NONLINEAR FORCED RESPONSE OF CIRCULAR CYLINDRICAL SHELLS

by

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(ABSTRACT)

A combination of the Galerkin procedure and the method of multiple scales is used to analyze the nonlinear forced response of circular cylindrical shells in the presence of internal (autoparametric) resonances. If ω_f and a_f denote the frequency and amplitude of a flexural mode and ω_b and a_b denote the frequency and amplitude of the breathing mode, the steady-state response exhibits a saturation phenomenon when $\omega_b \approx 2\omega_f$ if the shell is excited by a harmonic load having a frequency Ω near ω_b. As the amplitude f of the excitation increases from zero, a_b increases linearly whereas a_f remains zero until a threshold is reached. This threshold is a function of the damping coefficients and $\omega_b - 2\omega_f$. Beyond this threshold a_b remains constant (i.e., the breathing mode saturates) and the extra energy spills over into the flexural mode. In otherwords, although the breathing mode is directly excited by the load, it absorbs a small amount of the input energy (responds with a small amplitude) and passes the rest of the input energy into the flexural mode (responds with a large amplitude).

TO MY PARENTS

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CHAPTER ONE

EQUATIONS OF MOTION

1.1. Introduction

It was in the nineteenth century that shells began to be subjected to some serious studies. The complications arising in the shell structures came, contrary to the case of beams and plates, from the coupling between different degrees of freedom in a shell due to its curvature resulting in a set of coupled differential equations governing the deformations. Among the pioneers in this field was Rayleigh [1], who in his famous "Theory of Sound" proposed a hypothesis that for sufficiently thin shells the contribution of extensional energy to the total energy must be negligible. Thus he proposed a solution to one of the rather lively debates at the end of the nineteenth century concerning the order of importance of flexural and extensional energy terms in the theory of thin shells. He also used Rayleigh's principle to calculate the approximate natural frequencies of a thin shell from the identification of the strain and kinetic energies.

Despite the above mentioned work of Rayleigh, the foundations of the deformation of thin shell were laid down by Love [2] in a paper published in 1881 following Rayleigh's first article on this subject. Love argued that for a sufficiently thin shell, the extensional energy is the predominant term over the flexural energy, because the former is proportional to the thickness while the latter is proportional to the cube of thickness.

Although the general shell equations can be regarded as known since 1888, and the solution procedure for the free vibrations of finite cylindrical shells was given by Love [2], numerical solutions of the frequency equations began to appear only in the 1930's. Flugge [3] was among the first researchers to do it for cylindrical shells and Forsberg [4] for spherical shells. Arnold and Warburton [5] gave a detailed solution of the boundary-value problem of the free vibration of finite cylindrical shells. Their results matched experimental results extremely well, indicating the soundness of the mathematical model and analysis used. The effect of edge conditions on the natural frequencies of free vibrations of cylindrical shells was studied by Forsberg [4]. The development of modern computers and numerical methods, especially the finite-element method, made it possible for a lot of research to be carried out in this field.

Recently, the problem of the nonlinear vibration of shells has received considerable attention. The sources of the nonlinearities in the governing equations may be geometric, or inertial, or material, or any combination. The geometric nonlinearity stems from nonlinear strain-displacement relations (e.g., mid-plane stretching, large curvatures and large rotations), the inertial nonlinearity may be caused by the presence of concentrated or distributed masses, and the material nonlinearity occurs when the stresses are nonlinear functions of the strains. The nonlinearities appear in the governing partial-differential equation and they may appear in the boundary conditions. However, most of the existing studies of other than composite shells deal with geometric nonlinearities. Except for the studies of Dowell and Ventres [6] and Ginsberg [7], all existing studies deal with linear boundary conditions.

A number of nonlinear governing equations have been proposed for the dynamic response of shells. They include the theories of Donnell [8], Novozhilov [9], Sanders [10] and Reissner [11]. The main differences among these theories are the approximations used in relating the strains and curvatures to the displacements. Donnell's theory is the most widely used of all these theories. Nash and Modeer [12], Nowinski and Ismail [13], Bhattacharya [14], Ramachandran [15] and Sinharay and Banerjee [16] neglected the second invariant of the middle surface strain and derived approximate nonlinear equations describing motions of various shells. This assumption was used by Berger [17] for plates.

The methods of solution of the nonlinear partial differential equations governing shell motion can be broadly classified into three approaches: purely numerical methods, perturbation methods, and a combination of the Galerkin procedure with either perturbation or numerical methods.

An example of the purely numerical approach is the work of Stricklin, Martinez, Tillerson, Hong and Haisler [18] who used the matrix displacement method to predict the nonlinear response of shells of revolution. They used the rather typical energy approach to identify the mass and stiffness matrices of the shell along with a Runge-Kutta procedure to perform the integration in the time domain.

In the second approach, perturbation methods are applied to the partial differential equations and boundary conditions to derive linear sets of equations [7,19], which are solved in succession.

The third and most commonly used approach consists of expanding the dependent variables in terms of a linear combination of the shapes of the deflection with time varying coefficients. These temporal coefficients are treated as generalized coordinates. The Galerkin procedure is used to derive a set of nonlinear ordinary differential equations. Mente [20] numerically solved a set of n nonlinear equations arising from the Galerkin procedure. Atluri [21] employed the method of multiple scales to analyze free oscillations of shells in the absence of internal resonances and Reissner [11], Chen and Babcock [19], and Hui [22] employed the Lindstedt-Poincaré technique.

Since the problem is governed by partial-differential equations, the response, in general, consists of many modes. In fact, using the Galerkin procedure one obtains an infinite set of nonlinear coupled equations describing the time variation of the amplitudes of the infinitely many modes. All existing studies truncate the infinite set of equations to a finite number and many of them keep only one mode.

In the single-mode analyses, researchers focused on either free oscillations or primary-resonance responses to harmonic excitations. The major issue in the problem of free oscillations is the dependence of the frequency on the amplitude of oscillation and whether the nonlinearity is of the hardening type (i.e., the frequency increases with amplitude) or the softening type (i.e., the frequency decreases with amplitude). Since the problem reduces to that of a second-order differential equation with quadratic and cubic nonlinearities, the nonlinearity may be of the softening or hardening type, depending on the quadratic and cubic coefficients, and hence the mode under study [23]. For example, Chu [24] and Nowinski [25] obtained solutions for modes that exhibited a hardening nonlinearity, whereas Chen and Babcock [19] and Hui [22] obtained solutions that exhibited a hardening or softening nonlinearity, depending on the considered mode. In the experimental field, Olson [26], Evensen [27] and Matsuzaki and Kobayashi [28] observed only a softening type nonlinearity, whereas Chen and Babcock [19] observed both softening and hardening behavior, depending on the mode.

Although single-mode analyses can provide information on the type of nonlinearity and can predict some of the nonlinear phenomena exhibited in the response of shells to a harmonic excitation, such as, multiple solutions, jumps, and subharmonic and superharmonic resonances, it cannot predict combinational resonances and what is generally referred to as modal interactions [23]; the latter may provide a coupling or an energy exchange among the system's modes. This coupling can dominate the response of systems having some modes that are involved in internal (autoparametric) resonances, which occur when the linear natural frequencies are commensurable or nearly commensurable. The first studies of modal interactions in the response of shells were initiated by McIvor [29,30], Goodier and McIvor [31], McIvor and Sonstegard [32], and McIvor and Lovell [33]. They analyzed the response of cylindrical and spherical shells to radial and nearly radial impulses, taking into account the coupling of breathing and flexural modes when they are in the ratio of two-to-one. Integrating the governing ordinary-differential equations numerically they found that the energy is continuously exchanged between the internally resonant modes. They also linearized the equation governing the breathing mode and substituted the harmonic breathing response into the equation governing the flexural mode to obtain a Mathieu-type equation, whose solutions indicate the regions of instability of the breathing mode. Bieniek, Fan and Lackman [34] used Donnell's equations and determined an axially symmetric response. Then, they analyzed the stability of this response to asymmetric modes by deriving a Mathieu-type equation.

Here, we analyze the nonlinear response of an infinitely long cylindrical shell to a harmonic excitation when the frequency of the fundamental breathing mode is approximately twice the frequency of a flexural mode. We use the method of multiple scales to fully account for the nonlinear interaction, including the influence of the flexural mode on the breathing mode. We demonstrate the saturation phenomenon in the response of internally resonant shells. This phenomenon was first found by Nayfeh, Mook, and Marshall [35] in the response of ships.

1.2. Equilibrium Approach

Following McIvor [29] and Goodier and McIvor [31], we consider the case of plane strain in which the strain parallel to the generators of the shell is everywhere zero. Thus, the deformation of the shell is identical in every plane perpendicular to the shell axis, and the shell can be considered as being in plane motion. In such a plane, we consider a point P on the undeformed shell midsurface with the polar coordinate (a,θ) which after a time t moves to P^* with the polar coordinates r and ϕ, as shown in Fig. 1. Figure 2 shows an edge of an element of unit width of the shell in both the deformed and undeformed configurations. Let the coordinates of an element of the undeformed midsurface be (a,θ) and $(a,\theta + d\theta)$ and those of the deformed element be (r,ϕ) and $(r + \delta r, \phi + \delta\phi)$. Then, the undeformed length along the

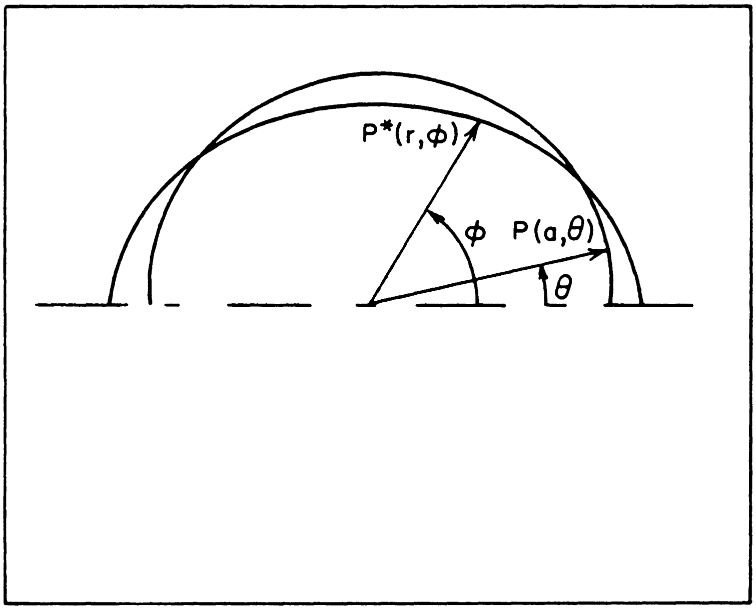

Figure 1. Polar coordinates of a point of the shell which was initially at P and at P* at t*.

