}_X(\sigma+c_1(B)) & \longrightarrow & Q_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle ev} & & \downarrow{\scriptstyle ev} & & \downarrow & & \\
0 & \longrightarrow & K_2 & \longrightarrow & \oplus'^{l}_{i=1}\mathcal{O}_X((n_i+1)\sigma+D_i)\otimes\mathcal{O}_X(\sigma+c_1(B)) & \xrightarrow{F_0} & \oplus'^{k}_{j=1}\mathcal{O}_X((m_j+1)\sigma+D_j)\otimes\mathcal{O}_X(\sigma+c_1(B)) & \longrightarrow & Q_2 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array} \tag{2.154}$$

Each column in the diagram is defines the Fourier-Mukai transform of the (direct sum of) line bundles by means of the resolution of the Poincare sheaf. Therefore in the second row $\mathcal{A}$ and $\mathcal{N}$ are the sheaves generated by the "fiberwise" global sections of the sheaves $\oplus'\mathcal{O}_X((n_j+1)\sigma+D_j)$ and $\oplus'\mathcal{O}_X((m_j+1)\sigma+D_j)$, respectively. The evaluation maps simply takes the global section, and evaluates the sheaf at each point. Finally, the map F_0 is simply the map induced by the monad map F itself (from (2.152)) on the line bundles with positive definite relative degree (which also acts on the "fiberwise" global sections too).

The most important parts of this diagram are the induced maps between the kernels and co-kernels, K_1, Q_1 and K_2, Q_2, respectively. The kernel and co-kernel of these maps give a rather explicit presentation of the spectral data, so we will give them specific names,

$$0 \longrightarrow \bar{\mathcal{L}} \longrightarrow K_1 \longrightarrow K_2 \longrightarrow \mathcal{L} \longrightarrow 0, \tag{2.155}$$

$$0 \longrightarrow \mathcal{M} \longrightarrow Q_1 \longrightarrow Q_2 \longrightarrow 0. \tag{2.156}$$

(Note that the final map in the second line above must be surjective, otherwise it will be in contradiction with the commutativity of the middle two columns in (2.154)).

Now, by careful diagram chasing, one can prove that the Fourier-Mukai transform of V can be given by the following (more consise) diagram,

$$\begin{array}{ccccccccccc} & & 0 & & & & & & & & \\ & & \downarrow & & & & & & & & \\ & & \mathcal{L} & & & & & & & & \\ & & \downarrow & & & & & & & & \\ 0 & \longrightarrow & \mathcal{J} & \longrightarrow & \Phi^1(V) & \longrightarrow & \oplus_{i=1}^{\prime\prime l}\Phi^1(\mathcal{O}_X(n_i\sigma + D_i)) & \longrightarrow & \oplus_{j=1}^{\prime\prime k}\Phi^1(\mathcal{O}_X(m_j\sigma + D_j)) & \longrightarrow & 0 \\ & & \downarrow & & & & & & & & \\ & & \mathcal{M} & & & & & & & & \\ & & \downarrow & & & & & & & & \\ & & 0 & & & & & & & & \end{array} \tag{2.157}$$

This construction is similar in spirit to the spectral data derived for monads in [67] and we will return to this in Section 2.5.2.

To make this abstract formalism more concrete, it is helpful to consider an explicit example. Let us take X_3 to be a Weierstrass elliptically fibered threefold over $\mathbb{P}^2$, realized as a hypersurface in a toric variety, given by the following "charge data" (i.e. in GLSM notation):

y	x	z	x_0	x_1	x_2	p
3	2	1	0	0	0	6
9	6	0	1	1	1	18

Here the holomorphic zero section is determined by the divisor $z = 0$. As an explicit monad bundle over this manifold, consider the following short exact sequence:

$$0 \longrightarrow V \longrightarrow \mathcal{O}_X(2,3) \oplus \mathcal{O}_X(1,6) \oplus \mathcal{O}_X(0,1)^{\oplus 3} \xrightarrow{F} \mathcal{O}_X(3,12) \longrightarrow 0. \tag{2.158}$$

We first need to find the Fourier-Mukai of the line bundles. This can be done using the tools outlined in before and we simply summarize the results here:

$$\Phi(\mathcal{O}_X(D)) = \mathcal{O}_\sigma(K_B + D)[-1], \tag{2.159}$$

$$0 \longrightarrow \Phi^0(\mathcal{O}_X(2\sigma - K_B)) \longrightarrow \mathcal{O}_X(\sigma - 2K_B) \oplus \mathcal{O}_X(\sigma) \oplus \mathcal{O}_X(\sigma + K_B) \xrightarrow{ev} \mathcal{O}_X(4\sigma - 2K_B) \longrightarrow 0, \tag{2.160}$$

$$0 \longleftarrow \Phi^0(\mathcal{O}_X(\sigma - 2K_B)) \longrightarrow \mathcal{O}_X(\sigma - 3K_B) \oplus \mathcal{O}_X(\sigma - K_B) \longrightarrow \mathcal{O}_X(3\sigma - 3K_B) \longrightarrow 0, \tag{2.161}$$

$$0 \longrightarrow \Phi^0(\mathcal{O}_X(3\sigma - 4K_B)) \longrightarrow \mathcal{O}_X(\sigma - 5K_B) \oplus \cdots \oplus \mathcal{O}_X(\sigma - K_B) \xrightarrow{ev} \mathcal{O}_X(5\sigma - 5K_B) \longrightarrow 0, \tag{2.162}$$

where the middle bundles in the each of the short exact sequences above are the "fiberwise" global section of the line bundles in (2.152) denoted as $\mathcal{A}$ and $\mathcal{N}$ (twisted with $\mathcal{O}(\sigma + c_1(B))$). With this we have determined the columns of (2.154). By explicitly performing the fiber restrictions it can also be verified that

$$\begin{aligned} &\oplus_{i=1}^{\prime\prime l}\Phi^1(\mathcal{O}_X(n_i\sigma + D_i)) = \mathcal{O}_\sigma(-2)^{\oplus 3}, \\ &\oplus_{i=1}^{\prime\prime l}\Phi^1(\mathcal{O}_X(m_i\sigma + D_i)) = 0, \end{aligned}$$

and the map F_0 is a "part" of the monad map F,

$$\mathcal{O}_X(2,3) \oplus \mathcal{O}_X(1,6) \xrightarrow{F_0} \mathcal{O}_X(3,12) \ ,$$
$$F_0 = \begin{pmatrix} zf_9 \\ x + f_6 z^2 \end{pmatrix}. \tag{2.163}$$

Obviously F_0 is singular on $\{f_9 = 0\} \cap \{x + f_6 z^2 = 0\}$.

The final task will be determining the explicit kernels and co-kernels: K_1, K_2, Q_1 and Q_2. This is local question, so we can assume we are in a affine patch with $y \neq 0$ and $x_1 \neq 0$ for example. Then it is not too hard to show that free part of K_1 is generated by

$$K_1 \sim \alpha z \begin{pmatrix} x + f_6 z^2 \\ -f_9 z \end{pmatrix}. \tag{2.164}$$

Naively, it may look like that over $f_9 = 0$, the kernel K_1 jumps, but this is at the presheaf level, one can actually show that

$$K_1 \simeq \pi^* \mathcal{O}_{P^2}(-3). \tag{2.165}$$

Similarly, one can compute the K_2,

$$K_2 = \begin{pmatrix} (x + f_6 z^2)\frac{1}{l_{3,3}} \\ -z f_9 \frac{1}{l_{2,6}} \end{pmatrix} \tag{2.166}$$

where $\frac{1}{l_{3,3}}$ and $\frac{1}{l_{2,6}}$ are the local generators of the line bundles $\mathcal{O}_X(3,3)$ and $\mathcal{O}_X(2,6)$. By checking the degrees, K_2 is fixed to be the line bundle $\mathcal{O}_X(1,-3)$. Again naively it might appear that K_2 jumps over $\{f_9 = 0\} \cap \{x + f_6 z^2 = 0\}$, but this is at the presheaf level as before, and K_2 is indeed free.

With this information in hand, we can determine $\mathcal{L}$ and $\bar{\mathcal{L}}$ in (2.154),

$$0 \longrightarrow \bar{\mathcal{L}} \longrightarrow \mathcal{O}_X(0,-3) \otimes \mathcal{O}_X(1,3) \xrightarrow{\Psi_0} \mathcal{O}_X(1,-3) \otimes \mathcal{O}_X(1,3) \longrightarrow \mathcal{L} \longrightarrow 0. \tag{2.167}$$

By computing the induced map Ψ_0, one finds

$$\bar{\mathcal{L}} = 0, \tag{2.168}$$
$$\mathcal{L} = \mathcal{O}_\sigma(-6). \tag{2.169}$$

As the next step, it remains to determine Q_1 and Q_2. For the former, one should note that the morphism on the "fiberwise" global sections i.e. $\mathcal{A} \xrightarrow{F_0} \mathcal{N}$ is generically rank 4, so it is surjective unless $f_9 = 0$. Over this locus, we obtain the following "defining" sequence for Q_1,

$$\begin{aligned} 0 \ &\to \ (\mathcal{O}_X \oplus \mathcal{O}_X(0,6))|_{f_9} \otimes \mathcal{O}_X(1,3) \\ &\hookrightarrow \ (\mathcal{O}_X \oplus \mathcal{O}_X(0,3) \oplus \mathcal{O}_X(0,6) \oplus \mathcal{O}_X(0,12))|_{f_9} \otimes \mathcal{O}_X(1,3) \to Q_1 \to 0. \end{aligned} \tag{2.170}$$

This turns out to be,

$$Q_1 \simeq (\mathcal{O}_X(0,12) \oplus \mathcal{O}_X(0,3))_{f_9=0} \otimes \mathcal{O}_X(1,3). \tag{2.171}$$

On the other hand, Q_2 can be identified easily with $\mathcal{O}_X(4,12)|_{\{f_9=0\}\cap\{x+f_6z^2=0\}} \otimes \mathcal{O}_X(1,3)$. So $\mathcal{M}$ will be given by,

$$\begin{aligned} 0 \longrightarrow \mathcal{M} \longrightarrow &(\mathcal{O}_X(0,12) \oplus \mathcal{O}_X(0,3))_{f_9=0} \otimes \mathcal{O}_X(1,3) \\ &\longrightarrow \mathcal{O}_X(4,12)|_{\{f_9=0\}\cap\{x+f_6z^2=0\}} \otimes \mathcal{O}_X(1,3) \longrightarrow 0. \end{aligned} \tag{2.172}$$

Therefore, $\mathcal{M}$ will be a torsion sheaf supported on $f_9 = 0$ with rank 2 when restricted on the support. So $\mathcal{J}$ in (2.154) can be given explicitly as,

$$0 \longrightarrow \mathcal{O}_\sigma(-6) \longrightarrow \mathcal{J} \longrightarrow \mathcal{M} \longrightarrow 0, \tag{2.173}$$

and we can see the support of $\mathcal{J}$ is in the divisor class $\sigma + 18D$ where the $18D$ is the support of the sheaf $\mathcal{M}$. Finally the support of the $\Phi^1(V)$, i.e. the spectral cover, is in the class

$$[S] = 4\sigma + 18D. \tag{2.174}$$

Explicitly we find that the spectral cover is reducible and non-reduced and given by the algebraic expression

$$S : (f_9)^2 z^4 = 0. \tag{2.175}$$

With this spectral data in hand we are now in a position to compare to the well-known results for the topology of smooth spectral cover bundles derived in Section 2.1. Before beginning this computation we must first observe that from the definition of the monad in (2.158), the Chern class of V is given by,

$$c(V) = 1 + 18\sigma D + 48f - 162w, \tag{2.176}$$

where f is the fiber class, and w is the class of a point. Now if one compares this to the topological constraints reviewed in (2.16), it follows that $\eta = 18D$ and hence

$$[S] = 4\sigma + 18D, \tag{2.177}$$
$$c_3(V) = 2\lambda\eta(\eta - 4c_1(B)). \tag{2.178}$$

The first one is always true whether or not the spectral cover is degenerate or what spectral sheaf we choose, so it is not surprising to get a correct answer. The second equation however implies that $\lambda = -\frac{3}{4}$. If we then insert this value into the formula for the $c_2(V)$ given in (2.16), it yields

$$c_2(V)_{expected} = 18\sigma D + 45f, \tag{2.179}$$

which is obviously wrong. This discrepancy has arisen because the chosen monad bundle manifestly does not correspond to a smooth spectral cover (and must correspond to a different component of the moduli space of bundles over X_3).

A comparison to existing techniques for FM transforms of monad bundles
It should be noted that several existing papers in the literature [36, 67] have laid out useful algorithms for explicitly computing the FM transforms of monad bundles of the form

$$0 \longrightarrow V \longrightarrow \mathcal{F} \xrightarrow{F} \mathcal{N} \longrightarrow 0, \tag{2.180}$$

where $\mathcal{F}$ and $\mathcal{N}$ are direct sum of line bundles as mentioned before.

In particular, [36] utilizes the simple and useful observation that the "fiber-wise" global sections of the twisted vector bundle $V \otimes \mathcal{O}_X(\sigma)$ contain information about the spectral cover. Specifically, the zeros of these sections along the fiber are coincident with the points where the spectral cover intersects the fibers. So one can consider the kernel of the map F in the following sequence,

$$0 \longrightarrow \pi^*\pi_*(V \otimes \mathcal{O}_X(\sigma)) \longrightarrow \pi^*\pi_*(\mathcal{F} \otimes \mathcal{O}_X(\sigma)) \xrightarrow{F} \pi^*\pi_*(\mathcal{N} \otimes \mathcal{O}_X(\sigma)) \longrightarrow 0, \tag{2.181}$$

where the morphism π is the usual projection of the elliptic fibrations.[18] Therefore wherever the rank of the kernel drops, must be the position of the spectral cover.

This approach, though explicit and computationally tractable, has some drawbacks. The obvious one is that it cannot immediately provide information about the spectral sheaf. The other problem is that it is possible and quite common that the spectral cover may wrap components of some non-generic elliptic fibers (i.e. when the restriction of the vector bundle on those non-generic fibers is unstable). In such cases it is possible that the number of global sections of the twisted vector bundle on these fibers jump instead of dropping, and since the algorithm sketched above is designed to detect where the kernel drops, it cannot find these vertical components of the spectral cover.[19]

To solve the first problem in [67], it was conjectured that the cokernel, $\mathcal{L}$, of the following evaluation map can provide a defining relation for the spectral sheaf,

$$0 \longrightarrow \pi^*\pi_*(V \otimes \mathcal{O}_X(\sigma)) \xrightarrow{ev} V \otimes \mathcal{O}_X(\sigma) \longrightarrow \mathcal{L} \longrightarrow 0. \tag{2.182}$$

[18] To derive this sequence the flatness of π and stability of V are necessary.

[19] As long as one wants to find the spectral cover only, it is still possible to use this algorithm, but with other twists to find the missing components. We have employed this technique in recent work [20], but in practice it can be very slow for Calabi-Yau threefolds.

However, although $\mathcal{L}$ is supported over the spectral cover, it is not the spectral sheaf generally (in particular when some of the line bundles in the monad have zero relative degree zero).

In our approach, we simply use the resolution of the Poincare sheaf to compute the Fourier-Mukai transforms directly, and is clear from (2.154) that this yields something very similar in spirit to the approaches mentioned above.

2.5.3 An extension bundle defined on an elliptic fibration with fibral divisors

In a similar spirit to the previous sections, it should be noted that a generic bundle chosen over an elliptic threefold with fibral divisors will unfortunately *not necessarily* correspond to a smooth spectral cover with the topology we derived in Section 2.2. However, we can verify that in some simple cases the explicit examples we construct do produce smooth spectral covers with the expected form. Moreover, the techniques outlined in the previous subsections for explicitly computing FM transforms carry over smoothly into this new geometric setting.

For simplicity, we will fix the Calabi-Yau geometry explicitly from the start to be given by an anticanconical hypersurface in the following toric variety:

$$\begin{array}{ccccccc|c} X & Y & Z & E & x_1 & x_2 & x_3 & p \\ 3 & 2 & 1 & 0 & 0 & 0 & 0 & 6 \\ 9 & 6 & 0 & 0 & 1 & 1 & 1 & 18 \\ 8 & 5 & 0 & 1 & 0 & 1 & 1 & 16 \end{array} \tag{2.183}$$

Note that here we denote the single exceptional (i.e. fibral) divisor in this geometry as E and the divisor class of x_1 is $D - E$ with D being the hyperplane divisor in the base, $B_2 = \mathbb{P}^2$. The image of E on the base is a line homologous to the hyperplane, here denoted D. Over D all of the fibers are degenerate of the Kodaira type I_2. Also one can show that E satisfies

$$E^2 = -2\sigma \cdot D + 7D \cdot E - 6f. \tag{2.184}$$

To illustrate a Fourier-Mukai transform here we can begin by choosing the simple rank two bundle defined by extension of two line bundles chosen in (2.142) (there in the case of a Weierstrass threefold)

$$0 \longrightarrow \mathcal{O}_X(-\sigma + D_b) \longrightarrow V_2 \longrightarrow \mathcal{O}_X(\sigma - D_b) \longrightarrow 0.$$

The calculation follows along exactly the same lines as outlined in previous sections, the only interesting point here is the existence of the (-2) curves. As we saw in the Weierstrass case, requiring a non-degenerate spectral cover, implies that $2D_b - c_1(B)$ must be effective. So in the present case, the Fourier-Mukai transform of V_2 is given by,

$$\Phi(V_2) = \mathcal{O}_{2\sigma+2D_b+c_1(B)}(\sigma + D_b).$$

In this case, the number of (-2)-curves in the spectral cover induced by the exceptional divisor is $\kappa := D \cdot_{B_2} (2D_b + c_1(B))$. So clearly the line bundle over the spectral cover is trivial with respect to the (-2)-curves, since $c_1(\mathcal{L}) = \sigma + D_b$.

From this starting point though, it is clear that we choose a new spectral sheaf with some of these exceptional divisors "turned on," and apply the inverse Fourier-Mukai transform. This will allow us to see how to modify a simple vector bundles line the one above so that its Fourier-Mukai transform will have some non-trivial dependence on the fibral (-2)-curves.

To this end, recall that the Fourier-Mukai transform above is given by a short exact sequence,

$$0 \longrightarrow \mathcal{O}_X(-\sigma + K_b - D_b) \longrightarrow \mathcal{O}_X(\sigma + D_b) \longrightarrow \Phi(V_2) \longrightarrow 0.$$

Now if we twist the above sequence with the $\mathcal{O}_X(E)$, then we obtain a Fourier-Mukai transform of a new stable rank two bundle $\tilde{V}_2$ with spectral line bundle,

$$c_1(\mathcal{L}) = \sigma + D_b + \sum_{i=1}^{\kappa} e_i. \qquad (2.185)$$

So twisting with $\mathcal{O}_X(E)$ turns on all of the exceptional divisors with multiplicity one.

Now it is possible to apply an inverse Fourier-Mukai transform. We will omit the details here from brevity and simply state the result, namely a defining sequence for a new bundle $\tilde{V}_2$,

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_X(-\sigma + D_b) \ &\longrightarrow \ \tilde{V}_2 \longrightarrow \mathcal{O}_X(\sigma - D_b + D - E) \\ &\longrightarrow \ \mathcal{O}_{D-E}(-\sigma + D_b + D + K_B) \longrightarrow 0. \end{aligned} \qquad (2.186)$$

Note that $D - E$ is an effective divisor. We can easily compute the Chern character of $\tilde{V}_2$ from the exact sequence above (and using GRR),

$$ch(\tilde{V}_2) = 2 - \sigma\,(2D_b + c_1(B)) + E \cdot (2D_b + c_1(B)) - D_b^2 + D \cdot (K_B - 2D_b)\,. \qquad (2.187)$$

This is in agreement with the topological equations derived above with $\beta_i = 1$, $\kappa = D \cdot (2D_b + c_1(B))$ and $\zeta = -(2D_b + c_1(B))$.

2.5.4 A bundle defined via extension on a CY threefold with rk(MW)=1

Once again in the case of an elliptic manifold with more than one section (and a holomorphic zero-section) we can illustrate the techniques of an FM transform via a simple rank two vector bundle defined via an extension,

$$0 \longrightarrow \mathcal{O}_X(-\sigma - S_1 + D_b) \longrightarrow V_2 \longrightarrow \mathcal{O}_X(\sigma + S_1 - D_b) \longrightarrow 0, \tag{2.188}$$

where here S_1 is the Shioda map (see Section 2.3) associated to the second section to the elliptic fibration.

Following the same pattern as in the Weierstrass case, we first compute the extension group,

$$Ext^1(\mathcal{O}_X(\sigma + S_1 - D_b), \mathcal{O}_X(-\sigma - S_1 + D_b)) = H^1(X, \mathcal{O}_X(-2\sigma - 2S_1 + 2D_b)). \tag{2.189}$$

To use the Leray spectral sequence we need to know the derived direct images of $\mathcal{O}_X(-2\sigma_1)$. With the help of Koszul sequence for σ_1 one obtains

$$R\pi_*\mathcal{O}_X(-2\sigma_1) = (K_B \oplus K_B^{-1})[-1]. \tag{2.190}$$

So we see that the extension group decomposes into two subgroups,

$$\begin{aligned} &Ext^1(\mathcal{O}_X(\sigma + S_1 - D_b), \mathcal{O}_X(-\sigma - S_1 + D_b)) = \\ &\qquad H^0(B, \mathcal{O}_B(2D_b + c_1(B)) \oplus \mathcal{O}_B(2D_b + 3c_1(B))). \end{aligned} \tag{2.191}$$

We expect that these two subgroups determine the complex structure of the spectral cover, and if we choose a generic element (assuming $2D_b + 3c_1(B)$ is effective), the spectral cover must be smooth, and the topological formulas derived in Section 2.3 must be valid.

Before computing the Fourier-Mukai transform of this bundle, it is useful to consider the Chern character of the bundle given in (2.188),

$$ch(V_2) = 2 - (3c_1(B) + 2D_b)\sigma - (3c_1(B) + 2D_b)S_1 + D_b^2 - 2c_1(B)^2. \tag{2.192}$$

From this form, we expect that if the topological formulas given in Section 2.3 are satisfied, the divisor class of S must be $2\sigma + 2D_b + 3c_1(B)$, and $c_1(\mathcal{L}) = \sigma - S_1 + c_1(B) + D_b$.

Now we can compute the Fourier-Mukai explicitly (along the same lines as in previous sections) and obtain

$$\Phi(\mathcal{O}_X(\sigma + S_1 - D_b)) = \mathcal{O}_X(-\sigma - S_1 - 2c_1(B) - D_b), \tag{2.193}$$
$$\Phi(\mathcal{O}_X(-\sigma - S_1 + D_b)) = \mathcal{O}_X(\sigma - S_1 + c_1(B) + D_b)[-1]. \tag{2.194}$$

Therefore the Fourier Mukai transform of V_2 is simply given by the following torsion sheaf,

$$\Phi(V_2) = \mathcal{O}_{2\sigma+2D_b+3c_1(B)}(-\sigma - S_1 + c_1(B) + D_b)[-1]. \tag{2.195}$$

In this carefully engineered example then, we are once again able to confirm the results derived in Section 2.3, but we emphasize again that the topological formulas derived will not generally satisfied by a randomly chosen bundle on the elliptic threefold.

2.6 Small instanton transitions and spectral covers

An application of the tools we have developed in Sections 2.2 is to consider small instanton transitions [135] (i.e. M5-brane/Fixed plane transitions in the language of Heterotic M-theory [111]) involving spectral cover bundles. This subject was first explored in depth in [45, 135] and there a simple form for such transitions were found for smooth spectral covers within Weierstrass models. Within that geometric setting, the authors categorized possible small instanton transitions involving spectral covers as a) Gauge group changing or b) Chirality changing depending on which components of the effective curve class

$$W = W_B\sigma + a_f f \tag{2.196}$$

(wrapped by the 5-brane) are "absorbed" into the bundle on the a boundary brane. Here σ is the holomorphic section of the Weierstrass 3-fold, W_B is a curve within the base B_2 and f the fiber class. The authors concluded that in the case that a part of the 5-brane wrapping the fiber class is absorbed into the bundle this can result in case a) above while if a curve in the base is involved (i.e W_B above) then the transition will induce a chirality change in the Heterotic effective theory, while in the case of purely "vertical" transitions (involving detaching a part of a_f above) the chirality is unchanged.

In the following section we will demonstrate that the generalized geometric setting for elliptically fibered CY 3-folds and spectral covers that we have found in Sections 2.2 provides new possibilities for such 5-brane transitions. In particular, we will illustrate these possibilities in the case of a transition involving a 5-brane wrapping a curve that is part of a fibral divisor (in the geometric setting of Section 2.2).

2.6.1 New chirality changing small instanton transitions

Consider for simplicity the case that X_3 contains a single fibral divisor class, D_1. Suppose that the small instanton is localized on a component of the I_2 fibers, C_1 (as defined in Section 2.2) with class,

$$[C_1] = (D - D_1) \cdot D \tag{2.197}$$

where D is a divisor pulled back from the base, B_2 and D_1 is the fibral divisor. Recall that in the case of a CY 3-fold of the type described in Section 2.2 we can parameterize the topology of a general bundle V as

$$ch(V) = N - \left(\sigma\eta + \omega f + \sum \zeta D_1\right) + \frac{1}{2}c_3(V) \ . \tag{2.198}$$

As described in [135], if the 5-brane is moved to touch the E_8 fixed plane in a small instanton transition, this geometrically results first in a torsion sheaf V_{C_1} supported over C_1, which can be combined with the initial smooth $SU(N)$ bundle V to make a torsion free sheaf $\tilde{V}$:

$$0 \longrightarrow \tilde{V} \longrightarrow V \longrightarrow i_{C_1 *}\mathcal{F} \longrightarrow 0, \tag{2.199}$$

where $i_{C_1} : C_1 \hookrightarrow X$ is the inclusion of the curve mentioned above, and $\mathcal{F}$ is the sheaf supported over the curve C_1, wrapped by the 5-brane. The specific order of the sheaves in (2.199) is chosen to describe the *absorption* of the 5-brane.

The final step in the process of the small instanton transition is to consider, for specific choices for $\mathcal{F}$, whether it is possible to "smooth out" $\tilde{V}$, to a final smooth/stable vector bundle, $\hat{V}$ as in [135]. To this end, we consider choices of sheaf $\mathcal{F}$ above (corresponding to parts of the 5-brane class which can be "detached" and absorbed into $\tilde{V}$) and ask whether the resulting bundle can be smoothed. In the case of the single fibral divisor we are considering (i.e. I_2 fibers as in Section 2.2), the curve being wrapped by the 5-branes is topologically a $\mathbb{P}^1$ and we can take the sheaf supported over the 5-brane to be simply a line bundle. Below we explore two choices of this line bundle.

Case 1: $\mathcal{F} = \mathcal{O}_{C_1}(-1)$ From (2.199), the total Chen character of $\tilde{V}$ is,

$$ch(\tilde{V}) = ch(V) - [C_1] = ch(V) + D \cdot D_1 - f. \tag{2.200}$$

In addition, recall that the Fourier-Mukai transform of $\mathcal{O}_{C_1}(-1)$ is $\mathcal{I}_{C_1} = \mathcal{O}_{C_2}(-2)$. So one can apply the Fourier-Mukai functor to (2.199) to obtain,

$$0 \longrightarrow i_{C_2 *}\mathcal{O}(-2) \longrightarrow i_{\tilde{S}}\tilde{\mathcal{L}} \longrightarrow i_S\mathcal{L} \longrightarrow 0, \tag{2.201}$$

where $\Phi(V) = i_S\mathcal{L}[-1]$ and $\Phi(\tilde{V}) = \tilde{\mathcal{L}}$ are Fourier-Mukai transforms of V and $\tilde{V}$ which are torsion sheaves supported over the N-sheeted covers of the base, S and $\tilde{S}$ respectively. Taking the case that S is integral, and C_2 is one of the (-2)-curves which S wraps, then $\tilde{S} = S$, and we get,

$$c_1(\tilde{\mathcal{L}}) = c_1(\mathcal{L}) + e_1. \tag{2.202}$$

Note that $\tilde{\mathcal{L}}$ is singular over C_2 $(= e_1)$, as may be expected,[20] however, in the process of deforming $\tilde{V}$ to a smooth bundle, $\tilde{\mathcal{L}}$ may also be smoothed out to a line bundle $\hat{\mathcal{L}}$ with the same topology. In this case we can say from the topological data derived earlier in this section that the corresponding (hypothetically) smooth vector bundle $\hat{V}$ must have the following topology (see (2.198) above)

$$\zeta(\hat{V}) = \zeta(V) - D, \tag{2.203}$$

$$\omega(\hat{V}) = \omega(V) + f, \tag{2.204}$$

$$ch(\hat{V}) = ch(V) + D \cdot D_1 - f. \tag{2.205}$$

For these choices, $ch(\hat{V})$ is the same as $ch(\tilde{V})$. So we conclude this transition is topologically unobstructed. In this case we can see that the third Chern character doesn't change in this transition (also γ remains unchanged), therefore neither the chiral index or zero-mode spectrum are changed.

Case 2: $\mathcal{F} = \mathcal{O}_{c_1}(-2)$ As above, from (2.199) we compute the Chern character of $\tilde{V}$ as

$$ch(\tilde{V}) = ch(V) - [C_1] + 1w, \tag{2.206}$$

[20] Due to the flatness of the projection and the Poincare bundle in the definition of the FM functor we use here, singularity of the "vector bundle" and the spectral sheaf are closely correlated.

where w is dual to the zero cycles. Note that if $\tilde{V}$ can be smoothed, we expect $ch(\tilde{V}) = ch(\hat{V})$ for the final smooth bundle after the small instanton transition. Thus it is clear that both the second Chern class and chirality can change in this case,

$$c_2(\hat{V}) = c_2(V) + D \cdot (D - D_1), \tag{2.207}$$
$$\frac{1}{2}c_3(\hat{V}) = \frac{1}{2}c_3(V) + 1. \tag{2.208}$$

To address the question of smoothing, we simply apply the Fourier Mukai functor to (2.199) for the chosen $i_{C_1*}\mathcal{F}$ and assume V is already WIT_1,

$$0 \longrightarrow i_{\tilde{S}*}\tilde{\mathcal{L}} \longrightarrow i_{S*}\mathcal{L} \longrightarrow i_{C_1*}\mathcal{O}_{C_1} \longrightarrow 0, \tag{2.209}$$

and we noted that $\Phi(i_{C_1*}\mathcal{O}_{C_1}(-2)) = i_{C_1*}\mathcal{O}_{C_1}[-1]$.

Now it must be observed that as long as the above short exact sequence can exist, the sheaf $\tilde{V}$ is indeed WIT_1. Note that since an irreducible spectral cover never wraps C_1, then the existence of this sequence forces both S and $\tilde{S}$ to have vertical components that contain C_1. As a result then, we can choose to consider a small instanton transition in which the spectral cover of the initial bundle V is *reducible* with vertical (i.e fiber-directions) and horizontal components,

$$S = S_V \cup S_H, \tag{2.210}$$

where S_V contains C_1. For simplicity, we will illustrate this transition below in the case that the divisor class S_V is simply D, and $\mathcal{L}_V$ is a line bundle.

Note that although we are choosing the spectral cover to be reducible, it is not the case that V itself must be a reducible bundle. As a next step, we can consider what topological constraints must be in place for a stable degree zero vector bundle such that its Fourier Mukai transform $i_{S*}\mathcal{L}$ is made of a vertical and horizontal piece:

$$0 \longrightarrow i_{S_H*}\mathcal{L}_V \longrightarrow i_{S*}\mathcal{L} \longrightarrow i_{S_V*}\mathcal{L}_H \longrightarrow 0. \tag{2.211}$$

Following the same procedure as before we can derive the the topological data,

$$[S_H] = N\sigma + \eta - D, \tag{2.212}$$
$$[S_H] \cdot \left(c_1(\mathcal{L}_H) - \frac{1}{2}[S_H]\right) + D \cdot \left(c_1(\mathcal{L}_V) - \frac{1}{2}D\right)$$
$$= (N\sigma + \eta)\left(-\frac{1}{2}c_1(B)\right) - \frac{1}{2}c_3(V)f - \zeta e_1. \tag{2.213}$$

A solution for this equation can be given as,

$$c_1(\mathcal{L}_H) = -\frac{1}{2}(c_1(B) - [S_H]) + \gamma_H, \tag{2.214}$$
$$\gamma_H = \lambda_H (N\sigma - \eta + D + Nc_1(B)) + \delta\sigma, \tag{2.215}$$
$$c_1(\mathcal{L}_V) = -\zeta e + \lambda_V D - \delta\sigma, \tag{2.216}$$
$$\frac{1}{2}c_3(V) = \lambda_H \eta (\eta - Nc_1(B)) - \lambda_V + \frac{1}{2}D\cdot(D - c_1(B)). \tag{2.217}$$

After a tedious algebraic calculation, one can derive a formula for ω, but it is not necessary here. Finally if we require both V and $\tilde{V}$ have the same spectral cover,[21] then (2.209) implies the following relation between the vertical parts of the spectral sheaves,

$$\tilde{\mathcal{L}}_{\tilde{V}} = \mathcal{L}_V \otimes \mathcal{O}_{S_V}(-D + E). \tag{2.218}$$

Therefore we easily get the following relations between the parameters of $\tilde{V}$ and V,

$$\lambda_{\tilde{V}} = \lambda_V - 1, \tag{2.219}$$
$$\zeta_{\hat{V}} = \zeta_V - 1. \tag{2.220}$$

Moreover if we put $\delta_V = \delta_{\hat{V}} = 0$, we can see by the above arguments that,

$$\omega_{\hat{V}} = \omega_V + 1. \tag{2.221}$$

Finally, we arrive at a point where we can compare the above conditions on V and $\tilde{V}$ with the relations (2.207) derived before and observe that they are exactly the same. Thus, the transition is unobstructed and we have provided an example of a complete (i.e. smooth-able) chirality changing transition involving fibral curves.

We should emphasize that the above geometry is by no means general and many choices were made for simplicity of computation. Nonetheless, it serves to illustrate that the existence of fibral divisors in the elliptically fibered CY 3-fold will make new forms of small instantons possible. In particular, the example above is a chirality changing transition that is unique compared to those classified in [135] for Weierstrass form (in which "vertical" transitions changed only the gauge group and "horizontal" curves led to chirality change). In this example we find chirality change from new vertical curves for the 5-brane to wrap and the gauge group remains unchanged even though C_1 is a vertical curve.

[21] Note for simplicity we assumed $\mathcal{L}_H$ is independent of the (-2)-curves on the horizontal components S_H.

2.7 Reducible spectral covers and obstructions to smoothing

As illustrated by the examples in Section 2.5, there are many limitations to the analysis that we completed in Sections 2.1 to 2.3. First, the Picard number of the spectral cover maybe larger $1 + h^{1,1}(B_2)$ generically. This corresponds to spectral surfaces in which there exist more divisors than those inherited from the ambient Calabi-Yau threefold. Moreover, it is known that at higher co-dimensional loci in moduli space, this Picard group can in fact jump [2]. Second, as seen in the examples in previous sections, the spectral cover can be singular, and therefore one cannot predict the general form of $ch(i_{S*}\mathcal{L})$.

In these cases it may be possible to choose special sheaves $\mathcal{L}$ that "obstruct" the deformation of the spectral cover to a smooth one. In other words, the corresponding vector bundles lands on a different component[22] than the one that is analyzed in [89, 90]. In this section we briefly outline how such a situation might be realized in the case that spectral cover is reducible but reduced. This analysis has some similarity to examples analyzed in [71].

We begin with the spectral data $(\mathcal{L}, S)$ of a bundle V defined over a Weierstrass CY threefold $\pi : X \to B$, where

$$S := S_1 \cup_\Sigma S_2, \tag{2.222}$$
$$0 \longrightarrow \mathcal{L}_1 \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}_2 \longrightarrow 0. \tag{2.223}$$

As usual

$$ch(V) = N - (\sigma\eta + \omega) + \frac{1}{2}c_3(V), \tag{2.224}$$

$$\begin{aligned} -ch(\Phi(V)) = ch(\mathcal{L}) = (N\sigma + \eta) \quad &+ \quad (N\sigma + \eta)\left(-\frac{c_1(B)}{2}\right) \\ &+ \quad \left(\frac{1}{6}nc_1(B)^2 - \omega\right). \end{aligned} \tag{2.225}$$

[22]Note that this cannot happen for a vector bundle over an elliptically fibered K3 surface. This phenomenon only appears for CY manifolds of complex dimension 3 or higher.

Now we assume that

$$[S_1] = n_1\sigma + \eta_1, \tag{2.226}$$

$$[S_2] = n_2\sigma + \eta_2, \tag{2.227}$$

$$N = N_1 + N_2, \tag{2.228}$$

$$\eta = \eta_1 + \eta_2. \tag{2.229}$$

With these assumptions, the general for for $c_1(\mathcal{L}_1)$ and $c_1(\mathcal{L}_2)$ are given below,

$$c_1(\mathcal{L}_1) = \frac{1}{2}\left(-c_1(B) + [S_1]\right) + \gamma_1 + \alpha_1[S_2], \tag{2.230}$$

$$c_1(\mathcal{L}_2) = \frac{1}{2}\left(-c_1(B) + [S_2]\right) + \gamma_2 + \alpha_2[S_1], \tag{2.231}$$

$$\gamma_i = \lambda_i\left(N_i\sigma - \eta_i + N_i c_1(B)\right), \tag{2.232}$$

$$\frac{1}{2}c_3(V) = \sum_i N_i\lambda_i\eta_i\left(\eta_i - N_i c_1(B)\right). \tag{2.233}$$

The main difference of the equations above with the standard one is the existence of the terms $\alpha_1[S_2]$ and $\alpha_2[S_1]$. For consistency we demand,

$$\alpha_1 + \alpha_2 = 0. \tag{2.234}$$

Note the existence of such terms implies $(\mathcal{L}_i, S_i)$ are spectral data of vector bundles V_i with first Chern class,

$$c_1(V_i) = \alpha_i\left(N_1\eta_2 + N_2\eta_1 - N_1N_2c_1(B)\right). \tag{2.235}$$

It is next possible to compute ω as before,

$$\begin{aligned}\frac{1}{6}Nc_1(B)^2 - \omega &= \frac{Nc_1(B)^2}{8} \\ &+ \frac{c_1(B)^2}{24}(N_1^3 + N_2^3) + \frac{1}{8}\left(N_1\eta_1\left(\eta_1 - N_1c_1(B)\right) + N_2\eta_2\left(\eta_2 - N_2c_1(B)\right)\right) \\ &+ \frac{1}{2}\pi_{1*}\gamma_1^2 + \frac{1}{2}\pi_{2*}\gamma_2^2 \\ &+ \frac{1}{2}\Sigma\cdot\left(\alpha_1^2[S_2] + \alpha_2^2[S_1] + 2\alpha_1\gamma_1 + 2\alpha_2\gamma_2\right).\end{aligned} \tag{2.236}$$

After some algebra it can be shown that *only for* $\alpha_1 = -\alpha_2 = \pm\frac{1}{2}$ can the above equation be simplified to,

$$\begin{aligned}\frac{1}{6}Nc_1(B)^2-\omega &= \frac{Nc_1(B)^2}{8}\\ &+ \frac{c_1(B)^2}{24}N^3+\frac{1}{8}N\eta\left(\eta-Nc_1(B)\right)\\ &+ \frac{1}{2}\pi_{1*}\gamma_1^2+\frac{1}{2}\pi_{2*}\gamma_2^2\\ &+ \Sigma\cdot\left(\alpha_1\gamma_1+\alpha_2\gamma_2\right). \end{aligned} \tag{2.237}$$

This is almost the same as the standard formula expected from Section 2.1 if there exists a λ such that

$$\frac{1}{2}\pi_*\gamma^2=\frac{1}{2}\pi_{1*}\gamma_1^2+\frac{1}{2}\pi_{2*}\gamma_2^2+\Sigma\cdot\left(\alpha_1\gamma_1+\alpha_2\gamma_2\right). \tag{2.238}$$

We come now to our central claim in this section:
If the restriction of $\mathcal{L}$ to Σ is a trivial line bundle, then it is always possible to deform the "singular" spectral data to a "smooth" spectral data, such that it satisfies the generic formulae expected in (2.37) – (2.39). Otherwise it is impossible (generically). In particular if the restriction is a non-trivial degree zero line bundle, the deformation is obstructed.

First note that if $\mathcal{L}$ is defined as

$$0\longrightarrow\mathcal{L}_1\longrightarrow\mathcal{L}\longrightarrow\mathcal{L}_2\longrightarrow 0, \tag{2.239}$$

the restriction of $\mathcal{L}$ on S_1 and S_2 are

$$\begin{aligned}&\mathcal{L}_1\otimes K_{S_2}|_{S_1},\\ &\mathcal{L}_2,\end{aligned} \tag{2.240}$$

respectively. Therefore the line bundle induced over Σ lives in

$$Hom_\Sigma\left(\mathcal{L}_2,\mathcal{L}_1\otimes K_{S_2}|_{S_1}\right)\simeq Ext^1_X\left(i_{S_2*}\mathcal{L}_2, i_{S_1*}\mathcal{L}_1\right), \tag{2.241}$$

corresponding to extensions. Conversely, if we define $\mathcal{L}$ as,

$$0\longrightarrow\mathcal{L}_2\longrightarrow\mathcal{L}\longrightarrow\mathcal{L}_1\longrightarrow 0, \tag{2.242}$$

the restriction of $\mathcal{L}$ on S_1 and S_2 are

$$\begin{aligned}&\mathcal{L}_2\otimes K_{S_1}|_{S_2},\\ &\mathcal{L}_1,\end{aligned} \tag{2.243}$$

respectively. Therefore the line bundle induced over Σ lives in

$$Hom_{\Sigma}\left(\mathcal{L}_1, \mathcal{L}_2 \otimes K_{S_1}|_{S_2}\right) \simeq Ext^1_X\left(i_{S_1*}\mathcal{L}_1, i_{S_2*}\mathcal{L}_2\right), \tag{2.244}$$

corresponding to the opposite extensions. If we rewrite the left hand side of (2.241) as,

$$\begin{aligned} &H^0(\Sigma, \mathcal{F}), \\ &\mathcal{F} := \mathcal{L}_1 \otimes \mathcal{L}_2^* \otimes K_{S_2}|_{S_1}, \end{aligned} \tag{2.245}$$

then (2.244) can be written as,

$$H^0\left(\Sigma, \mathcal{F}^* \otimes K_{\Sigma}\right). \tag{2.246}$$

Therefore we see[23] if $\mathcal{F} \simeq \mathcal{O}_{\Sigma}$,then both extensions are possible, and we can deform the spectral data to generic "smooth" one described in [89].

We can indeed check that in this case there is a λ that satisfy (2.238). To show that we choose,

$$\alpha_1 = -\alpha_2 = -\frac{1}{2} \tag{2.247}$$

(the other choice corresponds to $\mathcal{F} \otimes K_{\Sigma} \simeq \mathcal{O}_{\Sigma}$). Notice in this case if $\gamma_1 = \gamma_2$ as a divisor in X then $\mathcal{F} \simeq \mathcal{O}_{\Sigma}$. This constraint is equivalent to,

$$N_1\lambda_1 = N_2\lambda_2, \tag{2.248}$$

$$\eta_1\lambda_1 = \eta_2\lambda_2. \tag{2.249}$$

Let us look at (2.238) more closely,

$$\begin{aligned} \frac{1}{2}\lambda^2 N\eta(\eta - Nc_1(B)) \;=\; & \frac{1}{2}\lambda_1^2 N_1\eta_1\left(\eta_1 - N_1c_1(B)\right) \\ - \; & \frac{1}{2}\lambda_1 N_2\eta_1\left(\eta_1 - N_1c_1(B)\right) + \frac{1}{2}\lambda_2^2 N_2\eta_2\left(\eta_2 - N_2c_1(B)\right) \\ + \; & \frac{1}{2}\lambda_2 N_1\eta_2\left(\eta_2 - N_2c_1(B)\right). \end{aligned} \tag{2.250}$$

The second terms in the 2nd and 3rd line cancel. To find λ we choose an ansatz $\lambda = \alpha\lambda_1\lambda_2$, and use the constraints above, we can see the solution is ,

$$\lambda = \frac{\lambda_1\lambda_2}{\lambda_1 + \lambda_2}. \tag{2.251}$$

[23] We could also choose $\mathcal{F}^* \otimes K_{\Sigma} \simeq \mathcal{O}_{\Sigma}$. But since the analysis would run along very similar lines, we choose to just focus on the first case.

