THE COMPACT-OPEN TOPOLOGY ON C(X)

by

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"Malheur à toi si tu dis que ton bonheur est mort parce que tu n'avais pas rêvé pareil à cela ton bonheur...

Le rêve de demain est une joie. Mais la joie de demain en est une autre; et rien heureusement ne ressemble au rêve qu'on s'en était fait car c'est différemment que vaut chaque chose".

A.Gide, "Les nourritures terrestres".

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(ABSTRACT)

This paper investigates the compact-open topology on the set $C_k(X)$ of continuous real-valued functions defined on a Tychonoff space X.

More precisely, we study the following problem: If P is a topological property, does there exist a topological property Q so that $C_k(X)$ has P if and only if X has Q ?

Characterizations of many properties are obtained throughout the thesis, sometimes modulo some "mild" restrictions on the space X.

The main properties involved are summarized in a diagram in the introduction.

DEDICATION

To my wife and children for their patience and love;

To and who made our trip possible;

To my parents, my brother and sisters all gone for ever;

To you , my only hope in the family,

I dedicate this paper.

ACKNOWLEDGMENTS

Thanks are due to my advisor, Dr. Robert A. McCoy who introduced me to Function Spaces. His expertness in the area was an invaluable source of inspiration throughout my research.

I will deeply miss the topology team of the department: Dr. C. E. Aull, Dr. R. F. Dickman, Dr. P. Fletcher, Dr. R. A. McCoy, Dr. J. C. Smith, Dr. R. Telgarsky and Ray Price. Their talks and jokes at the topology seminar were so exciting.

I am indebted to , , Dr. R. A. McCoy and who understood our financial problems.

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TABLE OF CONTENTS

INTRODUCTION

The concept of a continuous function from one topological space into another topological space was first studied by Fréchet [27]. The idea of topologizing the set of continuous functions arose from the notion of convergence of sequences of functions. Of special interest was the topologization of the set C(X) of continuous real-valued functions on a given topological space X. The topology of pointwise convergence (i.e. point-open topology) is among the first such topologies on C(X). However, in the early years of "Analysis Situs" a metric ρ (defined by $\rho(f,g) = \sup\{|f(x)-g(x)| : x \in X\}$) called the supremum metric was already defined on C(X) for a compact space X. This metric function space $C_\rho(X)$ was the keystone to famous results such as the Stone-Weierstrass theorem and the Ascoli-Arzelà theorem.

In 1945, Fox [28] defined the compact-open topology. Many results that were then true on the metric space $C_\rho(X)$ were generalized to this new topology. What attracted the interest of many mathematicians was that C(X) is a locally convex topological vector space when carrying the compact-open topology. This prompted an intense investigation of this new topology by algebraists, analysts, topologists, and others. It did not take long for this effort to pay off. The most remarkable results of that period described the relationship between the algebraico-topological structure of C(X) and the topological structure of X (see [35], [54]). The topological aspect of the compact-open topology in the fifties is characterized by the

considerable work of many mathematicians such as Arens ([2]), Nachbin ([51]), Shirota ([59]), Warner ([65]), and others. Since the sixties the number of function space investigators has grown so fast that, today, it is difficult if not impossible to locate them all. However, recent papers of A. V. Arhangelskii, D. J. Lutzer, R. A. McCoy and E. A. Michael are among those contributing to a great understanding of the topological behavior of C(X).

Let us denote by $C_k(X)$ (respectively by $C_p(X)$) the set C(X) topologized with the compact-open (respectively the point-open) topology. A subbase for $C_k(X)$ (respectively for $C_p(X)$) is given by the collection $([A,V]:A$ is compact (respectively A is finite) in X and V is open in $\mathbb{R})$, where $[A,V] = (f \in C(X):f(A) \subseteq V)$. Being a locally convex topological vector space, $C_k(X)$ (or $C_p(X)$) is connected and homogeneous (i.e. given $f \neq g$ there is a homeomorphism from $C_k(X)$ onto itself mapping f onto g).

At the heart of the theory of function spaces lies the fundamental problem of the duality between topological properties of X and those of $C_k(X)$ (or $C_p(X)$). Apparently X and $C_k(X)$ (or $C_p(X)$) are topologically two different objects. But in many instances they bear the same topological properties: e.g., "X is an $\aleph_0$-space if and only if $C_k(X)$ is an $\aleph_0$-space"; "X is cosmic if and only if $C_p(X)$ is cosmic" (Michael [47]).

This thesis studies the compact-open topology on C(X). Our main aim is the duality problem mentioned above. More precisely, we deal with the following question: given a topological property P, does

there exist a topological property Q so that $C_k(X)$ has P if and only if X has Q?

Generally speaking, since the topology of $C_k(X)$ is defined by compact sets of X, one would expect that a given property P on $C_k(X)$ will translate on X into a property involving compact sets of X. Among the results obeying this "law" are the following: "$C_k(X)$ is metrizable if and only if X is hemicompact (Arens [2]); "$C_k(X)$ is submetrizable if and only if X is almost σ-compact" (McCoy [40]). There are some exceptions such as: "$C_k(X)$ is separable if and only if X has a coarser separable metrizable topology" (Warner [65]).

The introduction of the so-called "generalized metric spaces" in the mid sixties made the theory of function spaces more challenging by increasing the number of properties to consider. In this paper we will show that many of these new properties on $C_k(X)$ are equivalent to longtime well-known properties.

This work is the first text in the theory of function spaces that unifies the various results scattered in several mathematical journals. New properties are studied and references to known results are provided throughout the thesis.

In chapter one we present some generalities on the compact-open topology.

Chapter two studies the main countability properties: separability, metrizability, first and second countability, cosmic and $\aleph_0$-spaces.

Chapter three deals with "generalized metric spaces". We show

that on $C_k(X)$ all properties between metrizability and Michael's q-spaces are equivalent; and that $C_k(X)$ is a σ-space if and only if it is hereditarily a strong Σ-space. We also improve a result of [40] by proving that $C_k(X)$ is submetrizable if and only if each of its points is a G_δ-set. This result is strengthened in theorem 3.1.10 where we provide a necessary and sufficient condition for $C_k(X)$ to admit a coarser separable metrizable topology.

In chapter four we characterize the Fréchet property on $C_k(X)$. For a q-space X, we show that $C_k(X)$ is a Fréchet space if and only if X is a hemicompact k-space.

Completeness properties ranging from Polish spaces to the Baire space property are investigated in chapter five. The main theorem states that $C_k(X)$ is almost Čech-complete if and only if it is completely metrizable. The proof of this theorem shows that this result is also true on $C_p(X)$.

In chapter six, the property of being a continuous image of the irrationals (known as analyticity) is studied on $C_k(X)$. A characterization of this property on $C_k(X)$ is obtained for a quasi-k-space X. Necessary conditions for $C_k(X)$ to be analytic are provided. We also show that for a q-space X, a necessary and sufficient condition for $C_k(X)$ to be analytic is that $C_p(X)$ be analytic.

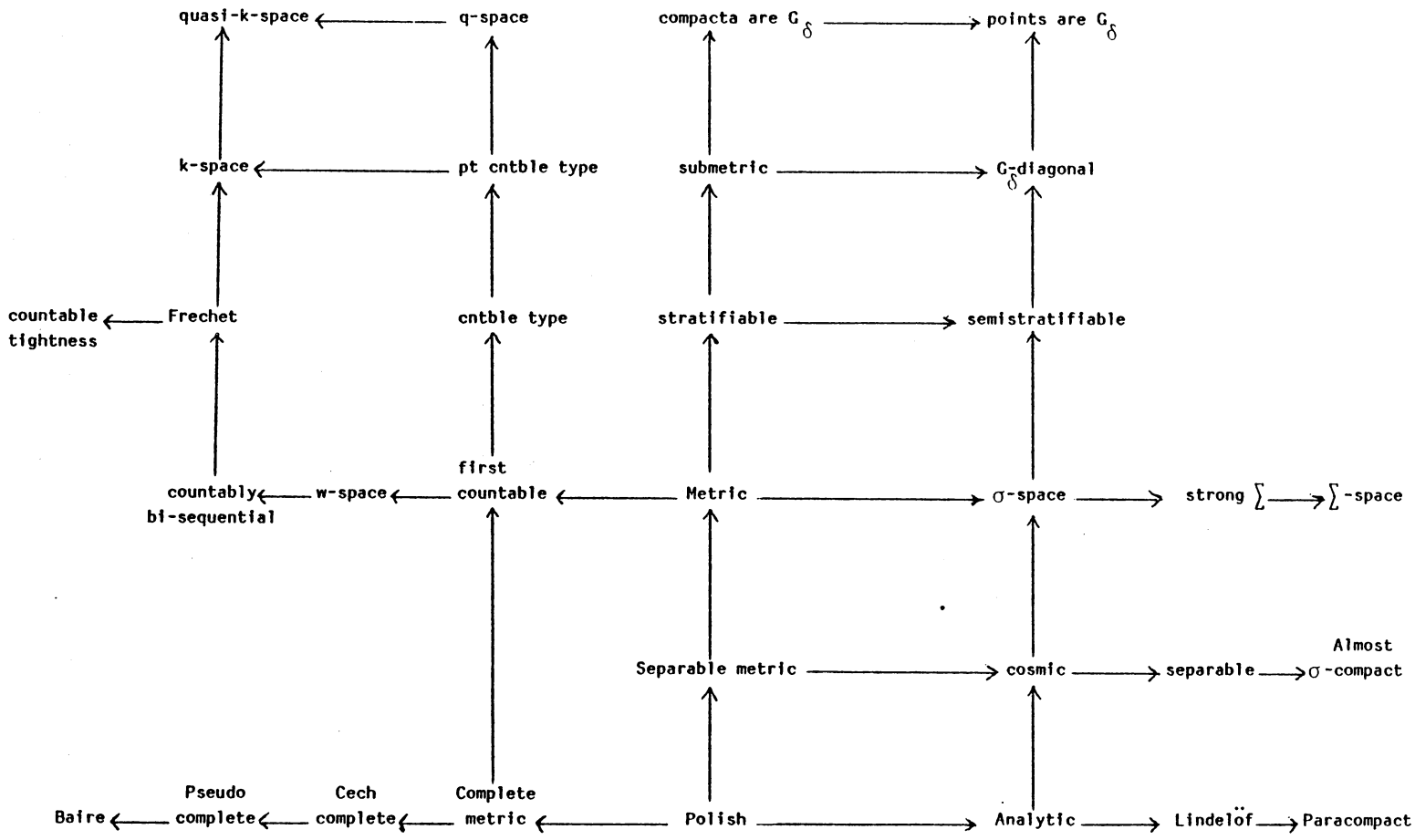
Chapter seven is concerned with covering properties on $C_k(X)$. Among the properties studied are: ccc, paracompactness and Lindelöf, hereditary separable and hereditary Lindelöf.

The diagram below describes the main properties involved in

this paper (some implications are omitted).

Results and definitions are numbered consecutively throughout the text. For instance, a reference of the type "by theorem 2.1.1" means that we refer to that theorem which is in chapter two. $\mathbb{R}$ will denote the set of real numbers with the usual topology and, unless otherwise stated, any subset of $\mathbb{R}$ carries its subspace topology. $\mathbb{N}$ represents the natural numbers and ω, ω_1 stand for the first infinite ordinal and the first uncountable ordinal respectively. If X is a set, $|X|$ is the cardinality of X.

In order to have "enough" functions in $C_K(X)$ to separate points of X, we will require all our spaces X to be Tychonoff spaces (i.e. completely regular T_1-spaces).



Footnote: 1) "$\longrightarrow$" means implies
2) The implications are for Tychonoff spaces

CHAPTER 1 : GENERALITIES

In this chapter we present some general facts about the compact-open topology.

If t and t' are two topologies on $C(X)$, $C_{t'}(X) \leqslant C_t(X)$ and $C_{t'}(X) = C_t(X)$ will respectively mean that t is finer than t' and t is equal to t'. From the definitions one sees that $C_p(X) \leqslant C_k(X)$. Recall that all spaces X are Tychonoff.

1. Topologies on C(X)

We have already observed that $C_p(X) \leqslant C_k(X)$. For the reverse inequality to hold we need the compact sets of X to be finite.

1.1.1. Theorem. $C_p(X) = C_k(X)$ if and only if each compact set in X is finite.

Proof. Suppose that $C_p(X) = C_k(X)$ and let A be a compact subset of X. We wish to show that A is finite. If $A = \emptyset$, we are done. So, assume $A \neq \emptyset$ and let θ denote the zero function in $C(X)$. Then $[A,(-1,1)]$ is open in $C_k(X) = C_p(X)$ and $\theta \in [A,(-1,1)]$. Therefore, there are finite sets $F_1, F_2, \ldots, F_m$ in X and corresponding open sets $V_1, V_2, \ldots, V_m$ in $\mathbb{R}$ so that $\theta \in [F_1,V_1] \cap \ldots \cap [F_m,V_m] \subseteq [A,(-1,1)]$. Now, let $F = F_1 \cup \ldots \cup F_m$. Then F is finite. We claim that $A \subseteq F$. To see this, suppose on the contrary that there is $x \in A-F$. Since X is Tychonoff, choose $f \in$

$C(X)$ such that $f(x) = 1$ and $f(F) = \{0\}$. Clearly then $f \in [F_1,V_1] \cap \ldots \cap [F_m,V_m] \subseteq [A,(-1,1)]$. But this is impossible because $f(x) \notin (-1,1)$.

For the converse suppose that each compact set in X is finite. To see that $C_p(X) = C_k(X)$, we only need to show that $C_k(X) \leqslant C_p(X)$. This is immediate for, if A is compact in X and V is open in $\mathbb{R}$, then the subbasic open set $[A,V]$ of $C_k(X)$ is clearly open in $C_p(X)$ since A is finite.

The next result presents a new base for the space $C_k(X)$. If $f \in C(X)$, A is compact in X and $\varepsilon > 0$ is a real number, we define $\langle f,A,\varepsilon\rangle = \{g \in C(X) : |f(x)-g(x)| < \varepsilon \text{ for all } x \text{ in } A\}$.

1.1.2. Theorem. The collection $\mathcal{B} = \{\langle f,A,\varepsilon\rangle : f \in C(X), A \text{ is compact in } X \text{ and } \varepsilon > 0\}$ is a base for the compact-open topology on $C(X)$.

Proof. We must show that every member of $\mathcal{B}$ is open in $C_k(X)$, and that if $f \in W$ where W is basic open in $C_k(X)$ then for some $W' \in \mathcal{B}$ we have $f \in W' \subseteq W$. So, let $f \in C(X)$, A compact in X and $\varepsilon > 0$. To see that $\langle f,A,\varepsilon\rangle$ is open in $C_k(X)$ it suffices to find W open in $C_k(X)$ with $f \in W \subseteq \langle f,A,\varepsilon\rangle$. To do this, we know that f is continuous on X and so, for each x in A there is an open neighborhood V_x of x in X such that $f(\bar{V}_x) \subseteq (f(x)-\varepsilon,f(x)+\varepsilon)$. Since the collection $\{V_x : x \in A\}$ is a cover of the compact set A by open sets of X then there is $N \in \mathbb{N}$ with $A \subseteq V_{x_1} \cup \ldots \cup$

V_{x_N}. Put $A_i = A \cap \bar{V}_{x_i}$ and $T_i = (f(x_i)-\varepsilon, f(x_i)+\varepsilon)$ for $i = 1,2,\ldots,N$. Then let $W = [A_1,T_1] \cap \ldots \cap [A_N,T_N]$. Clearly, $f \in W \subseteq \langle f,A,\varepsilon\rangle$ and W is open in $C_k(X)$. Next, let $[A_1,V_1] \cap \ldots \cap [A_n,V_n]$ be a non-empty basic open set in $C_k(X)$ with $f \in [A_1,V_1] \cap \ldots \cap [A_n,V_n]$. We have $f \in [A_i,V_i]$ for each $i = 1,2,\ldots,n$. Now fix i. Since $f(A_i)$ is compact and $f(A_i) \subseteq V_i$ with V_i open in $\mathbb{R}$, choose $\varepsilon_i > 0$ such that $(z-\varepsilon_i, z+\varepsilon_i) \subseteq V_i$ for all $z \in f(A_i)$. Then it is straightforward to see that $\langle f,A_i,\varepsilon_i\rangle \subseteq [A_i,V_i]$. This is true for each $1 \leq i \leq n$. So, let $A = \cup A_i$ and $\varepsilon = \min\{\varepsilon_i : i = 1,2,\ldots,n\}$. Then $f \in \langle f,A,\varepsilon\rangle \subseteq \cap\{[A_i,V_i] : i = 1,2,\ldots,n\}$.

Using theorem 1.1.2, one can prove that a net $(f_\alpha)_{\alpha\in\Gamma}$ in $C_k(X)$ converges to a function f in $C_k(X)$ if and only if the net $(f_\alpha|_A)_{\alpha\in\Gamma}$ converges uniformly to $f|_A$ for each compact set A in X.

We now define a metric on $C(X)$ called the supremum metric. We then show that the topology generated by this metric is finer than the compact-open topology.

For each pair (a,b) of real numbers let $d(a,b) = \min(1,|a-b|)$. Then d is a bounded metric on $\mathbb{R}$ and generates the usual topology of $\mathbb{R}$.

Next, put $\rho(f,g) = \sup\{d(f(x),g(x)) : x \in X\}$ for each $(f,g) \in C(X) \times C(X)$. Then ρ is a complete metric on $C(X)$. The topology induced by ρ is called the supremum metric topology on $C(X)$. If we denote this space by $C_\rho(X)$, then $C_\rho(X)$ is a completely metrizable function space.

We should point out that in general the supremum metric topology depends on the choice of the compatible metric d on $\mathbb{R}$.

The relationship between this topology and the compact-open topology is given in the next result.

1.1.3. Theorem. $C_k(X) \leqslant C_p(X)$ for any space X.

Proof. Let A be compact in X and V open in $\mathbb{R}$. We must show that $[A,V]$ is open in $C_p(X)$. So, let $f \in [A,V]$. Since $f(A)$ is compact and $f(A) \subseteq V$, choose $\varepsilon > 0$ so that $B(z,\varepsilon) \subseteq V$ for all $z \in f(A)$, where $B(z,\varepsilon) = \{y \in \mathbb{R} : \min\{1, |z-y|\} < \varepsilon\}$. Now, let $W = B(f,\frac{\varepsilon}{2})$ be the open ball about f with radius $\frac{\varepsilon}{2}$ in $C_p(X)$. It is easy to see that $f \in W \subseteq [A,V]$. Therefore $[A,V]$ is open in $C_p(X)$.

For compact spaces the converse is also true.

1.1.4. Theorem. $C_k(X) = C_p(X)$ if and only if X is compact.

Proof. Suppose that $C_k(X) = C_p(X)$. Let $W = B(\theta,1) = \{f \in C_p(X) : \rho(\theta,f) < 1\}$, where θ is the zero-function in $C_p(X)$ and ρ the supremum metric on $C(X)$. Since W is open in $C_p(X) = C_k(X)$, find compact sets $A_1, A_2, \ldots, A_n$ in X and corresponding open sets $V_1, \ldots, V_n$ in $\mathbb{R}$ so that $\theta \in [A_1,V_1] \cap \ldots \cap [A_n,V_n] \subseteq W$. Put $A = \cup A_i$. Suppose there is $x_0 \in X-A$. Since A is compact and X Tychonoff pick $g \in C(X)$ with $g(x_0) = 1$ and $g(A) = \{0\}$. Then $g \in W$. But, $\rho(\theta,g) = \sup\{d(0,g(x)) : x \in X\} \geqslant d(0,g(x_0)) = d(0,1) = \min\{1, |0 - 1|\} = 1$. Now, $\rho(\theta,g) \geqslant 1$ means that $g \notin W$, which is a contradiction. Therefore $X = A$, and so X is compact.

Conversely, suppose X compact. It only remains to show that $C_p(X) \lessdot C_k(X)$. Let $f \in C(X)$ and $\varepsilon > 0$. It suffices to prove that the open ball $B(f,\varepsilon)$ about f of radius ε is open in $C_k(X)$. To see this, notice first that the collection $(f^{-1}(U_x):x \in X)$ where $U_x = (f(x)-\frac{\varepsilon}{4},f(x)+\frac{\varepsilon}{4})$ is an open cover of the compact space X. Therefore, for some n we have $X = \cup(f^{-1}(U_{x_i}) : i = 1,2,3,\ldots,n)$. Next let $A_i = \overline{f^{-1}(U_{x_i})}$ and $V_i = (f(x_i)-\frac{\varepsilon}{3},f(x_i)+\frac{\varepsilon}{3})$ for each $i = 1,2,\ldots,n$. Put $W = \cap([A_i,V_i]: i = 1,2,3,\ldots,n)$. Then $f \in W \subseteq B(f,\varepsilon)$. Since W is open in $C_k(X)$ and f was arbitrary then $B(f,\varepsilon)$ is open in $C_k(X)$ and therefore $C_p(X) \lessdot C_k(X)$.

1.1.5. Corollary. If X is compact then $C_k(X)$ is completely metrizable.

2. Natural maps

In this section we study some natural maps on $C_k(X)$. We begin with the embedding of $\mathbb{R}$ into $C_k(X)$. For each real number r, let $c_r : X \longrightarrow \mathbb{R}$ be the constant map defined by $c_r(x) = r$ for all x in X.

1.2.1. Proposition. The map $i : \mathbb{R} \longrightarrow C_k(X)$ defined by $i(r) = c_r$ for each real number r is an embedding of $\mathbb{R}$ into $C_k(X)$. Moreover, $i(\mathbb{R})$ is a closed subset of $C_k(X)$.

Proof. The function i is clearly injective. To show that i is continuous and open it suffices to notice that if A is compact in X and V open in $\mathbb{R}$ then $i^{-1}([A,V]) = V$. Finally, to see that $i(\mathbb{R})$ is closed in $C_k(X)$, let $f \in C_k(X)-i(\mathbb{R})$. Since f is not constant there are x_1 and x_2 in X with $f(x_1) \neq f(x_2)$. Choose V_1 and V_2 disjoint open sets in $\mathbb{R}$ such that $f(x_1) \in V_1$ and $f(x_2) \in V_2$. Then $f \in [(x_1),V_1] \cap [(x_2),V_2] \subseteq C_k(X)-i(\mathbb{R})$. Since $W = [(x_1),V_1] \cap [(x_2),V_2]$ is open in $C_k(X)$ and f was arbitrary in $C_k(X)-i(\mathbb{R})$ then $C_k(X)-i(\mathbb{R})$ is open in $C_k(X)$. Therefore $i(\mathbb{R})$ is closed.