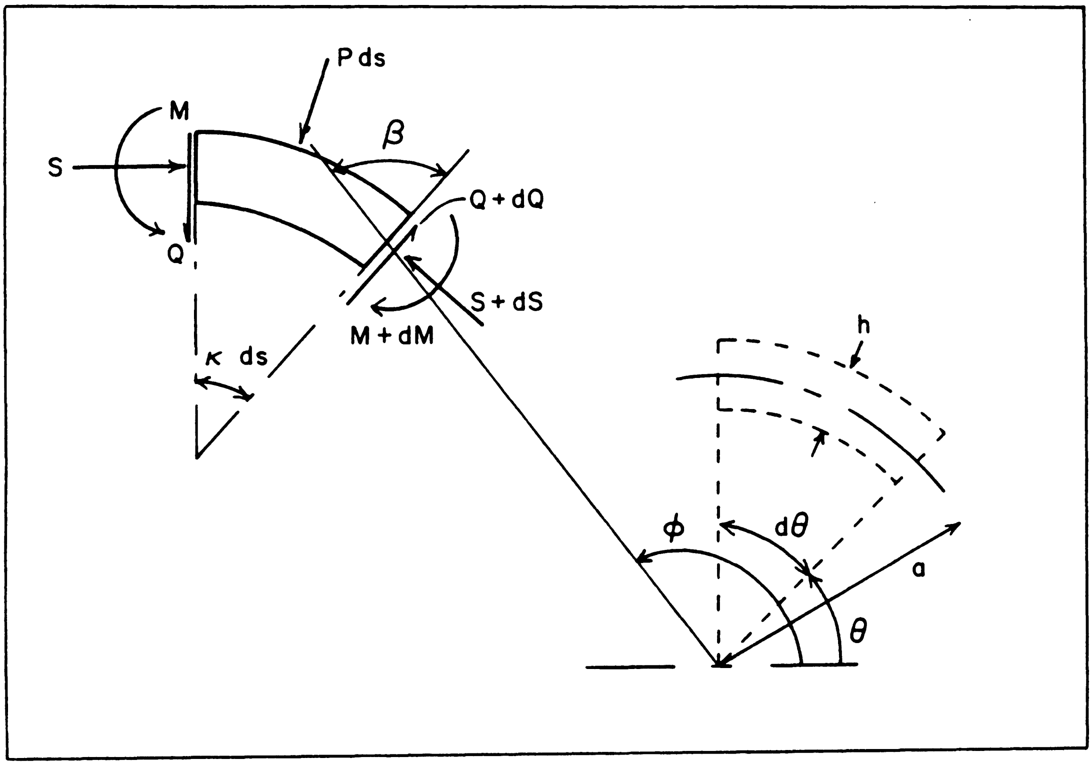

Figure 2. An edge of an element of the deformed and undeformed shell.

midsurface is $ds = a\,d\theta$ and the corresponding deformed length is

$$ds^* = \sqrt{(\delta r)^2 + (r\delta\phi)^2} \tag{1.1}$$

But

$$\delta r = \frac{\partial r}{\partial \theta} d\theta \quad \text{and} \quad \delta\phi = \frac{\partial \phi}{\partial \theta} d\theta \tag{1.2}$$

hence

$$ds^* = \left[\left(\frac{\partial r}{\partial \theta}\right)^2 + r^2\left(\frac{\partial \phi}{\partial \theta}\right)^2\right]^{\frac{1}{2}} d\theta \tag{1.3}$$

Therefore, the extensional strain is given by

$$\varepsilon_o = \frac{ds^* - ds}{ds} = a^{-1}(r'^2 + r^2\phi'^2)^{\frac{1}{2}} - 1 \tag{1.4}$$

where the prime denotes the partial derivative with respect to θ. Thus

$$ds^* = a\,d\theta(1 + \varepsilon_o) \tag{1.5}$$

It follows from differential geometry that the curvature is given by

$$\kappa = [\phi'(r^2\phi'^2 - rr'' + 2r'^2) + \phi''r'r][r'^2 + r^2\phi'^2]^{-3/2} \tag{1.6}$$

1.2.1. Equations of Equilibrium

Let S, Q, and M denote the normal force, transverse shear, and bending moment per unit width of the shell, respectively. They are functions of θ and t. Taking the directions normal and tangential to the deformed midsurface as a reference frame, we find from Fig. 2 that the equation of motion in the direction normal to the midsurface is

$$P(1 + \varepsilon_o) + a^{-1}Q' - \kappa(1 + \varepsilon_o)S = \rho h(-a_r\cos\beta + a_\phi\sin\beta) \tag{1.7}$$

and the equation of motion in the direction tangential to the midsurface is

$$a^{-1}S' + \kappa(1 + \varepsilon_o)Q = -\rho h(a_\phi\cos\beta + a_r\sin\beta) \tag{1.8}$$

where h is the shell thickness,

$$a_r = \ddot{r} - r\dot{\phi}^2 \ , \ a_\phi = r\ddot{\phi} + 2\dot{r}\dot{\phi} \tag{1.9}$$

$$\beta = \tan^{-1} \frac{r'}{r\phi'} \tag{1.10}$$

and the overdot denotes the partial derivative with respect to t. Neglecting rotary inertia and summing moments yields

$$M' + (1 + \varepsilon_o)\, a\, Q = 0 \tag{1.11}$$

1.2.2. Stress Resultants

The usual assumptions of thin shell theory are used here. Straight lines normal to the midsurface before deformation stay straight and normal to the midsurface after deformation, implying that the shear deformations are negligible. The thickness h of the shell is unchanged, and the normal stress is negligible. The ratio (h^2/a^2) is small.

Using the above assumptions and assuming plane strain, we find from Fig. 3 that

$$L_f = (1 - \frac{z}{a})ad\theta \tag{1.12}$$

and

$$L_f + \Delta L_f = (1 + \varepsilon_o)(1 - \kappa z)ad\theta \tag{1.13}$$

Therefore

$$\begin{aligned} \Delta L_f &= \left[(1 + \varepsilon_o)(1 - z\kappa) - (1 - \frac{z}{a})\right]ad\theta \\ &= \left[(1 + \varepsilon_o)z(\frac{1}{a} - \kappa) + \varepsilon_o(1 - \frac{z}{a})\right]ad\theta \end{aligned} \tag{1.14}$$

Hence

$$\varepsilon_\phi = \frac{\varepsilon_o(1 - \frac{z}{a}) + (1 + \varepsilon_o)z(\frac{1}{a} - \kappa)}{(1 - \frac{z}{a})}$$

which, upon expansion for small z/a, yields

$$\varepsilon_\phi = \varepsilon_o - z(1 + \varepsilon_o)(\kappa - \frac{1}{a})(1 + \frac{z}{a} + \frac{z^2}{a^2} + \ldots) \tag{1.15}$$

The shear force S is given by

$$S = -\frac{E}{1-\nu^2}\int_{-h/2}^{h/2} \varepsilon_\phi dz = -E_1 \int_{-h/2}^{h/2} \varepsilon_\phi dz \tag{1.16}$$

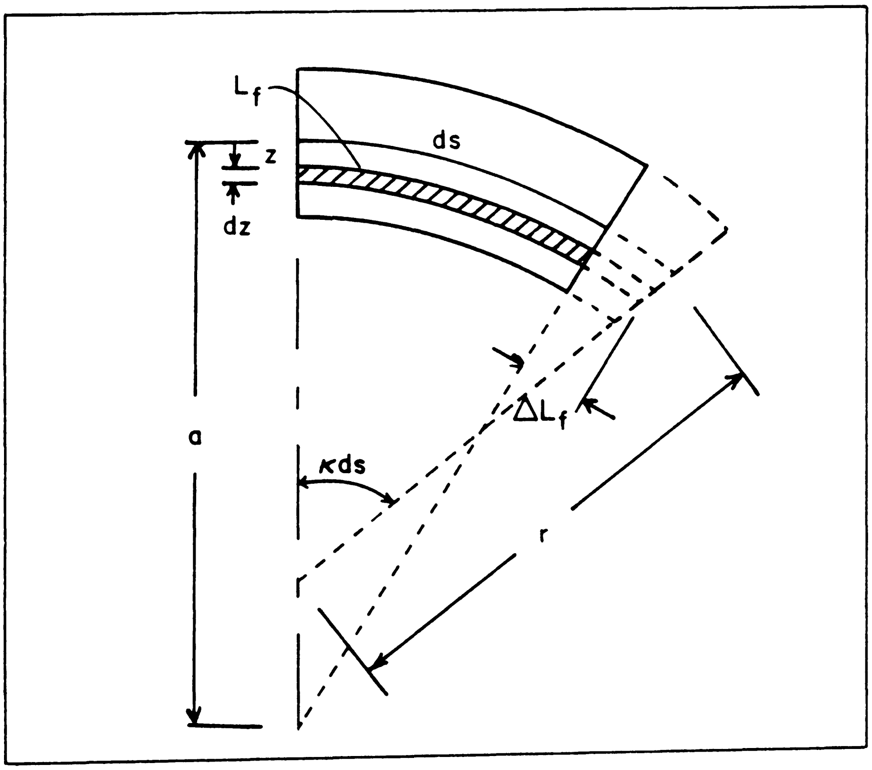

Figure 3. Deformed shell element.

where E is Young's modulus

ν is Poisson's ratio, and

$E_1 = E/(1-\nu^2)$

Substituting (1.15) into (1.16) yields

$$S = -E_1 h\varepsilon_0 + Da^{-1}(1+\varepsilon_0)(\kappa - a^{-1}) + \ldots \qquad (1.17)$$

where the flexural rigidity $D = Eh^3/(12(1-\nu^2))$

The bending moment M is given by

$$M = -E_1 \int_{-h/2}^{h/2} z\varepsilon_\phi dz = D(1+\varepsilon_0)(\kappa - a^{-1}) + \ldots \qquad (1.18)$$

