POISSON-LIE STRUCTURES ON INFINITE-DIMENSIONAL JET GROUPS AND THEIR QUANTIZATION

by

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(ABSTRACT)

We study the problem of classifying all Poisson-Lie structures on the group G_∞ of local diffeomorphisms of the real line $\mathbb{R}^1$ which leave the origin fixed, as well as the extended group of diffeomorphisms $G_{0\infty} \supset G_\infty$ whose action on $\mathbb{R}^1$ does not necessarily fix the origin.

A complete classification of all Poisson-Lie structures on the group G_∞ is given. All Poisson-Lie structures of coboundary type on the group $G_{0\infty}$ are classified. This includes a classification of all Lie-bialgebra structures on the Lie algebra $\mathcal{G}_\infty$ of G_∞, which we prove to be all of coboundary type, and a classification of all Lie-bialgebra strucutures of coboundary type on the Lie algebra $\mathcal{G}_{0\infty}$ of $G_{0\infty}$ which is the Witt algebra.

A large class of Poisson structures on the space V_λ of λ-densities on the real line is found such that V_λ becomes a homogeneous Poisson space under the action of the Poisson-Lie group G_∞.

We construct a series of finite-dimensional quantum groups whose quasiclassical limits are finite-dimensional Poisson-Lie factor groups of G_∞ and $G_{0\infty}$.

To my parents

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INTRODUCTION

Quantum groups have been introduced by Drinfel'd [D2] as deformations of universal enveloping algebras of Lie groups and of the algebra of functions on Poisson-Lie groups. The latter are Lie groups equipped with Poisson structures compatible with the group structure (from where the term Poisson-Lie group originates). In this approach of constructing quantum groups the first step is to analyze existence of Poisson-Lie structures on the corresponding Lie group. The question of classifying all Poisson-Lie structures on a given Lie group (if they exist) is very difficult. To give an idea of how difficult it is we present a list of groups for which the solution of the classification problem is known:
(a) Finite dimensional complex simple Lie groups [BD],
(b) The groups $GL(N)$, $SL(2)$, $GL(1|1)$ [Ku3,4,5],
(c) The 3-dimensional Heisenberg group [Ku6],
(d) The group of affine transformations of the line $Aff(1)$.
Note that all the groups mentioned above are *finite-dimensional.*

In the work presented here we study the problem of classifying all the Poisson-Lie structures on the group G_∞ of local diffeomorphisms of the real line $\mathbb{R}^1$ which leave the origin fixed, as well as the extended group of diffeomorphisms $G_{0\infty} \supset G_\infty$ whose action on $\mathbb{R}^1$ does not necessarily fix the origin.

The original problem was to decide whether such structures exist on the above described group G_∞. Surprisingly, there exist many Poisson-Lie structures on the groups G_∞ and $G_{0\infty}$. Moreover a complete classification of all Poisson-Lie structures on the group G_∞ is possible. For the group $G_{0\infty}$ the classification remains incomplete; a list of all Poisson-Lie structures of coboundary type on the group $G_{0\infty}$ is given.

The Lie algebras of the groups $G_{0\infty}$ and G_∞ are the Witt algebra $\mathcal{G}_{0\infty}$ and its principal subalgebra $\mathcal{G}_\infty \subset \mathcal{G}_{0\infty}$. We prove that there is a one-to-one correspondence between the Poisson-Lie structures on G_∞ and the Lie-bialgebra structures on $\mathcal{G}_\infty$.

The latter are shown to be *all* of coboundary type. All Lie-bialgebra structures of coboundary type on $\mathcal{G}_{0\infty}$ are classified, and there is a one-to-one correspondence between them and an explicitly listed family of the Poisson-Lie structures on $G_{0\infty}$. Thus a complete classification of all Lie-bialgebra structures on the principal subalgebra $\mathcal{G}_\infty$ of the Witt algebra $\mathcal{G}_{0\infty}$ is given and all Lie-bialgebra structures of coboundary type on the Witt algebra $\mathcal{G}_{0\infty}$ are classified.

The group G_∞ acts naturally on the space V_λ of λ-densities on $\mathbb{R}^1$. For each Poisson-Lie structure on the group G_∞ we determine a Poisson structure on V_λ such that V_λ becomes a homogeneous Poisson G_∞-space under the action of the Poisson-Lie group G_∞.

Finally, the quantization problem is addressed. We construct a series of finite-dimensional quantum groups whose quasi-classical limits are finite-dimensional Poisson-Lie factor groups of G_∞ and $G_{0\infty}$. The Poisson-Lie structures on these finite-dimensional groups are restrictions of the Poisson-Lie structures on G_∞.

We give now a brief guide to the organization of the text.

In Ch.I we introduce the basic concepts related to the Poisson-Lie theory and formulate the fundamental theorem of Drinfel'd relating Poisson-Lie groups and Lie-bialgebras.

In Ch.II we introduce the infinite-dimensional group G_∞ and a smooth structure on it.

In Ch.III we find all bialgebra structures on the Lie algebra $\mathcal{G}_\infty$ of G_∞.

In Ch.IV we find a class of Poisson-Lie structures on G_∞.

In Ch.V we show that there is a one-to-one correspondence between the Lie-bialgebra structures on $\mathcal{G}_\infty$ and the Poisson-Lie structures found on G_∞, and prove that the latter give a complete list of all Poisson-Lie structures on G_∞.

In Ch.VI we study Poisson-Lie structures on $G_{0\infty}$ which correspond to all Lie-bialgebra structures on $\mathcal{G}_{0\infty}$ of coboundary type.

Chapter VII is devoted to elements of representation theory for the Poisson-Lie group G_∞ on the homogeneous spaces V_λ.

In Ch.VIII we describe a series of quantum finite-dimensional groups.

Lastly, we discuss some open problems.

The bibliography does not pretend to give a complete list of works related to the subject. This is partly because the amount of literature devoted to Quantum and Poisson-Lie groups is so large that a complete list of references would have far exceeded the volume of this work, and partly because the subject is so quickly growing that a comprehensive bibliography is not a well-defined object. Therefore we give only a list of the works that served as guidelines to the author in trying to learn the subject and to begin being able to appreciate its beauty.

CHAPTER I

POISSON-LIE THEORY

In this chapter we introduce the basic objects to be studied: Poisson manifolds, Poisson-Lie groups, and Lie bialgebras, and prove basic results about them.

Let $\mathcal{M}$ be a finite-dimensional smooth manifold. A Poisson structure (bracket) on $\mathcal{M}$ is defined as a map $\{\ ,\ \}\colon C^\infty(\mathcal{M})\times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$, which makes $C^\infty(\mathcal{M})$ into a Lie algebra, and is a derivation with respect to each argument. That is, there exists a section $\omega \in \wedge^2 T_{\mathcal{M}}$, where $T_{\mathcal{M}}$ is the tangent bundle of $\mathcal{M}$, such that for any $f, g, h \in C^\infty(\mathcal{M})$ we have

$$(f,g) \mapsto \{f,g\} = \Big\langle \omega, df \wedge dg \Big\rangle,$$

and

(i) $\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0$ (Jacobi identity);

(ii) $\{f,gh\} = \{f,g\}h + \{f,h\}g$ (derivation property);

(iii) $\{f,g\} = -\{g,f\}$ (antisymmetry).

In local coordinates,

$$\{f,g\}(x) = \omega_{ij}(x)\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j},$$

where $\omega_x = \omega_{ij}(x)\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \in \wedge^2 T_x$ is a bi-vector field at the point $x \in \mathcal{M}$, and $\{\frac{\partial}{\partial x_i}\}$ is a basis of the tangent space T_x at $x \in \mathcal{M}$ in the local coordinates (x_i).

Here and throughout this text a summation is understood over repeated nonfixed indices unless stated otherwise.

The Jacobi identity (i) is equivalent to the following system of equations for the components $\omega_{ij}(x) = -\omega_{ji}(x)$:

$$\omega_{ij}\frac{\partial \omega_{kl}}{\partial x_i} + \omega_{ik}\frac{\partial \omega_{lj}}{\partial x_i} + \omega_{il}\frac{\partial \omega_{jk}}{\partial x_i} = 0. \tag{1.1}$$

Definition 1.1 (Poisson Manifold)[L]. A Poisson manifold is a smooth manifold with a Poisson structure.

A smooth map $F\colon \mathcal{M}_1 \to \mathcal{M}_2$, of two Poisson manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$, is said to be *Poisson* if

$$F^*(\{g,h\}_{\mathcal{M}_2}) = \{F^*(g), F^*(h)\}_{\mathcal{M}_1} \quad \text{for any } g,h \in C^\infty(\mathcal{M}_2),$$

where $(F^*(g))(x) \stackrel{def}{=} g(F(x))$, and $\{\ ,\ \}_{\mathcal{M}_1}$, $\{\ ,\ \}_{\mathcal{M}_2}$ are the Poisson brackets on $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively. Thus the above condition is equivalent to $\{g,h\}_{\mathcal{M}_2} \circ F = \{g \circ F, h \circ F\}_{\mathcal{M}_1}$.

If $\mathcal{M}_1$ and $\mathcal{M}_2$ are two Poisson manifolds with Poisson structures defined by $\omega_1 \in \wedge^2 T_{\mathcal{M}_1}$ and $\omega_2 \in \wedge^2 T_{\mathcal{M}_2}$, respectively, we define the direct product Poisson structure on $\mathcal{M}_1 \times \mathcal{M}_2$ as

$$\omega_1 \times \omega_2 \stackrel{def}{=} 1 \times \omega_2 + \omega_1 \times 1, \tag{1.2}$$

which is a map : $\wedge^2 T_{\mathcal{M}_1} \oplus \wedge^2 T_{\mathcal{M}_2} \hookrightarrow \wedge^2 T_{\mathcal{M}_1 \times \mathcal{M}_2}$. Here the space $C^\infty(\mathcal{M}_1 \times \mathcal{M}_2)$ is identified with the space $C^\infty(\mathcal{M}_1) \otimes C^\infty(\mathcal{M}_2)$(the reason being that a Poisson structure on $C^\infty(\mathcal{M}_1 \times \mathcal{M}_2)$ is uniquely defined by the one on $C^\infty(\mathcal{M}_1) \otimes C^\infty(\mathcal{M}_2)$). In more detail, for any function $f \in C^\infty(\mathcal{M}_1 \times \mathcal{M}_2)$, and for each $x \in \mathcal{M}_1$ and $y \in \mathcal{M}_2$ let us define the functions f^x on $\mathcal{M}_2$ and f^y on $\mathcal{M}_1$ as follows:

$$f^x(y) = f(x,y) \qquad \text{and} \qquad f^y(x) = f(x,y).$$

Then (1.2) means

$$\{f_1, f_2\}_{\mathcal{M}_1 \times \mathcal{M}_2}(x,y) = \{f_1^x, f_2^x\}_{\mathcal{M}_2}(y) + \{f_1^y, f_2^y\}_{\mathcal{M}_1}(x). \tag{1.3}$$

Definition 1.2 (Poisson-Lie group)[D1]. Let G be a Lie group. Let ω be a Poisson structure on G. The pair (G,ω) is said to be a Poisson-Lie group if the multiplication map $G \times G \to G$ is Poisson, where the manifold $G \times G$ is equipped with the direct product Poisson structure $\omega \times \omega$.

