FIBONACCI SEQUENCES

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I INTRODUCTION

Early in the thirteenth century, Leonardo de Pisa, or Fibonacci, wrote a notable work entitled _Liber Abacci_. Fibonacci had traveled extensively and gained knowledge regarding arithmetic systems used by merchants of different countries. He used this knowledge as a basis for the work, from which, for many centuries authors have gotten material for works on arithmetic and algebra. Pisa advocated and illustrated the Hindu Arabic system of numerals, and his _Liber Abacci_ did much to introduce this system into Europe. He discussed problems in arithmetic processes, barter, allegation, false position, cube roots, and many others.

A particular problem which appears may be stated simply as follows: Assume that rabbits reproduce at a rate such that one pair is born each month from each pair of adults not less than two months old. If one pair is present initially, and if none die, how many pairs will be present after one year? Fibonacci declared this number to be 377 by the following reasoning. Beginning with one pair, there will be two pairs at the end of the first month; at the end of the second month, the second pair will not yet have reproduced, so there will be 3 pairs; at the end of the third month, there will be 5 pairs and so on. The sequence of numbers he introduced is

$$1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377.$$

These numbers, with 0 and 1 prefixed, are the first 14 terms of the classical Fibonacci sequence. If the fourteenth number is not the last one, then the next number is $233 + 377 = 610$. In general, each number, or term, is the sum of the two which precede it. Denoting the nth term of this sequence by U_n, a "law of recurrence" may be written, i.e., U_n equals U_{n-1} plus U_{n-2}, the sum of the two terms which precede it. This can be written as an equation,

$$U_n = U_{n-1} + U_{n-2}, \quad n \geqslant 2,$$

and once U_0 and U_1 have been defined as 0 and 1 respectively, the sequence is determined. It is this sequence and the properties thereof which will be considered.

Two methods are available for studying the sequence. The first is to retain the law of recurrence and alter the definitions of U_0 and U_1; the second is to change the law of recurrence. The former method will be used throughout.

The recurrence relation and the principle of mathematical induction will be used to establish the majority of the properties. Sometimes one will be used and sometimes the other, and, in many cases, both.

II THE CLASSICAL FIBONACCI SEQUENCE

A. GENERAL PROPERTIES

The classical sequence is generated by preserving the recurrence relation

(1.1) $$U_n = U_{n-1} + U_{n-2}, \quad n \geqslant 2,$$

and defining $U_0 = 0$ and $U_1 = 1$. This gives the sequence

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ \cdots .$$

In order to find a particular term in the sequence when n is large, the n-1 terms preceding it would have to be written. However, a general formula for the nth term in the sequence can be developed. Rewriting (1.1) as

(1.1a) $$U_{n+2} - U_{n+1} - U_n = 0, \quad n \geqslant 2,$$

gives a linear, homogeneous, second order difference equation with initial conditions $U_0 = 0$, $U_1 = 1$. Assuming a solution of the form $U_n = \beta^n$ and substituting, the resulting equation is

$$\beta^n (\beta^2 - \beta - 1) = 0.$$

This equation has roots $\frac{1 + \sqrt{5}}{2}$ and $\frac{1 - \sqrt{5}}{2}$. Hence, the general solution is

$$U_n = C_1 \left(\frac{1 + \sqrt{5}}{2}\right)^n + C_2 \left(\frac{1 - \sqrt{5}}{2}\right)^n, \quad n \geqslant 2,$$

where C_1 and C_2 are constants. The constants are determined from the equations

$$U_0 = 0 = C_1 + C_2$$
$$U_1 = 1 = \left(\frac{1 + \sqrt{5}}{2}\right) C_1 + \left(\frac{1 - \sqrt{5}}{2}\right) C_2 .$$

The solution of these equations is $C_1 = -C_2 = \frac{1}{\sqrt{5}}$. Hence,

$$U_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right], \quad n = 0, 1, 2, \cdots .$$

Throughout all discussions, $\frac{1+\sqrt{5}}{2}$ will be represented by a, and $\frac{1-\sqrt{5}}{2}$ by b, so that

$$(1.2) \qquad U_n = \frac{1}{\sqrt{5}}(a^n - b^n), \quad n = 0, 1, 2, \cdots .$$

An immediate limiting property of the ratio of the nth term to the (n-j)th term in the sequence is

$$(1.3) \qquad \frac{U_n}{U_{n-j}} = \frac{a^n - b^n}{a^{n-j} - b^{n-j}} = \frac{a^j - \frac{b^n}{a^{n-j}}}{1 - \frac{b^{n-j}}{a^{n-j}}} \longrightarrow a^j \text{ as } n \longrightarrow \infty$$

since $a > 1$ and $-1 < b < 0$. The special case when $j = 1$ will be discussed later.

Some properties of the Fibonacci sequence follow. Some of the results will be proved, and the remaining ones can be proved by mathematical induction.

$$(1.4) \qquad U_{n-1}^2 + U_n^2 = U_{2n-1} .$$

To prove this relation, (1.2) is used along with the following identities.

$$\frac{1}{a} + a = \sqrt{5}, \quad \frac{1}{b} + b = -\sqrt{5}, \quad ab = -1 .$$

Now $U_{n-1}^2 + U_n^2 = \left[\frac{1}{\sqrt{5}} \; (a^{n-1} - b^{n-1})\right]^2 + \left[\frac{1}{\sqrt{5}} \; (a^n - b^n)\right]^2$

$$= \frac{1}{5}\left[a^{2n-2} - 2a^{n-1}b^{n-1} + b^{2n-2} + a^{2n} - 2a^n\,b^n + b^{2n}\right]$$

$$= \frac{1}{5}\left[a^{2n-1}\left(\frac{1}{a} + a\right) + b^{2n-1}\left(\frac{1}{b} + b\right) - 2a^{n-1}\,b^{n-1}\,(ab+1)\right]$$

$$= \frac{1}{\sqrt{5}}\left[a^{2n-1} - b^{2n-1}\right]$$

$$= U_{2n-1}$$

(1.5) $$U_{n+1}^3 + U_n^3 - U_{n-1}^3 = U_{3n}$$

The proof employs the identities

$$a^3 + 1 - \frac{1}{a^3} = 5, \quad b^3 + 1 - \frac{1}{b^3} = 5,$$

$$a^4\,b^2 + a^2\,b = 1, \quad a^2\,b^4 + ab^2 = 1,$$

and follows

$$U_{n+1}^3 + U_n^3 - U_{n-1}^3 = \left(\frac{a^{n+1} - b^{n+1}}{\sqrt{5}}\right)^3 + \left(\frac{a^n - b^n}{\sqrt{5}}\right)^3 - \left(\frac{a^{n-1} - b^{n-1}}{\sqrt{5}}\right)^3$$

$$= \frac{1}{5\sqrt{5}}\Big[a^{3n+3} - 3a^{2n+2}\,b^{n+1} + 3a^{n+1}\,b^{2n+2} - b^{3n+3} + a^{3n} - 3a^{2n}b^n$$

$$+ 3a^n\,b^{2n} - b^{3n} - a^{3n-3} + 3a^{2n-2}\,b^{n-1} - 3a^{n-1}\,b^{2n-2} + b^{3n-3}\Big]$$

$$= \frac{1}{5\sqrt{5}}\Big[a^{3n}\,(a^3 + 1 - \frac{1}{a^3}) - b^{3n}\,(b^3 + 1 - \frac{1}{b^3})$$

$$- 3a^{2n-2}\,b^{n-1}\,(a^4\,b^2 + a^2\,b - 1) + 3a^{n-1}\,b^{2n-2}\,(a^2\,b^4 + ab^2 - 1)\Big]$$

$$= \frac{1}{\sqrt{5}}\left[a^{3n} - b^{3n}\right]$$

$$= U_{3n} \quad .$$

A property which will be used frequently in subsequent proofs is

$$(1.6) \qquad U_n^2 - U_{n-t}\, U_{n+t} = (-1)^{n-t}\, U_t^2 \,, \quad n \geqslant t$$

Again, the left side is expanded according to (1.2) and the result simplified.

$$U_n^2 - U_{n-t}\, U_{n+t} = \left(\frac{a^n - b^n}{\sqrt{5}}\right)^2 - \left(\frac{a^{n-t} - b^{n-t}}{\sqrt{5}}\right)\left(\frac{a^{n+t} - b^{n+t}}{\sqrt{5}}\right)$$

$$= \frac{1}{5}\Big[a^{2n} - 2a^n b^n + b^{2n} - a^{2n} + a^{n+t} b^{n-t} + a^{n-t} b^{n+t} - b^{2n}\Big]$$

$$= \frac{1}{5}\Big[a^n b^n (a^t b^{-t} + a^{-t} b^t - 2)\Big]$$

$$= \frac{1}{5}\Big[(-1)^{n-t} (a^{2t} + b^{2t} - 2a^t b^t)\Big]$$

$$= \frac{1}{5}\Big[(-1)^{n-t} (a^t - b^t)^2\Big]$$

$$= (-1)^{n-t}\, U_t^2 \;.$$

In the special case when $t = 1$, $U_1 = 1$, and (1.6) reduces to

$$(1.6a) \qquad U_n^2 - U_{n-1}\, U_{n+1} = (-1)^{n-1} \,, \quad n \geqslant 1 \;.$$

An interesting property concerning the sum of n terms is

$$(1.7) \qquad U_1 + U_2 + \cdots + U_n = U_{n+2} - 1 \;.$$

This may be proved by induction as follows: If $n = 1$, (1.7) reduces to $U_1 = U_3 - 1$, which is seen to be correct. Assume (1.7) true for $n = k$ where k is fixed. Then

$$
\begin{aligned}
(U_1 + U_2 + \cdots + U_k) + U_{k+1} &= (U_{k+2} - 1) + U_{k+1} \\
&= U_{k+3} - 1 \\
&= U_{(k+1)+2} - 1,
\end{aligned}
$$

and the result is established for $n = k + 1$. Since (1.7) was true for 1, the above method shows (1.7) is true for $1 + 1 = 2$. Since it is true for $n = 2$, it must be true for $n = 3$, and so forth. This illustrates the principle of mathematical induction.