1.2.3. Approximate Equations

We introduce the dimensionless displacement w and time τ defined by

$$w = \frac{a-r}{a}, \quad \tau = \frac{ct}{a} \qquad (1.19)$$

and let

$$\psi = \phi - \theta \qquad (1.20)$$

where $c^2 = E/\rho(1-\nu^2)$

and ρ is the mass density

Substituting (1.19) and (1.20) into (1.4), (1.6) and (1.10) and

expanding the results for small w and ψ, we obtain

$$\varepsilon_0 = \psi' - w + \frac{1}{2}w'^2 - w\psi' + \ldots \qquad (1.21)$$

$$\kappa = \frac{1}{a}(1 + w + w'' + w^2 + 2ww'' + \frac{1}{2}w'^2 - 2\psi'w'' - w'\psi'') + \ldots \qquad (1.22)$$

and

$$\tan\beta = -w' - ww' + w'\psi' + \ldots \qquad (1.23)$$

It follows from (1.11) that

$$Q = -\frac{M'}{a(1+\varepsilon_0)} \qquad (1.24)$$

which upon substitution into (1.7) yields

$$P(1+\varepsilon_o) - \frac{1}{a^2}\left(\frac{M'}{1+\varepsilon_o}\right)' - \kappa(1+\varepsilon_o)S = \rho h(-a_r\cos\beta + a_\phi\sin\beta) \qquad (1.25)$$

Substituting (1.21) and (1.22) into (1.17) and (1.18) yields

$$S = -E_1h(\psi' - w - w\psi' + \frac{1}{2}w'^2) + \frac{D}{a^2}(w + w'') + \ldots \qquad (1.26)$$

$$M = \frac{D}{a}(1 + \psi' - w)(w + w'') + \ldots \qquad (1.27)$$

Substituting (1.26) and (1.27) into (1.25) and using (1.19) yields

$$P(1 + \psi' - w) - \frac{D}{a^3}(w'' + w^{iv}) + \frac{E_1h}{a}(\psi' - w + \frac{1}{2}w'^2 - w\psi' + \psi'w'' - ww''$$

$$+ \psi'^2 - w\psi') - \frac{D}{a^3}(w + w'') + \frac{E_1h}{a}\{-\ddot{w} - \dot{\psi}^2 + [(1-w)\ddot{\psi} - 2\dot{w}\dot{\psi}]w'\} = 0$$

or

$$\ddot{w} + \alpha^2(w^{iv} + 2w'' + w) - \psi' + w = w''(\psi' - w) - \dot{\psi}^2$$

$$+ \psi'^2 - 2w\psi' + w'\ddot{\psi} + \frac{1}{2}w'^2 + \frac{aP}{E_1h}(1 + \psi' - w) \qquad (1.28)$$

where $\alpha^2 = h^2/12a^2$. Substituting (1.19) and (1.25) into (1.8) and using (1.21)-(1.23), (1.26), and (1.27), we obtain

$$\ddot{\psi} - \psi'' + w' = w\ddot{\psi} + 2\dot{w}\dot{\psi} + w'w'' - w\psi'' - w'\psi' - w'\ddot{w} \qquad (1.29)$$

Equations (1.28) and (1.29) can be rewritten in a slightly different form. To this end, we obtain from (1.28) and (1.29) that

$$\ddot{w} = \psi' - w - \alpha^2(w^{iv} + 2w'' + w) + \frac{aP}{E_1h} + \ldots$$

$$\ddot{\psi} = \psi'' - w' + \ldots$$

which when substituted into the right-hand sides of (1.28) and (1.29) yields

$$\ddot{w} + \alpha^2(w^{iv} + 2w'' + w) - \psi' + w = w''(\psi' - w) - \dot{\psi}^2$$

$$+ \psi'^2 - 2w\psi' + w'\psi'' - \frac{1}{2}w'^2 + \frac{a(1-\nu^2)}{Eh}P(1 + \psi' - w) \qquad (1.30)$$

and

$$\ddot{\psi} - \psi'' + w' = w'w'' - 2w'\psi' + 2\dot{w}\dot{\psi} + \frac{a(1-\nu^2)}{Eh}w'P \qquad (1.31)$$

in agreement with those obtained by McIvor [29] and Goodier and McIvor [31]. The difference between (1.30) and (1.31) and equations (1.28) and (1.29) is in the cubic terms, which have been neglected.

1.3. Energy Approach

The kinetic energy per unit width of the shell is

$$T = \frac{1}{2}\rho ha\int_0^{2\pi}\left[\left(\frac{\partial r}{\partial t}\right)^2 + \left(r\frac{\partial\phi}{\partial t}\right)^2\right]d\theta \qquad (1.32)$$

which can be rewritten in dimensionless quantities as

$$T = \frac{1}{2}E_1 ah\int_0^{2\pi}[\dot{w}^2 + (1-w)^2\dot{\psi}^2]d\theta \qquad (1.33)$$

The potential energy consists of the strain energy and the potential of the applied loads. The strain energy U per unit width is

$$U = \frac{1}{2}\,\frac{E}{1-\nu^2}\int_0^{2\pi}\int_{-h/2}^{h/2}\varepsilon_\phi^2(a - z)dzd\theta \qquad (1.34)$$

Substituting (1.15) into (1.34) and using dimensionless variables yields

$$U = \frac{1}{2}E_1 ha\int_0^{2\pi}[\varepsilon_0^2 + \alpha^2a^2(\kappa - \frac{1}{a})^2(1 + \varepsilon_0)^2]d\theta \qquad (1.35)$$

Using (1.21) and (1.22) we rewrite (1.35) in terms of the variables ψ and w as

$$U = \frac{1}{2}E_1 ha\int_0^{2\pi}[(\psi'-w)^2 + \alpha^2(w'' + w)^2 + \psi'w'^2 + 2\psi'w^2 - 2w\psi'^2 - ww'^2]d\theta \qquad (1.36)$$

The potential due to an applied radial pressure loading P per unit width of the shell is

$$W = P\left[\frac{1}{2}\int_0^{2\pi} r^2 d\phi - \pi a^2\right] \tag{1.37}$$

Substituting (1.19) and (1.20) into (1.37) yields

$$W = \frac{1}{2} Pa^2 \int_0^{2\pi} (\psi' - 2w - 2w\psi' + w^2) d\theta \tag{1.38}$$

Using Hamilton's principle, we have

$$\delta\int_{\tau_1}^{\tau_2} \int_0^{2\pi} \{\dot{w}^2 + \dot{\psi}^2 - 2w\dot{\psi}^2 - \psi'^2 + 2w\psi' - w^2 - \alpha^2(w''^2 + 2ww'' + w^2) - w'^2\psi' - 2w^2\psi' + 2w\psi'^2 + ww'^2 + \frac{Pa}{E_1 h}(\psi' - 2w - 2w\psi' + w^2)\} d\theta d\tau \tag{1.39}$$

Using calculus of variation, we obtain exactly eqs. (1.30) and (1.31).

CHAPTER TWO

INEXTENSIONAL OSCILLATIONS

Goodier and McIvor restricted their analysis to impulses with durations much less than the period of the uniform radial mode of vibration, such a restriction made it possible to convert the problem into that of free vibration. Under the assumption of inextensionality and by simplifing the expression for the kinetic energy through neglecting the nonlinear coupling between modes, McIvor [29,30] and Goodier and McIvor [31] produced a numerical solution of the approximate equations of motion. In the present analysis, such assumptions concerning the energy are not used. In fact, these assumptions are not justified since the non-linear coupling produces second order terms in the equations of motion.

It was shown in Chapter 1 that the potential and kinetic energies per unit width of the shell are given by

$$U = \frac{1}{2}\frac{Eha}{1-\nu^2}\int_0^{2\pi}\left[(\psi' - w)^2 + \psi' w'^2 - ww'^2 + 2\psi' w^2 - 2w\psi'^2 + \alpha^2(w'' + w)^2\right]d\theta \tag{2.1}$$

and

$$T = \frac{1}{2}\frac{Eha}{1-\nu^2}\int_0^{2\pi}\left[\dot{w}^2 + (1 - 2w)\dot{\psi}^2\right]d\theta \tag{2.2}$$

Since the shell is closed, w and ψ must be periodic in θ with period 2π. Consequently, they can be expanded in Fourier series as

$$w = \eta_0(\tau) + \sum_{n=1}^{\infty}\left[\eta_n(\tau)\cos n\theta + \zeta_n(\tau)\sin n\theta\right] \tag{2.3}$$

and

$$\psi = \sum_{n=1}^{\alpha}\left[c_n(\tau)\cos n\theta + d_n(\tau)\sin n\theta\right] \tag{2.4}$$

The basic response of the shell is the radial, purely extensional (breathing), mode $\eta_o(\tau)$. We assume that the basic mode will be the breathing mode, which is purely extensional, perturbed by the inextensional modes. Thus, the deviation from the breathing circular mode must satisfy the inextensionality condition*, stated by McIvor [29] as

$$\psi' - w = 0 \tag{2.5}$$

Imposing this condition on (2.3) and (2.4) yields

$$\sum_{n=1}^{\infty} \{[\zeta_n + nc_n]\sin n\theta + [\eta_n - n\, d_n]\cos n\theta\} = 0 \tag{2.6}$$

Thus

$$c_n = -\frac{\zeta_n}{n}\,, \quad d_n = \frac{\eta_n}{n} \tag{2.7}$$

Hence, (2.3) and (2.4) become

$$w = \eta_o(\tau) + \sum_{n=2}^{\infty} [\eta_n(\tau)\cos n\theta + \zeta_n(\tau)\sin n\theta] \tag{2.8}$$

and

$$\psi = \sum_{n=2}^{\infty} [-\frac{\zeta_n}{n}\cos n\theta + \frac{\eta_n}{n}\sin n\theta] \tag{2.9}$$

The first harmonic is omitted in (2.8) and (2.9) because it corresponds to a rigid body translation in the inextensional model.