Let $L_x\colon G \to G$ and $R_x\colon G \to G$ be the left and right actions of G on itself defined by $y \mapsto xy$ and $y \mapsto yx$ respectively, where $x, y \in G$. Then for any two functions

$f_1, f_2 \in C^\infty(G)$ the compatibility between the product Poisson structure on $G \times G$ introduced by (1.2) and the Poisson structure on G can be witten as

$$\begin{aligned} \{f_1, f_2\}_G(xy) &= \{f_1, f_2\}_{G\times G}(xy) \\ &= \{f_1^x, f_2^x\}_G(y) + \{f_1^y, f_2^y\}_G(x) \\ &= \{f_1 \circ L_x, f_2 \circ L_x\}_G(y) + \{f_1 \circ R_y, f_2 \circ R_y\}_G(x), \end{aligned} \tag{1.4}$$

where a function in $C^\infty(G)$ evaluated at xy is considered once as a function on G, and once as a function on $G \times G$. In other words, we have

$$\left\langle \omega_{xy}, d_{xy}f_1 \wedge d_{xy}f_2 \right\rangle = \left\langle \omega_y, d_y(f_1 \circ L_x) \wedge d_y(f_2 \circ L_x) \right\rangle + \left\langle \omega_x, d_x(f_1 \circ R_y) \wedge d_x(f_2 \circ R_y) \right\rangle$$

$$= \left\langle \omega_y, d_{xy}f_1 \circ (L_x)_{*y} \wedge d_{xy}f_2 \circ (L_x)_{*y} \right\rangle + \left\langle \omega_x, d_{xy}f_1 \circ (R_y)_{*x} \wedge d_{xy}f_2 \circ (R_y)_{*x} \right\rangle$$

$$= \left\langle (L_x)_{*y} \otimes (L_x)_{*y}\omega_y, d_{xy}f_1 \wedge d_{xy}f_2 \right\rangle + \left\langle (R_y)_{*x} \otimes (R_y)_{*x}\omega_y, d_{xy}f_1 \wedge d_{xy}f_2 \right\rangle,$$

where we used the relations $d_y(f \circ L_x) = d_{xy}f \circ (L_x)_{*y}$ and $d_x(f \circ R_y) = d_{xy}f \circ (R_y)_{*x}$. Here $(L_x)_{*y}$ and $(R_y)_{*x}$ are the tangent maps to L_x and R_y evaluated at the points y and x respectively. Therefore we deduce

$$\omega_{xy} = [(L_x)_{*y} \otimes (L_x)_{*y}]\omega_y + [(R_y)_{*x} \otimes (R_y)_{*x}]\omega_x. \tag{1.5}$$

In local coordinates

$$\omega_{ij}(\xi) = \omega_{kl}(x)\frac{\partial \xi_i}{\partial x_k}\frac{\partial \xi_j}{\partial x_l} + \omega_{kl}(y)\frac{\partial \xi_i}{\partial y_k}\frac{\partial \xi_j}{\partial y_l}, \tag{1.6}$$

where $\xi = xy$.

If $e \in G$ is the identity of G, then (1.5) yields $2\omega_e = \omega_e$. Therefore $\omega_e = 0$. This implies that ω is not a symplectic structure since the rank of ω at the identity of G is zero, and we are dealing with more general Poisson manifolds.

Thus, equations (1.1) and (1.6) are the defining equations for a Poisson-Lie group. Many examples are known. The question of the classification of the Poisson-Lie structures is very difficult. Presently there is no general method of classifying all Poisson-Lie structures on a given Lie group. Some results are known for finite dimensional

Lie groups. The most general is the classification of all Poisson-Lie structures for complex simple Lie groups, due to Belavin and Drinfel'd [BD]. Up to now there is no known classification for an infinite-dimensional Lie group.

Remark. In the definitions above all manifolds were finite-dimensional ($\mathcal{M}$, respectively the group G). To extend these into the infinite-dimensional case, one needs two things: $T_{\mathcal{M}}$ and $C^\infty(\mathcal{M})$. Since we will study infinite-dimensional groups in this text, the infinite-dimensional aspects will be addressed at the moment they are introduced.

We now proceed with the definition of a Lie-bialgebra and formulate a theorem (again due to Drinfel'd) relating the concept of a Lie-bialgebra to the concept of a Poisson-Lie group.

Definition 1.3. A Lie-bialgebra $\mathcal{G}$ is a Lie algebra $\mathcal{G}$ equipped with a coalgebra map $\alpha\colon \mathcal{G} \to \mathcal{G} \wedge \mathcal{G}$ such that α is a 1-cocycle of $\mathcal{G}$ with values in the $\mathcal{G}$-module $\wedge^2\mathcal{G}$, where $\mathcal{G}$ acts on $\wedge^2\mathcal{G}$ by means of the adjoint representation, and α satisfies the co-Jacobi identity. Thus, $(\mathcal{G}, \alpha)$ is a Lie bialgebra iff

$$(i) \qquad \tau \circ \alpha = -\alpha$$

$$(ii) \qquad \alpha([X,Y]) = ad_X\alpha(Y) - ad_Y\alpha(X), \quad X, Y \in \mathcal{G},$$

$$(iii) \qquad [1 \otimes 1 \otimes 1 + (\tau \otimes 1)(1 \otimes \tau) + (1 \otimes \tau)(\tau \otimes 1)](1 \otimes \alpha) \circ \alpha = 0,$$

where the transposition map $\tau\colon \mathcal{G} \otimes \mathcal{G} \to \mathcal{G} \otimes \mathcal{G}$ is defined by $\tau(a \otimes b) = b \otimes a$ for any $a, b \in \mathcal{G}$.

Equation (iii) above has the following meaning. The coalgebra map $\alpha\colon \mathcal{G} \to \mathcal{G} \wedge \mathcal{G}$ satisfies (iii) if and only if its dual $\alpha^*\colon \mathcal{G}^* \wedge \mathcal{G}^* \to \mathcal{G}^*$ makes the dual space $\mathcal{G}^*$ into a Lie algebra. Let us see how this comes about. Let $l_1, l_2, l_3 \in \mathcal{G}^*$ be elements of the the dual space to $\mathcal{G}$. The the map $\alpha^*\colon \mathcal{G}^* \wedge \mathcal{G}^* \to \mathcal{G}^*$ is defined by

$$\alpha^*(l_1 \otimes l_2) \stackrel{def}{=} (l_1 \otimes l_2) \circ \alpha.$$

That is, for any $x \in \mathcal{G}$ we have $(\alpha^*(l_1 \otimes l_2))(x) = (l_1 \otimes l_2)(\alpha(x))$. Let also $\tau^*\colon \mathcal{G}^* \otimes \mathcal{G}^* \to \mathcal{G}^* \otimes \mathcal{G}^*$ be the dual of the transposition map τ defined by the formula

$$\tau^*(l_1 \otimes l_2) \stackrel{def}{=} (l_1 \otimes l_2) \circ \tau.$$

That is, for any $x \otimes y \in \mathcal{G} \otimes \mathcal{G}$ we have $(\tau^*(l_1 \otimes l_2))(x \otimes y) = (l_1 \otimes l_2)(y \otimes x) = l_1(y)l_2(x)$. We see that the above definition is equivalent to the following definition: $\tau^*(l_1 \otimes l_2) \stackrel{def}{=} l_2 \otimes l_1$. Then the Jacobi identity for α^* reads

$$\alpha^*(l_1 \otimes \alpha^*(l_2 \otimes l_3)) + \alpha^*(l_2 \otimes \alpha^*(l_3 \otimes l_1)) + \alpha^*(l_3 \otimes \alpha^*(l_1 \otimes l_2)) = 0,$$

which is equivalent to

$$\alpha^* \circ (1 \otimes \alpha^*)\Big[(l_1 \otimes l_2 \otimes l_3) + (l_2 \otimes l_3 \otimes l_1) + (l_3 \otimes l_1 \otimes l_2)\Big] = 0.$$

The above equation is then equivalent to

$$\alpha^* \circ (1 \otimes \alpha^*)\Big[1 \otimes 1 \otimes 1 + (1 \otimes \tau^*)(\tau^* \otimes 1) + (\tau^* \otimes 1)(1 \otimes \tau^*)\Big](l_1 \otimes l_2 \otimes l_3) = 0, \text{for any } l_1, l_2, l_3 \in \mathcal{G}^*.$$

Therefore we have

$$\alpha^* \circ (1 \otimes \alpha^*)\Big[1 \otimes 1 \otimes 1 + (1 \otimes \tau^*)(\tau^* \otimes 1) + (\tau^* \otimes 1)(1 \otimes \tau^*)\Big] = 0.$$

Taking the dual of this we obtain (iii).

In order to clarify the meaning of (ii) in the definition of a Lie bialgebra we recall here basic facts about the Chevalley-Eilenberg cohomology theory for Lie algebras.