On the other hand if we request $\gamma_1 = \gamma_2$ only over Σ, i.e.

$$S_1 \cdot S_2 \cdot \gamma_1 = \gamma_1|_\Sigma = \gamma_2|_\Sigma = S_1 \cdot S_2 \cdot \gamma_2, \tag{2.252}$$

then it means $\mathcal{F}$ is an element of $J(\Sigma)$ but it is not necessarily a trivial line bundle (as $g(\Sigma) \geq 1$ generally). In this case there is no solution for Λ generally.

In summary then, we have seen in this section that the properties of reducible spectral covers may indeed be quite distinct from their smooth cousins.

Chapter 3

Heterotic/Heterotic and Heterotic/F-theory Duality

This chapter is based on the paper [20] written in collaboration with L.B. Anderson, H. Feng and X. Gao. Compactifications of the Heterotic string and F-theory can lead to identical effective theories in the situation that the background geometries of the two theories both exhibit fibration structures [131]. Namely, Heterotic string theory compactified on a Calabi-Yau n-fold with an elliptic fibration

$$\pi_h : X_n \xrightarrow{\mathbb{E}} B_{n-1} \tag{3.1}$$

over a base manifold B_{n-1}, leads to the same effective physics as F-theory compactified on a Calabi-Yau $(n+1)$-fold with a $K3$ fibration over the *same* base manifold, B_{n-1}:

$$\pi_f : Y_{n+1} \xrightarrow{K3} B_{n-1} \tag{3.2}$$

In order to have a well-defined F-theory background, the $(n+1)$-fold Y_{n+1} must also be elliptically fibered, with compatible elliptic/K3 fibrations [131, 151].

In the context of (potential) Heterotic/Heterotic dualities and Heterotic/F-theory duality then, there are a number of natural questions that arise. Suppose that $(X_3, \pi : V \to X_3)$ and $(\tilde{X}_3, \pi : \tilde{V} \to \tilde{X}_3)$ are the requisite background geometries (i.e. (manifolds, vector bundles)) defining two TSD Heterotic theories, then these questions include:

- Can Target Space Dual pairs be found in which both X and $\tilde{X}$ are elliptically fibered as in (3.1)? In principle, these two fibrations need not be related in any obvious way, for example: two topologically distinct CY 3-folds $\pi : X_3 \to B_2$ and $\tilde{\pi} : \tilde{X}_3 \to \tilde{B}_2$, with distinct 2 (complex) dimensional base manifolds B_2, $\tilde{B}_2$ to their fibrations.

- If such elliptically fibered CY 3-fold geometries can be found within a TSD pair, this will in principle lead to two CY 4-folds, Y_4 and $\tilde{Y}_4$, as dual backgrounds for F-theory. It should follow by construction that these two geometries lead to the same 4-dimensional effective theory (or at least the same massless spectrum). How can this apparent duality be understood in the context of F-theory? How are Y_4 and $\tilde{Y}_4$ related?

For the first point, to our knowledge no explicit pairs of elliptically fibered TSD Heterotic geometries have yet appeared in the literature. However, at least one proposal for the latter point has been posited. In [42] it was proposed that if fibered Heterotic TSD pairs could be found, one possibility for the induced duality in F-theory would be the existence a CY 4-fold with a single elliptic fibration, but *more than one K3 fibration*:

$$\begin{array}{ccc} & Y_4 & \\ {}^{K3}\swarrow_{\pi_f} & & {}_{\tilde{\pi}_f}\searrow^{\widetilde{K3}} \\ B_2 & & \widetilde{B}_2 \end{array} \quad ,$$

where each fibration can be seen as the F-theory dual of one of the Heterotic vacua (associated to (X, V) or $(\tilde{X}, \tilde{V})$ respectively). Since by its very definition, F-theory requires that Y_4 is also elliptically fibered, this would imply that each $K3$-fiber appearing above is itself also elliptically fibered. Moreover, since the elliptic fibration of F-theory that determines the effective physics (i.e. gauge symmetry, matter spectrum, etc), in order for the two $K3$ fibrations to lead to identical effective theories, it would be expected that in fact in this scenario, Y_4 has only one elliptic fibration, but that it is compatible with two distinct $K3$ fibrations. If these compatible fibration structures were to exist it must be that the base of the elliptic fibration, $\rho : Y_4 \to B_3$, must have two different $\mathbb{P}^1$-fibrations:

$$\begin{array}{ccc} & Y_4 & \\ & \rho_f \downarrow \mathbb{E} & \\ & B_3 & \\ {}^{\mathbb{P}^1}\swarrow_{\sigma_f} & & {}_{\tilde{\sigma}_f}\searrow^{\mathbb{P}^1} \\ B_2 & & \widetilde{B}_2 \end{array}$$

The scenario above is one obvious way in which a "duality" of sorts could arise in F-theory. Of course, in this case the essential F-theory geometry is not changing, only the $K3$ fibrations which determine the Heterotic dual. This is clearly not the only possibility. As one alternative, it could prove that the F-theory duals of Heterotic TSD pairs are in fact two distinct CY 4-folds, Y_4 and $\tilde{Y}_4$ whose gauge symmetries, massless spectra and effective $\mathcal{N} = 1$ potentials are ultimately the same through non-trivial G-flux in the background geometry. We can summarize these two options for the induced duality in F-theory as follows

1. (Possibility 1) Heterotic TSD $\Leftrightarrow$ Multiple $K3$ fibrations in a single F-theory geometry (and hence manifestly leading to the same effective physics).

2. (Possibility 2) Heterotic TSD $\Leftrightarrow$ Two distinct pairs of manifolds and G-flux, (Y_4, G_4) and $(\tilde{Y}_4, \tilde{G}_4)$ which lead to the same effective physics in F-theory.

In this work we investigate the two questions listed above and provide explicit examples of Heterotic target space dual pairs with the requisite fibration structures to lead to F-theory dual theories. As we will outline in the following sections, substantial technical difficulties arise in explicitly computing the full F-theory duals of these Heterotic theories. In the present work we do not attempt to fully determine these dual F-theories and instead provide evidence for our primary conclusion: *multiple fibrations in F-theory cannot in general explain the dual physics of* $(0, 2)$ *TSD*.

To overcome some of the technical obstacles of Heterotic/F-theory duality, we begin our analysis by actually considering Heterotic/F-theory dual pairs in 6-dimensional effective theories rather than in 4-dimensions. In this context, the Heterotic duality is a trivial one – TSD pairs simply generate two bundles over K3 with the same second Chern class, and are thus trivially guaranteed to give rise to the same massless spectrum (see e.g. [35]). However, this very simple framework for Heterotic TSD pairs allows us to explicitly perform Fourier-Mukai transforms to render the data of a holomorphic, stable vector bundle over $K3$ into its spectral cover [89]. With this data, we are able to explicitly construct examples of F-theory duals and verify that in fact they cannot arise as multiple $K3$ fibrations of a CY 3-fold, Y_3, determining an F-theory background. The results of this study are presented in Section 3.3.3.

Turning once more to our primary area of interest in $\mathcal{N} = 1$ and Heterotic compactifications on CY 3-folds, we outline the essential ingredients determining the dual F-theory geometry. We find that in general a number of technical tools are missing for fully determining the F-theory physics. Some of these we have developed and will appear separately [21, 22] while others we leave to future work. However, we are able to indicate that in general the intermediate Jacobians of the dual F-theory geometries must play some role in the new "F-theory duality" whatever it may prove to be. This leads to the presence of essential data not associated to the complex structure of the CY 4-fold alone, but G-flux as well. In the singular limit such fluxes are well known to have the potential to dramatically change the effective physics through so-called T-brane solutions [14, 18, 51] and other possibilities.

In the following sections we will explore these ideas in detail. In particular, this chapter is organized in the following way. In Section 3.1 we review briefly the essential aspects of $(0, 2)$ Target Space Duality. In Section 3.2, we provide the first non-trivial examples to appear in the literature of Heterotic TSD pairs in which both CY 3-folds, X and $\tilde{X}$ are elliptically fibered. In these cases, the Heterotic geometries are smooth (consisting of smooth so-called CICY threefolds [49] and stable, holomorphic vector bundles defined via the monad construction [114] over them) and lead to well controlled, perturbative Heterotic theories. However we will demonstrate in this and subsequent sections that existing techniques in the literature to determine dual F-theory geometries, as outlined in Section 3.3.6 are insufficient to determine the geometry of Y_4 and $\tilde{Y}_4$ in these cases. However, we none-the-less still find some evidence indicating that multiple fibrations of Y_4 cannot be the F-theory manifestation of $(0, 2)$ TSD.

To make concrete the dual F-theory geometry we move to 6-dimensional examples in Section 3.3.3. More precisely, we consider Heterotic TSD theories consisting of pairs of bundles over $K3$ in which the second Chern class of both V and $\tilde{V}$ is taken to be 12. In this case, it is possible that the F-theory geometry, Y_3 is multiply fibered as described above. However after finding the spectral data (i.e. Fourier Mukai transform) of these bundles, we can explicitly construct the dual F-theory geometry and find that it does not in general agree with what can be obtained by multiple fibrations. We will argue further that the F-theory "image" of target space duality under the Heterotic/F-theory map shouldn't be purely geometric even in 6-dimensions, but rather it can be related to the intermediate Jacobian of the CY 3-fold. With these tools and observations in hand we return to the F-theory duals of four dimensional, $\mathcal{N} = 1$ Heterotic theories in Section 3.3.6.

Finally, in the Appendices we consider a handful of examples illustrating both the range of possibilities arising in Heterotic TSD dual geometries, as well as potential pitfalls that can arise in constructing dual pairs.

3.1 A brief review of (0,2) target space duality

Heterotic target space duality is best understood in the context of Heterotic string compactifications associated to $(0,2)$ gauged linear sigma models (GLSMs). It was first observed by Distler and Kachru in 1995 [65], and further studied by Blumenhagen [40, 41] with later a landscape study [42]. The GLSM provides a description of the complexified compact stringy Kähler moduli space which is divided into various phases [156]. The freedom to vary a Fayet-Illiopolos parameter links variety of distinct phases including the geometric phases (associated to target space geometries like Calabi-Yau threefolds X and holomorphic vector bundle V), non-geometric phase (commonly a Landau-Ginzburg phase), and a rich variety of hybrid phase. Described in the (0,2) GLSM language, target space duality is realized by exchanging two certain types of charges in theory, which is defined by (X,V) in the geometric phase, to give a different configuration $(\tilde{X},\tilde{V})$ from the original one while leaving the superpotential invariant and sharing a common Landau-Ginzburg phase. Meanwhile, in the geometric phases, this pair of theories (X,V) and $(\tilde{X},\tilde{V})$ preserve net number of moduli, the complete charged and singlet particle spectra.

In an Abelian GLSM, there exists multiple $U(1)$ gauge fields $A^{(\alpha)}$ with $\alpha = 1,...,r$, two sets of chiral superfields as $\{X_i|i=1,...,d\}$ with $U(1)$ charges $Q_i^{(\alpha)}$, and $\{P_l|l=1,...,\gamma\}$ with $U(1)$ charges $-M_l^{(\alpha)}$. Furthermore, there are two sets of Fermi superfields: $\{\Lambda^a|a=1,...,\delta\}$ with charges $N_a^{(\alpha)}$, and $\{\Gamma^j|j=1,...,c\}$ with charges $-S_j^{(\alpha)}$. These charges are given in order to realize the Calabi-Yau manifolds as complete intersection hypersurfaces in ambient space (Complete Intersection Calabi-Yau (CICY)) and stable, holomorphic vector bundles over them in some geometric phase. As a result, we will require the charges $Q_i^{(\alpha)} \geq 0$ and for each i, there exists at least one r such that $Q_i^{(\alpha)} > 0$. Similar assumption of (semi-)positivity will also hold for the charges $S_j^{(\alpha)}$ and $M_l^{(\alpha)}$. However, in some cases we will consider solutions in which charges $N_a^{(\alpha)}$ may be negative. Then the field content and charges of GLSM can be summarized in the following "charge matrix,"

$$\begin{array}{cccc||cccc}
\multicolumn{4}{c||}{x_i} & \multicolumn{4}{c}{\Gamma^j} \\
\hline
Q_1^{(1)} & Q_2^{(1)} & \dots & Q_d^{(1)} & -S_1^{(1)} & -S_2^{(1)} & \dots & S_c^{(1)} \\
Q_1^{(2)} & Q_2^{(2)} & \dots & Q_d^{(2)} & -S_1^{(2)} & -S_2^{(2)} & \dots & S_c^{(2)} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
Q_1^{(r)} & Q_2^{(r)} & \dots & Q_d^{(r)} & -S_1^{(r)} & -S_2^{(r)} & \dots & S_c^{(r)}
\end{array} \tag{3.3}$$

$$\begin{array}{cccc||cccc}
\multicolumn{4}{c||}{\Lambda^a} & \multicolumn{4}{c}{p_l} \\
\hline
N_1^{(1)} & N_2^{(1)} & \dots & N_\delta^{(1)} & -M_1^{(1)} & -M_2^{(1)} & \dots & -M_\gamma^{(1)} \\
N_1^{(2)} & N_2^{(2)} & \dots & N_\delta^{(2)} & -M_1^{(2)} & -M_2^{(2)} & \dots & -M_\gamma^{(2)} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
N_1^{(r)} & N_2^{(r)} & \dots & N_\delta^{(r)} & -M_1^{(r)} & -M_2^{(r)} & \dots & -M_\gamma^{(r)}
\end{array} \tag{3.4}$$

We can denote such starting point in the geometric phase as:

$$V_{N_1,\dots N_\delta}[M_1,\dots,M_\gamma] \longrightarrow \mathbb{P}_{Q_1,\dots Q_d}[S_1,\dots,S_c]. \tag{3.5}$$

Here anomaly cancellation condition requires the following linear and quadratic constraints for all $\alpha,\beta = 1,...,r$:

$$\begin{aligned}
\sum_{a=1}^{\delta} N_a^{(\alpha)} = \sum_{l=1}^{\gamma} M_l^{(\alpha)}, \qquad & \qquad \sum_{i=1}^{d} Q_i^{(\alpha)} = \sum_{j=1}^{c} S_j^{(\alpha)}, \\
\sum_{l=1}^{\gamma} M_l^{(\alpha)} M_l^{(\beta)} - \sum_{a=1}^{\delta} N_a^{(\alpha)} N_a^{(\beta)} \quad &= \quad \sum_{j=1}^{c} S_j^{(\alpha)} S_j^{(\beta)} - \sum_{i=1}^{d} Q_i^{(\alpha)} Q_i^{(\beta)}.
\end{aligned} \tag{3.6}$$

The GLSM is further described by a superpotential and a scalar potential, while the scalar potential has contributions from F-term and D-term:

$$\begin{aligned}
S &= \int d^2z d\theta \Big[\sum_j \Gamma^j G_j(x_i) + \sum_{l,a} P_l \Lambda^a F_a^l(x_i) \Big], \\
V_F &= \sum_j \big| G_j(x_i) \big|^2 + \sum_a \big| \sum_l p_l F_a^l(x_i) \big|^2, \\
V_D &= \sum_{\alpha=1}^{r} \Big(\sum_{i=1}^{d} Q_i^{(\alpha)} |x_i|^2 - \sum_{l=1}^{\gamma} M_l^{(\alpha)} |p_l|^2 - \xi^{(\alpha)} \Big)^2,
\end{aligned} \tag{3.7}$$

where the functions G_j and F_a^l are quasi-homogeneous polynomials with degrees shown in the following matrix:

$$\begin{array}{cccc}
\multicolumn{4}{c}{G^j} \\
\hline
S_1 & S_2 & \dots & S_c
\end{array}
\qquad
\begin{array}{cccc}
\multicolumn{4}{c}{{F_a}^l} \\
\hline
M_1 - N_1 & M_1 - N_2 & \dots & M_1 - N_\delta \\
M_2 - N_1 & M_2 - N_2 & \dots & M_2 - N_\delta \\
\vdots & \vdots & \ddots & \vdots \\
M_\gamma - N_1 & M_\gamma - N_2 & \dots & M_\gamma - N_\delta
\end{array} \tag{3.8}$$

Furthermore, the function F_a^l will be chosen to satisfy a transversality condition such that $F_a^l(x) = 0$ only when $x_i = 0$ for $i = 1, ..., d$.

The $\xi^{(\alpha)} \in \mathbb{R}$ in the D-term potential is the Fayet-Iliopoulos (FI) parameter which determines the structure of the vacuum. From the original Witten's paper [156], consider the simple case with only one $U(1)$ so there is only one ξ: If $\xi > 0$ then it's the geometric phase, described by a vector bundle over a Calabi-Yau manifold: $V_{N_1,...,N_\delta}[M_1, ..., M_\gamma] \longrightarrow X \equiv \mathbb{P}_{Q_1,...,Q_d}[S_1, ..., S_c]$, where the Calabi-Yau manifold is defined by complete intersection hypersurfaces in weighted projective space, i.e. CICY $X = \cap_{j=1}^c G_j$ with $G_j(x_i) = 0$, and the vector bundle is defined by

$$V = \frac{ker(F_a^l)}{im(E_i^a)} \tag{3.9}$$

with $rk(V) = (\delta - \gamma - r_{\mathcal{V}})$ through the monad on X:

$$0 \to \mathcal{O}_X^{\oplus r_{\mathcal{V}}} \xrightarrow{E_i^a} \bigoplus_{a=1}^{\delta} \mathcal{O}_X(N_a) \xrightarrow{F_a^l} \bigoplus_{l=1}^{\gamma} \mathcal{O}_X(M_l) \to 0, \tag{3.10}$$

for some integers δ and γ. If $\xi < 0$ then it's the Landau-Ginzburg phase described by a superpotential:

$$\mathcal{W}(x_i, \Lambda^a, \Gamma^i) = \sum_j \Gamma^j G_j(x_i) + \sum_a \Lambda^a F_a(x_i), \tag{3.11}$$

where it would be a hybrid-type non-geometric phases if there are multiple $U(1)'s$.

Now let us move to the target space duality. The first observation is an exchange/relabeling of G_j's and F_a's will leave the superpotential (3.11) invariant. This observation indicates that two distinct GLSMs could "share" a non-geometric phase in which the original role of G_j and F_a is obscured. So the full procedure of target space dual would be starting from a geometric phase, go to a Landau-Ginzburg phase, do a rescaling/relabeling of the fields, and go back to the geometric phase to get a new Calabi-Yau/vector bundle configuration.

If the Landau-Ginzburg phase exists, then the rescaling procedure is as follows, for a non-vanishing p_l and all $i = 1, \dots k$:

$$\tilde{\Lambda}^{a_i} := \frac{\Gamma^{j_i}}{\langle p_l \rangle}, \qquad \tilde{\Gamma}^{j_i} = \langle p_l \rangle \Lambda^{a_i}, \tag{3.12}$$

$$||\tilde{\Lambda}^{a_i}|| = ||\Gamma^{j_i}|| - ||P_l||, \qquad ||\tilde{\Gamma}^{j_i}|| = ||\Lambda^{a_i}|| + ||P_l||, \tag{3.13}$$

with $\sum_i ||G_{j_i}|| = \sum_i ||F_{a_i}{}^l||$ for anomaly cancellation. One thing to notice is that exchanging only one F with one G does nothing. So in all examples two or more F_a's are exchanged with two or more G_j's. It is clear that the "relabeling" of fields at the shared Landau-Ginzburg point can mix the degrees of freedom in $h^{2,1}(X)$ and $h^1(X, End_0(V))$ in the target space dual. In the landscape [42], the dual sides match in the number of charged matter, and the total number of massless gauge singlets, where the individual number of complex, Kähler and bundle moduli are interchanged as:

$$\begin{aligned} &h^*(X, \wedge^k V) = h^*(\widetilde{X}, \wedge^k \widetilde{V}), \qquad k = 1, 2, \cdots, rk(V) \\ &h^{2,1}(X) + h^{1,1}(X) + h^1_X(End_0(V)) = h^{2,1}(\widetilde{X}) + h^{1,1}(\widetilde{X}) + h^1_X(End_0(\widetilde{V})). \end{aligned} \tag{3.14}$$

Furthermore, more general target space duality are possible such that it can *also* change the dimension of $h^{1,1}(\tilde{X})$. For example, if there is only one column in G, which is not enough to make the exchange, then a blow up of $\mathbb{P}^1$ on the manifold will help. This procedure leads to a dual models with an additional U(1) action. In this case, it is necessary to re-write the initial GLSM in an equivalent/redundant way. It is always possible to introduce into the GLSM a new coordinate (i.e. a new Fermi superfield) y_1 with multi-degree $\mathcal{B}$ and a new hypersurface (i.e. a Chiral superfield with opposite charge to the new Fermi superfield) $G^{\mathcal{B}}$ corresponding to a homogeneous polynomial of multi-degree $\mathcal{B}$. Similar to (3.5), the above addition can be written

$$V_{N_1,\ldots N_\delta}[M_1, \ldots, M_\gamma] \longrightarrow \mathbb{P}_{Q_1,\ldots Q_d,\mathcal{B}}[S_1, \ldots, S_c, \mathcal{B}], \tag{3.15}$$

and the matrix form of such intermediate step should be:

$$\begin{array}{|ccccc||cccc|} \hline x_1 & \dots & x_d & y_1 & y_2 & \Gamma^1 & \dots & \Gamma^c & \Gamma^{\mathcal{B}} \\ \hline 0 & \dots & 0 & 1 & 1 & 0 & \dots & 0 & -1 \\ Q_1 & \dots & Q_d & \mathcal{B} & 0 & -S_1 & \dots & -S_c & -\mathcal{B} \\ \hline \end{array}$$
$$\begin{array}{|cccc||cccc|} \hline \Lambda^1 & \Lambda^1 & \dots & \Lambda^\delta & p_1 & p_2 & \dots & p_\gamma \\ \hline 0 & 0 & \dots & 0 & -1 & 0 & \dots & 0 \\ N_1 & N_2 & \dots & N_\delta & -M_1 & -M_2 & \dots & -M_\gamma \\ \hline \end{array} \tag{3.16}$$

Suppose that in an example there are two chosen map elements F^1_1 and F^1_2 that have been chosen to be interchanged with a defining relation S_1. In this case we can choose the redundant new coordinate, y_1, to have charge

$$||\mathcal{B}|| = ||F^1_1|| + ||F^1_2|| - S_1. \tag{3.17}$$

For the initial configuration, $||G_1|| + ||G_2|| = ||F_1^1|| + ||F_2^1||$ where G_1, G_2 are $S_1, \mathcal{B}$. Under the re-labelings required in (3.12) one can choose,

$$\tilde{N}_1 = M_1 - S_1 \ , \ \tilde{N}_2 = M_1 - \mathcal{B} \ , \ \tilde{S}_1 = ||F_1^1|| \ , \ \tilde{\mathcal{B}} = ||F_2^1||. \tag{3.18}$$

Applying the field redefinitions in (3.18) we arrive at last to the new configuration

x_1	...	x_d	y_1	y_2	$\tilde{\Gamma}^1$	...	Γ^c	$\tilde{\Gamma}^B$
0	...	0	1	1	-1	...	0	-1
Q_1	...	Q_d	$\mathcal{B}$	0	$-(M_1 - N_1)$	...	$-S_c$	$-(M_1 - N_2)$

$\tilde{\Lambda}^1$	$\tilde{\Lambda}^1$	...	Λ^δ	p_1	p_2	...	p_γ
1	0	...	0	-1	0	...	0
$M_1 - S_1$	$M_1 - \mathcal{B}$	...	N_δ	$-M_1$	$-M_2$	...	$-M_\gamma$

(3.19)

In subsequent sections we will consider primarily examples of this latter kind in which all three types of singlet moduli - Kähler, complex structure, and bundle moduli – are interchanged in the target space duality procedure. We turn to such an example next in which both X and $\tilde{X}$ are elliptically fibered.

3.2 A target space dual pair with elliptically fibered Calabi-Yau threefolds

Before we can begin to investigate the consequences of $(0, 2)$ target space duality for F-theory, a non-trivial first step is to establish if examples exist in which both halves of a TSD pair in turn lead to F-theory dual geometries. In this section we explicitly provide a first example of such a pair.

In the following example, we will find that the CY manifolds, X and $\tilde{X}$, consist of two Complete Intersection Calabi-Yau 3-folds (so-called "CICYs" [49, 116]), each of which is fibered over (a different) complex surface B_2. These two CICY 3-folds are related by a conifold transition [116] and can be constructed via the target space duality algorithm in which an additional $U(1)$ symmetry is added to the dual GLSM as in Section 3.1.

3.2.1 A tangent bundle deformation

To investigate these results, a simple starting point is given below – a dual pair for which the Calabi-Yau manifolds, X and $\tilde{X}$, are related by a conifold transition. Consider the following CICY 3-fold, described by a so-called "configuration matrix" [116]

$$X = \left[\begin{array}{c|cc} \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^2 & 1 & 2 \\ \mathbb{P}^2 & 1 & 2 \end{array} \right] . \tag{3.20}$$

Here the columns indicate the ambient space (a product of complex projective spaces) and the degrees of the defining equations in that space. The Hodge numbers are $h^{1,1}(X) = 3$ and $h^{2,1}(X) = 60$. Over this manifold, we choose a simple vector bundle built as a deformation of the holomorphic tangent bundle to X. In the present case we will choose this bundle to be a rank 6 smoothing deformation of the reducible bundle

$$V_{red} = \mathcal{O}^{\oplus 3} \oplus TX \ . \tag{3.21}$$

The smooth, indecomposable bundle will be defined[1] as a kernel $V \equiv ker(F^l_a)$ via the short exact sequence

$$0 \to V \to \bigoplus_{a=1}^{\delta} \mathcal{O}_{\mathcal{M}}(N_a) \xrightarrow{F^l_a} \bigoplus_{l=1}^{\gamma} \mathcal{O}_{\mathcal{M}}(M_l) \to 0, \tag{3.22}$$

which is the simple case of (3.10) when $E^a_i = 0$.

In the language of GLSM charge matrices, the manifold and rank 6 monad bundle (X, V) are given by the following charge matrix:

x_i								Γ^j		Λ^a								p_l	
1	1	0	0	0	0	0	0	−1	−1	1	1	0	0	0	0	0	0	−1	−1
0	0	1	1	1	0	0	0	−1	−2	0	0	1	1	1	0	0	0	−1	−2
0	0	0	0	0	1	1	1	−1	−2	0	0	0	0	0	1	1	1	−1	−2

(3.23)

The reason that this rank 6 bundle makes for a particularly simple choice of gauge bundle is that in this case the GLSM charges associated to the manifold and the bundle are identical (as can be seen above). As a result, anomaly cancellation conditions such as the requirement that

$$c_2(TX) = c_2(V) \tag{3.24}$$

[1] See [23] for discussions of this deformation problem and local moduli space.

(realized as (3.6) in the GLSM) are automatically satisfied.

Expanding the second Chern class of the manifold in a basis of $(1,1)$ forms J_r, $r = 1, \dots h^{1,1}$ we have

$$c_2(TX) = 3J_1J_2 + J_2^2 + 3J_1J_3 + 5J_2J_3 + J_3^2. \tag{3.25}$$

Following the standard $(0,2)$ target space duality procedure, it is easy to produce the TSD geometry $(\widetilde{X}, \widetilde{V})$. In this case, the duals we consider mix all three types of Heterotic geometry moduli and induce an additional $U(1)$ gauge symmetry in the GLSM. As an intermediate step we form the equivalent GLSM charge matrix with an additional $U(1)$ outlined in Section 3.1 (choosing $\mathcal{B} = 0$) and introduce a repeated entry in the monad bundle charges which does not change either the geometry of GLSM field theory.[2] This leads us to the following charge matrix with a new $\mathbb{P}^1$ row and a new column $\Gamma^{\mathcal{B}}$ as in (3.16):

x_i										Γ^j			Λ^a									p_l		
1	1	0	0	0	0	0	0	0	0	0	−1	−1	1	1	0	0	0	0	0	0	1	−1	−1	−1
0	0	1	1	1	0	0	0	0	0	0	−1	−2	0	0	1	1	1	0	0	0	2	−1	−2	−2
0	0	0	0	0	1	1	1	0	0	0	−1	−2	0	0	0	0	0	1	1	1	1	−1	−2	−1
0	0	0	0	0	0	0	0	1	1	−1	0	0	0	0	0	0	0	0	0	0	0	0	−1	0

(3.26)

Finally, we can perform field redefinitions in this intermediate geometry to obtain the final TSD. Here we choose two map elements – in this case, F_8^2 and F_9^2 – to be interchanged with a defining relation G_2 with degree $||S_2|| = \{1,2,2\}$. Such a choice satisfies the linear anomaly cancelation (3.17) since $||S_2|| + \mathbf{0} = ||F_8^2|| + ||F_9^2||$. In the intermediate configuration, applying the field redefinitions (3.18) gives:

$$\tilde{N}_8 = M_2 - S_2 = 0, \quad \tilde{N}_9 = M_2, \quad \tilde{S}_2 := F_9^2, \quad \tilde{\mathcal{B}} := F_8^2. \tag{3.27}$$

This leads us at last to the dual charge matrix associated to $(\widetilde{X}, \widetilde{V})$ with $h^{1,1}(\tilde{X}) = 4$ and $h^{2,1}(\tilde{X}) = 60$:

x_i										Γ^j			Λ^a									p_l		
1	1	0	0	0	0	0	0	0	0	0	−1	−1	1	1	0	0	0	0	0	0	1	−1	−1	−1
0	0	1	1	1	0	0	0	0	0	0	−1	−2	0	0	1	1	1	0	0	0	2	−1	−2	−2
0	0	0	0	0	1	1	1	0	0	−1	−1	−1	0	0	0	0	0	1	1	0	2	−1	−2	−1
0	0	0	0	0	0	0	0	1	1	−1	0	−1	0	0	0	0	0	0	0	1	0	0	−1	0

(3.28)

[2]See [42] for details of this argument.

Again, in the new configurations the anomaly cancellation condition are satisfied (as was proved in general to happen in [42]). To make sure they are true target space duals, we will show that these two different geometric phases preserve the net multiplicities of charged matter, and the total number of massless gauge singlets, while the individual number of complex, Kähler and bundle moduli are changed. First, it is clear that the low-energy gauge group G in the 4-dimensional gauge theory is given by the commutant of the structure group, H in $E_8 \times E_8$ of the bundles defined over the CY manifold. Here there is only one bundle (saturating the anomaly cancellation condition on $c_2(V)$). We choose to embed this structure group into one of the two E_8 factors and considering the other E_8 factor as an unbroken, hidden sector gauge symmetry.

In order to find the matter field representations, the adjoint **248** of E_8 must be decomposed under the subgroup $G \times H$. In the present case, the rank 6 bundles have $c_1 = 0$, so their structure groups are $H = SU(6)$. The charged matter spectrum can then be determined by the decomposition of E_8 into representations of the maximal subgroup $SU(2) \times SU(3) \times SU(6)$:

$$\begin{aligned}\mathbf{248}_{E_8} \to (\mathbf{3},\mathbf{1},\mathbf{1}) \;&\oplus\; (\mathbf{1},\mathbf{8},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{35}) \oplus (\mathbf{1},\mathbf{3},\overline{\mathbf{15}}) \\ &\oplus\; (\mathbf{1},\overline{\mathbf{3}},\mathbf{15}) \oplus (\mathbf{2},\mathbf{3},\mathbf{6}) \oplus (\overline{\mathbf{2}},\overline{\mathbf{3}},\overline{\mathbf{6}}) \oplus (\mathbf{2},\mathbf{1},\mathbf{20}). \qquad (3.29)\end{aligned}$$

As a result, the multiplicity of fields in the 4-dimensional theory transforming in representations of $SU(2) \times SU(3)$ is counted by those transforming in an $SU(6)$ representation over the CY. The latter are counted by the dimension of bundle valued cohomology groups, $H^*(X, \wedge^k V)$, for assorted values of k (see [6, 96] for details).

It is helpful to note that for a vector bundle V, on a Calabi-Yau 3-fold, X, the cohomology groups of the bundle and its dual are related by Serre duality as $H^i(X,V) = H^{3-i}(X,V^*)^*$ and when $H = SU(n)$, $H^*(X, \wedge^k V) \simeq H^*(X, \wedge^{n-k} V^*)$. Finally, a necessary condition for μ-stability of the vector bundle V is $h^0(X,V) = 0$ which is satisfied for tangent bundle deformations considered here (by direct computation).

With these observations in hand, the multiplicity of the charged chiral matter spectrum of these dual pair theories can be determined by computing corresponding vector bundle valued cohomology classes on the Calabi-Yau 3-fold:

$$\begin{array}{lll} (\mathbf{2},\mathbf{3})'s: & h^1(V)=57 & h^1(\widetilde{V})=57, \\ (\overline{\mathbf{2}},\overline{\mathbf{3}})'s: & h^1(V^*)=0 & h^1(\widetilde{V}^*)=0, \\ (\mathbf{1},\mathbf{3})'s: & h^1(\wedge^2 V)=115 & h^1(\wedge^2\widetilde{V})=115, \\ (\mathbf{1},\overline{\mathbf{3}})'s: & h^1(\wedge^2 V^*)=1 & h^1(\wedge^2\widetilde{V}^*)=1, \\ (\mathbf{2},\mathbf{1})'s: & h^1(\wedge^3 V)=2 & h^1(\wedge^3\widetilde{V})=2. \end{array} \tag{3.30}$$

Furthermore, the low-energy theory has massless gauge singlets, $(\mathbf{1},\mathbf{1})$, which are counted by $h^1(V\otimes V^*)=h^1(End_0(V))$. There are additional singlets, beyond those related to the complex structure and Kähler deformations of the Calabi-Yau 3-fold, which are counted by $h^{2,1}(X)$ and $h^{1,1}(X)$. The total number of singlet moduli are counted by:

$$\begin{aligned} h^{1,1}(X)+h^{2,1}(X)+h^1(End_0(V)) &= 3+60+292=355, \\ h^{1,1}(\widetilde{X})+h^{2,1}(\widetilde{X})+h^1(End_0(\widetilde{V})) &= 4+53+298=355. \end{aligned} \tag{3.31}$$

From the point of view of the massless Heterotic spectrum, it is clear that in the theories associated to the TSD geometries, (X,V) and $(\tilde{X},\tilde{V})$, all the degrees of freedom appear to match.

Moreover, we have chosen this pair of geometries to have a further special property. Each CY 3-fold appearing in the dual pair exhibits an elliptic fibration structure. As a result, by the arguments laid out in Section 3.1, we expect each Heterotic background in the pair to lead to its own F-theory dual.

A closer inspection yields the following elliptic fibration structures:

$$\pi_h: X \xrightarrow{\mathbb{E}} \mathbb{P}^2 \quad \text{and} \quad \tilde{\pi}_h: \tilde{X} \xrightarrow{\mathbb{E}} dP_1. \tag{3.32}$$

The fibrations of X and $\tilde{X}$ can be seen very explicitly from the form of the complete intersection descriptions of the manifolds (so-called "obvious" fibrations [17]). Below we use dotted lines to separate the "base" and "fiber" of the manifold:

$$X=\left[\begin{array}{c|cc} \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^2 & 1 & 2 \\ \hdashline \mathbb{P}^2 & 1 & 2 \end{array}\right], \qquad \widetilde{X}=\left[\begin{array}{c|c:cc} \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^2 & 0 & 1 & 2 \\ \hdashline \mathbb{P}^2 & 1 & 1 & 1 \\ \mathbb{P}^1 & 1 & 0 & 1 \end{array}\right], \tag{3.33}$$

where the base for the elliptically fibered X is $B_2=\mathbb{P}^2$ (the bottom row of the configuration matrix) while the dP_1 base for $\tilde{X}$ is given as $\widetilde{B_2} = \left[\begin{array}{c|c} \mathbb{P}^2 & 1 \\ \mathbb{P}^1 & 1 \end{array}\right]$.

Employing the techniques of [15, 16], we find that the fibrations in both X and $\tilde{X}$ in fact admit rational sections and as a result are elliptically fibered (as opposed to genus-one fibered only). Moreover, each fibration contains two rational sections (i.e. a higher rank Mordell-Weil group). In an abuse of notation, we will use σ_i to denote both the two sections to the elliptic fibration of X (respectively $\tilde{X}$) and the associated Kähler forms dual to the divisors. In terms of the basis of the Kähler $(1,1)$-forms J_r inherited from the ambient space factors $\mathbb{P}^n_r$ of each CICY 3-fold:

$$\begin{aligned} &\sigma_1(X) = -J_1 + J_2 + J_3, && \sigma_2(X) = 2J_1 - J_2 + 5J_3, \\ &\sigma_1(\widetilde{X}) = -J_1 + J_2 + J_3, && \sigma_2(\widetilde{X}) = 2J_1 - J_2 + 4J_3 + J_4. \end{aligned} \tag{3.34}$$

With a choice of zero section for each manifold from the set above, the CY 3-fold can in principle be put into Weierstrass form [61, 133]. For explicit techniques to carry out this process we refer the reader to [16].

In summary then, we have produced an explicit example of a TSD pair in which both sides are elliptically fibered manifolds, admitting 4-dimensional, $\mathcal{N} = 1$ F-theory duals in principle. This is an important point of principle, since we have demonstrated that *some* F-theory correspondence should exist for the dual F-theory EFTs. In practice however, it should be noted that explicitly determining the F-theory duals for the geometries given above is difficult. We will begin untangling this process explicitly in Section 3.3.

For now we close this example by observing an interesting feature of the TSD pair above: Since we began with a deformation of the tangent bundle, the associated $(0,2)$ GLSM admits a $(2,2)$ locus. However, in the TSD geometry the bundle we obtain is no longer manifestly a holomorphic deformation of the tangent bundle on $\tilde{X}$. It remains an open question whether this second theory admits a $(2,2)$ locus in some subtle way. For the moment, we will turn to one further TSD pair in which neither vector bundle is related to the tangent bundle.

3.2.2 More general vector bundles

Here we present a second example in which the same CY manifolds appear, but with different vector bundles. Once again, we start with the GLSM charge matrix determining the pair (X,V) as in (3.35) where in this time we have a rank 4 bundle with structure group $SU(4)$:

x_i								Γ^j		Λ^a						p_l	
1	1	0	0	0	0	0	0	−1	−1	1	0	0	0	0	1	−1	−1
0	0	1	1	1	0	0	0	−1	−2	0	1	1	0	0	2	−2	−2
0	0	0	0	0	1	1	1	−1	−2	0	0	0	1	1	1	−2	−1

(3.35)

In this case, the second Chern class of (X, V) is different from (3.25):

$$c_2(V) = 2J_1J_2 + J_2^2 + 2J_1J_3 + 4J_2J_3 + J_3^2. \tag{3.36}$$

However, in this case, $c_2(V) \leq c_2(TX)$ and thus it is expected that this bundle could be embedded into one factor of the $E_8 \times E_8$ Heterotic string, where another bundle V' is embedded into the second factor. By completing the geometry in this way, with $c_2(V) + c_2(V') = c_2(TX)$, the anomaly cancellation conditions can be satisfied (alternatively, NS5/M5 branes might be considered).

Following the standard procedure described above, the target space duality data is given by $(\widetilde{X}, \widetilde{V})$ with the following charge matrix:

x_i										Γ^j			Λ^a						p_l	
1	1	0	0	0	0	0	0	0	0	0	−1	−1	1	0	0	0	0	1	−1	−1
0	0	1	1	1	0	0	0	0	0	0	−1	−2	0	1	1	0	0	2	−2	−2
0	0	0	0	0	1	1	1	0	0	−1	−1	−1	0	0	0	1	0	2	−2	−1
0	0	0	0	0	0	0	0	1	1	−1	0	−1	0	0	0	0	1	0	−1	0

(3.37)

Here the second chern classes of the tangent bundle and the monad vector bundle are respectively:

$$\begin{aligned} c_2(TX) &= 3J_1J_2 + J_2^2 + 2J_1J_3 + 3J_2J_3 + J_1J_4 + 2J_2J_4 + 2J_3J_4, \\ c_2(V) &= 2J_1J_2 + J_2^2 + J_1J_3 + 2J_2J_3 + J_1J_4 + 2J_2J_4 + 2J_3J_4, \end{aligned} \tag{3.38}$$

which could also satisfy the c_2 matching condition with the addition of a hidden sector bundle.

In this background, the bundle structure group of $H = SU(4)$ breaks E_8 to $SO(10)$. As above, the charged matter content can be determined by the decomposition under $SO(10) \times SU(4)$:

$$\mathbf{248}_{E_8} \to (\mathbf{1}, \mathbf{15}) \oplus (\mathbf{10}, \mathbf{6}) \oplus (\overline{\mathbf{16}}, \overline{\mathbf{4}}) \oplus (\mathbf{16}, \mathbf{4}) \oplus (\mathbf{45}, \mathbf{1}). \tag{3.39}$$

The multiplicity of the spectrum is then determined via bundle-valued cohomology as:

$$\begin{aligned} \mathbf{16}'s: &\quad h^1(V) = 48, \quad h^1(\widetilde{V}) = 48, \\ \overline{\mathbf{16}}'s: &\quad h^1(V^*) = 0, \quad h^1(\widetilde{V}^*) = 0, \\ \mathbf{10}'s: &\quad h^1(\wedge^2 V) = 0, \quad h^1(\wedge^2 \widetilde{V}) = 0. \end{aligned} \tag{3.40}$$

Furthermore, the counting of the gauge singlets appearing in this TSD pair match as well:

$$\begin{aligned} h^{1,1}(X) + h^{2,1}(X) + h^1(End_0(V)) &= 3 + 60 + 159 = 222, \\ h^{1,1}(\widetilde{X}) + h^{2,1}(\widetilde{X}) + h^1(End_0(\widetilde{V})) &= 4 + 53 + 165 = 222. \end{aligned} \tag{3.41}$$

With these two examples in hand, it is clear that at least the first question outlined in at the beginning of this chapter can be answered in the positive. *Heterotic TSD pairs can indeed be found in which both halves of the dual pair exhibit elliptic fibrations.* However, it is clear that the manifolds in our examples above are not in simple Weierstrass form (and exhibit higher rank Mordell-Weil group) as a result, their F-theory dual geometries may be difficult to determine using standard tools. We review some of these tools in the subsequent Sections before returning to the two examples above in Section 3.3.6.

3.3 Inducing a duality in F-theory

3.3.1 Essential tools for Heterotic/F-theory duality

In type IIB superstring theory, the axion-dilaton transforms under $SL(2, Z)$ while leaving the action invariant. However, it is frequently assumed the string coupling g_s vanishes and the backreaction from 7-branes is ignored. As a result, many important non-perturbative aspects of the string compactification which are crucial both conceptually and phenomenologically, are missing. This is exactly where F-theory arises as a proper description of orientifold IIB theory with (p, q) 7-branes and varying finite string coupling (i.e. axion-dilaton). The classical $SL(2, Z)$ self-dual symmetry of Type IIB theory acting on the axion-dilaton is identified as the modular group of a one complex dimensional torus T^2 and as the complex structure of a fictitious elliptic curve. In this way, we formally attach an elliptic curve at each point of the type IIB space time and promote the 10-dimensional IIB theory to auxiliary 12-dimensional

F-theory. This structure defines a genus-one or elliptic fibration. The locus where the fiber degenerates is where the 7-brane is wrapped in the internal CY. F-theory realizes a remarkable synthesis of geometry and field theory in that the structure of the 7-branes/gauge sector, matter content and Yukawa couplings are all encoded in the geometry of the fibration structure and the back-reaction of these branes is taken into account.