Because of this proposition, it is clear that if P is a topological property inherited by closed subsets, then $C_k(X)$ has property P only if $\mathbb{R}$ has it. Consequently $C_k(X)$ is not pseudocompact for any space X. It also follows from this proposition that the cardinality of $C_k(X)$ is greater than or equal to the continuum.

One of the most useful tools in function spaces is the following concept of induced map. If $f : X \longrightarrow Y$ is a continuous map, then the induced map of f, denoted by $f^* : C_k(Y) \longrightarrow C_k(X)$ is defined by $f^*(g) = g \circ f$ for all g in $C_k(Y)$. The main properties of f^* are described in the next few pages.

1.2.2. Proposition. If $f : X \longrightarrow Y$ and $h : Y \longrightarrow Z$ are continuous functions, then $(h \circ f)^* = f^* \circ h^*$

Proof. We have $(h\circ f)^* : C_k(Z) \longrightarrow C_k(X)$ and $f^*\circ h^* : C_k(Z) \longrightarrow C_k(X)$. So, let $g \in C_k(Z)$. Then $(h\circ f)^*(g) = g\circ(h\circ f) = (g\circ h)\circ f = f^*(g\circ h) = f^*(h^*(g)) = (f^*\circ h^*)(g)$.

1.2.3. Proposition. If $f : X \longrightarrow Y$ is continuous and α, β are real numbers, then

(1) $f^*(\alpha g + \beta h) = \alpha f^*(g) + \beta f^*(h)$ for all g,h in $C_k(Y)$

(2) $f^*(gh) = f^*(g)f^*(h)$ for all g,h in $C_k(Y)$.

Proof. Straightforward.

1.2.4. Proposition. If $f: X \longrightarrow Y$ is continuous then $f^* : C_k(Y) \longrightarrow C_k(X)$ is continuous.

Proof. It suffices to show that if A is compact in X and V open in $\mathbb{R}$ then $(f^*)^{-1}([A,V])$ is open in $C_k(Y)$. But, $(f^*)^{-1}([A,V]) = [f(A),V]$ which is open in $C_k(Y)$ since $f(A)$ is compact in Y.

A continuous function $f : X \longrightarrow Y$ is called almost surjective if $f(X)$ is a dense subspace of Y. Almost surjective maps can be characterized by their induced maps.

1.2.5. Proposition. $f : X \longrightarrow Y$ is almost surjective if and

only if $f^* : C_k(Y) \longrightarrow C_k(X)$ is injective.

Proof. Suppose that f is almost surjective and let g_1 and g_2 be in $C_k(Y)$ with $f^*(g_1) = f^*(g_2)$. We wish to show that $g_1 = g_2$. Since they are continuous and real-valued it will suffice that they agree on the dense subspace $f(X)$ of Y. So, let $y \in f(X)$. Then there is $x \in X$ with $y = f(x)$. Now, $g_1(y) = g_1(f(x)) = f^*(g_1)(x) = f^*(g_2)(x) = g_2(f(x)) = g_2(y)$. Since this is true for all y in $f(X)$, we conclude that $g_1 = g_2$ on all of Y. Therefore f^* is injective.

Conversely suppose that $f^* : C_k(Y) \longrightarrow C_k(X)$ is injective. We must show that $f(X)$ is dense in Y. Suppose $f(X)$ is not dense in Y and let W be open in Y with $W \neq \emptyset$ and $W \cap f(X) = \emptyset$. Choose $y_0 \in W$. Since Y is Tychonoff, choose $g \in C_k(Y)$ such that $g(y_0) = 1$ and $g(Y-W) = \{0\}$. It follows that $g(f(X)) = \{0\}$. This means that $f^*(g)(x) = 0$ for all x in X. Therefore, if θ and θ_1 denote the zero functions of $C(X)$ and $C(Y)$ respectively then $f^*(g) = \theta = f^*(\theta_1)$. The injectivity of f^* implies that $g = \theta_1$ which is impossible since $g(y_0) = 1$. With this contradiction, we conclude that $f(X)$ must be dense in Y.

The dual of proposition 1.2.5 is also true.

1.2.6. Proposition. $f : X \longrightarrow Y$ is injective if and only if $f^*: C_k(Y) \longrightarrow C_k(X)$ is almost surjective.

Proof. Suppose that $f : X \to Y$ is injective. Let $\mathcal{R} = f^*(C_k(Y))$. Then we must show that $\mathcal{R}$ is dense in $C_k(X)$. Pick $g \in C_k(X)$, A compact in X and $\varepsilon > 0$ so that $W = \langle g, A, \varepsilon \rangle$. Now, let $Z = f(A)$. Then $f|_A : A \to Z$ is a homeomorphism since f is a continuous injection. Since Z is compact in the Tychonoff space Y, then the function $g \circ (f|_A)^{-1} : Z \to \mathbb{R}$ has a continuous extension $h : Y \to \mathbb{R}$. $h \in C(Y)$ and, if we put $t = h \circ f$, then $t = f^*(h) \in \mathcal{R}$. Moreover, $t = g$ on A. Therefore $t \in W \cap \mathcal{R}$. Hence $W \cap \mathcal{R} \neq \emptyset$; meaning that $\mathcal{R}$ is dense in $C_k(X)$.

For the converse, suppose that f^* is almost surjective and let x and y be two distinct points of X. We wish to show that $f(x) \neq f(y)$. Since X is Tychonoff, choose $g \in C(X)$ such that $g(x) \neq g(y)$ in $\mathbb{R}$. Let V and W be disjoint open neighborhoods of $g(x)$ and $g(y)$ respectively. Put $L = [\{x\}, V] \cap [\{y\}, W]$ which is a non-empty open set in $C_k(X)$. Since f^* is almost surjective, pick $h \in L \cap f^*(C_k(Y))$. Then $h = t \circ f$ for some $t \in C_k(Y)$, and $h(x) \neq h(y)$. But $h(x) = t(f(x))$ and $h(y) = t(f(y))$. So, $t(f(x)) \neq t(f(y))$. Therefore $f(x) \neq f(y)$. This shows that f is injective.

Our next goal is to characterize when f^* is an embedding. We need the following definitions. A continuous function $f : X \to Y$ is called a k-covering map provided each compact subspace of Y is contained in the image of some compact subspace of X. This concept is due to A. V. Arhangelskii. f will be called a compact-covering map if each compact subspace of Y is the image of a compact subspace of X. Clearly then compact-covering maps are k-covering maps.

The concept of k-covering maps is what we need for f^* to be an embedding.

1.2.7 Theorem. $f : X \longrightarrow Y$ is a k-covering map if and only if $f^* : C_k(Y) \longrightarrow C_k(X)$ is an embedding of $C_k(Y)$ into $C_k(X)$.

Proof. Assume that $f : X \longrightarrow Y$ is k-covering and let $\mathcal{R} = f^*(C_k(Y))$. By proposition 1.2.5, f^* is injective. It remains to show that $(f^*)^{-1} : \mathcal{R} \longrightarrow C_k(Y)$ is continuous. So, let A be compact in Y and V open in $\mathbb{R}$. We need to show that $f^*([A,V])$ is open in $\mathcal{R}$. Pick $g \in f^*([A,V])$. Then there is $h \in [A,V]$ such that $g = f^*(h) = h \circ f$. Since f is a k-covering map and A is compact in Y, choose C compact in X with $A \subseteq f(C)$. Let $W = [f^{-1}(A) \cap C, V] \cap \mathcal{R}$ which is open in $\mathcal{R}$. Then $g \in W \subseteq f^*([A,V])$.

Conversely suppose that $f^* : C_k(Y) \longrightarrow C_k(X)$ is an embedding. To see that $f : X \longrightarrow Y$ is k-covering, let A be compact in Y and $V = (-1,1)$. Then $f^*([A,V])$ is an open neighborhood of the zero function θ in $f^*(C_k(Y))$. So, choose compact sets $A_1, A_2, \ldots, A_n$ in X and corresponding open sets $V_1, V_2, \ldots, V_n$ in $\mathbb{R}$ such that $\theta \in [A_1, V_1] \cap \ldots \cap [A_n, V_n] \subseteq f^*([A,V])$. Put $K = A_1 \cup A_2 \cup \ldots \cup A_n$, which is compact in X. We must show that $A \subseteq f(K)$. Suppose by way of contradiction that there is $a \in A - f(K)$. Choose $g \in C(Y)$ such that $g(a) = 1$ and $g(f(K)) = \{0\}$. Then, $f^*(g)(K) = \{0\}$ implies that $f^*(g)(A_i) = \{0\}$ for each $i = 1,2,\ldots,n$. Therefore $f^*(g) \in [A_i, V_i]$ for each i. This implies that $f^*(g) \in f^*([A,V])$. So, $f^*(g) = f^*(g')$ for some $g' \in [A,V]$. Since f^* is injective, $g = g'$. But then $g(A) \subseteq V = (-1,1)$, which is impossible

since $g(a) = 1 \notin V$. Therefore, $A \subseteq f(K)$. This shows that f is k-covering.

1.2.8. Corollary. If $f : X \longrightarrow Y$ is a compact-covering map then $f^* : C_k(Y) \longrightarrow C_k(X)$ is an embedding.

A very natural question that arises is that of when f^* is a closed or an open embedding. Recall that $f^*(C_k(Y))$ is a subgroup of $C_k(X)$, f^* being a group homomorphism. The following fact is well-known (see [22]): If a subgroup of a topological group has a non-empty interior in the group then it is both an open and a closed subspace of the group. Therefore, the question of when f^* is an open embedding is a special case of that of when it is a closed embedding. More precisely, we have the following.

1.2.9. Proposition. Let $f : X \longrightarrow Y$ be continuous and $f^* : C_k(Y) \longrightarrow C_k(X)$ its induced map. If $f^*(C_k(Y))$ is open in $C_k(X)$ then it is also closed in $C_k(X)$ and therefore f^* is onto (since $C_k(X)$ is connected).

For closed embeddings we have the following.

1.2.10. Proposition. If $f : X \longrightarrow Y$ is a quotient map then $f^*(C_k(Y))$ is a closed subspace of $C_k(X)$.

Proof. Put $T = C_k(X) - f^*(C_k(Y))$. We need to show that T is open in $C_k(X)$. So, let $g \in T$. Since f is quotient, choose x and y in X such that $f(x) = f(y)$ but $g(x) \neq g(y)$. Let V and V' be two disjoint open neighborhoods in $\mathbb{R}$ for $g(x)$ and $g(y)$ respectively. If $W = [(x),V] \cap [(y),V']$, then $g \in W$ and W is open in $C_k(X)$. It remains to show that $W \subseteq T$. So, let $h \in W$. Since $V \cap V' = \emptyset$, $h(x) \neq h(y)$. If $h \notin T$, then $h \in f^*(C_k(Y))$ so that $h = f^*(t)$ for some $t \in C(Y)$. Therefore $h(x) = f^*(t)(x) = (t \circ f)(x) = t(f(x)) = t(f(y)) = f^*(t)(y) = h(y)$ which is impossible. Therefore $h \in T$. Thus, $g \in W \subseteq T$. This means that T is open in $C_k(X)$.

Let S be a closed subspace of X. If $i : S \longrightarrow X$ is the inclusion map, then $i^* : C_k(X) \longrightarrow C_k(S)$ is defined by $i^*(f) = f|_S$ for all $f \in C(X)$. Since i is injective, $i^*(C_k(X))$ is dense in $C_k(S)$ by proposition 1.2.6. In fact, more is true.

1.2.11. Proposition. Let S be a closed subspace of X and $i : S \longrightarrow X$ the inclusion map. If $\mathcal{R} = i^*(C_k(X))$ then $i^* : C_k(X) \longrightarrow \mathcal{R}$ is an open map.

Proof. Let θ be the zero function in $C_k(X)$ and $\theta' = \theta|_S$. Since i^* is a group homomorphism (by proposition 1.2.3), it suffices to show that if A is compact in X and V is open in $\mathbb{R}$ with $\theta \in [A,V]$ then $i^*([A,V])$ is an open neighborhood of θ' in $\mathcal{R}$. Now, there is no loss of generality by assuming that $V = (-1,1)$ and $A \cap S \neq \emptyset$. Let $K = A \cap S$ and put $W = [K,V]$

$\cap \mathbb{R}$. Then W is open in $\mathbb{R}$. Clearly $\theta' \in W$. We are through if we can show that $W \subseteq i^*([A,V])$. So let $h \in W$. Then $h = f|_S$ for some $f \in C(X)$. Now, let $T = \{x \in X : |f(x)| \geq 1\}$. If $T = \emptyset$, then $|f(x)| < 1$ for all x in X. Therefore $f \in [A,V]$. In this case $h \in i^*([A,V])$. If $T \neq \emptyset$, note first that $(T \cap A) \cap S = \emptyset$. Therefore, we can find $g : X \longrightarrow [0,1]$ continuous such that $g(T \cap A) = \{0\}$ and $g(S) = \{1\}$. Let $p = gf$. Then $p \in C(X)$ and for all $x \in S$ we have $p(x) = g(x)f(x) = 1.f(x) = f(x)$. Therefore, $p|_S = f|_S = i^*(f) = h$. It remains to show that $p \in [A,V]$. Note that $A = (T \cap A) \cup (A \cap S) \cup (A - ((T \cap A) \cup (A \cap S)))$. For all x in $T \cap A$, $p(x) = g(x)f(x) = 0.f(x) = 0 \in V$. So, $p(T \cap A) \subseteq V$. Next, by definition of T, $|f(x)| < 1$ for all $x \in X-T$. Since $(A \cap S) \cup (A - ((T \cap A) \cup (A \cap S))) \subseteq X-T$, then $p((A \cap S) \cup (A - ((T \cap A) \cup (A \cap S)))) \subseteq V$. Finally, we conclude that $p(A) \subseteq V$. Therefore $p \in [A,V]$. Consequently, $h \in i^*([A,V])$ for all $h \in W$. Thus $\theta' \in W \subseteq i^*([A,V])$ and so, $i^*([A,V])$ is an open neighborhood of θ' in $\mathbb{R}$.

Using the Tietze extension theorem, we obtain the following corollary.

1.2.12. Corollary. Let S be a closed subspace of a normal space X. If $i : S \longrightarrow X$ is the inclusion map, then $i^* : C_k(X) \longrightarrow C_k(S)$ is an open surjection.

Another kind of useful map on function spaces is the "sum function". Let $\{X_\alpha : \alpha \in \Gamma\}$ be a family of topological spaces. If $\oplus X_\alpha$

denotes their topological sum (i.e. free union or disjoint sum) then the sum function s is defined by $s : C_k(\oplus X_\alpha) \to \Pi(C_k(X_\alpha) : \alpha \in \Gamma)$ where $s(f) = \langle f|_{X_\alpha} \rangle_\alpha$ for each $f \in C_k(\oplus X_k)$. Here, $\Pi(C_k(X_\alpha : \alpha \in \Gamma)$ is the cartesian product of the $C_k(X_\alpha)$'s with the Tychonoff product topology. The sum function is a very nice map.

1.2.13. Proposition. The sum function $s : C_k(\oplus X_\alpha) \to \Pi(C_k(X_\alpha) : \alpha \in \Gamma)$ is a homeomorphism.

Proof. (1) s is a bijection: Define $v : \Pi(C_k(X_\alpha) : \alpha \in \Gamma) \to C_k(\oplus X_\alpha)$ by $v(\langle g_\alpha \rangle_{\alpha \in \Gamma}) = g_\beta$ on X_β for each $\beta \in \Gamma$ and each $\langle g_\alpha \rangle_\alpha$ in $\Pi(C_k(X_\alpha : \alpha \in \Gamma)$. It is easy to see that $v \circ s$ = the identity map on $C_k(\oplus X_\alpha)$ and $s \circ v$ = the identity map on $\Pi(C_k(X_\alpha) : \alpha \in \Gamma)$.

(2) s and s^{-1} are continuous: Let $p_\alpha : \Pi(C_k(X_\alpha) : \alpha \in \Gamma) \to C_k(X_\alpha)$ be the α^{th} projection, A compact in X_α and V open in $\mathbb{R}$. Then $[A,V] = s^{-1}(p_\alpha^{-1}([A,V]))$. This clearly implies that both s and s^{-1} are continuous maps.

Let us point out that several other kinds of important maps on function spaces such as exponential maps and product maps are not studied in this work. [26] is a good reference on this matter.

3. Separation axioms

The set C(X) is a subset of the set F(X) of all real-valued

functions on X. We topologize F(X) with the topology having as subbase the collection $\{[A,V]_F$: A is compact in X and V a bounded open interval in $\mathbb{R}\}$, where $[A,V]_F = \{f \in F(X) : \overline{f(A)} \subseteq V\}$. This topology will be called the compact-open topology on F(X) and will be denoted by $F_k(X)$. If one replaces "compact" with "finite" in the above subbase, one obtains a subbase for the topology of pointwise convergence on F(X). This is actually the product topology on $\mathbb{R}^X$. Note that $C_p(X)$ is dense in $\mathbb{R}^X$. The next result whose proof is in [46] describes the corresponding situation on $F_k(X)$.

1.3.1. Proposition. $C_k(X)$ is a dense subspace of $F_k(X)$.

Since the identity map id : $F_k(X) \longrightarrow \mathbb{R}^X$ is clearly continuous one deduces in a very natural way the following.

1.3.2. Proposition. $F_k(X)$ is a T_1-space.

In fact, more is true : $F_k(X)$ is a Tychonoff space. To see this we need two lemmas. Let us denote by $F_k(X,[0,1])$ the set $F(X,[0,1]) = \{f \in F(X) : f(X) \subseteq [0,1]\}$ as a subspace of $F_k(X)$.

1.3.3. Lemma. Let $g : \mathbb{R} \longrightarrow [0,1]$ be a continuous function. Then $g_* : F_k(X) \longrightarrow F_k(X,[0,1])$ defined by $g_*(f) = g \circ f$ for each f in $F_k(X)$, is a continuous function.

1.3.4. Lemma. Let A be a compact subset of X. Then φ : $F_k(X,[0,1]) \longrightarrow [0,1]$ defined by $\varphi(f) = \sup\overline{f(A)}$ for each f in $F_k(X,[0,1])$, is a continuous function.

1.3.5. Proposition. $F_k(X)$ is a Tychonoff space.

Proof. Let A be compact in X and V a bounded open interval in $\mathbb{R}$. It suffices to show that for each f in $[A,V]_F$ there exists a continuous function $\varphi : F_k(X) \longrightarrow [0,1]$ such that $\varphi(f) = 0$ and $\varphi(F_k(X)-[A,V]_F) = \{1\}$. To do that, let f be in $[A,V]_F$. Since $\overline{f(A)} \subseteq V$ and V is bounded then $\overline{f(A)}$ is a compact subset of $\mathbb{R}$. Since $\mathbb{R}$ is completely regular, choose $g : \mathbb{R} \longrightarrow [0,1]$ continuous with $g(\overline{f(A)}) = \{0\}$ and $g(\mathbb{R}-V) = \{1\}$. Define $\varphi : F_k(X) \longrightarrow [0,1]$ by $\varphi(h) = \sup(\overline{g(h(A))})$ for all $h \in F_k(X)$. Then define $\Psi : F_k(X,[0,1]) \longrightarrow [0,1]$ by $\Psi(t) = \sup\overline{(t(A))}$ for each $t \in F_k(X,[0,1])$. Let $g_* : F_k(X) \longrightarrow F_k(X,[0,1])$ be defined by $g_*(h) = g\circ h$ for all h in $F_k(X)$. By lemma 1.3.3 g_* is continuous, and by lemma 1.3.4 Ψ is also continuous. Clearly $\varphi = \Psi\circ g_*$ and so φ is continuous. Finally one sees that $\varphi(f) = 0$ and $\varphi(F_k(X)-[A,V]_F) = \{1\}$.

1.3.6. Corollary. $C_k(X)$ is a Tychonoff space.

We conclude this chapter with the relationship between locally compact, Lindelöf and first countable spaces.

4. Locally compact, Lindelöf and first countable spaces

A topological space X is σ-compact if X can be written as a countable union of compact subspaces.

X is hemicompact if $X = A_1 \cup A_2 \cup \ldots$ where each A_n is compact in X and if A is any compact subspace of X then A is contained in at least one A_n. The sequence $(A_n : n = 1,2,\ldots)$ will be called a fundamental system of compact subspaces of X.

X is a k-space if the open subspaces of X are precisely those subspaces whose intersections with each compact subspace of X are open in the compact subspace.

X is a $k_{\mathbb{R}}$-space if and only if the set $\{f \in F(X) : f|_A$ is continuous for each compact subset A of $X\} \subseteq C(X)$.

The following implications are true: hemicompact implies σ-compact implies Lindelöf. None of them is reversible. But, when X is locally compact (i.e. each point has a compact neighborhood) we get a positive result.

1.4.1. Proposition. Every locally compact Lindelöf space is hemicompact .

To generate local compactness from a hemicompact space, one

needs to have first countability.

1.4.2. Proposition. Let X be a first countable space. Then X is hemicompact if and only if X is locally compact and Lindelöf.

In fact, by modifying Arens' original proof of proposition 1.4.2, one may show that X is a hemicompact q-space if and only if X is a locally compact Lindelöf space (see next chapter for the definition of a q-space).

It is well known that every k-space is a $k_{\mathbb{R}}$-space. The converse is not true. But, for hemicompact spaces the two notions are identical.

1.4.3. Theorem. If X is a hemicompact $k_{\mathbb{R}}$-space then X is a k-space.

Proof. Let $\{K_n : n = 1,2,\ldots\}$ be a fundamental system of compact subspaces of X such that $K_n \subseteq K_{n+1}$ for each n. Let A be a subspace of X such that $A \cap K_n$ is closed in K_n for each n. We are through if we show that A is closed in X because of hemicompactness. If $A = X$ or $A = \emptyset$ there is nothing to prove. So, let $x_0 \notin A$. There is no loss of generality by assuming that $x_0 \in K_1$. Since K_1 is compact and $A \cap K_1$ is closed in K_1 with $x_0 \notin A \cap K_1$, choose $f_1 : K_1 \longrightarrow [0,1]$ continuous such

that $f_1(x_0) = 0$ and $f_1(A \cap K_1) = (1)$. Then define $g_1: K_1 \cup (K_2 \cap A) \longrightarrow$ $[0,1]$ by $g_1 = f_1$ on K_1 and $g_1(x) = 1$ for all x in $K_2 \cap A$. g_1 is clearly continuous on the compact subspace $K_1 \cup (K_2 \cap A)$ of K_2. By Tieze extension theorem, g_1 has a continuous extension $f_2: K_2 \longrightarrow [0,1]$. Similarly, we construct $g_2: K_2 \cup (K_3 \cap A) \longrightarrow [0,1]$ from f_2 by: $g_2 = f_2$ on K_2 and $g_2(x) = 1$ for each x in $K_3 \cap A$. Then the extension of g_2 on K_3 is f_3; and so on. Now, define $f: X \longrightarrow \mathbb{R}$ by $f = f_n$ on K_n for each n. Then f is continuous on each K_n, hence on each compact set of X. Since X is a $k_{\mathbb{R}}$-space, f must finally be continuous on all of X. Clearly $A = f^{-1}(1)$, which is closed in X.