Let $\{C_n, \delta\}_{n \in \mathbb{Z}_+}$ be the complex

$$C_0 \xrightarrow{\delta} C_1 \xrightarrow{\delta} \ldots \xrightarrow{\delta} C_n \xrightarrow{\delta} C_{n+1} \xrightarrow{\delta} \ldots .$$

Here $C_n = \mathrm{Hom}(\wedge^n \mathcal{G}, V)$ is the space of n-cochains, where V is a $\mathcal{G}$-module, i.e. a representation space for $\mathcal{G}$, and $C_0 = V$. The coboundary operator $\delta \colon C_n \to C_{n+1}$ is defined by the rule

(*)

$$(\delta\alpha^n)(X_1, \ldots, X_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} X_i.\alpha^n(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{n+1})$$
$$+ \sum_{i<j} (-1)^{i+j} \alpha^n([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{n+1}),$$

where $\alpha^n \colon \wedge^n \mathcal{G} \to V$ is an n-cochain. The hat over an argument means that it is being omitted, and the dot after X_i stands for the action of $\mathcal{G}$ on V. If we choose $V = \mathcal{G} \wedge \mathcal{G}$, then in the above formula the dot stands for the action induced by the

adjoint action of $\mathcal{G}$ on itself. Let $\alpha\colon \mathcal{G} \to \mathcal{G} \wedge \mathcal{G}$ be a 1-cochain. Then α is a 1-cocycle if and only if $(\delta\alpha)(X_1, X_2) = 0$ for any $X_1, X_2 \in \mathcal{G}$. From the definition of δ we obtain

$$(**) \qquad (\delta\alpha)(X_1, X_2) = X_1.\alpha(X_2) - X_2.\alpha(X_1) - \alpha([X_1, X_2]) = 0.$$

This is exactly equation (ii) in the definition of a Lie bialgebra. Therefore we will refer to (ii) as the 1-cocycle condition in the sequel. In the case when $\alpha = \delta\alpha^0$ is a 1-coboundary, equation (**) is automatically satisfied, since $\delta^2 = 0$. In this case $\alpha(X) = ad_X r$, where $r \in \mathcal{G} \wedge \mathcal{G} = C_0(\mathcal{G}, \mathcal{G} \wedge \mathcal{G})$.

Let $\{e_i\}$ be a basis of $\mathcal{G}$ and let us write α in this basis as $\alpha(e_n) = \alpha^n_{ij} e_i \wedge e_j$. Let C^{ij}_n be the structure constants of $\mathcal{G}$ defining the Lie structure on $\mathcal{G}$ by $[e_i, e_j] = C^{ij}_n e_n$. Property (i) in the definition of α implies that $\alpha^n_{ij} = -\alpha^n_{ji}$. Then the equation (iii) written in terms of α^n_{ij} becomes

$$(1.7) \qquad \alpha^n_{ij}\alpha^j_{sp} + \alpha^n_{pj}\alpha^j_{is} + \alpha^n_{sj}\alpha^j_{pi} = 0.$$

Similarly, equation (ii) expressed in terms of α^n_{ij} and the structure constants C^{ij}_n of $\mathcal{G}$ becomes

$$\begin{aligned} C^{ij}_n \alpha^n_{kl} e_k \wedge e_l &= \alpha^j_{kl}[e_i, e_k] \wedge e_l + \alpha^j_{kl} e_k \wedge [e_i, e_l] - \alpha^i_{kl}[e_j, e_k] \wedge e_l - \alpha^i_{kl} e_k \wedge [e_j, e_l] \\ &= \alpha^j_{kl} C^{ik}_p e_p \wedge e_l + \alpha^j_{kl} C^{il}_p e_k \wedge e_p - \alpha^i_{kl} C^{jk}_p e_p \wedge e_l - \alpha^i_{kl} C^{jl}_p e_k \wedge e_p \\ &= \alpha^j_{ml} C^{ml}_k e_k \wedge e_l + \alpha^j_{km} C^{im}_l e_k \wedge e_l - \alpha^i_{ml} C^{jm}_k e_k \wedge e_l - \alpha^i_{km} C^{jm}_l e_k \wedge e_l. \end{aligned}$$

This is equivalent to

$$(1.8) \qquad C^{ij}_n \alpha^n_{kl} = \alpha^j_{ml} C^{im}_k + \alpha^j_{km} C^{im}_l - \alpha^i_{ml} C^{jm}_k - \alpha^i_{km} C^{jm}_l.$$

Thus, these two systems of equations, (1.7) and (1.8), plus the Jacobi identity for the structure constants of the Lie algebra $\mathcal{G}$

$$C^{ij}_m C^{mk}_n + C^{jk}_m C^{mi}_n + C^{ki}_m C^{mj}_n = 0,$$

define a Lie-bialgebra structure on $\mathcal{G}$. Then we have the following result.

Theorem 1.4 (Drinfel'd). *The category of connected, simply connected finite dimensional Poisson-Lie groups is equivalent to the category of finite dimensional Lie-bialgebras.*

Proof. ($\Longrightarrow$) Let G be a Poisson-Lie group with a Poisson-Lie structure defined locally by

$$\{f,g\}(x) = \omega_{ij}(x)\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}, \tag{1.9}$$

for any $f, g \in C^\infty(G)$.

The differential of (1.9) is

$$d\{f,g\}(x) = \left[\frac{\partial \omega_{ij}(x)}{\partial x_n}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} + \omega_{ij}(x)\frac{\partial^2 f}{\partial x_n \partial x_i}\frac{\partial g}{\partial x_j} + \omega_{ij}(x)\frac{\partial f}{\partial x_i}\frac{\partial^2 g}{\partial x_n \partial x_j}\right] dx_n.$$

Evaluating at the identity e of G and using the previously derived fact $\omega_e = 0$ in the form $\omega_{ij}(e) = 0$, we obtain

$$d\{f,g\}(e) = \frac{\partial \omega_{ij}(e)}{\partial x_n}\frac{\partial f}{\partial x_i}\bigg|_e \frac{\partial g}{\partial x_j}\bigg|_e dx_n \in T_e^*G = \mathcal{G}^*,$$

i.e. an element of the dual space $\mathcal{G}^*$ of $\mathcal{G}$.

Let $\mathcal{G}$ be the Lie algebra of G and $\mathcal{G}^*$ its dual. Let us define a Lie algebra strucuture on $\mathcal{G}^*$ as follows. Let $\alpha \in \mathcal{G}^*$, $\alpha = \alpha_i dx_i$ and $\{dx_i\}$ be a basis for $\mathcal{G}^* = T_e^*G$. Then there is a function $f \in C^\infty(G)$ such that $\alpha_i = \frac{\partial f}{\partial x_i}\,|_e$, i.e. $df(e) = \alpha$. Let $\mathcal{A}$ be the class of all functions $f \in C^\infty(G)$ with the property $df(e) = \alpha$. Then $\mathcal{A}$ is an equivalence class:
(i) $f \sim f$,
(ii) if $f_1 \sim f_2$, i.e. $\frac{\partial f_1}{\partial x_i}\,|_e = \frac{\partial f_2}{\partial x_i}\,|_e$ then $f_2 \sim f_1$,
(iii) if $f_1 \sim f_2$ and $f_2 \sim f_3$ then $f_1 \sim f_3$. This follows from: $f_1 \sim f_2 \Longrightarrow \frac{\partial f_1}{\partial x_i}\,|_e = \frac{\partial f_2}{\partial x_i}\,|_e$ and $f_2 \sim f_3 \Longrightarrow \frac{\partial f_2}{\partial x_i}\,|_e = \frac{\partial f_3}{\partial x_i}\,|_e$. Therefore, $\frac{\partial f_1}{\partial x_i}\,|_e = \frac{\partial f_3}{\partial x_i}\,|_e \Longrightarrow f_1 \sim f_3$.

Let $\alpha, \beta \in \mathcal{G}^*$. Let $f, g \in C^\infty(G)$ be such that $df(e) = \alpha$ and $dg(e) = \beta$, and let $\mathcal{A}$ and $\mathcal{B}$ be the corresponding equivalence classes of functions of $C^\infty(G)$ such that if $f \in \mathcal{A} \Longrightarrow df(e) = \alpha$ and $g \in \mathcal{B} \Longrightarrow dg(e) = \beta$. Define $[\ ,\ ]_{\mathcal{G}^*} : \mathcal{G}^* \otimes \mathcal{G}^* \to \mathcal{G}^*$ by

$$[\alpha,\beta]_{\mathcal{G}^*} = d\{f,g\}(e) = \frac{\partial \omega_{ij}(e)}{\partial x_n}\frac{\partial f}{\partial x_i}\bigg|_e \frac{\partial g}{\partial x_j}\bigg|_e dx_n \in T_e^*G = \mathcal{G}^*.$$

Let $f' \in \mathcal{A}$ and $g' \in \mathcal{B}$ are another two elements of the equivalence classes $\mathcal{A}$ and $\mathcal{B}$ such that $df'(e) = \alpha$ and $dg'(e) = \beta$. Then

$$[\alpha,\beta]_{\mathcal{G}^*} = d\{f',g'\}(e) = \frac{\partial \omega_{ij}(e)}{\partial x_n}\frac{\partial f'}{\partial x_i}\bigg|_e \frac{\partial g'}{\partial x_j}\bigg|_e dx_n \in T_e^*G = \mathcal{G}^*.$$

Since $f \sim f'$ and $g \sim g'$ we have $\frac{\partial f}{\partial x_i}\mid_e = \frac{\partial f'}{\partial x_i}\mid_e$ and $\frac{\partial g}{\partial x_i}\mid_e = \frac{\partial g'}{\partial x_i}\mid_e$. Therefore the definition of $[\ ,\ ]_{\mathcal{G}^*}$ is independent of the choice of $f \in \mathcal{A}$ and $g \in \mathcal{B}$.