Substituting (2.8) and (2.9) into (2.1) and (2.2) gives the expressions for the potential and kinetic energies in terms of the η_n and ζ_n, which could be thought of as the generalized coordinates. Consequently, Lagrange's method can be used to derive the equations of motion. To this end, we note that

$$U = \frac{Eha\pi}{2(1-\nu^2)} [2\eta_o^2(1 + \alpha^2) + \sum_{n=2}^{\infty}[\alpha^2(n^2 - 1)^2 - \eta_o(n^2 - 2)](\eta_n^2 + \zeta_n^2)] \tag{2.10}$$

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* A thorough discussion of the inextensionality conditions is given by Lord Rayleigh [1].

$$T = \frac{Eha\pi}{2(1-\nu^2)} \left[2\dot{\eta}_o^2 + \sum_{n=2}^{\infty} \frac{n^2+1}{n^2} (\dot{\eta}_n^2 + \dot{\zeta}_n^2) - \sum_{n=2}^{\infty} \frac{2\eta_o}{n^2} (\dot{\eta}_n^2 + \dot{\zeta}_n^2)\right.$$

$$- \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \sum_{k=2}^{\infty} \frac{\dot{\zeta}_n \dot{\zeta}_m \eta_k}{nm} (\delta_{n-m,k} + \delta_{m-n,k} + \delta_{n+m,k})$$

$$- \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \sum_{k=2}^{\infty} \cdot \frac{\dot{\eta}_n \dot{\eta}_m \eta_k}{nm} (\delta_{n-m,k} + \delta_{m-n,k} - \delta_{n+m,k})$$

$$\left. + 2 \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \sum_{k=2}^{\infty} \frac{\dot{\zeta}_n \dot{\eta}_m \zeta_k}{nm} (\delta_{n+m,k} + \delta_{m-n,k} - \delta_{n-m,k})\right] \tag{2.11}$$

where δ is the Kronecker delta defined by

$$\delta_{\ell,s} = \begin{matrix} 1 & \text{if } \ell = s \\ 0 & \text{if } \ell \neq s \end{matrix}$$

The damping is assumed to be given by a dissipation function

$$D_f = \frac{1}{2} E_1 h\, a\pi\left[2\mu_o \dot{\eta}_o^2 + \sum_{n=2}^{\infty} \mu_n \frac{n^2 + 1}{n^2} (\dot{\eta}_n^2 + \dot{\xi}_n^2)\right] \tag{2.12}$$

The quantities η_o, η_n and ξ_n can be considered as generalized coordinates and when treated as such the use of Lagrange's equations yield the following equations of motion

$$\ddot{\eta}_o + \omega_o^2 \eta_o + 2\mu_o \dot{\eta}_o + \sum_{n=2}^{\infty} \left[\frac{1}{2n^2} (\dot{\eta}_n^2 + \dot{\zeta}_n^2) - \frac{1}{4} (n^2 - 2)(\eta_n^2 + \zeta_n^2)\right] = P_o(\tau) \tag{2.13}$$

$$\ddot{\eta}_n + \omega_n^2\eta_n + 2\mu_n\dot{\eta}_n - \frac{2}{n^2+1}\eta_o\ddot{\eta}_n - \frac{2}{n^2+1}\dot{\eta}_o\dot{\eta}_n - \frac{n^2-2}{\beta_n^2}\eta_o\eta_n$$

$$- \frac{1}{\beta_n^2}\sum_{m=2}^{\infty}\frac{\dot{\eta}_m}{mn}[\dot{\eta}_{(m-n)} + \dot{\eta}_{(n-m)} - \dot{\eta}_{(n+m)}] - \frac{1}{\beta_n^2}\sum_{m=2}^{\infty}\frac{\ddot{\eta}_m}{mn}[\eta_{(m-n)} + \eta_{(n-m)} - \eta_{(n+m)}]$$

$$+ \frac{1}{\beta_n^2}\sum_{m=2}^{\infty}\frac{\dot{\zeta}_m}{mn}[\dot{\zeta}_{(m+n)} + \dot{\zeta}_{(n-m)} - \dot{\zeta}_{(m-n)}] + \frac{1}{\beta_n^2}\sum_{m=2}^{\infty}\frac{\ddot{\zeta}_m}{mn}[\zeta_{(m+n)} + \zeta_{(n-m)} - \zeta_{(m-n)}]$$

$$+ \frac{1}{2\beta_n^2}\sum_{m=2}^{\infty}\frac{\dot{\zeta}_m}{m}\left[\frac{\dot{\zeta}_{(n-m)}}{n-m} + \frac{\dot{\zeta}_{(n+m)}}{n+m} + \frac{\dot{\zeta}_{(m-n)}}{m-n}\right] + \frac{1}{2\beta_n^2}\sum_{m=2}^{\infty}\frac{\dot{\eta}_m}{m}\left[\frac{\dot{\eta}_{(m-n)}}{m-n}\right.$$

$$\left. + \frac{\dot{\eta}_{(n+m)}}{n+m} - \frac{\dot{\eta}_{(n-m)}}{n-m}\right] = P_n(\tau) \qquad (2.14)$$

$$\ddot{\zeta}_n + \omega_n^2\zeta_n + 2\mu_n\dot{\xi}_n - (n^2 - 2)\frac{1}{\beta_n^2}\eta_o\zeta_n - \frac{2}{n^2+1}\eta_o\ddot{\zeta}_n - \frac{2}{n^2+1}\dot{\eta}_o\dot{\zeta}_n$$

$$- \frac{1}{\beta_n^2}\sum_{m=2}^{\infty}\frac{\dot{\zeta}_m}{mn}[\dot{\eta}_{(n-m)} + \dot{\eta}_{(m-n)} + \dot{\eta}_{(n+m)}] - \frac{1}{\beta_n^2}\sum_{m=2}^{\infty}\frac{\ddot{\zeta}_m}{mn}[\eta_{(n-m)} + \eta_{(m-n)}$$

$$+ \eta_{(n+m)}] + \frac{1}{\beta_n^2}\sum_{m=2}^{\infty}\frac{\dot{\eta}_m}{mn}[\dot{\zeta}_{(n+m)} + \dot{\zeta}_{(m-n)} - \dot{\zeta}_{(n-m)}] + \frac{1}{\beta_n^2}\sum_{m=2}^{\infty}\frac{\ddot{\eta}_m}{mn}[\zeta_{(n+m)}$$

$$+ \zeta_{(m-n)} - \zeta_{(n-m)}] - \frac{1}{\beta_n^2}\sum_{m=2}^{\infty}\frac{\dot{\eta}_m}{m}\left[\frac{\dot{\zeta}_{(n-m)}}{n-m} + \frac{\dot{\zeta}_{(m-n)}}{m-n} - \frac{\dot{\zeta}_{(n+m)}}{n+m}\right] = S_n(\tau) \qquad (2.15)$$

where

$$\omega_o^2 = 1 + \alpha^2 \qquad (2.16a)$$

$$\omega_n^2 = \frac{n^2(n^2-1)^2}{(n^2+1)}\alpha^2 \qquad (2.16b)$$

$$\beta_n^2 = \frac{n^2 + 1}{n^2} \qquad (2.16c)$$

$$P_o(\tau) = \frac{(1-\nu^2)}{2Eha\pi} Q_o(\tau) \tag{2.16d}$$

$$P_n(\tau) = \frac{(1-\nu^2)}{4Eha\pi} \frac{n^2}{n^2+1} Q_n(\tau) \tag{2.16e}$$

and

$$S_n(\tau) = \frac{(1-\nu^2)}{4Eha\pi} \frac{n^2}{n^2+1} R_n(\tau) \tag{2.16f}$$

CHAPTER THREE

PERTURBATION ANALYSIS

In this chapter, the method of multiple scales [36,37] is used to derive an asymptotically valid closed form solution for equations (2.14)-(2.16). Through the use of the method of multiple scales, the problem of free and forced inextensional oscillations of thin circular cylindrical shells is addressed in the presence and absence of internal resonances.

In the case of free vibrations with no internal resonance, the method of multiple scales shows that there are no modal interactions. On the other hand, if the frequencies are commensurable (i.e., there is internal resonance), the modal coupling is strongly pronounced. In the case of two-to-one internal resonance, the coupling between the breathing and a flexural mode in the case of free vibrations produces a continuous energy exchange between the coupled modes.

In the case of forced oscillations with a two-to-one internal resonance, the steady-state response exhibits a saturation phenomenon. Let the breathing mode be excited by a force having the frequency Ω near the natural frequency ω_b of the breathing mode. As the amplitude f of the excitation increases, the amplitude a_b of the breathing mode increases linearly with f, whereas the amplitude a_f of the flexural mode remains zero until the excitation amplitude exceeds a threshold ξ_2. Above this threshold, a_b does not increase further and a_f starts to increase. In otherwords, although the breathing mode is being directly excited, it absorbs a small amount of the input energy and spills the rest to the flexural mode which responds with a large amplitude.

3.1. General Treatment

Following the method of multiple scales, one seeks a uniformly valid expansion of the variables in the form

$$\eta_o(\tau;\varepsilon) = \varepsilon\eta_{o1}(T_o,T_1) + \varepsilon^2\eta_{o2}(T_o,T_1) + \dots \tag{3.1a}$$

$$\eta_n(\tau;\varepsilon) = \varepsilon\eta_{n1}(T_o,T_1) + \varepsilon^2\eta_{n2}(T_o,T_1) + \dots \tag{3.1b}$$

and

$$\zeta_n(\tau;\varepsilon) = \varepsilon\zeta_{n1}(T_o,T_1) + \varepsilon^2\zeta_{n2}(T_o,T_1) + \dots \tag{3.1c}$$

$$n = 2, 3, 4, \dots$$

where $T_o = \tau$ is a fast scale, $T_1 = \varepsilon\tau$ is a slow scale, and $\varepsilon << 1$ is a bookkeeping device used to indicate the smallness of the terms. The time derivatives transform according to

$$d/d\tau = D_o + \varepsilon D_1 + \dots \tag{3.2a}$$

and

$$d^2/d\tau^2 = D_o^2 + 2\varepsilon D_o D_1 + \dots \tag{3.2b}$$

where $D_n = \partial/\partial T_n$. Moreover, we order the amplitudes of the excitation and damping coefficients so that

$$P_o \to \varepsilon^2 P_o,\ P_n \to \varepsilon^2 P_n,\ S_n \to \varepsilon^2 S_n,\ \mu_o \to \varepsilon\,\mu_o \text{ and } \mu_n \to \varepsilon\,\mu_n$$