CHAPTER 2 : MAIN COUNTABILITY PROPERTIES

The main purpose of this chapter is the study of the so-called countability properties. These are properties involving some countable set in their definitions. We investigate properties such as separability, metrizability, first and second countability, and the properties of being a cosmic and an $\aleph_0$-space. On the function space $C_k(X)$, each property is fully characterized in terms of topological properties on X. Recall that all our spaces are Tychonoff and ω is the cardinality of the set $\mathbb{N}$.

Following Juhász [36], if X is a topological space, we define

$d(X) = \omega+\min\{|S|:S \subseteq X \text{ and } \bar{S} = X\}$=the density of X ;

$w(X) = \omega+\min\{|\mathcal{B}|:\mathcal{B} \text{ is a base of } X\}$ = the weight of X ;

$\pi w(X) = \omega+\min\{|\mathcal{B}|:\mathcal{B} \text{ is a } \pi\text{-base of } X\}$ = the π-weight of X ;

By a π-base for a topological space X we mean any collection $\mathcal{B}$ of non-empty open sets of X with the property that if V is any non-empty open set in X then V contains some member of $\mathcal{B}$.

A space X has countable local π-base if for each x in X there is a countable collection $\mathcal{B}_x$ of non-empty open sets such that any neighborhood of x contains some member of $\mathcal{B}_x$. This is clearly weaker than first countability.

From above, one easily sees that: X is separable if and only

if $d(X) \leqslant \omega$; X is second countable if and only if $w(X) \leqslant \omega$; X has countable π-base if and only if $\pi w(X) \leqslant \omega$.

Let us recall that if $\mathcal{P}$ is a topological property, then a space X has a $\mathcal{P}$-compression (or $\mathcal{P}$-contraction) if and only if there exist a topological space Y having property $\mathcal{P}$ and a continuous bijection from X onto Y. This is equivalent to say that the topology of X admits a smaller (i.e. coarser) topology having property $\mathcal{P}$. A submetrizable space is one that has a metrizable compression.

1. Separability

The first result describes a technique to generate topologies on $C(X)$ finer than the compact-open topology.

2.1.1. Theorem. Let $\mathcal{B}$ and $\mathcal{V}$ be bases for X and $\mathbb{R}$ respectively. If τ is the topology with subbase $\{[B,V]: B \in \mathcal{B}$ and $V \in \mathcal{V}\}$ then $C_k(X) \leqslant C_\tau(X)$.

Proof. Recall first that $[B,V] = \{f \in C(X): f(B) \subseteq V\}$. Now, let A be compact in X and $V \in \mathcal{V}$. It suffices to show that $[A,V] \in \tau$. To see this, let $f \in [A,V]$. Then $f(A) \subseteq V$. Let $a \in A$. Since $f(a) \in V$, by continuity of f there is $B_a \in \mathcal{B}$ with $a \in B_a$ and $f(B_a) \subseteq V$. The collection $\{B_a: a \in A\}$ is a cover of the compact set A by open subsets of X. Therefore, for some n, $A \subseteq B_{a_1} \cup B_{a_2} \cup \ldots \cup B_{a_n}$. Put $W = [B_{a_1}, V]$

$\cap \ldots \cap [B_{a_n}, V]$ which is open in $C_\tau(X)$. Then $f \in W \subseteq [A,V]$. Consequently $C_k(X) \leqslant C_\tau(X)$.

If $\mathcal{B}$ and $\mathcal{V}$ are countable bases in the above theorem then $C_\tau(X)$ will have countable base. Since the identity map id: $C_\tau(X) \longrightarrow C_k(X)$ is continuous then we have the following corollary.

2.1.2. Corollary. If X is second countable then $C_k(X)$ is separable.

The converse to this corollary is not true in general. In fact the next theorem describes when $C_k(X)$ is separable.

We need the following example: the hedgehog space with 2^ω spines. If $I = [0,1]$, let $\sim$ be the equivalence relation on $I \times I$ defined by $(a,b) \sim (c,d)$ if and only if either $(a,b) = (c,d)$ or $b = 0 = d$. For each $s \in I$, let $I_s = (s) \times I$. Then the hedgehog space is the space $H = \oplus\{I_s : s \in I\}/\sim$ with the quotient topology. It is easy to see that the identity map id: $\oplus\{I_s : s \in I\} \longrightarrow I \times I$ is continuous. Therefore the natural map p: $H \longrightarrow I \times I/\sim$ is continuous. Since $I \times I/\sim$ is separable metrizable then H has a separable metrizable compression.

A subset $\mathcal{F}$ of $C(X)$ is said to separate points of X provided for each $(x,y) \in X \times X$ with $x \neq y$ there exists an f in $\mathcal{F}$ such that $f(x) \neq f(y)$.

2.1.3. Theorem. The following statements are equivalent:

(1) $C_k(X)$ is separable

(2) $C_k(X)$ contains a countable subspace that separates points of X

(3) X admits a continuous injection into $\mathbb{R}^{\mathbb{N}}$

(4) X has a separable metrizable compression

(5) X is submetrizable and $|X| \leq 2^{\omega}$

(6) X is submetrizable and $d(X) \leq 2^{\omega}$.

Proof. (1) $\Rightarrow$ (2): If $C_k(X)$ is separable, let D be a countable dense subspace of $C_k(X)$. Then D separates points of X since X is Tychonoff. (2) $\Rightarrow$ (3): Suppose that $D = \langle f_1, f_2, \ldots \rangle$ is a countable subspace in $C_k(X)$ that separates points of X. Define $\varphi: X \longrightarrow \mathbb{R}^{\mathbb{N}}$ by $\varphi(x) = \langle f_1(x), f_2(x), \ldots \rangle$. If $p_n: \mathbb{R}^{\mathbb{N}} \longrightarrow \mathbb{R}$ is the nth projection, then $f_n = p_n \circ \varphi$. Since f_n and p_n are continuous, then so is φ. It remains to check that φ is injective. So let x and y be in X with $x \neq y$. Since D separates points of X there is $f_n \in D$ such that $f_n(x) \neq f_n(y)$. Therefore $\varphi(x) = \langle f_1(x), \ldots, f_n(x), \ldots \rangle \neq \langle f_1(y), \ldots, f_n(y), \ldots \rangle = \varphi(y)$.

(3) $\Rightarrow$ (4): Let $\varphi: X \longrightarrow \mathbb{R}^{\mathbb{N}}$ be a continuous injection. If $M = \varphi(X)$ then $M \subseteq \mathbb{R}^{\mathbb{N}}$ and so M is separable metrizable. Since $\varphi: X \longrightarrow M$ is a continuous bijection, then M is a separable metrizable compression for X.

(4) $\Rightarrow$ (5): Let $f: X \longrightarrow M$ be a continuous bijection and M a separable metrizable space. If D is a dense countable subspace of M then $|M| \leq |\{(a_n)_{n=1}^{\infty} : (a_n)_n \subseteq D\}| = 2^{\omega}$. Consequently $|X| = |M| \leq 2^{\omega}$.

(5) $\Rightarrow$ (6): This follows from the fact that $d(X) \leq |X|$. (6) $\Rightarrow$ (4): Suppose that X has a metrizable compression M and $d(X) \leq 2^{\omega}$. Let

$f: X \longrightarrow H^{\mathbb{N}}$ be a continuous injection of X into $H^{\mathbb{N}}$ where H is the hedgehog space (see [26] for this injection). If $f(X) = Z \subseteq H^{\mathbb{N}}$ then Z has a separable metrizable compression, which is also a compression for X.

(4) $\Rightarrow$ (1): Let $f: X \longrightarrow M$ be a continuous bijection and M a separable metrizable space. Then $f^*: C_k(M) \longrightarrow C_k(X)$ is almost surjective by proposition 1.2.6. By corollary 2.1.2, $C_k(M)$ is separable. Consequently $C_k(X)$ is separable.

The equivalence of (1) and (4) appears in Warner [65]. Vidossich [64] established (4) $\Leftrightarrow$ (5) $\Leftrightarrow$(6). See also McCoy [41].

2. metrizability and first countability

The main result of this section states that any property between metrizability and countable local π-base is equivalent to metrizability on $C_k(X)$. Let us mention some of them.

X is a developable space if it has a development. By a development of X we mean a sequence $(\mathcal{U}_n)_n$ of open covers of X such that $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$ for each n and $(st(x,\mathcal{U}_n))_n$ is a neighborhood base at x for each x in X. Here $st(x,\mathcal{U}_n) = \cup\{U \in \mathcal{U}_n: x \in U\}$. A regular Hausdorff developable space is called a Moore space.

X is semimetrizable if X admits a semimetric compatible with its topology. A semimetric is a metric without the triangle inequality.

X has a uniform base if it has a collection $\mathcal{B}$ of open sets such that for any $x \in X$, $(B_n)_{n \in \mathbb{N}}$ is a base at some point in X whenever $x \in B_n \in \mathcal{B}$ for all n.

X has a point-countable base if there exist a metrizable space Y and an open continuous surjection $f: Y \longrightarrow X$ such that each $f^{-1}(x)$ is second countable for all x in X.

X has a base of countable order (B.C.O.) if X admits a collection $\mathcal{B}$ of open sets such that if $B_1 \supseteq B_2 \supseteq \ldots$ is a sequence of distinct members of $\mathcal{B}$ each containing a fixed point x, then $(B_n)_{n \in \mathbb{N}}$ is a neighborhood base at x.

X has an orthobase $\mathcal{B}$ if $\mathcal{B}$ is a collection of open sets in X with the property that every infinite subfamily $\mathcal{B}'$ of $\mathcal{B}$ each member of which contains a given point of X, has an open intersection (i.e. $\cap\mathcal{B}'$ is open in X). Paracompact spaces with an orthobase are called protometrizable spaces.

For most of these notions see Aull [10].

2.2.1. Theorem. The following are equivalent:

(1) $C_k(X)$ has a countable local π-base

(2) $C_k(X)$ contains a dense subspace of countable local π-base

(3) $C_k(X)$ is first countable

(4) $C_k(X)$ is a developable (i.e. a Moore) space

(5) $C_k(X)$ is semimetrizable

(6) $C_k(X)$ has a uniform base

(7) $C_k(X)$ has a point countable base

(8) $C_k(X)$ has a base of countable order

(9) $C_k(X)$ has an orthobase

(10) $C_k(X)$ is protometrizable

(11) $C_k(X)$ is locally metrizable

(12) $C_k(X)$ contains a non-empty open metrizable subspace

(13) $C_k(X)$ is metrizable

(14) X is hemicompact.

Proof. Recall that a homogeneous space has countable local π-base if and only if it contains a dense subspace having countable local π-base. So (1) $\Leftrightarrow$ (2). Next, since $C_k(X)$ is connected then properties (3) - (13) are all stronger than (1) but weaker than (13). So we only need to show that (14) $\Rightarrow$ (13) $\Rightarrow$ (1) $\Rightarrow$ (14). (13) $\Rightarrow$ (1) is clear.

(14) $\Rightarrow$ (13): Suppose that X is hemicompact. Let $\{K_n: n=1,2,...\}$ be a fundamental system of compact subspaces of X. The trivial map $p: \oplus\{K_n: n=1,2,...\} \longrightarrow X$ is clearly compact-covering. By corollary 1.2.8 $p^*: C_k(X) \longrightarrow C_k(\oplus K_n)$ is an embedding of $C_k(X)$ into $C_k(\prod_n K_n)$. By proposition 1.2.13, $C_k(\oplus K_n) = \Pi\{C_k(K_n): n=1,2,...\}$. By corollary 1.1.5, each $C_k(K_n)$ is completely metrizable. Therefore $C_k(\oplus K_n)$ is metrizable and so, its subspace $C_k(X)$ is metrizable.

(1) $\Rightarrow$ (14): Suppose that $C_k(X)$ has countable local π-base. Let $\{W_n: n=1,2,...\}$ be a countable π-base at the zero function θ in $C_k(X)$. By definition every non-empty neighborhood of θ contains some

W_n. Now, there is no loss of generality if we choose each W_n basic open in $C_k(X)$, say $W_n = [A_{n1},V_{n1}] \cap \ldots \cap [A_{nk_n},V_{nk_n}]$ where each A_{ni} is compact in X and V_{ni} is open in $\mathbb{R}$. Put $K_n = A_{n1} \cup \ldots \cup A_{nk_n}$ for each n.

If K is any compact subspace of X, we wish to show that $K \subseteq K_n$ for some n. Let $W = [K,(-1,1)]$ which is an open neighborhood of θ. Suppose that $K \not\subseteq K_n$ for all n. Fix n and let $x \in K-K_n$. Since X is Tychonoff, choose $f \in C(X)$ with $f(x) = 1$ and $f(K_n) = \{0\}$. Clearly $f \notin W$. But $f(A_{ni}) \subseteq f(K_n) = \{0\}$ for each $i = 1,2,\ldots,k_n$. Therefore $f \in [A_{n1},V_{n1}] \cap \ldots \cap [A_{nk_n},V_{nk_n}]$, meaning that $f \in W_n$. Therefore, $W_n \not\subseteq W$. Since this is true for all n then no W_n is contained in W. This contradicts the definition of a π-base. Consequently $K \subseteq K_n$ for some n. Thus $(K_n: n = 1,2,\ldots)$ is a fundamental system of compact subsets in X. This means that X is hemicompact.

The next two corollaries follow from theorem 2.2.1 and propositions 1.4.1 and 1.4.2.

2.2.2. Corollary. Let X be locally compact. Then $C_k(X)$ is first countable if and only if X is Lindelöf.

2.2.3. Corollary. Let X be first countable. Then $C_k(X)$ is first countable if and only if X is locally compact and Lindelöf.

In theorem 2.2.1 we showed that $C_k(X)$ has point countable base

if and only if $C_k(X)$ is first countable. One wonders if "point countable base" can be weakened to "point countable π-base". The next example answers this question in the negative.

Let X be the set $\mathbb{R}$ of real numbers with the following topology: $A \subseteq X$ is open if and only if $X-A$ is countable or $0 \notin X-A$. This space is called the Fortissimo space on $\mathbb{R}$. Compact sets in X are finite. Therefore $C_k(X) = C_p(X)$ and X is not σ-compact. Corson [20] showed that $C_k(X)$ = $\sum$-product of 2^{ω}-copies of $\mathbb{R}$. It is shown in [23] that any subspace of a $\sum$-product of spaces with point-countable base has a point countable π-base. Therefore, $C_k(X)$ has point countable π-base. Since X is not hemicompact, $C_k(X)$ can not be first countable.

Let us mention that in the next chapter, theorem 2.2.1 will be extended to properties weaker than first countability.

3. Cosmic and $\aleph_0$-spaces

The concepts of cosmic and $\aleph_0$-spaces were introduced by Michael [47]. They may be interpreted as strengthenings of hereditary Lindelöf and hereditary separable spaces. We need the following definitions .

A family $\mathcal{B}$ of subsets of a topological space X is called a network for X if each open subset of X is a union of some subfamily of $\mathcal{B}$. A base for X is then a network whose members are open.

We then define the net-weight of X by

$$nw(X) = \omega + \min(|\mathcal{B}| : \mathcal{B} \text{ is a network of } X).$$

A cosmic (respectively a σ-)space is a regular space with a countable (respectively a σ-locally finite or σ-discrete) network.

A family $\mathcal{B}$ of subsets of X is called a k-network for X if for each compact set K in X and each open set V in X with $K \subseteq V$ there is n such that $K \subseteq B_1 \cup B_2 \cup \ldots \cup B_n \subseteq V$, where $B_i \in \mathcal{B}$ for each i.

The k-net-weight of X is defined by

$$knw(X) = \omega + \min(|\mathcal{B}| : \mathcal{B} \text{ is a k-network of } X).$$

The following inequalities are well known: $d(X) \leqslant nw(X) \leqslant knw(X) \leqslant w(X)$; and $nw(X) \leqslant |X|$. Notice also that for a regular space X one has $w(X) \leqslant 2^{d(X)}$.

An $\aleph_0$-(respectively an $\aleph$-)space is a regular space with a countable (respectively a σ-locally finite or σ-discrete) k-network.

It is clear that X is cosmic if and only if $nw(X) \leqslant \omega$; and similarly X is an $\aleph_0$-space if and only if $knw(X) \leqslant \omega$.

σ-spaces were studied in [52] and $\aleph$-spaces are due to O'Meara [56].

The main properties of cosmic and $\aleph_0$-spaces are described in the next propositions.

2.3.1. Proposition.

1) Every subspace of a cosmic (an $\aleph_0$-)space is a cosmic (an $\aleph_0$-)space;

2) Every $\aleph_0$-space is cosmic;

3) Every second countable space is an $\aleph_0$-space;

4) The following are equivalent:

a) X is cosmic

b) X is a continuous image of a separable metrizable space

c) X is a compression of a separable metrizable space

d) $C_p(X)$ is cosmic;

5) Every cosmic space is hereditarily separable and hereditarily Lindelöf;

6) The image of a cosmic space under a continuous map is cosmic;

7) The image of an $\aleph_0$-space under a continuous closed map is an $\aleph_0$-space;

8) $C_p(X)$ is an $\aleph_0$-space if and only if X is countable;

9) Cosmic and $\aleph_0$-properties are preserved by countable products and sums;

10) Every cosmic space is perfect (i.e. closed sets are G_δ).

Following Michael [47], a space X is a q-space if and only if for each point x in X there is a sequence $(V_n : n \in \mathbb{N})$ of neighborhoods of x such that if $x_n \in V_n$ for each n, then the set $(x_n : n = 1,2,\ldots)$ has a cluster point in X. Clearly, first countable spaces and locally compact spaces are all q-spaces.

The next result is an extension of a theorem in Michael [47].

2.3.2. Proposition. A topological space X is separable

metrizable if and only if X is an $\aleph_0$-space and a q-space.

In fact, it can be shown that X is metrizable if and only if X is an $\aleph$-space and a q-space. One then recalls that X is an $\aleph_0$-space if and only if it is a Lindelöf $\aleph$-space (if and only if X is a hereditarily separable $\aleph$-space).

2.3.3. Proposition. X is a σ-compact cosmic space if and only if X has a countable network made of compact subspaces.

Proof. Suppose that $\mathcal{P} = \{P_m : m = 1,2,\ldots\}$ is a countable network for X. We may choose each P_m closed. Let $X = K_1 \cup K_2 \cup \ldots$ where each K_n is compact in X. Then $\mathcal{L} = \{P_m \cap K_n : m,n \in \mathbb{N}\}$ is a countable network for X and each $P_m \cap K_n$ is compact.

For the converse, if $\mathcal{P} = \{P_m : m = 1,2,\ldots\}$ is a countable network for X with each P_m compact then $X = P_1 \cup P_2 \cup \ldots$. Therefore X is σ-compact. By proposition 2.3.2, each P_m is separable metrizable, hence cosmic. Now, a countable union of closed cosmic spaces is cosmic.

2.3.4. Corollary. Every σ-compact submetrizable space is cosmic.

2.3.5. Proposition. X is a hemicompact $\aleph_0$-space if and only if X has a countable k-network made of compact subspaces.

We now turn to cosmic and $\aleph_0$-space function spaces. By proposition 2.3.1, we already know that $C_p(X)$ is cosmic if and only if X is cosmic; and $C_p(X)$ is an $\aleph_0$-space if and only if X is countable. For $C_k(X)$ the situation is a bit different.

2.3.6. Lemma. If X is separable metrizable then $C_k(X)$ is an $\aleph_0$-space.

Proof. Let $\mathcal{B}$ and $\mathcal{V}$ be countable bases for X and $\mathbb{R}$ respectively. We may assume that $\mathcal{B}$ is closed under finite unions. Let $\mathcal{N} = \{[B,V]: B \in \mathcal{B}, V \in \mathcal{V}\}$. Then $\mathcal{N}$ is countable. By lemma 5.1 in [47] $C_k(X)$ will be an $\aleph_0$-space if we can show that whenever K is compact in $C_k(X)$, A compact in X and $V \in \mathcal{V}$ with $K \subseteq [A,V]$, then $K \subseteq N \subseteq [A,V]$ for some $N \in \mathcal{N}$. So, let K, A and V be as above with $K \subseteq [A,V]$. Put $W = \{x \in X: f(x) \in V, f \in K\}$. Then W is open in X and $A \subseteq W$. Since A is compact in X and $\mathcal{B}$ is a base for X closed under finite unions, choose $B \in \mathcal{B}$ with $A \subseteq B \subseteq W$. Put $N = [B,V]$. Then $N \in \mathcal{N}$ and $K \subseteq N \subseteq [A,V]$ as desired.

If $K(X)$ denotes the collection of non-empty compact subsets of X, let us topologize $K(X)$ with the Vietoris (i.e. finite) topology. Basic open sets in $K(X)$ are of the form $\langle V_1,\ldots,V_n\rangle = \{A \in K(X): A \subseteq V_1 \cup\ldots\cup V_n$ and $A \cap V_i \neq \emptyset$ for each $i\}$ where $V_1, V_2,\ldots,V_n$ are open sets in X. $K(X)$ is then called the hyperspace of compact sets of X. Throughout this work, $K(X)$ will only carry this topology.

2.3.7. Theorem. The following are equivalent:

(1) X is an $\aleph_0$-space

(2) X is a compact-covering image of a separable metrizable space

(3) $C_k(X)$ is an $\aleph_0$-space

(4) $C_k(X)$ is a cosmic space

(5) $K(X)$ is an $\aleph_0$-space

(6) $K(X)$ is a cosmic space.