Let $L_x\colon G \to G$ be a left translation by $x \in G$. Let $\omega_l(x) = (L_{x^{-1}})_*\omega(x)$, i.e., we translate ω back to the identity. Let also $\xi = x^{-1}y$. If $\{\frac{\partial}{\partial y_k}\}$ is a basis of T_yG then the action of the tangent map $(L_{x^{-1}})_*$ on a basis element of the tangent space T_yG is described by the formula

$$\left((L_{x^{-1}})_*\frac{\partial}{\partial y_k}\right)_\xi = \frac{\partial \xi_i}{\partial y_k}\bigg|_y \frac{\partial}{\partial \xi_i},$$

where $\{\frac{\partial}{\partial \xi_i}\}$ is a basis of $T_\xi G$. Therefore the action on the bi-vector $\omega(y)$ is

$$[(L_{x^{-1}})_* \otimes (L_{x^{-1}})_*]\omega(y) = \omega_{kl}(y)\frac{\partial \xi_i}{\partial y_k}\bigg|_y \frac{\partial \xi_j}{\partial y_l}\bigg|_y \frac{\partial}{\partial \xi_i} \wedge \frac{\partial}{\partial \xi_j}.$$

After evaluating at $y = x$ we obtain

$$(\omega_l)_{ij}(x) = \omega_{kl}(x)\frac{\partial \xi_i}{\partial y_k}\bigg|_{y=x} \frac{\partial \xi_j}{\partial y_l}\bigg|_{y=x}.$$

Therefore $\omega_l(x) = (\omega_l)_{ij}(x)e_i \wedge e_j$ is a map $\omega_l\colon G \to \mathcal{G} \wedge \mathcal{G}$, where $\{e_i\}$ is a basis for $\mathcal{G}$ at the identity. Define

$$\alpha^n_{ij} \equiv \frac{\partial(\omega_l)_{ij}}{\partial x_n}\bigg|_e = \left(\frac{\partial \omega_{kl}}{\partial x_n}\frac{\partial \xi_i}{\partial y_k}\bigg|_{y=x} \frac{\partial \xi_j}{\partial y_l}\bigg|_{y=x}\right)\bigg|_{x=e} = \beta^n_{kl}\chi^k_i\chi^l_j,$$

where

$$\beta^n_{kl} \equiv \frac{\partial \omega_{kl}}{\partial x_n}\bigg|_e \qquad \text{and} \qquad \chi^k_i \equiv \frac{\partial \xi_i}{\partial y_k}\bigg|_{y=x}\bigg|_{x=e}.$$

Differentiating $\omega_l(x)$ at the identity we obtain a map $\alpha\colon \mathcal{G} \to \mathcal{G} \wedge \mathcal{G}$. The map α endows $\mathcal{G}$ with a Lie-bialgebra structure. Indeed,

$$\begin{aligned}
&\alpha^n_{ij}\alpha^i_{kl} + \alpha^n_{ik}\alpha^i_{lj} + \alpha^n_{il}\alpha^i_{jk} = \\
&\qquad = \beta^n_{sp}\chi^s_i\chi^p_j\beta^i_{qr}\chi^q_k\chi^r_l + \beta^n_{sp}\chi^s_i\chi^p_k\beta^i_{qr}\chi^q_l\chi^r_j + \beta^n_{sp}\chi^s_i\chi^p_l\beta^i_{qr}\chi^q_j\chi^r_k = \\
&\qquad\qquad = (\beta^n_{sp}\beta^i_{qr} + \beta^n_{sq}\beta^i_{rp} + \beta^n_{sr}\beta^i_{pq})\chi^s_i\chi^p_j\chi^q_k\chi^r_l.
\end{aligned}$$

But

$$\beta^n_{sp}\beta^i_{qr} + \beta^n_{sq}\beta^i_{rp} + \beta^n_{sr}\beta^i_{pq} = 0,$$

which follows from

$$\omega_{ij}\frac{\partial \omega_{kl}}{\partial x_i} + \omega_{ik}\frac{\partial \omega_{lj}}{\partial x_i} + \omega_{il}\frac{\partial \omega_{jk}}{\partial x_i} = 0$$

after differentiating and evaluating at the identity of G, and using again $\omega_{ij}(e) = 0$. Thus α satisfies the co-Jacobi identity (1.7). That α is a 1-cocycle can be seen from the following argument. From (1.5) we have

$$\omega(xy) = [(L_x)_* \otimes (L_x)_*]\omega(y) + [(R_y)_* \otimes (R_y)_*]\omega(x)$$

where $\omega(x) = \omega_{ij}(x)\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$. Pulling back the left hand side to the identity element by a left translation, we obtain

$$\begin{aligned}[(L_{(xy)^{-1}})_* \otimes (L_{(xy)^{-1}})_*]\omega(xy) = {} & [(L_{(xy)^{-1}})_*(L_x)_* \otimes (L_{(xy)^{-1}})_*(L_x)_*]\omega(y) + \\ & [(L_{(xy)^{-1}})_*(R_y)_* \otimes (L_{(xy)^{-1}})_*(R_y)_*]\omega(x).\end{aligned}$$

From this we have

$$\omega_l(xy) = \omega_l(y) + Ad_{y^{-1}}\omega_l(x),$$

and this shows that $\omega_l(x) = (\omega_l)_{ij}(x)e_i \wedge e_j$ is a 1-cocycle $\omega_l \colon G \to \mathcal{G} \wedge \mathcal{G}$ on the group G. To this 1-cocycle on G corresponds exactly one 1-cocycle on the Lie algebra $\mathcal{G}$ of G as we now show.

Let $X \in \mathcal{G}$ and let $\varphi_X(t) = e^{tX}$ be the flow that X generates on G. Then the map $\alpha \colon \mathcal{G} \to \mathcal{G} \wedge \mathcal{G}$ is defined by $\alpha(X) = \frac{d}{dt}(\omega_l(e^{tX}))\,|_{t=0}$. From this we deduce

$$\begin{aligned} 0 = \frac{d}{dt}(\omega_l(e^{-tX}e^{tX}))\,\Big|_{t=0} &= \frac{d}{dt}\Big[\omega_l(e^{-tX}) + Ad_{e^{-tX}}\omega_l(e^{tX})\Big]\,\Big|_{t=0} \\ &= \alpha(-X) + \alpha(X). \end{aligned}$$

Let $Y \in \mathcal{G}$ and $\varphi_Y(s) = e^{sY}$ be the flow it generates on G. Since

$$\begin{aligned} \alpha([X,Y]) &= \frac{d}{dt}\left[\frac{d}{ds}\omega_l(e^{tX}e^{sY}e^{-tX})\,\Big|_{s=0}\right]\,\Big|_{t=0} \\ &= \frac{d}{dt}\Big[\alpha(e^{tX}Ye^{-tX})\Big]\,\Big|_{t=0} \\ &= \frac{d}{dt}\Big[\alpha(Y + t[X,Y] + \dots)\Big]\,\Big|_{t=0}, \end{aligned}$$

we have

$$\begin{aligned}
\alpha([X,Y]) &= \frac{d}{dt}\left[\frac{d}{ds}\omega_l(e^{tX}e^{sY}e^{-tX})\Big|_{s=0}\right]\Big|_{t=0} \\
&= \frac{d}{dt}\left[\frac{d}{ds}\left[\omega_l(e^{-tX}) + Ad_{e^{tX}}\omega_l(e^{tX}e^{sY})\right]\Big|_{s=0}\right]\Big|_{t=0} \\
&= \frac{d}{dt}\left[\frac{d}{ds}\left[\omega_l(e^{-tX}) + Ad_{e^{tX}}\omega_l(e^{sY}) + Ad_{e^{tX}}Ad_{e^{-sY}}\omega_l(e^{tX})\right]\Big|_{s=0}\right]\Big|_{t=0} \\
&= \frac{d}{dt}\left[Ad_{e^{tX}}\alpha(Y) + Ad_{e^{tX}}ad_{(-Y)}\omega_l(e^{tX})\right]\Big|_{t=0} \\
&= ad_X\alpha(Y) - Ad_e ad_Y\alpha(X) - ad_X ad_Y\omega_l(e) \\
&= ad_X\alpha(Y) - ad_Y\alpha(X).
\end{aligned}$$

Therefore

$$\alpha([X,Y]) = ad_X\alpha(Y) - ad_Y\alpha(X). \tag{1.10}$$

This proves the first part of the theorem.

($\Longleftarrow$) Assume now that $\alpha\colon \mathcal{G} \to \mathcal{G}\wedge\mathcal{G}$ is a 1-cocycle that satisfies

$$[1\otimes 1\otimes 1 + (\tau\otimes 1)(1\otimes\tau) + (1\otimes\tau)(\tau\otimes 1)](1\otimes\alpha)\circ\alpha = 0.$$

Since G is connected and simply connected α can be integrated to get a 1-cocycle $\omega_\alpha\colon G\to\mathcal{G}\wedge\mathcal{G}$ on G. That is, ω_α satisfies

$$\omega_\alpha(xy) = \omega_\alpha(y) + Ad_{y^{-1}}\omega_\alpha(x). \tag{1.11}$$

Let us define $\omega(x) = [(L_x)_*\otimes(L_x)_*]\omega_\alpha(x)$. Then from (1.11) we obtain that ω satisfies

$$\omega(xy) = [(L_x)_*\otimes(L_x)_*]\omega(y) + [(R_y)_*\otimes(R_y)_*]\omega(x). \tag{1.12}$$

Therefore $\omega(e) = 0$. In local coordinates $\omega(x) = \omega_{ij}(x)\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}$, where $\{\frac{\partial}{\partial x_i}\}$ is a basis of T_xG.

Define a map $\{\ ,\ \}\colon C^\infty(G)\times C^\infty(G)\to C^\infty(G)$ by

$$\{f,g\}(x) = \omega_{ij}(x)\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}, \qquad \text{for any} \quad f,g\in C^\infty(G).$$

Then for any $f,g,h\in C^\infty(G)$ we have

$$\{\{f,g\},h\}+\{\{g,h\},f\}+\{\{h,f\},g\}=\Omega_{jkl}\frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_k}\frac{\partial h}{\partial x_l}, \tag{1.13}$$

where

$$\Omega_{jkl}=\omega_{ij}\frac{\partial\omega_{kl}}{\partial x_i}+\omega_{ik}\frac{\partial\omega_{lj}}{\partial x_i}+\omega_{il}\frac{\partial\omega_{jk}}{\partial x_i}.$$