Induction can be used to establish

$$(1.8) \qquad U_1 - U_2 + U_3 - \cdots + (-1)^{n-1} U_n = (-1)^{n-1} U_{n-1} + 1, \; n \geqslant 1$$

For $n = 1$, $U_1 = U_0 + 1 = 1$. Assume (1.8) true for $n = k$. Then

$$
\begin{aligned}
(U_1 - U_2 + \cdots + (-1)^{k-1} U_k) + (-1)^k U_{k+1} &= \left[(-1)^{k-1} U_{k-1} + 1\right] \\
&\quad + (-1)^k U_{k+1} \\
&= (-1)^k (U_{k+1} - U_{k-1}) + 1 \\
&= (-1)^k U_k + 1,
\end{aligned}
$$

and so (1.8) is true for all n.

Listed below are some results all of which can be proved by the principle of mathematical induction.

$$(1.9a) \qquad U_1 + U_3 + U_5 + \cdots + U_{2n-1} = U_{2n}$$

$$(1.9b) \qquad U_2 + U_4 + U_6 + \cdots + U_{2n} = U_{2n+1} - 1$$

$$(1.9c) \qquad U_1 + U_4 + U_7 + \cdots + U_{3n-2} = \frac{1}{2} U_{3n}$$

(1.9d) $U_2 + U_5 + U_8 + \cdots + U_{3n-1} = \frac{1}{2}(U_{3n+1} - 1)$

(1.9e) $U_3 + U_6 + U_9 + \cdots + U_{3n} = \frac{1}{2}(U_{3n+2} - 1)$

(1.9f) $U_1 + U_5 + U_9 + \cdots + U_{4n-3} = U_{2n-1} U_{2n}$

(1.9g) $U_3 + U_7 + U_{11} + \cdots + U_{4n-1} = U_{2n} U_{2n+1}$

(1.9h) $U_2 + U_6 + U_{10} + \cdots + U_{4n-2} = U_{2n}^2$

(1.9i) $U_4 + U_8 + U_{12} + \cdots + U_{4n} = U_{2n} U_{2n+2}$

(1.9j) $U_1^2 + U_2^2 + U_3^2 + \cdots + U_n^2 = U_n U_{n+1}$

Each time a property of the sequence has been established, new properties can be derived. For example, (1.6a) with n replaced by n + 1 reads

(1.6b) $$U_n U_{n+2} - U_{n+1}^2 = (-1)^{n+1} .$$

Multiplying (1.6b) by U_{n+1} yields a new property,

(1.10) $$U_n U_{n+1} U_{n+2} = U_{n+1}^3 + (-1)^{n+1} U_{n+1} .$$

Also, properties such as

(1.11) $$U_{n+1}^3 = U_n^3 + U_{n-1}^3 + 3U_{n-1} U_n U_{n+1}$$

can be deduced by using the recursion formula (1.1) alone, for

$$U_{n+1}^3 = (U_n + U_{n-1})^3 = U_n^3 + 3U_n^2 U_{n-1} + 3U_n U_{n-1}^2 + U_{n-1}^3$$
$$= U_n^3 + U_{n-1}^3 + 3\, U_n U_{n-1} (U_n + U_{n-1})$$
$$= U_n^3 + U_{n-1}^3 + 3\, U_{n-1} U_n U_{n+1} \ .$$

Other properties of the classical Fibonacci sequence will be proved for more general cases and the obvious reductions noted. However, not all properties are simple algebraic equations. The theorems which follow will give a few of the more interesting ones. The first, appearing in [19], is

<u>Theorem 1.1</u> For any number x to be a term of the classical Fibonacci sequence, it is necessary and sufficient that

(1.12) either $5x^2 + 4$ or $5x^2 - 4$

is a perfect square.

The proof of the theorem will depend on

<u>Lemma 1.1</u> If x and y, (suppose $y \geqslant x$), are two consecutive terms of the classical Fibonacci sequence, it is necessary and sufficient that

(1.13) either $y^2 - xy - x^2 = +1$ or $y^2 - xy - x^2 = -1$.

To show that (1.13) is a necessary condition, let $y = U_{n+1}$ and let $x = U_n$. Employing (1.6a) with n replaced by n+1 yields

$$U_{n+1}^2 - U_n U_{n+2} = (-1)^n \ .$$

Substitution according to (1.1) gives

$$U_{n+1}^2 - U_n U_{n+1} - U_n^2 = y^2 - xy - x^2 = (-1)^n \ ,$$

which says that either $y^2 - xy - x^2 = +1$ or $y^2 - xy - x^2 = -1$ (depending on whether n is even or odd, respectively).

Next, suppose $y^2 - xy - x^2 = \pm 1$. Solve this equation for y and neglect the root which would make $x > y$. Then

$$y = \frac{x}{2} + \frac{1}{2}\sqrt{5x^2 \pm 4} \quad .$$

If $x = 0$, $y = 1$ and if $x = 1$, $y = 1$ or $y = 2$, so that these two terms of the sequence satisfy (1.13). Let $z = y - x$ and substitute into (1.13). Then

$$x^2 - xz - z^2 = \mp 1 \quad ,$$

so x and z satisfy one of the relations in (1.13). If $z > 1$, let $z_1 = x - z$, so z_1 and z satisfy one of the relations. If $z_1 > 1$, let $z_2 = z - z_1$, etc. In this manner, a sequence

$$\ldots,\ z_2,\ z_1,\ z,\ x,\ y,\ \ldots$$

is constructed. If at any stage $z_j = 1$, (1.13) is sufficient. If the sequence is carried far enough, the smallest number will be +1, for otherwise there would be an infinite number of positive integers less than x and greater than 0, which is false. Hence, (1.13) is a sufficient condition.

To prove the theorem, (1.12) is first shown to be necessary. Let y be the term following x. Then the lemma must hold, and therefore

$$y = \frac{1}{2}x + \frac{1}{2}\sqrt{5x^2 \pm 4} \quad .$$

But y being a term of the sequence means y is rational and consequently $5x^2 \pm 4$ is a perfect square.

On the other hand, knowing $5x^2 \pm 4$ is a perfect square, let

$$y = \frac{1}{2}x + \frac{1}{2}\sqrt{5x^2 \pm 4} \quad .$$

Regardless of whether x is an odd or even integer, y is an integer, so the conditions of the lemma are fulfilled. Thus, x is a term of the sequence, and (1.12) is sufficient.

Theorems 1.2 and 1.3 can be found in [20]. They are

Theorem 1.2 If k is an integer greater than 1, then a positive integer t, less than k and depending on k, can be found such that

$$U_{n_1+n_2+\dots+n_k-t} < U_{n_1} U_{n_2} \dots U_{n_k} < U_{n_1+n_2+\dots+n_k-t+1}$$

whenever each of the $n_1, n_2, \dots, n_k$ is $> n_0$, n_0 depending on k.

Define t to be the greatest integer in

$$\left(\frac{k-1}{2}\right)\frac{\log 5}{\log a} + 1 \;,$$

so that

$$\frac{1}{2}(k-1)\,\frac{\log 5}{\log a} < t < \frac{1}{2}(k-1)\,\frac{\log 5}{\log a} + 1 \;.$$

Employ (1.2) to write

$$\begin{aligned} U_{n_1} U_{n_2} \cdots U_{n_k} &= \left(\frac{1}{\sqrt{5}}\right)^k (a^{n_1} - b^{n_1})(a^{n_2} - b^{n_2}) \cdots (a^{n_k} - b^{n_k}) \\ &= \frac{1}{5^{k/2}}\, a^{n_1+n_2+\dots+n_k} \left[1 - \left(\frac{b}{a}\right)^{n_1}\right]\left[1 - \left(\frac{b}{a}\right)^{n_2}\right] \cdots \\ &\quad \left[1 - \left(\frac{b}{a}\right)^{n_k}\right] , \end{aligned}$$

and

$$U_{n_1+n_2+\ldots+n_k-t+1} = \frac{1}{\sqrt{5}} \left(a^{n_1+n_2+\ldots+n_k-t+1} - b^{n_1+n_2+\ldots+n_k-t+1}\right)$$

$$= \frac{1}{5^{\frac{1}{2}}} a^{n_1+n_2+\ldots+n_k-t+1} \left[1 - \left(\frac{b}{a}\right)^{n_1+n_2+\ldots+n_k-t+1}\right].$$

Therefore,

$$\lim_{n_1,n_2,\ldots,n_k\to\infty} \frac{U_{n_1} U_{n_2} \cdots U_{n_k}}{U_{n_1+n_2+\ldots+n_k-t+1}} = \frac{1}{5^{\frac{1}{2}(k-1)}} a^{t-1} ,$$

and $\frac{1}{5^{\frac{1}{2}(k-1)}} a^{t-1} < \frac{1}{5^{\frac{1}{2}(k-1)}} a^{\frac{1}{2}(k-1)\frac{\log 5}{\log a}} = \frac{5^{\frac{1}{2}(k-1)}}{5^{\frac{1}{2}(k-1)}} = 1$

Hence, $\frac{U_{n_1} U_{n_2} \cdots U_{n_k}}{U_{n_1+n_2+\ldots+n_k-t+1}} < 1$

if each of $n_1, n_2, \ldots , n_k$ is $> N_0$.

Similarly, $\lim_{n_1,n_2,\ldots n_k\to\infty} \frac{U_{n_1} U_{n_2} \cdots U_{n_k}}{U_{n_1+n_2+\ldots+n_k-t}} = \frac{1}{5^{\frac{1}{2}(k-1)}} a^{t} ,$

and $\frac{1}{5^{\frac{1}{2}(k-1)}} a^{t} > \frac{1}{5^{\frac{1}{2}(k-1)}} a^{\frac{1}{2}(k-1)\frac{\log 5}{\log a}} = 1$.

Therefore, $\frac{U_{n_1} U_{n_2} \cdots U_{n_k}}{U_{n_1+n_2+\ldots+n_k-t}} > 1$

if each of $n_1, n_2, \ldots , n_k$ is $> M_0$. Hence, if each of $n_1, n_2, \ldots , n_k$ is $> n_0 = \max (M_0,N_0)$,

$$U_{n_1+n_2+\ldots+n_k-t} < U_{n_1} U_{n_2} \cdots U_{n_k} < U_{n_1+n_2+\ldots+n_k-t+1} .$$

Corollary 1.2 Put $n_1 = n_2 = \ldots = n_k = n$ in Theorem 1.2. Then for $n > n_0$,

$$U_{kn-t} < U_n^k < U_{kn-t+1}$$

where t is defined in the theorem.

Theorem 1.3 There lie exactly mk Fibonacci numbers between U_n^k and U_{n+m}^k, n being greater than n_0.