Proof. (1) $\Rightarrow$ (2): follows from theorem 11.4 of [47]. (2) $\Rightarrow$ (3): Let $f: M \longrightarrow X$ be a compact-covering map from a separable metrizable space M onto X. By corollary 1.2.8, $f^*: C_k(X) \longrightarrow C_k(M)$ is an embedding of $C_k(X)$ into $C_k(M)$. By lemma 2.3.6, $C_k(M)$ is an $\aleph_0$-space. It follows that $C_k(X)$ is also an $\aleph_0$-space. (3) $\Rightarrow$ (4): clear. (4) $\Rightarrow$ (1): Suppose that $C_k(X)$ is cosmic with a countable network $\mathcal{A} = \{A_1, A_2, \ldots\}$. Let $A_n' = \{x \in X: f(x) > 0, \text{ for each } f \in A_n\}$ for each n. We claim that $\mathcal{A}' = \{A_n': n = 1,2,\ldots\}$ is a countable k-network for X. To see this, let K be compact in X and V open in X with $K \subseteq V$. Since X is Tychonoff choose f_0 in $C_k(X)$ such that $f_0(K) = \{1\}$ and $f_0(X-V) = \{0\}$. Let $W = \{f \in C_k(X): f(x) > 0 \text{ for all } x \in K\}$. Then W is open in $C_k(X)$ and $f_0 \in W$. Therefore, for some n, $f_0 \in A_n \subseteq W$. Now, if $x \in K$ and $f \in A_n$ then $f(x) > 0$. This is true for all $f \in A_n$. Therefore $x \in A_n'$. Hence $K \subseteq A_n'$. It remains to show that $A_n' \subseteq V$. Suppose on the contrary that there is $x_0 \in A_n' - V$. Since $x_0 \in X-V$, then $f_0(x_0) = 0$. But this contradicts the fact that f_0

$\in A_n$ and $x_0 \in A_n'$. Therefore $K \subseteq A_n' \subseteq V$ and so, $\mathcal{A}'$ is a countable k-network for X. Thus, X is an $\aleph_0$-space.

(1) $\Leftrightarrow$ (5): This is corollary 9.4 in [47]. (5) $\Rightarrow$ (6): is clear. (6) $\Rightarrow$ (1): Suppose K(X) cosmic with a countable network $\mathcal{W} = \{W_n: n = 1,2,...\}$. Let $A_n = \cup\{K: K \in W_n\}$ for each n. To see that $\{A_n: n = 1,2,...\}$ is a k-network for X, let A be compact in X and V open in X with $A \subseteq V$. If $\mathcal{W} = \{K \in K(X): K \subseteq V\}$, then $\mathcal{W}$ is open in K(X) and $A \in \mathcal{W}$. Therefore for some n, we have $A \in W_n \subseteq \mathcal{W}$. It follows that $A \subseteq A_n \subseteq V$. Thus, $\{A_n: n = 1,2,...\}$ is a countable k-network for X and so, X is an $\aleph_0$-space.

Theorem 2.3.7 appears in [47]. See McCoy [41] for a more general setting.

The next result is sometimes useful in function spaces.

2.3.8. Proposition. X is cosmic if and only if there are separable metrizable spaces M and M' and continuous bijections f and g such that $M \xrightarrow{f} X \xrightarrow{g} M'$.

Proof. Sufficiency follows from 2.3.1(4). For necessity, let X be cosmic. By proposition 2.3.1 (4), there are a metrizable space M and a continuous bijection $f: M \longrightarrow X$ (see [47]). Now, X is cosmic if and only if $C_p(X)$ is cosmic. Therefore, there is a separable metrizable space M"

and a continuous bijection φ: M" $\longrightarrow$ $C_p(X)$. Then the induced map φ^*: $C_p(C_p(X)) \longrightarrow C_p(M")$ can easily (see proof of theorem 1.2.7) be shown to be an embedding of $C_p(C_p(X))$ into $C_p(M")$.

Next, since M" is separable metrizable let D be a countable dense subset of M". Let D" be D with the discrete topology. The natural map i:D" $\longrightarrow$ M" is almost surjective. Therefore i^*:$C_p(M") \longrightarrow C_p(D")$ is one-to-one and continuous. Now, D" discrete implies that $C_p(D")$ is homeomorphic to $\mathbb{R}^D$. Since D is countable, $\mathbb{R}^D$ is separable metrizable. It follows that $i^*(C_p(M"))$ is separable metrizable.

Now, let e: X $\longrightarrow$ $C_p(C_p(X))$ be defined by $e(x) = e_x$: $C_p(X) \longrightarrow \mathbb{R}$ for each $x \in X$, where $e_x(f) = f(x)$ for all $f \in C_p(X)$. It can be shown that e is an embedding. We then have the following situation: $X \xrightarrow{e}$ $C_p(C_p(X)) \xrightarrow{\varphi^*} C_p(M") \xrightarrow{i^*} C_p(D") = \mathbb{R}^D$. Put $M' = i^*(\varphi^*(e(X)))$. Then M' is separable metrizable and $g = i^* \circ \varphi^* \circ e$:X $\longrightarrow$ M' is a continuous bijection.

4. Second countability

The properties of second countability and separable metrizability are identical on $C_k(X)$ because of corollary 1.3.6 and Urysohn's metrization theorem. Let us point out that a homogeneous space has countable π-base if and only if it contains a dense subspace having a countable π-base. The concept of having countable π-base is

weaker than second countability.

2.4.1. Theorem. The following assertions are equivalent:

(1) $C_k(X)$ has countable π-base

(2) $C_k(X)$ is second countable

(3) X is a hemicompact $\aleph_0$-space

(4) X admits a countable k-network of compact sets

(5) X is a hemicompact $\aleph$-space

(6) X is a hemicompact cosmic space

(7) X is a hemicompact σ-space

(8) X is a hemicompact submetrizable space.

Proof. (3) $\Leftrightarrow$ (4) is proposition 2.3.5. (3) $\Leftrightarrow$ (5): X is an $\aleph_0$-space if and only if X is a Lindelöf $\aleph$-space. (3) $\Rightarrow$ (6) $\Rightarrow$ (7): are clear. (7) $\Rightarrow$ (6): X is cosmic if and only if X is a Lindelöf σ-space. (6) $\Rightarrow$ (8): follows from proposition 2.3.8. (8) $\Rightarrow$ (2): Let X be hemicompact submetrizable with a fundamental system of compact sets $\{A_n : n=1,2,\ldots\}$. Then each A_n is separable metrizable. By theorems 2.1.3 and 2.2.1 each $C_k(A_n)$ is separable metrizable. Now, as in the proof of (14) $\Rightarrow$ (13) of theorem 2.2.1 $C_k(X)$ can be embedded in the countable product $\prod_n C_k(A_n)$ of separable metrizable spaces. Therefore $C_k(X)$ is second countable.

(2) $\Rightarrow$ (1): is clear. (1) $\Rightarrow$ (3): Suppose that $C_k(X)$ has countable π-base. Then $C_k(X)$ has countable local π-base, hence is metrizable by theorem 2.2.1. Since spaces with countable π-bases are separable then $C_k(X)$ is in fact separable metrizable. The result

follows from theorems 2.2.1 and 2.3.7.

2.4.2. Corollary. The following are equivalent for a locally compact space X:

(1) $C_k(X)$ is second countable

(2) X is Lindelöf submetrizable

(3) X is a countable union of compact metrizable subsets

(4) X is an $\aleph_0$-space

(5) X is a cosmic space

(6) X is second countable.

CHAPTER 3 : OTHER COUNTABILITY PROPERTIES

This chapter is a continuation of the study of countability properties. In recent years several new countability properties have been introduced in topology. In this chapter we show that on $C_k(X)$ many of them are equivalent to already well-known properties. We show for instance that $C_k(X)$ is a q-space if and only if it is metrizable.

1. Submetrizable branch

Following [9], a space X is called an E_0-space (respectively E_1-space) if each point in the space is a G_δ-set (respectively, each point is the intersection of a countable collection of closed neighborhoods). For T_3-spaces the two notions are clearly identical. E_0-spaces are sometimes called pointwise perfect spaces. Submetrizable spaces are E_0-spaces. Notice that X is an E_0-space if and only if it admits an E_0-space compression.

We call a space almost σ-compact if it contains a dense σ-compact subspace.

3.1.1. Theorem. The following assertions are equivalent:

(1) $C_k(X)$ is an E_0-space

(2) each point of $C_k(X)$ is a zero-set

(3) each compact subset of $C_k(X)$ is G_δ in $C_k(X)$

(4) each countably compact subset of $C_k(X)$ is G_δ in $C_k(X)$

(5) each pseudocompact subset of $C_k(X)$ is G_δ in $C_k(X)$

(6) $C_k(X)$ has a G_δ-diagonal

(7) $C_k(X)$ has a zero-set diagonal

(8) $C_k(X)$ has a first countable T_1-compression

(9) $C_k(X)$ has a developable T_1-compression

(10) $C_k(X)$ is submetrizable

(11) X is almost σ-compact.

Proof. We only need to prove (1) $\Rightarrow$ (11) $\Rightarrow$ (10) because each property in (2)-(9) is weaker than submetrizable but stronger than being an E_0-space.

(1) $\Rightarrow$ (11): Suppose that $C_k(X)$ is an E_0-space. If θ denotes the zero function in $C_k(X)$, then there is a sequence $(W_n: n = 1,2,\ldots)$ of open neighborhoods of θ in $C_k(X)$ such that $\{\theta\} = \cap(W_n: n = 1,2,\ldots)$. Then for each n, there are compact sets $A_{1n},\ldots,A_{k_n n}$ in X and corresponding open sets $V_{1n},\ldots,V_{k_n n}$ in $\mathbb{R}$ such that $\theta \in [A_{1n},V_{1n}] \cap \ldots \cap [A_{k_n n},V_{k_n n}] \subseteq W_n$. Let $A_n = A_{1n} \cup \ldots \cup A_{k_n n}$ for each n. To see that

$\cup A_n$ is dense in X, suppose on the contrary that there is x_0 in $X-\overline{\cup A_n}$. By complete regularity choose f in $C_k(X)$ so that $f(x_0) = 1$ and $f(\overline{\cup A_n}) = \{0\}$. Then $f \in W_n$ for each n. Therefore $f = \theta$. But this is impossible since $f(x_0) = 1$. Hence $X = \overline{\cup A_n}$ and so X is almost σ-compact.

(11) $\nRightarrow$ (10): Let $D = \cup A_n$ be a dense σ-compact subspace in X. The natural map $p: \oplus A_n \longrightarrow X$ is almost surjective. By proposition 1.2.5 $p^*: C_k(X) \longrightarrow \Pi\{C_k(A_n): n = 1,2,\ldots\}$ is a continuous injection. If $M = p^*(C_k(X))$ then M is metrizable and $p^*: C_k(X) \longrightarrow M$ is a bijection. Thus $C_k(X)$ is submetrizable.

The equivalence of (8), (10), and (11) were established in [40].

3.1.2. Corollary. If X is a locally compact paracompact space, then $C_k(X)$ is metrizable whenever it is an E_0-space.

Proof. Suppose that $C_k(X)$ is an E_0-space. Since X is locally compact paracompact then $X = \oplus\{A_\alpha: \alpha \in \Gamma\}$ where each A_α is hemicompact and locally compact. Therefore, $C_k(X) = \Pi\{C_k(A_\alpha): \alpha \in \Gamma\}$. This product is an E_0-space only for a countable Γ. Now, each $C_k(A_\alpha)$ is metrizable and

so, $C_k(X)$ is metrizable being a countable product of metrizable spaces.

In the next few pages we study subspaces of $C_k(X)$ which are G_δ-sets in $F_k(X)$. The following theorem of [45] provides a necessary condition for $C_k(X)$ to contain such sets.

3.1.3. Theorem. If $C_k(X)$ contains a non-empty G_δ subset of $F_k(X)$ then X is a topological sum of a σ-compact space and a discrete space.

Proof. Let $W = \bigcap_n W_n$ be a non-empty G_δ-subset of $F_k(X)$ where W_n is open in $F_k(X)$ for each n, and $W \subseteq C_k(X)$. Let $f \in W$. For each n, there are compact sets $A_{1n},\ldots,A_{k_n n}$ in X and bounded open intervals $V_{1n},\ldots,V_{k_n n}$ in $\mathbb{R}$ such that $f \in [A_{1n},V_{1n}]_F \cap \ldots \cap [A_{k_n n},V_{k_n n}]_F \subseteq W_n$. Let $A_n = A_{1n} \cup \ldots \cup A_{k_n n}$ for each n. Put $A = \bigcup_n A_n$ and $B = X-A$. Then A is σ-compact and, as shown in [45], B is closed and discrete. Therefore $X = A \oplus B$.

3.1.4. Corollary. If $C_k(X)$ contains a non-empty G_δ subset of $F_k(X)$ then X is paracompact.

3.1.5. Corollary. If $C_k(X)$ contains a non-empty G_δ subset of

$F_k(X)$ then X is Lindelöf if and only if X is σ-compact.

The proof of theorem 3.1.3 shows that X is a topological sum of a compact space and a discrete space(hence, is paracompact and locally compact) whenever $C_k(X)$ contains a non-empty open subset of $F_k(X)$.

We use theorem 3.1.3 to establish the next result which is essentially in [46]. This theorem should be compared with theorem 3.1.1.

3.1.6. Theorem. The following are equivalent:

(1) Each pseudocompact subset of $C_k(X)$ is G_δ in $F_k(X)$

(2) each countably compact subset of $C_k(X)$ is G_δ in $F_k(X)$

(3) each compact subset of $C_k(X)$ is G_δ in $F_k(X)$

(4) each point of $C_k(X)$ is G_δ in $F_k(X)$

(5) X is σ-compact.

Proof. (1) ⇒ (2) ⇒ (3) ⇒ (4): clear. (4) ⇒ (5): Suppose that each point of $C_k(X)$ is G_δ in $F_k(X)$. By theorem 3.1.3 $X = A \oplus B$ where A is σ-compact and B is discrete. Now $C_k(X)$ is homeomorphic to $C_k(A) \times C_k(B)$. By theorem 3.1.1 $C_k(X)$ is submetrizable. Therefore $C_k(B)$ is

submetrizable. Since B is discrete $C_k(B)$ is homeomorphic to $\mathbb{R}^B$. But $\mathbb{R}^B$ is submetrizable only when B is countable. Therefore $X = A \oplus B$ where A and B are both σ-compact. Thus X is σ-compact.

(5) $\Rightarrow$ (1): Suppose X is σ-compact with $X = \bigcup_n A_n$ where each A_n is compact and $A_n \subseteq A_{n+1}$ for each n. Let K be a pseudocompact subspace of $C_k(X)$. Since $C_k(X)$ is submetrizable, then K is compact. Next, let $f \in K$ and $n \in \mathbb{N}$. If $x \in A_n$, let $V(f,n,x) = (f(x)-\frac{1}{n}, f(x)+\frac{1}{n})$. By continuity of f, for each $x \in A_n$ there is a closed subset $A(f,n,x)$ of A_n which is a neighborhood of x in A_n such that $f(A(f,n,x)) \subseteq V(f,n,x)$. By compactness of A_n, choose finite subset $S(f,n)$ of A_n such that $A_n \subseteq \bigcup\{A(f,n,x): x \in S(f,n)\}$. Put $W(f,n) = \bigcap\{[A(f,n,x), V(f,n,x)]_F: x \in S(f,n)\}$. Then $W(f,n)$ is a basic open set in $F_k(X)$. Now let n be in $\mathbb{N}$. Since K is compact, there is a finite subset K_n of K such that $K \subseteq \bigcup\{W(f,n): f \in K_n\}$. Now, put $W_n = \bigcup\{W(f,n): f \in K_n\}$. Then by a straightforward argument one shows that $K = \bigcap_n W_n$.

3.1.7. Corollary. If X has no isolated point, then $C_k(X)$ contains a non-empty G_δ subset of $F_k(X)$ if and only if each point of $C_k(X)$ is G_δ in $F_k(X)$.

The question of which countably compact subspaces in $C_p(X)$ are compact was studied by Grothendieck [32]. (see also Pryce [57], Arhangelskii [4]). For $C_k(X)$ we have the following two results.

3.1.8. Theorem. If X is almost σ-compact and K is a subspace of $C_k(X)$, then the following are equivalent:

(1) K is compact

(2) K is sequentially compact

(3) K is countably compact

(4) K is pseudocompact.

Proof. By theorem 3.1.1 $C_k(X)$ is submetrizable. Since submetrizable Tychonoff pseudocompact spaces are metrizable hence compact, then clearly (4) ⇒ (1). The other implications are trivial.

3.1.9. Theorem. Let X be a $k_{\mathbb{R}}$-space and K a closed subspace of $C_k(X)$. Then the following are equivalent:

(1) K is compact

(2) K is sequentially compact

(3) K is countably compact

(4) K is pseudocompact.

Proof. Again we only need (4) ⇒ (1): Since X is a $k_{\mathbb{R}}$-space, then the

uniformity of uniform convergence on compact sets of X is complete in $C_k(X)$ (see [66]). Therefore, $C_k(X)$ is Dieudonné complete in the sense of [26]. Now, a closed pseudocompact subspace of a Dieudonné complete space is compact.

We now turn to subcosmic spaces (i.e. spaces with cosmic compressions; or equivalently: spaces with separable metrizable compressions) and almost cosmic spaces (i.e. spaces with dense cosmic subspaces).

3.1.10. Theorem. $C_k(X)$ is a subcosmic space if and only if X is almost σ-compact and $knw(X) \leqslant 2^{\omega}$.

Proof. Suppose that $C_k(X)$ is subcosmic. By theorem 2.1.3 $C_k(X)$ is submetrizable and $|C(X)| \leqslant 2^{\omega}$. It then follows that X is almost σ-compact (by theorem 3.1.1)and $knw(X) = nw(C_k(X)) \leqslant 2^{\omega}$.

Conversely suppose that X is almost σ-compact with $knw(X) \leqslant 2^{\omega}$. Then $C_k(X)$ is submetrizable and $d(C_k(X)) \leqslant nw(C_k(X)) = knw(X) \leqslant 2^{\omega}$. By theorem 2.1.3, $C_k(X)$ has a separable metrizable compression.

3.1.11. Corollary. If X is separable then $C_k(X)$ has a separable metrizable compression.

Proof. If X is separable then it is almost σ-compact and, as pointed out before, $Knw(X) \leqslant w(X) \leqslant 2^{\omega}$.

Almost cosmic function spaces can be characterized as follows.

3.1.12. Theorem. $C_k(X)$ is almost cosmic if and only if X is subcosmic.

Proof. Suppose $C_k(X)$ almost cosmic and let D be a dense cosmic subset of $C_k(X)$. $C_k(X)$ is separable because D is a dense separable subset of $C_k(X)$. By theorem 2.1.3 X has a separable metrizable compression.

Conversely, let X be subcosmic with a separable metrizable compression M. If $f: X \longrightarrow M$ is a continuous bijection, then $f^*: C_k(M) \longrightarrow C_k(X)$ is almost surjective and $f^*(C_k(M))$ is a dense cosmic subset of $C_k(X)$.

In fact, this proof reveals more than is stated.

3.1.13. Corollary. The following are equivalent:

(1) $C_k(X)$ is almost cosmic

(2) $C_k(X)$ is separable

(3) X is subcosmic

(4) X has an $\aleph_0$-space compression

(5) X has a separable metrizable compression.

3.1.14. Corollary. If X is cosmic then $C_k(X)$ is separable and subcosmic.

In theorem 3.1.1 we proved that X is almost σ-compact if and only if $C_k(X)$ is submetrizable. We want now to present the dual result. We need a lemma.

3.1.15. Lemma. Let X be a k-space. Then the map $e: X \longrightarrow C_k(C_k(X))$ defined by $e(x) = e_x: C_k(X) \longrightarrow \mathbb{R}$ with $e_x(f) = f(x)$ for each f in $C_k(X)$ and each x in X is a continuous injection.

Proof. The function e is clearly an injection. To show continuity, let $\mathcal{Q}$ be compact in $C_k(X)$ and V open in $\mathbb{R}$. We must show that $e^{-1}([\mathcal{Q},V])$ is open in X. So, let $x_0 \in e^{-1}([\mathcal{Q},V])$. Then $e(x_0) = e_{x_0} \in [\mathcal{Q},V]$. Now, if $f \in \mathcal{Q}$ then $e_{x_0}(f) = f(x_0) \in V$. Therefore $f \in [(x_0),V]$. This is true for each f in $\mathcal{Q}$. Therefore $\mathcal{Q} \subseteq [(x_0),V]$.

Put $\mathcal{Q}^* = \{x \in X: f(x) \in V, f \in \mathcal{Q}\}$. Since X is a k-space then

$\mathcal{A}^*$ is open in X (see lemma 9.2 of [47]). Moreover $x_0 \in \mathcal{A}^*$. It remains to show that $\mathcal{A}^* \subseteq e^{-1}([\mathcal{A},V])$. So, let $x \in \mathcal{A}^*$. To see that $e_x \in [\mathcal{A},V]$, let $f \in \mathcal{A}$. Then $f(x) = e_x(f) \in V$ by definition of $\mathcal{A}^*$. Since this holds for each f in $\mathcal{A}$ then clearly $e_x \in [\mathcal{A},V]$. Consequently $x_0 \in \mathcal{A}^* \subseteq e^{-1}([\mathcal{A},V])$; meaning that $e^{-1}([\mathcal{A},V])$ is open in X. Thus e is continuous.

In fact it can be shown that e is a closed embedding.

3.1.16. Theorem ([40]). Let X be a k-space. Then $C_k(X)$ is almost σ-compact if and only if X is submetrizable.

Proof. Suppose that $C_k(X)$ is almost σ-compact. By theorem 3.1.1 $C_k(C_k(X))$ is submetrizable. Since X is a k-space, by lemma 3.1.15 $e: X \longrightarrow C_k(C_k(X))$ is a continuous injection of X into a submetrizable space. Therefore X is submetrizable.

Conversely suppose that X is submetrizable and let M be a metrizable compression for X and $f: X \longrightarrow M$ a continuous bijection. Then $f^*: C_k(M) \longrightarrow C_k(X)$ is almost surjective. Therefore, it is sufficient to show that $C_k(M)$ is almost σ-compact. For each n, let $\mathcal{B}_n$ be a locally finite open refinement of $\mathcal{V}_n = \{B(x,\frac{1}{n}): x \in M\}$ where $B(x,\frac{1}{n})$ is the open

ball about x of radius $\frac{1}{n}$. Let F_n be a partition of unity subordinated to $\mathcal{B}_n$. Then each F_n is a subset of $C_k(M)$. If $F_n^* = F_n \cup \{1\}$ where 1 is the constant function mapping M onto 1, define $F_n' = \{rf_1 \cdot f_2 \cdots f_k : k \in \mathbb{N}$ with $k \leq n$, r real number such that $|r| \leq 1$, and $f_1,\ldots,f_k \in F_1^* \cup \ldots \cup F_n^*\}$ and put $F_n'' = \{f_1 + \ldots + f_k : k \in \mathbb{N}$ with $k \leq n$ and $f_1,\ldots,f_k \in F_n'\}$. Let $F = \bigcup_n F_n''$. Then, using the stone-Weierstrass theorem one can show (see theorem 3 of [40]) that F is dense in $C_k(M)$. Moreover F_n'' is compact in $C_k(M)$ for each n. Therefore, $C_k(M)$ is almost σ-compact.