There is no description of F-theory as a fundamental theory, but rather, as duals to other theories. A concrete example would be an 8-dimensional duality [151] i.e. F-theory conpactified on $K3$ is dual to type IIB on S^2 with 24 7-branes turned on, which is also dual to Heterotic on T^2. The duality between F-theory and Heterotic is described further as F-theory compactified on $K3$ fibered Calabi-Yau $(n+1)$-fold is dual to $E_8 \times E_8$ Heterotic string compactified on Calabi-Yau n-fold which is elliptically fibered on the same $(n-1)$-fold base:

- Heterotic: $\pi_h : X_n \xrightarrow{\mathbb{E}} B_{n-1}$ elliptic fibration,
- F-theory: $\pi_f : Y_{n+1} \xrightarrow{K3} B_{n-1}$ where $\rho_f : Y_{n+1} \xrightarrow{\mathbb{E}} B_n$, $\sigma_f : B_n \xrightarrow{\mathbb{P}^1} B_{n-1}$.

The paired Heterotic/F-theory geometries given above involves both elliptic and $K3$ fibered manifolds. In particular, the F-theory geometry, Y_{n+1} must be compatibly $K3$ and elliptically fibered. The requirement of these two fibration implies that Y_{n+1} should also be elliptically fibered over a complex n-dimensional base, B_n which is in turn rationally fibered. The existence of a section in any two of the fibrations structures is enough to guarantee the existence of a section in the third fibration (i.e. if ρ_f and σ_f both admit sections then so does the fibration π_f).

With different number of n's, there are theories in different dimensions. Specifically, $n = 1, 2, 3$ will lead to $8D$, $6D$, and $4D$ respectively. When $n = 1$, the $(n-1)$-fold base B_{n-1} is a point, when $n = 2$ it is a $\mathbb{P}^1$. In 4D case, the duality can be written as:

$$\begin{array}{ccc} Y_4 & \xrightarrow{\mathbb{E}} & B_3 \\ {\scriptstyle K3}\downarrow & & \downarrow{\scriptstyle \mathbb{P}^1} \\ B_2 & \xrightarrow{=} & B_2 \end{array} \tag{3.42}$$

By the fibration structure of the CY 4-fold (3.42), the base B_3 must be $\mathbb{P}^1$-fibered. As in [89], such a $\mathbb{P}^1$ bundle can be defined as the projectivization of two line bundles,

$$B_3 = \mathbb{P}(\mathcal{O} \oplus \mathcal{L}) \ , \tag{3.43}$$

where $\mathcal{O}$ is the trivial bundle and $\mathcal{L}$ is a general line bundle on B_2. In this case the topology of B_3 is completely fixed by the choice of line bundle $\mathcal{L}$ and we can defined a $(1,1)$-form on B_2 as $T = c_1(\mathcal{L})$. A special case would be $R = c_1(\mathcal{O}(1))$ where $\mathcal{O}(1)$ is a bundle that restricts to the usual $\mathcal{O}(1)$ on each $\mathbb{P}^1$ fiber. They satisfy the relation $R(R+T) = 0$ in cohomology, which indicates the two corresponding sections don't intersect to each other. This kind of twist allows us to matching the degrees of freedom in the 4-dimensional Heterotic/F-theory dual pairs.

In the $E_8 \times E_8$ Heterotic side, the vector bundle can be decomposed as $V = V_1 \oplus V_2$, and the curvature splits as

$$c_2(V) = \frac{1}{30}\mathrm{Tr}F_i^2 = \eta_i \wedge \sigma + \xi_i, \tag{3.44}$$

where η_i, ξ_i are pullback of 2-forms and 4-forms on B_2, σ is the Poincare dual to the section of the elliptic fibration $\pi_h : X_3 \xrightarrow{\mathbb{E}} B_2$. For any CY 3-fold in Weierstrass form as described above, $c_2(TX_3) = 12c_1(B_2)\wedge\omega_{\hat{0}}+(c_2(B_2)+11c_1(B_2)^2)$ [89]. The Heterotic Bianchi identity requires $\eta_1 + \eta_2 = 12c_1(B_2)$, which enable us to parameterize a solution as

$$\eta_{1,2} = 6c_1(B_2) \pm T', \tag{3.45}$$

where T' is a (1,1) form on B_2. By studying the 4D effective theories of these dual Heterotic/F-theory compactifications it is straightforward to determine that the defining (1, 1)-forms T, T' are identical to each other $T = T'$. Then the (1, 1)-form T is referred to as the so-called "twist" of the $\mathbb{P}^1$-fibration and is the crucial defining data of the simplest classes of Heterotic/F-theory dual pairs. Moreover, this duality map dependences on a particular method of constructing Mumford poly-stable vector bundles, the spectral cover construction.

3.3.2 Spectral cover construction

To find the dual F-theory model of a specific Heterotic model, we need a description of the moduli space of stable degree zero vector bundles over elliptically fibered manifolds, (the standard formulation works for Weierstrass fibration, but it can be generalized to other types of elliptic fibrations, as mentioned in the last chapter) in terms of two "pieces," which are called spectral data. This can be done by Fourier-Mukai transform that we already described.[3]

[3]Please note that we restrict ourselves to $SU(N)$ (degree zero, and stable) vector bundles over a Weierstrass elliptic fibration.

In principle, it is possible to find the F-theory dual by using the spectral data. We review this briefly in the following. Assume there are no NS5 branes, and suppose we have two vector bundles (V_1, V_2) over a Weierstrass elliptically fibered manifold X, then Heterotic anomaly cancellation requires,

$$c_2(V_1) + c_2(V_2) = c_2(X).$$

Then second Chern classes (which can be computed by Grothendieck-Riemann-Roch, if we have the spectral data [89]), can be written generally as,

$$\begin{aligned} c_2(V_i) &= \sigma\eta_i + \omega_i, \\ \eta_i &= 6C_1(B_H) \pm T, \end{aligned} \tag{3.46}$$

where η_i is a divisor in the base (B_H), and ω_i is the intersection of two divisor in B_H. Also by using the same method it is not too hard to show that the divisor class of the spectral cover of V_i is given by

$$[\mathcal{S}] = n_i\sigma + \eta_i. \tag{3.47}$$

Now, the first statement about the Heterotic and F-theory duality is that the topology of the base manifold of the F-theory Calabi-Yau is fixed by the "twist" T in (3.46) as,

$$B_F = \mathbb{P}\left(\mathcal{O}_{B_H} \oplus \mathcal{O}_{B_H}(T)\right). \tag{3.48}$$

The second statement is that the complex structure of $\mathcal{S}$, (partially) fix the complex structure of the Calabi-Yau in the F-theory side. It's easier to describe this with an example. Suppose we have a $SU(2)$ bundle V, and it's spectral cover is non-degenerate,

$$S = a_0 z^2 + a_2 x. \tag{3.49}$$

Since one of the E_8 factors breaks to E_7, we should have an E_7 singularity in the F-theory geometry, which is described by the following Weierstrass equation,

$$\begin{aligned} Y^2 &= X^3 + F(u,z)x + G(u,z), \\ F &= \Sigma_{i=1}^{8} F_i(z)u^i, \\ G &= \Sigma_{i=1}^{12} G_i(z)u^i, \end{aligned} \tag{3.50}$$

where u is the affine coordinate of the $\mathbb{P}^1$ fiber of (3.48), and z is the "collective" coordinate for B_H. Now, the conjectured duality tells us the corresponding E_7 singularity should be located near $u = 0$, therefore $F_0 = F_1 = F_2 = 0$, and $G_0 = \cdots = G_4 = 0$. Also a_0 is identified with G_5, and a_2 with F_3.

The other vector bundle (which is embedded in the other E_8 factor) determines the singularity near $u \to \infty$, and higher polynomials ($F_5, \dots$ and $G_7 \dots$) are determined by the spectral cover of that vector bundle (which we didn't write here). The middle polynomials F_4 and G_6 are determined by the Heterotic Weierstrass equation.

The last piece of data which is remained is the spectral sheaf $\mathcal{L}$, which is an element of the Picard group $Pic(S)$ (if one assumes S is smooth). The "space of line bundles" itself is made by two pieces, the "discrete" part $H^{1,1}(S)$, and the "continuous" part which is $J(S)$ (space of degree zero (flat) line bundles),

$$0 \to J(S) \to Pic(S) \to H^{1,1}_{\mathbb{Z}}(S) \to 0. \tag{3.51}$$

In 6D theories, the discrete part can be fixed uniquely by using Fourier-Mukai transform, and the Jacobian of the curve is mapped to the intermediate Jacobian of the Calabi-Yau threefold in F-theory. In type IIA or M-theory language, the "space of three forms," $H^3(X, \mathbb{R})/H^3(X, \mathbb{Z})$, is described by intermediate Jacobian [28, 59, 89].

The situation in 4D theories are even more complicated. In such cases, it is possible to have non-trivial 4-form fluxes which can be introduced in various (equivalent) ways. One way is to define is as 4-form induced by the field strength of the 3-form in M-theory limit. Another way is to define as a (1,1)-form flux over the 7-branes wrapping the divisors in the base times another (1,1)-form localized around the 7-brane locus [62, 70]. In general, the 4-flux data is parameterized by the Deligne cohomology. [4]

$$0 \to J^2(\hat{X}_4) \to H^4_D(\hat{X}_4, \mathbb{Z}(2)) \to H^{2,2}_{\mathbb{Z}}(\hat{X}_4) \to 0, \tag{3.52}$$

where $\hat{X}_4$ is the resolved geometry in M-theory limit, J^2 is the intermediate Jacobian,

$$J^2(\hat{X}_4) = H^3(\hat{X}_4, \mathbb{C})/(H^{3,0}(\hat{X}) \oplus H^{2,1}(\hat{X}_4)), \tag{3.53}$$

which corresponds to the space of flat 3-forms in M theory. The third, and most difficult part of the Heterotic/F-theory duality, is that the continuous part of the spectral sheaf data, $J(S)$, maps to $J^2(X_4)$, and discrete part, $H^{1,1}(S)$ (which is determined by the divisor class (first Chern class) of the spectral line bundle), maps to the discrete part of the 4-flux data $H^{2,2}(\hat{X}_4)$.

[4] See the lectures [154], and references there.

3.3.3 Warm up: Heterotic/F-theory duality in 6-dimensions

In this section, we'll begin in earnest the process of attempting to determine the induced duality in F-theory given by TSD and whether the multiple fibrations conjecture outlined in previous sections could be a viable realization. In this simpler context both the geometry of the F-theory compactification as well as the process of reparameterizing (i.e. performing a FM-transform) of the Heterotic data are more readily accomplished.

To begin, it should be noted that in the context of Heterotic target space duals we will consider smooth geometries (i.e. smooth bundles over $K3$ manifolds) in the large volume, perturbative limit of the theory. We will consider solutions without NS 5-branes so that the 6-dimensional theory exhibits a single tensor multiple (associated to the Heterotic dilaton) (see [148] for a review). Within the context of 6-dimensional F-theory EFTs with a single tensor and a Heterotic dual it is clear that we must restrict ourselves to CY 3-folds that are elliptically fibered over Hirzebruch surfaces:

$$\pi_f : Y_3 \to \mathbb{F}_n \tag{3.54}$$

with $n \leq 12$ [131, 132].

It is our goal in this section to test the multiple fibrations conjecture in the context of target space duality. At the level of GLSMs TSD in Heterotic compactifications on $K3$ works mechanically exactly as in the case of CY 3-folds. However, the associated geometry is dramatically simpler. It is clear that the two TSD GLSMS will parameterize at best two different descriptions of a $K3$ surface and that the process must by construction preserve the second chern class of the vector bundle V over $K3$ (see [42] for a proof valid for either CY 2- or 3-folds). Since the massless spectrum of the 6-dimensional Heterotic theory compactified on a smooth $K3$ is entirely determined by the rank and second Chern class of V,[5](see e.g. [35]) it is clear that TSD is only a simple re-writing of the same geometry and 6-dimensional EFT.

However, there remains something interesting to compare to in that it can still be asked: *Does the concrete process of Heterotic TSD duality in 6-dimensions correspond to exchanging $K3$ fibrations in the dual F-theory geometry?* In the context of F-theory 3-folds that are elliptically fibered over a Hirzebruch surface as described above, there is in fact only one geometry where multiple $K3$ fibrations can arise. It was established in the very first papers on F-theory [131, 132], that in order to have

[5]And the moduli space of stable sheaves over $K3$ with fixed Chern character has only one component.

different $K3$ fibrations within a CY threefold with a perturbative Heterotic dual, the base twofold must be F_0. Indeed, the remarkable observation of Morrison and Vafa was that the existence of two $K3$ fibrations in $\pi_f : Y_3 \to \mathbb{F}_0$ (only a simple relabeling in F-theory) was dual to a highly non-perturbative Heterotic/Heterotic duality discovered by Duff, Minasian and Witten [86].

With these observations in mind, we can immediately make several observations. To begin, we must recall that a base manifold for the F-theory fibration of $\mathbb{F}_n$ in this context is correlated to bundles with $c_2 = 12 \pm n$ in the $E_8 \times E_8$ Heterotic dual. Thus

- For any purely perturbative Heterotic TSD pair in 6-dimensions with $c_2(V) = c_2(\tilde{V}) \neq 12$, $(0, 2)$ TSD *cannot correspond to multiple fibrations in F-theory* (since as described above, such multiple $K3$ fibrations arise only for $n = 0$).

- With the point above, we have established that in general the multiple fibrations conjecture outlined in introduction the *must be false in general* at least in the 6-dimensional Heterotic theories.

- This demonstrates that not all TSD pairs can be described by F-theory multiple fibrations, but the converse question, namely – *can multiple fibrations in F-theory give rise to dual Heterotic TSD pairs?* – in principle remains open.

Thus, in this section we chose to look at this last point in closer detail by considering an example TSD pair over $K3$ in which $c_2(V) = c_2(\tilde{V}) = 12$ (the so-called symmetric embedding), corresponding to an $\mathbb{F}_n = \mathbb{P}^1 \times \mathbb{P}^1$ base in F-theory. This will at least make it possible in principle for the two duals to consider.

3.3.4 Spectral cover of monads

To begin, we observe that since the vector bundles defined by GLSMs (in their geometric phases) are usually presented as monads [114], we must deal with how to convert this description of a bundle into one compatible with Heterotic/F-theory duality. As discussed in Section 3.3, it is necessary to perform an FM transform to compute the spectral cover is this case. As a result, here we briefly review the method first introduced in [36].

Suppose we are given a general monad such as,

$$0 \to \mathcal{V} \to \mathcal{H} \xrightarrow{F} \mathcal{N} \to 0, \tag{3.55}$$

where $\mathcal{H}$ and $\mathcal{F}$ are direct sum of line bundles of appropriate degrees. If we assume $\mathcal{V}$ is stable and degree zero, then from the previous subsection we know that its restriction over a generic elliptic fiber will look like $\oplus_i \mathcal{O}(p_i - \sigma)$. So if we twist the whole monad by $\mathcal{O}(\sigma)$,

$$\tilde{\mathcal{V}} := \mathcal{V} \otimes \mathcal{O}(\sigma)|_E = \bigoplus_i \mathcal{O}(p_i). \tag{3.56}$$

Each factor has only one global section over the fiber, and it becomes zero exactly at the point p_i which is the intersection of the spectral cover with the fiber. So the idea is try to find the global section of the twisted vector bundle over elliptic fibers, then check at what points the dimension of the vector space generated by global sections drop. To illustrate how this can be done explicitly, first consider twisting the full monad sequence by $\mathcal{O}(\sigma)$,

$$0 \to \tilde{\mathcal{V}} \to \tilde{\mathcal{H}} \xrightarrow{F} \tilde{\mathcal{N}} \to 0 \tag{3.57}$$

and follow this by next taking the action of the left exact functor π_* on the above sequence ($\bar{F}$ is the induced map, corresponding to F):

$$0 \to \pi_*\tilde{\mathcal{V}} \to \pi_*\tilde{\mathcal{H}} \xrightarrow{\bar{F}} \pi_*\tilde{\mathcal{N}} \to R^1\pi_*\tilde{\mathcal{V}} \to \dots \tag{3.58}$$

If we assume the vector bundle is semistable over every elliptic fiber,[6] then $R^1\pi_*\tilde{\mathcal{V}}$ is identically zero, because its presheaf is locally of the form $H^1(E, \mathcal{O}(p_i))$. Now consider the action of the right exact functor π^* over the last sequence, since the elliptic fibration map π is a flat morphism, it doesn't have higher left derived functors (we have $Tor_{1S}(M, R) = 0$ due to the flatness, where S is the ring that corresponds to $\mathcal{O}_B$, R corresponds to $\mathcal{O}_X$, and M is the free module corresponding to $\mathcal{V}$ (see e.g. [153], Chapter 3)). So we get,

$$0 \to \pi^*\pi_*\tilde{\mathcal{V}} \to \pi^*\pi_*\tilde{\mathcal{H}} \xrightarrow{\bar{F}} \pi^*\pi_*\tilde{\mathcal{N}} \to 0. \tag{3.59}$$

Note, $\pi^*\pi_*\tilde{\mathcal{V}}$ is vector bundle that its fibers over a point p are generated by the global sections of $\tilde{\mathcal{V}}$ over elliptic curve E_b, where $b = \pi(p)$. So (3.59) tells us that

[6] this need not be true always, as we'll see. It is only necessary that vector bundle be semistable over generic elliptic fibers.

if we find the global sections of $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{N}}$, then the kernel of the induced map F is isomorphic to $\pi^*\pi_*\tilde{\mathcal{V}}$. So the locus where the rank of the kernel drop, coincides with the spectral cover. To clarify these rather abstract ideas, in the following subsection, we explicitly compute the spectral cover of two examples which will be used in the final subsection.

Examples

To begin we assume that the $K3$ can be written in the following simple toric form,

$$\begin{array}{|ccccc|c|}\hline & & x_i & & & \Gamma^j \\ \hline 3 & 2 & 1 & 0 & 0 & -6 \\ 6 & 4 & 0 & 1 & 1 & -12 \\ \hline \end{array} \tag{3.60}$$

• **Example 1** The first example is the following $SU(2)$ monad,

$$\begin{array}{|cccc|cc|}\hline & \Lambda & & & p & \\ \hline 1 & 1 & 2 & 3 & -3 & -4 \\ 1 & 5 & 3 & 7 & -9 & -7 \\ \hline \end{array} \tag{3.61}$$

The second Chern class of this monad is 12. The map F of the monad is given by the following generic matrix,

$$F \sim \begin{bmatrix} xf_4 + z^2 f_8 & ax + z^2 g_4 & zf_6 & f_2 \\ by + xzg_2 + z^3 g_6 & z^3 h_2 & cx + z^2 h_4 & dz \end{bmatrix} \tag{3.62}$$

where subscripts indicate the degree of homogeneous polynomials over $\mathbb{P}^1$. With this choice, it can be verified that the kernel of $\bar{F}$ in (3.59) takes the following generic form,

$$\begin{bmatrix} x - z^2 \frac{-f_2h_4+df_6}{cf_2} & 0 \\ 0 & x\frac{cf_2}{f_2h_4-df_6} + z^2 \\ -\frac{b}{c}y - xz\frac{f_2g_2-bf_4}{cf_2} - z^3\frac{f_2g_6-df_8}{cf_2} & -z^3\frac{f_2h_2-dg_4}{f_2h_4-df_6} + -xz\frac{ad}{df_6-f_2h_4} \\ -x^2\frac{f_4}{f_2} + yz\frac{bf_6}{cf_2} - xz^2\frac{cf_8+f_4h_4-g_2f_6}{cf_2} - z^4\frac{f_8h_4-f_6g_6}{cf_2} & x^2\frac{ac}{df_6-f_2h_4} - xz^2\frac{cg_4+ah_4}{f_2h_4-df_6} - z^4\frac{-f_6h_2+g_4h_4}{f_2h_4-df_6} \end{bmatrix} \tag{3.63}$$

where f_i, g_i, h_i are polynomials in terms of base coordinates with degree i, and a, b, c, d are constant. The common factor of the minors of (3.63) is,

$$cf_2x + (f_2h_4 - df_6)\, z^2. \tag{3.64}$$

From the previous discussion naively we might conclude that (3.64) must be the spectral cover. However, in the Fourier-Mukai discussion it was noted that the divisor class of the spectral cover should be $2\sigma + 12D$. So, in the expression above we are clearly missing a degree 6 polynomial in (3.64). The correct spectral cover should be

$$S = F_6\left(cf_2x + (f_2h_4 - df_6)\, z^2\right). \tag{3.65}$$

But why then is F_6 is missing? The reason is that in the previous subsection we assumed the vector bundles are semistable over every fiber. This not necessarily true. It is possible to start from a stable bundle, and modify it in a way that it becomes semistable over every fiber [90].

To see clearly what happens, let us first find the elliptic fibers such that vector bundle over them is unstable. Note that from (3.64) it can be seen that the spectral cover is a non-degenerate two sheeted surface, and over generic E, $V|_E = \mathcal{O}(p-\sigma) \oplus \mathcal{O}(q-\sigma)$, where $p+q = 2\sigma$. So $V|_E$ does not have global section over almost every fiber except when $p = q = \sigma$. These points are on the intersection of $z = 0$ and the spectral cover, which are the zeros of f_2F_6. The idea then is to see if we can find the elliptic fibers which over the vector bundle (not its twisted descendant) can have global section. So all we need to do to find F_6 is to study the kernel of the induced map in the following sequence:

$$0 \to \pi_*\mathcal{V} \to \pi_*\mathcal{H} \overset{F_{ind}}{\to} \pi_*\mathcal{N} \to R^1\pi_*\mathcal{V} \to \ldots \tag{3.66}$$

where the induced map, F_{ind}, in the above case is a 7×7 matrix in terms of the base coordinates. Generically it's rank is 7, excepts over the zeros of f_2F_6, so we can read the missing polynomial from this form. Please note that the above computations are local, globally $\pi_*\mathcal{V} = 0$. Because $\mathcal{V}$ is locally free, and π is a flat morphism, the pushforward of $\mathcal{V}$ should also be torsion-free (see [104], Ch. III.9.2).

Interestingly, when we repeat the same analysis after twisting with $\mathcal{O}(\sigma)$ and $\mathcal{O}(2\sigma)$, the rank of the corresponding induced map drops over F_6, and nowhere respectively. This means that the bundle over those fibers takes the following form,

$$\mathcal{V}|_{E_{\text{missing}}} = \mathcal{O}(p) \oplus \mathcal{O}(-p), \tag{3.67}$$

therefore $h^0(E, \mathcal{V}\otimes\mathcal{O}(\sigma)) \geq 2$, and the rank of the kernel in (3.59) (which generically is the same as the rank of the bundle, in this case, 2) doesn't drop over the points of these fibers. So the algorithm suggested in [36] doesn't find them. But it can be seen from the definition of the Fourier-Mukai transform that these (whole) elliptic fibers are in the support of the spectral sheaf. In summary, a detailed analysis along the lines sketched above shows that the missing component is given by,

$$F_6 \;=\; h_2cf_2^2 - cdf_2g_4 + adf_2h_4 - ad^2f_6, \tag{3.68}$$

where each in the expression polynomial above is defined from the monad map in (3.62).

• **Example 2** The second example is interesting because its spectral cover is degenerate (in this case, a non-reduced scheme):

$$\begin{array}{|cccc|cc|}\hline \multicolumn{4}{|c|}{\Lambda} & \multicolumn{2}{c|}{p} \\ \hline 0 & 1 & 2 & 3 & -2 & -4 \\ 2 & 1 & 3 & 7 & -6 & -7 \\ \hline \end{array} \tag{3.69}$$

The second Chern class of this monad is $c_2(V) = 6$, so from the previous discussion, it is clear that the divisor class of its spectral cover must be $2\sigma + 6D$. The number of global sections of $\mathcal{H}$ and $\mathcal{N}$ is seven and six respectively, which tells us that the kernel of F_{ind} is at least one dimensional over almost every elliptic fiber. Since $\mathcal{V}$ is a stable, rank two bundle with $c_1(\mathcal{V}) = 0$, we conclude either $\mathcal{V}|_E = \mathcal{O} \oplus \mathcal{O}$ or $\mathcal{V}|_E = \epsilon_2$.[7] In both cases the spectral cover must have the following general form,

$$S = F_6z^2. \tag{3.70}$$

In fact, by computing the F_{ind} directly, we can see the rank of the kernel is always one, so $\mathcal{V}|_E = \epsilon_2$. As before we can compute the kernel of (3.59), to generate the spectral cover,

$$\begin{bmatrix} z & 0 \\ 0 & z^2 \\ -\frac{xz}{f_3} - z^3\frac{f_4}{f_3} & z^3\frac{f_5}{f_3} \\ a\frac{x^2}{f_3} + xz^2\frac{f_4}{f_3} + z^4\frac{f_8}{f_3} & byz + xz^2\frac{g_5}{f_3} + z^4\frac{f_9}{f_3} \end{bmatrix}. \tag{3.71}$$

Then we look at the minors, and common factor should be spectral cover. However, as in the previous examples, the algorithm in [36] miss the polynomial F_6. The reason is similar to the previous example, the bundle is unstable over the zeros of F_6. It can be shown that the correct spectral cover is indeed,

$$S = (F_3)^2z^2, \tag{3.72}$$

where F_3 is the entry $(1,3)$ of the monad's map (i.e. the map between the line

[7] By $\mathcal{V}|_E = \epsilon_2$ we mean the unique non-trivial extension of the trivial bundles over the elliptic curve.

bundles $\mathcal{O}_X(2,3)$ and $\mathcal{O}_X(2,6)$).[8]

3.3.5 Counterexamples of the conjecture

Here we return to the main goal of this section. Suppose we have two target space dual GLSMs that describe different stable bundles over elliptic $K3$ surfaces. The goal is to check whether their F-theory dual geometries can be related via a change in $K3$- fibrations (i.e. a change in $\mathbb{P}^1$ fibrations in the two-fold base of the CY 3-fold). Generally the base of the F-theory threefold will be a Hirzebruch surface F_n, where n is given by the twist in (3.46). The only situation that can accommodate such multiple fibration structures is when $n = 0$. So here we focus on this case and demand that the second Chern class of both Heterotic vector bundles be 12.

We assume that one of the target space dual geometries is given by monads on the toric $K3$ (3.60). We also write the elliptically fibered $K3$ in Weierstrass form,

$$y^2 = x^3 + f_8(x_1, x_2)xz^4 + f_{12}(x_1, x_2)z^6, \tag{3.73}$$

where x_1 and x_2 are the coordinates of the base $\mathbb{P}^1$. To find explicit examples for target space duality, recall there are several constraints that must be met. First, it is necessary to have a well defined GLSM. This means the first Chern class of both bundles should be zero, and the second Chern class of both bundles (or sheaves) should be 12.

In addition we must make sure that the hybrid phase in which we do the TSD "exchange" of G and F actually exists. In the process of generating the TSD pairs, it may happen that singularities arise the in bundle or manifold (we expect that crepant resolutions should exist for the manifold and that the singularities in the "bundle" should be codimension 2 in the base, so that the sheaf is torsion free). In addition to these constraints for the GLSM, there is another practical requirement for finding the F-theory geometry: we prefer to work with $SU(N)$ bundles which have a non-degenerate spectral cover. If this is not satisfied it is still possible to find the F-theory dual, but we should remember that the form of the spectral sheaves can

[8]After a thorough computation we can show that,

$$0 \longrightarrow \mathcal{J} \longrightarrow FM^1(V) \longrightarrow \mathcal{O}_\sigma \longrightarrow 0,$$

where $\mathcal{J}$ is a torsion sheaf supported over $\{z = 0\} \cup \{F_3 = 0\}$, and its rank over $\{F_3 = 0\}$ is 2. $\mathcal{J}$ can be computed explicitly, but it is outside of the scope of this chapter.

be vastly more rich/complex in these cases. This enhanced data in the Picard group will not be manifest in the spectral cover, or in the complex structure moduli of the dual F-theory geometry. Instead it will be related in the dual F-theory to nilpotent Higgs bundles over singular curves [14, 29, 67, 71].

It is straightforward to find many GLSMs where at least one of the bundles (say $\mathcal{V}_1$) has $SU(N)$ structure, and spectral cover is non-degenerate. For example, consider (3.61) once again,

Λ				p	
1	1	2	3	−3	−4
1	5	3	7	−9	−7

with Chern class

$$C_2(V_1) = 5\sigma^2 + 22\sigma D + 23D^2 = 12 \ . \tag{3.74}$$

It should be noted that the algorithm for determining the spectral cover, using the methods of [36], was sketched above, but when the spectral cover becomes reducible (which can still be reduced), it is not guaranteed that those methods will find the full spectral cover (i.e. usually some (vertical) components will be missed). One can find these components by closer examination of the morphisms that define the bundle and elliptic fibration, as we saw in the last subsection. The spectral cover (schematically) is then given by (3.65)

$$S = F_6\left(f_2 x + f_6 z^2\right) .$$

Note that (3.61) by itself is not a well defined linear sigma model, therefore we need another bundle such that it's structure group embedded in the other E_8 factor. This second bundle must also have GLSM description over the same $K3$, and it's second Chern class should be,

$$C_2(V_2) = 6\sigma^2 + 24\sigma D + 21D^2 = 12. \tag{3.75}$$

Since the existence of this bundle with above properties may not be quite obvious, we turn now to constructing appropriate examples explicitly.

Example 1

We can construct an example $\mathcal{V}_2$ (though not the most general such bundle) as a direct sum of two bundles, each defined by the monad in (3.69) (which we denote it here by $\mathcal{V}_0$), with $c_2(\mathcal{V}_0) = 6$:

$$\mathcal{V}_2 = \mathcal{V}_0 \oplus \mathcal{V}_0. \tag{3.76}$$

For this monad bundle, the spectral cover was found to be of the form given in (3.72). In addition, the rank 1 sheaf on the spectral cover can be readily constrained. Here $FM^0(\mathcal{V}_0)$ is zero by results in [104] (see Section III.12, the final theorem). So we actually have the following short exact sequence:

$$FM^1(\mathcal{V}_2) = FM^1(\mathcal{V}_0) \oplus FM^1(\mathcal{V}_0) \tag{3.77}$$

where the support of $FM^1(\mathcal{V}_2)$, which is the spectral cover of $\mathcal{V}_2$, is the union of the spectral covers associated to the two copies of $\mathcal{V}_0$. The resulting spectral cover is a non-reduced scheme, which can be realized by the following polynomial[9]

$$S(V_2) = (F_3)^2\,(G_3)^2 z^4. \tag{3.78}$$

Before turning to the F-theory dual of this geometry, let us first construct a target space dual model for the above GLSM. To do that we add new chiral fields, in a way that after integrating them out, we return to the initial model. This can done by adding "repeated entries" to the charge matrix of the $K3$, and can lead to multiple TSD geometries (all still of the same topological type of manifold and bundle, of course). One possibility is

x						Γ		Λ_1					p_1			Λ_2				p_2	
3	2	1	0	0	0	−6	0	1	1	2	3	2	−3	−4	−2	0	1	2	3	−2	−4
3	2	0	1	1	1	−6	−2	0	4	1	5	6	−6	−3	−7	2	0	1	4	−4	−3

(3.79)

This Heterotic geometry ($K3$ manifold and bundle) has point-like singularities in the would-be bundle – that is, it is a rank 2 torsion free sheaf rather that a vector bundle [66].

With this pair of TSD bundles over $K3$ in hand, we are now in a position to consider the dual F-theory geometry. In this case we will ask the key question: *are the two GLSMs/geometries (i.e. that defined by $\mathcal{V}_1$ and $\mathcal{V}_2$, and its TSD dual in (3.79)), realized as different fibrations of a single F-theory geometry?*

By the results of the previous subsection, the complex structure of the

[9] Generally the two $\mathcal{V}_0$ in the above construction can be related by a continuous deformation, so we consider F_3 and G_3 as different generic degree 3 polynomials.

Calabi-Yau threefold can be readily determined:

$$\begin{aligned} Y^2 &= X^3 + F\,(u_1, u_2, x_1, x_2)\,XZ^4 + G\,(u_1, u_2, x_1, x_2)\,z^6, \\ F(u_1, u_2, x_1, x_2) &= u_1^4 u_2^4 f_8(x_1, x_2) + u_1^3 u_2^5 F_6(x_1, x_2) f_2(x_1, x_2), \\ G(u_1, u_2, x_1, x_2) &= u_1^7 u_2^5\,(F_3(x_1, x_2))^2\,(G_3(x_1, x_2))^2 \\ &\quad + u_1^6 u_2^6 f_{12}(x_1, x_2) + u_1^5 u_2^7 F_6(x_1, x_2) f_6(x_1, x_2). \end{aligned} \tag{3.80}$$

As frequently happens with degenerate spectral data, we find that the apparent F-theory gauge symmetry seems in contradiction with what is expected from the Heterotic theory we have engineered. By inspection of the discriminant of (3.80), it is straightforward to see that there appears to be an E_7 symmetry on $u_1 \to 0$, and an apparent E_8 singularity above the curve $u_1 \to \infty$. This might seem in contradiction with the expected gauge symmetry of $SO(12)$ in the hidden sector (determined as the commutant of the $SU(2) \times SU(2)$ structure group defined by the reducible bundle in (3.76)). However, in the case of degenerate spectral covers, we naturally expect that T-brane type solutions [14, 29] may well arise in the dual F-theory geometry. That is, we expect a nilpotent $SU(2) \times SU(2)$ Higgs bundle over the 7-brane which wraps around this curve ($u_2 = 0$) and breaks the space time gauge group to $SO(12)$ as expected (see [18] for a similar construction).

Next, as demonstrated in [132], changing the $K3$-(resp. $\mathbb{P}^1$-)fibration in the F-theory geometry simply amounts to exchanging the vertical $\mathbb{P}^1$ (whose coordinates are u_1 and u_2) with the horizontal $\mathbb{P}^1$ which is the base in the initial Heterotic $K3$ surface in (3.60). This means that the vertical $\mathbb{P}^1$ becomes the base of a dual Heterotic $K3$-surface. To determine gauge groups in the dual Heterotic theory, the discriminant curve must be considered. This is shown in Figure 3.1. The figure on the right hand side, shows the discriminant of the F-theory Calabi-Yau 3-fold. The line at the top is the locus of the E_7 singularity and the one at the bottom corresponds to E_8. The curve is the locus of the I_1 singularities. It intersects eight times with the III^* curve, where on six of them it has triple point singularities. These six points are exactly the zeros of F_6 in Figure 3.1, which naively correspond to point-like instantons that are responsible for the vertical components in spectral cover of $\mathcal{V}_1$ (not taking into account the spectral sheaf/T-brane data). Similarly the I_1 curve intersects with II^* at two sets of three points, which are the zeros of F_3 and G_3 in (3.78), and over all of them the curve has double point singularities. We can expand the polynomials in (3.80) in terms of x_1 and x_2, and read the dual Heterotic complex structure from there. Clearly we see that the elliptic $K3$ in the new Heterotic dual

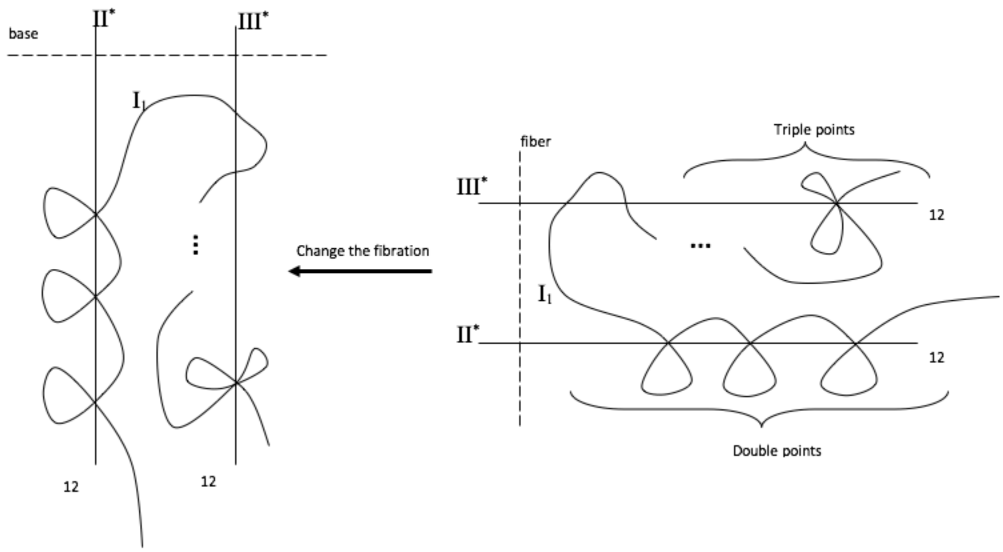


Figure 3.1: The vertical dotted line on the right hand side is the "vertical $\mathbb{P}^1$." After change of fibration on the left hand side the same $\mathbb{P}^1$ will be the base of the dual Heterotic $K3$.

must be singular. In particular, it exhibits singular E_8 and E_7 located at $u_1 = 0$ and $u_1 \to \infty$ respectively (with expected instanton number of 12 on each locus). This is a highly non-perturbative limit of the string theory. This exchange of gauge symmetry with singularities in the base $K3$ surface arising in the Heterotic theory seems to be a generic feature of exchanging F-theory fibrations [15]. As a result, it seems is impossible to get something which is purely smooth/perturbative on both sides like (3.79) from such a change of fibrations. This shows that at least some of the TSD dual pairs cannot be seen simply as different fibrations of the F-theory geometry. We explore these possibilities a little more in one further example.

Example 2

In this example, the starting geometry/bundles, are the same as before, but here we present another TSD geometry that can also be described easily by spectral cover.

So once again we take as our starting point the manifold/bundle:

x_i					Γ^j	Λ^a				p_l	
3	2	1	0	0	−6	1	1	2	3	−3	−4
6	4	0	1	1	−12	1	5	3	7	−9	−7

(3.81)

and embed it into a larger GLSM by adding a new gauge field, and fermionic and chiral fields:

x_i								Γ^j			Λ^a						p_l			
3	2	1	0	0	0	0	0	−6	0	0	1	1	2	3	3	3	−3	−4	−3	−3
6	4	0	1	1	0	0	1	−12	0	−1	1	5	3	7	8	9	−9	−7	−8	−9
0	0	0	0	0	1	1	0	0	−1	0	0	0	0	0	0	0	−1	0	0	0

(3.82)

After performing the combinatoric "exchange" (i.e. the usual TSD procedure), this yields the new TSD geometry

x_i								Γ^j			Λ^a					p_l		
3	2	1	0	0	0	0	0	−6	0	0	1	1	2	3	3	−3	−4	−3
6	4	0	1	1	0	0	1	−12	−1	0	1	5	3	7	8	−9	−7	−8
0	0	0	0	0	1	1	0	0	−1	−1	0	0	0	0	1	−1	0	0

(3.83)

The advantage of this new example is that, it is possible to compute the spectral cover of both sides easily[10] and they are both reducible but still reduced

$$S_1 = F_6(f_2 X + f_6 Z^2), \tag{3.84}$$

$$S_2 = F_7(f_1 X + f_5 Z^2). \tag{3.85}$$

As in the previous example, we can readily construct the F-theory geometry of both sides, and check whether they are related by exchanging the fibration or not. The

[10] In the previous example the base, $\mathbb{P}^1$, was defined as a conic inside $\mathbb{P}^2$. However, the spectral cover equations would be in terms of the ambient space coordinates and imposing the non-linear relations between the coordinates to define the $\mathbb{P}^1$ makes the situation somewhat obscure.

Weierstrass polynomials, F and G, of the dual F-theories is given by,

$$F_1 = \mathcal{O}(u_1^5) + u_1^4 u_2^4 f_1^8(v_1, v_2) + u_1^3 u_2^5 F_6(v_1, v_2) f_2(v_1, v_2), \tag{3.86}$$

$$G_1 = \mathcal{O}(u_1^7) + u_1^6 u_2^6 g_1^{12}(v_1, v_2) + u_1^5 u_2^7 F_6(v_1, v_2) f_6(v_1, v_2), \tag{3.87}$$

$$\tag{3.88}$$

$$F_2 = \mathcal{O}(v_1^5) + v_1^4 v_2^4 f_2^8(u_1, u_2) + v_1^3 v_2^5 F_7(u_1, u_2) f_1(u_1, u_2), \tag{3.89}$$

$$G_2 = \mathcal{O}(v_1^7) + v_1^6 v_2^6 g_2^{12}(u_1, u_2) + v_1^5 v_2^7 F_7(u_1, u_2) f_5(u_1, u_2). \tag{3.90}$$

As in the previous example, the change in fibration can be realized in F_1 and G_1 by re-expanding these polynomials in terms of v_1 and v_2. Then if the dual (F-theory) geometries are related through changing the fibration, after this rearrangement, F_1 and G_1 must be equal to F_2 and G_2.

But since F_2 and G_2 have an order three zero at $v_1 = 0$, it means that $f_1^8(v_1, v_2)$ and $g_1^{12}(v_1, v_2)$ must have an order three zero at $v_1 = 0$. Recall that these two polynomials are the f and g of the dual Heterotic $K3$ surface, so the above argument tells us if the TSD geometries are related to the different fibrations of the same geometry in F-theory, both TSD Calabi-Yau 2-folds must have an E_7 singularity at some point on the base. Thus once again, we see that exchange of fibration leads to a perturbative/non-perturbative duality in Heterotic [132] and not the apparent correspondence arising from TSD.

In summary, if we start with two perfectly smooth TSD geometries, they cannot be related through different $K3$-fibrations of a single F-theory 3-fold. But if we allow both $K3$ surfaces to be singular, and at the same time put bundles/small instantons over them, they might be dual to a single geometry in F-theory.[11]

Having determined that the multiple fibrations are not describing the TSD exchange in 6-dimensions, we can take a step back and ask *what F-theory correspondence is induced by TSD in 6-dimensions?* Since the spectral covers in (3.84) and (3.85) are relatively simple, we can try to roughly figure out some generalities about the F-theory duals of each of them. Let us start with the first one. The topology of the vector bundle fixes the dimension of the moduli space of the bundle,

$$h^1(V_1 \otimes V_1^*) = 42. \tag{3.91}$$

[11] But we should recall that the GLSM is only a perturbative formulation and clearly lacks information about the full string theory in such a context.

We can describe them in terms of the spectral data as follows,

$$dim(\mathcal{M}_V) = dim(cplx(C)) + dim(Jac(C)) + 6pts + 6 \times dim(Jac(E)) + gluing, \quad (3.92)$$

where C is the irreducible smooth curve defined by $f_2X + f_6Z^2$, by $6pts$ we mean the degrees of freedom in choosing the location of the six points defined by the zero set of $F_6 = 0$, and over them we have 6 elliptic curves (whose Jacobians must also be taken into account), and finally "gluing" denotes the degrees of freedom associated with the choice of spectral sheaf at the intersection of the 6 vertical fiber with C. The genus of C can be computed easily,

$$g(C) = 9. \quad (3.93)$$

Therefore, the dimension of the Jacobian and the complex structure of C must be 9. On the other hand, obviously, Jacobian of E is 1-dimensional, and the contribution of the "gluing" is 12-dimensional (each vertical fiber intersects C at 2 points). Therefore the total dimension of the Moduli space is,

$$dim(\mathcal{M}_V) = 9 + 9 + 6 + 6 + 12 = 42. \quad (3.94)$$

Now, to obtain the F-theory EFT we must use the spectral data as explained before, and infer the form of the complex structure of the CY 3-fold. From this procedure it can be seen that there are 6 (4,6,12) points in the F-theory geometry. Since the Heterotic dual is a perturbative model, we should consider these singularities as the singular limit of the following deformations,

$$F_1 = \mathcal{O}(u_1^5) + u_1^4u_2^4f_1^8(v_1, v_2) + u_1^3u_2^5(F_6(v_1, v_2)f_2(v_1, v_2) + \epsilon F_8(v_1, v_2)), \quad (3.95)$$

$$G_1 = \mathcal{O}(u_1^7) + u_1^6u_2^6g_1^{12}(v_1, v_2) + u_1^5u_2^7(F_6(v_1, v_2)f_6(v_1, v_2) + \lambda F_{12}(v_1, v_2)), \quad (3.96)$$

where ϵ and λ correspond to deforming the Higgs field over the 7-branes [14, 33]. Therefore we can deform these two theories into each other by continuously deforming the Higgs bundle. This reflects the fact the moduli space of the vector bundles on $K3$ is connected. Phrased differently, the existence of apparent $(4, 6, 12)$ points in the putative dual F-theory indicates that such solutions can *only be dual to the expected perturbative Heterotic theories* in the case that T-brane solutions arise. This has been seen before in [29] and is a substantial hint that G-flux must play an important role in the non-trivial F-theory correspondence expected in 4-dimensional compactifications.