By Corollary 1.2, for $n > n_0$

$$U_{kn-t} < U_n^k < U_{kn-t+1} \quad .$$

Replace n by n + 1 to get

$$U_{kn+k-t} < U_{n+1}^k < U_{kn+k-t+1} \quad .$$

Hence, between U_n^k and U_{n+1}^k lie the k Fibonacci numbers

$$U_{kn-t+1}, \quad U_{kn-t+2}, \cdots, \quad U_{kn-t+k-1}, \quad U_{kn-t+k} \quad .$$

Similarly, replace n by n + m to get

$$U_{kn+km-t} < U_{n+m}^k < U_{kn+km-t+1} \ ,$$

and between U_n^k and U_{n+m}^k lie the km Fibonacci numbers

$$U_{kn-t+1}, \quad U_{kn-t+2}, \cdots, \quad U_{kn-t+km-1}, \quad U_{kn-t+km} \quad .$$

The next two theorems will be stated without proof. The proofs, which employ concepts of number theory can be found in [2] and [12], respectively.

Theorem 1.4 Let the greatest common divisor (m,n) of m and n be 1, 2, or 5. Then U_{mn} is divisible by $U_m U_n$.

Theorem 1.5 If U_{mn} is divisible by $U_m U_n$, then (m,n) = 1,2, or 5 (the converse of Theorem 1.4).

B. WAYS OF EXPRESSING U_n

$$(1.14)\qquad U_n = \frac{1}{2^{n-1}}\left\{\binom{n}{1} + \binom{n}{3}5 + \binom{n}{5}5^2 + \dots + \binom{n}{2m-1}5^m + \dots\right\}.$$

From (1.2), $U_n = \dfrac{(a^n - b^n)}{\sqrt{5}}$, so

$$U_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

$$= \frac{1}{2^n\sqrt{5}}\left[\sum_{x=0}^{n}\binom{n}{x}(\sqrt{5})^x - \sum_{x=0}^{n}\binom{n}{x}(-\sqrt{5})^x\right]$$

$$= \frac{1}{2^n\sqrt{5}}\left[\sum_{x=0}^{n}\binom{n}{x}(\sqrt{5})^x\left\{1 - (-1)^x\right\}\right]$$

For x even, the terms in the summation vanish; for x odd, the term in braces is equal to 2. Hence,

$$U_n = \frac{1}{2^{n-1}\sqrt{5}}\left[\sum_{\substack{x=1\\ x\ \text{odd}}}^{n}\binom{n}{x}(\sqrt{5})^x\right],\quad \text{or,}$$

$$U_n = \frac{1}{2^{n-1}}\left\{\binom{n}{1} + \binom{n}{3}5 + \binom{n}{5}5^2 + \dots + \binom{n}{2m-1}5^m + \dots\right\}.$$

$$(1.15)\qquad U_{n+1} = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \dots\ .$$

The proof is by induction. The result is trivial for n = 0 and n = 1. Hence, assume (1.15) true for n = k - 1 and n = k. Then

$$U_k = \binom{k-1}{0} + \binom{k-2}{1} + \binom{k-3}{2} + \dots$$

$$U_{k+1} = \binom{k}{0} + \binom{k-1}{1} + \binom{k-2}{2} + \dots .$$

Add these to obtain

$$U_k + U_{k+1} = \binom{k}{0} + \left[\binom{k-1}{0} + \binom{k-1}{1}\right] + \left[\binom{k-2}{1} + \binom{k-2}{2}\right] + \dots .$$

Since $\binom{k}{0} = \binom{k+1}{0}$, $\binom{k-1}{0} + \binom{k-1}{1} = \binom{k}{1}$, ... , this gives

$$U_k + U_{k+1} = \binom{k+1}{0} + \binom{k}{1} + \binom{k-1}{2} + \dots$$

$$= U_{k+2} ,$$

and the induction is complete.

(1.16)

$$U_n = \begin{vmatrix} 1 & -1 & 1 & -1 & 1 & -1 & \dots \\ 1 & 1 & 0 & 1 & 0 & 1 & \dots \\ 0 & 1 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & 1 & 0 & 1 & \dots \\ . & . & . & . & . & . & . \end{vmatrix} , \quad n \geqslant 2,$$

where the determinant is of order (n-1).

The proof is by induction. For $n = 2$ and $n = 3$, the result is obvious. Now, assume that U_{k-1} and U_{k-2} can be expressed as determinants of order (k-2) and (k-3) respectively, each having the form (1.16). Consider the (k-1) by (k-1) determinant

$$D = \begin{vmatrix} 1 & -1 & 1 & -1 & 1 & -1 & \cdots \\ 1 & 1 & 0 & 1 & 0 & 1 & \cdots \\ 0 & 1 & 1 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & 1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} .$$

Expand D about the first column to get

$$D = \begin{vmatrix} 1 & 0 & 1 & 0 & 1 & \cdots \\ 1 & 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} - \begin{vmatrix} -1 & 1 & -1 & 1 & -1 & \cdots \\ 1 & 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} ,$$

where each of the determinants are of order $(k-2)$. Subtract the second row from the first row in the first determinant and incorporate the negative sign into the first row of the second determinant to get

$$D = \begin{vmatrix} 0 & -1 & 1 & -1 & 1 & \cdots \\ 1 & 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} + \begin{vmatrix} 1 & -1 & 1 & -1 & 1 & \cdots \\ 1 & 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} .$$

By the induction hypothesis, the second determinant is U_{k-1} . Expand the first determinant about the first column and again incorporate the negative sign into the first row. This gives

$$D = \begin{vmatrix} 1 & -1 & 1 & -1 & 1 & \cdots \\ 1 & 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 1 & \cdots \\ . & . & . & . & . & . \end{vmatrix} + U_{k-1},$$

where the determinant is of order (k-3). But by the induction assumption, this determinant is U_{k-2}, and so

$$D = U_{k-2} + U_{k-1} = U_k \quad ,$$

and the proof is complete.

(1.17)
$$U_n = \frac{\sinh n\alpha}{\cosh \alpha} \qquad \text{(n even)},$$

$$U_n = \frac{\cosh n\alpha}{\cosh \alpha} \qquad \text{(n odd)},$$

where $\sinh \alpha = \frac{1}{2}$.

For $\sinh \alpha = \frac{1}{2}$, $2 \cosh \alpha = \sqrt{5}$, so that by (1.2),

$$U_n = \frac{1}{2 \cosh \alpha} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right]$$

$$= \frac{1}{2 \cosh \alpha} \left[(\sinh \alpha + \cosh \alpha)^n - (\sinh \alpha - \cosh \alpha)^n \right]$$

$$= \frac{1}{\cosh \alpha} \left[\frac{e^{n\alpha} - (-1)^n e^{-n\alpha}}{2} \right]$$

Hence,

$$U_n = \frac{\sinh n\alpha}{\cosh \alpha} \qquad \text{(n even)},$$

$$U_n = \frac{\cosh n\alpha}{\cosh \alpha} \qquad \text{(n odd)} \quad .$$

C. THE GOLDEN RATIO

In (1.3) it was shown that the ratio of two terms of the classical Fibonacci sequence approached a limit as n increased. Here will be considered a special case, i.e., the limit of the ratio of two consecutive terms of the sequence. Put $j = 1$ in (1.3) to obtain

$$\frac{U_n}{U_{n-1}} \longrightarrow a \quad \text{as} \quad n \rightarrow \infty .$$

Throughout the discussion above, a has represented the number $\frac{(1 + \sqrt{5})}{2}$. But this irrational number, commonly called phi, is known as the golden ratio, and has the value

$$\varphi = a = 1.61803398 \cdots .$$

There are several interesting ways to express φ, two of which are given below. They appear in [6].

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}} \quad . \tag{1.18}$$

$$\varphi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}} \quad . \tag{1.19}$$

The ancient Greeks were familiar with the golden ratio; there is little doubt that it was consciously used by some Greek architects and sculptors, particularly in the structure of the Parthenon. The U. S. mathematician Mark Barr had this in mind when he gave the ratio the name of phi. It is the first Greek letter in the name of the great Phidias who is believed to have used the golden proportion frequently in his sculpture.

Many medieval Renaissance mathematicians such as Kepler became intrigued by phi, almost to the point of obsession. Coxeter [6] quotes Kepler as follows: "Geometry has two great treasures: one is the theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious jewel." Renaissance writers spoke of the ratio as a "divine proportion", or, following Euclid, as "extreme and mean ratio." The term "golden section" did not come into use until the 19th century.

There is a classic geometrical paradox that brings out strikingly how phi is related to the classical sequence: If a square of 64 square units is dissected as shown in figure 1, the four pieces can be put together again to make a rectangle of 65 square units (figure 2).

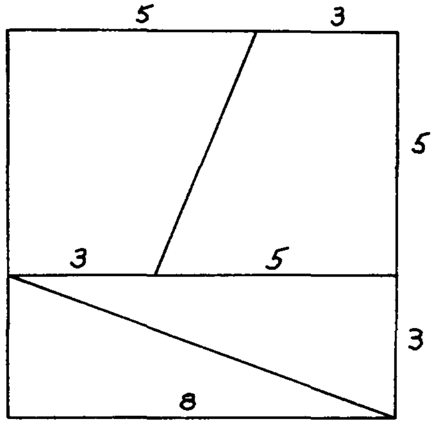


Figure 1

The paradox is explained by the fact that the pieces do not fit exactly along the long diagonal where there is a narrow space equal to 1 square unit (see figure 3). Note that the lengths of the line segments in these figures are terms of the classical sequence. The only way to cut the square so there is no gain of area in the rectangle is to cut it with segment lengths taken from the sequence,

$$1,\ \varphi,\ \varphi+1,\ 2\varphi+1,\ 3\varphi+2,\ 5\varphi+3,\ \ldots,$$

or, rewriting, the sequence

$$1,\ \varphi,\ \varphi^2,\ \varphi^3,\ \varphi^4,\ \varphi^5,\ \ldots\ .$$

This is an interesting sequence in which the ratio between any two consecutive terms is constant.