We must point out that the hypothesis that X be a k-space in theorem 3.1.16 cannot be removed. Indeed, let X be the Fortissimo space defined in the remark following corollary 2.3.2. Corson [20] showed that $C_k(X)$ is almost σ-compact. Since 0 is not G_δ in X clearly X is not submetrizable. X is not a k-space of course.

The dual result of theorem 3.1.10 holds for a k-space X.

3.1.17 Theorem. If X is a k-space, then $C_k(X)$ is almost σ-compact and $knw(C_k(X)) \leq 2^\omega$ if and only if X has a separable metrizable compression.

Proof. Sufficiency follows from theorem 2.1.3 and the previous inequalities. For necessity, assume $C_k(X)$ almost σ-compact with $knw(C_k(X)) \leqslant 2^{\omega}$. Then $knw(X) = nw(C_k(X)) = knw(C_k(X)) \leqslant 2^{\omega}$ and, X is submetrizable by theorem 3.1.16. That X has a separable metrizable compression follows from theorem 2.1.3 since $d(X) \leqslant knw(X)$.

The proof of the next result follows from previous theorems.

3.1.18. Theorem. $C_k(X)$ is a separable subcosmic space if and only if X is a separable subcosmic space.

We now turn to σ-spaces and Σ-spaces. Following Nagami [53], a space X is a (strong) Σ-space if there exists a σ-discrete collection $\mathcal{F}$, and a cover $\mathcal{C}$ of X by closed countably compact (compact) sets, such that, whenever $C \in \mathcal{C}$ and $C \subseteq U$ with U open, then $C \subseteq F \subseteq U$ for some $F \in \mathcal{F}$.

Every strong Σ-space is clearly a Σ-space. The converse fails. But any property that makes countably compact subsets compact will make a Σ-space a strong Σ-space.

Now, as properties on $C_k(X)$ we have the following results.

3.1.19. Theorem. If X is a $K_{\mathbb{R}}$-space then $C_k(X)$ is a strong Σ-space if and only if $C_k(X)$ is a Σ-space.

Proof. Since X is a $k_{\mathbb{R}}$-space, by theorem 3.1.9 every closed countably compact subset of $C_k(X)$ is compact. Therefore, $C_k(X)$ is a Σ-space if and only if it is a strong Σ-space.

3.1.20. Theorem. If X is almost σ-compact then the following are equivalent:

(1) $C_k(X)$ is a σ-space

(2) $C_k(X)$ is a strong Σ-space

(3) $C_k(X)$ is a Σ-space.

Proof. We only need (3) $\Rightarrow$ (1): Since X is almost σ-compact, $C_k(X)$ has G_δ-diagonal (theorem 3.1.1). Now, a G_δ-diagonal Σ-space is a σ-space ([15]).

3.1.21. Theorem. If $C_k(X)$ is a hereditarily strong Σ-space then X is almost σ-compact.

Proof. Let $\mathcal{Q} = \{f \in C_k(X): f \text{ is not } G_\delta \text{ in } C_k(X)\}$. Since $C_k(X)$ is a hereditarily strong Σ-space then each subset of $\mathcal{Q}$ is an F_σ-set in $C_k(X)$ by lemma 2.8 of [11]. Now, if $C_k(X) = \mathcal{Q}$ then $C_k(X)$ is an E_0-space, hence contradicting the definition of $\mathcal{Q}$. Consequently there exists $f \in$

$C_k(X)-\mathcal{Q}$. But then f is G_δ in $C_k(X)$. By homogeneity of $C_k(X)$ each point of $C_k(X)$ is G_δ in $C_k(X)$. Therefore $C_k(X)$ is an E_0-space and so X is almost σ-compact by theorem 3.1.1.

3.1.22. Theorem. $C_k(X)$ is a σ-space if and only if $C_k(X)$ is a hereditarily strong Σ-space.

Proof. This follows from theorems 3.1.20 and 3.1.21.

We conclude this section with the following theorem which puts together many properties. The hypothesis of metrizability in this theorem cannot be weakened to first countability as the Sorgenfrey line is a counterexample.

3.1.23. Theorem. If X is metrizable then the following are equivalent:

(1) $C_k(X)$ is an $\mathcal{N}_0$-space

(2) $C_k(X)$ is an $\mathcal{N}$-space

(3) $C_k(X)$ is a σ-space

(4) $C_k(X)$ is a perfect space

(5) $C_k(X)$ is an E_0-space

(6) $C_k(X)$ is a subcosmic space

(7) X is separable

(8) X is cosmic

(9) X is an $\aleph_0$-space.

2. Countable type branch

Under "countable type branch" we group the main topological properties between first countability and Michael's q-spaces. We show that every q-space function space is metrizable. We need the following definitions.

A subset S of a topological space X is said to have countable character in X if there is a sequence $\langle W_n : n \in \mathbb{N} \rangle$ of open subsets of X such that $S \subseteq W_n$ for each n and, if W is any open set in X containing S then $W_n \subseteq W$ for some n.

X is a space of countable type ([3]) if and only if each compact set in X is contained in a compact set of countable character in X. If each point is contained in a compact set of countable character we say that X is a space of point(wise) countable type ([3]). We have that first countable implies countable type.

X is a p-space (i.e. plumed or feathered space) if there is in βX (the stone-Čech compactification of X) a sequence $\langle \mathcal{U}_n : n \in \mathbb{N} \rangle$ of open covers of X such that for any x in X, the set $\cap\{st(x,\mathcal{U}_n) : n = 1,2,\ldots\}$ is contained in X. This concept is due to A.V. Arhangel'skii. Every metrizable space is a p-space.

X is an r-space ([47]) if each point of X has a sequence $\langle V_n : n \in \mathbb{N} \rangle$ of neighborhoods with the property that if $x_n \in V_n$ for each n then the set $\{x_n : n = 1,2,\ldots\}$ is contained in a compact subset of X. We have the following implications: point countable type implies r-space implies q-space.

X is an M-space ([50]) if there are a metrizable space Y and a surjective quasi-perfect map (i.e. a continuous closed map in which inverse images of points are countably compact) from X onto Y. Every metrizable space is an M-space and the class of q-spaces contains the M-spaces. Note that M-space implies Σ-space. Arhangelskii has shown that in the class of paracompact spaces the concepts of q-space, M-space, p-space and point countable type are equivalent.

The proof of the next lemma is straightforward.

3.2.1. Lemma. Let X be a regular space, D a dense subset of X and A a compact subset of D. Then A has countable character in D if and only if A is of countable character in X.

3.2.2. Theorem. $C_k(X)$ is a space of point countable type if and only if $C_k(X)$ contains a dense subspace of point countable type.

Proof. This follows from the above lemma and the homogeneity of $C_k(X)$.

In fact, more is true.

3.2.3. Theorem. The following are equivalent:

(1) $C_k(X)$ is a q-space

(2) $C_k(X)$ is an M-space

(3) $C_k(X)$ is an r-space

(4) $C_k(X)$ is a p-space

(5) $C_k(X)$ is of point countable type

(6) $C_k(X)$ is of countable type

(7) $C_k(X)$ is metrizable

(8) X is hemicompact.

Proof. Since properties (1)-(8) are all weaker than metrizability but stronger than being a q-space, we only need to prove that (1) $\Rightarrow$ (8): Suppose that $C_k(X)$ is a q-space. If θ is the zero function of $C_k(X)$ then by definition there is a sequence $\langle W_n(\theta): n = 1,2,\ldots\rangle$ of neighborhoods of θ such that if $f_n \in W_n(\theta)$ for each n then $\langle f_n: n = 1,2,\ldots\rangle$ clusters in $C_k(X)$. Now, for each n there are compact sets $A_{1n},\ldots,A_{k_n n}$ in X and corresponding open sets $V_{1n},\ldots,V_{k_n n}$ in $\mathbb{R}$ such that $\theta \in [A_{1n},V_{1n}] \cap \ldots \cap [A_{k_n n},V_{k_n n}) \subseteq W_n(\theta)$. Let $A_n = A_{1n} \cup \ldots \cup A_{k_n n}$ for each n. To see that $X = \bigcup_n A_n$, let $x_0 \in X - \bigcup A_n$. By complete regularity

define f_n in $C_k(X)$ such that $f_n(x_0) = n$ and $f_n(A_n) = \{0\}$ for each n. Since each $f_n \in W_n(\theta)$ then by above $(f_n : n = 1,2,\ldots)$ clusters in $C_k(X)$. But this is impossible as $f_n(x_0) = n$ for all n. Therefore $X = \bigcup_n A_n$ and so by theorem 3.1.1 $C_k(X)$ is an E_1-space. Now, an E_1-space that is a q-space is first countable ([60]). The conclusion that X is hemicompact follows from theorem 2.2.1.

For locally M-spaces (i.e. spaces in which each point has a neighborhood that is an M-subspace) we have the following.

3.2.4. Theorem. The following statements are equivalent:

(1) $C_k(X)$ is paracompact and locally M

(2) $C_k(X)$ is submetrizable and locally M

(3) $C_k(X)$ is metrizable.

Proof. (3) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1): are clear. (1) $\Rightarrow$ (3): This follows from theorem 3.2.3 and the fact that paracompact locally M-spaces are M-spaces ([37]). (2) $\Rightarrow$ (3): Let W be an M-subspace that is a neighborhood of the zero function θ of the submetrizable space $C_k(X)$. Then W is a submetrizable M-space. By theorem 5.1.3 of [15] W is metrizable. It then follows that $C_k(X)$ is locally metrizable at θ. By homogeneity it is locally metrizable everywhere. By theorem 2.1.1

$C_k(X)$ must be metrizable.

CHAPTER 4 : FRECHET FUNCTION SPACES

We characterize the Fréchet property on $C_k(X)$. In the first section we study the countable tightness of $C_k(X)$. For further results see [43] and [31]. Let us recall that a space X is Fréchet (respectively, has countable tightness) if and only if for each point x in X and $A \subseteq X$ with $x \in \bar{A}$ there is a sequence $(x_n)_n \subseteq A$ such that $(x_n)_n$ converges to x (respectively, $x \in \overline{\{x_n : n = 1,2,...\}}$). X is a strict-Fréchet (i.e. w-space in the sense of [33]) space if for each point x and any sequence of subsets $(A_n : n = 1,2,...)$ in X with $x \in \bar{A}_n$ for each n there exists a sequence of points $(x_n)_n$ in X such that $x_n \in A_n$ for each n and $(x_n)_n$ converges to x. X is strong-Fréchet (i.e. countably bi-sequential in the sense of [48]) if for each x in X and any decreasing sequence of subsets $(A_n : n = 1,2,...)$ in X with $x \in \bar{A}_n$ for all n there is $x_n \in A_n$ such that $(x_n)_n$ converges to x.

We have the following implications: First-countable implies w-space implies countably bi-sequential implies Fréchet implies sequential implies k-space. Spaces of point-countable type are k-spaces. We must mention that hereditarily separable spaces are of countable tightness.

1. Countable tightness

If $\mathcal{U}$ is a cover for X, we say that $\mathcal{U}$ is a c-cover of X if each compact subset in X is contained in at least one member of $\mathcal{U}$. It is clear that for a space X to be compact it is necessary and sufficient that every open c-cover of X contain a finite open c-subcover. c-covers are crucial in characterizing countable tightness of $C_k(X)$.

4.1.1. Theorem. $C_k(X)$ has countable tightness if and only if every open c-cover of X has a countable open c-subcover.

Proof. Suppose that $C_k(X)$ has countable tightness. Let $\mathcal{U} = \langle U_\alpha : \alpha \in \Gamma \rangle$ be an open c-cover for X. For each compact K in X there is $\alpha_K \in \Gamma$ such that $K \subseteq U_{\alpha_K}$. By complete regularity of X there is $f_{n,K} \in C_k(X)$ such that $f_{n,K}: X \longrightarrow [\frac{1}{n}, n]$, $f_{n,K}(K) = \{\frac{1}{n}\}$ and $f_{n,K}(X - U_{\alpha_K}) = \{n\}$ for each n and each K compact in X. Let $\mathcal{A} = \{f_{n,K} : K \text{ is compact in } X \text{ and } n \in \mathbb{N}\}$. If θ is the zero function in $C_k(X)$ let $\langle \theta, K', \varepsilon \rangle$ be any basic neighborhood of θ in $C_k(X)$. Since $\varepsilon > 0$, choose $n \in \mathbb{N}$ so that $\frac{1}{n} < \varepsilon$. Then $f_{n,K'} \in \langle \theta, K', \varepsilon \rangle$. This means that $\theta \in \bar{\mathcal{A}}$. By countable tightness of $C_k(X)$ let $H \subseteq \mathcal{A}$ such that H is countable with $\theta \in \bar{H}$. Then there exist $n_1, n_2, \ldots, n_k, \ldots$ in $\mathbb{N}$ and corresponding compact sets $K_1, K_2, \ldots, K_k, \ldots$ in X with $H = \{f_{n_1 K_1}, f_{n_2 K_2}, \ldots\}$. Let $\mathcal{V} = \{U_{\alpha_{K_i}} : i =$

$1,2,\ldots$). Then $\mathcal{V}$ is a countable subcover of $\mathcal{U}$. It remains to show that $\mathcal{V}$ is a c-cover. So, let K be any compact set in X. Since $[K,(-1,1)]$ is an open neighborhood of θ in $C_k(X)$ and $\theta \in \bar{H}$, there is an $i \in \mathbb{N}$ such that $f_{n_i,K_i} \in [K,(-1,1)] \cap H$. Now, to see that $K \subseteq U_{\alpha_{K_i}}$, suppose on the contrary that there is $x_0 \in K - U_{\alpha_{K_i}}$. Then by definition of f_{n_i,K_i} we must have $f_{n_i,K_i}(x_0) = n_i \geq 1$. But this contradicts the fact that $f_{n_i,K_i} \in [K,(-1,1)]$ with $x_0 \in K$. Consequently $K \subseteq U_{\alpha_{K_i}}$.

Conversely suppose that every open c-cover of X has a countable open c-subcover. We must show that $C_k(X)$ has countable tightness. Because of homogeneity of $C_k(X)$ we localize the problem at the zero function θ. So, let $\mathcal{A} \subseteq C_k(X)$ such that $\theta \in \bar{\mathcal{A}}$. We must find H countable so that $H \subseteq \mathcal{A}$ and $\theta \in \bar{H}$. For each $n \in \mathbb{N}$ and each compact K in X we have $\theta \in \bar{\mathcal{A}} \cap [K,(-\frac{1}{n},\frac{1}{n})]$. Therefore there is $f_{n,K} \in \mathcal{A} \cap [K,(-\frac{1}{n},\frac{1}{n})]$. Now, let $W(n,K) = \{x \in X: |f_{n,K}(x)| < \frac{1}{n}\}$. Then each $W(n,K)$ is open in X and, for each n, the collection $\mathcal{W}_n = \{W(n,K): K \text{ is compact in } X\}$ is an open c-cover of X. By hypothesis there is $\mathcal{V}_n = \{W(n_i,K_i): i = 1,2,\ldots\}$ a c-subcover of $\mathcal{W}_n$ for each n. Put $H = \{f_{n_i,K_i}: n,i \in \mathbb{N}\}$. It is straightforward to check that $\theta \in \bar{H}$.

4.1.2. Corollary. If $C_k(X)$ has countable tightness then X^n is Lindelöf for each $n \in \mathbb{N}$.

4.1.3. Corollary. Let X be locally compact. Then $C_k(X)$ has countable tightness if and only if $C_k(X)$ is metrizable.

4.1.4. Corollary. If X is a σ-space and $C_k(X)$ has countable tightness then X is cosmic.

Since cosmic $\aleph$-spaces are $\aleph_0$-spaces, the next corollary is a consequence of corollary 4.1.4.

4.1.5. Corollary. Let X be an $\aleph$-space. Then $C_k(X)$ has countable tightness if and only if X is an $\aleph_0$-space.

The converse to corollary 4.1.4 is not true in general. Indeed example 15 of [43] gives a countable (hence cosmic) space X whose $C_k(X)$ does not have countable tightness. Theorem 4.1.1 appears in [44] for $C_p(X)$.

2. The Fréchet property

In [58] Pytkeev announced that he has a proof of the following: $C_k(X)$ is a k-space if and only if $C_k(X)$ is Fréchet. It is unfortunate that we are unable to prove this result. Using the machinery in [30] the problem can be reduced to showing that $C_k(X)$ has countable tightness whenever it is a k-space.

However we can prove the following.

4.2.1. Theorem. The following assertions are equivalent:

(1) $C_k(X)$ is a w-space

(2) $C_k(X)$ is countably bi-sequential

(3) $C_k(X)$ is Fréchet

(4) For every sequence $\langle \mathcal{U}_n : n = 1,2,\ldots\rangle$ of open c-covers of X there is a sequence $\langle U_n : n = 1,2,\ldots\rangle$ of open subsets of X such that $U_n \in \mathcal{U}_n$ for each n and, if K is any compact set in X, there is $N \in \mathbb{N}$ such that $K \subseteq U_n$ for all $n > N$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3): obvious. (4) $\Rightarrow$ (1): Suppose that (4) is true. To see that $C_k(X)$ is a w-space, let θ denote the zero function in $C_k(X)$. Let $\langle F_m : m \in \mathbb{N}\rangle$ be a sequence of subsets of $C_k(X)$ such that $\theta \in \bar{F}_m$ for each m. We must show that there is $f_m \in F_m$ for each m with $\langle f_m\rangle_m$ converging to θ. First fix $m \in \mathbb{N}$ and let $\mathcal{A} = \langle A_1, A_2, \ldots, A_k\rangle$ be a finite

collection of compact sets in X. Since $\theta \in \bar{F}_m$ there is $f_{m,\mathcal{A}} \in F_m \cap [A_1,(-\frac{1}{m},\frac{1}{m})] \cap \ldots \cap [A_k,(-\frac{1}{m},\frac{1}{m})]$. If we put $W(m,\mathcal{A}) = \{x \in X: |f_{m,\mathcal{A}}(x)| < \frac{1}{m}\}$ then $W(m,\mathcal{A})$ is open in X and the collection $\mathcal{W}_m = \{W(m,\mathcal{A}): \mathcal{A}$ is a finite collection of compact sets in $X\}$ is an open c-cover of X. This is true for each $m \in \mathbb{N}$. Now $\{\mathcal{W}_m: m = 1,2,\ldots\}$ is a sequence of open c-covers of X. By hypothesis there is $W_m = W(m,\mathcal{A}_m) \in \mathcal{W}_m$ for each m such that if K is compact in X then for some M we have $K \subseteq W_m$ for all $m \geq M$. Now, for each m, let $f_m = f_{m,\mathcal{A}_m}$. Then $f_m \in F_m$. To see that $(f_m)_m$ converges to θ, let $W = [A_1,V_1] \cap \ldots \cap [A_k,V_k]$ be a basic open set containing θ in $C_k(X)$. First there is an $m \in \mathbb{N}$ such that $(-\frac{1}{m},\frac{1}{m}) \subseteq V_i$ for each $i = 1,2,\ldots,k$. By above there is $M \in \mathbb{N}$ such that for all $m \geq M$, $A_1 \cup A_2 \cup \ldots \cup A_k \subseteq W_m = W(m,\mathcal{A}_m)$. But, if $m \geq M$ and $x \in A_i$ for $1 \leq i \leq k$ then $x \in W_m = W(m,\mathcal{A}_m)$ so that $|f_m(x)| < \frac{1}{m}$. Therefore $f_m(x) \in (-\frac{1}{m},\frac{1}{m}) \subseteq V_i$. Since this is true for each $x \in A_i$, it follows that $f_m \in [A_i,V_i]$ for each $1 \leq i \leq k$. Consequently $f_m \in W$. Therefore $(f_m)_m$ converges to θ.

(3) $\Rightarrow$ (4): Suppose that $C_k(X)$ is Fréchet and let $\{\mathcal{U}_n : n = 1,2,\ldots\}$ be a sequence of open c-covers of X. Define a new sequence $\{\mathcal{W}_n: n = 1,2,\ldots\}$ as follows: let $\mathcal{W}_1 = \mathcal{U}_1$ and for each $n>1$ let $\mathcal{W}_n$ be an open c-cover of X refining both $\mathcal{W}_{n-1}$ and $\mathcal{U}_n$. Now, for each $n \in \mathbb{N}$ and

each compact set A in X, let $W(n,A) \in \mathcal{W}_n$ be such that $A \subseteq W(n,A)$. Since X is Tychonoff there is $f_{n,A}: X \longrightarrow [\frac{1}{n},1]$ continuous such that $f_{n,A}(A) = \{\frac{1}{n}\}$ and $f_{n,A}(X-W(n,A)) = \{1\}$. Let $\mathcal{F} = \{f_{n,A}: n \in \mathbb{N}$ and A is compact in $X\}$. To see that $\theta \in \bar{\mathcal{F}}$, let $W = [A_1,V_1] \cap \ldots \cap [A_k,V_k]$ be a basic open neighborhood of θ in $C_k(X)$. Put $A = A_1 \cup A_2 \cup \ldots \cup A_k$. Choose $n \in \mathbb{N}$ so that $\frac{1}{n} \in V_i$ for each $1 \leqslant i \leqslant k$. Then $f_{n,A}(A_i) \subseteq V_i$. Therefore $f_{n,A} \in [A_i,V_i]$ for each $1 \leqslant i \leqslant k$. Thus $f_{n,A} \in W$; meaning that $\theta \in \bar{\mathcal{F}}$. Since $C_k(X)$ is Fréchet there are sequences $(n_i)_i \subseteq \mathbb{N}$ and $(A_i)_i$ where each A_i is compact in X such that $(f_{n_i,A_i})_i$ converges to θ.

Next, let $i_0 = 1$. If $n_{i_0} > 1$, we know that $\mathcal{W}_{n_{i_0}}$ refines $\mathcal{W}_i$ for each $i = 1,2,\ldots,n_{i_0}-1$. Therefore there is $W_i \in \mathcal{W}_i$ such that $W(n_{i_0},A_{i_0}) \subseteq W_i$. Then put $W_{n_{i_0}} = W(n_{i_0},A_{i_0})$. Now, let i_1 be the first i such that $n_{i_1} > n_{i_0}$. If $n_i > n_{i_0}+1$ for each $i = n_{i_0}+1,\ldots,n_{i_1}-1$, let $W_i \in \mathcal{W}_i$ such that $W(n_{i_1},A_{i_1}) \subseteq W_i$. Then put $W_{n_1} = W(n_{i_1},A_{i_1})$. Keep defining the $W_{n_{i_j}}$'s inductively as above.