Substituting (3.1) and (3.2) into (2.14)-(2.15) yields

$$\left[D_o^2\eta_{o1} + \omega_o^2\,\eta_{o1}\right]\varepsilon + \left[D_o^2\eta_{o2} + \omega_o^2\eta_{o2} + 2\mu_o D_o\eta_{o1} + 2D_o D_1\eta_{o1}\right.$$

$$\left. + \sum_{n=2}^{\infty}\left\{\frac{1}{2n^2}\left((D_o\eta_{n1})^2 + (D_o\zeta_{n1})^2\right) - \frac{(n^2-2)}{4}\left(\eta_{n1}^2 + \zeta_{n1}^2\right)\right\}\right]\varepsilon^2 = \varepsilon^2 P_o(\tau) \tag{3.3}$$

$$[D_o^2\eta_{n1} + \omega_n^2\eta_{n1}]\varepsilon + \{D_o^2\eta_{n2} + \omega_n^2\eta_{n2} + 2\mu_n D_o\eta_{n1} + 2D_oD_1\eta_{n1} - \frac{2}{(n^2+1)}[(D^2\eta_{n1})\eta_{o1}$$

$$+ (D_o\eta_{o1})(D_o\eta_{n1})] - \frac{(n^2-2)}{\beta_n^2}\eta_{o1}\eta_{n1} - \frac{1}{\beta_n^2}\sum_{m=2}^{\infty}\frac{D_o\eta_{m1}}{mn}[D_o\eta_{(m-n)1}$$

$$+ D_o\eta_{(n-m)1} - D_o\eta_{(n+m)1}] - \frac{1}{\beta_n^2}\sum_{m=2}^{\infty}\frac{D_o^2\eta_{m1}}{mn}[\eta_{(m-n)1} + \eta_{(n-m)1}$$

$$- \eta_{(n+m)1}] + \frac{1}{\beta_n^2}\sum_{m=2}^{\infty}\frac{D_o\zeta_{m1}}{mn}[D_o\zeta_{(m+n)1} + D_o\zeta_{(n-m)1} - D_o\zeta_{(m-n)1}]$$

$$+ \frac{1}{\beta_n^2}\sum_{m=2}^{\infty}\frac{D_o^2\zeta_{m1}}{mn}[\zeta_{(n+m)1} + \zeta_{(n-m)1} - \zeta_{(m-n)1}]$$

$$+ \frac{1}{2\beta_n^2}\sum_{m=2}^{\infty}\frac{D_o\zeta_{mo}}{m}[\frac{D_o\zeta_{(n-m)1}}{(n-m)} + \frac{D_o\zeta_{(n+m)1}}{(n+m)} + \frac{D_o\zeta_{(m-n)1}}{(m-n)}]$$

$$+ \frac{1}{2\beta_n^2}\sum_{m=2}^{\infty}\frac{D_o\eta_{m1}}{m}[\frac{D_o\eta_{(m-n)1}}{(m-n)} + \frac{D_o\eta_{(n+m)1}}{(n+m)} - \frac{D_o\eta_{(n-m)1}}{(n-m)}]\}\varepsilon^2 = \varepsilon^2 P_n(\tau) \tag{3.4}$$

$$[D_o^2\zeta_{n1} + \omega_n^2\zeta_{n1}]\varepsilon + \{D_o^2\zeta_{n2} + 2\mu_n D_o\zeta_{n1} + \omega_n^2\zeta_{n2} + 2D_oD_1\zeta_{n1} - \frac{2}{n^2+1}[(\eta_{o1}D_o^2\zeta_{n1}$$

$$+ (D_o\eta_{o1})(D_o\zeta_{n1})] - \frac{n^2 - 2}{\beta_n^2}\eta_{o1}\zeta_{n1} - \frac{1}{\beta_n^2}\sum_{m=2}^{\infty}\frac{D_o\zeta_{m1}}{mn}[D_o\eta_{(n-m)1}$$

$$+ D_o\eta_{(m-n)1} + D_o\eta_{(n+m)1}] - \frac{1}{\beta_n^2}\sum_{m=2}^{\infty}\frac{D_o^2\zeta_{m1}}{mn}[\eta_{(n-m)1} + \eta_{(m-n)1} + \eta_{(n+m)1}]$$

$$+ \frac{1}{\beta_n^2}\sum_{m=2}^{\infty}\frac{D_o\eta_{m1}}{mn}[D_o\zeta_{(n+m)1} + D_o\zeta_{(m-n)1} - D_o\zeta_{(n-m)1}]$$

$$+ \frac{1}{\beta_n^2}\sum_{m=2}^{\infty}\frac{D_o^2\eta_{m1}}{mn}[\zeta_{(n+m)1} + \zeta_{(m-n)1} - \zeta_{(n-m)1}]$$

$$- \frac{1}{\beta_n^2}\sum_{m=2}^{\infty}\frac{D_o\eta_{m1}}{m}[\frac{D_o\zeta_{(n-m)1}}{(n-m)} + \frac{D_o\zeta_{(m-n)1}}{(m-n)} - \frac{D_o\zeta_{(n+m)1}}{(n+m)}]\}\varepsilon^2 = \varepsilon^2 S_n(\tau) \tag{3.5}$$

Equating the coefficients of ε on both sides of (3.3)-(3.5) yields

$$D_o^2\eta_{o1} + \omega_o^2\eta_{o1} = 0 \qquad (3.6a)$$

$$D_o^2\eta_{n1} + \omega_n^2\eta_{n1} = 0 \qquad (3.6b)$$

$$D_o^2\zeta_{n1} + \omega_n^2\zeta_{n1} = 0 \qquad (3.6c)$$

The solutions of (3.6) can be expressed as

$$\eta_{o1} = A_o(T_1)e^{i\omega_o T_o} + \text{c.c.} \qquad (3.7a)$$

$$\eta_{n1} = A_n(T_1)e^{i\omega_n T_o} + \text{c.c.} \qquad (3.7b)$$

$$\zeta_{n1} = B_n(T_1)e^{i\omega_n T_o} + \text{c.c.} \qquad (3.7c)$$

where c.c. stands for the complex conjugate of the preceding terms.

Equating the coefficients of ε^2 on both sides of (3.3)-(3.5) yields

$$D_o^2\eta_{o2} + \omega_o^2\eta_{o2} = -\ 2D_oD_1\eta_{o1} - 2\mu_nD_o\eta_{o1} - \sum_{n=2}^{\infty} \{\frac{1}{2n^2}\,[(D_o\eta_{n1})^2 + (D_o\zeta_{n1})^2]$$

$$-\ \frac{1}{4}\,(n^2 - 2)(\eta_{n1}^2 + \zeta_{n1}^2)\} + P_o(\tau) \qquad (3.8a)$$

$$D_o^2\eta_{n2}+\omega_n^2\eta_{n2}=-2D_oD_1\eta_{n1}-2\mu_nD_o\eta_{n1}+\frac{2}{(n^2+1)}\{(D_o^2\eta_{n1})\eta_{o1}+(D_o\eta_{o1})(D_o\eta_{n1})\}$$

$$+\frac{(n^2-2)}{\beta_n^2}\eta_{o1}\eta_{n1}+\frac{1}{\beta_n^2}\sum_{m=2}^{\infty}\frac{D_o\eta_{m1}}{mn}[D_o\eta_{(m-n)1}+D_o\eta_{(n-m)1}-D_o\eta_{(n+m)1}]$$

$$+\frac{1}{\beta_n^2}\sum_{m=2}^{\infty}\frac{D_o^2\eta_{m1}}{mn}[\eta_{(m-n)1}+\eta_{(n-m)1}-\eta_{(n+m)1}]$$

$$-\frac{1}{\beta_n^2}\sum_{m=2}^{\infty}\frac{D_o\zeta_{m1}}{mn}[D_o\zeta_{(m+n)1}+D_o\zeta_{(n-m)1}-D_o\zeta_{(m-n)1}]$$

$$-\frac{1}{\beta_n^2}\sum_{m=2}^{\infty}\frac{D_o^2\zeta_{m1}}{mn}[\zeta_{(n+m)1}+\zeta_{(n-m)1}-\zeta_{(m-n)1}]$$

$$-\frac{1}{2\beta_n^2}\sum_{m=2}^{\infty}\frac{D_o\zeta_{m1}}{m}[\frac{D_o\zeta_{(n-m)1}}{(n-m)}+\frac{D_o\zeta_{(n+m)1}}{(n+m)}-\frac{D_o\zeta_{(m-n)1}}{(m-n)}]$$

$$-\frac{1}{2\beta_n^2}\sum_{m=2}^{\infty}\frac{D_o\eta_{m1}}{m}[\frac{D_o\eta_{(m-n)1}}{(m-n)}+\frac{D_o\eta_{(n+m)1}}{(n+m)}-\frac{D_o\eta_{(n-m)1}}{(n-m)}]+P_n(\tau) \tag{3.8b}$$

and

$$D_o^2\zeta_{n2}+\omega_n^2\zeta_{n2}=-2D_oD_1\zeta_{n1}-2\mu_nD_o\zeta_{n1}+\frac{2}{n^2+1}[\eta_{o1}D_o^2\zeta_{n1}+(D_o\eta_{o1})(D_o\zeta_{n1})]$$

$$+\frac{n^2-2}{\beta_n^2}\eta_{o1}\zeta_{n1}+\frac{1}{\beta_n^2}\sum_{m=2}^{\infty}\frac{D_o\zeta_{m1}}{mn}[D_o\eta_{(n-m)1}+D_o\eta_{(m-n)1}+D_o\eta_{(n+m)1}]$$

$$+\frac{1}{\beta_n^2}\sum_{m=2}^{\infty}\frac{D_o^2\zeta_{m1}}{mn}[\eta_{(n-m)1}+\eta_{(m-n)1}+\eta_{(n+m)1}]$$

$$-\frac{1}{\beta_n^2}\sum_{m=2}^{\infty}\frac{D_o\eta_{m1}}{mn}[D_o\zeta_{(n+m)1}+D_o\zeta_{(m-n)1}-D_o\zeta_{(n-m)1}]$$

$$-\frac{1}{\beta_n^2}\sum_{m=2}^{\infty}\frac{D_o^2\eta_{m1}}{mn}[\zeta_{(n+m)1}+\zeta_{(m-n)1}-\zeta_{(n-m)1}]$$

$$+\frac{1}{\beta_n^2}\sum_{m=2}^{\infty}\frac{D_o\eta_{m1}}{m}[\frac{D_o\zeta_{(n-m)1}}{(n-m)}+\frac{D_o\zeta_{(m-n)1}}{(m-n)}-\frac{D_o\zeta_{(n+m)1}}{(n+m)}]+S_n(\tau) \tag{3.8c}$$

Next we substitute (3.7) into (3.8) and eliminate the secular and small-divisor terms. Since small-divisor terms are dependent on the case under study, each case will be considered separately. In the following sections the analysis will eliminate the secular and small-divisor terms from (3.8) depending on the resonance conditions being studied and continue the solution procedure.

3.2. Undamped Free Oscillations with No Internal Resonances

Equations (3.8) can be used in the case of undamped free oscillations if the forcing functions $P_o(\tau)$, $P_n(\tau)$ and $S_n(\tau)$ and the damping coefficients μ_o and μ_n are set equal to zero.