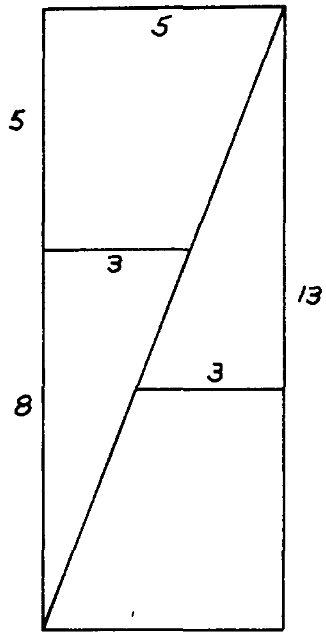


Figure 2

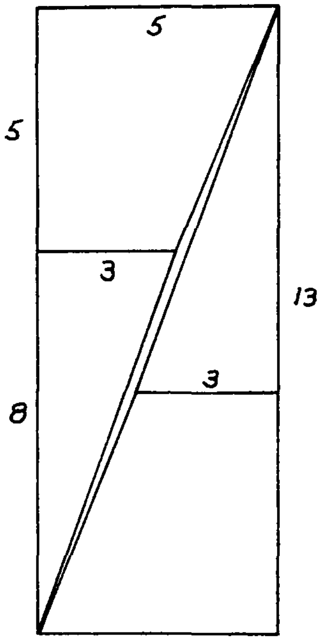


Figure 3

III A GENERALIZED FIBONACCI SEQUENCE

A generalized Fibonacci sequence can be constructed by retaining the law of recurrence but redefining the first two terms. The nth term of this new sequence will be denoted by F_n and the terms F_1 and F_2 will be defined by p' and p' + q' respectively, where p' and q' are arbitrary integers. Hence,

(2.1) $F_n = F_{n-1} + F_{n-2}$, $n \geqslant 3$, $F_1 = p'$, $F_2 = p' + q'$

generates the sequence

(2.1a) $p', p'+q', 2p'+q', 3p'+2q', 5p'+3q', 8p'+5q', \cdots$.

The special case when p' = 0, q' = 1 yields the classical sequence (1.1a). Notice that in this special case, $F_1 = U_0$, $F_2 = U_1, \ldots, F_n = U_{n-1}$, ... , or, the subscripts of the U's are 1 less than the subscripts of the F's. Therefore, all properties of the generalized sequence F_n, n = 1, 2, ... , will be true for U_n, n = 0, 1, 2, ... , as a special case.

In order to obtain a formula for the nth term of the generalized sequence, the difference equation

$$F_{n+2} - F_{n+1} - F_n = 0$$

is again solved, but with initial conditions $F_1 = p'$, $F_2 = p' + q'$. The general solution is

$$F_n = c_1\, a^n + c_2\, b^n ,$$

and the equations

$$p' = c_1 a + c_2 b$$
$$p' + q' = c_1 a^2 + c_2 b^2$$

determine c_1 and c_2 as $\frac{1}{\sqrt{5}}$ $(p' - q'b)$ and $-\frac{1}{\sqrt{5}}$ $(p' - q'a)$, respectively. Substitute to obtain

(2.2) $$F_n = \frac{1}{\sqrt{5}} (p'-q'b) a^n - \frac{1}{\sqrt{5}} (p'-q'a)b^n .$$

Now define

(2.2a) $$f' = 2(p'-q'b), \ g' = 2(p'-q'a)$$

so that (2.2) becomes

(2.3) $$F_n = \frac{1}{2\sqrt{5}} (f' a^n - g' b^n).$$

Let $\frac{1}{4} f'g' = p'^2 - p'q'-q'^2$ be denoted by e'. Then the properties of the classical sequence can be generalized. For example,

(1.4) $$U^2_{n-1} + U^2_n = U_{2n-1} , \quad n \geqslant 1$$

becomes

(2.4) $$F^2_{n-1} + F^2_n = (2p' - q') F_{2n-1} - e' U_{2n-1} , \ n \geqslant 2,$$

and

(1.6) $$U^2_n - U_{n-t} U_{n+t} = (-1)^{n-t} U^2_t , \quad n \geqslant t$$

becomes

(2.5) $$F^2_n - F_{n-t} F_{n+t} = (-1)^{n-t} e' F^2_t , \quad n \geqslant t + 1.$$

These and other properties will occur as special cases of the complex sequence. The purpose of introducing the sequence (2.1) is to deduce results due to Horadam [10] concerning Fibonacci number triples.

By requiring p' and q' to be integers, a "Pythagorean" Theorem involving the terms of the generalized sequence is developed in [9] and follows.

Theorem 2.1 $(2\,F_{n+1}\,F_{n+2})^2 + (F_n\,F_{n+3})^2 = (2\,F_{n+1}F_{n+2} + F_n^2)^2$.

The theorem is proved by working with the second term on the left and then adding the first term. Now

$$\begin{aligned}(F_n\,F_{n+3})^2 &= F_n^2\,F_{n+3}^2 = F_n^2\,(F_{n+1} + F_{n+2})^2\\ &= F_n^2\,F_{n+1}^2 + 2\,F_n^2\,F_{n+1}\,F_{n+2} + F_n^2\,F_{n+2}^2\\ &= F_n^2\,F_{n+1}^2 + 2\,F_n^2\,F_{n+1}\,(F_n + F_{n+1}) + F_n^2\,(F_n + F_{n+1})^2\\ &= F_n^2\,F_{n+1}^2 + 2F_n^3\,F_{n+1} + 2F_n^2\,F_{n+1}^2 + F_n^4 + 2F_n^3\,F_{n+1} + F_n^2\,F_{n+1}^2\\ &= 4\,F_n^2\,F_{n+1}^2 + 4\,F_n^3\,F_{n+1} + F_n^4\\ &= 4\,F_n^2\,F_{n+1}\,(F_{n+1} + F_n) + F_n^4\\ &= 4\,F_n^2\,F_{n+1}\,F_{n+2} + F_n^4 \quad .\end{aligned}$$

Hence,

$$(2\,F_{n+1}\,F_{n+2})^2 + (F_n\,F_{n+3})^2 = 4\,F^2_{n+1}\,F^2_{n+2} + 4\,F^2_n\,F_{n+1}\,F_{n+2} + F^4_n$$
$$= (2\,F_{n+1}\,F_{n+2} + F^2_n)^2 \,.$$

Using the theorem, any four consecutive terms of the sequence will generate three integers such that the sum of squares of two of them will equal the square of the third.

The problem to be considered is: given a Pythagorean triple, α, β, γ, (mutually prime integers with $\alpha^2 + \beta^2 = \gamma^2$), can an n, p' and q' be found such that the integers whose squares appear in Theorem 2.1 are these α, β and γ? The answer is yes, and so Pythagorean triples may be thought of as Fibonacci triples.

To establish the result, first notice that all Pythagorean triples are given by $x^2 - y^2$, $2xy$, $x^2 + y^2$, where $x > y$ and x and y are mutually prime but not simultaneously odd. This avoids repetitions and assures that $x + y$ is odd. Next is proved

<u>Theorem 2.2</u> All Pythagorean triples are Fibonacci triples.

Put $p' = x-y$ and $q' = 2y-x$ in (2.1a) to obtain the sequence

(2.6) $$x - y,\ y,\ x,\ x + y,\ \dots\ .$$

For $n = 1$, Theorem 2.1 becomes

$$(2\,F_2\,F_3)^2 + (F_1\,F_4)^2 = (2\,F_2\,F_3 + F_1^2)^2 \,,$$

and substitution yields

$$(x^2 - y^2)^2 + (2xy)^2 = (x^2 + y^2)^2 .$$

For example, $p', q' = 2,1$ gives the triple 3, 4, 5, $p', q' = 3,2$ gives the triple 5, 12, 13, $p', q' = 4,1$ gives the triple 15, 8, 17, and $p', q' = 4,3$ gives the triple 7, 24, 25, where $n = 1$ in each case.

An interesting observation can be made concerning Pythagorean triples and the generalized sequence. Let $F_{p',q'}$ denote the generalized sequence for particular values of p' and q'. For example, (2.6) is the sequence $F_{x-y,2y-x}$. From (2.1a) and the remarks following (2.1a), F_n can be written in the form

$$F_n = p' U_n + q' U_{n-1} \quad . \tag{2.7}$$

If $x = F_{n+2}$ and $y = F_{n+1}$ in Theorem 2.1, then

$$x^2 - y^2 = F^2_{n+2} - F^2_{n+1} = (F_{n+2} - F_{n+1}) (F_{n+2} + F_{n+1}) = F_n F_{n+3}$$

and

$$\begin{aligned} x^2 + y^2 = F^2_{n+2} + F^2_{n+1} &= F_{n+2} (F_n + F_{n+1}) + F_{n+1} (F_{n+2} - F_n) \\ &= F_{n+2} F_n + F_{n+2} F_{n+1} + F_{n+1} F_{n+2} - F_n F_{n+1} \\ &= 2 F_{n+1} F_{n+2} + F_n (F_{n+2} - F_{n+1}) \\ &= 2 F_{n+1} F_{n+2} + F^2_n \end{aligned}$$

From (2.7), $x = p' U_{n+2} + q' U_{n+1}$ and $y = p' U_{n+1} + q' U_n$. Solve these equations for p' and q' to get

$$p' = \frac{(x U_n - y U_{n+1})}{(U_n U_{n+2} - U^2_{n+1})}$$

$$q' = \frac{(y U_{n+2} - x U_{n+1})}{(U_n U_{n+2} - U^2_{n+1})} \quad .$$

The denominators are simply (1.6a) with n replaced by n + 1, and so they may be replaced by $(-1)^{n+1}$ to obtain

$$p' = (-1)^n (y U_{n+1} - x U_n)$$

$$q' = (-1)^n (x U_{n+1} - y U_{n+2})$$

When n = 1, the values of p' and q' are those in (2.6). If n is allowed to take on all positive integral values, an infinite sequence of sequences $F_{p',q'}$ is obtained, where

$p',q' = x-y,\ 2y-x;\ 2y-x,\ 2x-3y;\ 2x-3y,\ 5y-3x;\ 5y-3x;\ 5x-8y;\ \ldots$

corresponding to n = 1, 2, 3, 4, ..., respectively. A given Pythagorean triple may be derived from any of these sequences provided that the correct value of n is associated with it. For example, the triple 5, 12, 13, is obtained from the sequences $F_{1,1}$, $F_{1,0}$, $F_{0,1}$, $F_{1,-1}$, ... (i.e., x = 3, y = 2) when n = 1, 2, 3, 4, ... respectively.