Now, let A be any compact set in X. Since $(f_{n_i,A_i})_i$ converges to θ, there is $N \in \mathbb{N}$ such that $f_{n_i,A_i} \in [A,(-1,1)]$ for all $i \geqslant N$. If

$i > n_{i_N}$ then for some $p \geq N$ we have $n_{i_p} < i \leq n_{i_{p+1}}$. Since $n_{i_{p+1}} > n_{i_N} > N$, by above $f_{n_{i_{p+1}}, A_{n_{i_{p+1}}}} \in [A,(-1,1)]$. It then follows that $A \subseteq W(n_{i_{p+1}}, A_{i_{p+1}}) \subseteq W_i$. Now, for each n, let $U_n \in \mathcal{U}_n$ be such that $W_n \subseteq U_n$. Then by above for any compact set A in X there is an $N_A \in \mathbb{N}$ so that $A \subseteq U_n$ for all $n \geq N_A$. Therefore, $\{U_n : n = 1,2,\ldots\}$ is the right sequence and so, (4) is satisfied.

For a q-space X we have a nicer property on X for $C_k(X)$ to be Fréchet.

4.2.2. Theorem. Let X be a q-space. Then $C_k(X)$ is Fréchet if and only if X is a hemicompact k-space.

Proof. If X is hemicompact then $C_k(X)$ is metrizable (by theorem 2.2.1) hence Fréchet.

For the converse assume that $C_k(X)$ is Fréchet. By corollary 4.1.2 X is Lindelöf. Now, a Lindelöf q-space is of point countable type and so, X is a k-space. Next, let $C_k(X,I) = \{f \in C(X) : f(X) \subseteq [0,1]\}$. Then $C_k(X,I)$ is a subspace of $C_k(X)$. Since $C_k(X)$ is Fréchet then so is $C_k(X,I)$. Now, for a paracompact space of point countable type X,

$C_k(X,I)$ is a k-space if and only if $X = A \oplus B$ where A is a discrete space and B a locally compact Lindelöf space (see [49] p. 305). Since locally compact Lindelöf spaces are hemicompact (see proposition 1.4.1) then $C_k(B)$ is metrizable. On the other hand, A is discrete and Lindelöf (because X is Lindelöf) and so must be a countable space. This implies that $C_k(A)$ which is homeomorphic to $\mathbb{R}^A$, is in fact a metrizable space. Now, $X = A \oplus B$ implies that $C_k(X)$ is homeomorphic to $C_k(A) \times C_k(B)$ (see proposition 1.2.13) which is metrizable by above. That X is hemicompact follows from theorem 2.2.1.

4.2.3. Corollary. If $C_k(X)$ is Fréchet then every closed q-subset of X is a hemicompact k-space.

Proof. Let S be a closed subset of X that is a q-subspace. If $i: S \longrightarrow X$ is the inclusion map then (X being Lindelöf, hence normal) by corollary 1.2.12 the induced map $i^*: C_k(X) \longrightarrow C_k(S)$ is an open surjection. Since Fréchet is preserved by open maps then $C_k(S)$ is a Fréchet space. By theorem 4.2.2 S must be a hemicompact k-space.

4.2.4. Theorem. The following are equivalent:

(1) $C_k(X)$ is a Fréchet $\mathcal{N}$-space

(2) $C_k(X)$ is a Fréchet space with a point-countable k-network

(3) X is hemicompact.

Proof. (1) $\Rightarrow$ (2) is clear. (2) $\Rightarrow$ (3): If $C_k(X)$ is Fréchet with a point-countable k-network, by theorem 4.2.1 $C_k(X)$ is countably bi-sequential. But a regular countably bi-K space (this is weaker than countably bi-sequential) with a point-countable k-network has a point countable base (see corollary 3.6 of [34]). The result follows from theorem 2.2.1. (3) $\Rightarrow$ (1): This is obvious.

4.2.5. Corollary. $C_k(X)$ is a Fréchet $\aleph_0$-space if and only if X is a hemicompact $\aleph_0$-space.

4.2.6. Corollary. If $C_k(X)$ is Fréchet then every closed $\aleph$-subset of X is a hemicompact $\aleph_0$-space.

Proof. Let S be a closed subset of X. Since $C_k(X)$ is Fréchet then X is Lindelöf by corollary 4.1.2. Therefore S is Lindelöf. Now, if S is an $\aleph$-space then it becomes an $\aleph_0$-space so that $C_k(S)$ is an $\aleph_0$-space. As in the proof of corollary 4.2.3 it can be seen that $C_k(S)$ is Fréchet. The result then follows from corollary 4.2.5.

4.2.7. Corollary. Let X be an $\aleph$-space. Then $C_k(X)$ is Fréchet if and only if $C_k(X)$ is separable metrizable.

We close this chapter with a characterization of Lasnev function spaces. By a Lasnev space we mean a continuous closed image of a metrizable space. Note that every Lasnev space is Fréchet. Metric spaces are of course Lasnev. For function spaces we have the next result.

4.2.8. Theorem. $C_k(X)$ is a Lasnev space if and only if X is hemicompact.

Proof. Since sufficiency is clear we only prove necessity. If $C_k(X)$ is Lasnev then it is Fréchet, hence countably bi-sequential. A countably bi-sequential Lasnev space is metrizable (see corollary 9.10 in [48]).

CHAPTER 5 : COMPLETENESS PROPERTIES

In [39] it was shown that $C_p(X)$ is completely metrizable if and only if it is Čech-complete. Herein, we prove a similar result on $C_k(X)$. We discuss some necessary conditions for $C_k(X)$ to be Baire. Using a theorem of Anderson-Kadec (see [13]) we describe the "geometry" of Polish function spaces.

By a Polish space we mean a separable completely metrizable space. A Tychonoff space that is a G_δ-subset of any Hausdorff space in which it is densely embedded is called a Čech-complete space. Sieve-complete spaces are precisely the continuous open images of Čech-complete spaces (see [62]). An almost Čech-complete space ([1]) is one that has a dense Čech-complete subspace. We have the following implications: Polish implies completely metrizable implies Čech-complete implies Sieve-complete implies almost Čech-complete.

A property weaker than almost Čech-completeness is that of pseudocompleteness introduced in [1]. A space X is pseudocomplete if it admits a sequence $(\mathcal{B}_n : n = 1,2,\ldots)$ of π-bases such that if $B_n \in \mathcal{B}_n$ and $\bar{B}_{n+1} \subseteq B_n$ for each n, then $\bigcap_n B_n \neq \emptyset$. X is called a Baire space (respectively a space of second category in itself) if and only if the intersection of any countable collection of open dense subsets of X is

dense (respectively is non-empty) in X. Clearly then every Baire space is of second category in itself. For homogeneous spaces such as $C_k(X)$ (or $C_p(X)$) the two concepts are equivalent. Note that pseudocomplete spaces are Baire. It is well known that $\mathbb{R}^X$ is pseudocomplete. For $F_k(X)$ the situation is similar.

5.1. Lemma ([46]). $F_k(X)$ is a pseudocomplete space.

Proof. Let $\mathcal{B}$ be the standard base for $F_k(X)$ as described in chapter one. For each n, let $B_n \in \mathcal{B}$ with $\bar{B}_{n+1} \subseteq B_n$. We claim that $\bigcap_n B_n \neq \emptyset$. To see this, let $x \in X$. Define $V_n(x) = \cap\{ V_{ni} : x \in A_{ni}$ for each $i=1,2,\ldots,k_n\}$ where A_{ni} is compact in X for each i and V_{ni} is a bounded open interval in $\mathbb{R}$ such that $B_n = [A_{n1},V_{n1}]_F \cap \ldots \cap [A_{nk_n},V_{nk_n}]_F$. It is clear that $V_n(x) \neq \emptyset$ for each n. Moreover $V_{n+1}(x) \subseteq V_n(x)$ for all n. Since the $V_n(x)$'s are bounded, then $\bigcap_n V_n(x) \neq \emptyset$. Pick $y \in \bigcap_n V_n(x)$. Define $f(x) = y$. Then $f:X \longrightarrow \mathbb{R}$ defined this way is clearly well-defined, $f \in F_k(X)$ and, by construction, $f \in \bigcap_n B_n$. Thus $\bigcap_n B_n \neq \emptyset$.

Now, for each n, let $\mathcal{B}_n = \mathcal{B}$. Then $\{\mathcal{B}_n : n=1,2,\ldots\}$ is the required sequence of π-bases for pseudocompleteness.

We use lemma 5.1 to prove the next theorem which is the

analogue of theorem 1 of [25].

5.2. Theorem. If $C_k(X)$ contains a dense G_δ-subset of $F_k(X)$ then X is a $k_{\mathbb{R}}$-space.

Proof. Let D be a dense G_δ-subset of $F_k(X)$ such that $D \subseteq C_k(X)$. Suppose that X is not a $k_{\mathbb{R}}$-space. Let $f \in F_k(X)$ be such that $f|_A$ is continuous for each compact set A of X but f is not continuous on all of X. Define $\Psi: F_k(X) \longrightarrow F_k(X)$ by $\Psi(g) = f+g$ for all $g \in F_k(X)$. Then Ψ is a homeomorphism. Since $f \notin C_k(X)$, $\Psi(C_k(X)) \cap C_k(X) = \emptyset$. Since $\Psi(D)$ is a dense G_δ-subset of $F_k(X)$ in $\Psi(C_k(X))$ then $\Psi(D)$ and D are two disjoint dense G_δ-subsets of the Baire space $F_k(X)$. This is impossible. Therefore X must be a $k_{\mathbb{R}}$-space.

5.3. Theorem. The following are equivalent:

(1) $C_k(X)$ is almost $\check{C}$ech-complete

(2) $C_k(X)$ is sieve complete

(3) $C_k(X)$ is $\check{C}$ech-complete

(4) $C_k(X)$ is completely metrizable

(5) X is a hemicompact k-space.

Proof. (4)⇒(3)⇒(2)⇒(1): follow from the definitions. (5)⇒(4): suppose that X is hemicompact k-space. Let $(A_n : n = 1,2,...)$ be a sequence of compact sets in X satisfying the definition of a hemicompact space. The trivial map $p: \bigoplus_n A_n \longrightarrow X$ which is compact-covering is in fact quotient since X is a k-space. By proposition 1.2.10, $p^*: C_k(X) \longrightarrow \prod_n C_k(A_n)$ is a closed embedding of $C_k(X)$ into the completely metrizable space $\prod_n C_k(A_n)$. Therefore $C_k(X)$ is completely metrizable.

(1)⇒(5): Suppose that D is a dense $\check{C}$ech-complete subset of $C_k(X)$. Since D is dense in $F_k(X)$ then it is a G_δ-subset of $F_k(X)$. By theorem 5.2 X is a $k_{\mathbb{R}}$-space. On the other hand, D being $\check{C}$ech-complete is of point countable type. By theorems 3.2.2 and 3.2.3, X is hemicompact. By proposition 1.4.3 a hemicompact $k_{\mathbb{R}}$-space is a k-space.

The equivalence of (4) and (5) were obtained by [65] (see also [12]).

5.4. Corollary. The following are equivalent:

(1) $C_k(X)$ is a pseudocomplete σ-space

(2) $C_k(X)$ is a pseudocomplete q-space

(3) X is a hemicompact k-space.

Proof. (3)⇒(2): This follows from the above theorem and the definitions. (2)⇒(1): If $C_k(X)$ is a q-space then it is metrizable by theorem 3.2.3, hence a σ-space. (1)⇒(3): Suppose that $C_k(X)$ is a pseudocomplete σ-space. Every σ-space that is Baire has a dense metrizable subset (see Van Douwen [63]). Therefore, $C_k(X)$ is metrizable (by theorem 3.2.3) and pseudocomplete. But, such a space has a dense completely metrizable subspace (see [1]). It follows that $C_k(X)$ is almost Čech-complete.

5.5. Proposition. If $C_k(X)$ is Baire then every closed $\aleph_0$-subset of X is hemicompact.

Proof. Let A be a closed $\aleph_0$-subset of X. If $i : A \longrightarrow X$ is the inclusion map. Then the induced map $i^*:C_k(X) \longrightarrow C_k(A)$ is almost surjective and, $i^*:C_k(X) \longrightarrow i^*(C_k(X))$ is an open map by proposition 1.2.11. Since open maps preserve the Baire property, then $i^*(C_k(X))$ is a dense Baire subset of $C_k(A)$. It follows that $C_k(A)$ is itself a Baire space. Now, $C_k(A)$ is an $\aleph_0$-space, hence a σ-space. By Van Douwen's result mentioned in the proof of (1)⇒(3) of corollary 5.4, $C_k(A)$ is metrizable. Thus A is hemicompact.

5.6. Corollary. Let X be separable metrizable. Then $C_k(X)$ is

Baire if and only if $C_k(X)$ is metrizable.

When X is pseudocompact we can characterize the Baire property on $C_k(X)$ as follows.

5.7. Theorem. Let X be pseudocompact. Then $C_k(X)$ is Baire if and only if X is compact.

Proof. When X is compact, $C_k(X)$ is completely metrizable. Conversely suppose X not compact. For each n, let $W_n = U\{[(x),(n,n+1)]:x \in X\}$. Then each W_n is dense and open in $C_k(X)$. Clearly, $\cap_n W_n = \emptyset$. Thus $C_k(X)$ is not Baire.

5.8. Corollary. If $C_k(X)$ is Baire then every closed pseudocompact subset of X is compact.

5.9. Theorem ([46]). If $C_k(X)$ is Baire then X is locally compact at each point at which it is locally a q-space.

Proof. Suppose that X is locally a q-space at x_0. Let $(U_n:n=1,2,...)$ be a sequence of open neighborhoods of x_0 such that $\bar{U}_{n+1} \subseteq U_n$ for each n with the property that if $x_n \in U_n$ for each n then $(x_n:n=1,2...)$ clusters

in X. Suppose that none of the $\bar{U}_n$'s is compact. Then, for each n, G_n = $\cup\{[a,(n,n+1)]:a \in \bar{U}_n\}$ is a dense open subset of $C_k(X)$. To see that $\cap_n G_n$ = $\emptyset$ suppose there is $f \in G_n$ for each n. Then $f \in [(a_n),(n,n+1)]$ where $a_n \in \bar{U}_n$ for each n. Since $\bar{U}_{n+1} \subseteq U_n$ for each n then the sequence $\{a_n:n = 1,2,\ldots\}$ must cluster in X. But this is impossible since f is continuous and $f(a_n) \in (n,n+1)$. Consequently $\cap_n G_n=\emptyset$. This destroys the Baireness of $C_k(X)$. Thus, for some n, $\bar{U}_n$ must be compact.

5.10. Corollary. If X is a q-space and $C_k(X)$ is Baire then X is locally compact.

We have a partial converse to corollary 5.10.

5.11. Theorem. If X is a paracompact locally compact space then $C_k(X)$ is pseudocomplete (hence Baire).

Proof. If X is paracompact locally compact then $X = \oplus_\alpha A_\alpha$ where each A_α is locally compact Lindelöf. Therefore $C_k(X) = \Pi_\alpha C_k(A_\alpha)$ is pseudocomplete since each $C_k(A_\alpha)$ is completely metrizable.

5.12. Corollary. If X is a paracompact q-space then the

following are equivalent:

(1) $C_k(X)$ is Baire

(2) $C_k(X)$ is pseudocomplete

(3) X is locally compact.

5.13. Theorem. Let X be a q-space. Then the following are equivalent:

(1) $C_k(X)$ is a Baire space of countable tightness

(2) $C_k(X)$ is completely metrizable

(3) X is locally compact Lindelöf.

Using corollary 5.12 we get the following result generalizing proposition 5.5.

5.14. Proposition. If $C_k(X)$ is a Baire space then every closed $\aleph$-subset of X has a σ-locally finite k-network of compact sets.

Proof. Let A be a closed $\aleph$-subset of X. If $i : A \longrightarrow X$ is as in the proof of proposition 5.5 then as shown there, $C_k(A)$ is Baire. Now, by Lemma 2.2 in [61], it suffices to show that any closed subset of A that is a paracompact q-subset, is locally compact. So, let B be such a subset of A. Then $C_k(B)$ is Baire. Since B is paracompact q-space, then by corollary 5.12 B is indeed locally compact.

Polish function spaces can be characterized as follows.

5.15. Theorem. The following are equivalent:

(1) $C_k(X)$ contains a dense Polish subspace

(2) $C_k(X)$ is a Polish space

(3) $C_k(X)$ is homeomorphic to $\mathbb{R}^{\mathbb{N}}$

(4) $C_k(X)$ contains a dense copy of $\mathbb{R}^{\mathbb{N}}$

(5) X is a hemicompact cosmic k-space.

Proof. (1)$\Leftrightarrow$(2): This follows from the definition and theorem 5.3. (2)$\Leftrightarrow$(3): $\mathbb{R}^{\mathbb{N}}$ is Polish, hence (3)$\Rightarrow$(2). (2)$\Rightarrow$(3) follows from the theorem of Anderson-Kadec stating that locally convex linear Polish spaces are homeomorphic to $\mathbb{R}^{\mathbb{N}}$. (3)$\Leftrightarrow$(4): This is clear since (1)$\Leftrightarrow$(3). (2)$\Leftrightarrow$(5): This can be seen by putting together theorems 5.3 and 2.4.1.

5.16. Corollary. Let X be locally compact. Then $C_k(X)$ is Polish if and only if X is Polish.

If X is the space of irrational numbers with the usual topology, then X is a Čech-complete space that is not locally compact. X is a Polish space such that $C_k(X)$ is not Polish. Therefore, the

hypothesis of local compactness in corollary 5.16 can not be weakened to
$\check{C}$ech-completeness.

CHAPTER 6 : ANALYTICITY

A topological space X is an analytic space (or a souslin space) if X is a continuous image of a Polish space or, equivalently a continuous image of the space of irrational numbers (with its usual topology). Note that the latter is homeomorphic to the product space $\mathbb{N}^{\mathbb{N}}$. We have the following implications: Polish implies analytic implies cosmic.

If X is σ-compact, say $X = \bigcup_n K_n$ where each K_n is compact, then X is analytic if and only if each K_n is metrizable. A quasi-K space is one in which the closed sets are precisely those subsets whose intersections with every countably-compact subset K are closed in K. Clearly, k-spaces are quasi-K. As shown in [48], regular q-spaces are quasi-K spaces. In theorem 6.3.2, we characterize the analyticity of $C_k(X)$ for a quasi-K space X. But first of all we need the concept of a Borel structure on a set.

1. Borel structures

Let X be a non-empty set. A Borel structure on X is any non-empty collection $\mathcal{B}$ of subsets of X that is closed under taking complements and countable unions. This should be compared with the concept of a topological structure (i.e. topology) on a set X. If $\mathcal{B}$ is a Borel structure on X then X together with $\mathcal{B}$, denoted by $(X,\mathcal{B})$, is

called a Borel (or a measurable) space. Members of $\mathcal{B}$ are named Borel sets.

If $(X,\mathcal{T})$ is a topological space then the "smallest" (in the sense of inclusion) Borel structure on X containing all open sets of $(X,\mathcal{T})$ is called the Borel structure generated by $(X,\mathcal{T})$. This structure will be denoted by $\mathcal{B}_{\mathcal{T}}$. Borel structures on subsets and products are defined similarly to their topological counterparts.

If $(X,\mathcal{B})$ and $(Y,\mathcal{D})$ are two Borel spaces, a map $f : (X,\mathcal{B}) \longrightarrow (Y,\mathcal{D})$ is measurable if and only if for each $D \in \mathcal{D}$, $f^{-1}(D) \in \mathcal{B}$. f is a Borel isomorphism if it is bijective and if f and f^{-1} are both measurable. If $\mathcal{B}$ and $\mathcal{B}'$ are two Borel structures on X, $\mathcal{B}$ is isomorphic to $\mathcal{B}'$ means that the identity map $id : (X,\mathcal{B}) \longrightarrow (X,\mathcal{B}')$ is a Borel isomorphism.

A Borel space $(X,\mathcal{B})$ is analytic if $\mathcal{B}$ is isomorphic to $\mathcal{B}_{\mathcal{T}}$ for some analytic topology $\mathcal{T}$ on X. Let us point out that whenever we say that X is analytic with no mention of the structure, it means that X is analytic as a topological space.

It is clear that the Borel space $(X,\mathcal{B}_{\mathcal{T}})$ is analytic whenever the topology $\mathcal{T}$ of X is analytic. The converse is not true as the Sorgenfrey line is a counterexample. But when the topology of X is metrizable, Frolik [29] has the following.

6.1.1. Proposition. If $(X,\mathcal{T})$ is a metrizable topological space then $(X,\mathcal{T})$ is analytic if and only if $(X,\mathcal{B}_{\mathcal{T}})$ is analytic.

The next theorem is a special case of a more general theorem known as Hurewicz theorem (see [16]).

6.1.2. Theorem. Let X be analytic. Then X is σ-compact if and only if X does not contain a closed copy of $\mathbb{N}^{\mathbb{N}}$.

Borel sets in a Polish space are analytic topological subspaces (see Dellacherie [24]).

6.1.3. Theorem. Every Borel set in a Polish space is a compression of a closed subset of $\mathbb{N}^{\mathbb{N}}$.

2. Necessary conditions

6.2.1. Theorem. If X is a $k_{\mathbb{R}}$-space, then there exists no continuous surjection from X onto $C_k(X)$.

Proof. Suppose by way of contradiction that there is a continuous surjection $\varphi : X \longrightarrow C_k(X)$. Define $f : X \longrightarrow \mathbb{R}$ by $f(x) = 1+\varphi(x)(x)$ for each x in X. To see that f is continuous on X it suffices to show that it is continuous on each compact subset of X. So, let A be compact in X. Then on A f can be written as:

$$A \longrightarrow AxA \longrightarrow AxC_k(X) \longrightarrow \mathbb{R}.$$

$$x \mapsto (x,x) \mapsto (x,\varphi(x)) \mapsto 1+\varphi(x)(x)$$

Since the compact-open topology is jointly continuous on compact sets then clearly fl_A is continuous. Now, $f \in C_k(X)$ and φ is surjective. Therefore $f = \varphi(x_0)$ for some $x_0 \in X$. But then $1+\varphi(x_0)(x_0) = f(x_0) = \varphi(x_0)(x_0)$ which is impossible.