To show this we compute the l.h.s. of (1.13):

$$\begin{aligned}
\{\{f,g\},h\}+\{\{g,h\},f\}+\{\{h,f\},g\}&=\\
&=\omega_{ij}\frac{\partial}{\partial x_i}\left(\omega_{kl}\frac{\partial f}{\partial x_k}\frac{\partial g}{\partial x_l}\right)\frac{\partial h}{\partial x_j}+\mathrm{cycl}(f,g,h)\\
&=\omega_{ij}\frac{\partial\omega_{kl}}{\partial x_i}\frac{\partial f}{\partial x_k}\frac{\partial g}{\partial x_l}\frac{\partial h}{\partial x_j}+\omega_{ij}\omega_{kl}\frac{\partial^2 f}{\partial x_i\partial x_k}\frac{\partial g}{\partial x_l}\frac{\partial h}{\partial x_j}+\\
&\quad+\omega_{ij}\omega_{kl}\frac{\partial f}{\partial x_k}\frac{\partial^2 g}{\partial x_i\partial x_l}\frac{\partial h}{\partial x_j}+\mathrm{cycl}(f,g,h)\\
&=\left[\omega_{ij}\frac{\partial\omega_{kl}}{\partial x_i}+\omega_{ik}\frac{\partial\omega_{lj}}{\partial x_i}+\omega_{il}\frac{\partial\omega_{jk}}{\partial x_i}\right]\frac{\partial f}{\partial x_k}\frac{\partial g}{\partial x_l}\frac{\partial h}{\partial x_j}+\\
&\quad+(\omega_{ij}\omega_{kl}+\omega_{il}\omega_{jk})\frac{\partial^2 f}{\partial x_i\partial x_k}\frac{\partial g}{\partial x_l}\frac{\partial h}{\partial x_j}+\mathrm{cycl}(f,g,h).
\end{aligned} \tag{1.14}$$

But

$$\begin{aligned}
\left[\omega_{ij}\omega_{kl}+\omega_{il}\omega_{jk}\right]\frac{\partial^2 f}{\partial x_i\partial x_k}&=\left[\omega_{kj}\omega_{il}+\omega_{kl}\omega_{ji}\right]\frac{\partial^2 f}{\partial x_i\partial x_k}\\
&=-\left[\omega_{il}\omega_{jk}+\omega_{ij}\omega_{kl}\right]\frac{\partial^2 f}{\partial x_i\partial x_k}.
\end{aligned}$$

Therefore

$$\left[\omega_{ij}\omega_{kl}+\omega_{il}\omega_{jk}\right]\frac{\partial^2 f}{\partial x_i\partial x_k}=0, \tag{1.15}$$

and the same is true for the remainig two terms of the same form in (1.14) obtained after cyclic permutation of f,g,h.

Next we show that Ω_{jkl} is a 1-cocycle on the group G. Let $\xi=xy$, for $x,y\in G$. Then

$$\Omega_{jkl}(\xi)=\omega_{ij}(\xi)\frac{\partial\omega_{kl}}{\partial\xi_i}+\omega_{ik}(\xi)\frac{\partial\omega_{lj}}{\partial\xi_i}+\omega_{il}(\xi)\frac{\partial\omega_{jk}}{\partial\xi_i}$$

$$
\begin{aligned}
&=\left[\omega_{mn}(x)\frac{\partial\xi_i}{\partial x_m}\frac{\partial\xi_j}{\partial x_n}+\omega_{mn}(y)\frac{\partial\xi_i}{\partial y_m}\frac{\partial\xi_j}{\partial y_n}\right]\frac{\partial\omega_{kl}}{\partial\xi_i}+\mathrm{cycl}(j,k,l)\\
&=\omega_{mn}(x)\frac{\partial\xi_j}{\partial x_n}\frac{\partial}{\partial x_m}\omega_{kl}(xy)+\omega_{mn}(y)\frac{\partial\xi_j}{\partial y_n}\frac{\partial}{\partial y_m}\omega_{kl}(xy)+\mathrm{cycl}(j,k,l)\\
&=\omega_{mn}(x)\frac{\partial\xi_j}{\partial x_n}\frac{\partial}{\partial x_m}\left[\omega_{sp}(x)\frac{\partial\xi_k}{\partial x_s}\frac{\partial\xi_l}{\partial x_p}+\omega_{sp}(y)\frac{\partial\xi_k}{\partial y_s}\frac{\partial\xi_l}{\partial y_p}\right]+\\
&\quad+\omega_{mn}(y)\frac{\partial\xi_j}{\partial y_n}\frac{\partial}{\partial y_m}\left[\omega_{sp}(x)\frac{\partial\xi_k}{\partial x_s}\frac{\partial\xi_l}{\partial x_p}+\omega_{sp}(y)\frac{\partial\xi_k}{\partial y_s}\frac{\partial\xi_l}{\partial y_p}\right]+\mathrm{cycl}(j,k,l)\\
&=\Omega_{nsp}(x)\frac{\partial\xi_j}{\partial x_n}\frac{\partial\xi_k}{\partial x_s}\frac{\partial\xi_l}{\partial x_p}+\Omega_{nsp}(y)\frac{\partial\xi_j}{\partial y_n}\frac{\partial\xi_k}{\partial y_s}\frac{\partial\xi_l}{\partial y_p}+\\
&\quad+\omega_{mn}(x)\omega_{sp}(x)\left[\frac{\partial\xi_j}{\partial x_n}\frac{\partial^2\xi_k}{\partial x_m\partial x_s}\frac{\partial\xi_l}{\partial x_p}+\frac{\partial\xi_l}{\partial x_n}\frac{\partial\xi_j}{\partial x_s}\frac{\partial^2\xi_k}{\partial x_m\partial x_p}+\mathrm{cycl}(j,k,l)\right]+\\
&\quad+\omega_{mn}(y)\omega_{sp}(y)\left[\frac{\partial\xi_j}{\partial y_n}\frac{\partial^2\xi_k}{\partial y_m\partial y_s}\frac{\partial\xi_l}{\partial y_p}+\frac{\partial\xi_l}{\partial y_n}\frac{\partial\xi_j}{\partial y_s}\frac{\partial^2\xi_k}{\partial y_m\partial y_p}+\mathrm{cycl}(j,k,l)\right]+\\
&\quad+\omega_{mn}(x)\omega_{sp}(y)\left[\frac{\partial\xi_j}{\partial x_n}\frac{\partial^2\xi_k}{\partial x_m\partial y_s}\frac{\partial\xi_l}{\partial y_p}+\frac{\partial\xi_l}{\partial x_n}\frac{\partial\xi_j}{\partial y_s}\frac{\partial^2\xi_k}{\partial x_m\partial y_p}+\mathrm{cycl}(j,k,l)\right]+\\
&\quad+\omega_{mn}(y)\omega_{sp}(x)\left[\frac{\partial\xi_j}{\partial y_n}\frac{\partial^2\xi_k}{\partial y_m\partial x_s}\frac{\partial\xi_l}{\partial x_p}+\frac{\partial\xi_l}{\partial y_n}\frac{\partial\xi_j}{\partial x_s}\frac{\partial^2\xi_k}{\partial y_m\partial x_p}+\mathrm{cycl}(j,k,l)\right].
\end{aligned}
$$

The terms proportional to $\omega_{mn}(x)\omega_{sp}(x)$ and $\omega_{mn}(y)\omega_{sp}(y)$ are all zero due to the identity (1.15). The terms proportional to $\omega_{mn}(x)\omega_{sp}(y)$ and $\omega_{mn}(y)\omega_{sp}(x)$ also add up to zero due to identities as, for example, the one shown below

$$
\left[\omega_{mn}(x)\omega_{sp}(y)+\omega_{sp}(y)\omega_{mn}(x)\right]\frac{\partial\xi_j}{\partial x_n}\frac{\partial\xi_l}{\partial y_p}\frac{\partial^2\xi_k}{\partial x_m\partial y_s}=0.
$$

As a result we obtain that Ω_{jkl} satisfies

$$
\Omega_{jkl}(xy)=\Omega_{nsp}(x)\frac{\partial\xi_j}{\partial x_n}\frac{\partial\xi_k}{\partial x_s}\frac{\partial\xi_l}{\partial x_p}+\Omega_{nsp}(y)\frac{\partial\xi_j}{\partial y_n}\frac{\partial\xi_k}{\partial y_s}\frac{\partial\xi_l}{\partial y_p}.
$$

Writing this in an invariant notation we have that $\Omega(x)=\Omega_{jkl}(x)\frac{\partial}{\partial x_j}\wedge\frac{\partial}{\partial x_k}\wedge\frac{\partial}{\partial x_l}$ satisfies

$$
\Omega(xy)=[(L_x)_*\otimes(L_x)_*\otimes(L_x)_*]\Omega(y)+[(R_y)_*\otimes(R_y)_*\otimes(R_y)_*]\Omega(x). \tag{1.16}
$$

From (1.16) it follows that $\Omega(e)=0$. Pulling Ω back to the identity by a left translation we have

$$
\Omega_l(x)=[(L_{x^{-1}})_*\otimes(L_{x^{-1}})_*\otimes(L_{x^{-1}})_*]\Omega(x),
$$

and from (1.16) we obtain that $\Omega_l(x)$ satisfies the cocycle condition

$$
\Omega_l(xy)=\Omega_l(y)+Ad_{y^{-1}}\Omega_l(x). \tag{1.17}
$$

Thus $\Omega_l\colon G \to \mathcal{G} \wedge \mathcal{G}$ is a 1-cocycle on G. In coordinates

$$\Omega_l(x) = \Omega_{jkl}(x) \frac{\partial \xi_p}{\partial y_j}\bigg|_{y=x} \frac{\partial \xi_q}{\partial y_k}\bigg|_{y=x} \frac{\partial \xi_s}{\partial y_l}\bigg|_{y=x} e_p \wedge e_q \wedge e_s,$$

where $\xi = x^{-1}y$ and $\{e_i\}$ is a basis of $\mathcal{G}$. For the derivative of Ω_l we have

$$\frac{\partial}{\partial x_n}\Omega_l(x)\bigg|_e = \frac{\partial}{\partial x_n}\Omega_{jkl}(x)\bigg|_e \chi_p^j \chi_q^k \chi_s^l e_p \wedge e_q \wedge e_s, \quad \text{where} \quad \chi_p^j \equiv \frac{\partial \xi_p}{\partial y_j}\bigg|_{y=x}\bigg|_{x=e},$$

which gives rise to a 1-cocycle $\beta\colon \mathcal{G} \to \mathcal{G} \wedge \mathcal{G} \wedge \mathcal{G}$, defined by $\beta(e_n) = \frac{\partial \Omega_l}{\partial x_n}\Big|_e$, for each basis element $e_n \in \mathcal{G}$. But

$$\frac{\partial \Omega_{jkl}}{\partial x_n}(e) = \alpha_{js}^n \alpha_{kl}^s + \alpha_{ks}^n \alpha_{lj}^s + \alpha_{ls}^n \alpha_{jk}^s, \quad \text{where} \quad \alpha_{ij}^n = \frac{\partial \omega_{ij}}{\partial x_n}\bigg|_e.$$