It is worth commenting briefly also on another branch of the theory visible from this singular limit. We can increase the number of tensor multiplets in the 6-dimensional YM theory by performing small instanton transitions (i.e. moving NS5/M5 branes off the E_8 fixed plane in the language of Heterotic M-theory). For bundles described as spectral covers, this small instanton limit is visible by the spectral cover becoming reducible and vertical components (corresponding to small instantons) appearing (note that this limit must also set all gluing data to zero). Naively it seems that this limit appears different for the TSD pair of bundles defined by (3.84) and (3.85) since they exhibit different degree polynomials defining their vertical components (i.e. F_6 vs. F_7). However, this is simply a statement that the mapping of moduli in this case may exchange what are spectral cover deformations in one description with data associated to the Jacobian of the spectral cover (i.e. gluing data in this singular case). To really obtain the same point in moduli space, we must consider a scenario in which both halves of the TSD gain the same number of tensor multiplets (i.e. we pull either 6 or 7 5-branes into the bulk). In this case it would be intriguing to analyze the dual F-theory geometry – which would correspond to blowing up the base of the elliptic fibration. We expect in this case that the F-theory 3-fold will still be $K3$ fibered but no longer of such a simple form. In particular, the elliptic fibration over a Hirzebruch surface would be modified to become a more general conic bundle [15]. We will return to questions of a similar geometric nature in the following section.

Let us briefly summarize the results of our 6-dimensional investigation. We have seen that after exchanging $K3$-fibrations within the F-theory geometry, the dual Heterotic $K3$ surface must become singular, and therefore perturbative smooth Heterotic geometries arising in TSD pairs cannot in general be realized as different fibrations within F-theory. On the other hand we saw that the dual F-theory EFTS arising from the chosen TSD pairs must crucially rely on data from the intermediate Jacobian of the CY 3-fold – so-called T-brane solutions – in order to give rise to the same physical theories. Starting from such points we can deform back to smooth points in the CY 3-fold moduli space and identify the theories. Any possible correspondences within the tensor branch of the 6-dimensional theories must involve more complicated $K3$-fibrations (i.e. conic bundles) and we leave this exploration to future work.

3.3.6 F-theory duals of 4-dimensional Heterotic TSD pairs

In Section 3.2 we provided a non-trivial example of a Heterotic TSD pair in which both X and $\tilde{X}$ were elliptically fibered. It is now natural to ask – *what are the*

F-theory duals of these Heterotic theories? As we will explain below, this example (and others like it that we have found) seem to force beyond the arena of "standard" Heterotic/F-theory duality (as in the canonical reference [89]) by including several important features in the dual geometries. In this section, we will not try to solve all the obstacles that arise at once. Instead, we will outline what can be determined about the dual F-theory geometries and where new tools will be needed to fully probe this correspondence. Many of these we are currently developing [21, 22] and we hope to definitively answer these questions in future work.

As a first step towards determining the dual F-theory geometry, the data of the Heterotic bundle must be taken through a Fourier-Mukai transform to be presented as spectral data (see the discussion in Section 3.3.3). However in this we immediately encounter several problems. The first of these is that unlike in the case of Heterotic/F-theory dual pairs studied in the literature to date, neither of these Heterotic CY elliptic 3-folds is in Weierstrass form.

To be specific we focus on the examples in Section 3.2 (though similar obstacles will arise in general in this context). Recall that each of the CY 3-folds listed in (3.33) admitted two rational sections. Those for X in (3.33) lie in the following classes

$$[\sigma_1(X)] = -D_1 + D_2 + D_3, \qquad [\sigma_2(X)] = 2D_1 - D_2 + 5D_3,$$

where D_i are a basis of divisors on X (inherited from the ambient space hyperplanes by restriction). By "rational" it is meant that these divisors are isomorphic to blow ups of the base manifold (in this case P^2). The first difficulty with this example is that the standard Fourier-Mukai transformation with Poincare bundle is not applicable here. The reason is we need the zero section to intersect exactly one point on *every fiber*, but both of the sections described above wrap around a finite number of rational curves (which are components of reducible fibers). We have shown [21] that in specific situations one can use flop transitions to make one of the sections holomorphic, and since derived categories are invariant under the flop transitions (i.e. there is a specific Fourier-Mukai functor for flops), it is still possible to define the spectral data in the "flopped" geometry. However the example given in (3.33) proves to be too complicated to be analyzed in this manner since σ_1 and σ_2 wrap around 27 and 127 rational curves respectively, rendering the necessary birational transformations (i.e. flops) impractical.

In principle, one might hope to bypass this difficulty by transitioning X directly to its Weierstrass form (by blowing down the reducible components of fibers), following the Deligne procedure outlined in [15, 16]. However, this poses difficulties in a Heterotic theory in that it is unclear how the Heterotic bundle data should be appropriately mapped to this singular limit of X.

Nonetheless, if we choose σ_1 as the zero section, it can be demonstrated that the spectral cover in the singular Weierstrass limit, has the same divisor class as before (this is seen by taking the FM transform before blowing down the reducible fiber components). In other words, if we write the second Chern class as

$$c_2(V) = 36\sigma_1 H + 14 S_{sh} H + 156 f, \tag{3.97}$$

where H is the (pull-back of the) hyperplane divisor in the base, S_{sh} is the divisor corresponding to the Shioda map [145, 146, 152] for non-trivial Mordell-Weil group,

$$S_{sh} = \sigma_2 - \sigma_1 - 18H \tag{3.98}$$

and f is the fiber class. In terms of these divisors, the class of the spectral cover, S, in the singular limit will be,

$$[S] = 6\sigma_1 + 36H. \tag{3.99}$$

We might hope to get some information about the F-theory geometry just from the spectral cover alone. Naively, we may write the algebraic formula for the spectral cover whose class is given in (3.99) as

$$S = f_{36} z^3 + f_{30} xz + f_{27} y, \tag{3.100}$$

where f_i are generic polynomials of degree i over $\mathbb{P}^2$. A generic deformation of the spectral cover of the form (3.100) can be obtained by counting the degrees of freedom in the polynomials f_{36}, f_{30}, f_{27} which contain 703, 496 and 406 parameters, respectively. Immediately we see that these numbers much higher than the dimension of the vector bundle moduli space in our example, which is 292-dimensional. Thus, we can see that the FM-transform of the monad in (3.23) is certainly *not* a generic spectral cover. This is not too surprising. We have seen examples of the spectral cover of monads in Section 3.3.3 and there it was clear that the polynomials are not generic, rather they are dictated by the monad's map (see also [36, 67]). In principle, a similar story happens in the current case. We expect that the spectral cover may also be non-reduced/reducible [36]. However, regardless of its explicit form, the question arises, why is the spectral cover forbidden from assuming a generic form? That is, given an explicit starting point (in which the polynomials are determined by the monad map as in (3.23)) why is the deformation space restricted?

We expect that the answer to this lies with the other half of the spectral data of this monad, that is, the rank 1 sheaf [89] supported over the spectral cover in (3.99) and (3.100). It has been observed previously [72] that the Picard group of S may "jump" at higher co-dimensional loci in moduli space – i.e. so-called Noether-Lefschetz loci. This phenomenon could "freeze" the moduli of the spectral cover to a sub-space compatible with the form of the monad map (see also [4]). In terms of the 4-dimensional, $\mathcal{N} = 1$ EFT, the reduction in the apparent number of singlets (i.e. the non-generic form of the spectral cover) is a symptom of existence of a specific superpotential – arising from the Gukov-Vafa-Witen form [102]:

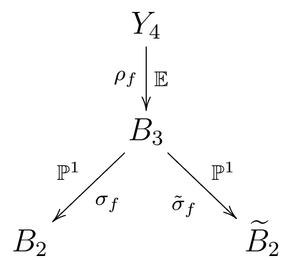
$$W \sim \int_X H \wedge \Omega \tag{3.101}$$

where $H \sim dB + \omega_3^{YM} - \omega_3^{Lorentz}$, and $\omega_3 = F \wedge A - \frac{1}{3} A \wedge A \wedge A$ is the Chern Simons 3-form (and the associated Lorentz quantity built from the spin connection in $\omega_3^{Lorentz}$ and Ω is the holomorphic $(0,3)$ form on X. The existence of this superpotential arises from the presence of the gauge bundle (rather than from quantized flux) (see [8, 10, 72] for related discussions) but none-the-less stabilizes vector bundle moduli.

As a result, in the dual F-theory EFT, we also expect the existence of a superpotential. Geometrically, since the spectral cover determines part of the complex structure moduli of the Calabi-Yau fourfold, it is clear that the dual of the bundle data given in (3.23) should include a specific G-flux that stabilizes the moduli through the GVW superpotential. It should be noted that there is another way to see the requirement for this flux: since there are no $D3$ branes in the F-theory dual (since we have chosen $c_2(X) = c_2(V)$ in the Heterotic theory), G-flux is also necessary for anomaly cancellation.

Although we have not yet explicitly calculated the FM transform of the Heterotic bundle or determined the dual F-theory geometry, the arguments above show that whatever the F-theory geometry, G-flux must play a prominent role and therefore it cannot be ignored. A similar set of arguments can also be made about the F-theory dual of the Heterotic TSD geometry $(\tilde{X}, \tilde{V})$. In this case as well, the naive deformations of the spectral cover are much larger than the dimension of the vector bundle moduli space, and therefore we conclude that Noether-Lefschetz loci/G-flux should be in play.

Despite the fact that flux must be involved in the putative F-theory duality, it still remains to be asked whether the dual F-theory 4-folds might still exhibit multiple fibration structure? That is could the geometric scenario described in the introduction with these compatible elliptic/$\mathbb{P}^1$ (and hence $K3$) fibrations exist?



On this front, once again we see that we must quickly leave behind the "standard" geometry of Heterotic/F-theory duality. As reviewed in Section 3.3, if the Heterotic CY 3-fold is in Weierstrass form, the construction of [89] generates a threefold base, B_3 (see (3.43)) for the CY 4-fold that is a $\mathbb{P}^1$-bundle over the base B_2 (which is the base of the dual elliptically fibered CY 3-fold and $K3$-fibered 4-fold). The topology of this bundle (i.e. B_3 itself) is determined by the second Chern class of the Heterotic bundle $c_2(V)$. In this context then, we can ask whether or not such a base could admit two different descriptions as a $\mathbb{P}^1$ bundle? While multiply fibered $\mathbb{P}^1$ bundles certainly exist (for example the "generalized Hirzebruch" toric 3-fold defined as the zero twist over $\mathbb{F}_n$ or the n-twist over $\mathbb{F}_0$ [15, 34]), it is easy to demonstrate that

$$h^{1,1}(B_3) = 1 + h^{1,1}(B_2) \tag{3.102}$$

for any $\mathbb{P}^1$ bundle. As a result, it is clear that *there exists no multiply fibered $\mathbb{P}^1$-bundles compatible with the B_2 and $\tilde{B}_2$ arising in Section 3.2* since for those manifolds $B_2 = \mathbb{P}^2$ and $\tilde{B}_2 = dP_1$. Hence $h^{1,1}(B_2) < h^{1,1}(\tilde{B}_2$ and $h^{1,1}(B_3) \neq h^{1,1}(\tilde{B}_3)$ for 3-fold bases constructed as $\mathbb{P}^1$-bundles.

From the results above we would be tempted to conclude that the hypothesis we set out to test in the literature (i.e. is Heterotic TSD dual to multiply fibered geometries in F-theory?) is manifestly false. However, we must first recall that the construction of $K3$ fibrations in terms of $\mathbb{P}^1$-bundle bases B_3 as commonly used in the literature is not the only possible structure. More general 3-fold bases, B_3 are possible which are $\mathbb{P}^1$ fibrations *but not $\mathbb{P}^1$ bundles.* These fibrations degenerate (as multiple $\mathbb{P}^1$s) over higher co-dimensional loci in the base B_2 and are known as "conic bundles" in the literature (see e.g. [143]).

If we consider this more general class of bases for CY 4-folds it seems that some possibilities remain. For example, the following threefold

$$B_3 = \left[\begin{array}{c|cc} \mathbb{P}^2 & 0 & 1 \\ \mathbb{P}^1 & 1 & 0 \\ \mathbb{P}^2 & 1 & 1 \end{array} \right] \tag{3.103}$$

is manifestly fibered over both $\mathbb{P}^2$ and dP_1 as required. However, it is unclear that the generic "twist" of such a fibration is compatible with the topology of the bundles defined in Section 3.2. It is possible to generalize simple constructions like the one above to accommodate more general twists by choosing more general toric ambient spaces. However, in each case we hit a new problem in that the stable degeneration limits of $\mathbb{P}^1$ bundles such as that in (3.103) are not yet understood in the literature (though we are considering such geometries in separate work [22]). As a result, it is a non-trivial task to determine whether such a geometry might arise in the F-theory duals of the examples outlined in Section 3.2. To check this we need precise spectral data. But as explained before, finding the Fourier-Mukai transforms of the Heterotic bundles, while possible in principle, is beyond our current computational limits for the bundles in Section 3.2.

For now though, we can conclude that whatever the F-theory correspondence induced from $(0, 2)$ target space duality may be, it must expand the current understanding of Heterotic/F-theory duality both via the crucial inclusion of G-fluxes (including possibly limits and T-brane solutions) and via more general geometry – in particular $K3/\mathbb{P}^1$-fibrations – than has previously been considered.

Chapter 4

F-theory on General Conic Bundle Bases

This chapter is based on an upcoming paper written in collaboration with L.B. Anderson, J. Gray, P.K. Oehlmann and N. Raghuram. As mentioned in the introduction, it is the goal of the present chapter to extend the $\mathbb{P}^1$ fibration geometries B_n (which is the base of a Calabi-Yau $(n+1)$-fold),

$$B_n \to^{\mathbb{P}^1} B_{n-1}, \tag{4.1}$$

in two important ways in the context of 6- and 4-dimensional effective theories arising from Heterotic string theory and F-theory:

- In 4-dimensional compatifcations of F-theory, we consider new classes of $\mathbb{P}^1$-bundles, defined as the projectivization of a *general rank 2 vector bundle* over B_2. The Hartshorne/Serre construction [88] guarantees that any rank 2 vector bundle over a complex surface can be described via an extension sequence

  $$0 \to L_1 \to V_2 \to L_2 \otimes \mathcal{I}_z \to 0 \tag{4.2}$$

  where L_1 and L_2 are line bundles over B_2 and $\mathcal{I}_z$ is an ideal sheaf associated to a set of points $\{z\}$ on B_2.

  We will build more general base manifolds $\mathcal{B}_3$ for F-theory as the projectivization, $\mathbb{P}(\pi : V_2 \to B_2)$, of the rank 2 bundles shown in (4.2). We will see in this case that rational sections can appear in the $\mathbb{P}^1$-bundle which have important consequences in Heterotic/F-theory duality.

- We will further analyze $\mathbb{P}^1$ fibrations which degenerate over some sublocus[1] $\Delta_b \subset B_{n-1}$. In the mathematics literature, the Sarkisov program (see e.g. [142, 143]) has led to a systematic classification of such objects in terms of their birational geometry. The simplest example of a fiber which could degenerate consists of a $\mathbb{P}^1$ fiber which is described as a conic in $\mathbb{P}^2$: $\mathbb{P}^2[2]$. Over higher-codimensional loci in the base manifold, the defining equation of such a fiber can clearly factor into a product of two linear functions in $\mathbb{P}^2$, leading to a degeneration of the $\mathbb{P}^1$ fiber into *two* distinct $\mathbb{P}^1$s over Δ_b.

In each of these cases above we will review the geometry of $\mathbb{P}^1$-fibrations in as general a context as possible and comment on the effective physics of both F-theory and Heterotic string theory defined over the relevant dual geometries (as in (3.42)).

We find that moving away from the standard set of $\mathbb{P}^1$-fibered bases $\mathcal{B}_n$ (and hence $K3$-fibered CY manifolds), the effective physics in Heterotic/F-theory duality can become very different from that seen in the standard situation [89]. In particular, we demonstrate that they behave differently under weak-coupling Heterotic limits (i.e. under stable degeneration [83]) and can lead to previously unexplored structure in the dual Heterotic geometry.

The structure of this chapter is as follows. In Section 4.1 we provide the basic geometric ingredients for the present study – namely the known properties and categorizations of $\mathbb{P}^1$-bundles and fibrations. In Section 4.2 we apply these insights to the study of 6-dimensional F-theory compactifications on elliptically fibered surfaces. We reframe the standard perturbative and non-perturbative Heterotic/F-theory duality in terms of properties of the $\mathbb{P}^1$-fibration. Most of this section is review, however, even in the well-understood arena we find new phenomena possible with the so-called "jumping effect" of $\mathbb{P}^1$-bundles in Section 4.2.3. By analyzing the structure of $\mathbb{P}^1$-bundles over $\mathbb{P}^1$ we consider the possibility that for special values of the complex structure of the Calabi-Yau threefold the base complex surface can "jump" between distinct complex manifolds– for example between the Hirzebruch surfaces $\mathbb{F}_0$ and $\mathbb{F}_2$ (two geometries which are diffeomorphic as real manifolds, but distinct as complex manifolds). Next in Section 4.3 we consider simple 4-dimensional compactifications of F-theory arising from more general $\mathbb{P}^1$-fibered 3-dimensional base manifolds, $\mathcal{B}_3$. Finally, in Sections 4.4 and 4.5 we consider monodromy in the $\mathbb{P}^1$-fibrations and its consequences for dual physical theories.

[1] Note we will refer to the discriminant of the $\mathbb{P}^1$-fibration as $\Delta_b \subset B_{n-1}$ to distinguish it from the discriminant of the CY elliptic fibration, $\Delta_f \subset \mathcal{B}_n$.

4.1 $\mathbb{P}^1$-fibrations in a nutshell

4.1.1 $\mathbb{P}^1$-bundles

In the case that the base, B_{n-1}, to the F-theory $K3$-fibration has a trivial Brauer group, it is known that $\mathbb{P}^1$-bundles over B_{n-1} are in $1-1$ correspondence with the projectivization of rank 2 vector bundles over B_{n-1} [104]. Intuitively, projectivization of the fiber space of a rank m vector bundle, $\pi : V \to B_{n-1}$, is simply the transition from a non-compact $\mathbb{C}^m$-dimensional fiber to its compactification,[2] $\mathbb{P}^{m-1}$, where the fiber coordinates are identified up to a scale that is chosen by $\wedge^m V$, point by point over the base [104]. The resulting $\mathbb{P}^{m-1}$-fibered manifold is denoted $\mathbb{P}(V)$. As an example relevant to the present work, the projectivization of any rank 2 vector bundle, $\mathbb{P}(V_2)$ is a smooth, n-dimensional manifold with a nowhere degenerate $\mathbb{P}^1$-fibration.

The simplest example of this is to consider a sum of two line bundles

$$V_2 = L_1 \oplus L_2 \tag{4.3}$$

over B_{n-1}. In the case that B_{n-1} is a toric manifold and V_2 is abelian as in (4.3), the $\mathbb{P}^1$-fibered manifold $\mathcal{B}_n$ obtained by projectiviation is also manifestly toric by construction. It is this class of geometries that initiated the first systematic studies of Heterotic/F-theory duality. Examples include:

- In 6-dimensional dual compactifications of Heterotic/F-theory, the dimension of the base of the elliptic fibration is $n = 2$ in (3.42) and the "shared" base to the Heterotic elliptic fibration and F-theory K3 fibration is simply $\mathbb{P}^1$. In this case the projectivization of a sum of line bundles yields the well-known Hirzebruch surfaces [88]

$$\mathbb{P}\left(\pi : \mathcal{O} \oplus \mathcal{O}(n) \to \mathbb{P}^1\right) = \mathbb{F}_n \tag{4.4}$$

which provided the context for the first 6-dimensional studies of Heterotic/F-theory duality (see e.g. [35]).

Likewise, over any complex surface, the projectiviation of a sum of line bundles provides a simple $\mathbb{P}^1$ bundle threefold, $\mathcal{B}_3$. This geometry was first outlined

[2] Note that since we consider complex manifolds only, throughout this work $\mathbb{P}^m$ refers to *complex* projective space.

by Friedman, Morgan and Witten [89] and crucially used in their explicit matching of degrees of freedom, anomaly cancellation, etc in Heterotic/F-theory duality. The projectivization of a sum of line bundles over a *toric* complex surface was later employed in the literature to systematically generate large classes of 3-fold bases for elliptically fibered Calabi-Yau 4-folds [7, 103]. A simple example is given below for a toric $\mathbb{P}^1$ bundle defined over $\mathbb{P}^2$ (i.e. a threefold analog of a Hirzebruch surface). The GLSM charge data is

$$\begin{array}{ccccc} x_0 & x_1 & y_0 & y_1 & y_2 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 0 & n & 1 & 1 & 1 \end{array} \tag{4.5}$$

This manifold is the projectiviation $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-n))$. Although the projectivization of a sum of line bundles has led to a well-understood Heterotic/F-theory dual pair, it is far from the only possibility. At this point we move beyond the simple cases considered in the F-theory literature to date.

As already described, vector bundles over complex surfaces can be simply classified thanks to the Serre construction [88]. A general rank 2 vector bundle over B_2 takes the form:

$$0 \to L_1 \to V_2 \to L_2 \otimes \mathcal{I}_z \to 0 \tag{4.6}$$

where L_1 and L_2 are line bundles over B_2 and $\mathcal{I}_z$ is an ideal sheaf associated to a (possibly empty) co-dimension 2 subscheme – i.e. a set of points $\{z\}$ on B_2. Since the projectivization of a bundle is invariant under twists by a line bundle – i.e. $\mathbb{P}(V) \simeq \mathbb{P}(L \otimes V)$ for any line bundle L [104] – we can without loss of generality write this sequence as

$$0 \to \mathcal{O} \to V_2 \to \mathcal{O}(D) \otimes I_z \to 0. \tag{4.7}$$

In the case that the ideal sheaf is non-trivial, the resulting $\mathbb{P}^1$ bundle, $\mathcal{B}_3 = \mathbb{P}(V_2)$, can have very different properties to those described for Abelian bundles above (i.e. the case of trivial extension class and vanishing $\{z\}$). One important feature that we will explore further in subsequent sections is that the $\mathbb{P}^1$ bundle need not have two holomorphic sections. Instead, it may admit only *rational* sections (which are birational but not diffeomorphic to the base, B_2).

We begin our exploration of these geometries by considering when the extension class in (4.7) is non-trivial. The extension class is determined by

$$Ext^1(\mathcal{O}(D) \otimes \mathcal{I}_z, \mathcal{O}) = Ext^1(\mathcal{O}, \mathcal{I}_z^\vee \otimes \mathcal{O}(-D)), \tag{4.8}$$

$$\mathcal{I}_z^\vee = \mathcal{O}_{B_n} \oplus det(\mathcal{N}_z B_2)|_z[-1]. \tag{4.9}$$

and $\mathcal{N}_z B_2$ is the normal bundle to z. Using the Leray sequence, it follows that

$$\begin{aligned} 0 &\rightarrow Ext^1_{B_2}(\mathcal{O}_{B_2}, \mathcal{O}_{B_n}(-D)) \rightarrow Ext^1_{B_2}(\mathcal{O}(D) \otimes \mathcal{I}_z, \mathcal{O}) \rightarrow Ext^0_z(\mathcal{O}_z, det(\mathcal{N}_z B_2)) \\ &\rightarrow Ext^2_{B_2}(\mathcal{O}_{B_2}, \mathcal{O}_{B_2}(-D)) \rightarrow \dots \end{aligned} \quad (4.10)$$

As a result, it is clear that even for divisors D such that $h^i(\mathcal{O}(-D)) = 0$, it is still possible for a non-zero extension class to exist. Note in the case that $h^i(\mathcal{O}(-D)) = 0$, this must in fact be the case, since if the extension were zero, V_2 would not be locally free (i.e. a vector bundle).

The projectivization $\mathbb{P}(V_2) = \mathcal{B}_3$ comes equipped with a canonical section to the $\mathbb{P}^1$ fibration (dual to the so-called "tautological" line bundle [104]). That is, there exists a divisor, S, on $\pi : \mathcal{B}_3 \rightarrow B_2$ and associated line bundle $\mathcal{O}(S)$ such that

$$\pi_*(\mathcal{O}(S)) = V \quad (4.11)$$

and the intersection of S with the generic fiber of the $\mathbb{P}^1$ fibration is the hyperplane within that fiber (i.e. a single point). Thus, S is a section to the $\mathbb{P}^1$-fibration (i.e. it induces a map $\sigma : B_2 \rightarrow \mathcal{B}_3$ such that $\pi \circ \sigma = id_{B_2}$). The intersection structure of the manifold guarantees that

$$S^2 = c_1(V) \cdot S. \quad (4.12)$$

The addition of this divisor completes the Picard group of B_3 and the Kähler structure of $\mathcal{B}_3$ is simply induced from that of the fiber and the base with

$$h^{1,1}(\mathcal{B}_3) = 1 + h^{1,1}(B_2). \quad (4.13)$$

Moreover, the Chern classes are readily computed to be

$$\begin{aligned} c_1(B_3) &= c_1(B_2) - D + 2S, && (4.14) \\ c_2(B_3) &= -c_1(B_2) \cdot D + 2c_1(B_2) \cdot S + c_2(B_2) - D \cdot S + S^2 + \pi^*[z], && (4.15) \\ c_3(B_3) &= -c_1(B_2) \cdot D \cdot S + c_1(B_2) \cdot S^2 + 2c_2(B_2) \cdot S. && (4.16) \end{aligned}$$

It is worth noting that while the first Chern class remains unchanged from the case of $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(D))$, the second Chern class of B_3 has picked up a contribution from the ideal sheaf. We will return to the physical significance of this term for anomaly cancellation in Heterotic/F-theory duality in Section 4.3.

Before concluding this section, we return now to the issue of whether or not there exist other sections to the $\mathbb{P}^1$-fibration, besides S described in (4.11)? It was also be crucial to determine whether or not these sections are rational or holomorphic – that is whether the zero-locus of the section is strictly birational to the base, B_2, or exactly diffeomorphic to it.

To this end it is useful to recall a simple result from Hartshorne [104]:

Theorem 4.1. *Let $\mathbb{P}(V)$ be defined as above and let $g : S \to B_2$ be any morphism. Then defining a morphism $f : S \to \mathbb{P}(V)$ is equivalent to specifying a surjective morphism of sheaves on S, $g * V \to \mathcal{L} \to 0$ where $\mathcal{L}$ is an invertible sheaf on S.*

In our case we will apply this theorem to the case that S is in fact a *holomorphic* section (i.e. a true section for *every* fiber) and hence g is a diffeomorphism. In this setting then, there is a $1-1$ correspondence between such sections and surjective morphisms of the form $V \to L \to 0$ with L a line bundle on B_2. If there are two such holomorphic sections *that are homologically inequivalent* then it is clear that there are two surjections $V \to L \to 0$ and $V \to L' \to 0$ which implies that V splits as $V = L \oplus L'$. Thus, the existence of two distinct holomorphic sections reduces the $\mathbb{P}^1$ bundle to the projectivization of the familiar form $\mathbb{P}(L_1 \oplus L_2)$.

In this case, we can without loss of generality write $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(D))$ and the two defining sections of the $\mathbb{P}^1$ fibration are the well known S_0, S_∞ introduced in [89]. Here S_0 is the canonical section defined in (4.11) above and $S_\infty = S_0 + \pi^* D$ is defined so that $S_0 \cdot S_\infty = 0$.

For the case that the additional sections are *rational*, we must tread more carefully. Here as we will see in examples in Section 4.2.3, multiple sections can exist for the projectivization of non-trivial extensions of the form (4.7). We will return to the geometry above in more detail, as well as its physical consequences in later Sections.

4.1.2 Conic bundles

In this section we consider the geometry of more general $\mathbb{P}^1$-fibrations in the base manifolds, $\mathcal{B}_n$ of F-theory. As has been done elsewhere in the literature, we will simply refer to such geometries as "conic bundles" after the simplest prototype of such a fibration in which the fiber is a degree two hypersurface inside $\mathbb{P}^2$. For example the following threefold

$$\mathcal{B}_3 = \left[\begin{array}{c|c} \mathbb{P}^2 & 2 \\ \mathbb{P}^2 & 2 \end{array} \right] \qquad (4.17)$$

is a $\mathbb{P}^1$ fibration over $\mathbb{P}^2$ whose fiber is described as a conic in $\mathbb{P}^2$. As mentioned before, this fiber can degenerate over higher-codimensional loci in the base $\mathbb{P}^2$. On this locus $\Delta_b \in B_2$, the conic in $\mathbb{P}^2$ can factor into *two* linear equations and hence, lead to a fiber consisting of not one, but two $\mathbb{P}^1$s.

More formally, we define a conic bundle as [139, 142]

Definition: A *conic bundle* is a proper flat morphism $\pi : \mathcal{B}_n \to B_{n-1}$ of smooth varieties such that it is of relative dimension one (i.e. the fiber is 1-(complex) dimensional) and the anti-canonical divisor $-K_{\mathcal{B}_n}$ is relatively ample.

Roughly, the term "relatively ample" above refers to the property that $-K_{\mathcal{B}_n}$ restricted to the fiber is ample.

In general, a conic bundle as defined above may have a variety of algebraic descriptions. However, the Sarkisov program [142] has characterized these manifolds in terms of birational minimal models whose fibers are conics in $\mathbb{P}^2$. In general, a "standard form" for a conic is characterized by

1. A discriminant locus, $\Delta_b \subset B_{n-1}$ (over which the fiber degenerates).
2. A generic twist to the $\mathbb{P}^1$ fibration (playing the role of $\mathcal{O}(D)$ in the usual case of the projectivization of two line bundles: $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(D))$).
3. A two-sheeted cover of B_{n-1} defined as the multisection defined by the hyperplane in the $\mathbb{P}^2[2]$ fiber in the standard form.

We will return to a number of these characterizing features as we study examples in later sections.

It is worth noting here that since our primary motivation in this work is the study of Heterotic/F-theory duality, we will be interested in conic bundles that *admit sections*. This in turn is equivalent to the statement that the F-theory elliptically fibered Calabi-Yau manifold, Y_{n+1}, possesses a $K3$-fibration with section. It is in this context that we are sure that the 8-dimensional Heterotic/F-theory duality [151] extends to a lower dimensional duality relating the effective theories. However, this choice causes us to diverge from the birational "standard models" for conic bundles described above, since in general fibers of the form $\mathbb{P}^2[2]$ admit multi-sections only, rather than true sections. We will also see in Section 4.5 that for some corners of moduli space, rational sections can be tuned even for such "standard" conic bundles.

It is clear that for general $\mathbb{P}^1$-fibrations we can characterize these fibrations in terms of those that can be simply related (i.e. via small resolutions) to $\mathbb{P}^1$-bundles and those that cannot. Morally, this is a question of whether one of the curves in the degenerate fibers can be shrunk to zero size, leading to a true $\mathbb{P}^1$ bundle.

A key feature that determines this property is whether or not the fibration exhibits monodromy over some higher-codimensional locus in B_2, that is, whether or not the multiple $\mathbb{P}^1$-components of singular fibers are homologically equivalent or not. We will explore this property in detail in Sections 4.4 and 4.5. For now however, it should be noted that this is monodromy of the $\mathbb{P}^1$-fibration is a phenomenon that is intrinsic to 4-dimensional compactifications of F-theory. Although conic bundles can exist for 6-dimensional compactifications of F-theory (i.e. conic bundles defined over $\mathbb{P}^1$), the degenerate fibers occur at most at points in the $\mathbb{P}^1$ base and thus, do not exhibit monodromy. In that setting the total space of the $\mathbb{P}^1$ fibration is a complex surface that is *always* birational to a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$ (i.e. a blow up of a Hirzebruch surface). Examples of this type have been studied in the literature [50] (see also Section 4.2 below).

Unlike in the case of $\mathbb{P}^1$-bundles, more general $\mathbb{P}^1$-fibrations can have topology that varies more widely from that of the simple case outlined in Section 4.1.1. Indeed, $h^{1,1}(\mathcal{B}_3)$ can be much larger than the minimal case of $1 + h^{1,1}(B_2)$ seen in (4.13). We will provide examples of such geometries in subsequent sections.

4.2 $\mathbb{P}^1$-fibered bases in 6-dimensional

F-theory compactifications In this section we undertake a simple "warm-up" and study 6-dimensional compactifications before moving on to 4-dimensional compactifications of F-theory. In particular, we consider the geometry and effective physics associated to $\mathbb{P}^1$-fibered base manifolds $\mathcal{B}_2$ for elliptically fibered Calabi-Yau threefolds, $\pi : Y_3 \to \mathcal{B}_2$. We will review several well known possibilities in this context and show that even in this simple setting, unexpected new phenomena are possible.

4.2.1 $\mathbb{P}^1$-bundles and $\mathbb{P}^1$-fibrations over $\mathbb{P}^1$

Here we consider complex surfaces $\mathcal{B}_2$ that are $\mathbb{P}^1$-fibered. In this context $\pi : \mathcal{B}_2 \to \mathbb{P}^1$. As argued above, every non-degenerate $\mathbb{P}^1$ fibration can be written as projectivization of a rank 2 vector bundle defined over the base (in this case $\mathbb{P}^1$). Since every vector bundle splits as a sum of line bundles over $\mathbb{P}^1$, it is clear that the most general $\mathbb{P}^1$-bundle takes the form

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)). \tag{4.18}$$

That is, the only $\mathbb{P}^1$-bundle surfaces are Hirzebruch surfaces.

In the case that we allow the $\mathbb{P}^1$-fibration to degenerate over points in the $\mathbb{P}^1$ base, this class can be extended to a wide range of surfaces, all birational to Hirzebruch surfaces [130]. For example, $\mathcal{B}_2 = dP_2$ can be viewed as a blow up of $dP_1 = \mathbb{F}_1$ and presented here as a co-dimension 2 complete intersection:

$$\mathcal{B}_2 = \left[\begin{array}{c|cc} \mathbb{P}^2 & 1 & 1 \\ \mathbb{P}^1 & 1 & 0 \\ \mathbb{P}^1 & 0 & 1 \end{array} \right]. \tag{4.19}$$

Alternatively, many simple toric descriptions for "conic bundles" over $\mathbb{P}^1$ exist in this context, for example

$$\begin{array}{ccccc} p & x_0 & x_1 & y_0 & y_1 \\ \hline 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & n & 1 & 1 \\ 1 & 0 & q & 1 & 0 \end{array} \tag{4.20}$$

is a simple toric blow-up of $\mathbb{F}_n$ [50].

The chains of possible blow ups can be vast (see [130] for explicit enumerations of them in the toric context). Indeed the *vast majority* of 2-dimensional bases for elliptically fibered Calabi-Yau threefolds fall into this latter category [100].

4.2.2 Dual Heterotic/F-theory geometry

Since it will be useful in subsequent sections we briefly review here the well-known dictionary of Heterotic/F-theory duality in 6-dimensions.

The perturbative $E_8 \times E_8$ Heterotic theory compactified on a $K3$ surface is fully specified by two poly-stable vector bundles, V_i, $i = 1, 2$ satsifying $c_1(V_i) = 0$ and $c_2(V_1) + c_2(V_2) = 24$. The massless spectrum of the theory is fixed by the topology of V_i, specifically the second Chern class which we can parameterize as $c_2(V_{1/2}) = 12 \pm n$ where $0 \leq n \leq 12$.

In the very first work on F-theory, a precise dictionary was established between Calabi-Yau threefold backgrounds of F-theory of the form $\pi : Y_3 \rightarrow \mathbb{F}_n$ and pertubative Heterotic theories with bundles of the form described above. In the so-called "stable degeneration" limit of F-theory, the elliptically fibered $K3$ surfaces degenerates into a fiber product of two rational elliptically fibered surfaces (i.e. dP_9s), glued together along a shared $\mathbb{P}^1$-base: $K3 \rightarrow dP_9 \cup_{\mathbb{P}^1} dP_9$. This limit consists of a "cylinderizing" of the $\mathbb{P}^1$-base in which the poles of the 2-sphere support E_8 gauge symmetries. The data of the Heterotic gauge bundles is encoded in the complex structure of the two dP_9-fibered "halves" of the degenerate Calabi-Yau threefold via the spectral cover construction [69].

In the non-perturbative limit of the theory, the addition of $NS5$ branes to the Heterotic theory increases the number of tensor multiplets [27, 50, 131, 132]. In the dual F-theory geometry this process consists of blowing up the base manifold (here $\mathbb{F}_n$ initially to likewise increase the number of tensors. For *smooth* $K3$ surfaces the number of 5-branes is limited by

$$c_2(V_1) + c_2(V_2) + m = 24, \tag{4.21}$$

where m is the number of 5-branes.

Importantly, in 6-dimensional theories, small instanton transitions, in which 5-branes are emitted/absorbed by the $S^1/\mathbb{Z}_2$ fixed planes are always possible. That is for every theory that contains additional tensor multiplets contains a limit in which the corresponding cycles in Y_3 are sent to zero size (i.e. blown-down).

From the point of view of $\mathbb{P}^1$ fibrations in the base of the F-theory geometry we see that the structure is simple. Either a $\mathbb{P}^1$ fibration is of the simple form already studied in the literature (i.e. $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n)) = \mathbb{F}_n$) or it is simply a blow up of this case. This latter possibility encompasses the conic bundles in this context – that is those $\mathbb{P}^1$ fibrations which degenerate over points in the $\mathbb{P}^1$-base. No monodromy is possible for such degenerates and they can always be limited back to the standard case.

The basic physical summary then is as follows:

In 6-dimensional Heterotic F-theory duality, base surfaces, $\mathcal{B}_2$, that are $\mathbb{P}^1$-bundles are simply Hirzebruch surfaces and lead to perturbative Heterotic string compactifications over $K3$. In the case that the $\mathbb{P}^1$-fiber degenerates at m points over the base $\mathbb{P}^1$, this leads to an increase in m tensor multiplets in the 6-dimensional theory and a non-perturbative Heterotic theory compactified over a (possible singular) $K3$ surface with

m $NS5$ branes.

The results above seems complete and it might be tempting to conclude from them that there is nothing new to observe about $\mathbb{P}^1$ fibered geometries in 6-dimensional string compactifications. However, even in this relatively simple context intriguing phenomena are possible.

As one example, first studied in [132], a single $\mathbb{P}^1$-fibered base manifold, $\mathcal{B}_2$, may admit *more than one* distinct $\mathbb{P}^1$-fibration. In such cases the F-theory effective physics is invariant under the choice of $\mathbb{P}^1$ fibration. However, this simple observation on the F-theory side can lead to novel structure in the dual Heterotic theory. The two possible interpretations of fiber and base in $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ lead to a highly non-trivial strong/weak coupling Heterotic duality studied by Duff, Minasian and Witten [86]. This effect was generalized in [15] to include more general $\mathbb{P}^1$ fibrations (i.e. conic bundles) and the corresponding Heterotic duality includes a rich array of possible interchanges between the Heterotic dilaton and the additional tensor multiplets arising from 5-branes.

To conclude we consider one last phenomenon that is made visible by considering $\mathbb{P}^1$-bundles in the F-theory geometry. It is possible for one $\mathbb{P}^1$-fibered manifold to "jump" between two distinct complex surfaces, with interesting consequences in the dual Heterotic geometry. We examine this effect below.

4.2.3 Jumping phenomena

In this section we consider a previously unexplored effect which is visible even in the relatively understood arena of 6-dimensional dual compactifications of Heterotic string theory/F-theory. We will consider a base, $\mathcal{B}_2$, to the F-theory elliptically fibered threefold, Y_3, that is itself defined as the projectivizataion, $\mathbb{P}(V)$ of a rank 2 vector bundle $\pi : V \to \mathbb{P}^1$.

To begin, let us consider a vector bundle defined via a non-trivial extension of two line bundles over $\mathbb{P}^1$. For example, since $H^1(\mathbb{P}^1, \mathcal{O}(-2)) = \mathbb{C}$, there is a 1-dimensional family of non-trivial extensions of the form

$$0 \to \mathcal{O} \to V \to \mathcal{O}(2) \to 0. \tag{4.22}$$

However, since it is well known that *every* vector bundle over $\mathbb{P}^1$ splits as a sum of line bundles [53], it is clear that V in (4.22) can *also* be written as an Abelian sum of two line bundles. In this case, it is easy to verify that

$$V = \mathcal{O}(1) \oplus \mathcal{O}(1) \tag{4.23}$$

has the same first Chern class (the only topological invariant in this case) as the extensional bundle defined in (4.22). However, a non-trivial extensional class is not the only possibility to be considered in (4.22). If the extensional class is chosen to be trivial then the extension splits as the sum $\mathcal{O} \oplus \mathcal{O}(2)$. Thus in this case, the moduli space of the simple extension bundle in (4.22) in fact consists of a disjoint point and a line. As the extension class is smoothly varied to zero the bundle "jumps" from one holomorphic type to another. These two sums of line bundles are distinct as holomorphic objects, but isomorphic as real bundles. This interesting effect is a well-known phenomena in the moduli space of bundles over $\mathbb{P}^1$.

Now, to extend this observation to the geometry of $\mathbb{P}^1$-fibrations central to this work, consider the projectivization of the bundle given in (4.22). For non-zero values of the extension class, it can be verified that $\mathbb{P}(V) = \mathbb{F}_0$ and can be simply written as a co-dimension 2 complete intersection manifold

$$\left[\begin{array}{c|cc} \mathbb{P}^3 & 1 & 1 \\ \mathbb{P}^1 & 1 & 1 \end{array} \right]. \tag{4.24}$$

Without loss of generality, the defining equations can be written as [116]

$$z_0 w_0 + z_1 w_1 = 0, \tag{4.25}$$

$$z_2 w_0 + \left[\sum_{i=0}^{2} a_i z_i + \epsilon z_3 \right] w_1 = 0, \tag{4.26}$$

where $\{z_0, z_1, z_2, z_3\}$ are the homogeneous coordinates on the ambient $\mathbb{P}^3$ and $\{w_0, w_1\}$ those of the ambient $\mathbb{P}^1$. As pointed out in [116], for generic values of the defining equations with $\epsilon \neq 0$ this surface is a smooth description of $\mathbb{F}_0$. However, for the special value of $\epsilon = 0$, the surface "jumps" to become $\mathbb{F}_2$. In each case the surface described by (4.24) is in fact rigid. However, as expected from the bundles which defined this surface, the moduli space consists of two infinitesimally close but distinct points, one for each of $\mathbb{F}_0$ and $\mathbb{F}_2$. Similarly to the observation above regarding bundles, these two surfaces are the same as real manifolds, but differ in their complex structure.

What happens then, when a Calabi-Yau manifold is defined as an elliptic fibration over such a "jumping" base surface? It is straightforward to show that the

Calabi-Yau threefold remains smooth and well behaved for all values of ϵ in (4.25). In this case, the topology of the Calabi-Yau threefold cannot vary as the complex structure is changed and indeed, we find that it does not. The only thing that changes in the structure of the elliptic threefold is that its cone of effective divisors changes as this modulus is varied. As the ϵ-parameter is varied to cause $\mathbb{F}_0$ to jump to $\mathbb{F}_2$, a curve of self intersection -2 becomes effective in (4.24) inducing a new effective divisor in the Calabi-Yau threefold Y_3 (which could be used as a locus on which to support gauge symmetry in the F-theory compactification).

In summary, using the structure of $\mathbb{P}^1$ bundles we have demonstrated that it is possible for the base manifolds of Calabi-Yau elliptic fibrations to non-trivially "jump" to distinct complex surfaces, while leaving the topological family of Calabi-Yau threefolds unchanged. From the point of view of F-theory, the changing effective cone of Y_3 allows for new singular limits/gauge enhancements in F-theory.