6.2.2. Corollary. $C_k(\mathbb{N}^{\mathbb{N}})$ is not an analytic space.

Proof. Suppose $C_k(\mathbb{N}^{\mathbb{N}})$ analytic and let $\varphi : \mathbb{N}^{\mathbb{N}} \longrightarrow C_k(\mathbb{N}^{\mathbb{N}})$ be a continuous surjection. Then this contradicts the above theorem since $\mathbb{N}^{\mathbb{N}}$ is a $k_{\mathbb{R}}$-space.

6.2.3. Proposition. If $C_k(X)$ is analytic then $C_k(X)$ contains a closed copy of $\mathbb{N}^{\mathbb{N}}$.

Proof. Suppose that $C_k(X)$ is analytic. If X is discrete then $C_k(X) = \mathbb{R}^X$ is analytic if and only if X is countable. Since $\mathbb{R}^X (=\mathbb{R}^{\mathbb{N}})$ is not σ-compact, by theorem 6.1.2 $C_k(X)$ contains a closed copy of $\mathbb{N}^{\mathbb{N}}$. If X is not discrete, let $A = (x_1,x_2,\ldots) \cup (x)$ where $(x_n)_n$ is a sequence in X converging to some x in X (this can be done in any cosmic space, See

[8]). Since A is a compact metrizable subset of X then $C_k(A)$ is homeomorphic to $\mathbb{R}^{\mathbb{N}}$ (by theorem 5.15 in chapter five). It then follows that $C_k(A)$ is not σ-compact.

Next, let $i : A \longrightarrow X$ be the inclusion map. By corollary 1.2.12, $i^*: C_k(X) \longrightarrow C_k(A)$ is a continuous surjection. If $C_k(X)$ is σ-compact, then so will be $C_k(A)$, thence contradicting the preceding argument. So, by theorem 6.1.2, $C_k(X)$ has a closed copy of $\mathbb{N}^{\mathbb{N}}$.

6.2.4. Theorem. Let $\mathcal{P}$ be a topological property preserved by either continuous or open maps but not satisfied by $C_k(\mathbb{N}^{\mathbb{N}})$. If X is analytic and $C_k(X)$ has $\mathcal{P}$ then X is a σ-compact space.

Proof. Suppose that X is not σ-compact. By theorem 6.1.2, X has a closed copy of $\mathbb{N}^{\mathbb{N}}$. If $i : \mathbb{N}^{\mathbb{N}} \longrightarrow X$ is the inclusion map then $i^*: C_k(X) \longrightarrow C_k(\mathbb{N}^{\mathbb{N}})$ is a continuous open surjection. Now, let $\mathcal{P}$ be a property preserved by either continuous or open maps. If $C_k(X)$ has $\mathcal{P}$ then so does $C_k(\mathbb{N}^{\mathbb{N}})$, hence contradicting the hypothesis.

6.2.5. Corollary. If $C_k(X)$ is analytic or Baire or Fréchet then every closed analytic subset of X is σ-compact.

6.2.6. Proposition. If $C_k(X)$ is analytic then X does not contain a closed copy of $\mathbb{N}^{\mathbb{N}}$.

Proof. Suppose on the contrary that there is a closed copy of $\mathbb{N}^{\mathbb{N}}$ in X. Again, if $i : \mathbb{N}^{\mathbb{N}} \longrightarrow X$ is the inclusion map then $i^* : C_k(X) \longrightarrow C_k(\mathbb{N}^{\mathbb{N}})$ is a continuous surjection, hence making $C_k(\mathbb{N}^{\mathbb{N}})$ analytic. This contradicts corollary 6.2.2.

Notice that σ-compactness is stronger than the property of not containing a closed copy of $\mathbb{N}^{\mathbb{N}}$.

Our next goal is to refine proposition 6.2.6 by showing that X is σ-compact whenever $C_k(X)$ is analytic. But first we need the following lemmas.

6.2.7. Lemma ([19]). Let X be separable metrizable. Then X is Polish if and only if there are a Polish space Y and a map $p : K(Y) \longrightarrow K(X)$ satisfying:

(1) if $K \subseteq C$ then $p(K) \subseteq p(C)$

(2) if $K \in K(X)$ then there exists $C \in K(Y)$ such that $K \subseteq p(C)$.

Since $\mathbb{R}$ is homeomorphic to $(0,\omega)$ then $C_k(X)$ is clearly homeomorphic to its subspace $C_k^+(X) = \{f \in C(X) : f(X) \subseteq (0,\omega)\}$.

Similarly, $F_K(X)$ is homeomorphic to $F_K^+(X) = \{f \in F_K(X): f(X) \subseteq (0,\infty)\}$.

6.2.8. Lemma. If $C_K(X)$ is analytic then there exists a map $\Gamma : \mathbb{N}^{\mathbb{N}} \to F_K^+(X)$ satisfying:

(1) if $\beta \leqslant \sigma$ then $\Gamma(\beta) \geqslant \Gamma(\sigma)$

(2) if $f \in C_K^+(X)$ then $f \geqslant \Gamma(\sigma)$ for some $\sigma \in \mathbb{N}^{\mathbb{N}}$.

Proof. Let $\varphi : \mathbb{N}^{\mathbb{N}} \to C_K^+(X)$ be a continuous surjection. Define $\Gamma : \mathbb{N}^{\mathbb{N}} \to F_K^+(X)$ by $\Gamma(\sigma)(x) = \inf\{\varphi(\beta)(x): \beta \leqslant \sigma\}$ for all x in X and each $\sigma \in \mathbb{N}^{\mathbb{N}}$; where $\beta = (\beta(1),\beta(2),\ldots) \leqslant \sigma = (\sigma(1),\sigma(2),\ldots)$ means that $\beta(n) \leqslant \sigma(n)$ for all $n \in \mathbb{N}$. Clearly Γ satisfies (1) and (2).

The next result is contained in a theorem of Calbrix [16].

6.2.9. Theorem. If $C_K(X)$ is analytic then X is analytic.

Proof. Since $C_K(X)$ is analytic then X is clearly cosmic. Let $\mathcal{T}$ be the topology on X. Choose a coarser separable metrizable topology $\mathcal{T}'$ on X with the following properties:

(1) $(X,\mathcal{T})$ and $(X,\mathcal{T}')$ have the same Borel structure

(2) $(X,\mathcal{T})$ is analytic if and only if $(X,\mathcal{T}')$ is analytic.

This can be done in any cosmic space (see [17]). The idea is to show

that $(X,\mathcal{T}')$ is analytic. To see this, let (Y,ρ) be a compact metric space (with metric ρ) in which $(X,\mathcal{T}')$ is densely embedded and let $Z = (Y,\rho)-(X,\mathcal{T}')$. Then Z is separable metrizable. We claim that Z is Polish. By lemma 6.2.7 it suffices to construct a map $\rho : K(\mathbb{N}^{\mathbb{N}}) \longrightarrow K(Z)$ satisfying the conditions of that lemma.

First, let $\Gamma : \mathbb{N}^{\mathbb{N}} \longrightarrow F_k^+(X)$ be defined as in lemma 6.2.8. Then define $\Psi : \mathbb{N}^{\mathbb{N}} \longrightarrow K(Y)$ by $\Psi(\sigma) = \cap(Y-B_x : x \in X)$ where $B_x = (y \in Y : \rho(x,y) < \Gamma(\sigma)(x))$ is the open ball about x of radius $\Gamma(\sigma)(x)$.

If A is compact in Z, the map $f : Y \longrightarrow \mathbb{R}$ defined by $f(y) = \rho(A,y)$ for all $y \in Y$, is continuous and $f|_X \in C_k^+(X)$. By construction of Γ there is $\sigma \in \mathbb{N}^{\mathbb{N}}$ such that $f|_X \gg \Gamma(\sigma)$. Clearly then $A \subseteq \Psi(\sigma)$.

Now, define $\rho : K(\mathbb{N}^{\mathbb{N}}) \longrightarrow K(Z)$ by $\rho((\beta : \beta \leqslant \sigma)) = \Psi(\sigma)$. Then ρ satisfies (1) and (2) of lemma 6.2.7. Consequently Z is Polish, hence G_δ in (Y,ρ). But then $(Y,\rho)-Z = (X,\mathcal{T}')$ is F_σ in the compact space (Y,ρ). Therefore $(X,\mathcal{T}')$ is a σ-compact metrizable space. That $(X,\mathcal{T}')$ is analytic follows from the fact that a countable union of closed analytic spaces is analytic. Now, $(X,\mathcal{T}')$ is analytic if and only if $(X,\mathcal{T})$ is analytic.

6.2.10. Corollary. If $C_k(X)$ is analytic then X is a σ-compact $\aleph_0$-space.

Proof. Suppose $C_k(X)$ analytic. Clearly X is an $\aleph_0$-space and, by theorem 6.2.9 X is also analytic. By corollary 6.2.5, X must be σ-compact.

In fact, Calbrix [16] showed that X is σ-compact whenever $C_p(X)$ is analytic.

3. Characterizations

We impose some mild restrictions on X to obtain a characterization of the analyticity of $C_k(X)$.

It is always true that if $C_k(X)$ is analytic then so is $C_p(X)$. The converse is not true in general but we do have the following.

6.3.1. Theorem. Let X be a q-space. Then $C_k(X)$ is analytic if and only if $C_p(X)$ is analytic.

Proof. Suppose that $C_p(X)$ is analytic. Then $C_p(X)$ is cosmic, implying that X is also cosmic. Now a cosmic M-space is separable metrizable (see theorem 5.1.2 of [15] or see [18] cited in [15]).

Let $\{B_n : n \in \mathbb{N}\}$ be a countable base for X, closed under finite unions. Choose $(r_1, r_2, \ldots)$ a dense sequence in $\mathbb{R}$ and, for each $(f,g) \in C(X) \times C(X)$, define the following pseudometric $d(f,g) = \sum_{k,n}$

$\frac{1}{2^{k+n}} \left| \left(\sup_{x \in B_n} |f(x)-r_k| \wedge 1 \right) - \left(\sup_{x \in B_n} |g(x)-r_k| \wedge 1 \right) \right|$, where $a \wedge b$ means $\min(a,b)$.

As shown in [19], d is a metric on $C(X)$. Moreover the metric space $C_d(X)$ and the spaces $C_k(X)$ and $C_p(X)$ all have the same Borel structure; and the identity map $id : C_d(X) \longrightarrow C_k(X)$ is continuous.

Now, since $C_p(X)$ is analytic then the Borel structure generated by the topology of pointwise convergence is analytic. It then follows that the Borel structure generated by the metric d is analytic. By proposition 6.1.1 the metric space $C_d(X)$ is analytic. Consequently $C_k(X)$ is also analytic.

6.3.2. Theorem. Let X be a quasi-K space. Then $C_k(X)$ is analytic if and only if X is a σ-compact $\aleph_0$-space.

Proof. The condition is necessary by corollary 6.2.10. To establish sufficiency, suppose X is a σ-compact $\aleph_0$-space. Since X is quasi-K then each countably compact subset of X is metrizable, and is hence compact ([18]). Therefore X is a K-space. Let (M,ρ) be a separable metric space (with metric ρ) and $\varphi : (M,\rho) \longrightarrow X$ a compact-covering map, which is indeed quotient. By proposition 1.2.10, $\varphi^* : C_k(X) \longrightarrow C_k(M)$ is a closed embedding of $C_k(X)$ into $C_k(M)$. We are through if we can show that

$\varphi^*(C_k(X))$ is contained in an analytic subspace of $C_k(M)$.

Let ρ' be a metric on X inducing a coarser separable metrizable topology (see proposition 2.3.8). Without loss of generality we may assume that the metric ρ on M is such that the composite function $(M,\rho) \xrightarrow{\varphi} X \xrightarrow{id} (X,\rho')$ is uniformly continuous. Since X is σ-compact, let $\langle A_n : n \in \mathbb{N}\rangle$ be a sequence of compact sets in X such that $X = \bigcup_n A_n$. Put $\mathcal{Q} = \{g \in C_k(M) : g \text{ is uniformly continuous on each } \varphi^{-1}(A_n)\}$.

Note that for each $g \in C_k(X)$, $\varphi^*(g) = g \circ \varphi$ is uniformly continuous on each $\varphi^{-1}(A_n)$. Therefore $\varphi^*(C_k(X)) \subseteq \mathcal{Q}$. We claim that $\mathcal{Q}$ is an analytic subspace of $C_k(M)$. To see this let $S = \langle s_1, s_2, s_3, \ldots\rangle$ be a sequence in (M,ρ) such that $S \cap \varphi^{-1}(A_n)$ is dense in $\varphi^{-1}(A_n)$ for each n. Then define $\Psi : C_k(M) \to \mathbb{R}^S$ by $\Psi(g) = \langle g(s_1), g(s_2), \ldots\rangle$ for each $g \in C_k(M)$. As shown in [19] (see proof of theorem 3.7) Ψ is a Borel isomorphism of $C_k(M)$ onto its image $\Psi(C_k(M))$. Moreover $\Psi(\mathcal{Q}) = \bigcap_n \{(r_{s_k}) \in \mathbb{R}^S : \limsup_{p\to\infty} \{|r_{s_l} - r_{s_m}| : s_l \in \varphi^{-1}(A_n),\ \rho(s_l,s_m) < \frac{1}{p}\} = 0\}$. Therefore $\Psi(\mathcal{Q})$ is a Borel set in the Polish space $\mathbb{R}^S$. By theorem 6.1.3, $\Psi(\mathcal{Q})$ is an analytic subspace of $\Psi(C_k(M))$. Let d be the metric on $C(M)$ defined as in the proof of theorem 6.3.1 and denote by $(\mathcal{Q},d)$ the set $\mathcal{Q}$ as a subspace of the metric space $C_d(M)$. Note first that the Borel structure

generated by the topology of $\mathcal{Q}$ as a subspace of $C_k(M)$ is an analytic Borel structure. This is because $\Psi|_{\mathcal{Q}} : \mathcal{Q} \longrightarrow \Psi(\mathcal{Q})$ is a Borel isomorphism. Since $\mathcal{Q}$ and $(\mathcal{Q},d)$ have the same Borel structure then the metric space $(\mathcal{Q},d)$ is in fact an analytic measurable (i.e. Borel) space. By proposition 6.1.1 $(\mathcal{Q},d)$ is an analytic metric space. Since the identity map $id : (\mathcal{Q},d) \longrightarrow \mathcal{Q} \subseteq C_k(M)$ is continuous then $\mathcal{Q}$ is clearly an analytic subspace of $C_k(M)$. It follows that $\varphi^*(C_k(X))$ is analytic, being a closed subspace of $\mathcal{Q}$. That $C_k(X)$ is analytic follows from the fact that it is homeomorphic to $\varphi^*(C_k(X))$.

The quasi-K space hypothesis on X in theorem 6.3.2 is not imperative. Indeed, it is shown in [16] that if X is the classical Arens space then $C_k(X)$ is analytic. X is not a quasi-K space.

Theorem 6.3.2 generalizes theorem 3.7 of [19] (see also Calbrix [16]).

For q-spaces (note that regular q-spaces are quasi-K) we get a stronger result.

6.3.3. Theorem. If X is a q-space then $C_k(X)$ is analytic if and only if X is a σ-compact metrizable space.

Proof. Sufficiency follows from theorem 6.3.2 since σ-compact metrizable

spaces are $\aleph_0$-spaces. For necessity suppose $C_k(X)$ analytic. Then X is already σ-compact and $\aleph_0$-space (corollary 6.2.10). To see that X is metrizable it suffices to recall proposition 2.3.2.

If we strengthen the q-space hypothesis to local compactness in theorem 6.3.3 we obtain the following corollary.

6.3.4. Corollary. Let X be locally compact. Then the following are equivalent:

(1) $C_k(X)$ is Polish

(2) $C_k(X)$ is analytic

(3) $C_k(X)$ is cosmic

(4) X is Polish.

We conclude this chapter with a result that shows that whenever one adds a completeness property to an analytic function space then one obtains a Polish function space.

6.3.5. Theorem. $C_k(X)$ is analytic Baire if and only if $C_k(X)$ is Polish.

Proof. Sufficiency is clear since every Polish space is analytic and Baire. For necessity suppose that $C_k(X)$ is analytic and Baire. Then

$C_k(X)$ contains a dense completely metrizable subspace (see [38]). Therefore $C_k(X)$ is almost Čech-complete. By theorem 5.3 in chapter five, $C_k(X)$ is a completely metrizable space. That $C_k(X)$ is separable follows from the fact that analyticity is stronger than separability.

CHAPTER 7 : COVERING PROPERTIES

We study various kinds of covering properties on $C_k(X)$.

1. Countable Chain Condition (ccc)

A space X has the countable chain condition (i.e. ccc) if and only if any collection of pairwise disjoint non-empty open sets in X is countable. Separable spaces and hereditarily Lindelöf spaces have ccc.

We introduce a linear property due to Arhangelskii [6] which is weaker than both ccc and Lindelöf.

Let (G,*) be a topological group with an identity element e. We say that (G,*) is $\aleph_0$-bounded if for every neighborhood W of e there is a countable subset A_W of G such that $G = \{x*w : x \in A_W \text{ and } w \in W\}$.

Arhangelskii has characterized $\aleph_0$-bounded topological groups as being groups topologically isomorphic to a subgroup of a group having ccc. This linear property on $C_k(X)$ (note that $C_k(X)$ is a topological group with the usual addition of functions) has a nice description in terms of topological properties on X.

7.1.1. Theorem. The following statements are equivalent:

(1) $C_k(X)$ is $\aleph_0$-bounded

(2) each compact subset of X is metrizable

(3) each hemicompact subset of X is an $\aleph_0$-space

(4) each σ-compact subset of X is a cosmic space

(5) X is a compact-covering image of a metrizable space

(6) X is a compact-covering image of a completely metrizable space.

Proof. (1)$\Leftrightarrow$(2): Suppose that $C_k(X)$ is $\aleph_0$-bounded and let A be a compact subset of X. If $i : A \to X$ is the inclusion map then $i^* : C_k(X) \to C_k(A)$ is almost surjective. Since $\aleph_0$-boundedness is preserved by group homomorphisms then $i^*(C_k(X))$ is a dense $\aleph_0$-bounded subgroup of the metrizable group $C_k(A)$. It then follows that $C_k(A)$ is also $\aleph_0$-bounded (see [6]). Now, $\aleph_0$-bounded metrizable groups are separable. Therefore, by theorem 2.1.1 A is submetrizable, hence metrizable because of compactness.

Conversely, suppose that each compact set in X is metrizable. Let $\langle A_\alpha : \alpha \in \Gamma \rangle$ be the collection of all non-empty compact sets of X. The natural map $p : \bigoplus_{\alpha \in \Gamma} A_\alpha \to X$ is compact-covering and so, the induced map $p^* : C_k(X) \to \prod_\alpha C_k(A_\alpha)$ is a topological isomorphism mapping $C_k(X)$ onto $p^*(C_k(X))$. Since each A_α is metrizable, then $C_k(A_\alpha)$ is separable. Therefore $\prod_\alpha C_k(A_\alpha)$ has ccc. It follows that $C_k(X)$ is $\aleph_0$-bounded.

(2)⇔(3): If each compact subset of X is metrizable let A be a hemicompact subset of X. Take $\{A_n : n=1,2,\ldots\}$ a fundamental system of compact subsets of A. Then $A = \cup A_n$ with each A_n compact, hence metrizable. It then follows that A is cosmic (being a countable union of closed cosmic subsets). By theorem 2.4.1, A is an $\aleph_0$-space.

Conversely, if every hemicompact subset of X is an $\aleph_0$-space then compact sets are $\aleph_0$-spaces. Now, locally compact $\aleph_0$-spaces are metrizable.

(2)⇔(4): If every σ-compact subset of X is cosmic then so is every compact subset. Since compact cosmic spaces are $\aleph_0$-spaces, the conclusion follows from the above argument.

For the converse, suppose that each compact subset of X is metrizable. If $A = \cup A_n$ is a σ-compact subset in X with each A_n compact, then each A_n is metrizable, hence cosmic. Again A is cosmic being a countable union of closed cosmic spaces.

(6)⇒(5): This is clear. (5)⇒(2): Let M be a metrizable space and $f : M \longrightarrow X$ a compact-covering map. If A is a compact subset of X then there is a compact set K in M such that A = f(K). Now, $f|_K : K \longrightarrow A$ is a continuous surjection. Since K is compact metrizable then so is A.

(2)⇒(6): Suppose that each compact set in X is metrizable. If $\{A_\alpha : \alpha \in \Gamma\}$ is the collection of all nonempty compact subsets of X then

$\oplus_{\alpha\in\Gamma} A_\alpha$ is a completely metrizable space and the natural map $p : \oplus_\alpha A_\alpha \to X$ is compact-covering.

Since $\aleph_0$-boundedness is weaker than ccc then the next result is a corollary of theorem 7.1.1.

7.1.2. Corollary. If $C_k(X)$ has ccc then each compact subset of X is metrizable.

The converse to this corollary is not true in general. Indeed, let X be the ordinal space $[0,\omega_1)$ with the order topology. Then each compact set in X is metrizable. Let S be the set of isolated points of X. For each $\alpha \in S$, let $W_\alpha = [\{x \in X : x < \alpha\},(0,1)] \cap [\{\alpha\},(1,2)]$. Then each W_α is open in $C_k(X)$ and, if $\alpha < \beta$ in S, $W_\alpha \cap W_\beta = \emptyset$ because if $f \in W_\alpha \cap W_\beta$ then $f(\alpha) \in (1,2)$ and $f(\alpha) \in (0,1)$ which is impossible. So, $(W_\alpha : \alpha \in S)$ is an uncountable collection of pairwise disjoint open sets in $C_k(X)$. Therefore, $C_k(X)$ does not have ccc.

However, we have the following partial converse.

7.1.3. Theorem. If X is submetrizable then $C_k(X)$ has ccc.

Proof. Let X be a submetrizable space. Choose $\{W_\alpha : \alpha \in \Gamma\}$ a collection of pairwise disjoint non-empty basic open sets in $C_k(X)$. Then for each $\alpha \in \Gamma$, $W_\alpha = \cap\{[A,V] : A \in \mathcal{A}_\alpha \text{ and } V \in \mathcal{V}_\alpha\}$ where $\mathcal{A}_\alpha$ is a finite family of compact sets in X and $\mathcal{V}_\alpha$ a finite family of open sets in $\mathbb{R}$.

Pick $\Gamma_0 \subseteq \Gamma$ with $|\Gamma_0| \leq 2^\omega$ and let $Y = \cup\{\cup\mathcal{A}_\alpha : \alpha \in \Gamma_0\}$. Then Y is a submetrizable space and $d(Y) \leq 2^\omega$. By theorem 2.1.3 $C_k(Y)$ is separable.