By assumption the map α satisfies the co-Jacobi identity. Therefore we conclude that $\frac{\partial \Omega^{jkl}}{\partial x_n}(e) = 0$, and from here it follows that $\beta = 0$. Since there is a one-to-one correspondence between the 1-cocycles on the Lie algebra and 1-cocycles on the group we conclude that $\Omega = 0$ because its infinitesimal part is zero. Hence, to every Lie-bialgebra structure on the Lie algebra $\mathcal{G}$ defined by the map $\alpha\colon \mathcal{G} \to \mathcal{G} \wedge \mathcal{G}$ corresponds a unique Poisson-Lie structure on the group G.■

Theorem 1.5. *Let G be a Poisson-Lie group. Then the map $\varphi\colon G \to G$ defined by $\varphi(x) = x^{-1}$ is an anti-Poisson map.*

Proof. We prove the statement in a neighbourhood of the identity element of G. Let $\xi_i = \xi_i(x, y)$, for $i = 1, \ldots, n$, be the coordinate functions of $\xi = xy$. The multiplicativity condition reads

$$\omega_{ij}(\xi) = \omega_{kl}(x) \frac{\partial \xi_i}{\partial x_k} \frac{\partial \xi_j}{\partial x_l} + \omega_{kl}(y) \frac{\partial \xi_i}{\partial y_k} \frac{\partial \xi_j}{\partial y_l}. \tag{1.18}$$

After solving $\xi_i = \xi_i(x, y)$ with respect to the coordinates of y we have $y_i = y_i(x, \xi)$. We differentiate the identities

$$y_i \equiv y_i(x, \xi(x, y)), \quad \text{for} \quad i = 1, \ldots, n,$$

with respect to y_k for each $k = 1, \ldots, n$ to obtain

$$\delta_i^k = \frac{\partial y_i}{\partial \xi_l}\bigg|_{(x,\xi)} \frac{\partial \xi_l}{\partial y_k}\bigg|_{(x,y)}. \tag{1.19}$$

Let $\varphi\colon G \to G$ be the map defined by $\varphi(x) = x^{-1}$, which is given in coordinates by the functions $\varphi_i = \varphi_i(x)$. The we have

$$0 = \xi_i(x, \varphi(x)), \quad \text{for} \quad i = 1, \dots, n. \tag{1.20}$$

We differentiate (1.20) with respect to x_k to obtain

$$0 = \frac{\partial \xi_l}{\partial x_k}\bigg|_{(x,\varphi(x))} + \frac{\partial \xi_i}{\partial y_l}\bigg|_{(x,\varphi(x))} \frac{\partial \varphi_l}{\partial x_k}. \tag{1.21}$$

After multiplying both sides of (1.18) by $\frac{\partial y_m}{\partial \xi_i}\Big|_{(x,\xi)} \frac{\partial y_n}{\partial \xi_j}\Big|_{(x,\xi)}$ and summing over i, j we get

$$\frac{\partial y_m}{\partial \xi_i}\bigg|_{(x,\xi)} \frac{\partial y_n}{\partial \xi_j}\bigg|_{(x,\xi)} \omega_{ij}(\xi) = \frac{\partial y_m}{\partial \xi_i}\bigg|_{(x,\xi)} \frac{\partial y_n}{\partial \xi_j}\bigg|_{(x,\xi)} \omega_{kl}(x) \frac{\partial \xi_i}{\partial x_k} \frac{\partial \xi_j}{\partial x_l} + \omega_{mn}(y), \tag{1.22}$$

where we used (1.19).

We now set $\xi = e = x\varphi(x)$ in (1.22), and obtain

$$0 = \frac{\partial y_m}{\partial \xi_i}\bigg|_{(x,e)} \frac{\partial y_n}{\partial \xi_j}\bigg|_{(x,e)} \omega_{kl}(x) \frac{\partial \xi_i}{\partial x_k}\bigg|_{(x,\varphi(x))} \frac{\partial \xi_j}{\partial x_l}\bigg|_{(x,\varphi(x))} + \omega_{mn}(\varphi(x)). \tag{1.23}$$

Using (1.21) the above equality is equivalent to

$$0 = \frac{\partial y_m}{\partial \xi_i}\bigg|_{(x,e)} \frac{\partial y_n}{\partial \xi_j}\bigg|_{(x,e)} \omega_{kl}(x) \frac{\partial \xi_i}{\partial y_p}\bigg|_{(x,\varphi(x))} \frac{\partial \varphi_p}{\partial x_k} \frac{\partial \xi_j}{\partial y_s}\bigg|_{(x,\varphi(x))} \frac{\partial \varphi_s}{\partial x_l} + \omega_{mn}(\varphi(x)).$$

Now using again (1.19) as

$$\delta_i^k = \frac{\partial y_i}{\partial \xi_l}\bigg|_{(x,e)} \frac{\partial \xi_l}{\partial y_k}\bigg|_{(x,\varphi(x))},$$

we finally conclude that

$$\omega_{mn}(\varphi(x)) = -\omega_{kl}(x) \frac{\partial \varphi_m}{\partial x_k} \frac{\partial \varphi_n}{\partial x_l}. \blacksquare$$

In the following chapters we will adapt the above given definitions for the case where G will stand for particular infinite-dimensional groups, and address the classification problem. We will show that theorems analogous to Theorems 1.4 and 1.5 above also hold in this case.

CHAPTER II

THE GROUP OF INFINITE-JETS G_∞ AND ITS LIE ALGEBRA

In this chapter we recall the necessary background on infinite-dimensional manifolds needed to justify the results obtained in the subsequent chapters. This includes the definitions of projective and inductive limits of families of topological vector spaces. We illustrate this definitions on the example of the space $\mathbb{R}^\infty$. Then we introduce a group G_∞ as an infinite-dimensional manifold modeled on the space $\mathbb{R}^\infty$ and also introduce a smooth structure on it. We also define a concept of a tangent manifold TG_∞ which allows us to carry out the Poisson-Lie theory, discussed in the previous chapter, to this particular infinite-dimensional group. Finally we derive the Lie algebra of this group.

The group G_∞ of infinite-jets.

Let us consider the space of formal power series in the variable u

$$\mathcal{X}(u) = \sum_{i=1}^{\infty} x_i u^i.$$

We think of each $\mathcal{X}$ as representing a point $x = (x_1, x_2, \dots)$ on the infinite-dimensional subspace G_∞ of $\mathbb{R}^\infty$: $G_\infty = \{x \in \mathbb{R}^\infty \mid x_1 \neq 0\}$, where x_i's are the coordinate functions of the point.

Define a multiplication map $m_\infty \colon G_\infty \times G_\infty \to G_\infty$ on G_∞ induced by the substitution of formal power series. For any two $\mathcal{X}(u), \mathcal{Y}(u)$ define

$$(\mathcal{X}\mathcal{Y})(u) = \mathcal{X}(\mathcal{Y}(u))$$

to be the product of $\mathcal{X}(u)$ and $\mathcal{Y}(u)$.

Obviously the induced multiplication makes G_∞ a group with an identity $e = (1, 0, 0, \dots)$. In local coordinates

$$\xi_k = \sum_{i=1}^{k} x_i \sum_{(\sum_{\alpha=1}^{i} j_\alpha)=k} y_{j_1} \cdots y_{j_i}. \tag{2.1}$$

The first several formulae are given below

$$\begin{aligned}
\xi_1 &= x_1 y_1 \\
\xi_2 &= x_1 y_2 + x_2 y_1^2 \\
\xi_3 &= x_1 y_3 + x_2 2 y_1 y_2 + x_3 y_1^3 \\
\xi_4 &= x_1 y_4 + x_2 (y_2^2 + 2 y_1 y_3) + x_3 3 y_1^2 y_2 + x_4 y_1^4 \\
&\vdots \\
\xi_n &= x_1 y_n + x_n y_1^n + y_{n-1} 2 y_1 x_2 + x_{n-1}(n-1) y_1^{n-2} y_2 + \mathrm{O}(< n-1), \quad n > 3 \\
&\vdots
\end{aligned}$$

where $\xi = xy$.

The group G_∞ can be made into a smooth manifold modeled on the complete locally convex topological vector space $\mathbb{R}^\infty$ in the following way.

The space $\mathbb{R}^\infty$ and smooth structures on it.

We will begin with two definitions. Namely, the definitions of inverse (projective) and direct (inductive) limits of topological vector spaces, which will establish the background for our discussion of infinite-dimensional groups and Poisson-Lie structures on them.

Definition 2.1. (see, e.g. [Sa]) Let $(V_n, f_{n+1,n})_{n\in\mathbb{N}}$ be a family of topological vector spaces and maps, where $f_{n+1,n}\colon V_{n+1} \to V_n$ are continuous. Then the pair $(V_\infty; (f_{\infty,n})_{n\in\mathbb{N}})$ is called an inverse (projective) limit of $(V_n, f_{n+1,n})_{n\in\mathbb{N}}$ if
(i) V_∞ is a topological vector space; each $f_{\infty,n}\colon V_\infty \to V_n$ is a continuous map, and $f_{n+1,n} \circ f_{\infty,n+1} = f_{\infty,n}$ for each $n \in \mathbb{N}$.
(ii) If W is a topological vector space and $g_n\colon W \to V_n$ are continuous maps satisfying $f_{n+1,n} \circ g_{n+1} = g_n$ for every $n \in \mathbb{N}$, then there exists a unique continuous map $g\colon W \to V_\infty$ which satisfies $g_n = f_{\infty,n} \circ g$, $\forall\, n \in \mathbb{N}$.

Part (ii) of the above definition simply asserts that V_∞ is unique up to an isomorphism. Next, we give an example, which is the main space we will work with.