In the dual Heterotic theory however, the "jump" is even more striking. In jumping from an $\mathbb{F}_0$ to an $\mathbb{F}_2$ base we have jumped from an $E_8 \times E_8$ theory with gauge bundles whose second chern classes are $c_2(V_1) = c_2(V)_2 = 12$ to one in which $c_2(V_1) = 14$ and $c_2(V_2) = 10$. The matter spectrum of the theory depends on the topology of the V_i and hence can also jump in this process! It is natural to ask – what geometric/physcial mechanism can cause such a dramatic shift in the Heterotic theory?

Clearly, the solution to this problem relies on the form of the gauge symmetry tuned in F-theory, the sections to the $\mathbb{P}^1$ fibration and their roles in the stable degeneration limit. We will return to this important question of the dual Heterotic theory to this "jumping phenomena" after first building techniques for handling rational sections and stable degeneration limits in the subsequent sections.

4.3 General $\mathbb{P}^1$-bundle bases for 4-dimensional F-theory compactifications

In this section, we try to study the general features of a $\mathbb{P}^1$ bundle over a complex surface B_2 defined as the projectivization of a rank two vector bundle $\mathbb{P}(V)$. In particular, since the existence of a Heterotic dual depends on the existence of sections (that can define two can define two "homogeneous coordinates" over each fiber), we explore the properties of the sections of such $\mathbb{P}^1$ bundles. It turns out unless V is a split of two line bundles, the standard Heterotic/F-theory duality doesn't work.

By section, we mean an effective divisor that intersects almost all fibers at one point. By using the theorem4.1 (mentioned above) we conclude:

1. If $i : S \hookrightarrow \mathbb{P}(V)$ is a holomorphic section i.e., when the morphism g in 4.1 is an isomorphism, then there is a surjection $g^*V \simeq V \to \mathcal{L}_2$ for some line bundle $\mathcal{L}_2$ on S. By Serre's construction [88], we can "complete" this surjection into an extension on $S \simeq B_2$,

$$0 \to \mathcal{L}_1 \to V \to \mathcal{L}_2 \to 0. \tag{4.27}$$

   Therefore, $\mathbb{P}(V)$ has a holomorphic section if and only if V is an extension of two line bundles.

2. If S is a rational section, i.e. when g is a birational morphism. Again from the theorem, we must have a surjection $g^*V \to \mathcal{L}_2$ on S. Similarly, we can "complete" this surjection into an extension on S,

$$0 \to \mathcal{L}_1 \to g^*V \to \mathcal{L}_2 \to 0. \tag{4.28}$$

   By applying the pushforward functor Rg_* on the short exact sequence above, we cannot get (4.29) on B_2 because this contradicts the reasoning above ($S \neq B_2$). The only possibility that remains (by the Serre construction [88]) is to get the following short exact sequence on B_2,

$$0 \to \mathcal{N}_1 \to V \to \mathcal{N}_2 \otimes \mathcal{I}_z \to 0, \tag{4.29}$$

   where $\mathcal{N}_1$ and $\mathcal{N}_2$ are two line bundles on B_2, and $\mathcal{I}_z$ is the ideal sheaf of a set of points on B_2. Therefore without loss generality, we assume,

$$\mathcal{N}_1 \simeq g^*\mathcal{L}_1, \tag{4.30}$$

$$\mathcal{L}_2 = (g^*\mathcal{O}_{B_2}(D)) \otimes \mathcal{O}_S(e), \tag{4.31}$$

   for some divisor $D \in Pic(B_2)$ and some exceptional (-1)-curves in S. Therefore if $B_3 = \mathbb{P}(V)$ where V is given by (4.30) with $z \neq 0$, then B_3 has a rational section. In other words, S corresponds with the blow up of B_2 on z.

We will study each case in the following. From now on we take S to be a divisor corresponding to $c_1(\mathcal{O}_{\mathbb{P}(V)}(1))$.

- Type I: In this case we assume V is givven by (4.29). We assume $\mathcal{L}_1 \simeq \mathcal{O}_{B_2}$ and $\mathcal{L}_2 \simeq \mathcal{O}_{B_2}(D)$,

$$0 \to \mathcal{O} \to V \to \mathcal{O}(D) \to 0.$$

First note that, $h^0(\mathcal{O}(S)) = h^0(V) \geq 1$. So S is an effective divisor. We need the self-intersection S^2. Using,

$$0 \to TS \to TB_3|_S \to \mathcal{O}_S(S) \to 0, \tag{4.32}$$

then,

$$\begin{aligned} c_1(S) + S^2 = c_1(TB_3) \cdot S \quad &\Rightarrow \quad c_1 + S^2 = S \cdot (c_1 - D + 2S) \\ &\Rightarrow \quad S^2 = DS. \end{aligned} \tag{4.33}$$

Usually, for finding the Heterotic dual, one needs two sections, S_0 and S_∞ such that $S_0 \cdot S_\infty = 0$. This enables us to define the homogeneous coordinates on the $\mathbb{P}^1$ fibers. From the intersection relation above, we may guess $S_\infty = S - D$. But, $h^0(\mathcal{O}(S-D)) = h^0(V \otimes \mathcal{O}(-D))$, and it is easy to show that if the defining S.E.S is non-split, then $h^0(\mathcal{O}(S - D)) = 0$ (at least when $B_2 \simeq \mathbb{P}^2$, $\mathbb{F}^n$ and their blow-ups) i.e., *If the defining S.E.S is non-split, then $S_\infty = S - D$ is not an effective divisor, and it cannot be a section.*

This means the usual Heterotic/F-theory duality procedure cannot be applied. However, one might guess it may be possible to "rewrite" as a sum of two line bundles different from $\mathcal{O}$ and $\mathcal{O}(D)$ (up to a common twist). If this is possible, it means there are other sections for $\mathbb{P}(V)$. So we must check whether there are other sections or not? We use the following procedure to check for other sections,

1. Find divisors in $Pic(B_2)$ D_1 and D_2 such that: $D_1 + D_2 = D, \quad D_1 \cdot D_2 = 0$.
2. Check whether $H^0(B_2, V^* \otimes \mathcal{O}(D_2)) \neq 0$.

If both conditions are satisfied, define $S_0 = S - D_1$. We can show S_0 is a holomorphic section.

If in addition, $H^0(B_2, V^* \otimes \mathcal{O}(D_1)) \neq 0$, then there is also another section $S_\infty = S - D_2$, such that $S_0 \cdot S_\infty = 0$.

However note if S_∞ is a section, then by (4.1) there is a surjection, $V \to \mathcal{O}(D_1)$, but remember, the existence of S_0 also means the morphism $\mathcal{O}(D_1) \to V$ is injective. Therefore *when both S_0 and S_∞ sections, then V must be a direct sum of two line bundles i.e. $V = \mathcal{O}(D_1) \oplus \mathcal{O}(D_2)$.*

Before proceeding further, we should justify the criteria mentioned above. As already explained, the existence of a holomorphic section is equivalent to the surjection $V \to \mathcal{L}'$. Since V is a vector bundle (rather than being a singular coherent sheaf), we can complete the sequence as,

$$0 \to \mathcal{L}'' \to V \to \mathcal{L}' \to 0. \tag{4.34}$$

By comparing the original Chern classes of V, and the Chern classes that can be derived from the S.E.S above, we get the first condition,

$$D_1 + D_2 = D, \quad D_1 \cdot D_2 = 0, \tag{4.35}$$

$$D_2 = c_1(\mathcal{L}'), \quad D_1 = c_1(\mathcal{L}''). \tag{4.36}$$

On the other hand, the surjection $V \to \mathcal{L}'$ means $Hom(V, \mathcal{L}') = H^0(B_2, V^* \otimes \mathcal{L}') \neq 0$. Conversely, if the first condition above is satisfied, any non-zero morphism $V \to \mathcal{L}'$ must be surjective. Therefore,

$$V \to \mathcal{L}' \quad \text{surjective} \Leftrightarrow H^0(B_2, V^* \otimes \mathcal{L}') \neq 0. \tag{4.37}$$

Therefore these two conditions mean there is a section. We call it S_0, and it must satisfy,

$$S_0^2 = (D_2 - D_1)S_0. \tag{4.38}$$

From this we can find, $S_0 = S - D_1$. Interestingly one can find,

1. $\mathcal{O}(1)|_{S_0} = \mathcal{O}(D_2) = \mathcal{L}'$. As the theorem predicts.
2. $h^0(\mathcal{O}(S)) = h^0(V \otimes \mathcal{O}(-D_1))$. Note that $V \otimes \mathcal{O}(-D_1) \simeq V^* \otimes \mathcal{O}(D_2)$. Therefore,

$$\text{Effectiveness of } S_0 \Leftrightarrow H^0(B_2, V^* \otimes \mathcal{L}') \neq 0. \tag{4.39}$$

Note that this procedure can also be interpreted as a way of finding other descriptions of the initial V in terms of extension/direct sum of other line bundles. For vector bundles over a rational curve $\mathbb{P}^1$, this is a well-known fact. But, to our knowledge, for a general surface B_2, this has not been explored at least in string theory literature.

4.3.1 Examples: Type I

We illustrate the procedure above with two examples.

Example 1: $B_2 = \mathbb{F}_1$.

In this example, the initial defining V is given by a non-split extension, but it turns out there is also another description. V is defined as,

$$0 \to \mathcal{O} \to V \to \mathcal{O}(3,1) \to 0, \tag{4.40}$$
$$h^*(V) = (4,1,0). \tag{4.41}$$

We can check,

$$h^1(\mathcal{O}(-3,-1)) = 3, \quad \text{So the sequence } \textbf{can be} \text{ non-split}, \tag{4.42}$$
$$D_1 = (2,0), \quad D_2 = (1,1), \tag{4.43}$$
$$h^0(V \otimes \mathcal{O}(-2,0)) = 1, \quad \text{So } S_0 = S - D_1 \text{ is a section}, \tag{4.44}$$
$$h^0(V \otimes \mathcal{O}(-1,-1)) = 1, \quad \text{So } S_\infty = S - D_2 \text{ is a section}, \tag{4.45}$$
$$h^1(\mathbb{F}_1, \mathcal{O}(D_1 - D_2)) = 1, \tag{4.46}$$

we may naively conclude that V can be written as a non-split extension of $\mathcal{O}(D_1)$ and $\mathcal{O}(D_2)$ such that $\mathbb{P}(V)$ has two holomorphic sections. But note if this is true $h^*(V) = (3,0,0)$ and this contradicts with (4.40). In fact V is a direct sum

$$V = \mathcal{O}(2,0) \oplus \mathcal{O}(1,1). \tag{4.47}$$

This example confirms our reasoning. We can derive the intersection relations as before,

$$S^2 = (3t+f)S, \tag{4.48}$$
$$S_0 S_\infty = 0, \tag{4.49}$$
$$S_0^2 = (t-f)S_0, \tag{4.50}$$
$$S_\infty^2 = -(t-f)S_\infty. \tag{4.51}$$

Also note that $H^0(B_3, K_{B_3}^{-1}) = 18$, so this B_3 is a good base for a Weierstrass model, because the anticanonical line bundle is effective. Overall this is a good model for Heterotic/F-theory duality, and no unusual phenomena happen. In the next example, however, the next example is not as easy as this one.

Example 2: $B_2 = \mathbb{F}_0$.
This time we start with,

$$0 \to \mathcal{O} \to V \to \mathcal{O}(3,-1) \to 0. \tag{4.52}$$

We can check,

$$h^1(\mathcal{O}(-3,1)) = 4, \quad \text{So the sequence } \textbf{can be} \text{ non-split}, \tag{4.53}$$
$$D_1 = (2,-2), \quad D_2 = (1,1), \tag{4.54}$$
$$h^0(V \otimes \mathcal{O}(-2,2)) = 1, \quad \text{So } S_0 = S - D_1 \text{ is a section}, \tag{4.55}$$
$$h^0(V \otimes \mathcal{O}(-1,-1)) = 0, \quad \text{So } S_\infty = S - D_2 \text{ is } \textbf{not} \text{ a section}, \tag{4.56}$$
$$h^1(\mathbb{F}_0, \mathcal{O}(D_1 - D_2)) = 4. \tag{4.57}$$

One can show the solutions D_1 and D_2 are unique. However, even with D_1 and D_2 the vector bundle cannot be split (otherwise S_∞ would be effective). So there are no ways to rewrite this vector bundle as a direct sum, and in any event $\mathbb{P}(V)$ has only one section, and hence, we cannot (globally) define the two homogeneous coordinates we need on every fiber. One can also show $h^0(B_3, K_{B_3}^{-1}) = 21$. Therefore, it is possible to define a Weierstrass model over this base, and therefore we can have a well defined F-theory model. Then the question is, does this model has a Heterotic dual?! This is one of the main questions of this work.

- Type II: This time we consider the more general vector bundle over B_2,

$$0 \to \mathcal{O} \to V \to \mathcal{O}(D) \otimes \mathcal{I}_z \to 0.$$

Similar to the previous case, $S = c_1(\mathcal{O}_{\mathbb{P}(V)}(1))$, is an effective divisor. But it wraps around a finite number of P^1 fibers. So it is a rational section. One can also derive the self-intersection as before,

$$S^2 = DS - \pi^*[z], \tag{4.58}$$

where $\pi : \mathbb{P}(V) \to B_2$ is the projection map. We can rewrite the relation above as $S^2 = (D+e)S$, where e is the (-1)-curve in the rational section.

Contrary to the previous case, "non-splitness" of the S.E.S doesn't require $S - D$ to be non-effective. In other words, $S_0 = S$, $S_\infty = S - D$ could be both effective.

$$S_0 \cdot S_\infty = -[z]. \tag{4.59}$$

Question 1: Does $\mathbb{P}(V)$ have any holomorphic sections. In other words, is it possible to redefine the vector bundle V as a S.E.S of Type I?

The conditions that should be checked in this case are similar to the previous case,

1. Find divisors in B_2 D_1 and D_2 such that: $D_1 + D_2 = D, \quad D_1 \cdot D_2 = [z]$.
2. Check whether $H^0(B_2, V^* \otimes \mathcal{O}(D_2)) \neq 0$.

If a solution exist, it means we can redefine the vector bundle V as extension of two line bundles. So the problem reduces to the Type I problem, and if there are cases with 2 sections, it means V is a direct sum.

Question 2: Suppose there are no solutions to the conditions above. Then similar to the previous case, one can ask how many sections we can have.

By the same reasoning as Type I, we should solve the following conditions,

1. Find divisors in B_2 D_1 and D_2 such that: $D_1 + D_2 = D, \quad D_1 \cdot D_2 = [z']$.
2. Check whether $H^0(B_2, V^* \otimes \mathcal{O}(D_2) \otimes \mathcal{I}_{z'}) \neq 0$.

If there is a solution, then it is possible to rewrite the vector bundle as,

$$0 \to \mathcal{O}(D_1) \to V \to \mathcal{O}(D_2) \otimes \mathcal{I}_{z'} \to 0. \tag{4.60}$$

Again, $S_0 = S - D_1$ is an effective divisor. So there is another rational section. We may define $S_\infty = S - D_2$ as before, and then we get,

$$S_0 \cdot S_\infty = -[z'], \tag{4.61}$$
$$S_0^2 = (D_2 - D_1) \cdot S_0 - [z'], \tag{4.62}$$
$$S_\infty^2 = -(D_2 - D_1) S_\infty - [z']. \tag{4.63}$$

The effectiveness of the divisor S_∞ doesn't necessarily put the strong conditions, in contrast to Type I, on the cohomology of $\mathcal{O}(D_2 - D_1)$.

4.3.2 Examples: Type II

Example 1 $B_2 = \mathbb{P}^2$.

In this example we start with a non-split defining sequence,

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to V \to \mathcal{O}_{\mathbb{P}^2}(H) \otimes \mathcal{I}_p \to 0, \tag{4.64}$$

where p is a point. Note that the extension class can be non-zero even though the base is $\mathbb{P}^2$,

$$RHom(\mathcal{O}(H)\otimes\mathcal{I}_p,\mathcal{O}) = RHom(\mathcal{O},\mathcal{O}(-H)\otimes\mathcal{I}_p^\vee), \tag{4.65}$$

but,

$$\mathcal{I}_p^\vee = \mathcal{O}\oplus\mathcal{O}_p[-1], \tag{4.66}$$

so,

$$Ext^1(\mathcal{O}(H)\otimes\mathcal{I}_p,\mathcal{O}) = Ext^0(\mathcal{O},\mathcal{O}_p) = \mathbb{C}. \tag{4.67}$$

To find sections we have to repeat the analysis mentioned previously,

$$D_1 + D_2 = H, \tag{4.68}$$
$$0 \le D_1\cdot D_2 \le 1. \tag{4.69}$$

But these equations do not have any non-trivial solutions, thus $\mathbb{P}(V)$ *is not a direct sum of two line bundles, and three rational section with the same divisor class* $[S]$. One can rewrite V as,

$$0\to\mathcal{O}_{\mathbb{P}^2}(-H)\to\mathcal{O}_{\mathbb{P}^2}^{\oplus 3}\to V\to 0. \tag{4.70}$$

This means, $\mathbb{P}(V)$ is simply a hypersurface with divisor class $\bar{S}+H$ in $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}^{\oplus 3})$ Where $\bar{S}$ is just the (negative) divisor class of the tautological line bundle. In other words,

$$\mathbb{P}(V) = \left[\begin{array}{c|c} \mathbb{P}^2 & 1 \\ \mathbb{P}^2 & 1 \end{array}\right]. \tag{4.71}$$

It is easy to see $S = c_1(\mathcal{O}(1,0))$, and the sequence (4.70) is simply the pushforward of the Koszul sequence to the base $\mathbb{P}^2$. It is also simple to check that the section is actually a dP_1 surface, confirming the previous claims.

Example 2: $B_2 = \mathbb{P}^2$. As another example of the same type, consider the defining non-split sequence as follows,

$$0\to\mathcal{O}\to V\to\mathcal{I}_p\to 0. \tag{4.72}$$

In this case, we have only one rational section with self-intersection, $S^2 = -[p]$, and there are no other ways to find other sections. Similar to the last example, one can rewrite the sequence as,

$$0\to\mathcal{O}_{\mathbb{P}^2}(-2H)\to\mathcal{O}_{\mathbb{P}^2}(-H)\oplus\mathcal{O}_{\mathbb{P}^2}(-H)\oplus\mathcal{O}_{\mathbb{P}^2}\to V\to 0. \tag{4.73}$$

This means $\mathbb{P}(V)$ is just a hypersurface with divisor class $\bar{S}+2H$ in the toric variety $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(-H) \oplus \mathcal{O}_{\mathbb{P}^2}(-H) \oplus \mathcal{O}_{\mathbb{P}^2})$,

$$\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & 1 & 1 & 1 & 2 \end{array} \tag{4.74}$$

In both of these examples, we can have well defined F-theory models, but to find the Heterotic dual in addition to the problems of non-split bundles in Type I, we have a rational section. In other words, over some specific point in the base, the section not only doesn't define homogeneous coordinates; rather, it wraps the whole fiber. So is there a Heterotic dual?

We finish this section with the following example,

Example 3: $B_2 = \mathbb{P}^2$. As in type I, it may be possible to rewrite the defining vector bundle as the direct sum of two line bundles. For example, let us define V as,

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to V \to \mathcal{O}_{\mathbb{P}^2}(2H) \otimes \mathcal{I}_p \to 0, \tag{4.75}$$

The only solution for $D_1 + D_2 = 2H$ and $D_1 \cdot D_2 = 1$ is $D_1 = D_2 = H$.

So even though the defining sequence is non-split, we can show,

$$V = \mathcal{O}_{\mathbb{P}^2}(H) \oplus \mathcal{O}_{\mathbb{P}^2}(H). \tag{4.76}$$

So,

$$\mathbb{P}(V) = \mathbb{P}^1 \times \mathbb{P}^2. \tag{4.77}$$

For this example it is, or course, possible to find a Heterotic dual.

4.4 Conic bundles without monodromy

We now turn to 4D F-theory models where the base of the elliptic fibration is a conic bundle. As mentioned previously, the fiber at generic points in the conic bundle base is essentially a $\mathbb{P}^1$, but the fiber may degenerate at codimension one or higher in the base. Section 4.2 discussed the physics associated with conic bundles in 6-dimensional F-theory models. We now wish to understand the physics associated with conic bundles in 4-dimensional F-theory models. While the 6D analysis should give significant insights into the 4D physics, there are new features in 4D situations that need to be considered. Chief among these is the possibility of a monodromy that exchanges the components in degenerate fibers, which we discuss in Section 4.5. However, there may be other differences between the physics in six and four dimensions, so it is worth analyzing 4D conic bundle models without the added complication of monodromy. Therefore, this section focuses on 4D F-theory models where the base of the elliptic fibration is a three-dimensional conic bundle that lacks monodromy.

We consider a specific example of a conic bundle $\mathcal{B}_3$ described by the complete intersection configuration matrix

$$\left[\begin{array}{c|cc} \mathbb{P}^2_{(1)} & 1 & 1 \\ \mathbb{P}^1_{(2)} & 1 & 0 \\ \hdashline \mathbb{P}^2_{(3)} & 0 & 1 \end{array}\right]. \tag{4.78}$$

Here, $\mathbb{P}^2_{(3)}$ serves as the base B_2 of the conic bundle. We choose the $\mathbb{P}^n$ coordinates to be

$$\mathbb{P}^2_{(1)} : [x_0 : x_1 : x_2], \qquad \mathbb{P}^1_{(2)} : [y_0 : y_1], \qquad \mathbb{P}^2_{(3)} : [z_0 : z_1 : z_2]. \tag{4.79}$$

The equations describing the complete intersection can then be written as

$$P_1 \equiv l_0(y_0, y_1)x_0 + l_1(y_0, y_1)x_1 + l_2(y_0, y_1)x_2 = 0, \tag{4.80}$$

$$P_2 \equiv m_0(z_0, z_1, z_2)x_0 + m_1(z_0, z_1, z_2)x_1 + m_2(z_0, z_1, z_2)x_2 = 0, \tag{4.81}$$

where the l_i and m_i are linear expressions in the y_i and z_i, respectively.

It will be helpful to list some topological data for this complete intersection. We let J_1, J_2, and J_3 be the harmonic (1,1) forms descending from $\mathbb{P}^2_{(1)}$, $\mathbb{P}^1_{(2)}$ and $\mathbb{P}^2_{(3)}$, respectively. Triple intersection numbers on $\mathcal{B}_3$ can then be calculated as

$$D_1 \cdot D_2 \cdot D_3 = \int [D_1] \wedge [D_2] \wedge [D_3] \wedge (J_1 + J_2) \wedge (J_1 + J_3)\,. \tag{4.82}$$

Since the base of the conic bundle is simply a $\mathbb{P}^2$, the Chern classes for B_2 are given by the standard formulas:

$$c_1(B_2) = 3J_3, \qquad c_2(B_2) = 3J_3^2. \tag{4.83}$$

The Chern classes for the conic bundle $\mathcal{B}_3$ are

$$c_1(B_3) = J_1 + J_2 + 2J_3, \qquad c_2(B_3) = 2J_1J_2 + 3J_1J_3 + 2J_2J_3 + J_3^2. \tag{4.84}$$

4.4.1 Sections

We are interested in exploring the Heterotic duals to F-theory models constructed using the conic bundle $\mathcal{B}_3$. In the standard Heterotic/F-theory duality, the base of the F-theory elliptic fibration is a $\mathbb{P}^1$ bundle with two sections. Gauge groups supported on these two sections in the F-theory geometry are dual to the gauge groups coming from the two E_8 factors in the Heterotic model. This suggests that sections play a similarly important role when the F-theory elliptic fibration is constructed over a conic bundle. We should therefore determine the possible sections of the conic bundle described above.

Any section must satisfy certain topological criteria. Let Σ be the divisor class of a section, $\hat{D}^b_\alpha$ with $\alpha = 1, \ldots h^{1,1}(B_2)$ be the basis divisor classes for the base B_2 conic bundle, and D^b_α be the pullbacks of $\hat{D}^b_\alpha$ to the full conic bundle $\mathcal{B}_3$. Σ must satisfy the Oguiso condition

$$\Sigma \cdot \prod_{k=1}^{n-1} D^b_{\alpha_k} = \prod_{k=1}^{n-1} \hat{D}^b_{\alpha_k} \tag{4.85}$$

for all $n-1$ tuples $(\alpha_1, \ldots \alpha_{n-1})$, where each α_k is an integer from 1 to $h^{1,1}(B_{n-1})$. Additionally, Σ must satisfy a modified version of the condition from [129] that accounts for the non-CY nature of $\mathcal{B}_3$:

$$\Sigma \cdot (\Sigma - c_1(B_n)) \cdot \prod_{k=1}^{n-2} D^b_{\alpha_k} = -c_1(B_{n-1}) \cdot \Sigma \cdot \prod_{k=1}^{n-2} D^b_{\alpha_k}. \tag{4.86}$$

If we let $\Sigma = aJ_1 + bJ_2 + cJ_3$, the equations reduce to

$$a + b = 1, \qquad a\,(a + 2b - 1) + 2c\,(a + b - 1) = 0. \tag{4.87}$$

There are two classes of solutions to these equations:

$$\text{(i)} : \Sigma = J_1 + cJ_3, \qquad \text{(ii)} : \Sigma = J_2 + cJ_3. \tag{4.88}$$

We want two sections to the conic bundle, which may have different divisor classes. Based on the analysis of [89], we let the classes of the two sections be Σ and $\Sigma + t$, subject to the constraint

$$\Sigma \cdot (\Sigma + t) = 0. \tag{4.89}$$

The class t, known as the twist, is a divisor class in the base. The condition can be thought of as asking that the divisors corresponding to the E_8 factors do not intersect; this ensures there is no matter jointly charged under gauge factors contained in distinct E_8 factors, as seen in typical, non-singular Heterotic models.

For the example at hand, we can write the twist as $t = t_3 J_3$. Then, if we assume that $\Sigma = J_1 + cJ_3$, $\Sigma \cdot (\Sigma + t) = 0$ leads to the conditions

$$\Sigma \cdot (\Sigma + t) \cdot J_1 = t_3 + c\,(2 + c + t_3) = 0, \tag{4.90}$$
$$\Sigma \cdot (\Sigma + t) \cdot J_1 = t_3 + c\,(2 + c + t_3) = 0, \tag{4.91}$$
$$\Sigma \cdot (\Sigma + t) \cdot J_3 = 1 + 2c + t_3 = 0. \tag{4.92}$$

There are no intergral solutions to the conditions, suggesting that one cannot have $\Sigma \cdot (\Sigma + t) = 0$ with $\Sigma = J_1 + cJ_3$. However, if we let $\Sigma = J_2 + cJ_3$, we obtain the conditions

$$\Sigma \cdot (\Sigma + t) \cdot J_1 = t_3 + c\,(2 + c + t_3) = 0, \tag{4.93}$$
$$\Sigma \cdot (\Sigma + t) \cdot J_1 = c\,(c + t_3) = 0, \tag{4.94}$$
$$\Sigma \cdot (\Sigma + t) \cdot J_3 = 2c + t_3 = 0, \tag{4.95}$$

which is satisfied when $c = t_3 = 0$.

In summary, we can have $\Sigma \cdot (\Sigma + t) = 0$ if

$$\Sigma = J_2, \qquad t = 0. \tag{4.96}$$

The sections essentially specify a particular point on $\mathbb{P}^1_{(2)}$ that does not vary with the position in the base $\mathbb{P}^2_{(3)}$. For instance, one could choose the two sections to be

$$[y_0 : y_1] = [1 : 0], \tag{4.97}$$

and

$$[y_0 : y_1] = [0 : 1], \tag{4.98}$$

with the $[x_0 : x_1 : x_2]$ coordinates determined by plugging the values for y_0 and y_1 into Equations (4.80) and (4.81).

4.4.2 Degenerations

At generic points in $B_2 = \mathbb{P}^2_{(3)}$, the fiber is simply a $\mathbb{P}^1$. We can think of the coordinates $[y_0 : y_1]$ as parametrizing the $\mathbb{P}^1$ fiber. Plugging specific values for $[y_0 : y_1]$

and $[z_0 : z_1 : z_2]$ into $P_1 = P_2 = 0$ gives us two linear equations for $[x_0 : x_1 : x_2]$; the solution of these equations is a single point in $\mathbb{P}^2_{(1)}$. The defining relations thus assign a particular point in $\mathbb{P}^2_{(1)}$ to each point in $\mathbb{P}^1_{(2)}$. In a vague sense, the defining relations provide an embedding of $\mathbb{P}^1_{(2)}$ within $\mathbb{P}^2_{(1)}$. The fiber is therefore essentially the $\mathbb{P}^1$ described by $[y_0 : y_1]$, at least at most points in the base.

This story changes at particular loci in B_2, where the fiber consists of two intersecting $\mathbb{P}^1$ curves. These degenerations occur because, for, special values of $[y_0 : y_1]$ and $[z_0 : z_1 : z_2]$, $P_1 = 0$ and $P_2 = 0$ are no longer independent lines in $\mathbb{P}^2_{(1)}$. As a result, there are two ways of solving the defining relations. First, one can leave $[y_0 : y_1]$ unrestricted and solve for the x_i, just as done at generic points in the base. Second, one can fix $[y_0 : y_1]$ to take the sepcial value and solve $P_2 = 0$ (or $P_1 = 0$) for the x_i. Each of these solutions gives us a different $\mathbb{P}^1$*component*, and the fiber consists of two components intersecting at a single point.

To determine the specific locus in B_2 where this degeneration occurs, we need to find the conditions such that the lines $P_1 = 0$ and $P_2 = 0$ in $\mathbb{P}^2_{(1)}$ are no longer independent lines. Recall that P_1 and P_2 are defined in Equations (4.80) and (4.81), respectively. Since $l_0(y)$, $l_1(y)$, and $l_2(y)$ are linear expressions in the $\mathbb{P}^1_{(2)}$ coordinates, they cannot be independent, and one of them must be a linear combination of the others. Let us assume that $l_2(y)$ can be written as

$$\alpha l_0(y) + \beta l_1(y)$$

for some complex numbers α and β. Then, P_1 can be rewritten as

$$P_1 = l_0(y)x_0 + l_1(y)x_1 + (\alpha l_0(y) + \beta l_1(y))\, x_2. \tag{4.99}$$

In order for P_2 to describe the same line in $\mathbb{P}^2_{(1)}$, we require that

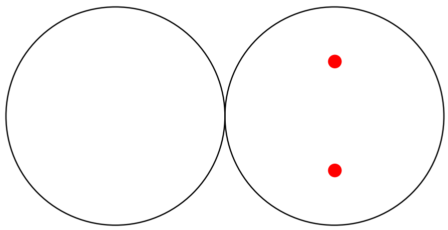
$$\Delta_b \equiv m_2(z) - \alpha m_0(z) - \beta m_1(z) = 0. \tag{4.100}$$

The degeneration locus $\Delta_b = 0$ is therefore a line in the base $B_2 = \mathbb{P}^2_{(3)}$. We can also obtain explicit expressions for the fiber components. The first component is specified by the equations

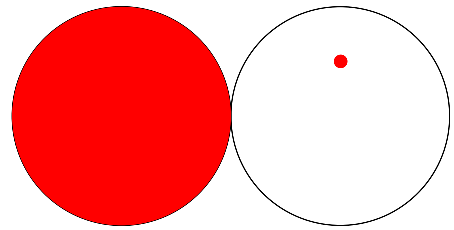
$$m_0(z)x_0 + m_1(z)x_1 + (\alpha m_0(z) + \beta m_1(z))\, x_2 = 0, \quad l_0(y)x_0 + l_1(y)x_1 + l_2(y)x_2 = 0, \tag{4.101}$$

with $[y_0 : y_1]$ unrestricted. The second component is specified by

$$m_0(z)l_1(y) - m_1(z)l_0(y) = 0, \qquad m_0(z)x_0 + m_1(z)x_1 + m_2(z)x_2 = 0. \tag{4.102}$$



(a) codimension one behavior



(b) codimension two behavior

Figure 4.1: Illustration of the degenerate fibers for the conic bundle analyzed in Section 4.4. The red dots mark the points on the fiber hit by the sections. The filled component for the codimension two behavior indicates that one of the sections wraps a component.

The degenerate fibers along $\Delta_b = 0$ are illustrated in Figure 4.1. The figure also depicts the behavior of the two sections chosen in the fashion discussed above. At most points along $\Delta_b = 0$, both sections will hit the component described by (4.101) and will miss the component described by (4.102). But at codimension two loci along $\Delta_b = 0$, the $[y_0 : y_1]$ picked out by the definition of a section may satisfy

$$m_0(z)l_1(y) - m_1(z) - l_0(y) = 0.$$

When this happens, the section wraps the (4.102) component. There does not seem to be any monodromy effect that identifies the two components in the degenerate fiber. Fibering one of the components over $\Delta_b = 0$ should therefore give a valid divisor, and the degenerations should give extra contributions $h^{1,1}(\mathcal{B}_3)$. We should also be able to shrink one of these divisors while leaving the volume of the fiber non-zero. In fact, this particular conic bundle seems similar to a blowup of $\mathbb{P}^1 \times \mathbb{P}^2$, with the component in (4.102) acting as the exceptional divisor. Based on these observations, one would expect that, for an F-theory model with $\mathcal{B}_3$ as the base of the elliptic fibration, the degenerations should correspond to bulk M5-branes in a dual Heterotic model.

4.5 Conic bundles with monodromy

Having considered a conic bundle without monodromy, we now analyze a conic bundle that does exhibit monodromy. For this example, let us take the conic fiber to be a quadratic curve in $\mathbb{P}^2$. We want the F-theory model built from this conic bundle to be dual to an $E_8 \times E_8$ Heterotic compactification. The conic bundle should therefore admit two sections, which act as the F-theory divisors corresponding to the two E_8 factors. In light of this requirement, we use a conic bundle specified by the defining relation

$$p = \sum_{i,j=1}^{3} l_i C_{ij} l_j = C_{11} l_1^2 + 2C_{12} l_1 l_2 + 2C_{13} l_1 l_3 + 2C_{23} l_2 l_3 = 0, \tag{4.103}$$

where $[l_1 : l_2 : l_3]$ are the coordinates of the $\mathbb{P}^2$ in which the conic fiber is embedded. The C_{ij} may depend on the position in the conic bundle base. Since we are interested in F-theory compactifications to 4D, the conic bundle base is complex two-dimensional, and the conic bundle is complex three-dimensional.

This conic bundle admits two particularly useful sections: $l_1 = l_2 = 0$ and $l_1 = l_3 = 0$. These are not the only sections, as there are additional sections such as $C_{11} l_1 + C_{12} l_2 + C_{13} l_3 = l_2 = 0$ and $C_{11} l_1 + C_{12} l_2 + C_{13} l_3 = l_3 = 0$. But the sections $l_1 = l_2 = 0$ and $l_1 = l_3 = 0$ do not intersect, and we therefore choose them to be the divisors corresponding to the E_8 factors.

4.5.1 Degenerations

The discriminant locus of this conic bundle, found from $\det(C)$, is

$$\Delta_b \equiv C_{23} \left(C_{11} C_{23} - 2C_{12} C_{13} \right) = 0. \tag{4.104}$$

The fiber therefore degenerates along the codimension one loci $C_{11}C_{23} - 2C_{12}C_{13} = 0$ and $C_{23} = 0$, which intersect at the codimension two loci $C_{12} = C_{23} = 0$ and $C_{13} = C_{23} = 0$. We analyze the fibers at these loci individually. The degenerate fibers along the codimension one loci are illustrated in Figure 4.2.

Codimension one behavior First, we consider the locus $C_{11}C_{23} - 2C_{12}C_{13} = 0$. Note that

$$C_{23} p + \left(2C_{12} C_{13} - C_{11} C_{23} \right) l_1^2 = 2 \left(C_{13} l_1 + C_{23} l_2 \right) \left(C_{12} l_1 + C_{23} l_3 \right). \tag{4.105}$$

Therefore, at generic points along $2C_{12}C_{13} - C_{11}C_{23} = 0$ (away from $C_{23} = 0$), the conic splits into two components given by $C_{13} l_1 + C_{23} l_2 = 0$ and $C_{12} l_1 + C_{23} l_3 = 0$. We will refer to these components as c_2 and c_3, respectively. These intersect at a point on the fiber given by $C_{13}l_1+C_{23}l_2 = C_{12}l_1+C_{23}l_3 = 0$, or $[l_1 : l_2 : l_3] = [-C_{23} : C_{13} : C_{12}]$. The two sections hit different components: the section $l_1 = l_2 = 0$ hits c_2, while the section $l_1 = l_3 = 0$ hits c_3. At least naively, this situation is reminiscent of F-theory constructions dual to Heterotic/M-theory models with an M5 brane in the bulk. In particular, one can imagine wrapping D3 branes on either c_2 or c_3; the dual analogue of these D3 branes would be M2 branes stretched between the bulk M5 brane and one of the two E_8 walls. Shrinking the codimension one component formed by fibering c_2 over $2C_{12}C_{13} - C_{11}C_{23} = 0$ would then correspond moving the M5 brane through the bulk to one of the E_8 walls. On the other hand, shrinking the component formed by fibering c_3 would correspond to moving the M5 brane to the other E_8 wall. Of course, the monodromy effects discuss below may complicate this interpretation.

At generic points along $C_{23} = 0$,the fiber degenerates to two $\mathbb{P}^1$'s as well. Specifically, when $C_{23} = 0$, the defining relation becomes

$$l_1 \left(C_{11}l_1 + 2C_{12}l_2 + 2C_{13}l_3\right) = 0. \tag{4.106}$$

This factorization indicates that the fiber splits into the components $l_1 = 0$, which we refer to as $\tilde{c}_a$, and $C_{11}l_1 + C_{12}l_2 + C_{13}l_3 = 0$, which we refer to as $\tilde{c}_b$. The two components intersect at points on the fibers given by $l_1 = C_{12}l_2 + C_{13}l_3 = 0$. Both the sections hit $\tilde{c}_a$ and miss $\tilde{c}_b$. This section behavior makes a Heterotic/M-theory interpretation of this degeneration more challenging, even at a naive level. On the F-theory side, one can still imagine either wrapping D3 branes on one of the $\tilde{c}$'s component or shrinking the codimension one component found by fibering $\tilde{c}_a$ or $\tilde{c}_b$ over $C_{23} = 0$. But suppose we attempted to interpret the of the $\tilde{c}_b$ component in Heterotic/M-theory. A D3-brane on a shrunken $\tilde{c}_b$ curve should correspond to a light M2 brane in the dual picture. However, since neither of the sections intersects $\tilde{c}_b$, it seems difficult to interpret a D3-brane on $\tilde{c}_b$ stretched between an M5 brane and an E_8 wall. In particular, on which E_8 wall would the M2 brane end? Said another way, if one shrinks the $\tilde{c}_b$ component, towards which E_8 wall would the dual M5 brane move? Thus, while the $C_{23} = 0$ degeneration may have some M5 brane interpretation, it does not fit as neatly into the standard bulk M5 brane story. Again, monodromy would likely further complicate any M5 brane interpretation.

Codimension two behavior At the intersections between $C_{23} = 0$ and $C_{11}C_{23} - 2C_{12}C_{13} = 0$, the conic bundle described by (4.103) is in fact singular. This fact can be made apparent by rewriting p as

$$l_1 \left(C_{11}l_1 + 2C_{12}l_2 + 2C_{13}l_3\right) + C_{23}l_2l_3. \tag{4.107}$$

(a) $C_{11}C_{23} - 2C_{12}C_{13} = 0$ (b) $C_{23} = 0$

Figure 4.2: Degenerate fibers at codimension one for the conic bundle with monodromy. The points represent the two sections $l_1 = l_2 = 0$ and $l_1 = l_3 = 0$.

The defining relation has the structure of a conifold with singularities at

$$C_{23} = C_{13} = l_1 = l_2 = 0,$$

and

$$C_{23} = C_{12} = l_1 = l_3 = 0.$$

Let us first focus on the behavior at $C_{23} = C_{13} = 0$. Since the base of the conic bundle is two-dimensional, we can assume that C_{12} and C_{11} are non-zero at $C_{23} = C_{13} = 0$, which will be true if the C_{ij} are sufficiently general. To resolve the singularity, we introduce a new $\mathbb{P}^1$ with coordinates $[x_1 : x_2]$ constrained by the equations

$$\begin{pmatrix} l_1 & l_2 \\ C_{23}l_3 & C_{11}l_1 + 2C_{12}l_2 + 2C_{13}l_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0. \tag{4.108}$$

This resolution does not change the fiber away from $C_{23} = C_{13} = 0$, as the matrix above is non-zero. At $C_{23} = C_{13} = 0$, however, the fiber has three components γ_a, γ_b, and γ_c:

$$\gamma_a : \qquad [l_1 : l_2 : l_3] = [0 : l_2 : l_3] \qquad [x_1 : x_2] = [1 : 0], \tag{4.109}$$

$$\gamma_b : \qquad [l_1 : l_2 : l_3] = \left[-\frac{2C_{12}}{C_{11}} l_2 : l_2 : l_3 \right] \qquad [x_1 : x_2] = [C_{11} : 2C_{12}], \tag{4.110}$$

$$\gamma_c : \qquad [l_1 : l_2 : l_3] = [0 : 0 : 1] \qquad [x_1 : x_2] \text{ unrestricted}. \tag{4.111}$$

γ_c intersects γ_a and γ_b at a point, and γ_a and γ_b do not intersect. While the section $l_1 = l_3 = 0$ hits γ_a at a single point and misses γ_b and γ_c, the other section, $l_1 = l_2 = 0$, now wraps γ_c.

In preparation for the monodromy analysis, it is worth investigating how the c_2, c_3, $\tilde{c}_a$, and $\tilde{c}_b$ components split into γ_a, γ_b, γ_c at $C_{23} = C_{13} = 0$. Based on the expressions for $\tilde{c}_a$ and $\tilde{c}_b$ read off from (4.106), one can see that

$$\tilde{c}_a \to \gamma_a + \gamma_c, \qquad \tilde{c}_b \to \gamma_b, \tag{4.112}$$

as one approaches $C_{13} = 0$ along $C_{23} = 0$. The components c_3, which is given by $C_{12}l_1 + C_{23}l_3 = 0$ away from $C_{13} = C_{23} = 0$, becomes $C_{12}l_1$. Therefore, c_3 splits into the components with $l_1 = 0$, namely γ_a and γ_c. The expression $C_{13}l_1 + C_{23}l_2 = 0$ for c_2, meanwhile, seems ill-defined at $C_{13} = C_{23} = 0$. But at points along $C_{11}C_{23} - C_{12}C_{13} = 0$ and not at $C_{23} = 0$, one can rewrite the expression for c_2 as

$$c_{11}l_1 + 2c_{12}l_2 = 0.$$

This new expression is well-defined as $C_{23} \to 0$ and in fact corresponds to the expression for γ_b. Therefore, c_2 should become γ_2 as $C_{23} \to 0$. In summary, we have

$$c_2 \to \gamma_b, \qquad c_3 \to \gamma_a + \gamma_c. \tag{4.113}$$

A similar story holds at $C_{23} = C_{12} = 0$. To resolve the singularity here, we introduce another $\mathbb{P}^1$ with coordinates $[y_1 : y_2]$ and impose the equations

$$\begin{pmatrix} l_1 & l_3 \\ C_{23}l_2 & C_{11}l_1 + 2C_{12}l_2 + 2C_{13}l_3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0. \tag{4.114}$$

The fiber is essentially unchanged away from $C_{23} = C_{12} = 0$, but the fiber at $C_{23} = C_{12} = 0$ has three components $\tilde{\gamma}_a$, $\tilde{\gamma}_b$, $\tilde{\gamma}_c$:

$$\tilde{\gamma}_a: \quad [l_1 : l_2 : l_3] = [0 : l_2 : l_3] \qquad [y_1 : y_2] = [1 : 0], \tag{4.115}$$

$$\tilde{\gamma}_b: \quad [l_1 : l_2 : l_3] = \left[-\frac{2C_{13}}{C_{11}}l_3 : l_2 : l_3\right] \qquad [y_1 : y_2] = [C_{11} : 2C_{13}], \tag{4.116}$$

$$\tilde{\gamma}_c: \quad [l_1 : l_2 : l_3] = [0 : 1 : 0] \qquad [y_1 : y_2] \text{ unrestricted.} \tag{4.117}$$

As before, $\tilde{\gamma}_c$ intersects $\tilde{\gamma}_a$ and $\tilde{\gamma}_b$ at a point, and $\tilde{\gamma}_a$ and $\tilde{\gamma}_b$ do not intersect. Now, the section $l_1 = l_2 = 0$ hits $\tilde{\gamma}_a$ at a single point, while the section $l_1 = l_3 = 0$ wraps $\tilde{\gamma}_c$. The components $\tilde{c}_a$ and $\tilde{c}_b$ along $C_{23} = 0$ still split as

$$\tilde{c}_a \to \tilde{\gamma}_a + \tilde{\gamma}_c, \qquad \tilde{c}_b \to \tilde{\gamma}_b. \tag{4.118}$$

However, an analysis similar to that for $C_{23} = C_{13} = 0$ reveals that the components c_2 and c_3 along $C_{11}C_{23} - 2C_{12}C_{13} = 0$ split as

$$c_2 \to \tilde{\gamma}_a + \tilde{\gamma}_c, \qquad c_3 \to \tilde{\gamma}_b. \tag{4.119}$$

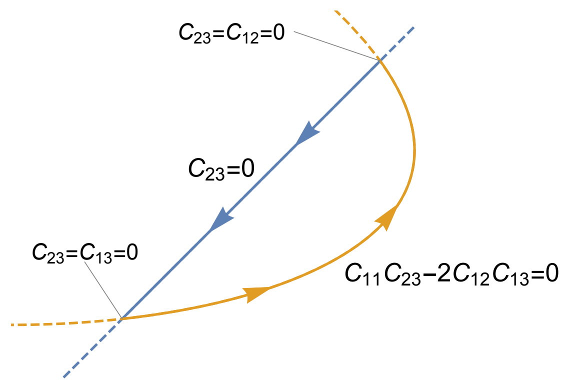


Figure 4.3: Illustration of the closed path used to demonstrate monodromy.