Now, for each $\alpha \in \Gamma$, let $W'_\alpha = \cap\{[A,V]' : A \in \mathcal{A}_\alpha \text{ and } V \in \mathcal{V}_\alpha\}$ where $[A,V]' = \{f \in C_k(Y) : f(A) \subseteq V\}$. Then $\{W'_\alpha : \alpha \in \Gamma_0\}$ is a collection of pairwise disjoint non-empty open sets in $C_k(Y)$ since $\cup\mathcal{A}_\alpha \subseteq Y$ for each $\alpha \in \Gamma_0$. It follows that Γ_0 is countable because $C_k(Y)$ is separable. Therefore, Γ is itself countable.

The next corollary follows from theorems 3.1.15 and 7.1.3.

7.1.4. Corollary. If X is a k-space and $C_k(X)$ is almost σ-compact then $C_k(X)$ has ccc.

7.1.5. Proposition. If each compact subset of $C_p(C_k(X))$ is metrizable then $C_k(X)$ has ccc.

Proof. This follows from the fact that a space Y has ccc whenever each compact subset of $C_p(Y)$ is metrizable (see [7]).

7.1.6. Theorem. Let X be a σ-compact space. Then the following are equivalent:

(1) $C_k(X)$ is separable

(2) $C_k(X)$ is almost σ-compact

(3) $C_k(X)$ has ccc

(4) X is cosmic (analytic)

(5) X is submetrizable.

Proof. It suffices to notice that a σ-compact submetrizable space is cosmic (analytic) and compact sets of X are metrizable whenever $C_k(X)$ is almost σ-compact.

We must point out that for any space X, $C_p(X)$ has ccc. This is because $C_p(X)$ is dense in the product space $\mathbb{R}^X$, which has ccc. For $C_k(X)$ the situation is different. Indeed $C_k(X)$ does not have ccc for a non-metrizable compact space X.

Now, let X be the Fortissimo space introduced earlier (see remark following corollary 2.2.3). Since $C_k(X) = C_p(X)$ then $C_k(X)$ has

ccc. However X is not an E_0-space, hence not submetrizable. Therefore the converse to theorem 7.1.3 is false.

2. Paracompactness and the Lindelöf property

One of the best covering properties that $C_k(X)$ can carry is the Lindelöf property. In [42] some necessary conditions are studied for $C_p(X)$ (hence for $C_k(X)$) to be Lindelöf (see also [7]).

Whenever X is an $\aleph_0$-space, $C_k(X)$ is also an $\aleph_0$-space, hence (hereditarily) Lindelöf. Apart from that, very little is known on the Lindelöf property on $C_k(X)$. Note first that $C_p(X)$ is Lindelöf if and only if it is paracompact. This is because a paracompact ccc space is Lindelöf. Now, if X is a non-metrizable compact space then $C_k(X)$ is a paracompact space that is not Lindelöf. But, when X is submetrizable one has the following result.

7.2.1. Theorem. Let X be submetrizable. Then $C_k(X)$ is Lindelöf if and only if $C_k(X)$ is paracompact.

7.2.2. Corollary. If X is cosmic and $C_k(X)$ is paracompact then

$C_k(X)$ is Lindelöf.

7.2.3. Proposition. If $C_k(X)$ contains a dense Lindelöf subspace then every hemicompact subset of X is an $\aleph_0$-subspace.

Proof. This follows from the fact that the property of having a dense Lindelöf subset is preserved by continuous almost surjective maps.

The proof of the next proposition is straightforward. Recall that a locally compact $\aleph_0$-space is separable metrizable.

7.2.4. Proposition. The following are equivalent for a locally compact paracompact space X:

(1) $C_k(X)$ is separable

(2) $C_k(X)$ is Lindelöf

(3) $C_k(X)$ is separable metrizable

(4) $C_k(X)$ has ccc

(5) $C_k(X)$ is Polish

(6) X is Polish

(7) X is separable metrizable.

A concept a little stronger than Lindelöf is that of strong paracompactness (sometimes called hypocompactness) (see [26]). On $C_k(X)$ this notion is equivalent to the Lindelöf property.

7.2.5. Theorem. $C_k(X)$ is strongly paracompact if and only if $C_k(X)$ is Lindelöf.

Proof. $C_k(X)$ is connected as mentioned earlier. If it is strongly paracompact then it must be Lindelöf (by exercise 6.1.D, p.442 in [26]). The converse is obvious.

7.2.6. Theorem. If $C_k(X)$ is Lindelöf then X^n has countable tightness for each $n \in \mathbb{N}$.

Proof. We only prove the case n=1 (the general case is similar). Let $A \subseteq X$ and $x_0 \in \bar{A}$. We must show that there is a countable set $B \subseteq A$ such that $x_0 \in \bar{B}$. Suppose on the contrary that no such B exists in A. Let $S = \{f \in C_k(X) : f(x_0) = 0\}$. Then S is a closed subset of $C_k(X)$. Therefore S is Lindelöf. Next, for each $a \in A$ let $W_a = [\{a\}, \mathbb{R}-\{1\}]$. Then $\{W_a : a \in A\}$ is an open cover of S. therefore, there is a

countable set $C \subseteq A$ with $S \subseteq \cup\{W_a : a \in C\}$. Since $x_0 \notin \bar{C}$, choose $f \in C(X)$ such that $f(x_0) = 0$ and $f(\bar{C}) = \{1\}$. Then $f \in S$ but $f \notin \cup\{W_a : a \in C\}$ which is contradictory. Consequently $\{W_a : a \in A\}$ is a cover of S by open sets of $C_k(X)$ with no countable subcover. But this is impossible since S is Lindelöf. Therefore, for some B countable we must have $x_0 \in \bar{B}$. This means that X has countable tightness.

Notice the duality: If $C_k(X)$ has countable tightness then X^n is Lindelöf for each $n \in \mathbb{N}$ (see corollary 4.1.2).

7.2.7. Theorem. If $C_k(X)$ is Lindelöf then every discrete family of open sets in X is countable.

Proof. Let $\mathcal{V} = \{V_\alpha : \alpha \in \Gamma\}$ be a discrete family of open sets in X. For each $\alpha \in \Gamma$, let $x_\alpha \in V_\alpha$ and define $\Psi : C_k(X) \to \mathbb{R}^\Gamma$ by $\Psi(f) = \langle f(x_\alpha)\rangle_{\alpha \in \Gamma}$ for all f in $C_k(X)$. Then Ψ is continuous. To see that Ψ is onto, let $\langle y_\alpha\rangle_{\alpha \in \Gamma}$ be in $\mathbb{R}^\Gamma$. By complete regularity of X choose $f_\alpha \in C(X)$ such that $f_\alpha(x_\alpha) = y_\alpha$ and $f_\alpha(X - V_\alpha) = \{0\}$. Then, define $f(x) = \sum_{\alpha \in \Gamma} f_\alpha(x)$ for each x in X. Then f is a well-defined continuous

function and $\Psi(f) = \langle y_\alpha \rangle_{\alpha \in \Gamma}$. Now, since $C_k(X)$ is Lindelöf then so is $\mathbb{R}^\Gamma$. But then Γ must be countable.

Similarly one can show the following.

7.2.8. Theorem ([42]). If X is normal and $C_k(X)$ is Lindelöf then every discrete family of closed sets in X is countable.

7.2.9. Corollary. If X is a normal σ-space and $C_k(X)$ is Lindelöf then X is cosmic.

Proof. Let $\mathcal{B} = \cup \mathcal{B}_n$ be a σ-discrete closed network for X, where each $\mathcal{B}_n$ is a discrete family. By theorem 7.2.8 each $\mathcal{B}_n$ is countable. Therefore $\mathcal{B}$ is a countable network of X.

7.2.10. Corollary. If X is a normal $\aleph$-space, then $C_k(X)$ is Lindelöf if and only if X is an $\aleph_0$-space.

7.2.11. Corollary. If X is paracompact and $C_k(X)$ is Lindelöf then X is Lindelöf.

7.2.12. Corollary. If X is metrizable then $C_k(X)$ is Lindelöf if and only if X is Lindelöf.

As pointed out in [42] $C_k([0,\omega_1))$ is Lindelöf. However, $C_k([0,\omega_1))$ does not have ccc as we have seen in section 1 of this chapter.

We now turn to Lindelöf Σ-spaces.

7.2.13. Theorem. Let X be almost σ-compact. Then $C_k(X)$ is a Lindelöf Σ-space if and only if X is an $\aleph_0$-space.

Proof. Sufficiency follows from the fact that $\aleph_0$-spaces are Lindelöf Σ-spaces. For necessity, suppose that $C_k(X)$ is a Lindelöf Σ-space. If X is almost σ-compact, then $C_k(X)$ becomes a submetrizable Lindelöf Σ-space. Therefore it is cosmic. By theorem 2.3.7, X is an $\aleph_0$-space.

7.2.14. Corollary. If X is cosmic and $C_k(X)$ is a Lindelöf Σ-space then X is an $\aleph_0$-space.

The next result whose proof is in [42] provides a necessary condition for $C_k(X)$ to be a Lindelöf Σ-space.

7.2.15. Proposition. If $C_k(X)$ is a Lindelöf Σ-space then X has a countable subset that intersects every non-empty open set having compact complement in X.

By theorem 7.2.13, $C_k(X)$ is not a Lindelöf Σ-space for every separable non $\aleph_0$-space X. Notice that Lindelöf Σ is preserved by continuous maps (see Arhangelskii [7]).

7.2.16. Proposition. If $C_k(X)$ is a Lindelöf Σ-space then every almost σ-compact subset of X is subcosmic.

Proof. Let D be an almost σ-compact subset of X. If $i : D \longrightarrow X$ is the inclusion map, then $i^* : C_k(X) \longrightarrow C_k(D)$ is almost surjective. Now, $C_k(D)$ contains a dense submetrizable Lindelöf Σ-space. This subset is cosmic. Therefore $C_k(D)$ is separable. By corollary 3.1.12, D is subcosmic.

For a normal space X we get a stronger result.

7.2.17. Proposition. If X is normal and $C_k(X)$ is a Lindelöf Σ-space then every almost σ-compact subset of X is an $\aleph_0$-subspace.

Proof. Let D be an almost σ-compact subset of X. It is enough to show that $\bar{D}$ is an $\aleph_0$-subset of X. To see this, let $i : \bar{D} \longrightarrow X$ be the inclusion map. By corollary 1.2.12, $i^* : C_k(X) \longrightarrow C_k(\bar{D})$ is a continuous surjection because of normality. It then follows that $C_k(\bar{D})$ is a Lindelöf Σ-space. Since $\bar{D}$ is almost σ-compact, by theorem 7.2.13, $\bar{D}$ is an $\aleph_0$-space.

7.2.18. Corollary. Let X be normal. If $C_k(X)$ is a Lindelöf Σ-space then every countable subset of X is an $\aleph_0$-subset of X.

3. Hereditarily Lindelöf and hereditarily separable properties

The problem of when $C_k(X)$ is hereditarily Lindelöf or hereditarily separable became attractive when Zenor [67] studied these properties for the topology of pointwise convergence. Indeed the next result is a consequence of theorem 6 in [67].

7.3.1. Theorem. The following are equivalent:

(1) $C_p(X,\mathbb{R}^\omega)$ is hereditarily Lindelöf (respectively, hereditarily separable)

(2) $C_p(X,Y)$ is hereditarily Lindelöf (respectively, hereditarily separable) for every separable metrizable space Y

(3) X^ω is hereditarily separable (respectively, hereditarily Lindelöf).

The key in proving this theorem lies in the next lemma.

7.3.2. Lemma. Let X, Y and Z be topological spaces with Y second countable, and let $f : X\times Z \longrightarrow Y$ be a function satisfying:

(1) f is continuous on X;

(2) Z has the weakest topology for which f is continuous on Z.

Then

a) if X^ω is hereditarily Lindelöf then Z is hereditarily separable;

b) if X^ω is hereditarily separable then Z is hereditarily Lindelöf.

This lemma is a very powerful tool in connecting $C_k(X)$ with the hyperspace K(X) of compact sets of X introduced in chapter two.

7.3.3. Theorem.

(1) If $K(X)^\omega$ is hereditarily Lindelöf then $C_k(X)$ is hereditarily

separable;

(2) If $K(X)^{\omega}$ is hereditarily separable then $C_k(X)$ is hereditarily Lindelöf.

Proof. Define $\Psi : K(X) \times C_k(X) \longrightarrow K(\mathbb{R})$ by $\Psi(A,f) = f(A)$ for each $(A,f) \in K(X) \times C_k(X)$. We will be through if we can show that Ψ satisfies the hypothesis of lemma 7.3.2.

1) Fix f in $C_k(X)$ and let $\Psi_f : K(X) \longrightarrow K(\mathbb{R})$ be defined by $\Psi_f(A) = f(A)$ for each $A \in K(X)$. We must show that Ψ_f is continuous. Let $W = \langle V_1, V_2, \ldots, V_n \rangle$ be a non-empty basic open set in $K(\mathbb{R})$. To see that $\Psi_f^{-1}(W)$ is open in $K(X)$, let $A \in \Psi_f^{-1}(W)$. Then $A \in \langle f^{-1}(V_1), \ldots, f^{-1}(V_n) \rangle \subseteq \Psi_f^{-1}(W)$ and $\langle f^{-1}(V_1), \ldots, f^{-1}(V_n) \rangle$ is open in $K(X)$. So, $\Psi_f^{-1}(W)$ is open in $K(X)$.

2) Fix A in $K(X)$ and define $\Psi_A : C_k(X) \longrightarrow K(\mathbb{R})$ by $\Psi_A(f) = f(A)$ for all $f \in C_k(X)$. To see that $C_k(X)$ has the weakest topology making Ψ continuous on $C_k(X)$, it suffices to show that Ψ_A is continuous for each $A \in K(X)$. So, let $W = \langle V_1, V_2, \ldots, V_n \rangle$ be a nonempty basic open set in $K(\mathbb{R})$. To see that $\Psi_A^{-1}(W)$ is open in $C_k(X)$, let $f \in \Psi_A^{-1}(W)$. Then $A \subseteq f^{-1}(V_1 \cup \ldots \cup V_n)$ and $f(A) \cap V_i \neq \emptyset$ for each $i=1,\ldots,n$. Let x_i

$\in A$ such that $x_i \in f^{-1}(V_i)$ for each $i=1,\ldots,n$. Put $L = [(x_1),V_1] \cap \ldots \cap [(x_n),V_n] \cap [A,V_1 \cup \ldots \cup V_n]$. Then L is open in $C_k(X)$ and $f \in L \subseteq \Psi_A^{-1}(W)$. This is true for each f in $C_k(X)$. Therefore $\Psi_A^{-1}(W)$ is open in $C_k(X)$. This means that Ψ_A is continuous.

The next result is nearly a converse to theorem 7.3.3.

7.3.4. Theorem.

1) If $C_k(X)$ is hereditarily separable then X and $K(X)$ are hereditarily Lindelöf.

2) If $C_k(X)$ is hereditarily Lindelöf then X and $K(X)$ are hereditarily separable.

Proof. That X is hereditarily Lindelöf (respectively, hereditarily separable) whenever $C_k(X)$ is hereditarily separable (respectively, hereditarily Lindelöf) follows from theorem 5 (respectively, theorem 5*) of [67].

(1): To see that $K(X)$ is hereditarily Lindelöf, let $\mathcal{V}$ be a countable base (made of basic open sets) for $K(\mathbb{R})$. Then the collection $\mathcal{B} = \{\langle f,V\rangle : f \in C_k(X) \text{ and } V \in \mathcal{V}\}$ where $\langle f,V\rangle = \{A \in K(X) : f(A) \in V\}$ is a base for the Vietoris topology on $K(X)$ (see [55] lemma 3). Now,

let $\mathcal{A}$ be a subset of $K(X)$ and let $\mathcal{U}$ be a cover of $\mathcal{A}$ by open sets of $K(X)$. We may assume that elements of $\mathcal{U}$ are of the form $Q(f,V)$. To see that $\mathcal{U}$ has a countable subcover for $\mathcal{A}$ it suffices to show that every subset $\mathcal{B}'$ of $\mathcal{B}$ contains a countable subset $\mathcal{B}''$ such that $U\mathcal{B}'' = U\mathcal{B}'$. To see this let $\mathcal{B}' = \{Q(f_\alpha,V_\alpha) : f_\alpha \in C_k(X), V_\alpha \in \mathcal{V}\} \subseteq \mathcal{B}$. For each $V \in \mathcal{V}$, put $F_V = \{f_\alpha : V_\alpha = V\}$ and denote F_V by $F_V = \{g_\gamma : \gamma \in \Gamma_V\}$. Since $C_k(X)$ is hereditarily separable, there is $\Gamma'_V \subseteq \Gamma_V$ such that Γ'_V is countable and $\{g_\gamma : \gamma \in \Gamma'_V\}$ is dense in F_V. Let $F_0 = \{G_\gamma : \gamma \in U\{\Gamma'_V : V \in \mathcal{V}\}\}$. Then F_0 is countable and dense in $U\{F_V : V \in \mathcal{V}\}$. By lemma 5 of [55], $U\mathcal{B}' = U\{Q(f_\alpha,V_\alpha) : V_\alpha \in \mathcal{V}\} = U\{Q(g_\gamma,V) : \gamma \in \Gamma'_V \text{ and } V \in \mathcal{V}\}$. So, let $\mathcal{B}''$ $= \{Q(g_\gamma,V) : \gamma \in \Gamma'_V, V \in \mathcal{V}\}$. Then $\mathcal{B}'' \subseteq \mathcal{B}'$ and $U\mathcal{B}' = U\mathcal{B}''$.

(2): To finish , assume $C_k(X)$ hereditarily Lindelöf and let $\mathcal{A}$ be a subset of $K(X)$. We must show that $\mathcal{A}$ is separable. For each $S \in \mathcal{A}$ and each $V = \langle V_1,\ldots,V_n\rangle$ in $\mathcal{V}$, the set $F(S,V) = \{f \in C_k(X) : f(S) \in V\}$ is open in $C_k(X)$. Let $W = U\{F(S,V) : S \in \mathcal{A}\}$. Then W is open in $C_k(X)$. By hypothesis, W is a Lindelöf subset of $C_k(X)$. Since the $F(S,V)$'s cover W then there is a countable set $\mathcal{A}_V \subseteq \mathcal{A}$ such that $W = U\{F(S,V) : S \in \mathcal{A}_V\}$. This is true for each $V \in \mathcal{V}$. Now, let $\mathcal{A}_0 = U\{\mathcal{A}_V : V \in \mathcal{V}\}$. Then $\mathcal{A}_0$ is dense in $\mathcal{A}$ (see lemma 4 of [55]) and countable. Therefore $\mathcal{A}$ is separable as desired.

7.3.5. Theorem. If $K(X)$ is hereditarily separable or

hereditarily Lindelöf then $C_k(X)$ has ccc.

Proof. Let $\langle\langle f_\alpha, A_\alpha, \varepsilon_\alpha\rangle : \alpha \in \Gamma\rangle$ be a pairwise disjoint family of non-empty basic open sets in $C_k(X)$, where $f_\alpha \in C_k(X)$, A_α is compact in X and $\varepsilon_\alpha > 0$ for each $\alpha \in \Gamma$. It is straightforward to show that $\mathcal{A} = \langle A_\alpha : \alpha \in \Gamma\rangle$ is a discrete subspace of $K(X)$ (see proof of theorem 3 of [55]). If $K(X)$ is hereditarily separable or hereditarily Lindelöf then Γ must be countable.

If X is the Butterfly space of [47], it is shown in [14] that $K(X)$ is neither hereditarily separable nor normal. Since X is cosmic, $C_k(X)$ has ccc. Therefore, the converse to theorem 7.3.5 is false. By theorem 7.3.4 $C_k(X)$ is neither hereditarily separable nor hereditarily Lindelöf. It then follows that theorem 7.3.1 does not have an analogue for the compact-open topology.

We must mention that the main part of theorem 7.3.4 is a special case of a more general result in [55], and that theorem 7.3.3 is a refinement of theorem 2.2 of [41].

We now investigate some special cases involving hereditary Lindelöf and hereditary separability.

7.3.6. Proposition. The following statements are equivalent:

(1) $C_k(X)$ is a hereditarily Lindelöf Σ-space

(2) $C_k(X)$ is a Lindelöf hereditarily strong Σ-space

(3) $C_k(X)$ is a Lindelöf σ-space

(4) X is an $\aleph_0$-space.

Proof. We only need to prove (1)$\Rightarrow$(4): suppose $C_k(X)$ hereditarily Lindelöf Σ-space. Then it is a perfect Σ-space. By theorem 3.1.1 (perfect spaces are E_0-spaces) $C_k(X)$ is a submetrizable Σ-space. Therefore it is a σ-space. Now, a Lindelöf σ-space is cosmic. The result follows from theorem 2.3.7.

7.3.7. Proposition. Let X be a Σ-space. Then $C_k(X)$ is a hereditarily separable Σ-space if and only if X is an $\aleph_0$-space.

Proof. We only need necessity. Suppose $C_k(X)$ hereditarily separable Σ-space. Then X is hereditarily Lindelöf by theorem 7.3.4 and so X is a Lindelöf Σ-space. Now, each separable subset of $C_k(X)$ must be subcosmic (This follows from corollary 5 of Arhangelskii [5]). But $C_k(X)$ is itself separable. Therefore $C_k(X)$ is a submetrizable Σ-space, hence a σ-space. Note that it is easy to see that a hereditarily separable

σ-space is cosmic. By theorem 2.3.7, X must be an $\aleph_0$-space.

By taking a non-metrizable compact space X, one sees that in proposition 7.3.7, hereditary separability can not be weakened to countable tightness.

In the next proposition the concept of semistratifiable space is used as defined in [21]. Every σ-space is semistratifiable and semistratifiable implies E_0-space (in fact perfect space).

7.3.8. Proposition. Let X be a q-space. Then the following are equivalent:

(1) $C_k(X)$ is hereditarily separable

(2) $C_k(X)$ is Lindelöf semistratifiable

(3) X is separable metrizable.

Proof. (3)⇒(2): If X is separable metrizable then $C_k(X)$ is an $\aleph_0$-space, hence Lindelöf and semistratifiable. (2)⇒(1): In semistratifiable spaces, Lindelöf is equivalent to hereditarily separable ([21]). (1)⇒(3): Suppose that $C_k(X)$ is hereditarily separable. Then X is Lindelöf (by theorem 7.3.4) and submetrizable (by theorem 2.1.3). A Lindelöf q-space is an M-space and a submetrizable M-space is metrizable

([15]). Finally, a Lindelöf metrizable space is separable.

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