Consider the family $(\mathbb{R}^n, \pi_{n+1,n})_{n\in\mathbb{N}}$, where the projections $\pi_{n+1,n}\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ are defined by

$$\pi_{n+1,n}(x_1, \dots, x_n, x_{n+1}) = (x_1, \dots, x_n),$$

and $\{x_i\}_{i=1}^n$ are the standard coordinates on $\mathbb{R}^n$. Then the above family has an inverse limit $(\mathbb{R}^\infty, \pi_{\infty,n})$ where $\pi_{\infty,n}\colon \mathbb{R}^\infty \to \mathbb{R}^n$ are defined by

$$\pi_{\infty,n}(x_1, \dots, x_n, x_{n+1}, \dots) = (x_1, \dots, x_n).$$

Clearly the maps $\pi_{n+1,n}$ are continuous in the (e.g. Euclidean) topology on $\mathbb{R}^n$. The inverse limit topology on $\mathbb{R}^\infty$ is defined by declaring subsets of the form $\pi_{\infty,n}^{-1}(\mathcal{O}_n)$, for open $\mathcal{O}_n \in \mathbb{R}^n$, to be open in $\mathbb{R}^\infty$ and to form a basis for the topology on $\mathbb{R}^\infty$. We also have $\pi_{n+1,n} \circ \pi_{\infty,n+1} = \pi_{\infty,n}$ for every $n \in \mathbb{N}$. If W is another candidate for a projective limit and $g_n\colon W \to \mathbb{R}^n$, $\forall\, n \in \mathbb{N}$, then we define $g\colon W \to \mathbb{R}^\infty$ by $(g(x))_n = (g_n(x))_n$, where $(g(x))_n$ is the n-th component of $g(x) \in \mathbb{R}^\infty$, and $x \in W$.

The dual concept is the concept of a direct (inductive) limit of a family of topological vector spaces.

Definition 2.2. (see, e.g. [Sa]) Let $(V_n, f_{n,n+1})_{n\in\mathbb{N}}$ be a family of topological vector spaces and continuous maps $f_{n,n+1}\colon V_n \to V_{n+1}$. Then the pair $(V_\infty; (f_{n,\infty})_{n\in\mathbb{N}})$ is called a direct (inductive) limit of $(V_n, f_{n,n+1})_{n\in\mathbb{N}}$ if
(i) V_∞ is a topological vector space, each $f_{n,\infty}\colon V_n \to V_\infty$ is a continuous map, and $f_{n+1,\infty} \circ f_{n,n+1} = f_{n,\infty}$ for every $n \in \mathbb{N}$;
(ii) if W is a topological vector space and $g_n\colon V_n \to W$ are continuous maps for every $n \in \mathbb{N}$, satisfying $g_n = g_{n+1} \circ f_{n,n+1}$, $\forall\, n \in \mathbb{N}$, then there exists a unique map $g\colon V_\infty \to W$ which satisfies $g_n = g \circ f_{n,\infty}$.

Going back to our example, let us consider the family $(\mathbb{R}^n, \iota_{n,n+1})_{n\in\mathbb{N}}$, where $\iota_{n,n+1}\colon \mathbb{R}^n \to \mathbb{R}^{n+1}$ is defined by

$$\iota_{n,n+1}(x_1, \dots, x_n) = (x_1, \dots, x_n, 0).$$

This family has a direct limit $(\mathbb{R}_0^\infty, \iota_{n,\infty})$, where $\mathbb{R}_0^\infty$ is a vector subspace of $\mathbb{R}^\infty$ consisting of all sequences containing only finite number of non-zero terms, and $\iota_{n,\infty}\colon \mathbb{R}^n \to \mathbb{R}_0^\infty$ is defined by

$$\iota_{n,\infty}(x_1, \dots, x_n) = (x_1, \dots, x_n, 0, 0, \dots).$$

The direct (inductive) limit topology on $\mathbb{R}_0^\infty$ is defined by declaring a subset $\mathcal{O} \subset \mathbb{R}_0^\infty$ to be open if and only if $\iota_{n,\infty}^{-1}(\mathcal{O})$ is open in $\mathbb{R}^n$ for every $n \in \mathbb{N}$. Then $\iota_{n,\infty}$ is continuous and $\iota_{n,\infty} = \iota_{n+1,\infty} \circ \iota_{n,n+1}$, $\forall\, n \in \mathbb{N}$. If W is another topological vector space and $g_n\colon \mathbb{R}^n \to W$, let us define $g\colon \mathbb{R}_0^\infty \to W$ as follows

(i) $g(0) = 0$;
(ii) if $x \in \mathbb{R}_0^\infty$ is non-zero, let n be the largest index for which $x_n \neq 0$, so that there is a unique $\overline{x} \in \mathbb{R}^n$ such that $\iota_{n,\infty}(\overline{x}) = x$. Then $g_n = g \circ \iota_{n,\infty}$.

Now we give the proof of the following important fact.

Lemma 2.3. [Sa] *The topological dual of* $\mathbb{R}^\infty$ *is isomorphic to* $\mathbb{R}_0^\infty$: $\mathbb{R}^{\infty *} \cong \mathbb{R}_0^\infty$.

Proof. Let $\alpha\colon \mathbb{R}^\infty \to \mathbb{R}$ be a linear map.
(i) First we show that if α depends on finite number of components of its argument, i.e. if $\exists n \in \mathbb{N}$ such that if $x, y \in \mathbb{R}^\infty$ and $x_k = y_k$ for $k \leq n$ then $\alpha(x) = \alpha(y)$, then α is continuous. Define $\alpha_n\colon \mathbb{R}^n \to \mathbb{R}$ by $\alpha_n = \alpha \circ \iota_{n,\infty}$. Use $(\iota_{n,\infty} \circ \pi_{\infty,n})(x_1, \dots, x_n, x_{n+1}, \dots) = (x_1, \dots, x_n, 0, 0, \dots)$. Then $\alpha = \alpha_n \circ \pi_{\infty,n}$. Therefore α is continuous, since α_n and $\pi_{\infty,n}$ are. Thus $\alpha \in \mathbb{R}^{\infty *}$.
(ii) Now assume that α depends on infinitely many components of its argument. Then $\forall\, n \in \mathbb{N}\ \exists x_{(n)}, y_{(n)} \in \mathbb{R}^\infty$ such that $x_{(n)k} = y_{(n)k}$ for every $k \leq n$ but $\alpha(x_{(n)}) \neq \alpha(y_{(n)})$. Consider $\{z_{(n)}\}$, where $z_{(n)} \in \mathbb{R}^\infty$ for each $n \in \mathbb{N}$ and such that

$$z_{(n)} = \frac{x_{(n)} - y_{(n)}}{\alpha(x_{(n)} - y_{(n)})}.$$

Then $z_{(n)} \to 0$ as $n \to \infty$, but $\alpha(z_{(n)}) = 1$. Therefore α is not continuous.

The isomorphism $\mathbb{R}_0^\infty \cong \mathbb{R}^{\infty *}$ is given by the following map. For each $\alpha \in \mathbb{R}_0^\infty$ define

$$F_\alpha(x) = \sum_{k=1}^{\infty} \alpha_k x_k \qquad \text{for} \qquad x \in \mathbb{R}^\infty.$$

Note that the above sum is finite, since $\alpha \in \mathbb{R}_0^\infty$.■

Remark. Similarly, $\mathbb{R}_0^{\infty *} \cong \mathbb{R}^\infty$. We will not need this fact.

Definition 2.4. Let V,W be topological vector spaces and let $\mathcal{O} \subset V$ be open. The map $f\colon \mathcal{O} \to W$ is said to be of class C^1 if for every $x \in \mathcal{O}$ and every $v \in V$ the limit

$$Df(x; v) = \lim_{t \to 0} \frac{1}{t}[f(x + tv) - f(x)]$$

exists, and the map $Df\colon \mathcal{O} \times V \to W$ is continuous.

For us it will be important that the derivative defined above has the following property which we state without proof: for every $x \in \mathcal{O}$, $Df(x;v)$ is linear and continuous in v.

Definition 2.5. Let V,W be topological vector spaces and let $\mathcal{O} \subset V$ be open. The map $f\colon \mathcal{O} \to W$ is said to be of class C^{k+1} for $k \geq 1$ if it is of class C^k and if, for every $v_1, \dots, v_k \in V$, the map $\mathcal{O} \to W$ given by $x \mapsto D^k f(x; v_1, \dots, v_k)$ is of class C^1. Here the map $D^{k+1}f\colon \mathcal{O} \times V^{k+1} \to W$ is then defined by

$$D^{k+1}f(x; v, v_1, \dots, v_k) = \lim_{t \to 0} \frac{1}{t}[D^k f(x + tv; v_1, \dots, v_k) - D^k f(x; v_1, \dots, v_k)].$$

If f is of class C^k for each $k \geq 1$ then it is said to be smooth, or of class C^∞.

Let $\mathcal{O} \subset \mathbb{R}^\infty$ be open, and let $f\colon \mathcal{O} \to \mathbb{R}$ be C^1. Then for every $x \in \mathcal{O}$ the map defined by $v \mapsto Df(x;v)$ is linear and continuous from $\mathbb{R}^\infty \to \mathbb{R}$. Thus, it is an element of $\mathbb{R}_0^\infty$. By choosing v to point at the direction of the coordinate axes we conclude that at each point $x \in \mathcal{O}$ the only partial derivatives of f that are non-zero are $\frac{\partial f}{\partial x_i}$, $i \leq n$ for some $n \in \mathbb{N}$. This remains true if f is of class C^k, $k \geq 1$ or C^∞ and is shown by applying the same argument to the higher order directional derivatives $D^k f\colon \mathcal{O} \times (\mathbb{R}^\infty)^k \to \mathbb{R}^\infty$ which are all linear and continuous maps from $(\mathbb{R}^\infty)^k \to \mathbb{R}$ at each point $x \in \mathcal{O}$. Therefore we define the space $C^\infty(\mathbb{R}^\infty)$ of smooth functions on $\mathbb{R}^\infty$ to be the space of smooth functions depending only upon a finite number of variables.

Since in what follows we will encounter maps of the form $f : V \to \mathbb{R}^\infty$, for some locally convex topological vector space V, we give next a criterion for these maps to be smooth.

Lemma 2.6. [Sa] *Let V, $W = \mathbb{R}^\infty$ be locally convex topological vector spaces, and $\mathcal{O} \subset V$ an open subset of V. The map $f : \mathcal{O} \to \mathbb{R}^\infty$ is a smooth map if and only if each $f_n = \pi_{\infty,n} \circ f$, $n \in \mathbb{N}$, is a smooth map $f_n : \mathcal{O} \to \mathbb{R}^n$.*

Proof. If $f : \mathcal{O} \to \mathbb{R}^\infty$ is a smooth map then each f_n is smooth since $\pi_{\infty,n}$ are smooth.