4.5.2 Monodromy

This conic bundle in fact admits a monodromy that identifies components of the degenerate conic. To see monodromy, we follow a closed path on the base that starts along $C_{11}C_{23} - 2C_{12}C_{13} = 0$, goes to $C_{23} = C_{12} = 0$, continues to $C_{23} = C_{13} = 0$ along $C_{23} = 0$, and returns to the starting point along $C_{11}C_{23} - 2C_{12}C_{13} = 0$. The path is illustrated in Figure 4.3. We then choose a component at the starting point, say c_3, and track this components as we follow the path. If, after completing the path, we find that the tracked component has changed from c_3 to c_2, then we will demonstrated monodromy. With this general procedure in mind, let us start at some point along $C_{11}C_{23} - 2C_{12}C_{13} = 0$ and use c_3 as the tracked component. As we move towards $C_{23} = C_{12} = 0$, the component c_3 becomes $\tilde{\gamma}_b$, according to the analysis performed above. We also know that $\tilde{\gamma}_b$ is identified $\tilde{c}_b$ at $C_{23} = C_{12} = 0$, so our tracked component is now $\tilde{c}_b$. We then move along $C_{23} = 0$ towards $C_{23} = C_{13} = 0$. At $C_{23} = C_{13} = 0$, $\tilde{c}_b$ becomes γ_b, which is in turn identified with c_2. Finally, we turn to our starting along $C_{11}C_{23} - 2C_{12}C_{13} = 0$. In the end, the tracked component has changed from c_3 and c_2, indicating that these two components are exchanged under monodromy. A similar argument shows that $\tilde{c}_a$ and $\tilde{c}_b$ are also exchanged under monodromy.

Because of this monodromy, the degenerate fibers should not give extra contributions to the $h^{1,1}$ of the conic bundle. For instance, if one attempted to consistently shrink the c_2 components, monodromy would force the c_3 components to shrink simultaneously as well. This fact represents a fundamental difference from two-dimensional conic bundles, where the degenerations always give an extra contribution to $h^{1,1}$.

The key feature of this conic bundle that allows for this monodromy seems to be the $C_{23} = C_{12} = 0$ and $C_{23} = C_{13} = 0$ intersections. Both of these intersections are needed to form the closed path above. So far, we have been too much about the specific forms of the C_{ij}. But in particular situations, C_{23} and C_{12} may never intersect, C_{23} and C_{13} may never intersect. In these cases, the model will not have monodromy, and there may be extra contributions to $h^{1,1}$. For example, if $[C_{23}]$ is trivial, the degenerations along $C_{23} = 0$ are absent, and the conic bundle does not have the monodromy described above. There are therefore independent divisors corresponding to the c_2 and c_3 components, and the degeneration along $C_{11}C_{23} - 2C_{12}C_{13} = 0$ gives a contribution to $h^{1,1}$.

In fact, one can transform (4.103) to remove the $C_{23} = 0$ degeneration locus. If we let $l_1 = C_{23}l'_1$, we can rewrite p as

$$C_{23}\left(C_{23}C_{11}{l'_1}^2 + 2C_{12}l'_1l_2 + 2C_{13}l'_1l_3 + 2l_2l_3\right). \tag{4.120}$$

Dividing through by C_{23} leads to a new conic bundle of the form

$$p' = C_{23}C_{11}{l'_1}^2 + 2C_{12}l'_1l_2 + 2C_{13}l'_1l_3 + 2l_2l_3 = 0, \tag{4.121}$$

which has a discriminant

$$\Delta'_b = C_{23}C_{11} - 2C_{12}C_{13}. \tag{4.122}$$

This conic bundle admits sections $l'_1 = l_2 = 0$ and $l'_1 = l_3 = 0$. Since

$$p' - \Delta'_b{l'_1}^2 = 2\left(l_2 + C_{12}l'_1\right)\left(l_3 + C_{13}l'_1\right), \tag{4.123}$$

the fiber at $\Delta'_b = 0$ splits into two components, each of which is hit by one of the sections. Additionally, this behavior does not change at codimension two. We would therefore expect that this new conic bundle does not have monodromy. In turn, the naive M5 brane interpretation of the $C_{11}C_{23} - 2C_{12}C_{13} = 0$ locus mentioned previously should be directly applicable hear, unmodified by monodromy considerations. The fact that we can convert the conic bundle of (4.103) to a new conic bundle monodromy may lead to new insights into the origin of the monodromy.

4.6 Heterotic dual

One of the main questions of this chapter is the existence and the properties of the possible Heterotic dual for F-theory set-ups that are studied in the previous sections.

The standard statement in the literature is that there is a heterotic dual once the $\mathbb{P}^1$ bundle has a section. However, as we mentioned in the first chapter and will review in a little more details in the following subsections, in the standard Heterotic/F-theory duality one actually needs two holomorphic section. One for each E_8 plane in Heterotic M-theory. But we saw in this chapter that we may have $\mathbb{P}^1$-fibrations with only one rational or holomorphic section, or conic bundles with an irreducible 2-section. It is also possible to have a "combination" of such situations. So one naturally asks whether there is a Heterotic dual for such F-theory set-ups, and if they exist, what are the properties of them.

In the following subsections we first review the standard Heterotic/F-theory duality via stable degeneration. Then we will repeat this process in more general $\mathbb{P}^1$-fibrations. It turns out when there is only one (possibly rational) section there is not any Heterotic dual but such set-ups are in the same complex structure moduli with F-theory set-ups which are dual to very special Heterotic models. Especially when the section is rational the dual Heterotic model Calabi-Yau is an elliptically fibered three-fold over the rational section.[3]

When there is a 2-section only, we don't have a clear solution. The stable degeneration looks like to give a $Spin(32)/\mathbb{Z}_2$ bundle in Heterotic side rather than a $E_8 \times E_8$ bundle. However the problem is still open.

4.6.1 Review

We already mentioned the specific stable degeneration process that one can use to identify the complex structure moduli of a $K3$ with the Wilson lines of an $E_8 \times E_8$. Let us review the standard stable degeneration again for a Calab–Yau threefold. Consider the following Calabi-Yau threefold WSF X_3 over a base B_2,

$$y^2 = x^3 + fx + g, \tag{4.124}$$

one fiber this threefold over a disc (or an affine line) Δ parameterized by a complex variable t,

$$\Pi : \chi \to \Delta, \quad \text{fibers } X_3(t) \text{ are WSF threefold parametrized by } t. \tag{4.125}$$

To be able to use the Clemens-Schmid sequence (we are not going to do it here, but in principle this is necessary), we need to transform χ into a stable degeneration i.e.

[3] More clearly, the base of the Heterotic Calabi-Yau three-fold is not the base of the $\mathbb{P}^1$-bundle in F-theory rather, it is surface defined by the section of the $\mathbb{P}^1$-bundle.

the central fiber is reducible with normal crossing singularity (semi-stable), and it doesn't have infinitesimal automorphisms (stable).

Suppose $B_{n+1} = \mathbb{P}(\mathcal{O}_{B_n} \oplus \mathcal{O}_{B_n}(D))$, and let's assume the $\mathbb{P}^1$ fiber coordinates are z_0 and z_1. To make the fibration χ into a (semi-) stable degeneration, we blow up $t = 0, z_0 = 0$,

$$z_0 \rightarrow e z_0, \tag{4.126}$$
$$t \rightarrow et. \tag{4.127}$$

Since χ is Calabi-Yau, f and g must transform as,

$$f \rightarrow e^4 f, \tag{4.128}$$
$$g \rightarrow e^6 g. \tag{4.129}$$

Consider the general form of f for example,

$$f = z_0^8 f_0 + z_0^7 z_1 f_1 + \cdots + z_0^4 z_1^4 f_4 + \cdots + z_1^8 f_8, \tag{4.130}$$

to satisfy 4.128 we require,

$$f_i = t^{i-4} f_i', \quad \text{for } i \geq 4, \tag{4.131}$$
$$f_i \quad \text{generic in } t \text{ for } i \leq 4. \tag{4.132}$$

After the transformation 4.126, we get

$$f = z_0^8 e^4 f_0 + z_0^7 z_1 e^3 f_1 + \cdots + z_0^4 z_1^4 f_4 + \cdots + z_1^7 z_0 t^3 f_7' + z_1^8 t^4 f_8'. \tag{4.133}$$

So at the the original $t = 0$, we get two new directions $e = 0$ and $t = 0$, we describe each one of these cases separately,

1. $t = 0, \quad z_0 = 1$. On this locus z_1 and e cannot be zero at the same time, so it defines a $\mathbb{P}^1$,

$$f = e^4 f_0 + z_1 e^3 f_1 + \cdots + z_1^4 f_4. \tag{4.134}$$

   Similar things happens to g. These new f and g are WSF of a $d\mathbb{P}_9$, with base is (e, z_1).

2. $e = 0, \quad z_1 = 1$. On this locus t and z_0 cannot be zero at the same time, so it defines a $\mathbb{P}^1$,

$$f = z_0^4 f_4 + \cdots + z_0 t^3 f_7' + t^4 f_8'. \tag{4.135}$$

   So again we get another $d\mathbb{P}_9$ over $e = 0$.

3. Over the intersection $t = e = 0$, we get the central fiber given by f_4 and g_6.

4.6.2 General $\mathbb{P}^1$-fibrations

Now we start analyzing the other possibilities.

What is the problem with one section?

The problem in this case is that there is no section at infinity relative to S. In other words, imagine f is given by (4.133) but there is no z_1, and we only have the zero section z_0. The consequence is that in the first dP_9 (4.135) is behaving quite normal, but the second dP_9 (4.134) doesn't have z_1. So if we tune f and g such that we would get a certain singularity over in z_1 in the standard case, the second dP_9 remains smooth. Since there is no z_1. Therefore a heterotic one would wonder whether there is a heterotic dual in this case.

In this subsection we give a concrete algebraic description of the $\mathbb{P}^1$-bundle. In particular, both split and non-split case can be described in a more "unified" way. This is compatible with the fact that the topology of the $\mathbb{P}^1$ bundle doesn't depend on the extension.

6D theory

As a toy model, we start with the following example,

$$B_2 = \mathbb{P}(V_2), \tag{4.136}$$

$$0 \to \mathcal{O}_{P^1} \to V_2 \to \mathcal{O}_{P^1}(2) \to 0. \tag{4.137}$$

When the extension is zero, we have two holomorphic section S_2 and S_{-2} with self intersection 2 and -2 respectively. When the extension is non-zero the section S_{-2} disappears.

We can rewrite V_2 as,

$$0 \to \mathcal{O}_{P^1}(-2) \to \mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}(-1) \oplus \mathcal{O}_{P^1}(1) \to V_2 \to 0. \tag{4.138}$$

Note that the first map $\mathcal{O}_{P^1}(-2) \to \mathcal{O}_{P^1}$ can be identified with an element of the extension group $Ext^1(\mathcal{O}_{P^1}(2), \mathcal{O}_{P^1})$. This short exact sequence we can rewrite $\mathbb{P}(V_2)$ as a hypersurface,

$$\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 1 & 2 \end{array} \tag{4.139}$$

with the following defining equation,

$$F = f_2 x_0 + f_1 x_1 + f_3 x_2 = 0. \tag{4.140}$$

One can show $S_2 \simeq x_0 = 0$. In case the extension is zero i.e., $f_2 \to 0$ we can see the other section can be identified with $S_{-2} \simeq x_1 = x_2 = 0$. However, when $f_2 \neq 0$, we don't have S_{-2}.

Before continuing, let us see what happens to S_{-2} when $f_2 \neq 0$. Note locally S_2 correspond to the following point in the $\mathbb{P}^2$,

$$(0, -f_3, f_1). \tag{4.141}$$

Relative to this the point at infinity would be,

$$(1, -f_3 - \epsilon \frac{f_2}{f_1}, f_1 + (-1+\epsilon)\frac{f_2}{f_3}), \tag{4.142}$$

for some constant ϵ. So we can see when $f_2 = 0$ this point at infinity can define a section globally, but when $f_2 \neq 0$ this section will be ill-defined over $f_3 = 0$ or $f_1 = 0$. Therefore the fiberwise Heterotic/F-theory duality wouldn't work over $f_3 = 0$ or $f_1 = 0$. This means the adiabatic argument for extending the 8D Heterotic/F-theory duality to lower dimensions cannot work in this case. We can repeat the process of stable degeneration as before,

$$\begin{aligned} f &\in H^0(-4K_B) = H^0(\mathcal{O}(8,4)), && (4.143) \\ f &= x_0^8 f_{0,0,4} + x_0^7 \sum_{i=0}^{1} x_1^i x_2^{1-i} f_{1,i,5-2i} + x_0^6 \sum_{i=0}^{2} x_1^i x_2^{2-i} f_{2,i,6-2i} + x_0^5 \sum_{i=0}^{3} x_1^i x_2^{3-i} f_{3,i,7-2i} \\ &\quad + x_0^4 \sum_{i=0}^{4} x_1^i x_2^{4-i} f_{4,i,8-2i} \\ &\quad + x_0^3 \sum_{i=0}^{5} x_1^i x_2^{5-i} f_{5,i,9-2i} + x_0^2 \sum_{i=0}^{6} x_1^i x_2^{6-i} f_{6,i,10-2i} + x_0 \sum_{i=0}^{7} x_1^i x_2^{7-i} f_{7,i,11-2i} \\ &\quad + \sum_{i=0}^{8} x_1^i x_2^{8-i} f_{8,i,12-2i}, \\ g &\in H^0(-6K_B) = H^0(\mathcal{O}(12,6)). && (4.144) \end{aligned}$$

The stable degeneration can be achieved by parameterizing f and g over a disk with parameter t, such that over $t = x_0 = 0$ f vanishes with order 4 and g vanishes with order 6. So,

$$f_{i,j,k} \rightarrow t^{i-4} f_{i,j,k}, \quad \text{for } i \geq 4, \tag{4.145}$$
$$g_{i,j,k} \rightarrow t^{i-6} g_{i,j,k}, \quad \text{for } i \geq 6. \tag{4.146}$$

We blow up the central fiber as before,

$$t \rightarrow et, \tag{4.147}$$
$$x_0 \rightarrow ex_0, \tag{4.148}$$
$$x \rightarrow e^2 x, \tag{4.149}$$
$$y \rightarrow e^3 y. \tag{4.150}$$

At the locus where previously was the central fiber $t = 0$, now we have two branches $et = 0$ (4.147). Each one ($e = 0$ and $t = 0$) now becomes a dP_9, and the intersection $e = t = 0$ is just the central fiber.

- **The dP_9 over $e = 0$:** On this locus, (t, x_0) are homogeneous coordinate of the base of dP_9, and the base of the dP_9 fibration is given by,

$$F' = f_1 x_1 + f_3 x_2 = 0, \tag{4.151}$$

where (x_1, x_2) are now homogeneous coordinates of a $\mathbb{P}^1$. *Note that this is isomorphic to* $\mathbb{P}^1$. The defining f and g for this dP_9 is,

$$f' = x_0^4 \sum_{i=0}^{4} x_1^i x_2^{4-i} f_{4,i,8-2i} + x_0^3 t \sum_{i=0}^{5} x_1^i x_2^{5-i} f_{5,i,9-2i} + x_0^2 t^2 \sum_{i=0}^{6} x_1^i x_2^{6-i} f_{6,i,10-2i} + x_0 t^3 \sum_{i=0}^{7} x_1^i x_2^{7-i} f_{7,i,11-2i} + t^4 \sum_{i=0}^{8} x_1^i x_2^{8-i} f_{8,i,12-2i}, \tag{4.152}$$

$$g' = x_0^6 \sum_{i=0}^{6} x_1^i x_2^{6-i} g_{6,i,12-2i} + x_0^5 t \sum_{i=0}^{7} x_1^i x_2^{7-i} g_{7,i,13-2i} + \cdots + t^6 \sum_{i=0}^{1} 2 x_1^i x_2^{12-i} g_{12,i,18-2i}. \tag{4.153}$$

So in this case everything is normal. One can tune a singularity over x_0, and this dP_9 ($e = 0$) describe the moduli space of an E_8 bundle over the central fiber.

- **The dP_9 over $t = 0$:** On this locus $x_0 \to 1$, and the coordinate e replaces the old x_0 in the defining equations. In other words (e, x_1, x_2) are homogeneous coordinates of a $\mathbb{P}^2$, and the base of the elliptic fibration on this locus (i.e.,$t = 0$) is given by the "old" defining equation,

$$F'' = f_2 e + f_1 x_1 + f_3 x_2 = 0. \tag{4.154}$$

In this case $e = 0$ is a holomorphic section with self intersection 2. The defining f and g in this branch are,

$$f'' = e^4 f_{0,0,4} + e^3 \sum_{i=0}^{1} x_1^i x_2^{1-i} f_{1,i,5-2i} + e^2 \sum_{i=0}^{2} x_1^i x_2^{2-i} f_{2,i,6-2i} + \cdots + \sum_{i=0}^{4} x_1^i x_2^{4-i} f_{4,i,8-2i}, \tag{4.155}$$

$$g'' = e^6 g_{0,0,6} + e^5 \sum_{i=0}^{1} x_1^i x_2^{1-i} g_{1,i,7-2i} + e^4 \sum_{i=0}^{2} x_1^i x_2^{2-i} g_{2,i,8-2i} + \cdots + \sum_{i=0}^{6} x_1^i x_2^{6-i} g_{6,i,12-2i}. \tag{4.156}$$

Where $f \in H^0(-2K_B)$ and $g \in H^0(-3K_B)$. So this branch $(t = 0)$ is also another dP_9 fibration. But this branch doesn't behave normally. Note that we cannot put singularity over $e = 0$ in (4.155),(4.156). Because $e = 0$ is the central fiber in this branch, and (at least for the standard Het/F duality) we are supposed to keep the central fiber fixed and smooth.

In the standard situations, we put singularity in the section at infinity relative to $e = 0$ i.e., if $e = 0$ is the zero section, we need a section at infinity say e_∞ such that $e \cdot s_\infty = 0$. But in this branch $t = 0$, there is not a well defined e_∞ (even though there is still a local one (4.142)). So there is a problem, since we cannot make the second dP_9 singular while keeping the central fiber smooth.

However, we saw when $f_2 = 0$ in (4.140),(4.142), there is a globally defined section at infinity corresponding to $x_1 = x_2 = 0$. So one can perform a **crepant** birational transformation,[4]

$$\begin{aligned} x_1 &\to w x_1, \\ x_2 &\to w x_2. \end{aligned} \tag{4.157}$$

So after this transformation (x_1, x_2) are homogeneous coordinate of a $\mathbb{P}^1$ and (e, w) are another $\mathbb{P}^1$ coordinates,

[4]In this particular example, this birational transformation is simply an isomorphism, but this is not a generic behavior.

$$\begin{array}{cccccc|c} e & w & x_1 & x_2 & u_1 & u_2 & F''' \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 1 & 2 \end{array} \tag{4.158}$$

$$F''' = x_1 f_1 + x_2 f_3 = 0. \tag{4.159}$$

Now the defining WSF (4.155) and (4.156) in this dP_9 becomes,

$$\begin{aligned} f''' &= e^4 f_{0,0,4} + e^3 w \sum_{i=0}^{1} x_1^i x_2^{1-i} f_{1,i,5-2i} + e^2 w^2 \sum_{i=0}^{2} x_1^i x_2^{2-i} f_{2,i,6-2i} \\ &\quad + e w^3 \sum_{i=0}^{3} x_1^i x_2^{3-i} f_{3,i,7-2i} + w^4 \sum_{i=0}^{4} x_1^i x_2^{4-i} f_{4,i,8-2i}, \end{aligned} \tag{4.160}$$

$$\begin{aligned} g''' &= e^6 g_{0,0,6} + e^5 w \sum_{i=0}^{1} x_1^i x_2^{1-i} g_{1,i,7-2i} + e^4 w^2 \sum_{i=0}^{2} x_1^i x_2^{2-i} g_{2,i,8-2i} + \dots \\ &\quad + w^6 \sum_{i=0}^{6} x_1^i x_2^{6-i} g_{6,i,12-2i}. \end{aligned} \tag{4.161}$$

So now we have a "normal dP_9" over $t = 0$ when $f_2 = 0$, and we can have a standard Heterotic dual.

Conclusion: Since the two situations (i.e., when V_2 is split or non-split) are related by continuous deformation we may conclude, even when there is only one holomorphic section, there is a Heterotic dual which corresponds to the standard Heterotic dual when $f_2 = 0$. However this is somewhat too fast. When the extension is zero (i.e., $f_2 = 0$) there is a (-2)-cutve in the F-theory base. But after the turning the extension on (i.e., $f_2 \neq 0$) this curve becomes non-holomorphic. So the corresponding 7-brane, that wraps around this (-2)-curve, becomes non-BPS. So the correct conclusion is that when the extension is zero we have a standard Heterotic dual, and the F-theory complex moduli is frozen at $f_2 = 0$ (due to a D-term potential in the $N = 2$ type IIB language). But in 6D theories it is always possible to Higgs the effective gauge theory over the (-2)-curve completely (in geometric terms it means deform the 7-brane completely so there will be no singularity over the (-2)-curve). After this Higgsing there will be no obstruction in turning on the deformation $f_2 \neq 0$. This situation is schemetically illustrated in the Figure 4.5.

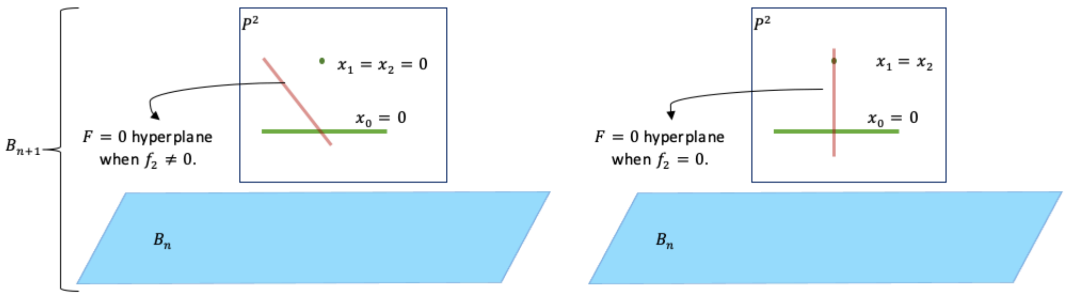


Figure 4.4: The left figure shows the generic case that there is only one section (the intersection of $x_0 = 0$ with the red line). This correspond to the projectivization of a non-split rank two bundle. The right hand side, corresponds to the case where the extension is zero, and there is another section at $x_1 = x_2 = 0$. In this case there is a standard Heterotic dual.

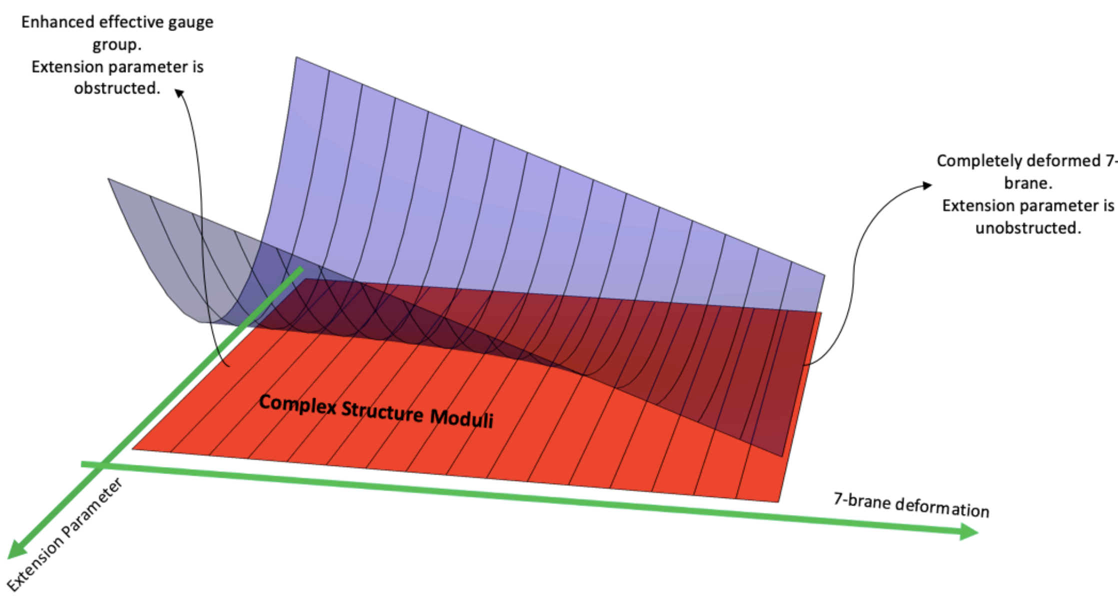


Figure 4.5: The expected form of the potential energy in terms of the complex structure parameters.

4D theory

In 4D theories, in addition to the standard story, there are three other possibilities. i) A $\mathbb{P}^1$ bundle with only one holomorphic section. ii) A $\mathbb{P}^1$ bundle with only one rational section. iii) Conic bundles and it's birational cousins. We try to figure out the Heterotic dual in each one of these cases.

Case i) A $\mathbb{P}^1$ bundle with only one holomorphic section

The stable degeneration process in this case is similar to the 6D theory. So we will not repeat it again. Consider the following example,

$$0 \to \mathcal{O}_{F_0} \to V_2 \to \mathcal{O}_{F_0}(3,-1) \to 0. \tag{4.162}$$

It is possible to rewrite V_2 as,

$$0 \to \mathcal{O}(-2,-1) \to \mathcal{O} \oplus \mathcal{O}(1,-1) \oplus \mathcal{O}(0,-1) \to V_2 \to 0, \tag{4.163}$$

where the first map $\mathcal{O}(-2,-1) \to \mathcal{O}$ can be identified with an element of the extension group. Similar to the 6D case this suggests that it is possible to realize $\mathbb{P}(V_2)$ as hypersurface,

$$\begin{array}{ccccccc|c} x_0 & x_1 & x_2 & u_1 & u_2 & v_1 & v_2 & \text{F} \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & \text{-1} & 0 & 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \end{array} \tag{4.164}$$

$$F = x_0 f_{2,1} + x_1 f_{3,0} + x_2 f_{0,0} = 0. \tag{4.165}$$

Similar to the 6D example, when $f_{2,1} \neq 0$ there is only one section. After the stable degeneration, one of the dP_9's cannot become singular while keeping the central fiber fixed. However, when $f_{2,1} = 0$, there is another section at $x_1 = x_2 = 0$ and one can repeat the same birational transformation as (4.157) (which is again isomorphism for this example) to derive the full, standard Heterotic dual model in this complex structure locus. The infinitesimal deformation $f_{2,1} \neq 0$ makes the 7-brane over this second becomes non-BPS. Therefore this complex structure deformation is stabilized. As in the 6D example one may expect to be able to turn on the deformation $f_{2,1} \neq 0$ after Higgsing the effective gauge fields over this 7-brane. However we should be cautious about this. Because it is possible to turn on specific fluxes over the 7-brane to stabilize the 7-brane deformation (corresponding to GVW superpotential). The reader can refer to the discussions in Chapter 2 about this.

Case ii) A $\mathbb{P}^1$ bundle with only one rational section

The calculations in this situation is also almost similar to the previous cases, with minor, but important differences. Take our well known example again,

$$0 \to \mathcal{O}_{P^2} \to V_2 \to \mathcal{O}_{P^2}(H) \otimes \mathcal{I}_p \to 0. \tag{4.166}$$

As already mentioned, one can rewrite this bundle as,

$$0 \to \mathcal{O}(-1) \to \mathcal{O}^{\oplus 3} \to V_2 \to 0. \tag{4.167}$$

This suggest a realization $\mathbb{P}(V_2)$ as hypersurface as before,

$$\mathbb{P}(V_2) = \left[\begin{array}{c|c} \mathbb{P}^2 & 1 \\ \mathbb{P}^2 & 1 \end{array} \right],$$
$$F = u_1 x_0 + u_2 x_1 + \epsilon u_3 x_2 = 0, \tag{4.168}$$

where $(u1, u2, u3)$ are the coordinates of the base. The maps $\mathcal{O}(-1) \to \mathcal{O}^{\oplus 3}$ are given by the coefficients of x_0, x_1 and x_2 in F. So we can identify the term ϵu_3 with an element in the extension group. For completeness we can rewrite the defining polynomials for WSF,

$$f = x_2^8 f_{0,8} + x_2^7 \sum_j^1 x_0^j x_1^{1-j} f_{j,7} + \cdots + x_2^4 \sum_j^4 x_0^j x_1^{4-j} f_{j,4} + \ldots x_2 \sum_j^7 x_0^j x_1^{7-j} f_{j,1} + \sum_j^8 x_0^j x_1^{8-j} f_{j,0}, \tag{4.169}$$

$$g = x_2^{12} g_{0,12} + x_2^{11} \sum_j^1 x_0^j x_1^{1-j} g_{j,11} + \cdots + x_2^6 \sum_j^6 x_0^j x_1^{6-j} g_{j,6} + \ldots x_2 \sum_j^{11} x_0^j x_1^{11-j} g_{j,1} + \sum_j^{12} x_0^j x_1^{12-j} g_{j,0}, \tag{4.170}$$

Similar as before we can choose x_2 as the (rational) zero section, and try to do the stable degeneration by fibering the WSF over t as,

$$f_{i,j} \to t^{4-j} f_{i,j}, \quad \text{for } j \leq 4, \tag{4.171}$$
$$g_{i,j} \to t^{6-j} g_{i,j}, \quad \text{for } j \leq 6. \tag{4.172}$$

Then we make the following birational transformation to make the degeneration (semi)stable,

$$t \to et, \tag{4.173}$$
$$x_2 \to ex_2, \tag{4.174}$$
$$x \to e^2 x, \tag{4.175}$$
$$y \to e^3 y. \tag{4.176}$$

Again over $t = 0$ and $e = 0$, we will get two dP_9's. The one over $e = 0$ is "normal", but the one over $t = 0$ cannot become singular while keeping the central fiber fixed. The first novelty with this case is that the central fiber, $e = t = 0$, which is dual to the purported Heterotic geometry, is now a WSF Calabi-Yau over dP_1 rather than $B_2 = \mathbb{P}^2$.[5] We are not going to repeat the stable degeneration since the calculation are similar to the one done in 6D.

Note when $\epsilon = 0$ in (4.168) $\mathbb{P}(V_2)$ becomes **singular and non-flat**. This can be explained again from the extension point of view. First remember,

$$\epsilon \to 0, \quad \Rightarrow \quad V_2 \to \mathcal{O} \oplus \mathcal{O}(H) \otimes \mathcal{I}_p. \tag{4.177}$$

However, the second term above i.e., $\mathcal{O}(H) \otimes \mathcal{I}_p$ is the cokernel of the map $\mathcal{O}(-1) \to \mathcal{O}^{\oplus 2}$ where the two maps are given by u_1 and u_2. Therefore when $\epsilon = 0$ and $u_1 = u_2 = 0$ rank of V_2 jumps to three. B_3 will be singular in this case.

Anyways, similar to the 6D and Case i), we can go to the limit $\epsilon \to 0$. The we will find another section at $x_0 = x_1 = 0$. One blows up this locus as before,

$$x_0 \to wx_0, \tag{4.178}$$
$$x_1 \to wx_1. \tag{4.179}$$

After this birational transform we get a **smooth, flat** $\mathbb{P}^1$-bundle over dP_1. We will have two holomorphic sections $w = 0$ and $x_2 = 0$ which don't intersect with each other, and therefore we find a standard Heterotic dual over dP_1 rather than $B_2 = \mathbb{P}^2$ which is defined as,

$$B_2 - simeq dP_1 = \left[\begin{array}{c|c} \mathbb{P}^1 & 1 \\ \mathbb{P}^2 & 1 \end{array} \right], \tag{4.180}$$
$$F|_{\epsilon=0} = u_1 x_0 + u_2 x_1 = 0. \tag{4.181}$$

The second novelty is that, this birational transformation is not isomorphism anymore. It increases the $h^{1,1}(B_3)$ by one. Also in terms of the Calabi-Yau WSF this birational transformation increases $h^{1,1}(X_4)$ by one, and reduces the $h^{3,1}(X_4)$ by one. In other words, we are doing a conifold transition to reach to a geometry in F-theory that can have a Heterotic dual!

Case iii) Conic bundles and it's birational cousins

[5] In general the base of the "claimed" Calabi-Yau three-fold in the F-theory have to be isomorphic to the rational section in the $\mathbb{P}^1$-bundle.

This is the most exotic case. Let's start with the minimal conic bundle given as the following simple defining relation,[6]

$$B_3 = \left[\begin{array}{c|c} \mathbb{P}^2 & 2 \\ \mathbb{P}^2 & d \end{array} \right], \tag{4.182}$$

$$F = \sum_{i+j+k=2} x_1^i x_2^j x_3^k l_{i,j,k}. \tag{4.183}$$

We can repeat the same procedure for stable degeneration. First we fix our choice for the 2-section,

$$S = x_1. \tag{4.184}$$

The defining equation for WSF is given by

$$\begin{aligned} f &= S^4 f_{0,0,4(3-d)} + S^3 \sum_j^1 x_2^j x_3^{1-j} f_{1,j,4(3-d)} \\ &\quad + S^2 \sum_j^2 x_2^j x_3^{2-j} f_{2,j,4(3-d)} + S \sum_j^3 x_2^j x_3^{3-j} f_{3,j,4(3-d)} + \sum_j^4 x_2^j x_3^{4-j} f_{4,j,4(3-d)}, \end{aligned} \tag{4.185}$$

$$\begin{aligned} g &= S^6 g_{0,0,6(3-d)} + S^5 \sum_j^1 x_2^j x_3^{1-j} g_{1,j,6(3-d)} + S^4 \sum_j^2 x_2^j x_3^{2-j} g_{2,j,6(3-d)} + S^3 \sum_j^3 x_2^j x_3^{3-j} g_{3,j,6(3-d)} \\ &\quad + S^2 \sum_j^4 x_2^j x_3^{4-j} g_{4,j,6(3-d)} + S \sum_j^5 x_2^j x_3^{5-j} g_{5,j,6(3-d)} + \sum_j^6 x_2^j x_3^{6-j} g_{6,j,6(3-d)}, \end{aligned} \tag{4.186}$$

One can try to repeat the stable degeneration for this case. However in this case the central fiber in the family of the Calabi-Yaus, even though the fibration is still a Kulikov model, has only one component. In other words, after the stable degeneration we get only one dP_9 fibration. To do this we fiber WSF over t in the same way as before,

$$f_{i,j,k} \rightarrow t^i f_{i,j,k}, \tag{4.187}$$

$$g_{i,j,k} \rightarrow t^i g_{i,j,k}, \tag{4.188}$$

$$t \rightarrow et, \tag{4.189}$$

$$S \rightarrow eS. \tag{4.190}$$

[6] It is possible to find more complicated examples. But the as long as their behavior under stable degeneration is concerned, this example is good enough.

After repeating a procedure similar to the 6D case, one can see over $t = 0$, instead of a dP_9 we only have the central fiber (the purported Heterotic WSF) given by $f_{0,0,4(3-d)}$ and $g_{0,0,6(3-d)}$. Over $e = 0$ we find a dP_9 where the base of the dP_9 is given by (t, S), and the base of the this dP_9 fibration is given by

$$B_2 = \left[\begin{array}{c|c} \mathbb{P}^1 & 2 \\ \mathbb{P}^2 & d \end{array} \right], \tag{4.191}$$

$$F_2 := F|_{x_1=0} = \sum_{j+k=2} x_2^j x_3^k l_{0,j,k}, \tag{4.192}$$

where the homogeneous coordinates in the fiber $\mathbb{P}^1$ is (x_2, x_3). The base of the purported Heterotic dual is then B_2, with the WSF equation is given by $f_{0,0,4(3-d)}$ and $g_{0,0,6(3-d)}$.

Question: What is the Heterotic dual in this case? The dP_9 above clearly corresponds to a single E_8 bundle over a Calabi-Yau WSF given by $f_{0,0,4(3-d)}$ and $g_{0,0,6(3-d)}$ over B_2. But this Calabi-Yau is singular. Note B_2 is a double cover of $\mathbb{P}^2$ with a branch locus of degree $2d$. Then one can push forward the singular Calabi-Yau to a non-Calabi-Yau WSF over $\mathbb{P}^2$, but then the spectral cover will be $SO(32)$. Is this non-Calabi-Yau $SO(32)$ set-up corresponds to the Heterotic dual?!

Similar to the cases with a 1-section, we can tune the complex structure of B_3 is specific ways, and then do some birational transformation. We don't have a complete physical interpretation as in the previous case yet, but the outcome seems to be nice.

Suppose we tune all of the terms proportional to x_1^2 to zero. Over this locus in the moduli space we still have a smooth 2-section $S = x_1$, but the locus $x_2 = x_3 = 0$ is a new 1-section. Note that the conic now has double point singularities over $x_2 = x_3 = l_{1,j,k} = 0$. We can blow up the locus $x_2 = x_3 = 0$,

$$x_2 \rightarrow wx_2, \tag{4.193}$$

$$x_3 \rightarrow wx_3. \tag{4.194}$$

After this B_3 becomes,

$$B_3' = \begin{array}{ccccccc|c} w & x_1 = S & x_2 & x_3 & u_1 & u_2 & u_3 & F \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & d \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}, \tag{4.195}$$

$$F = \sum_{i+j+k=2, i\leq 1} S^i w^{1-i} x_2^j x_3^k l_{i,j,k}. \tag{4.196}$$

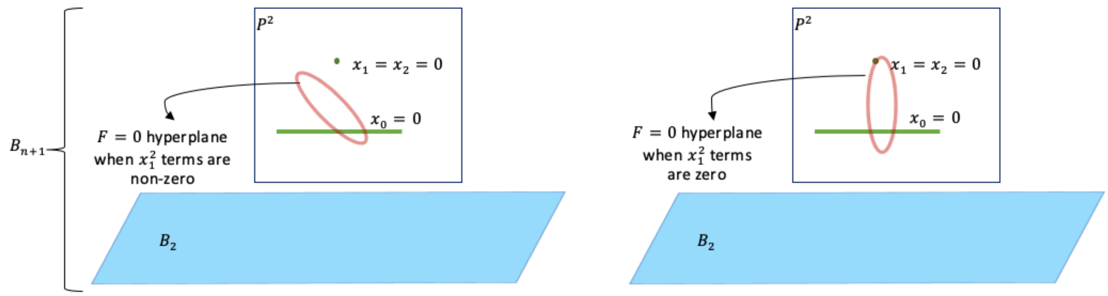


Figure 4.6: In the left hand side there is only one section, but in the right hand side after tuning the complex structure we get a *singular* conic which has a 1-section.

Interestingly, in this equation $w = 0$ is rational 1-section which wraps around d^2 (-1)-curves. Note that $S \cdot w = 0$. We can rewrite (4.185) and (4.186) in this new geometry,

$$\begin{aligned} f &= S^4 f_{0,0,4(3-d)} + S^3 w \sum_j^1 x_2^j x_3^{1-j} f_{1,j,4(3-d)} \\ &\quad + S^2 w^2 \sum_j^2 x_2^j x_3^{2-j} f_{2,j,4(3-d)} + S w^3 \sum_j^3 x_2^j x_3^{3-j} f_{3,j,4(3-d)} + w^4 \sum_j^4 x_2^j x_3^{4-j} f_{4,j,4(3-d)}, \end{aligned} \tag{4.197}$$

$$\begin{aligned} g &= S^6 g_{0,0,6(3-d)} + S^5 w \sum_j^1 x_2^j x_3^{1-j} g_{1,j,6(3-d)} + S^4 w^2 \sum_j^2 x_2^j x_3^{2-j} g_{2,j,6(3-d)} \\ &\quad + S^3 w^3 \sum_j^3 x_2^j x_3^{3-j} g_{3,j,6(3-d)} \\ &\quad + S^2 w^4 \sum_j^4 x_2^j x_3^{4-j} g_{4,j,6(3-d)} + S w^5 \sum_j^5 x_2^j x_3^{5-j} g_{5,j,6(3-d)} + w^6 \sum_j^6 x_2^j x_3^{6-j} g_{6,j,6(3-d)}. \end{aligned} \tag{4.198}$$

We can now perform stable degeneration relative to $w = 0$. Even though $w = 0$ is a rational 1-section, similar to the Case ii), we still get only one dP_9. This is quite interesting because it means this F-theor model is completely determined by only one E_8 stable vector bundle over an elliptically fibered Calabi-Yau three-fold with

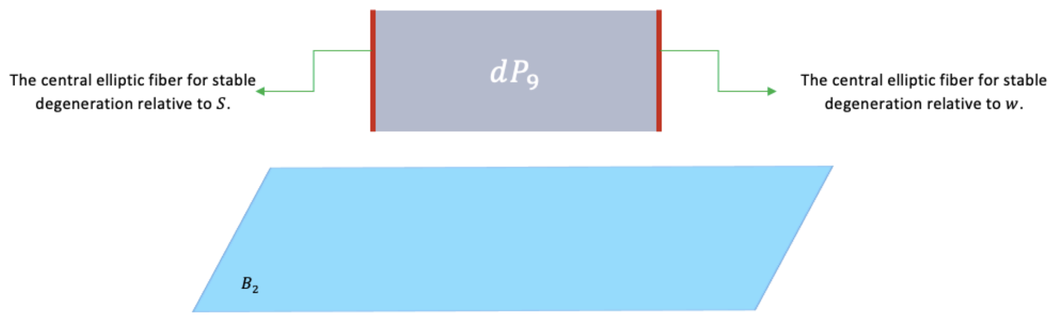


Figure 4.7: When we perform the stable degeneration procedure in this geometry (after the birational transformation) relative to S (2-section) and w (1-section), we get a single dP_9 fiber in both cases, the only difference is that the central fiber will be different in each case.

base identified with $w = 0$ locus in B_3. We are still not quite sure what should be the corresponding Heterotic model. It certainly cannot be the usual Heterotic M-theory set-up. My be it corresponds to CHL string or something similar to that where the two E_8 planes are identified.

Anyways, For generic points of the moduli space this dP_9 is isomorphic to the dP_9 we find when we do the stable degeneration relative to the 2-section S. Of course in this second case the dP_9 corresponds to the an E_8 bundle over a WSF Calabi-Yau which is an elliptic fibration over dP_{d^2}. So we can end up with the following conjecture.