Substituting (3.7) into (3.8) (with no forcing considered) and eliminating secular terms, we have

$$- 2i\omega_o A_o' e^{i\omega_o T_o} = 0 \tag{3.9a}$$

$$- 2i\omega_n A_n' e^{i\omega_n T_o} = 0 \tag{3.9b}$$

and

$$- 2i\omega_n B_n' e^{i\omega_n T_o} = 0 \tag{3.9c}$$

Thus

$$A_o' = 0 \quad \text{or} \quad A_o = \text{constant} \tag{3.10a}$$

$$A_n' = 0 \quad \text{or} \quad A_n = \text{constant} \tag{3.10b}$$

$$B_n' = 0 \quad \text{or} \quad B_n = \text{constant} \tag{3.10c}$$

Consequently, the solutions are given by

$$\eta_o(\tau;\varepsilon) = \varepsilon c_{o1} \cos\omega_o\tau + \varepsilon c_{o2} \sin\omega_o\tau \tag{3.11a}$$

$$\eta_n(\tau;\varepsilon) = \varepsilon c_{n3} \cos\omega_n\tau + \varepsilon c_{n4} \sin\omega_n\tau \tag{3.11b}$$

$$\zeta_n(\tau;\varepsilon) = \varepsilon c_{n5}\cos\omega_n\tau + \varepsilon c_{n6}\sin\omega_n\tau \tag{3.11c}$$

where the c_{mn} are constants. Substituting (3.11) into (2.9) and (2.10) yields

$$w(\tau) = \varepsilon\Big[c_{01}\cos\omega_0\tau + c_{02}\sin\omega_0\tau + \sum_{n=2}^{\infty}\{(c_{n3}\cos\omega_n\tau + c_{n4}\sin\omega_n\tau)\cos n\theta + (c_{n5}\cos\omega_n\tau + c_{n6}\sin\omega_n\tau)\sin n\theta\}\Big] \tag{3.12a}$$

and

$$\psi(\tau) = \varepsilon\Big[\sum_{n=2}^{\infty}\{-\frac{1}{n}(c_{n5}\cos\omega_n\tau + c_{n6}\sin\omega_n\tau)\cos n\theta + \frac{1}{n}(c_{n3}\cos\omega_n\tau + c_{n4}\sin\omega_n\tau)\sin n\theta\}\Big] \tag{3.12b}$$

where the c_{nm} can be determined from the initial conditions.

As expected, the undamped free response of the shell in the absence of internal resonances is a periodic one with amplitudes depending on the initial conditions. Such a response is basically the same as that predicted by the linear analysis.

3.3. Undamped Free Oscillations with a Two-to-One Internal Resonance

Depending on the value of α^2, many resonances may occur. Among them is the interesting case of two-to-one internal resonance which will be studied here in the case of free oscillations, and later on in the case of forced oscillations.

For the case $\alpha^2 = 2.0918\times10^{-4}$, ω_6 will be equal to 0.4993 and $\omega_0 = 1.0001$ which is a condition of two-to-one internal resonance; that is $2\omega_6 \approx \omega_0$. In the following, we consider the general case of a two-to-one internal resonance $2\omega_n \approx \omega_0$ and introduce a detuning parameter σ

according to

$$2\omega_n = \omega_o + \varepsilon\sigma \tag{3.13}$$

Numerical results, though, will be given for the particular case of n = 6. In the general case, the solvability conditions arising from the elimination of secular and small-divisor terms are

$$-2i\omega_o A_o' + 4\omega_o \Lambda_1 A_n^2 e^{i\sigma T_1} + 4\omega_o \Lambda_1 B_n^2 e^{i\sigma T_1} = 0 \tag{3.14}$$

$$-2i\omega_n A_n' + 4\omega_n \Lambda_2 A_o \bar{A}_n e^{-i\sigma T_1} = 0 \tag{3.15}$$

$$-2i\omega_n B_n' + 4\omega_n \Lambda_2 A_o \bar{B}_n e^{-i\sigma T_1} = 0 \tag{3.16}$$

where the overbar denotes the complex conjugate and

$$4\omega_o \Lambda_1 = \frac{1}{4}(n^2 - 2) + \frac{\omega_n^2}{2n^2} \tag{3.17a}$$

$$4\omega_n \Lambda_2 = \frac{1}{n^2+1}\left[n^2(n^2 - 2) - 2(\omega_n^2 + \omega_n\omega_o)\right] \tag{3.17b}$$

Let

$$A_o = \frac{1}{2} a_o(T_1) e^{i\beta_o(T_1)} \tag{3.18a}$$

$$B_n = \frac{1}{2} b_n(T_1) e^{i\nu_n(T_1)} \tag{3.18b}$$

and

$$A_n = \frac{1}{2} a_n(T_1) e^{i\beta_n(T_1)} \tag{3.18c}$$

separate real and imaginary parts in (3.14)-(3.16) and obtain

$$a_o' + \Lambda_1 a_n^2 \sin\gamma_2 + \Lambda_1 b_n^2 \sin\gamma_3 = 0 \tag{3.19}$$

$$a_n' - \Lambda_2 a_o a_n \sin\gamma_2 = 0 \tag{3.20}$$

$$b_n' - \Lambda_2 a_o b_n \sin\gamma_3 = 0 \tag{3.21}$$

$$a_o \beta_o' + \Lambda_1 a_n^2 \cos\gamma_2 + \Lambda_1 b_n^2 \cos\gamma_3 = 0 \tag{3.22}$$

$$a_n\beta_n' + \Lambda_2 a_o a_n \cos\gamma_2 = 0 \tag{3.23}$$

$$b_n\nu_n' + \Lambda_2 a_o b_n \cos\gamma_3 = 0 \tag{3.24}$$

where

$$\gamma_2 = \beta_o - 2\beta_n - \sigma T_1 \tag{3.25}$$

and

$$\gamma_3 = \beta_o - 2\nu_n - \sigma T_1 \tag{3.26}$$

Eliminating γ_2 and γ_3 from (3.19)-(3.21) yields

$$\Lambda_2 a_o a_o' + \Lambda_1 a_n a_n' + \Lambda_1 b_n b_n' = 0 \tag{3.27}$$

Integrating (3.27) yields

$$(\Lambda_2/\Lambda_1)a_o^2 + a_n^2 + b_n^2 = E \tag{3.28}$$

where E is a constant of integration.

Since all the terms on the left-hand side of (3.28) are positive, then E must also be positive, which means that the motion described is a bounded one. Moreover, (3.28) is a statement of the principle of the conservation of energy. The energy is in a state of continuous transfer between the breathing and flexural modes as a result of the strong coupling of the system due to internal resonance in this case.

Figure 4 shows the time history of the response for the case of free oscillations when $\alpha^2 = 2.0918 \times 10^{-4}$, which leads to a two-to-one internal resonance for n = 6, i.e., $\omega_o \simeq 2\omega_6$. A fifth- and sixth-order Runge-Kutta-Verner integration scheme was used to numerically integrate equations (3.19)-(3.24). Figure 4 shows the response to be dominated by the flexural mode. The numerical results satisfy relation (3.28) obtained by the perturbation analysis. The perturbation analysis yields

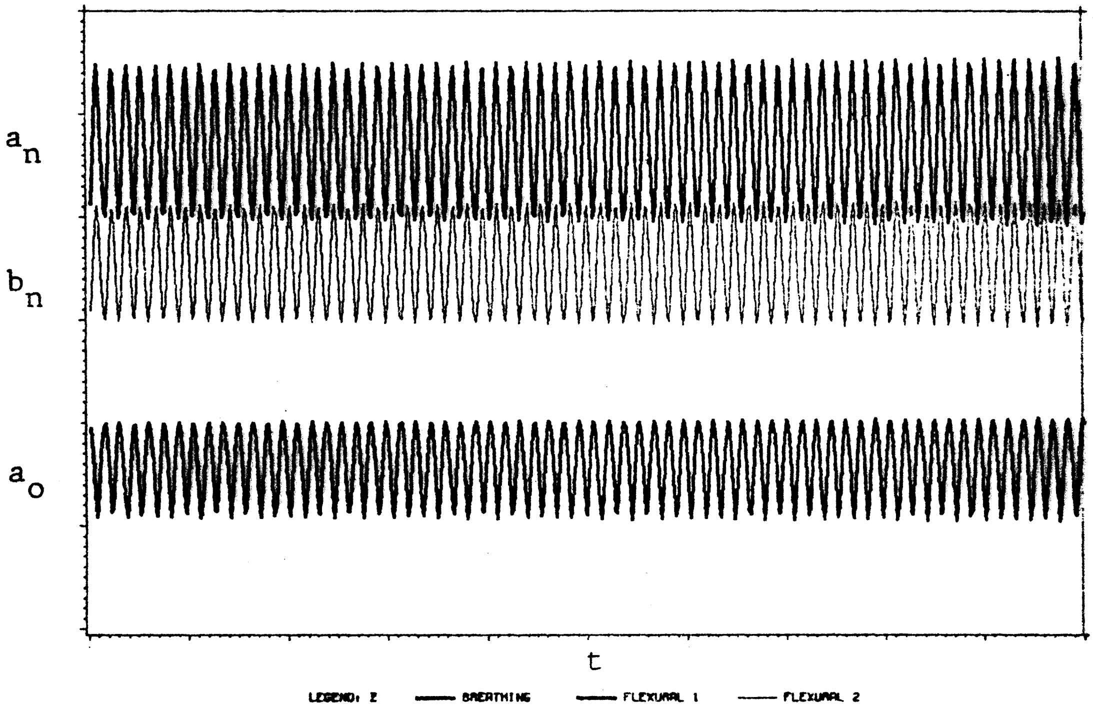

Figure 4. A typical time history of the free oscillation.

closed form expressions for the response amplitudes, which can be expressed in terms of Jacobi elliptic functions [23]. In contrast, one needs to use the so-called "black box" numerical scheme to determine the time history every time the systems parameters, or initial conditions change.