IV A COMPLEX FIBONACCI SEQUENCE

Suppose the recurrence relation is retained and the first two terms are defined as complex numbers. This willproduce a complex sequence defined by

$$(3.1) \qquad H_n = H_{n-1} + H_{n-2},\ n \geqslant 3, \qquad H_1 = p + iq,\ H_2 = r + is,$$

where H_n denotes the nth term in the sequence, p, q, r, and s are arbitrary integers, and $i = (0,1)$. The sequence generated is then

$$(3.1a) \qquad (p+iq),\ (r+is),\ (p+r) + i(q+s),\ (p+2r) + i(q+2s),\ (2p+3r) + i(2q+3s),\ \ldots\ .$$

By employing the previous methods with initial conditions given by (3.1), the nth term of the complex sequence is found to be

$$(3.2) \qquad H_n = \frac{1}{\sqrt{5}}\left[(p+pb-rb) + i(q+qb-sb)\right]\ a^n - \frac{1}{\sqrt{5}}\left[(p+pa-ra) + i(q+qa-sa)\right]\ b^n\ .$$

Define

$$(3.2a) \qquad f = 2\left[(p+pb-rb) + i(q+qb-sb)\right], \text{ and} \qquad g = 2\left[(p+pa-ra) + i(q+qa-sa)\right],$$

so that (3.2) becomes

$$(3.3) \qquad H_n = \frac{1}{2\sqrt{5}}\ (f\ a^n - g\ b^n),$$

and $\frac{1}{4}\, fg = (p^2+pr-q^2-qs+s^2-r^2) + i\,(2pq+ps+qr-2rs)$, which will be represented by e.

Notice that when $r = s = 0$ and $p = p'$, $q = p' + q'$, then $f = f'$, $g = g'$ and $e = e'$, so H_n becomes F_n as a special case. Therefore, properties developed for the complex sequence are valid for both the generalized and classical sequences.

The following properties are readily established.

(3.4) $\dfrac{H_n}{H_{n-j}} \longrightarrow a^j$ as $n \to \infty$, since

$$\frac{H_n}{H_{n-j}} = \frac{fa^n - gb^n}{fa^{n-j} - gb^{n-j}} = \frac{a^j - \dfrac{gb^n}{fa^{n-j}}}{1 - \dfrac{gb^{n-j}}{fa^{n-j}}} \quad \text{, and } a > 1 \text{ ,}$$

$-1 < b < 0$, so $\frac{b}{a} \to 0$ as $n \to \infty$.

(3.5) $\dfrac{H_n}{U_n} \longrightarrow \dfrac{f}{2}$ as $n \to \infty$ since $\dfrac{H_n}{U_n} = \dfrac{\frac{1}{2\sqrt{5}}\,(fa^n - gb^n)}{\frac{1}{\sqrt{5}}\,(a^n - b^n)}$,

and $\frac{f}{2} = (1 + b)\, H_1 - b\, H_2$ by (3.2a).

(3.6) $$H_{n+2} - 2\,H_n - H_{n-1} = 0, \qquad n \geqslant 2 \text{ .}$$

From (3.1), $H_{n+2} - 2\,H_n - H_{n-1} = H_{n+1} + H_n - 2H_n - H_{n-1}$

$$= H_n + H_{n-1} + H_n - 2\,H_n - H_{n-1} = 0 \text{ .}$$

(3.6a) $$H_{n+1} - 2H_n - H_{n-2} = 0, \quad n \geqslant 3 .$$

Consider $H_{n+2} - H_{n-1} = H_{n+1} + H_n - (H_n - H_{n-2}) = H_{n+1} - H_{n-2}$.

Then $H_{n+2} - 2H_n - H_{n-1} = H_{n+1} - 2H_n - H_{n-2} = 0$ by (3.6).

(3.7a) $$\sum_{j=0}^{n-1} H_{2j+1} = H_{2n} + H_1 - H_2 .$$

By induction, for n = 1, $H_1 = H_1$, so assume the result for n = k. Then for n = k + 1,

$$\begin{aligned}(H_1 + \cdots + H_{2k-1}) + H_{2k+1} &= (H_{2k} + H_1 - H_2) + H_{2k+1} \\ &= H_{2k+2} + H_1 - H_2 \\ &= H_{2(k+1)} + H_1 - H_2 ,\end{aligned}$$

and (3.7a) is true for all n.

(3.7b) $$\sum_{j=1}^{n} H_{2j} = H_{2n+1} - H_1 .$$

By induction, for n = 1, $H_2 = H_3 - H_1$ which is (3.1) with n = 3. Hence, assume the result for n = k.

Then for n = k + 1,

$$\begin{aligned}(H_2 + H_4 + \cdots + H_{2k}) + H_{2k+2} &= H_{2k+1} - H_1 + H_{2k+2} \\ &= H_{2(k+1)+1} - H_1 .\end{aligned}$$

Hence, the result is true for all n.

(Note that (3.7a) and (3.7b) are generalizations of (1.9a) and (1.9b)).

Since $\sum_{j=1}^{n} (H_{2j-1} - H_{2j}) = \sum_{j=1}^{n} H_{2j-1} - \sum_{j=1}^{n} H_{2j}$,

employing (3.7a) and (3.7b) gives $(H_{2n} + H_1 - H_2) - (H_{2n+1} - H_1)$

$$= - H_{2n-1} + 2H_1 - H_2 .$$

Thus,

$$\sum_{j=1}^{n} (H_{2j-1} - H_{2j}) = - H_{2n-1} + 2H_1 - H_2 , \tag{3.7c}$$

has been proved.

Let $S_n = \sum_{j=1}^{n} H_j$ and $T_n = \sum_{j=1}^{n} S_j$. Then

$$S_n = H_{n+2} - H_2 , \quad \text{and} \tag{3.8a}$$

$$T_n = H_{n+4} - (n+2) H_2 - H_1 . \tag{3.8b}$$

(3.8a) is established by induction as follows:

For n = 1, $H_1 = H_3 - H_2$ which is true, so assume the result for n = k; for n = k + 1,

$$(H_1 + H_2 + \dots + H_k) + H_{k+1} = H_{k+2} - H_2 + H_{k+1}$$

$$= H_{(k+1)+2} - H_2 .$$

(3.8b) requires use of (3.8a) for proof.

Now

$$T_n = \sum_{j=1}^{n} S_j = \sum_{j=1}^{n} (H_{j+2} - H_2) = \sum_{j=1}^{n} H_{j+2} - n H_2,$$

and since

$$\sum_{j=1}^{n} H_{j+2} = \sum_{j=3}^{n+2} H_j = (H_{n+4} - H_2) - H_1 - H_2 ,$$

$$T_n = (H_{n+4} - H_2) - H_1 - H_2 - n H_2$$

$$= H_{n+4} - 2 H_2 - n H_2 - H_1$$

$$= H_{n+4} - (n + 2) H_2 - H_1 \quad .$$

Rewriting (3.1a) in the form

$(p+iq)$, $(r+is)$, $(p+iq) + (r+is)$, $(p+iq) + 2(r+is)$, $2(p+iq) +3 (r+is)$, $\cdots$,

it is clear that since the sequence is additive, the coefficeints of $(p+iq)$, from the second term onward, and the coefficients of $(r+is)$, from the first term onward, both form a classical Fibonacci sequence. Therefore, H_n can be expressed in terms of U_n and U_{n+1} as

(3.9a) $$H_{n+1} = (H_2 - H_1)U_n + H_1 U_{n+1} ,$$ and

(3.9b) $$H_{n+2} = H_1 U_n + H_2 U_{n+1} \quad .$$

Replace n by t in (3.9a) and (3.9b) to get

$$H_{t+3} = H_{t+2} + H_{t+1} = H_2\, U_t + H_3\, U_{t+1} \ ,$$

$$H_{t+4} = H_{t+3} + H_{t+2} = H_3\, U_t + H_4\, U_{t+1}, \ \cdots \ .$$

And in general,

$$(3.9c) \qquad H_{n+t} = H_{n-1}\, U_t + H_n\, U_{t+1} \quad , \ n \geqslant 2 \ .$$

In the ensuing results, the following identities have been employed:

$$a + \frac{1}{a^2} = 2 \quad , \qquad b + \frac{1}{b^2} = 2$$

$$a + \frac{1}{a} = \sqrt{5} \quad , \qquad b + \frac{1}{b} = -\sqrt{5} \ .$$

<u>Theorem 3.1</u> $H^2_{n-1} + H^2_n = (3\, H_1 - H_2)\, H_{2n-1} - e\, U_{2n-1}$.

By induction, for $n = 2$, $H^2_1 + H^2_2 = (3H_1 - H_2)\, H_3 - e\, U_3$. To see that this is correct, replace H_3 by $H_2 + H_1$ and $e\, U_3$ by $2\,(H^2_1 + H_1\, H_2 - H^2_2)$. Then

$$\begin{aligned}(3\, H_1 - H_2)\, H_3 - e\, U_3 &= (3\, H_1 - H_2)(H_1 + H_2) - 2(H_1^2 + H_1\, H_2 - H_2^2) \\ &= 3\, H_1^2 + 2\, H_1\, H_2 - H_2^2 - 2H_1^2 - 2H_1\, H_2 + 2H_2^2 \\ &= H^2_1 + H^2_2 \ .\end{aligned}$$

Hence, the theorem is true for $n = 2$, Next, assume the result for $n = k - 1$ and $n = k$. Then

$$H_{k-2}^2 + H_{k-1}^2 = (3\,H_1 - H_2)\,H_{2k-3} - e\,U_{2k-3}$$

$$H_{k-1}^2 + H_k^2 = (3\,H_1 - H_2)\,H_{2k-1} - e\,U_{2k-1}\ .$$

Since $H_{2k+1} = H_{2k} + H_{2k-1} = 3\,H_{2k-1} - H_{2k-3}$, it follows that

$$3\,(H_{k-1}^2 + H_k^2) - (H_{k-2}^2 + H_{k-1}^2) = (3\,H_1 - H_2)\,H_{2k+1} - e\,U_{2k+1}$$

Simplify the left side of this equation to get

$$\begin{aligned}
3\,H_{k-1}^2 + 3\,H_k^2 - H_{k-2}^2 - H_{k-1}^2 &= 2\,H_{k-1}^2 + 3\,H_k^2 - H_{k-2}^2 \\
&= 2\,H_{k-1}^2 + 3\,H_k^2 - (H_k - H_{k-1})^2 \\
&= 2\,H_{k-1}^2 + 3\,H_k^2 - H_k^2 + 2\,H_k\,H_{k-1} - H_{k-1}^2 \\
&= H_{k-1}^2 + 2\,H_k^2 + 2\,H_{k-1}\,H_k \\
&= (H_{k+1} - H_k)^2 + 2\,H_k^2 + 2\,(H_{k+1} - H_k)f_k \\
&= H_{k+1}^2 - 2\,H_{k+1}\,H_k + H_k^2 + 2\,H_k^2 + 2H_{k+1}\,H_k - 2H_k^2 \\
&= H_{k+1}^2 + H_k^2\ .
\end{aligned}$$

Therefore, $H_{k+1}^2 + H_k^2 = (3\,H_1 - H_2)\,H_{2k+1} - e\,U_{2k+1}$, and the induction is complete.