Conjecture: The moduli space of an E_8 bundles over an elliptic WSF CY3 with $B_2 \to \mathbb{P}^2$ a double cover with branch locus of degree $2d$ is dual to the moduli space of an E_8 bundles over an elliptic WSF CY3 with $B_2 = dP_{d^2}$.

Chapter 5

Conclusions and Future directions

5.1 Research summary

The main goal in this dissertation was to generalize the constraints of the standard heterotic/F-theory duality procedure and/or the standard way of defining the moduli space of stable vector bundles over elliptically fibered Calabi-Yau's. Then we investigated the consequences both in Heterotic string and F-theory.

In the first part of this we have generalized the famous spectral cover construction of Friedman, Morgan and Witten [57, 89, 90] to the case of elliptic Calabi-Yau threefolds with higher rank Picard group (i.e. containing either fibral divisors or multiple sections to the elliptic fibration). In particular, the well-established work of [57, 89] provided a simple formula for the Chern classes of bundles associated to smooth (i.e. reduced and irreducible) spectral covers in Weierstrass CY 3-folds:

$$c_1(V) = 0, \tag{5.1}$$

$$c_2(V) = \eta\sigma - \frac{N^3 - N}{24}c_1(B_2)^2 + \frac{N}{2}\left(\lambda^2 - \frac{1}{4}\right)\eta \cdot (\eta - Nc_1(B_2))\,, \tag{5.2}$$

$$c_3(V) = 2\lambda\sigma\eta \cdot (\eta - Nc_1(B_2))\,. \tag{5.3}$$

We have utilized the techniques of Fourier-Mukai functors to generalize these formula to bundles defined over geometries with fibral divisors and higher rank Mordell-Weil. In the case of I_n type singular fibers we find that $c_1(V)$ and $c_3(V)$ are unchanged and in the case of I_2 fibers we find a correction to the second Chern class of the form

$$c_2(V) = \sigma \cdot \eta + \omega_{std} + \left(\zeta_1 \cdot \mathcal{S} + \sum_{i=2}^{k} \beta_i \right)^2 + \sum_{i=2}^{k} \beta_i^2 + \zeta_1 \cdot D_1, \tag{5.4}$$

where D_1 is the new fibral divisor, ζ_1 is an effective class pulled back from the base, B_2, β_i are integers and the divisor $\mathcal{S}$ is a component of the discriminant locus of the fibration (supporting the I_2 fibers) in the base. Here

$$\omega_{std} = -\frac{N^3 - N}{24} c_1(B_2)^2 + \frac{N}{2} \left(\lambda^2 - \frac{1}{4} \right) \eta \cdot (\eta - N c_1(B_2)) \,. \tag{5.5}$$

Similarly, in the case of an additional, holomorphic zero section we find

$$c_2(V) = \sigma \cdot \eta - \beta_1(\eta + N D_{11}) \cdot S_1 + \left(\omega_{std} - \frac{1}{2} \beta_1^2 (\eta + N D_{11}) S_1^2 \right) f, \tag{5.6}$$

where β_1 is integer, S_1 is the Shioda map of the new section and D_{11} is a divisor in B_2 determined by the triple intersection numbers involving the sections.

In the case that the additional sections are rational rather than holomorphic (and hence can wrap reducible components of fibers over higher-codimensional loci in the base), there remain open questions about how best to define a Fourier-Mukai functor that can accommodate the singular fibers (and a section which wraps some of them). As a result, we cannot yet determine how these topological formulae will change. However, we can see in this case that interesting new results are possible since we expect not only the second Chern class but the chiral index to change as well. We have outlined in this work several ways forward on this important problem, and we hope to return to it in future work.

Within heterotic/F-theory duality, the constrained geometric arena –i..e Weierstrass from for both the heterotic strings and F-theory Calabi-Yau backgrounds – has long been a frustrating obstacle to studying new phenomena. Within effective heterotic theories, for example, there are several interesting effects that are believed to have interesting F-theory duals, including perhaps novel mechanisms for moduli stabilization such as the linking of the bundle and complex structure moduli in the heterotic theory through the condition of holomorphy [9, 10, 11, 13] and potentially new 4-dimensional $\mathcal{N} = 1$ dualities including heterotic threefolds admitting multiple elliptic fibrations (and hence leading to multiple, related dual F-theory fourfolds) [8, 15, 16], the F-theory duals of heterotic target space duality [20] or F-theory duals [43, 54] of known "standard model like" heterotic compactifications (including [12]). However, in all cases, these theories have crucially involved decidedly non-Weierstrass geometry on the heterotic side. These questions have formed the motivation for the present work. We believe that here we have taken important first steps towards extending the geometries for which explicit heterotic/F-theory duals can be constructed.

Next, in the TSD project, we have taken a first step towards exploring the consequences of $(0,2)$ target space duality for heterotic/F-theory duality. In an important proof of principle, we have illustrated that heterotic TSD pairs exist in which both halves of the geometry exhibit Calabi-Yau threefolds with elliptic fibrations. As a result, it is clear that some F-theory correspondence should be induced in these cases. We take several steps to explore the properties of this putative duality. First, we consider the conjecture made previously in the literature that the F-theory realization of TSD could be multiple $K3$ fibrations of the same elliptically fibered Calabi-Yau 4-fold background of F-theory. To explore this possibility in earnest, we begin in 6-dimensional compactifications of heterotic string theory/F-theory and demonstrate that in general multiple fibrations within F-theory CY backgrounds cannot correspond to the (topologically trivial) TSD realizable for bundles on $K3$ surfaces. Finally, we provide a sketch of the open questions that arise when attempting to directly compute the F-theory duals of 4-dimensional heterotic TSD geometries. In particular, we demonstrate that multiple $K3$ fibrations in F-theory cannot account for $(0,2)$ TSD in the case that the threefold base, B_3 of the F-theory elliptic fibration takes the form normally assumed – that of a $\mathbb{P}^1$ bundle over a two (complex) dimensional surface, B_2.

Finally, we have taken a deeper look at $\mathbb{P}^1$-fibrations within F-theory with a goal of formulating more general possibilities than have been considered thus far in the literature. The majority of work in heterotic/F-theory duality has restricted consideration to n-fold bases $\mathcal{B}_n$ to the Calabi-Yau elliptic fibration that are of the form $\mathcal{B}_n = \mathbb{P}(\pi : \mathcal{O} \oplus \mathcal{O}(D) \to B_{n-1})$.

Here we demonstrate that even in the context of bases $\mathcal{B}_n$, which are $\mathbb{P}^1$-bundles more general choices of a vector bundle over $\mathbb{B}_{n-1}$ are possible in both 6- and 4-dimensional compactifications of F-theory and they can come equipped with rational rather than holomorphic sections in 4-dimensions. This new base geometry is of interest in enhancing the range of Calabi-Yau backgrounds in F-theory and hence the possible effective fields that can be obtained. Since F-theory has proven to sweep out vast swathes of the string landscape, it has played a key role in recent investigations into the string Swampland.

In addition, the novel geometry we have built in this dissertation demonstrates that there are many new "weakly coupled" limits of F-theory that should – in principle – be related to heterotic string theory. We hope these results have shed some light on this familiar duality may be extended.

5.2 Future works

In the first project, there remain, however, important open questions. First, as mentioned above, we require new and more robust tools to address the general case of a higher rank Mordell-Weil group with rational generators studied in Section 2.4. In addition, as illustrated in the explicit examples constructed in Section 2.5 all the formulas we have derived in this work have been limited by the restriction of *smoothness* of the spectral cover. In general, many examples in the literature (see e.g., [29]) have demonstrated that smooth vector bundles do not necessarily correspond to smooth spectral covers. Indeed, this observation has been a powerful tool in determining the effective physics of T-brane solutions in F-theory [14, 18, 19, 51]. By placing the constraint of smoothness on the spectral data, we are clearly losing information about general components of the bundle moduli space (as illustrated in Section 2.7). Finally, there remain interesting open questions about how to determine the full Picard groups of spectral covers (since these are surfaces of general type, this is a notoriously hard problem in algebraic geometry, see e.g., [4]) and a number of interesting possibilities remaining to be explored related to higher co-dimensional behavior in moduli spaces (i.e. so-called "jumping" phenomena or Noether-Lefschetz problems [72]).

One approach to the problem of singular covers above might arise through a recursive approach. As noted above, the only general topological formulas derived (here and in the literature overall) are for vector bundles realized (modulo the Picard number problem) by smooth spectral covers. In the case that the spectral cover is a union of several components that can be smooth or non-reduced or vertical, the main obstacle is providing a general form for the Chern character of the spectral sheaf (which is clearly a hard problem in the algebraic geometry of singular surfaces). However, we might hope to avoid this difficult question by deriving a "recursive" algorithm to resolve the singularities of the spectral cover that could work in general. For example, if the spectral cover is degenerate, it is still possible to find a locally free resolution (with length one) of the spectral sheaf. We might hope to use Fourier-Mukai transforms to study the vector bundles associated with this resolution. If one can argue that the "degree of the degeneracy" drops in each step, then this process will terminate at some point.

All of these problems deserve further attention and are necessary for a general study fo heterotic/F-theory duality. We hope to continue to explore them in future work.

Also, there are some future directions that naturally lead on from our TSD, most importantly to explicitly determine the F-theory mechanism that generates dual theories from potentially disparate 4-fold geometries. We hope to understand this correspondence in future work. The present study has shed light on these questions, however, and highlighted areas where the current state-of-the-art in the literature is insufficient to determine the dual heterotic/F-theory geometries.

As noted in Section 3.3.6, it is clear that new tools will be needed to fully determine this duality. The new geometric features that must be understood in heterotic/F-theory duality in this context clearly extend beyond the canonical assumptions made in [89], and new tools must be developed. These include the following open problems in heterotic/F-theory duality

- Heterotic compactifications on elliptic threefolds with higher rank Mordell-Weil group (as in the examples in Section 3.2).
- F-theory compactifications on threefold bases that are $\mathbb{P}^1$ fibered, but not $\mathbb{P}^1$ bundles. In other words, F-theory on elliptic fibrations with *conic bundle* (see e.g. [143]) bases.
- F-theory duals of degenerate (i.e., non-reduced and reducible) heterotic spectral covers. These seem to be a ubiquitous feature in the context of $(0,2)$ target space duality since the spectral data of monad bundles appear to be generically singular [36].
- 4-dimensional T-brane solutions of F-theory (expected to arise in the context of degenerate spectral covers above [14, 18, 19]).

This last point seems to be an essential part of the story for 4-dimensional heterotic/F-theory pairs since degenerate spectral data naturally arise for monad bundles (and hence geometries arising from $(0,2)$ GLSMs). Moreover, the arguments of Section

3.3.6 make it clear that the degrees of freedom of an expected dual F-theory 4-fold must be constrained by the flux to match the moduli count of the heterotic theory. Several of these "missing ingredients" are currently being studied (see [20] for generalizations of heterotic geometries in heterotic/F-theory duality and [22] for a study of F-theory on conic bundles). We hope that the present work illustrates the need for these new tools and demonstrates that there remain many interesting open questions within the context of 4-dimensional heterotic/F-theory duality. We will return to these open questions in future work.

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Appendices

Appendix A

Basic Definitions

In order to be self contained, we briefly review some of the main mathematical objects that are frequently used in string theory compactification, for more detailed information refer to [96, 116, 117, 123].

- **Complex manifold**

  Intuitively, complex manifolds are topological spaces that locally look like flat complex space $\mathbb{C}^n$ for some n. More precisely,

  Definition: Consider a real $2n$ dimensional manifold M. Then there is an atlas $\{U_i, \psi_i\}$ of open sets (which cover the manifold), and local coordinates. If we can "complexify" the local coordinates, which means finding homomorphisms $\psi_i : U_i \to \mathbb{C}^n$, such that for any (non-empty) $U_i \cap U_j$, $\psi_i o \psi_j^{-1} : \psi_j(U_i \cap U_j) \to \psi_i(U_i \cap U_j)$ is a holomorphic map from $\mathbb{C}^n$ to itself, then M is called a complex manifold of dimension n.

  In order to give a necessary and sufficient condition for when a real manifold is complex, one first defines an almost complex structure J which is a (n, n)- tensor on M (consider M as a real manifold) such that $J^2 = -1$. This means it's possible to define local complex coordinates. More concretely, choose a patch U we have $2n$ real coordinates $\{x_1, \ldots x_n, y_1 \ldots y_n\}$, then J acts on coordinate basis as

  $$J(\partial_{x_i}) = \partial_{y_i}, \qquad J(\partial_{y_i}) = -\partial_{x_i}. \tag{A.1}$$

  So by defining local complex coordinates,

$$z_j = x_j + iy_j, \qquad \bar{z}_j = x_j - iy_j \tag{A.2}$$

we get $J\partial_{z_i} = i\partial_{z_i}$, $J\partial_{\bar{z}_i} = -i\partial_{\bar{z}_i}$. Then M being a complex manifold is equivalent to being able define complex coordinates in each patch such that under coordinate transformations the almost complex structure stays diagonal (integrability). In this situation J is called a complex structure tensor.

The necessary and sufficient condition for J to be complex structure is that the following tensor becomes zero (see [123] Theorem 8.12 for a proof),

$$N(v,w) = [v,w] + J[v,Jw] + J[jv,w] - [Jv,Jw], \tag{A.3}$$

where v,w are arbitrary vector fields. This is called *Nijenhuis* tensor.

- **Intersection numbers:**

As it is clear from the name it is the number of intersection points between cycles in M, so by Poincare duality we may be able to express the intersection number of divisors as the integration of the corresponding dual $(1,1)$-forms. For example, consider $\mathbb{P}^n$. All of the divisors in this space can be written as mH, where H is the hyperplane divisor corresponding to the vanishing locus of any linear polynomial. Then the intersection number of n different divisors,

$$[m_1H]\cdot[m_2H]\cdots[m_nH] \tag{A.4}$$

can be written as the integral

$$\int_{\mathbb{P}^n}(m_1\omega)\wedge(m_2\omega)\wedge\cdots\wedge(m_n\omega) = m_1m_2\cdots m_n Vol(\mathbb{P}^n) \tag{A.5}$$

where ω is the Kahler form of the projective space. So if we normalize the integral so that $\int J^n = 1$, then we can use the integral above to say the intersection number is $m_1m_2\cdots m_n$. As another example, consider the product of two projective spaces $\mathbb{P}^{n_1}\times\mathbb{P}^{n_2}$. Similar to previous case we normalize the integral $\int \omega_1^{n_1}\omega_2^{n_2} = 1$ where ω_1 and ω_2 are Kahler forms of the two projective spaces. Then the intersection numbers can be computed as

$$\begin{aligned}[m_1H_1]\cdot[m_2H_1] \quad \cdots \quad & [m_{n_1}H_1]\cdot[l_1H_2]\cdots[l_{n_2}H_2] \\ = \quad & m_1\cdots m_{n_1}\cdot l_1\cdots l_{n_2}\int_{\mathbb{P}^{n_1}\times\mathbb{P}^{n_2}}\omega_1^{n_1}\wedge\omega_2^{n_2}\end{aligned} \tag{A.6}$$

where H_1, and H_2 are the hyperplane divisors. For a general toric variety it's possible to figure out the intersection numbers from the toric data. The reader can refer to [55] for more information.

- **Blow up**

In this subsection we try illustrate the blow up process by a simple example (see [104] I.4 , also II.7 for more abstract definitions). Consider $\mathbb{P}^2$ with homogeneous coordinates (x, y, z). We choose the patch $z = 1$, and consider the following hyperserface inside $\mathbb{P}^2 \times \mathbb{P}^1$,

$$xu_1 - yu_2 = 0, \tag{A.7}$$

where u_1 and u_2 are the homogeneous coordinates of $\mathbb{P}^1$. We see from this equation whenever $(x, y) \neq (0, 0)$, a single point in $\mathbb{P}^1$ is fixed, however when $(x, y) = (0, 0)$, there is no constraint on u_1 and u_2. So we see that (A.7) correspond to surface which generically seems to be the same as the original $\mathbb{P}^2$ plane, but the origin is replaced by a whole $\mathbb{P}^1$.

This $\mathbb{P}^1$ is called an **exceptional divisor** E, and it can be shown since we've blown up a generic smooth point in $\mathbb{P}^2$, it's self intersection is -1,

$$E.E = -1. \tag{A.8}$$

This exceptional divisor can also be seen as the projectivization of the normal bundle to the point $(0, 0, 1)$ i.e. the origin.

To see how blow up can be used to "smooth out" singularities consider a curve with double point singularity (node, or cusp) at the origin, as an example (in patch $z = 1$),

$$y^2 = -x^2(x - 1). \tag{A.9}$$

In this case we have a node (see Fig (A.1)). Now we rewrite $x = ex'$, and $y = ey'$, then it's clear from (A.7) that $e = 0$ is the locus of the exceptional divisor. Then the above equation becomes,

$$e^2 \left(y'^2 + x'^2(ex' - 1) \right) = 0. \tag{A.10}$$

The order 2 zero at $e = 0$ indicates the order of singularity. so if we remove this factor from the equation the rest of that will be a smooth curve.

More formally we can describe what we did as a morphism,

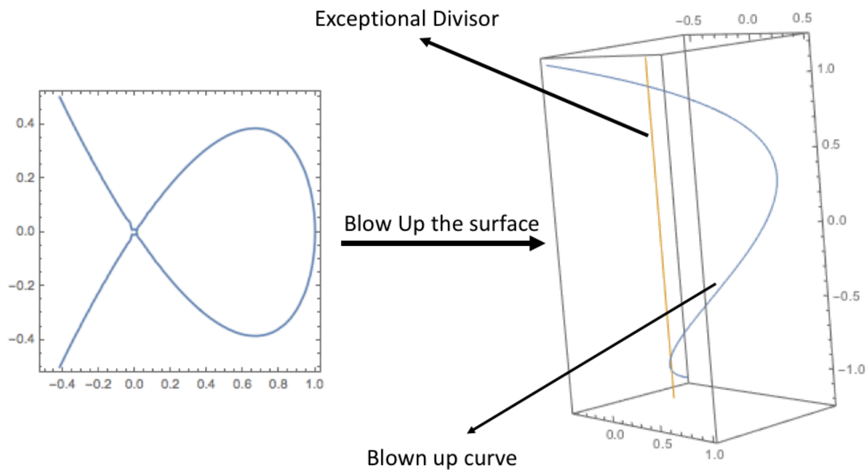


Figure A.1: *The curve on the left correspond to the curve* $y^2 + x^2(x-1) = 0$ *in* $\mathbb{P}^2$ *inside the patch* $z = 1$*, after adding a exceptional divisor in the origin, we get a reducible curve, one component correspond to the exceptional divisor (the* e^2 *factor above) shown as a orange line, and the other irreducible component is called the strict transform of the original curve on right curve. Note if we look at the curve on the right from top (or shrink the exceptional divisor to zero), its image over the horizontal plane will be the same as the curve on the left.*

$$\rho : \tilde{\mathbb{P}^2} \to \mathbb{P}^2, \tag{A.11}$$

where $\tilde{\mathbb{P}^2}$ is the blown up projective plane (A.7) (this is just the first Hirzebruch surface $\mathbb{F}_1$). Then the strict transformation of the curve will be,

$$\tilde{C} = \rho^* C - 2E, \tag{A.12}$$

where C is just the divisor class of the curve in the projective plane. The factor 2 represents the double point singularity.

- **Vector bundles**
  - **Definition** Consider a compact manifold M (real or complex), we can cover it with open sets and local coordinates $\{U_i, \psi_i\}$. Intuitively, a vector

bundle locally looks like a product $U_i \times W$ where W is a vector space with fixed dimension. To get a non-trivial vector bundle over M, we need to glue these local structures.

Again we need to define this more precisely. A vector bundle is given by a projection,

$$\pi : V \to M, \tag{A.13}$$

where V is the total space of the bundle, M is the base manifold, and $\pi^{-1}(x) \sim W$ for any point x in the base manifold. Similar to the definition of manifolds, there are homomorphisms (called local trivializations) $\phi_i : V \to U_i \times W = \pi^{-1}(U_i)$, and similar to coordinate transformation between patches, we need to define "transition functions" on $U_i \cap U_j$ as $t_{ij} = \phi_i \circ \phi_j^{-1} : U_j \times W \to U_i \times W$. Over any point $x \in U_i \cap U_j$, $t_{ij}(x)$ is just a homomorphism inside the vector space W. In principle the transition functions can be elements of Lie groups G in various representations. G is called the structure group of the bundle, and the rank of the bundle $rk(V)$ is the dimension of W.

- **Section** Sections are defined as maps $S : M \to V$. Locally this means over each open patch in the base manifold there is a map $S_i : U_i \to W$ such that for any $x \in U_i$, $S_i(x)$ is a unique vector in W. These local maps then glue together by the transition functions as $S_i = t_{ij} S_j$ to make a global section.

- **Connection and Curvature** Similar to the tangent vectors of a manifold, we can define the parallel transport of elements in the vector bundle. To do this consider a local frame over U_i (a basis of the vector space in $U_i \times W$) $\{e_1 \dots e_p\}$, then parallel transport of e_i in direction μ in the base manifold is given defined by the connections:

$$\nabla_\mu e_i = A^j_{\mu i} e_j. \tag{A.14}$$

Note that A_μ is a one form with values in (adjoint representation of) the structure group. Also it's clear that under the "local" transformations $e'_i = g(x)_{ji} e_j$, the connections transform in the following non-covariant way,

$$A' = g^{-1} A g + g^{-1} dg. \tag{A.15}$$

The corresponding curvature, which is covariant under these transformations, is defined similar to curvature of manifolds,

$$F_{\mu\nu} = [\nabla_\mu, \nabla_\nu], \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{A.16}$$

- **Gauge theory** There is a clear correspondence between physical gauge theories and vector bundles. Structure groups, vector bundles, connections, curvature, and the transformations g, correspond to the gauge group, matter field, gauge fields, field strength and gauge transformations respectively.

- **Holomorphic bundles** Suppose $\pi : V \to M$ is a complex vector bundle (which means the fibers are isomorphic to a $\mathbb{C}$-linear space) over a complex manifold. Then V is called holomorphic if the transition functions are holomorphic relative to the complex coordinates. It can be shown for every holomorphic bundles, we can choose a gauge such that $A_{\bar{a}}$ components of the connection becomes zero. In other words $\nabla_{\bar{a}} = \partial_{\bar{a}}$. Also (if we can define a hermitian inner product on the fibers of V) the $(2,0)$ and $(0,2)$ components of the field strength are zero for holomorphic bundles (see [117] section 4.3 and appendix 4.B, also [96] 15.6 for more intuitive/-physical discussion),

$$F_{ab} = F_{\bar{a}\bar{b}} = 0. \tag{A.17}$$

If V_1 and V_2 are two bundles with structure groups G_1 and G_2, we can define the direct sum and direct product of bundles as $G_1 \oplus G_2$ and $G_1 \otimes G_2$ structure groups respectively.

- **Cohomology** There are various ways to define cohomology groups. We only briefly discuss the de Rham and Dolbeault cohomology here (see [104] III.4 for Cech cohomology).

  Generally when there is a complex as,

$$0 \to A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \xrightarrow{d^2} \ldots \tag{A.18}$$

such that $d^{i+1} \circ d^i = 0$, then cohomology groups are defined as,

$$H^i = \frac{Ker\,(d^i : A^i \to A^{i+1})}{Im\,(d^{i-1} : A^{i-1} \to A^i)} \tag{A.19}$$

  - **de Rham cohmology**

One example of cohomology is the de Rham cohomology over a real manifold M defined by the differential operator d on a complex of differential forms,

$$0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \dots \tag{A.20}$$

where Ω^n is the space of n-forms, and the corresponding cohomology groups $H^n(X, \mathbb{R})$ are the space of closed n-forms modulo the exact ones. The dimensions of these groups are called Betti numbers b_n

– **Dolbeault cohomology**

On a complex manifold we can decompose the differential operators into holomorphic and anti-holomorphic parts $d = \partial + \bar{\partial}$, where $\partial^2 = \bar{\partial}^2 = 0$, and correspondingly, the decompose into the direct sum of mixed (p, q)-forms,

$$\Omega^n = \bigoplus_{p+q=n} \Omega^{(p,q)}, \tag{A.21}$$

where elements of $\Omega^{(p,q)}$ can be written as

$$\omega_{a_1 \dots a_p \bar{a}_1 \dots \bar{a}_q} dz^{a_1} \dots dz^{a_p} d\bar{z}^{\bar{a}_1} \dots d\bar{z}^{\bar{a}_q}. \tag{A.22}$$

Since $\bar{\partial}^2 = 0$, we can define the Dolbeault cohomology relative to $\bar{\partial}$,

$$0 \to \Omega^{(p,0)} \xrightarrow{\bar{\partial}} \Omega^{(p,1)} \xrightarrow{\bar{\partial}} \Omega^{(p,2)} \xrightarrow{\bar{\partial}} \dots. \tag{A.23}$$

$$H^{p,q}_{\bar{\partial}}(X) := H^{q}_{\bar{\partial}}(X, \Omega^{(p,0)}) = \frac{Ker\left(\bar{\partial} : \Omega^{(p,q)} \to \Omega^{(p,q+1)}\right)}{Im\left(\bar{\partial} : \Omega^{(p,q-1)} \to \Omega^{(p,q)}\right)}. \tag{A.24}$$

The dimension of the cohomology group $H^{p,q}_{\bar{\partial}}(X)$ is called the **Hodge number** $h^{p,q}$. If the manifold M is also compact, we get the following relations,

$$b_n = \sum_{p+q=n} h^{p,q}. \tag{A.25}$$

Similarly on a holomorphic vector bundle V, the connection also decomposes (by complexity of the bundle) $\nabla = \nabla^{(1,0)} + \nabla^{(0,1)}$, and by holomorphicity condition, $(\nabla^{(1,0)})^2 = (\nabla^{(0,1)})^2 = 0$. So we may define the cohomology groups $H^n(X, V)$ with respect to the differential operator

$\bar{\nabla} = \nabla^{(0,1)}$. The elements $\psi^x_{\bar{a}_1 \ldots \bar{a}_n}$ of this group are $(0, n)$-forms with values in V. The upper index x is the vector bundle index which correspond to a representation of the structure (gauge) group. Also these elements are $\nabla^{(0,1)}$ closed but not exact.

To get a little physical intuition consider n-forms living in the internal compact manifold with an index transforming in some representation of the gauge group. One way to see how these fields arise in string theory is from the fact that the space of zero modes of gauginos decompose into subspaces isomorphic to the space of differential n-forms. Since they are zero modes, they must be $\bar{\nabla}$-closed. However $\bar{\nabla}^2 = 0$, so we always have a gauge freedom,

$$\psi^x \to \psi^x + \left(\bar{\nabla}\Lambda\right)^x, \tag{A.26}$$

where Λ is an arbitrary $n-1$-form. Since the theory is invariant under this "gauge transformation," it justifies to consider only the elements of the cohomology groups as the space of physical solution of the massless Dirac equation (discussed in detail in [96] chapters 13 to 16).

- **Chern classes**

Characteristic classes (including Chern classes) are elements of the cohomology groups that are invariant under smooth deformations, and measure the "non-triviality" of the bundles. There are various ways to define the Chern classes, here we use the differential geometric definition. Here we restrict ourselves to the complex vector bundles $\pi : V \to M$ with curvature 2-form F, and rank n.

  - **Chern class**

    The total Chern class is defined as

$$c(V) = det\left(1 + i\frac{F}{2\pi}\right) \tag{A.27}$$

We can expand this order by order to get,

$$\begin{aligned}
c(V) &= 1 + c_1(V) + c_2(V) + \ldots, && \text{(A.28)}\\
c_1(V) &= \frac{i}{2\pi}\,(trF)\,, && \text{(A.29)}\\
c_2(V) &= \frac{1}{2}\left(\frac{i}{2\pi}\right)^2 (tr\,(F\wedge F) - tr(F)\wedge tr(F))\,, && \text{(A.30)}\\
&\vdots\\
c_n(V) &= \left(\frac{i}{2\pi}\right)^n det\,(F) && \text{(A.31)}
\end{aligned}$$

– **Chern character**

$$\begin{aligned}
ch(V) &= tr\left(e^{i\frac{F}{2\pi}}\right), && \text{(A.32)}\\
ch_0(V) &= n, && \text{(A.33)}\\
ch_1(V) &= i\frac{F}{2\pi} = c_1(V), && \text{(A.34)}\\
ch_2(V) &= -\frac{1}{4\pi^2}tr\,(F\wedge F) = \frac{1}{2}\left(c_1(V)^2 - 2c_2(V)\right), && \text{(A.35)}\\
\ldots & && \text{(A.36)}
\end{aligned}$$

– **Properties**

There are important identities for Chern classes/characters of direct sum and direct product of vector bundles. One can figure out these identities by trying to understand what is the corresponding curvature 2-form,

$$\begin{aligned}
V &= V_1 \oplus V_2,\\
F_V &= \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix}, && \text{(A.37)}\\
W &= V_1 \otimes V_2,\\
F_w &= F_1 \otimes 1 + 1 \otimes F_2 && \text{(A.38)}
\end{aligned}$$

Then by the definition, the following identities hold,

$$\begin{aligned}
c(V) &= c\,(V_1)\wedge c(V_2), && \text{(A.39)}\\
ch(V) &= ch(V_1) + ch(V_2), && \text{(A.40)}\\
ch(W) &= ch(V_1)\wedge ch(V_2). && \text{(A.41)}
\end{aligned}$$

The first two relations will be true even for non-trivial extensions,

$$0 \to V_1 \to V \to V_2 \to 0. \tag{A.42}$$

– **Atiyah-Singer index theorem** Suppose V is a holomorphic vector bundle over a compact complex manifold M of complex dimension m, then the **Euler Characteristic** of V is defined as

$$\chi\,(M,V) = \sum_{i=1}^{m}(-1)^m h^m(M,V). \tag{A.43}$$

The following theorem connects the Euler characteristic and Chern classes,

$$\chi(M,V) = \int_M ch(V)td(M) \tag{A.44}$$

where $td(M)$ is the total Todd class, and it's relation with Chern classes of the tangent bundle TM is given as,

$$\begin{aligned} td_0(M) &= 1, \\ td_1(M) &= \frac{1}{2}c_1(M), \\ td_2(M) &= \frac{1}{12}\left(c_1(M)^2 + c_2(M)\right), \\ &\cdots \end{aligned} \tag{A.45}$$

This theorem is important because it gives the chirality of the effective field theories in terms of the topological quantities of the extra dimensional manifold and the gauge bundles over it.

– **Stability of V**

Another important quantity that is defined for complex vector bundles over compact Kahler manifolds of complex dimension m is the slope of bundle,

$$\mu(V) = \frac{1}{rank(V)} \int c_1(V) \wedge \omega^{m-1} \tag{A.46}$$

where ω is the Kahler class of the complex manifold.

Definition A holomorphic vector bundle V over a compact Kahler manifold is called slope (Mumford) stable is for every sub-sheaf $\mathcal{F} \subset V$, $\mu(\mathcal{F}) < \mu(V)$.

We have seen that in string theory compactification 4-dimensional supersymmetry puts the following constraint on the holomorphic bundles,

$$g^{a\bar{b}} F_{a\bar{b}} = 0. \tag{A.47}$$

A theorem by Donaldson-Uhlenbeck-Yau [85, 149], states that for any holomorphic vector bundle over a compact Kahler manifold, the above condition is satisfied if and only if the bundle is poly-stable.[1] A poly-stable bundle is simply a direct sum of stable bundles, all with the same slope: $V = \bigoplus_i V_i$ with $\mu(V_i) = \mu(V)\ \forall i$.

[1]To be more precise, the connection of the vector bundle must satisfy $g^{a\bar{b}} F_{a\bar{b}} = \frac{-i}{Vol(M)} \mu(V) 1$ if and only is it's stable. However in Heterotic compactifications we are restricted to the case of zero first Chern class, and therefore zero slope (see e.g. [117] Ch. 4.B).

Appendix B

Basics of Derived Categories

Since the Fourier-Mukai functor, which we use a lot in this dissertation, is a special integral transform, we devote this appendix on reviewing some key points about them. For more details, see [32, 117].

• **Hom$_{\mathcal{A}}$** First of all note that any functor between two categories $F : \mathcal{A} \to \mathcal{B}$ induces a map between the space of morphisms

$$Hom_{\mathcal{A}}(A, B) \to Hom_{\mathcal{B}}(F(A), F(B)), \tag{B.1}$$

where A, B are arbitrary objects of the category $\mathcal{A}$ (i.e. the map is "functorial"). In case the categories are additive the set of morphisms form an abelian group, and in the cases we are concerned in this dissertation they are actually $\mathbb{C}-$vector spaces. Abelian categories are particular additive categories that for any functor one can define kernel and cokernel. The specific category we need in this dissertation is **Coh**(X), i.e. the category of coherent sheaves over a variety X, and the categories derived from that.

• **Fully faithful functor** A functor $F : \mathcal{A} \to \mathcal{B}$ is called full if the map (B.1) is surjective and it is called faithful if it is injective. So a fully faithful functor induces an isomorphism in (B.1).

• **Morphism of functors** Consider any two objects A,$B \in \mathcal{A}$, and a morphism $f : A \to B$ between them, then a *morphism of functors* $\theta : F \to G$ between the two functors $F, G : \mathcal{A} \to \mathcal{B}$ is defined with the following commutative diagram,

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \downarrow{\scriptstyle \theta_A} & & \downarrow{\scriptstyle \theta_B} \\ G(A) & \xrightarrow{G(f)} & G(B) \end{array} \tag{B.2}$$

• **Left and right adjoint** A functor $G : \mathcal{B} \to \mathcal{A}$ is a right adjoint of $F : \mathcal{A} \to \mathcal{B}$, written as $F \dashv G$ if

$$Hom_{\mathcal{B}}(F(A), B) \sim Hom_{\mathcal{A}}(A, G(B)), \tag{B.3}$$

where $A \in \mathcal{A}$ and $B \in \mathcal{B}$ are any arbitrary objects. In particular one can see

$$Hom_{\mathcal{B}}(F(A), F(B)) \sim Hom_{\mathcal{A}}(A, G \circ F(B)).$$

Since this isomorphism is functorial (i.e. it's true for any A and B), we get a functor morphism,

$$g : id_{\mathcal{A}} \longrightarrow G \circ F. \tag{B.4}$$

Then it is not too hard to prove the existence of the following commutative diagram (for any two objects A_1 and A_2 in $\mathcal{A}$),

$$\begin{array}{ccc} Hom_{\mathcal{A}}(A_1, A_2) & \xrightarrow{g_{A_2}} & Hom_{\mathcal{A}}(A_1, G \circ F(A_2)) \\ & \searrow^{F} & \downarrow{\wr} \\ & & Hom_{\mathcal{B}}(F(A_1), F(A_2)) \end{array} . \tag{B.5}$$

A similar diagram can be drawn for the left adjoint functor.

• **Equivalence of categories** A functor $F : \mathcal{A} \to \mathcal{B}$ is called an equivalence if there are functors $G, H : \mathcal{B} \to \mathcal{A}$ such that $G \circ F \sim id_{\mathcal{A}}$ and $F \circ H \sim id_{\mathcal{B}}$.

It is now easy to see from (B.5) that if a functor is fully faithful and has both a left and right adjoint then it is an equivalence.

• **Serre functor** In case the *Hom* group of some abelian category is actually a vector space, we can define a Serre factor $S : \mathcal{A} \to \mathcal{A}$ as follows,

$$Hom_{\mathcal{A}}(A, B) \xrightarrow{\sim} Hom_{\mathcal{A}}(B, S(A))^*, \tag{B.6}$$

where A, B are arbitrary objects (so the isomorphism is functorial).

• **Triangulated category**

An additive category $\mathcal{A}$ is called triangulated if there is a "shift functor,"

$$T : \mathcal{A} \to \mathcal{A}, \tag{B.7}$$

and a set of "distinguished triangles,"

$$A \to B \to C \to A[1]. \tag{B.8}$$

These distinguished triangles are constrained to satisfy four axioms. Since we will not use them extensively in this dissertation we will not mention them here [105].Here it is probably enough to mention that the composition of any two consecutive morphisms is zero, a fact that can be proven directly from the axioms. Later when we introduce derived categories of complexes, the shift functor corresponds to shifting the complex to right or left, and the distinguished triangles correspond to exact triangle of complexes.

• **Category of complexes** Suppose $\mathcal{A}$ is an abelian category. Then one defines the category of complex $C(\mathcal{A})$, which it's objects are complexes of objects in $\mathcal{A}$,

$$A^\bullet := \cdots \longrightarrow A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \longrightarrow \ldots \tag{B.9}$$

such that $d^i \circ d^{i-1} = 0$. Note that we defined the complex in cohomological way. The morphisms in $C(\mathcal{A})$ between two objects $h : A^\bullet \to B^\bullet$ are defined by a collection of morphisms $\{h^i\}$ in $\mathcal{A}$ as,

$$\begin{array}{ccccccc} \ldots & \longrightarrow & A^{i-1} & \xrightarrow{d_A^{i-1}} & A^i & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle h^{i-1}} & & \downarrow{\scriptstyle h^i} & & \\ \ldots & \longrightarrow & B^{i-1} & \xrightarrow{d_B^{i-1}} & B^i & \longrightarrow & \ldots \end{array} \tag{B.10}$$

which must be commutative. There are several remarks that must be mentioned,

i) One can define the shift functor $T : C(\mathcal{A}) \to C(\mathcal{A})$ naturally in this category as,

$$\begin{aligned} A^\bullet[1] &:= T(A^\bullet), \\ (A^\bullet[1])^i &= A^{i+1}, \qquad d^i_{A^\bullet[1]} = -d^{i+1}_{A^\bullet}. \end{aligned} \tag{B.11}$$

ii) As usual we can define cohomology for complexes,

$$\mathcal{H}^i(A^\bullet) = \frac{Ker(d^i)}{Im(d^{i-1})}. \tag{B.12}$$

Two complexes $A^\bullet$, $B^\bullet$ are said to be ***Quasi-Isomorphic*** if all of their cohomologies are isomorphic.

iii) It is easy to show that because $\mathcal{A}$ is abelian then $C(\mathcal{A})$ is also abelian. Therefore in particular one can have a short exact sequence, $0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$. It is also possible to consider this S.E.S as a triangle,

$$\begin{array}{ccc} A^\bullet & \longrightarrow & B^\bullet \\ & {}_{[1]}\nwarrow \qquad \swarrow & \\ & C^\bullet & \end{array} \tag{B.13}$$

with trivial morphism $C^\bullet \to A^\bullet[1]$. But it doesn't mean $C(\mathcal{A})$ is a triangulated category, because the triangles mentioned above don't satisfy the axioms of the distinguished triangles.

iv) One can define a "homomorphism complex" in the following way,

$$Hom^\bullet(A^\bullet, B^\bullet) := \bigoplus_i Hom_{C(\mathcal{A})}(A^\bullet, B^\bullet[i]),$$
$$f^i \in Hom^n(A^\bullet, B^\bullet) \Rightarrow df^i := d^{i+n}_{B^\bullet} \circ f^i + (-1)^{n+1} \circ d^i_{A^\bullet}. \tag{B.14}$$

• **Homotopy category** A morphism $f : C^\bullet(\mathcal{A}) \to C^\bullet(\mathcal{A})$ is called homotopic to zero $f \sim 0$, if there is a collection of morphisms $\{h^i\}$ in $\mathcal{A}$ such that,

$$\begin{array}{ccccccccc} \dots & \longrightarrow & A^{i-1} & \xrightarrow{d_A^{i-1}} & A^i & \xrightarrow{d_A^i} & A^{i+1} & \longrightarrow & \dots \\ & & \downarrow{\scriptstyle f^{i-1}} & \swarrow{\scriptstyle h^i} & \downarrow{\scriptstyle f^i} & \swarrow{\scriptstyle h^{i+1}} & \downarrow{\scriptstyle f^{i+1}} & & \\ \dots & \longrightarrow & B^{i-1} & \xrightarrow[d_B^{i-1}]{} & B^i & \xrightarrow[d_B^i]{} & B^{i+1} & \longrightarrow & \dots \end{array} \tag{B.15}$$

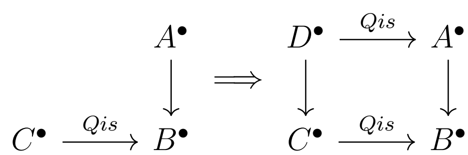
such that,

$$f^i = d^{i-1}_B \circ h^i + h^{i+1} \circ d^i_A. \tag{B.16}$$

Two morphisms $f, gC^\bullet(\mathcal{A}) \to C^\bullet(\mathcal{A})$ are called homotopically equivalent if $f - g \sim 0$ (one can easily check that this actually an equivalence relation). Then one defines the homotopy category $K(\mathcal{A})$ as a category whose objects is the same as $C(\mathcal{A})$, $Ob(C(\mathcal{A})) = Ob(K(\mathcal{A}))$, and

$$Hom_{K(\mathcal{A})}(A^\bullet, B^\bullet) := Hom_{C(\mathcal{A})}(A^\bullet, B^\bullet)/\sim . \tag{B.17}$$

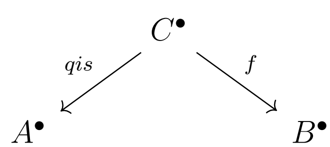
The most important point about the homotopy category for us is that we can complete a square as follows,



(B.18)

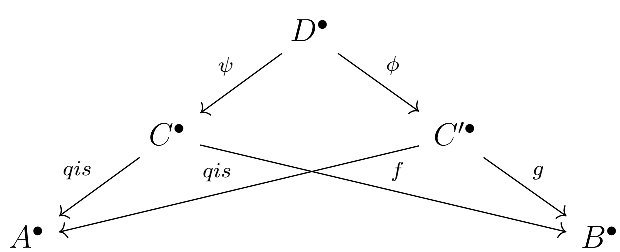
This is will be essential for defining the morphisms in the derived category.

• **Derived category** It is derived from the homotopy category by localizing with the "ideal of quasi-isomorphisms." More clearly $Ob(D(\mathcal{A})) := Ob(K(\mathcal{A}))$, and morphisms in $D(\mathcal{A})$ between two objects $A^\bullet$, $B^\bullet$ are of the form,



(B.19)

In general f descends to a morphism in the homotopy category. As a result if f is also a quasi-isomorphism, then the corresponding morphism in the derived category is isomorphism. So *in $\mathcal{A}$, if cohomology of two complex is isomorphic, then the complexes themselves are isomorphic.* We should also emphasize that the representation (B.19) is not unique, i.e. $C^\bullet$ is not unique, and two such "roofs" (one with $C^\bullet$ at the top and one with $C'^\bullet$ at the top) are equivalent if,



(B.20)

where by (B.18) $D^\bullet$ exists, and both ψ and ϕ are quasi-isomorphisms. In the same way one can define the combination of morphisms in a derived category. Suppose

$f \in Hom_{D(\mathcal{A})}(A^\bullet, B^\bullet)$, $g \in Hom_{D(\mathcal{A})}(B^\bullet, C^\bullet)$ be two different morphisms,

$$f := \left(\begin{array}{ccc} & D^\bullet & \\ {\scriptstyle\psi}\swarrow & & \searrow{\scriptstyle f_0} \\ A^\bullet & & B^\bullet \end{array} \right), \quad g := \left(\begin{array}{ccc} & D'^\bullet & \\ {\scriptstyle\phi}\swarrow & & \searrow{\scriptstyle g_0} \\ B^\bullet & & C^\bullet \end{array} \right), \tag{B.21}$$

then using (B.18) we can define the combination $g \circ f$ as

$$g \circ f = \left(\begin{array}{ccccccc} & & & D''^\bullet & & & \\ & & {\scriptstyle\phi'}\swarrow & & \searrow{\scriptstyle f'_0} & & \\ & D^\bullet & & & & D'^\bullet & \\ {\scriptstyle\phi}\swarrow & & \searrow{\scriptstyle f_0} & & {\scriptstyle\phi}\swarrow & & \searrow{\scriptstyle g_0} \\ A^\bullet & & & C^\bullet & & & B^\bullet \end{array} \right). \tag{B.22}$$

Since both ϕ and ϕ are quasi-isomorphism, then $\phi \circ \phi'$ is also quasi-isomorphism, hence the above diagram is a legitimate morphism in the derived category.