3.4. Damped Forced Oscillations of the Breathing Mode in the case of a Two-To-One Internal Resonances

We consider the case of damped forced response under harmonic excitation of the breathing mode given by $P_o = 2F\cos\Omega t$, where $\Omega \approx \omega_o$. In this case, the excitation can be put in complex notation as $P_o = F\exp(i\Omega T_o) + c.c.$ The resonance conditions are given by

$$2\omega_n \approx \omega_o \quad \text{and} \quad \Omega \approx \omega_o$$

or

$$2\omega_n = \omega_o + \varepsilon\sigma_2 \tag{3.29a}$$

and

$$\Omega = \omega_o + \varepsilon\sigma_1 \tag{3.29b}$$

where σ_1 and σ_2 are detuning parameters. In this case, the solvability conditions arising from the elimination of secular and small-divisor terms are

$$2A_o' + 2\mu_o A_o + 4i\Lambda_1(A_n^2 + B_n^2)e^{i\sigma_2 T_1} + if\, e^{i\sigma_1 T_1} = 0 \tag{3.30}$$

$$2A_n' + 2\mu_n A_n + 4i\Lambda_2 A_o \bar{A}_n e^{-i\sigma_2 T_1} = 0 \tag{3.31}$$

$$2B_n' + 2\mu_n B_n + 4i\Lambda_2 A_o \bar{B}_n e^{-i\sigma_2 T_1} = 0 \tag{3.32}$$

where $F = \omega_o f$. Substituting the polar form (3.18) into (3.30)-(3.32) and separating real and imaginary parts yields

$$a_o' + \mu_o a_o + \Lambda_1 a_n^2 \sin\gamma_2 + \Lambda_1 b_n^2 \sin\gamma_3 - f\sin\gamma_1 = 0 \quad (3.33)$$

$$a_n' + \mu_n a_n - \Lambda_2 a_o a_n \sin\gamma_2 = 0 \quad (3.34)$$

$$b_n' + \mu_n b_n - \Lambda_2 a_o b_n \sin\gamma_3 = 0 \quad (3.35)$$

$$a_o \beta_o' + \Lambda_1 a_n^2 \cos\gamma_2 + \Lambda_1 b_n^2 \cos\gamma_3 + f\cos\gamma_1 = 0 \quad (3.36)$$

$$a_n \beta_n' + \Lambda_2 a_o a_n \cos\gamma_2 = 0 \quad (3.37)$$

$$b_n \nu_n' + \Lambda_2 a_o b_n \cos\gamma_3 = 0 \quad (3.38)$$

where

$$\gamma_1 = \sigma_1 T_1 - \beta_o \quad (3.39)$$

$$\gamma_2 = \beta_o - 2\beta_n - \sigma_2 T_1 \quad (3.40)$$

$$\gamma_3 = \beta_o - 2\nu_n - \sigma_2 T_1 \quad (3.41)$$

Steady-state solutions correspond to $a_o' = a_n' = b_n' = 0$ and $\gamma_n' = 0$. It follows from (3.39)-(3.41) that $\beta_o' = \sigma_1$ and $\beta_n' = \nu_n' = \frac{1}{2}(\sigma_1 - \sigma_2)$. Hence, steady-state solutions correspond to the solutions of

$$\mu_o a_o + \Lambda_1 a_n^2 \sin\gamma_2 + \Lambda_1 b_n^2 \sin\gamma_3 - f\sin\gamma_1 = 0 \quad (3.42)$$

$$\mu_n a_n - \Lambda_2 a_o a_n \sin\gamma_2 = 0 \quad (3.43)$$

$$\mu_n b_n - \Lambda_2 a_o b_n \sin\gamma_3 = 0 \quad (3.44)$$

$$\sigma_1 a_o + \Lambda_1 a_n^2 \cos\gamma_2 + \Lambda_1 b_n^2 \cos\gamma_3 + f\cos\gamma_1 = 0 \quad (3.45)$$

$$\frac{1}{2}(\sigma_1 - \sigma_2) a_n + \Lambda_2 a_o a_n \cos\gamma_2 = 0 \quad (3.46)$$

$$\frac{1}{2}(\sigma_1 - \sigma_2) b_n + \Lambda_2 a_o b_n \cos\gamma_3 = 0 \quad (3.47)$$

There are two possible solutions. First,

$$a_n = b_n = 0 \text{ and } a_o = f(\mu_o^2 + \sigma_1^2)^{-\frac{1}{2}} \tag{3.48}$$

which is essentially the linear solution. Second, $a_n \neq 0$ and $b_n \neq 0$. Third, $a_n \neq 0$ and $b_n = 0$. Fourth, $a_n \neq 0$ and $b_n \neq 0$. The last case includes the second and third solutions as special cases. Then, it follows from (3.43) (3.46), (3.44), and(3.47) that

$$a_o = a_o^* = \Lambda_2^{-1}[\mu_n^2 + \frac{1}{4}(\sigma_1 - \sigma_2)^2]^{\frac{1}{2}} \tag{3.49}$$

$$\tan\gamma_2 = \tan\gamma_3 = [2\mu_n/(\sigma_2 - \sigma_1)] \tag{3.50}$$

Then, it follows from (3.42) and (3.45) that

$$a_n^2 + b_n^2 = (\Lambda_1\Lambda_2)^{-1}\{\frac{1}{2}\sigma_1(\sigma_1 - \sigma_2) - \mu_o\mu_n \pm [f^2\Lambda_2^2$$

$$- (\sigma_1\mu_o + \frac{1}{2}\mu_n(\sigma_1 - \sigma_2)^2)^2]^{\frac{1}{2}}\} \tag{3.51}$$

Equation (3.49) shows that the amplitude a_o of the directly excited breathing mode is independent of the amplitude f of the excitaiton. It depends only on the damping of the flexural modes and the detuning parameters σ_1 and σ_2. On the other hand, the amplitudes a_n and b_n of the flexural mode are strongly dependent on the excitation amplitude f.

To determine the stablity of the steady-state solutions, we let

$$A_o = \frac{1}{2}(p_1 - iq_1)e^{i\nu_1 T_1} \tag{3.52}$$

$$A_n = \frac{1}{2}(p_2 - iq_2)e^{i\nu_2 T_1} \tag{3.53}$$

$$B_n = \frac{1}{2}(p_3 - iq_3)e^{i\nu_2 T_1} \tag{3.54}$$

where

$$\nu_1 = \sigma_1 \quad \text{and} \quad \nu_2 = \frac{1}{2}(\sigma_1 - \sigma_2)$$

in (3.30)-(3.32), separate real and imaginary parts, and obtain

$$p_1' + \nu_1 q_1 + \mu_o p_1 + 2\Lambda_1(p_2 q_2 + p_3 q_3) = 0 \tag{3.55}$$

$$q_1' - \nu_1 p_1 + \mu_0 q_1 - \Lambda_1(p_2^2 + p_3^2 - q_2^2 - q_3^2) = f \qquad (3.56)$$

$$p_2' + \nu_2 q_2 + \mu_n p_2 + \Lambda_2(q_1 p_2 - q_2 p_1) = 0 \qquad (3.57)$$

$$q_2' - \nu_2 p_2 + \mu_n q_2 - \Lambda_2(p_1 p_2 + q_1 q_2) = 0 \qquad (3.58)$$

$$p_3' + \nu_2 q_3 + \mu_n p_3 + \Lambda_2(q_1 p_3 - q_3 p_1) = 0 \qquad (3.59)$$

$$q_3' - \nu_2 p_3 + \mu_n q_3 - \Lambda_2(p_1 p_3 + q_1 q_3) = 0 \qquad (3.60)$$

Equations (3.55)-(3.60) are alternates to (3.33)-(3.41) and have the general form

$$\underline{x}' + \underline{f}(\underline{x}) = 0 \qquad (3.61)$$

where x and f are column vectors having six components.

The steady-state solutions (i.e., fixed points) of (3.61) correspond to the solutions of $\underline{f}(\underline{x}) = 0$. If $\underline{x}_0$ is such a solution, then its local stability can be determined by letting $\underline{x} = \underline{x}_0 + \underline{x}_1$, where $\underline{x}_1$ is small compared with $\underline{x}_0$ and linearizing the resulting equation in $\underline{x}_1$. The result is

$$\underline{x}_1' + \nabla \underline{f}(\underline{x}_0) \cdot \underline{x}_1 = 0 \qquad (3.62)$$

Hence, the local stability of the fixed point $\underline{x}_0$ with respect to a small perturbation $x_1 \propto \exp(\lambda T_1)$ is determined by the zeros of the characteristic equation

$$|\nabla \underline{f}(\underline{x}_0) + \lambda I| = 0 \qquad (3.63)$$

In our case, the characteristic equation takes the form

$$\begin{vmatrix} \lambda+\mu_o & \nu_1 & 2\Lambda_1 q_2 & 2\Lambda_1 p_2 & 2\Lambda_1 q_3 & 2\Lambda_1 p_3 \\ -\nu_1 & \lambda+\mu_o & -2\Lambda_1 p_2 & 2\Lambda_1 q_2 & -2\Lambda_1 p_3 & 2\Lambda_1 q_3 \\ -\Lambda_2 q_2 & \Lambda_2 p_2 & \lambda+\mu_n+\Lambda_2 q_1 & \nu_2-\Lambda_2 p_1 & 0 & 0 \\ -\Lambda_2 p_2 & -\Lambda_2 q_2 & -\nu_2-\Lambda_2 p_1 & \lambda+\mu_n-\Lambda_2 q_1 & 0 & 0 \\ -\Lambda_2 q_3 & \Lambda_2 p_3 & 0 & 0 & \lambda+\mu_n+\Lambda_2 q_1 & \nu_2-\Lambda_2 p_1 \\ -\Lambda_2 p_3 & -\Lambda_2 q_3 & 0 & 0 & -\nu_2-\Lambda_2 p_1 & \lambda+\mu_n-\Lambda_2 q_1 \end{vmatrix} = 0 \tag{3.64}$$

To investigate the stability of the linear solution (3.48), we put $p_2 = p_3 = q_2 = q_3 = 0$ in (3.64) and after some algebraic manipulations obtain

$$[(\lambda + \mu_o)^2 + \nu_1^2][(\lambda + \mu_n)^2 + \nu_2^2 - \Lambda_2^2 a_o^2]^2 = 0 \tag{3.65}$$

because $a_o^2 = p_1^2 + q_1^2$. Hence,

$$\lambda = - \mu_o \pm i\nu_1, \ - \mu_n \pm (\Lambda_2^2 a_o^2 - \nu_2^2)^{\frac{1}{2}} \ - \mu_n \pm (\Lambda_2^2 a_o^2 - \nu_2^2)^{\frac{1}{2}} \tag{3.66}$$

Consequently, the linear solution is stable if and only if

$$\Lambda_2^2 a_o^2 \le \nu_2^2 + \mu_n^2 \tag{3.67}$$

which, in conjunction with (3.49), implies that the linear solution is stable if $a_o \le a_o^*$ and unstable if $a_o > a_o^*$.

To study the stability of the nonlinear solution (3.49)-(3.51) when $b_n = 0$, we let $p_3 = q_3 = 0$ in (3.64), use (3.49), and obtain

$$[(\lambda + \mu_n)^2 + \mu_n^2]\{\lambda^4 + 2(\mu_o + \mu_n)\lambda^3 + [\mu_o^2 + 4\mu_o\mu_n + \nu_1^2$$

$$+ 4\Lambda_1\Lambda_2 a_n^2]\lambda^2 + [2\mu_n\mu_o^2 + 2\mu_n\nu_1^2 + 4\Lambda_1\Lambda_2(\mu_o + \mu_n)a_n^2]\lambda$$

$$+ 4\Lambda_1\Lambda_2 a_n^2[\Lambda_1\Lambda_2 a_n^2 + \mu_o\mu_n - \nu_1\nu_2]\} = 0 \qquad (3.68)$$

Hence, either $\lambda = 0$ or $-2\mu_n$ or

$$\lambda^4 + 2(\mu_o + \mu_n)\lambda^3 + [\mu_o^2 + 4\mu_o\mu_n + \nu_1^2 + 4\Lambda_1\Lambda_2 a_n^2]\lambda^2$$

$$+ [2\mu_n\mu_o^2 + 2\mu_n\nu_1^2 + 4\Lambda_1\Lambda_2(\mu_o + \mu_n)a_n^2]\lambda$$

$$+ 4\Lambda_1\Lambda_2 a_n^2[\Lambda_1\Lambda_2 a_n^2 + \mu_o\mu_n - \nu_1\nu_2] = 0 \qquad (3.69)$$

The necessary and sufficient condition that none of the roots of (3.69) has a positive real part is

$$\Lambda_1\Lambda_2 a_n^2 + \mu_o\mu_n - \nu_1\nu_2 > 0 \qquad (3.70)$$

which, in conjunction with (3.51), implies that the solution corresponding to the positive sign is stable whereas the solution corresponding to the negative sign is unstable.