Notice that Theorem 3.1 is a generalization of (2.4), and hence, a generalization of (1.4).

Theorem 3.2 $$H_{n+1}^2 - H_{n-1}^2 = (3\ H_1 - H_2)\ H_{2n-1} - e\ U_{2n}\ .$$

The theorem is proved by using Theorem 3.1 and replacing H_{2n-1} by $H_{2n+1} - H_{2n}$ to get

$$\begin{aligned} H_{n-1}^2 + H_n^2 &= (3\ H_1 - H_2)(H_{2n+1} - H_{2n}) - e\ (U_{2n+1} - U_{2n}) \\ &= H_{n+1}^2 + H_n^2 - \left[(3\ H_1 - H_2)\ H_{2n-1} - e\ U_{2n}\right] \end{aligned}$$

Hence, $H_{n+1}^2 - H_{n-1}^2 = (3\ H_1 - H_2)\ H_{2n-1} - e\ U_{2n}$.

Theorem 3.3 $$H_{n-1}\ H_{n+1} - H_n^2 = (-1)^n\ e\ .$$

By induction, for $n = 2$,

$$\begin{aligned} H_1\ H_3 - H_2^2 &= (p+iq)\left[(p + iq) + (r + is)\right] - (r + is)^2 \\ &= (p + iq)^2 + (p + iq)\ (r + is) - (r + is)^2 \\ &= (-1)^2\ e\ . \end{aligned}$$

Assume the theorem for $n = k - 1$ and $n = k$. Now

$$\begin{aligned} H_k\ H_{k+2} - H_{k+1}^2 &= (H_{k-2} + H_{k-1})\ (H_k + H_{k+1}) - (H_{k-1} + H_k)^2 \\ &= H_{k-2}H_k + H_{k-1}H_k + H_{k-2}H_{k+1} + H_{k-1}\ H_{k+1} - H_{k-1}^2 \\ &\quad - 2\ H_{k-1}\ H_k - H_k^2 \\ &= (H_{k-2}\ H_k - H_{k-1}^2) + (H_{k-1}\ H_{k+1} - H_k^2) + (H_{k-2}\ H_{k+1} \\ &\quad - H_{k-1}\ H_k). \end{aligned}$$

But $H_{k-2}\, H_{k+1} - H_{k-1}\, H_k = H_{k-2}\,(H_{k-1}+ H_k) - H_{k-1}\,(H_{k-2}+ H_{k-1})$

$$= H_{k-2}\, H_{k-1} + H_{k-2}\, H_k - H_{k-1}\, H_{k-2} - H^2_{k-1}$$

$$= H_{k-2}\, H_k - H^2_{k-1} \quad .$$

Hence, $H_k\, H_{k+2} - H^2_{k+1} = 2\,(H_{k-2}\, H_k - H^2_{k-1}) + (H_{k-1}\, H_{k+1} - H^2_k)$.

By the induction assumption, this reduces to

$$H_k\, H_{k+2} - H^2_{k+1} = 2(-1)^{k-1}\, e + (-1)^k\, e$$

$$= \left[2(-1)^{k-1} + (-1)^k\right] e$$

$$= \left[2(-1)^{k+1} - (-1)^{k+1}\right] e$$

$$= (-1)^{k+1}\, e\ ,$$

and the proof is complete.

__Theorem 3.4__ $H_n\, H_{n+t+1} - H_{n-v}\, H_{n+t+v+1} = (-1)^{n-v}\, e\, U_v\, U_{t+v+1}$.

The proof employs (3.3).

$H_n\, H_{n+t+1} - H_{n-v}\, H_{n+t+v+1}$

$$= \left(\frac{fa^n - gb^n}{2\sqrt{5}}\right)\left(\frac{fa^{n+t+1} - gb^{n+t+1}}{2\sqrt{5}}\right) - \left(\frac{fa^{n-v} - gb^{n-v}}{2\sqrt{5}}\right)\left(\frac{fa^{n+t+v+1} - gb^{n+t+v+1}}{2\sqrt{5}}\right)$$

$$= \frac{1}{20}\Big\{ f^2\, a^{2n+t+1} - fga^{n+t+1}b^n - fga^n b^{n+t+1} + g^2\, b^{2n+t+1}$$

$$- f^2\, a^{2n+t+1} + fgb^{n-v}\, a^{n+t+v+1} + fga^{n-v} b^{n+t+v+1} - g^2 b^{2n+t+1} \Big\}$$

$$= \frac{1}{20}\left\{ fga^n\, b^n \left[a^{t+1}\,(-1 + b^{-v}\, a^v) + b^{t+1}\,(-1 + a^{-v}\, b^v)\right]\right\}$$

$$= \frac{1}{5}\left\{ (-1)^n \quad e \left[(a^v - b^v)(a^{t+1}\, b^{-v} - b^{t+1}\, a^{-v})\right]\right\}.$$

But $a^{t+1}\, b^{-v} - b^{t+1}\, a^{-v} = \dfrac{(a^{t+v+1} - b^{t+v+1})}{(ab)^v}$, so

$$H_n H_{n+t+1} - H_{n-v} H_{n+t+v+1} = (-1)^{n-v} \left(\frac{a^v - b^v}{\sqrt{5}}\right)\left(\frac{a^{t+v+1} - b^{t+v+1}}{\sqrt{5}}\right)$$

$$= (-1)^{n-v}\, U_v\, U_{t+v+1}$$

Theorem 3.5 $\quad H_n^3 + H_{n+1}^3 = 2\, H_n\, H_{n+1}^2 + (-1)^n\, e\, H_{n-1}$.

Apply Theorem 3.3 to the term involving $(-1)^n$ e to obtain

$$2H_n\, H_{n+1}^2 + (-1)^n\, e\, H_{n-1} = 2\, H_n\, H_{n+1}^2 + (H_{n-1}\, H_{n+1} - H_n^2)\, H_{n-1}$$

$$= 2H_n\, H_{n+1}\, (H_n + H_{n-1}) + (H_{n-1} H_{n+2} - H_n^2)(H_{n+1} - H_n)$$

$$= 2H_n^2\, H_{n+1} + 2H_{n-1} H_n H_{n+1} + H_{n-1}\, H^2_{n+1} - H_n^2 H_{n+1}$$

$$- H_{n-1}\, H_n\, H_{n+1} + H_n^3$$

$$= H_n^2\, H_{n+1} + H_{n-1}\, H_n\, H_{n+1} + H_{n-1}\, H_{n+1}^2 + H_n^3$$

$$= H_n\, H_{n+1}\, (H_n + H_{n-1}) + H_{n-1}\, H_{n+1}^2 + H_n^3$$

$$= H_n\, H_{n+1}^2 + H_{n-1}\, H_{n+1}^2 + H_n^3$$

$$= H_{n+1}^2\, (H_n + H_{n-1}) + H_n^3$$

$$= H_n^3 + H_{n+1}^3$$

Theorem 3.6 $\quad H_{n+1-t}\, H_{n+1+t} - H_{n+1}^2 = (-1)^{n-t}\, e\, U_t^2$

Use of (3.3) gives

$$H_{n+1-t}\, H_{n+1+t} - H_{n+1}^2 = \left(\frac{fa^{n+1-t} - gb^{n+1-t}}{2\sqrt{5}}\right)\left(\frac{fa^{n+1+t} - gb^{n+1+t}}{2\sqrt{5}}\right) - \left(\frac{fa^{n+1} - gb^{n+1}}{2\sqrt{5}}\right)^2$$

$$= \frac{1}{20}\Big[f^2 a^{2n+2} - fga^{n+1+t} b^{n+1-t} - fga^{n+1-t} b^{n+1+t} + g^2 b^{2n+2} - f^2 a^{2n+2} + 2\, fga^{n+1} b^{n+1} - g^2 b^{2n+2} \Big]$$

$$= \frac{1}{20}\Big[fg\, a^{n+1} b^{n+1} (-a^t b^{-t} - a^{-t} b^t + 2)\Big]$$

$$= \frac{1}{5}\Big[(-1)^{n+1} e\, (a^{-t} b^{-t})(-a^{2t} + 2a^t b^t - b^{2t})\Big]$$

$$= (-1)^{n+1-t} e\, (-1) \left(\frac{a^t - b^t}{\sqrt{5}}\right)^2$$

$$= (-1)^{n-t} e\, U_t^2 \quad .$$

Observe that theorem 3.3 is a special case of Theorem 3.6 when t = 1 and n is replaced by n - 1. Put n = t in Theorem 3.6 to derive

$$H_1\, H_{2n+1} - H_{n+1}^2 = e\, U_n^2 \text{ , or,}$$

(3.10)

$$H_{n+1}^2 + e\, U_n^2 = H_1\, H_{2n+1}$$

Finally, a proof is given for

<u>Theorem 3.7</u> $\dfrac{H_{n+t} + (-1)^t H_{n-t}}{H_n} = U_{t+1} + U_{t-1} = a^t + b^t$.

First is proved $\dfrac{H_{n+t} + (-1)^t H_{n-t}}{H_n} = a^t + b^t$

Apply (3.3) to get

$$\frac{H_{n+t} + (-1)^t H_{n-t}}{H_n} = \frac{(fa^{n+t} - gb^{n+t}) + (-1)^t (fa^{n-t} - gb^{n-t})}{(fa^n - gb^n)}$$

$$= \frac{fa^n (a^t + (-1)^t a^{-t}) - gb^n (b^t + (-1)^t b^{-t})}{fa^n - gb^n}$$

$$= \frac{fa^n (a^t + (ab)^t a^{-t}) - gb^n (b^t + (ab)^t b^{-t})}{fa^n - gb^n}$$

$$= \frac{fa^n (a^t + b^t) - gb^n (a^t + b^t)}{fa^n - gb^n}$$

$$= a^t + b^t .$$

Next, $U_{t+1} + U_{t-1} = a^t + b^t$, for

$$U_{t+1} + U_{t-1} = \left(\frac{a^{t+1} - b^{t+1}}{\sqrt{5}} \right) + \left(\frac{a^{t-1} - b^{t-1}}{\sqrt{5}} \right)$$

$$= \frac{1}{\sqrt{5}} \left[a^t (a + \frac{1}{a}) - b^t (b + \frac{1}{b}) \right]$$

$$= \frac{1}{\sqrt{5}} \left[\sqrt{5}\, a^t + \sqrt{5}\, b^t \right]$$

$$= a^t + b^t .$$

Hence, the theorem is true, showing the remarkable fact that the left-hand side is independent of the choice of p, q, r, or s.