Note, From now on we restrict ourselves to bounded derived categories, $D^b(\mathcal{A})$, which it's objects are isomorphic to complexes with bounded cohomology complexes.

• **Derived functor** ([32, 117]) If a functor $F : K(\mathcal{A}) \to K(\mathcal{B})$ between homotopy categories is compatible with quasi-isomorphisms, i.e. it sends quasi-isomorphisms to quasi-isomorphisms (or equivalently it sends acyclic complexes to acyclic complexes), then it naturally induces a functor on derived categories. But generally it may not happen, so we need to 'derive' a functor from F such that it is compatible with 'localization' of morphisms with quasi-isomorphisms. This functor is called derived functor RF, and here we briefly describe the most general way to define such functors, and then we clarify what that means by constructing some of the most common derived functors one encounters in geometric contexts and hence in this dissertation.

Definition B.1. Let $F : K^b(\mathcal{A}) \to K^b(\mathcal{B})$ be a functor between homotopy categories with the following properties:

a) F is exact.

b) There is a triangulated subcategory $K_F \subset K^b(\mathcal{A})$ such that,

i) There is functor $I : K^b(\mathcal{A}) \longrightarrow K_F$ which for any $M^\bullet \in K^b(\mathcal{A})$ there is *functorial* quasi-isomorphism $M^\bullet \longrightarrow I(M^\bullet)$.

ii) For any acyclic complex $J^\bullet \in K_F$, $I(J^\bullet)$ is also acyclic.

In this situation the derived functor of F is defined as,

$$RF(M^\bullet) := F(I(M^\bullet)). \tag{B.23}$$

Conditions (i), (ii) guarantee that RF maps quasi-isomorphisms to quasi-isomorphisms, so it naturally induces a functor in derived categories,

$$RF : D^b(\mathcal{A}) \longrightarrow D^b(\mathcal{B}), \quad R^iF(M^\bullet) := \mathcal{H}^i(F(I(M^\bullet))). \tag{B.24}$$

In summary, one starts with homotopy categories, and derives another type of category and functor from that, namely the "derived category" and "derived functor."

From now on, we restrict ourselves with categories of coherent sheaves $Coh(X)$ and quasi-coherent sheaves $Qcoh(X)$ over a variety X.

Derived direct image Here we want to find the derived functor of $f_* : Coh(X) \longrightarrow Coh(Y)$ induced from a projective (or at least proper) morphism of varieties $f : X \longrightarrow Y$. First we should mention several facts without proof,

i) f_* is a left exact functor, i.e. if we have a short exact sequence,

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

then $0 \longrightarrow f_*(A) \longrightarrow f_*(B) \longrightarrow f_*(C) \longrightarrow \ldots$ is only exact on the left two elements.

ii) $Qcoh(X)$ has enough injectives, i.e. any quasi-coherent sheaf $A \in Qcoh(X)$ can be imbeded in an injective quasi-coherent sheaf $0 \longrightarrow A \longrightarrow I$.

iii) The last point means if we consider A as a complex in $K(Qcoh(X))$ concentrated in the 0th element, then there is a quasi-isomorphism $A \longrightarrow I^\bullet$, where $I^\bullet$ is a complex of injectives. One can prove (using the Cartan-Eilenberg resolution) there is a functorial quasi-isomorphism $A^\bullet \xrightarrow{I} I(A^\bullet)$, where $A^\bullet$ is an arbitrary object in $K^b(Qcoh(X))$.

iv) Let $D^b_{Coh(X)}(Qcoh(X))$ be the derived category of complexes of quasi-coherent sheaves with coherent cohomology, then there is an isomorphism $D^b(X) := D^b(Coh(X)) \simeq D^b_{Coh(X)}(Qcoh(X))$.

v) As mentioned above, $Qcoh(X)$ has enough injectives, so one can prove $D^b_{Coh(X)}(Qcoh(X)) \simeq K^b_{Coh(X)}(\mathcal{I})$.

Based on the above properties, we get the following conclusions,

a) From i) and iii) we get that f_* maps distinguished triangles to distinguished triangles, i.e. it is an exact functor.

b) From comparing the last two points with the definition of right derived functors above, we can identify $K(\mathcal{I})$ with K_{f_*}.

So down to earth, if we have proper morphism of varieties $f : X \longrightarrow Y$, then we defined (right) direct image $Rf_* : D^b(X) \longrightarrow D^b(Y)$ in the following way,

1) For any complex of coherent sheaves $A^\bullet$ with bounded cohomology, we have an injective resolution $A^\bullet \longrightarrow I(A^\bullet)$.

2) We define

$$\begin{aligned} Rf_*(A^\bullet) &:= f_*(I(A^\bullet)), \\ R^i f_*(A^\bullet) &:= \mathcal{H}^i(f_*(I(A^\bullet))). \end{aligned} \tag{B.25}$$

Derived *Hom* functor and *Ext* groups Again we start by reviewing some facts about the *Hom* functor first,

i) From (B.14) and the definition of Homotopy category, it is easy to prove the following identity for any abelian category $\mathcal{A}$,

$$\mathcal{H}^i(Hom^\bullet_{C(\mathcal{A})}(A^\bullet, B^\bullet)) = Hom_{K(\mathcal{A})}(A^\bullet, B^\bullet[i]). \tag{B.26}$$

ii) One can prove a bounded bellow complex of injective sheaves is actually an injective complex. For the definition of an injective complex see the following:

Definition B.2. A complex in $\mathcal{I}^\bullet \in C(\mathcal{M}od(X))$ is called injective COMPLEX if the right exact functor $Hom^\bullet_{C(\mathcal{M}od(X))}(\ldots, \mathcal{I}^\bullet) : C(\mathcal{M}od(X)) \longrightarrow \mathcal{A}b$ maps any acyclic complex to another acyclic complex (or equivalently map any quasi-isomorphism to another quasi-isomorphism).

iii) If $\mathcal{I}^\bullet$ is both injective and acyclic, then $Hom^\bullet_{C(\mathcal{M}od(X))}(A^\bullet, \mathcal{I}^\bullet)$ is also acyclic for any $A^\bullet$.

From these facts and the points about $D^b(X)$ which mentioned previously, one can find the derived functor for the left exact functor $F_{A^\bullet} := Hom^\bullet_{C(\mathcal{M}od(X))}(A^\bullet, \ldots)$" with K_F in the definition as $K(I)$ as before,

$$RHom^{\bullet}_{C(\mathcal{M}od(X))}(A^{\bullet}, \dots) : D^b(X) \longrightarrow D^b(\mathcal{A}b). \qquad \text{(B.27)}$$

On the other hand, From the definition and the points mentioned above, and the following fact,

$$Hom_{D(Coh(X))}(A^{\bullet}, \mathcal{I}^{\bullet}) \simeq Hom_{K(Coh(X))}(A^{\bullet}, \mathcal{I}^{\bullet}), \qquad \text{(B.28)}$$

one can show if $A^{\bullet} \overset{qis^{\bullet}}{B}$, then then there is a functor isomorphism $F_{A^{\bullet}} \sim F_{B^{\bullet}}$. So if we consider $RHom$ as a functor on the first variable, it naturally induces a well defied functor in the derived category. Therefore,

$$RHom : D^0(X) \times D^b(X) \longrightarrow D(\mathcal{A}b), \qquad \text{(B.29)}$$

where $D^0(X)$ is the opposite category of $D(X)$.

Definition B.3. $Ext^i_{D(X)}(A^{\bullet}, B^{\bullet}) := R^i Hom(A^{\bullet}, B^{\bullet})$.

So far we only considered the global Hom functor, but in the case of sheaves one can define a local version [104] $\mathcal{H}om$,

$$R\mathcal{H}om_{\mathcal{O}_X} : D^0(X) \times D^b(X) \longrightarrow D^b(X), \qquad \text{(B.30)}$$

and similar to the global version one has local "ext" sheaves,

$$\mathcal{E}xt^i_{\mathcal{O}_X}(A^{\bullet}, B^{\bullet}) := R^i \mathcal{H}om_{\mathcal{O}_X}(A^{\bullet}, B^{\bullet}). \qquad \text{(B.31)}$$

Derived tensor product again we start by recalling some standard facts,

i) For any sheaf A, the functor $A \otimes \dots$ is right exact, and we say A is flat if $A \otimes \dots$ is exact.

ii) For any coherent sheaf A, there is a flat resolution of finite length

$$\cdots \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_0 \longrightarrow A \longrightarrow 0, \qquad \text{(B.32)}$$

where $\mathcal{F}_i$'s are flat sheaves.

iii) One can define the tensor product of two complexes $A^{\bullet} \otimes B^{\bullet}$ as a double complex.

iv) A flat complex is defined as complex $\mathcal{P}^\bullet$, which the functor $\mathcal{P}^\bullet \otimes \ldots$, maps acyclic complexes to acyclic complexes (or equivalently quasi-isomorphism to quasi-isomorphism).

v) A bounded above (in particular bounded) complex of flat sheaves is a flat complex. If we have bounded complex of coherent sheaves, $A^\bullet$, then (using point (ii)) one can find a quasi-isomorphism $\mathcal{P}^\bullet \longrightarrow A^\bullet$. If $\mathcal{P}^\bullet$ is both flat and acyclic, then $A^\bullet \otimes \mathcal{P}^\bullet$ is again acyclic for any complex $A^\bullet$.

From the facts given above, and comparison with the general definition, for the functor $F_{A^\bullet} := A^\bullet \otimes \ldots$ we can define the derived tensor product with $K_F = K(\mathcal{P}^\bullet)$,

$$RF_{A^\bullet} := A^\bullet \otimes^L \cdots : D^b(X) \longrightarrow D^b(X). \tag{B.33}$$

Note that the process of defining derived tensor product is symmetric, and we could define it using the first variable. Also please note that by the definition of flat complexes, and the way we defined derived tensor product, if there is a quasi-isomorphism $A^{bullet} \xrightarrow{qis} B^\bullet$, then we have a functor isomorphism $F_{A^\bullet} \sim F_{B^\bullet}$. So naturally the derived tensor product descends to a well defined functor in derived category relative to the first variable,

$$\cdots \otimes^L \cdots : D^b(X) \times D^b(X) \longrightarrow D^b(X). \tag{B.34}$$

Definition B.4.

$$\mathcal{T}or_i(A^\bullet, B^\bullet) := \mathcal{H}^{-i}(A^\bullet \otimes^L B^\bullet). \tag{B.35}$$

Derived pullback Finally we are at the position to define the left derived functor for the pullback of a morphism $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$. As before, we recall some basic facts and then compare with the general definition.

i) Recall that the pull back of a sheaf under f is defined as,

$$f^*(\mathcal{F}) := \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{F}. \tag{B.36}$$

ii) We have projective resolution for every coherent sheaf,

$$\cdots \longrightarrow \mathcal{P}^1 \longrightarrow \mathcal{P}^0 \longrightarrow \mathbb{F} \longrightarrow 0. \tag{B.37}$$

This induces a quasi-isomorphism for any bounded complex of coherent sheaves (at least bounded above) we get a quasi-isomorphism $\mathcal{P}^\bullet \xrightarrow{qis} \mathcal{F}^\bullet$.

So by combining these facts and what we learned for derived tensor product we can write,

$$Lf^*(\mathcal{F}^\bullet) := \mathcal{O}_X \otimes^L_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{F}^\bullet,$$
$$L^i f^*(\mathcal{F}^\bullet) := \mathcal{H}^i(Lf^*(\mathcal{F}^\bullet)). \tag{B.38}$$

• **Important identities** Here we collect the identities that are going to be useful in the calculations throughout this dissertation.

Lets start with the following general theorem,

Theorem B.5. *Suppose $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{C}$ be functors between abelian categories such that $G(K_F) \subset K_G$ (look at the definition of derived functors). Then we get the following identity,*

$$R(G \circ F) = RG \circ RF. \tag{B.39}$$

This theorem looks pretty simple, but it allows us to combine derived functors. Basically it says there is a spectral sequence,

$$E_2^{p,q} := R^p G(R^q(F)) \Longrightarrow E_\infty^{p+q} := R^{p+q} G \circ F. \tag{B.40}$$

Here we write some of the applications. First lets consider the direct image of a bounded complex,

$$R^i f_*(\mathcal{H}^j(\mathcal{F}^\bullet)) \Rightarrow R^{i+j} f_* \mathcal{F}^\bullet. \tag{B.41}$$

Obviously we can write a similar spectral sequence formula to compute the derived functor of complexes. Another example is the global section functor over a variety X, $\Gamma : Coh(X) \longrightarrow \mathcal{A}b$. The direct images of this functor are just the cohomology of sheaves [104], i.e. $R^i\Gamma(\mathcal{F}) = H^i(X, \mathcal{F})$. Now let combine this with the direct image functor induced by a proper morphism $f : X \longrightarrow Y$,

$$\Gamma_Y : Coh(Y) :\longrightarrow point, \quad \Gamma_X = \Gamma_Y \circ f_* : Coh(X) \longrightarrow point,$$
$$R\Gamma_X(\mathcal{F}) = R\Gamma_Y \circ Rf_*(\mathbb{F}),$$
$$E_2^{p,q} = H^p(Y, R^q f_* \mathcal{F}) \Rightarrow E_\infty^{p+q} = H^{p+q}(X, \mathcal{F}). \tag{B.42}$$

The last line is precisely the Leray spectral sequence. As the final example consider the relation between local extension $\mathcal{E}xt$, and the global extension Ext,

$$R\Gamma \circ R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = RHom_{D^b(X)}(\mathcal{F}^\bullet, \mathcal{G}^\bullet). \tag{B.43}$$

In particular if we apply this to concentrated complexes at zero position (i.e. a single coherent sheaf), we get the following famous "local to global" identity,

$$H^i(X, \mathcal{E}xt^j_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})) \Rightarrow Ext^{i+j}_X(\mathcal{F}, \mathcal{G}). \tag{B.44}$$

.

Theorem B.6 (Base change formula)**.** *Consider the following commutative diagram of proper morphisms,*

$$\begin{array}{ccc} X & \xrightarrow{g} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{g'} & Y' \end{array}$$

Then, in general, there is a morphism of functors ,

$$Lf'^* Rg'_* \longrightarrow Rf_* Lg^*. \tag{B.45}$$

In particular if f (g) is flat, then f' (g') is flat, and the above morphism is actually an isomorphism of functors.

One of the main properties of the Fourier-Mukai functor is its compatibility with base change, and therefore the theorem above will be very useful.

Definition B.7 (Dualizing Complex)**.** Consider a proper morphism $f_X \longrightarrow Y$, it's dualizing complex is defined as,

$$Hom_{D^b(Y)}(Rf_* \mathcal{F}^\bullet, \mathcal{G}^\bullet) = Hom_{D^b(X)}(\mathcal{F}^\bullet, f^! \mathcal{G}^\bullet). \tag{B.46}$$

In particular it satisfies the identities,

$$f^! \mathcal{G}^\bullet = Lf^* \mathcal{G} \otimes^L f^! \mathcal{O}_Y, \tag{B.47}$$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle h}\searrow & \downarrow{\scriptstyle g} \\ & & Z \end{array} \quad s.t. \quad h = g \circ f \Longrightarrow h^! = f^! \circ g^!. \tag{B.48}$$

So by the first identity we only need to know the dualizing complex of morphism relative to the structure sheaf.

Definition B.8. A morphism is called *Gorenstein* if the dualizing complex is a concentrated complex, i.e. $f^1 \mathcal{O}_Y = \Omega[k]$ for some $k \in \mathbb{Z}$.

There two specific cases that will be useful for us in this dissertation,

Flat Fibration In this case $f^!\mathcal{O}_Y = \omega_{X/Y}[n]$, where n is the relative dimension (i.e. the dimension of the fibers), and $\omega_{X/Y} = \omega_X \otimes f^*\omega_Y$.

Complete intersection By this we mean an inclusion morphism $f : X \hookrightarrow Y$ where X is a complete intersection of varieties in Y. In this case $f^!\mathcal{O}_Y = det(\mathcal{N})[-d]$, where $\mathcal{N}$ is the normal bundle, and d is the codimension of X is Y.

The definition above is called Grothendieck-Verdier duality, and it is a general form of Serre duality. There is also a local version of this duality,[1]

$$R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*\mathcal{F}^\bullet, \mathcal{G}^\bullet) = Rf_*R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}^\bullet, Lf^*\mathcal{G}^\bullet \otimes^L f^!\mathcal{O}_Y). \tag{B.49}$$

Definition B.9. One can define derived dual of a complex $\mathcal{F}^\bullet \in D^b(X)$ as,

$$\mathcal{F}^{\bullet\vee} := R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{O}_X). \tag{B.50}$$

Theorem B.10.

$$R\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \simeq R\mathcal{H}om(\mathcal{O}_X, \mathcal{F}^{\bullet\vee} \otimes^L \mathcal{G}^\bullet) \simeq \mathcal{F}^{\bullet\vee} \otimes^L \mathcal{G}^\bullet. \tag{B.51}$$

Theorem B.11. $Rf_* \dashv Lf^*$,

$$RHom_{D^b(Y)}(\mathcal{F}^\bullet, Rf_*\mathcal{G}^\bullet) \simeq RHom_{D^b(X)}(Lf^*\mathcal{F}^\bullet, \mathcal{G}^\bullet), \tag{B.52}$$
$$R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}^\bullet, Rf_*\mathcal{G}^\bullet) \simeq Rf_*R\mathcal{H}om_{\mathcal{O}_X}(Lf^*\mathcal{F}^\bullet, \mathcal{G}^\bullet). \tag{B.53}$$

Theorem B.12 (Projection Formula)**.**

$$Rf_*(Lf^*\mathcal{F}^\bullet \otimes^L \mathcal{G}^\bullet) = \mathcal{F}^\bullet \otimes^L Rf_*\mathcal{G}^\bullet. \tag{B.54}$$

Theorem B.13. *From theorem B.6 and the commutative diagram below for a* projective morphism f,

$$\begin{array}{ccc} f^{-1}(p) & \xhookrightarrow[i_f]{} & X \\ {\scriptstyle f_p}\downarrow & & \downarrow{\scriptstyle f} \\ p & \xhookrightarrow[i]{} & Y \end{array} \tag{B.55}$$

[1]The reader can find the proof of the following identities in [32] Appendix A and Appendix C.

we get the following results when $\mathcal{F} \in Coh(X)$. They will be very useful in many cases, and also give a rather clear intuitive picture about the direct images,

$$
\begin{aligned}
Li^* Rf_* \mathcal{F} &\longrightarrow Rf_{p*}(Li_f^* \mathcal{F}), \\
\phi^j : (Li^* Rf_* \mathcal{F})^j = Tor_{-j}^{i^{-1}\mathcal{O}_Y}(Rf_* \mathcal{F}, \mathcal{O}_p) &= R^j f_* \mathcal{F} \otimes \mathcal{O}_p \longrightarrow H^j(f_p^{-1}(p), i_f^* \mathcal{F}).
\end{aligned}
\tag{B.56}
$$

It is proved in [104] theorem III.12.10, that ϕ^j is isomorphism if and only if it is surjective, and $R^j f_* \mathcal{F}$ is locally free if and only if ϕ^{j-1} is surjective.

Appendix C

Integral Functors

In this section we briefly review the main features of integral functors, specially the Fourier Mukai functors which are the important special cases. (For more details, the interested reader can look at [32] and [117]).

Definition C.1. Let $D^b(X)$ and $D^b(Y)$ be the derived categories of the varieties X and Y. Consider the following morphisms,

$$\begin{array}{ccc} & X \times Y & \\ {}^{\pi_X}\swarrow & & \searrow^{\pi_Y} \\ X & & Y \end{array} \tag{C.1}$$

Then the integral functor $\Phi^{\mathcal{P}^\bullet}_{X \to Y}$ is defined in the following way,

$$\Phi^{\mathcal{P}^\bullet}_{X \to Y} : D^b(X) \longrightarrow D^b(Y),$$

$$\Phi^{\mathcal{P}^\bullet}_{X \to Y}(\dots) := R\pi_{Y*}\left(\pi_X^*(\dots) \otimes^L \mathcal{P}^\bullet\right), \tag{C.2}$$

where π_X and π_Y are projections to the corresponding factors, and $\mathcal{P}^\bullet$ is the kernel of the transform. Note that π_X is a flat morphism, so $L\pi_X^* = \pi_X^*$.[1] In particular if the

[1]Such functors are quite similar to the familiar integral transform of functions. Remember that to find the integral transform of $f(x)$ with $x \in \mathbb{R}^1$ we first consider it as a function in a product space space $\mathbb{R}^1 \times \mathbb{R}^1$. This is similar to the pull back π_X^* above. Then we multiply $f(x)$ with a kernel $K(x, y)$ which is the function in $\mathbb{R}^1 \times \mathbb{R}^1$, this part is similar to the tensor product in the formula above, finally we integrate over x, $g(y) = \int dx f(x)K(x, y)$, which is analogues to the push forward $R\pi_{Y*}$.

integral transform of a sheaf $\mathcal{E}$ (consider it as complex which is only non-zero at the zero entry, i.e. concentrated on the zero position) is concentrated the ith position, it is called a WIT_i sheaf.

Note that any integral functor is a composition of three exact functors in derived categories, derived inverse image, derived tensor product and derived direct image. So $\Phi^{\mathcal{P}^\bullet}_{X\to Y}$ is also an exact functor. In particular, to any short exact sequence there is an associated long exact sequence induced by that integral functor.

We are particularly interested in "relative" integral transforms. Suppose $\Phi^{\mathcal{K}}_{X\to Y} : D^b(X) \longrightarrow D^b(Y)$ is an integral transform, then for any variety T, the corresponding relative integral functor (relative to T) $\Phi^{\mathcal{K}^\bullet_T}_{X\times T\to Y\times T}$ is defined as

$$\begin{array}{ccccc} & & X\times Y\times T & & \\ & \swarrow^{\pi_{X\times T}} & \downarrow{\scriptstyle \pi_{X\times Y}} & \searrow^{\pi_{Y\times T}} & \\ X\times T & & X\times Y & & Y\times T \end{array}$$

$$\Phi^{\mathcal{K}^\bullet_T}_T(\dots) := R\pi_{Y\times T_*}\left(\pi^*_{X\times T}(\dots)\otimes^L \mathcal{K}^\bullet_T\right),$$

$$\mathcal{K}^\bullet_T := \pi^*_{X\times Y}\mathcal{K}^\bullet. \tag{C.3}$$

Now consider a morphism of varieties $f : S \longrightarrow T$, and the induced relative morphisms: $f_X : S\times X \longrightarrow T\times X$ and $f_Y : S\times Y \longrightarrow T\times Y$, then one can prove the following functorial isomorphism ([32]),

$$Lf^*_Y\Phi_T(\mathcal{E}^\bullet) \simeq \Phi_S\left(Lf^*_X\mathcal{E}^\bullet\right), \tag{C.4}$$

with $\mathcal{E}^\bullet \in D^b(X\times T)$. In particular if $j_t : t \longrightarrow T$ is the inclusion of a point t, then the identity above gives,

$$Lj^*_t\Phi_T(\mathcal{E}^\bullet) = \Phi_t(Lj^*_t\mathcal{E}^\bullet). \tag{C.5}$$

This has important consequences: first of all if $\mathcal{E}$ is a sheaf, one can prove (by checking the spectral sequences of the combined functors),

$$\Phi^{n_m}_t(j^*_t\mathcal{E}) \simeq j^*_t\Phi^{n_m}_T(\mathcal{E}), \tag{C.6}$$

where n_m is the maximal integer that either $\Phi^{n_m}_t$ or $\Phi^{n_m}_T$ is non-zero. Moreover, if both $\mathcal{E}$ and $\Phi^i_T(\mathcal{E})$ are flat over T, then $\mathcal{E}_t$ is WIT_i relative to Φ_t if and only if $\mathcal{E}$

is WIT_i relative to Φ_T ([32] Corollary 1.9 part 3). This is an important point, and when we want to describe the Fourier-Mukai transform of vector bundles which are unstable over some non-generic elliptic fibers, or when we need to deal with general coherent sheaves, it is going to help us.

Finally we mention that there are similar result for non-trivial fibration ([32] Chapter 6), which we discuss briefly later. For now, let's move on to review Fourier-Mukai functors briefly.

Definition C.2. A Fourier Mukai functor is an integral functor which is also an exact equivalence.

Probably the first important point about Fourier-Mukai functors is that any equivalence can be written as Fourier-Mukai.

Theorem C.3 (Orlov's representability theorem)**.** *Let X and Y be two smooth projective varieties, and let*

$$F : D^b(X) \longrightarrow D^b(Y)$$

be a fully faithful exact functor. If F admits right and left adjoint functors, then there exists an object $\mathcal{P}^\bullet \in D^b(X \times Y)$ unique up to isomorphism such that F is isomorphic to a Fourier Mukai functor $\Phi^{\mathcal{P}}_{X\to Y}$.

There is a partial inverse to this theorem, due to Bondal and Orlov, which states when an integral functor is indeed fully faithful, i.e. it puts constraints over the kernel of the transform ([32] Theorem 2.56).

Theorem C.4. *Let X and Y be smooth projective varieties. Consider $\Phi^{\mathcal{P}^\bullet}_{X\to Y} : D^b(X) \longrightarrow D^b(Y)$ with $\mathcal{P}^\bullet$ in $D^b(X \times Y)$. Then $\Phi^{\mathcal{P}^\bullet}_{X\to Y}$ is a fully faithful functor if and only if $\mathcal{P}^\bullet$ is a strongly simple object over X, i.e.*

$$\begin{aligned} &Hom^i_{D^b(Y)}\left(Lj^*_{x_1}\mathcal{P}^\bullet, Lj^*_{x_2}\mathcal{P}^\bullet\right) = 0 \quad unless \quad x_1 = x_2 \quad and \quad 0 \le i \le dimX; \\ &Hom^0_{D^b(Y)}\left(Lj^*_{x}\mathcal{P}^\bullet, Lj^*_{x}\mathcal{P}^\bullet\right) = \mathbb{C}. \end{aligned} \tag{C.7}$$

In addition, if $Lj^*_x\mathcal{P}^\bullet$ is a special object of $D^b(Y)$, i.e. $Lj^*_x\mathcal{P}^\bullet \otimes K_Y \simeq Lj^*_x\mathcal{P}^\bullet$, then $\Phi^{\mathcal{P}^\bullet}_{X\to Y}$ is an equivalence.

In particular if both X and Y are smooth Calabi-Yau varieties, and the kernel is a strongly simple object, then the corresponding integral functor is a Fourier-Mukai functor.

It is worth to mention another very important property of Fourier-Mukai functors, and that is these kind of integral functors are sensitive to smoothness and "Calabi-Yau'ness," and dimension. In other words, if two varieties X and Y are Fourier-Mukai partners (their derived category are equivalent), then X is smooth if and only if Y is smooth (proven by Serre's criterion on regular local rings of finite homological dimension), and X is Calabi-Yau if and only if Y is Calabi-Yau (this is proven by using Grothendieck-Verdier duality), and both of them must have the same dimension. There are also other geometrical constraints which are induced by the equivalence condition, but we ignore them here ([32] Theorem 2.37).

We finish this section by quickly deriving the inverse transform of a Fourier-Mukai functor $\Phi^{\mathcal{P}^\bullet}_{X\to Y}$. Since for an equivalence of categories, the adjoint functor is actually the inverse functor, we can find it easily for the Fourier Mukai functor as follows,

$$\begin{aligned} RHom_{D^b(Y)}(\Phi^{\mathcal{P}^\bullet}_{X\to Y}(\mathcal{F}^\bullet), \mathcal{G}^\bullet) &= RHom_{D^b(X\times Y)}\left(\pi_X^*\mathcal{F}^\bullet, \pi_Y^*\mathcal{G}^\bullet \otimes^L \mathcal{P}^{\bullet\vee} \otimes \pi_X^*\omega_X[n]\right) \\ &= RHom_{D^b(X)}\left(\mathcal{F}^\bullet, R\pi_{X*}\left(\pi_Y^*\mathcal{F}^\bullet \otimes^L \mathcal{P}^{\bullet\vee} \otimes \pi_X^*\omega_X[n]\right)\right) \\ &= RHom_{D^b(X)}\left(\mathcal{F}^\bullet, \Phi^{\mathcal{P}^{\bullet\vee}\otimes\pi_X^*\omega_X[n]}_{Y\to X}(\mathcal{G}^\bullet)\right), \end{aligned} \tag{C.8}$$

where $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ are generic objects of derived category of varieties X and Y, n is the dimension of both X and Y,[2] and ω_X is the canonical sheaf of X. Therefore the "inverse transform" is itself a Fourier Mukai functor,

$$\Phi^{\mathcal{P}^{\bullet\vee}\otimes\pi_X^*\omega_X[n]}_{Y\to X}. \tag{C.9}$$

[2] Actually uniqueness of the inverse implies the dimension of X and Y must be the same.

Appendix D

Exotic Examples in TSD

There are many exotic examples that we ecountered when we were trying to find "good" TSD pairs where both sides are elliptically fibered and the vector bundles are stable. These examples include cases with apparently perfect TSD pairs but with with different effective theory (Type A), or cases with naively different geometries, but in fact we ended up with the initial set up but in different algebraic description (Type B).

D.1 Type A

In this section we present some of the exotic cases we encountered during the search for finding "good examples" of stable, smooth vector bundles over bases that are Weierstrass elliptic fibrations. All of these examples pass the usual necessary conditions for stability such as $h^0(V) = 0$ and Bogomolov topological constraint, but either the spectrum charged hypermultiplets of the 4d effective are different or the total moduli is not conserved. By using careful Fourier-Mukai analysis we can show that the first example is indeed unstable, so it explains the discrepancy, but the other two are perfectly stable vector bundles,[1] and we are unable to explain the reason. In the third example that spectrum match on both sides one may suggest that the existence of the flux (which must exist due to the generically non-reduced spectral cover) may stabilize the moduli space.

[1] At least up to the point that we could actually check everything.

For example, consider the following GLSM,

x_i							Γ^j	Λ^a					p_l	
3	2	1	0	0	0	0	−6	1	0	1	1	1	−1	−3
0	0	−2	1	1	0	0	0	1	3	−2	2	1	−1	−4
0	0	−2	0	0	1	1	0	0	3	−2	3	0	−1	−3

(D.1)

with the second Chern classes of the Calabi-Yau (X) and the vector bundle (V) as:

$$\begin{aligned} c_2(X) &= 11\sigma^2 + 2\sigma D_1 + 2\sigma D_2 - 3D_1^2 - 4D_1D_2 - 3D_2^2 = 24\sigma D_1 + 24\sigma D_2 - 4D_1D_2, \\ c_2(V) &= 3\sigma^2 + 11\sigma D_1 + 9\sigma D_2 - D_1^2 - 6D_1D_2 - 6D_2^2 = 17\sigma D_1 + 15\sigma D_2 - 6D_1D_2, \end{aligned}$$

where σ, D_1, and D_2 are the section and base divisors correspondingly, with $D_1^2 = D_2^2 = 0$, and $D_1D_2 = f$ the class of the generic fiber f. Anomaly cancellation is not satisfied in the strong sense, but we can still make sense of it at least as a Heterotic string theory (may be not GLSM, but well defined as Heterotic string theory). Again we embed this GLSM into a larger one:

x_i									Γ^j			Λ^a					p_l	
3	2	1	0	0	0	0	1	0	−6	−1	0	1	0	1	1	1	−1	−3
0	0	−2	1	1	0	0	−3	1	0	3	−1	1	3	−2	2	1	−1	−4
0	0	−2	0	0	1	1	−2	1	0	2	−1	0	3	−2	3	0	−1	−3

(D.2)

with the degrees of the monad maps is as follows:

F^1					F^2				
0	1	0	0	0	2	3	2	2	2
0	−2	3	−1	0	3	1	6	2	3
1	−2	3	−2	1	3	0	5	0	3

(D.3)

After exchanging Γ^2, Γ^3 (degree $||G_2||$, $||G_3||$respectively) with F_1^1, F_2^1 respectively, and integrating out the repeated entries, the dual $(\widetilde{X}, \widetilde{V})$ can be written as follows:

x_i							Γ^j	Λ^a					p_l	
3	2	1	0	0	0	0	−6	1	0	1	1	1	−1	−3
0	0	−3	1	1	1	0	0	0	4	−2	2	1	−1	−4
0	0	−2	0	0	1	1	0	0	3	−2	3	0	−1	−3

(D.4)

The dual geometry is perfectly smooth and anomaly cancellation condition can also be make sense as before. However, the spectrum of charged scalars are not the same (i.e, $h^1(V) = 121$ while $h^1(\widetilde{V}) = 101$. So there should be a problem. We can argue that it is related to stability.

After a detailed calculation of Fourier-Mukai transform of V, it becomes clear that $FM^1(V)$ is of relative rank 1 and degree 2. On the other hand $FM^0(V)$ is also non-zero with relative rank and degree 1 and -1. It is well known that Fourier-Mukai transformation of a sheaf of relative rank and degree (n, d) is *complex* of relative rank and degree $(d, -n)$. So it is clear from the above data that the restriction of V on a generic elliptic fiber E is roughly of the form $\mathcal{O}_E(\sigma) \oplus \mathcal{V}_2$, where $\mathcal{V}_2$ is a rank 2 irreducible bundle of degree -1 on E. Obviously it tells us that the bundle must be unstable because it is unstable on generic fibers (even though it seems $h^0(V) = 0$). As a sanity check we can compute the rank of $\pi_* V$ and $\pi_*(V \otimes \mathcal{O}_X(\sigma))$, and they are 1 and 3 respectively. This is consistent because $h^*(\mathcal{V}_2) = (0, 1)$.[2] A similar statement can be made about the TSD set up.

D.2 Type B

In this section, we present an example of a TSD pair in which the geometries (X, V) and $(\tilde{X}, \tilde{V})$ are actually equivalent geometries, even though they are described by different algebraic descriptions (of manifolds and monad bundles). Another interesting feature in this case is that both sides of this "trivial" correspondence are elliptically fibered, however the base manifolds are two *different* Hirzebruch surfaces, $\mathbb{F}_0$ and $\mathbb{F}_2$. These base surfaces are distinct as complex manifolds but identical as real manifolds (and the elliptic CY 3-fold over these different surfaces is the *same* complex manifold). This demonstrates that even "trivial" TSD correspondences may involve interesting geometric structure.

In the following example the bundle $\widetilde{V}$ on $\widetilde{X}$ as a non-trivial rewriting of bundle V on X. Both of the CY 3-folds are weighted projective space $\mathbb{P}^{123}$ fibered Calabi-Yau 3-folds. For X the base is Hirzebruch surface $\mathbb{F}_0$, i.e, $B_2 = \mathbb{P}^3[2]$ while for $\widetilde{X}$ the base is $\widetilde{B_2} = \left[\begin{array}{c|cc} \mathbb{P}^3 & 1 & 1 \\ \mathbb{P}^1 & 1 & 1 \end{array}\right]$, which is generically $\mathbb{F}_0$ but at special complex structure moduli it "jumps" to become $\mathbb{F}_2$ [116]. A $(0, 2)$ target space map can be found that takes X to $\widetilde{X}$ (this can be achieved by adding a $\mathbb{P}^1$ to the configuration as usual for a $U(1)$-changing TSD pair). On this manifold, both the tangent bundle

[2] One can also get the same numbers from semi stable bundles with rank 3 and degree zero, so they are just necessary conditions.

as well as non-tangent bundles will be studied.

D.2.1 Non-trivial rewriting with tangent bundle

Let us first consider the case of a deformation of the tangent bundle. The GLSM charge matrix is a general deformation of $(X, V = TX + \mathcal{O}^{\oplus 2})$ and can be written as follows:

x_i							Γ^j		Λ^a							p_l	
3	2	1	0	0	0	0	−6	0	3	2	1	0	0	0	0	−6	0
0	0	−2	1	1	1	1	0	−2	0	0	−2	1	1	1	1	0	−2

(D.5)

Following the procedure described in previous section, we will end up with the new charge matrix of the target space dual $(\widetilde{X}, \widetilde{V})$ as:

x_i									Γ^j			Λ^a							p_l	
3	2	1	0	0	0	0	0	0	−6	0	0	3	2	1	0	0	0	0	−6	0
0	0	−2	1	1	1	1	0	0	0	−1	−1	0	0	−2	1	1	0	2	0	−2
0	0	0	0	0	0	0	1	1	0	−1	−1	0	0	0	0	0	1	0	0	−1

(D.6)

The number of both charged and uncharged geometric moduli of the theories on these two manifolds are the same, which suggests that they are indeed target space dual to each other. The number of degrees of freedom is given by:

$$\begin{aligned} h^*(V) = (0, 241, 1, 0) \quad & h^{1,1}(X) + h^{2,1}(X) + h^1(End_0(V)) = 3 + 243 + 1074 = 1320, \\ h^*(\widetilde{V}) = (0, 241, 1, 0) \quad & h^{1,1}(\widetilde{X}) + h^{2,1}(\widetilde{X}) + h^1(End_0(\widetilde{V})) = 3 + 243 + 1074 = 1320. \end{aligned} \tag{D.7}$$

Calculate twist of V and $\widetilde{V}$

Starting with the Heterotic theory, without loss of generality, the second Chern class can splits as (3.44) and the Heterotic Bianchi identity will implies further that η can be parameterized as (3.45) with the twist of the theory $T' = T$. In order to get the twist in our example, one can first calculate the second Chern class of V and $\widetilde{V}$ as

$$\begin{aligned} c_2(V) &= c_2\left(TX\right) = 11J_1^2 + 2J_1J_2 - 2J_2^2, \\ c_2\left(\widetilde{V}\right) &= c_2\left(\widetilde{TX}\right) = 11J_1^2 + 2J_1J_2 - 3J_2^2 + 2J_2J_3. \end{aligned} \tag{D.8}$$

For both X and $\widetilde{X}$, the section can be parameterized as $\sigma = J_1 - 2J_2$, and it satisfies the condition $\sigma^2 = -c_1(B)\sigma$. Then by applying (3.44), (3.45) we get:

$$\eta = 24J_2, \quad T = 12J_2 = 6c_1(B) \tag{D.9}$$

for both V and $\widetilde{V}$. This indicates that if we start from a deformation of the tangent bundle, after target space dual we will at least end up with a TSD bundle over the same manifold that is topologically equivalent.

Complex deformation of bundle moduli

We can further compare V and $\tilde{V}$ by analyzing the deformation of these vector bundles. Consider the difference of V and $\widetilde{V}$ defined on B_2 and $\widetilde{B}_2$ in the sequence separately, they are:

$$\begin{aligned} 0 \to V \to \mathcal{O}(0,1)^{\oplus 2} \to \mathcal{O}(0,2) \to 0, \\ 0 \to \widetilde{V} \to \mathcal{O}(0,0,1) \oplus \mathcal{O}(0,2,0) \to \mathcal{O}(0,2,1) \to 0, \end{aligned} \tag{D.10}$$

where V is the kernel of the map with two degree $||1||$ polynomial on $B_2 = \left[\begin{array}{c|c} \mathbb{P}^3 & 2 \end{array}\right]$, and $\widetilde{V}$ is the kernel of the map F with degree $||0,1||$ and $||2,0||$ on $\widetilde{B_2} = \left[\begin{array}{c|cc} \mathbb{P}^3 & 1 & 1 \\ \mathbb{P}^1 & 1 & 1 \end{array}\right]$. However for $(\widetilde{B}_2, \widetilde{V})$, if we first solve the polynomial of degree $||0,1||$ and put the constraint on the second map with degree $||2,0||$, the second map will exactly reduce to a degree $||2||$ polynomial on the manifold $\left[\begin{array}{c|cc} \mathbb{P}^3 & 1 & 1 \end{array}\right]$. So it seems that the bundle moduli in $(\widetilde{B}_2, \widetilde{V})$ are transformed to the complex moduli in (B_2, V). Then one would be interesting to ask whether it is possible that the complex structure and bundle moduli exchange in (X, V) and $(\widetilde{X}, \widetilde{V})$.

Before answering this question, there is an important observation as $\widetilde{B_2}$ is generically $\mathbb{F}_0$, but at a special point it "jumps" to become $\mathbb{F}_2$. Write down the defining equations for $\widetilde{B_2} = \left[\begin{array}{c|cc} \mathbb{P}^3 & 1 & 1 \\ \mathbb{P}^1 & 1 & 1 \end{array}\right]$ as:

$$\begin{aligned} z_0 w_0 + z_1 w_1 &= 0, \\ z_2 w_0 + \left(\sum_{i=0}^{2} a_i z_i + \epsilon z_3 \right) w_1 &= 0, \end{aligned} \tag{D.11}$$

with $[z_0, z_1, z_2, z_3] \in \mathbb{P}^3$ and $[w_0, w_1] \in \mathbb{P}^1$. If $\epsilon \neq 0$, this system defines $\mathbb{F}_0$. When $\epsilon = 0$, a $\mathbb{P}^1$ blows up at $(0,0,0,1) \in \mathbb{P}^3$, which makes it becomes $\mathbb{F}_2$. So the question about whether the complex structure and bundle moduli exchange in (B_2, V) and $(\widetilde{B}_2, \widetilde{V})$ changes to what happens for the geometric moduli of $(\widetilde{B}_2, \widetilde{V})$ when $\widetilde{B_2}$ becomes $\mathbb{F}_2$, and the same for tuning the map of bundle in the (B_2, V) system.

So in calculating the line bundle cohomology in the new system $(\widetilde{B}_2, \widetilde{V})$, we will not only count the dimension of the cohomology group appearing in the sequence but also their polynomial representations and the explicit map. More specifically, we will set $z_3 \in \mathbb{P}^3$ in our calculation to be zero to deform the $\widetilde{B}_2$ to be F_2 and see what happens. In this case, the line bundle cohomologies are $h^*(\mathcal{O}(0,1)) = \{2,0,0\}, h^*(\mathcal{O}(2,0)) = \{9,0,0\}, h^*(\mathcal{O}(1,1)) = \{12,0,0\}$ with and without turning the base manifold. Furthermore, we can check that the cohomology of the bundle $h^*(\widetilde{B}_2, \widetilde{V}) = \{4,5,0\}$ will not be effected by the tuning. On the other hand, we can also tune the complex structure of the map ($x_7 = 0$ in F) in defining the map of V in the (B_2, V) system. Again the deformation of map does not change the bundle valued cohomology $h^*(B_2, V) = \{1,2,0\}$.

D.2.2 Non-trivial rewriting with general vector bundle

Similarly, we can consider another example with the same manifolds but different bundles. Again, we start from the following manifold with charge matrix fo the form (X,V):

x_i							Γ^j		Λ^a					p_l	
3	2	1	0	0	0	0	−6	0	4	2	0	0	0	−6	0
0	0	−2	1	1	1	1	0	−2	0	−2	2	2	1	0	−3

(D.12)

The second Chern classes of (X, TX) and (X, V) are given by

$$\begin{aligned} c_2(TX) &= 11J_1^2 + 2J_1J_2 - 2J_2^2, \\ c_2(V) &= 8J_1^2 + 4J_1J_2 - 2J_2^2, \end{aligned} \tag{D.13}$$

which satisfy the c_2 condition $c_2(V) \leq c_2(TX)$. The target space dual of this theory is given by the form of $(\widetilde{X}, \widetilde{V})$:

x_i									Γ^j			Λ^a					p_l	
3	2	1	0	0	0	0	0	0	−6	0	0	4	2	0	0	0	−6	0
0	0	−2	1	1	1	1	0	0	0	−1	−1	0	−2	1	3	1	0	−3
0	0	0	0	0	0	0	1	1	0	−1	−1	0	0	1	0	0	0	−1

(D.14)

with second Chern class as:

$$\begin{aligned} c_2\left(\widetilde{TX}\right) &= 11J_1^2 + 2J_1J_2 - 3J_2^2 + 2J_2J_3, \\ c_2\left(\widetilde{V}\right) &= 8J_1^2 + 4J_1J_2 - 3J_2^2 + 2J_2J_3. \end{aligned} \tag{D.15}$$

Once again we get their twists of the base to be the same:

$$\eta = 20J_2, \quad T = 8J_2$$

for both V and $\widetilde{V}$. These result indicates that this target space dual is just a kind of rewriting of the origin (X, V).