3.5. General Discussion of the Saturation Phenomenon

The nonlinear solution of the flexural mode (3.51) (with $b_n = 0$) can be expressed as

$$a_n = [-\Gamma_1 \pm (\frac{f^2}{\Lambda_1^2} - \Gamma_2^2)^{\frac{1}{2}}]^{\frac{1}{2}} \qquad (3.71)$$

$$\Gamma_1 = \frac{1}{\Lambda_1\Lambda_2}(\mu_o\mu_n - \frac{1}{2}\sigma_1(\sigma_1 - \sigma_2)) \qquad (3.72)$$

$$\Gamma_2 = \frac{1}{\Lambda_1\Lambda_2}(\sigma_1\mu_o + \frac{1}{2}\mu_n(\sigma_1 - \sigma_2)^2) \qquad (3.73)$$

$$\xi_1 = f = \Lambda_1\Gamma_2 \qquad (3.74)$$

$$\xi_2 = f = \Lambda_1(\Gamma_1^2 + \Gamma_2^2)^{\frac{1}{2}} \tag{3.75}$$

Next, we determine the conditions under which (3.71) has real roots. To this end, we define the two critical values ξ_1 and ξ_2 of f as

When $\Gamma_1 \geq 0$, (3.71) has only one real solution if

$$\xi_2 = f > \Lambda_1(\Gamma_1^2 + \Gamma_2^2)^{\frac{1}{2}} \tag{3.76}$$

When $\Gamma_1 < 0$, (3.71) has one real solution if $f > \xi_2$ and has two real solutions if $\xi_1 < f < \xi_2$ if $f > \xi_2$. Next, we present numerical results for the case $\alpha^2 = 2.0918 \times 10^{-4}$, which yields $\omega_0 \approx 2\omega_6$.

Figure 5 shows a typical plot of the steady-state response amplitudes versus the forcing amplitude when $\Gamma_1 \geq 0$. The system's response is the linear response as long as the amplitude of forcing is less than ξ_2. Thus, the shell responds to the radial excitation by purely radial displacements (the breathing mode). On the otherhand, if the forcing amplitudes exceeds ξ_2, in which the response of the shell exhibits satration phenomenon, the directly excited breathing mode assumes a constant (saturated) value and spills the additional input energy into the flexural modes that increase rather rapidly with increasing f and eventually dominate the response.

Figure 6 shows the response in the case when $\Gamma_1 < 0$. When the excitation amplitude f lies in the interval $[\xi_1, \xi_2]$, there are three possible steady-state responses. Two of these responses are stable: the trivial response and the larger amplitude response. The response tha is attained physically depends on the initial conditions. If the excitation amplitude increases from zero, one observes only the breathing mode until f reaches ξ_2. As f increases beyond ξ_2, a_6 jumps up from zero to point c, producing a large wrinkling of the shell. As f

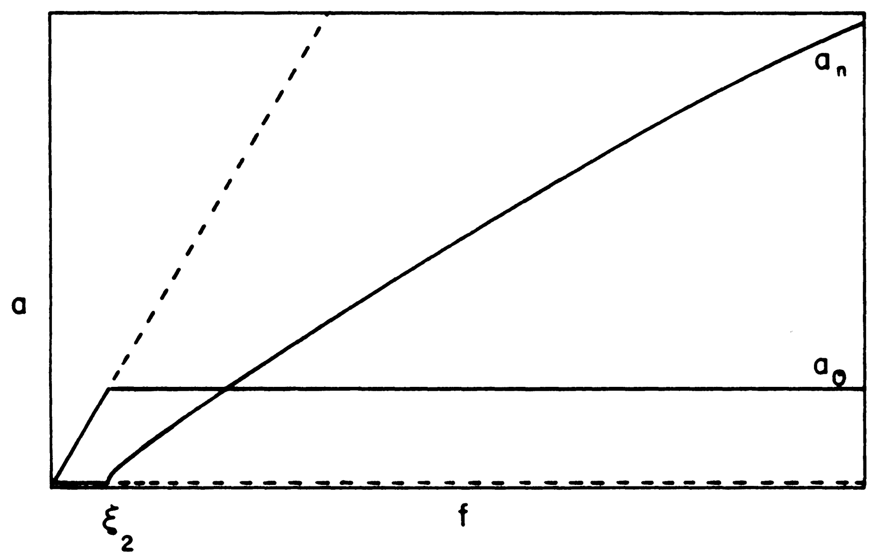

Figure 5. Modal response amplitudes as functions of the amplitude of the excitation when $\Gamma < 0$.

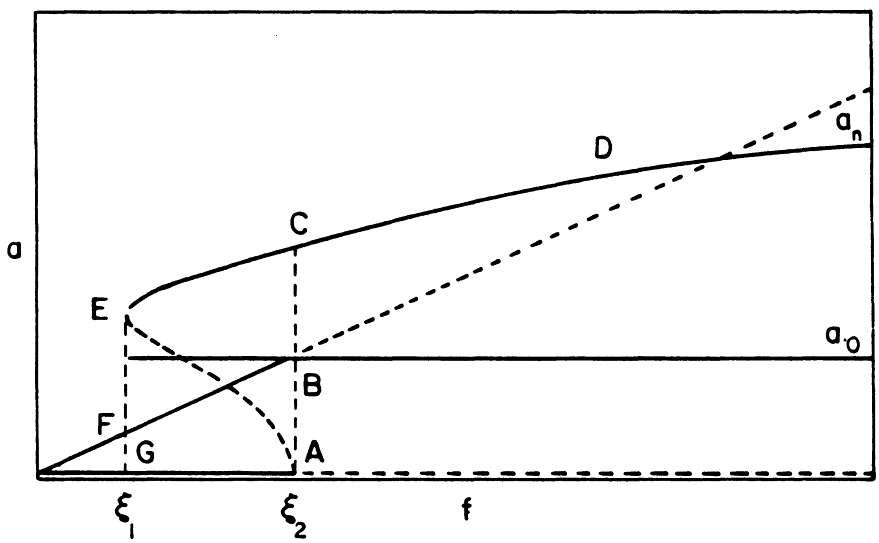

Figure 6. Modal response amplitudes as functions of the amplitude of the excitation when $\Gamma > 0$.

increases further, a_0 remains constant, whereas a_6 increases slowly along the curve ECD. If f decreases from a value corresponding to point D, a_6 decreases slowly along the curve DCE and a_0 remains constant until point E is reached. As f decreases below ξ_1, a_6 jumps down to zero and a_0 jumps down to point F. As f decreases further, a_6 remains zero and a_0 decreases linearly with f.

If the amplitude of the excitation is set at a value in the interval $[\xi_1,\xi_2]$ and the shell is initially undisturbed, the response corresponds to the linear solution, in which the shell is breathing without wrinkling. However, if the shell is disturbed, the shell may respond with the nonlinear solution, in which the amplitude of the breathing mode as well as the flexural mode increase dramatically, yielding a much larger response. These phenomena were also observed experimentally in the response of a simple model consisting of two beams and two concentrated masses by Haddow, Barr, and Mook [38] and in the vibration laboratory at VPI & SU.

The importance of such phenomena is apparent in design processes. It simply says that a designer can't design a cylindrical shell subjected to radial loads expecting only a purely radial, breathing mode response since flexural modes may be strongly activated giving rise to large moments. The results given in figures 5 and 6 are for a cylindrical shell with a thickness to radius ration of 1/20 which qualifies it as a thin shell. A 2.0 meter radius cylindrical shell with a 0.1 meter thickness and 0.01 damping coefficients made of steel is well behaved according to linear theory. However the present theory indicates that at a forcing level of about 1700 kg/m^2 (a loading that is not uncommon in nowadays structures) nonlinear effects start to appear and may produce unexpected large amplitude flexural oscillations. A designer or a practicing engineer can't afford to neglect nonlinear resonances. One can't but admire the words of Y. C. Fung, 1965, "nonhomogeneity and anistropy prevail, nonlinearity reveals itself almost everywhere" [39].

CHAPTER FOUR

CONCLUSIONS AND RECOMMENDATIONS

4.1. Conclusions

This work deals with the nonlinear extensional dynamic analysis of infinite length circular cylindrical shells in cases of free and forced oscillations.

Under the usual thin shells assumptions, the governing equations of motion are dervied using both te free body diagram and energy approaches. A Galerkin procedure is used to eliminate the spatial dependence in these equations by expanding the dependnet variables in infinite series of periodic temporal functions with time-dependent coefficients. Such a procedure transformed the partial differential equations into a set of infinite number of ordinary differential equations. These equations are given in their most general case. The method of multiple scales is used to derive the equations governing the amplitudes and phases of oscillation in different cases, taking into account the effect of modal interactions. The studied cases are:

A. The case of undamped free oscillations with no internal resonances.

The response is the linear response which is periodic with coefficients that depend on the initial conditions.

B. The case of undamped free oscillations with two-to-one internal resonances.

The response is dominated by the flexural modes and an energy exchange takes place between breathing and flexural modes in such a way that makes the total solution bounded.

C. The case of damped forced and externally resonant oscillation with two-to-one internal resonances.

The excitation is taken to be radial but the response is found to exhibit a flexural mode response above a certain level of excitation. The saturation and jump phenomena are demonstrated and an example of a cylindrical shells with apparently linear characteristics is given to show that the nonlinear effects may occur even at rather common levels of design loads.

4.2 Recommendations for Future Research

The area of nonlinear dynamic analysis of shells is yet to be discovered. Some topics which the author feels need to be studied are listed below:

a. The forced damped oscillations with two-to-one internal resonances where the flexural modes are directly excited.

b. The problem of combination resonance conditions.

c. The oscillations of finite-length cylindrical shells.

d. The extensional oscillations of cylindrical shells.

e. The oscillations of thick cylindrical shells.

f. The oscillations of composite shells

g. The effects of rotary inertia and shear deformation, and

h. The oscillations of arbitrary shaped shells.

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