V THE FIBONACCI SEQUENCE MODULO m

In order to make this discussion of the Fibonacci sequence more complete, results due to Wall [21] with additions by Mamangakis [14] are presented below. The problem is to determine the length of the period of the recurring sequence obtained by reducing a Fibonacci sequence by a modulus m. Some of the results will be proved and others will be stated without proof. The proofs can be found in the references given above.

The generalized sequence (2.1) is altered by taking $p' = d$, $p' + q' = c + d$, so that a new term, $F_0 = c$, may be prefixed without destroying the recurrence relation. Then (2.1) becomes

$$(4.1) \qquad F_n = F_{n-1} + F_{n-2}\,, \quad n \geqslant 2, \quad F_0 = c,\ F_1 = d,$$

and the sequence (2.1a) becomes

$$(4.1a) \qquad c,\ d,\ c + d,\ c + 2d,\ 2c + 3d,\ 3c + 5d,\ \ldots\ .$$

Reduce F_n modulo m, taking the least non-negative residues. Let h denote the length of the period of the repeating sequence that results. The letter p is reserved to designate a prime, but c, d, and m may be arbitrary integers, except that it is assumed, without loss of generality, that $(c, d, m) = 1$. Reference will be made to the classical sequence (1.1), and some of the properties developed above will be used. Let $k = k(m)$ denote the length of the period of U_n (mod m) in distinction from h, which depends on c and d as well as m.

For example, the values of U_n (mod 7) are

0, 1, 1, 2, 3, 5, 1, 6, 0, 6, 6, 5, 4, 2, 6, 1,

and then repeat, so k(7) = 16. Note that $U_8 \equiv 0$ (mod 7), so that the 16 terms in the period form two sets of 8 terms each, the terms of the second half being 6, or -1, times the corresponding terms of the first half. Theorem 4.7 will generalize this property and explain the relation k(7) = 16.

Theorem 4.1 F_n (mod m) forms a simply periodic sequence. That is, the sequence is periodic and repeats by returning to its starting values.

Proof The sequence repeats because there are only a finite number, m^2, of pairs of terms possible, and the recurrence of a pair results in recurrence of all following terms. From (4.1), $F_{n-1} = F_{n+1} - F_n$, so if $F_{t+1} \equiv F_{s+1}$ and $F_t \equiv F_s$ (mod m), then $F_{t-1} \equiv F_{s-1}, \dots ,$ $F_{t-s+1} \equiv F_1$, and $F_{t-s} \equiv F_0$, so the sequence is simply periodic.

A direct consequence of Theorem 4.1 is

Corollary 4.1 $U_k \equiv 0$ (mod m).

Theorem 4.2 If m has the prime factorization $m = \prod p_i^{e_i}$, and if h_i denotes the length of the period of F_n (mod $p_i^{e_i}$), then h = L.C.M. (h_i), the lowest common multiple of the h_i .

Proof The statement "h_i denotes the length of the period of F_n (mod $p_i^{e_i}$)" implies that the sequence F_n (mod $p_i^{e_i}$) repeats only after blocks of length αh_i. Also, the statement "h is the length of the period of F_n (mod m)" implies that F_n (mod $p_i^{e_i}$) repeats only after h terms for all values of i. Therefore, h is of the form αh_i for all values of i, and since any such number gives a period of F_n (mod m), then h = L.C.M. (h_i).

In view of Theorem 4.2, it is henceforth assumed that m is of the form $m = p^e$.

Theorems 4.3 through 4.7 establish properties of the classical Fibonacci sequence. The length of the period of this sequence modulo m is k. The relationship between the sequence U_n and the sequence F_n is contained in (2.7) which now reads

$$F_n = d\,U_n + c\,U_{n-1}\,, \tag{4.2}$$

so that F_n repeats after k terms. Hence, h is a divisor of k: $h \mid k$.

Theorem 4.3 The terms for which $U_n \equiv 0$ (mod m) have subscripts that form a simple arithmetic progression. That is, $n = xt$ for $x = 0, 1, 2, \ldots$, and some positive integer $t = t(m)$ gives all n with $U_n \equiv 0$ (mod m).

Proof First notice that any two consecutive terms of the classical sequence are relatively prime. This follows from the fact that

$$\frac{U_{n+1}}{U_n} = 1 + \frac{U_{n-1}}{U_n} ,$$

and $\frac{U_{n-1}}{U_n} < 1$ for $n \geqslant 3$. Next recall that as a special case of the complex sequence $(q = s = 0,\ p = 0,\ r = 1)$, $H_1 = U_0, \ldots, H_n = U_{n-1}$, and $H_{n+1} = U_n$. The obvious substitutions in (3.9c) give the relation

$$U_{n+s-1} = U_{n-2}\, U_s + U_{n-1}\, U_{s+1} ,$$

and, replacing n by n + 2 and s by s - 1, this becomes

$$U_{n+s} = U_n\, U_{s-1} + U_{n+1}\, U_s \quad . \tag{4.3}$$

Employing (4.3) along with $(U_n, U_{n+1}) = 1$, it is seen that $U_i \equiv 0 \pmod{m}$ and $U_j \equiv 0 \pmod{m}$ imply that $U_{i+j} \equiv 0 \pmod{m}$, and, with $i \geqslant j$, that $U_{i-j} \equiv 0 \pmod{m}$. The first congruence follows by setting $n = i$, $s = j$ in (4.3). Setting $n + t = i$, $n = j$ gives $U_{n+1}\, U_s \equiv 0 \pmod{m}$, and (U_n, U_{n+1}) along with $U_n \equiv 0 \pmod{m}$ gives the second congruence $U_s \equiv U_{i-j} \equiv 0 \pmod{m}$. Therefore, the subscripts n are of the form $n = xt$. Corollary 4.1 shows that U_0 is not the only $U_n \equiv 0 \pmod{m}$, so $t > 0$ and the proof is complete.

The next two theorems are stated without proof.

Theorem 4.4 If $m > 2$, k is an even number.

Theorem 4.5 If $k(p^2) \neq k(p)$, then $k(p^e) = p^{e-1}\, k(p)$. Also, if s is the largest integer for which $k(p^s) = k(p)$, then $k(p^e) = p^{e-s}\, k(p)$ for $e > s$.

In his paper, Wall [21] poses a question: "The most perplexing problem we have met in this study concerns the hypothesis $k(p^2) \neq k(p)$. We have run a test on a digital computer which shows that $k(p^2) \neq k(p)$ for all p up to 10,000; however, we cannot yet prove that $k(p^2) = k(p)$ is impossible." The following two theorems, due to Mamangakis [14] furnish a proof of the hypothesis $k(p^2) \neq k(p)$ under certain mild conditions. The proofs are omitted here.

<u>Theorem 4.5a</u> If p and q are relatively prime and pq occurs in U_n, then $k(p^2) \neq k(p)$.

<u>Theorem 4.5b</u> Let p and q be relatively prime, $e \leq s$, and $U_j = qp^s$ be the first multiple of p to occur in U_n. Then $k(p^e) = k(p)$ if and only if U_{j-1} has the same order mod p and mod p^e.

Theorems 4.6 and 4.7 furnish upper bounds for the function k(p).

<u>Theorem 4.6</u> If $m = p = 10x \pm 1$, then $k(p) \mid (p - 1)$.

<u>Theorem 4.7</u> If $m = p = 10x \pm 3$, then $k(p) \mid (2p + 2)$.

<u>Corollary 4.7</u> If $p = 10x \pm 3$, then $k(p) \equiv 0 \pmod 4$.

Next, the general sequence F_n is investigated. One of the most interesting properties of h, the length of the period of F_n (mod m), is that it is often independent of the choice of c and d. Theorems 4.8 through 4.12 describe this property.

Theorem 4.8 If $p = 10x \pm 3$, then $h(p^e) = k(p^e)$.

Proof The congruences which indicate that $F_n \pmod{m}$ repeats with period h may be written in the form

$$F_h - c = d\,U_h + c\,(U_{h-1} - 1) \equiv 0 \pmod{m},$$

and,

$$F_{h+1} - d = (d + c)\,U_h + d(U_{h-1} - 1) \equiv 0 \pmod{m}.$$

Considering c and d as given coefficients, the determinant of this system is $D = d^2 - cd - c^2$. With $m = p^e$, if $D \equiv 0 \pmod{p}$, then $4c^2 + 4cd + d^2 = (2c + d)^2 \equiv 5d^2 \pmod{p}$ since $c^2 + cd \equiv d^2 \pmod{p}$. Now $d \not\equiv 0 \pmod{p}$ simultaneously with $D \equiv 0 \pmod{p}$ since this would mean $c \equiv 0 \pmod{p}$, contradicting $(c,d,m) = 1$. Therefore, $D \equiv 0 \pmod{p}$ implies that 5 is a quadratic residue of p. But 5 is not a quadratic residue of primes $p = 10x \pm 3$, so for these p and $m = p^e$, $(D, m) = 1$, and the only solution to the system of congruences is $U_h \equiv 0$, $U_{h-1} \equiv 1 \pmod{p^e}$, which shows $k \mid h$. By the remark following (4.2), $h \mid k$. Therefore, $h = k$.

Corollary 4.8 Whenever $D = d^2 - cd - c^2$ satisfied $(D,m) = 1$, then $h = k$.

Finally, four theorems, stated without proof, will complete the discussion.

Theorem 4.9 If $m = 2^e$, then $h = k$, and if $m = 5^e$, then either $h = k$ or else $h = (1/5)\,k$, according as $D = d^2 - cd - c^2$ is not or is divisible by 5.

<u>Corollary 4.9</u> If $m = p^e$ and h is odd, then $p = 10x \pm 1$ or $m = 2$.

<u>Theorem 4.10</u> If $m = p^e$, $p > 2$, and if the pair (c,d) gives $h = 2t + 1$, then $k = 4t + 2$.

<u>Theorem 4.11</u> If $m = p^e$, $p > 2$, and if $k = 4t + 2$, then $h = 2t + 1$ for some (c,d). (The converse of Theorem 4.10).

<u>Theorem 4.12</u> If $m = p^e$, $p > 2$, $p \neq 5$, and h is even, then $h = k$.

VI APPLICATIONS

A. COMMUNICATIONS

Imagine a signaling system [7] that has only two signals, S_1 and S_2 (e.g., the dots and dashes in telegraphy). Messages are transmitted over some channel by first coding them into sequences of these two signals.

Suppose S_1 requires exactly t_1 units of time and S_2 exactly t_2 units of time to be transmitted. Let N_t denote the number of possible message sequences of duration t.

Consider first those messages of duration t which end with an S_1. Since S_1 takes t_1 units of time, this last signal must start at time $t - t_1$. Up to time $t - t_1$, there are N_{t-t_1} possible messages to which the last S_1 may be appended. Hence, the total number of messages of duration t which end with an S_1 is just N_{t-t_1}. Similarly, the total number of messages of duration t which end with an S_2 is given by N_{t-t_2}. But a message of duration t must end in either an S_1 or an S_2. Hence,

$$N_t = N_{t-t_1} + N_{t-t_2} \quad .$$

As an example, take $t_1 = 1$, and $t_2 = 2$, i.e., where one signal takes twice as long to be transmitted over the channel as the other. Then for this special case,

$$N_t = N_{t-1} + N_{t-2} \text{ , or,}$$

$$N_{t+2} - N_{t+1} - N_t = 0, \quad t = 0, 1, 2, \ldots \ .$$

By prescribing initial conditions $N_0 = 0$ and $N_1 = 1$, the solution is

$$N_t = \frac{1}{\sqrt{5}} (a^t - b^t), \ t = 0, 1, 2, \ldots,$$

which is the t th term of the classical Fibonacci sequence.

The capacity C of the channel is defined by Shannon [17] as follows:

$$(5.1) \qquad C = \lim_{t \to \infty} \frac{\log_2 N_t}{t},$$

where $\log_2$ denotes the logarithm to the base 2. (The base 2, rather than the more common bases 10 or $e = 2.718 \ldots$, is used in information theory. This is due to the following definition: if a single selection is to be made from a number of equally probable alternatives, and if information is transmitted which reduces the number of alternatives by a factor of 2, this amount of information is 1 <u>bit</u>, the unit of information. For example, 1 bit is transmitted in reducing the number of alternatives from 32 to 16, from 16 to 8, from 8 to 4, etc. Since $\log_2 2^n = n$, then $\log_2 32 = 5$, $\log_2 16 = 4$, $\log_2 8 = 3$, etc. The logarithm to the base 2 varies in steps equal to the amount of information transmitted.)

To calculate the capacity C, first note that the sequence N_t has the same limiting behavior as

$$\left\{ \frac{1}{\sqrt{5}} a^t \right\}.$$

This follows from (1.3). Hence, from (5.1),

$$C = \lim_{t \to \infty} \frac{\log_2 \frac{1}{\sqrt{5}} a^t}{t} .$$

But $\log_2 \frac{1}{\sqrt{5}} a^t = \log_2 \frac{1}{\sqrt{5}} + t \log a$, so that

$$C = \lim_{t \to \infty} \frac{\log_2 \frac{1}{\sqrt{5}}}{t} + \log_2 a$$

$$= \log_2 a.$$

C is found to be 0.7 approximately.

B. LEAF ARRANGEMENTS

The most frequently occurring type of leaf arrangement on erect or horizontal stems is the alternate or spiral, in which there is one leaf at each node [4]. Spiral leaf arrangements fall into certain definite patterns, the simplest being that in which the leaves are in two rows, on opposite sides of the stem. In this type, such as corn, the position of each leaf forms an angle of 180° with the position of the next nearest leaf on the stem. Spiral leaf arrangements are often expressed by fractions, or fractional divergences. In corn, the insertion of each leaf is ½ of the circumference of the stem from the insertion of the next leaf above or below. In hazel and beech, the leaves are in three rows on the stem, and adjacent leaves are separated from each other by 1/3 of a circumference, or 120°. Actually, tracing the spiral in the opposite direction, adjacent leaves are also separated by 2/3 of a circumference, but the smaller fraction is arbitrarily used. In oak and cherry, the leaves are in five rows. However, adjacent leaves are separated by 2/5 of a circumference, or by 3/5, tracing the spiral the long way, but not by 1/5. In rose and raspberry, the fraction is 3/8. Other common spiral leaf arrangements are 5/13 and 8/21. However, variations due to twisting and bending of stems sometimes occur.

It was noted in the 19th century that spiral leaf arrangements commonly fall into a sequence expressed by the fractions outlined above, namely

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{5}, \frac{3}{8}, \frac{5}{13}, \frac{8}{21}, \ldots \quad .$$

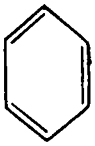
Both numerators and denominators form part of the classical sequence (1.1). Why the common leaf arrangements in plants should fall into this sequence has never been adequately determined.

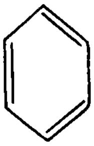
A partial explanation is that overlapping and shading of leaves is prevented in the spirals given above. If, for instance, oak had the leaves in a 1/5 spiral, with adjacent leaves separated from each other by 1/5 of a circumference, they would tend to overlap and to shade each other, cutting down on photosynthesis. The same would be even more true in a 1/8 or 1/13 spiral. Plants with relatively simple spirals, such as 1/2 or 1/3, often have narrow leaves and elongated internodes, thus preventing shading.

C. RESONANCE THEORY

Occasionally, in structural chemistry, no reasonable electronic picture can be drawn for a molecule which satisfactorily accounts for its observed properties. Sienko [18] defines resonance as "a situation in which no single electronic formula conforms both to observed properties and to the octet rule". The octet rule is a supposition that, when atoms combine, the bonds formed are such that each atom is surrounded by a complete octet of electrons.

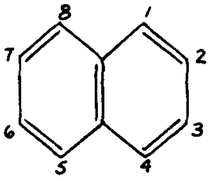
One of the most important applications of resonance theory has been the interpretation of the structure of aromatic compounds. Benzene is to be regarded as a resonating structure contributed to mainly by the two equivalent Kekulé forms

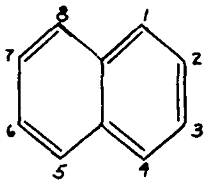


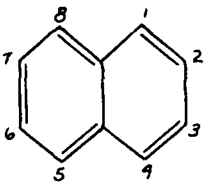


with only minor contributions from other structures. The result is a structure of great stability, in which all six carbon-carbon bonds are identical, being something between single and double bonds.

Aromatic molecules containing two or more fused rings can be represented by a greater number of resonating structures which contribute to the final condition of the molecule. The principal contributors are the possible Kekulé forms; thus for napthalene there are three, as follows, the first being probably the most important:





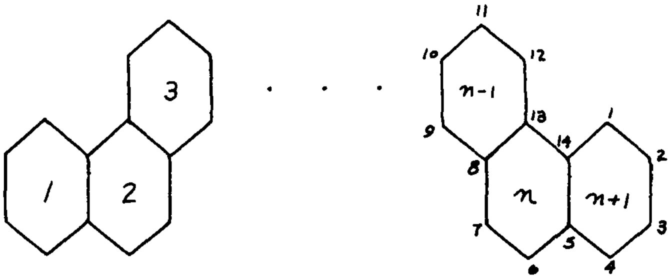


The outcome is that bonds between α and β carbons (1:2, 3:4, etc.) have more double bond character than those between β carbons (2:3, 6:7).

A fundamental property of compounds having resonating molecules is that the energy content is lower than that calculated from the laws of thermochemistry for any of the contributing structures. This energy deficiency (greater heat of formation) is called the resonance energy by Pauling [15]. For benzene, the resonance energy is about 37 kcal. per mol.

The higher condensed ring systems can be similarly represented as resonating among many valence-bond structures. The resonance energy increases in rough proportion to the number of hexagonal rings in the system. In addition, it is somewhat greater for the branched and angular ring systems than for the corresponding linear ones, the former resonating among more stable valence-bond structures than the latter [15].

The problem considered here is to calculate the number of these stable valence-bond structures for a given number of hexagonal rings in a system, no three rings being either collinear or mutually fused together. In order to do this, let N_n denote the number of structures possible with n hexagons, and consider the following structure:



where the (n + 1) st hexagon has been added. Notice that in N_n, either the bond 14:5 or the bonds 5:6 and 13:14 are formed. Hence, in N_{n+1}, first form the bonds 1:2 and 3:4. Then clearly there are N_n stable structures possible. However, to compute N_{n+1}, it is also possible to form the bonds 14:1, 2:3 and 4:5. This requires the bond 6:7 (in the nth hexagon), which in turn requires either the bond 8:13 or the bonds 8:9 and 13:12. But this requirement exhausts the number of possible structures in N_{n-1}. Also, the two cases above represent the only two ways in which bonds can be formed in the (n + 1)st hexagon. Evidently, this shows

$$N_{n+1} = N_n + N_{n-1}, \quad n \geq 2, \quad N_1 = 2, \quad N_2 = 3,$$

the initial values following from the preliminary remarks. Hence, $N_n = U_{n+2}$, the (n + 2)nd term in the classical Fibonacci sequence.

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ABSTRACT

FIBONACCI SEQUENCES

BY

CARL ALLAN PERSINGER

Early in the thirteenth century, Leonardo de Pisa, or, Fibonacci, introduced his famous rabbit problem, which may be stated simply as follows: assume that rabbits reproduce at a rate such that one pair is born each month from each pair of adults not less than two months old. If one pair is present initially, and if none die, how many pairs will be present after one year? The solution to the problem gives rise to a sequence $\{U_n\}$ known as the Classical Fibonacci Sequence. U_n is defined by the recurrence relation

$$U_n = U_{n-1} + U_{n-2}\ , \quad n \geq 2 \quad , U_0 = 0,\ U_1 = 1.$$

Many properties of this sequence have been derived.

A generalized sequence $\{F_n\}$ can be obtained by retaining the law of recurrence and redefining the first two terms as $F_1 = p'$, $F_2 = p' + q'$ for arbitrary real numbers p' and q'. Moreover, by defining $H_1 = p+iq$, $H_2 = r+is$, p,q,r and s real, a complex sequence is determined. Hence, all the properties of the classical sequence can be extended to the complex case.

By reducing the classical sequence by a modulus m, many properties of the repeating sequence that results can be derived.

The Fibonacci sequence and associated golden ratio occur in communication theory, chemistry, and in nature.