Let f_n be smooth for every $n \in \mathbb{N}$. From $f_n = \pi_{n+1,n} \circ f_{n+1}$ it follows, using the chain rule, that $Df_n = D\pi_{n+1,n} \circ Df_{n+1} = \pi_{n+1,n} \circ Df_{n+1}$, since $\pi_{n+1,n}$ is a linear

map for every $n \in \mathbb{N}$. Let $x \in \mathcal{O}$, $v \in V$, and let $\xi \in \mathbb{R}^\infty$ be such that $\pi_{\infty,m}(\xi) = Df_m(x;v)$ for every $m \in \mathbb{N}$ (by the definition of $\mathbb{R}^\infty$ as a projective limit). For every neighbourhood $U \ni \xi$ of ξ there is an $n \in \mathbb{N}$ and a neighbourhood $U_n \ni Df_n(x;v)$ of $Df_n(x;v)$ such that $\pi_{\infty,n}^{-1}(U_n) \subset U$ (by the definition of the topology on $\mathbb{R}^\infty$). Then there is an $\epsilon > 0$ such that

$$\frac{1}{t}\Big[f_n(x+tv) - f_n(x)\Big] \subset U_n, \qquad \text{for } |t| < \epsilon.$$

Since

$$\pi_{\infty,n}\Big[\frac{1}{t}\Big(f(x+tv) - f(x)\Big)\Big] = \frac{1}{t}\Big[f_n(x+tv) - f_n(x)\Big],$$

it follows that

$$\pi_{\infty,n}\Big[\frac{1}{t}\Big(f(x+tv) - f(x)\Big)\Big] \subset U_n,$$

and therefore

$$\frac{1}{t}\Big[f(x+tv) - f(x)\Big] \in U.$$

Since U was chosen to be arbitrary we conclude that

$$\xi = \lim_{t\to 0}\frac{1}{t}\Big(f(x+tv) - f(x)\Big).$$

Applying the same argument inductively to the higher derivatives $D^k f_n = \pi_{\infty,n} \circ D^k f$ of f_n we deduce that f is smooth.■

Let us consider the connected components $G_{>0} = \{x \in G_\infty \mid x_1 > 0\}$ and $G_{<0} = \{x \in G_\infty \mid x_1 < 0\}$ of the group G_∞. Then G_∞ is a disjoint union $G_\infty = G_{>0} \cup G_{<0}$. We model each of the two connected components $G_{>0}$ and $G_{<0}$ of G_∞ on the space $\mathbb{R}^\infty$. In the sequel we follow the theory of infinite dimensional Lie groups and the notation as it has been developed in [M]. (We will give the arguments for the component $G_{>0}$ which contains the identity. Clearly $G_{<0}$ is a mirror copy of $G_{>0}$ under the reflection map $x_1 \mapsto -x_1$.)

Let us introduce a local coordinate system $\varphi_{>0}$ on $G_{>0}$ by the smooth map

$$\varphi_{>0} : \mathbb{R}^\infty \to G_{>0} \tag{2.2}$$

defined by

$$\varphi_{>0}(x) = (e^{x_1}, x_2, x_3, \dots), \text{ where } x = (x_1, x_2, x_3, \dots) \in \mathbb{R}^\infty, \tag{2.3}$$

and its inverse $\varphi_{>0}^{-1} : G_{>0} \to \mathbb{R}^\infty$ is defined by

$$(2.3^*) \qquad \varphi_{>0}^{-1}(y) = (\ln y_1, y_2, y_3, \dots), \text{ where } y = (y_1, y_2, y_3, \dots) \in G_{>0}.$$

(Analogously one could define a local coordinate system $\varphi_{<0}$ on $G_{<0}$: $\varphi_{<0}(x) = (-e^{x_1}, x_2, x_3, \dots)$.) With this local coordinate system $G_{>0}$ becomes a smooth manifold (not a very interesting one from a topological point of view), such that the multiplication map $m_\infty : G_{>0} \times G_{>0} \to G_{>0}$ and the inverse map $i_\infty : G_{>0} \to G_{>0}$, defined by $x \mapsto x^{-1}$, are smooth maps.

To show that m_∞ is smooth we need to show that the map $\varphi_{>0}^{-1} \circ m_\infty \circ (\varphi_{>0} \times \varphi_{>0}) : \mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}^\infty$ is smooth. But each of the maps $\pi_{\infty,n} \circ \varphi_{>0}^{-1} \circ m_\infty \circ (\varphi_{>0} \times \varphi_{>0}) : \mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}^n$ defined by

$$(xy)_1 = x_1 + y_1$$

$$(xy)_k = e^{x_1} y_k + e^{k y_1} x_k + \sum_{i=2}^{k-1} x_i \sum_{\sum_{\alpha=1}^{i} j_\alpha = k} y'_{j_1} \dots y'_{j_i}, \quad \text{for } 2 \le k \le n,$$

and where

$$y'_{j_\alpha} = \begin{cases} e^{y_1}, & j_\alpha = 1 \\ y_{j_\alpha}, & j_\alpha \neq 1, \end{cases}$$

is smooth. Therefore we apply Lemma 2.6. Analogously, to show that the map i_∞ is smooth we need to show that the map $\varphi_{>0}^{-1} \circ i_\infty \circ \varphi_{>0} : \mathbb{R}^\infty \to \mathbb{R}^\infty$ is smooth. Again each of the maps $\pi_{\infty,n} \circ (\varphi_{>0}^{-1} \circ i_\infty \circ \varphi_{>0}) : \mathbb{R}^\infty \to \mathbb{R}^n$ defined by

$$(x^{-1})_1 = -x_1$$

$$(x^{-1})_k = -e^{-(k+1)x_1} x_k - e^{-x_1} \sum_{i=2}^{k-1} x_i \sum_{\sum_{\alpha=1}^{i} j_\alpha = k} y'_{j_1} \dots y'_{j_i}, \quad \text{for } 2 \le k \le n,$$

and where

$$y'_{j_\alpha} = \begin{cases} e^{-x_1}, & j_\alpha = 1 \\ (x^{-1})_{j_\alpha}, & j_\alpha \neq 1, \end{cases}$$

is smooth. Here $(x^{-1})_{j_\alpha}$, for $2 \le j_\alpha \le n-1$ are computed inductively starting with

$$\begin{aligned} (x^{-1})_1 &= -x_1 \\ (x^{-1})_2 &= -e^{-3x_1} x_2 \\ (x^{-1})_3 &= 2e^{-5x_1} x_2^2 - e^{-4x_1} x_3 \\ &\vdots \end{aligned}$$

Now we apply again Lemma 2.6.

In order to define a tangent space to $G_{>0}$ we need a notion of an equivalence class of paths passing through a point $x_0 \in G_{>0}$. Let $\gamma_1 : \mathbb{R} \to G_{>0}$ and $\gamma_2 : \mathbb{R} \to G_{>0}$ be two paths in $G_{>0}$ such that $\gamma_1(0) = 0 = \gamma_2(0)$. We say that γ_1 and γ_2 are equivalent if the corresponding paths

$$t \mapsto \varphi_{>0}^{-1}(\gamma_1(t)) \quad \text{and} \quad t \mapsto \varphi_{>0}^{-1}(\gamma_2(t))$$

in $\mathbb{R}^\infty$ have the same first derivatives at $t = 0$. The set of all equivalence classes of paths through the point $x_0 \in G_{>0}$ is defined to be the *tangent space* $T_{x_0}G_{>0}$. In fact there is a one-to-one correspondence between $T_{x_0}G_{>0}$ and the model space $\mathbb{R}^\infty$. Indeed, let $p_0 \in \mathbb{R}^\infty$ and $\varphi_{>0}(p_0) = x_0$. Then to every $p \in \mathbb{R}^\infty$ there corresponds the equivalence class of the path

$$t \mapsto \varphi_{>0}(p_0 + tp)$$

through the point $x_0 \in G_{>0}$. Using this correspondence $T_{x_0}G_{>0}$ can be given a structure of a locally convex topological vector space isomorphic to $\mathbb{R}^\infty$.

Define $TG_{>0} = \bigcup_{x \in G_{>0}} T_xG_{>0}$ to be the *tangent manifold* [M] of $G_{>0}$. The tangent manifold $TG_{>0}$ can be given a smooth structure. We model $TG_{>0}$ on the space $\mathbb{R}^\infty \times \mathbb{R}^\infty$. The local coordinate system is given by the smooth map

$$\psi_{>0} : \mathbb{R}^\infty \times \mathbb{R}^\infty \to TG_{>0},$$

defined by $(p, q) \mapsto$ {equivalence class of the path $t \mapsto \psi_{>0}(p + tq)$ through the point p}.

We recall that given two smooth manifolds $\mathcal{M}_1$, $\mathcal{M}_2$, a smooth map $f : \mathcal{M}_1 \to \mathcal{M}_2$ induces a continuous linear map $f'_x : T_x\mathcal{M}_1 \to T_{f(x)}\mathcal{M}_2$ between the corresponding tangent spaces (the derivative of the map f). This map induces a smooth map

$$f_* : T\mathcal{M}_1 \to T\mathcal{M}_2$$

between the corresponding tangent manifolds.

Definition 2.7. A smooth vector field is a smooth map $v : G_{>0} \to TG_{>0}$ defined by $x \mapsto v(x) \in T_xG_{>0}$ for every $x \in G_{>0}$.

In local coordinates

$$v = \sum_{i=1}^{\infty} v_i \frac{\partial}{\partial x_i}, \quad v_i \in C^\infty(G_{>0}).$$

The vector field v acts as a linear differential operator (derivation) on the ring $C^\infty(G_{>0}, \mathbb{R})$ of smooth functions

$$f : G_{>0} \to \mathbb{R}.$$

As we mentioned above this map induces the continous linear map

$$f'_x : T_x G_{>0} \to T_{f(x)}\mathbb{R} \quad \text{for each } x \in G_{>0}.$$

Because of the canonical isomorphism $T_{f(x)}\mathbb{R} \cong \mathbb{R}$ we could think of $f'_x(v(x))$ as a point in $\mathbb{R}$ for every $v(x) \in T_x G_{>0}$. Thus for every $v \in TG_{>0}$ and every $f : G_{>0} \to \mathbb{R}$ we obtain a map

$$X_v(f) : G_{>0} \to \mathbb{R}, \tag{2.4}$$

defined by

$$X_v(f)(x) \equiv f'_x(v(x)) \in T_{f(x)}\mathbb{R} \cong \mathbb{R}.$$

Clearly the map (2.4) has the properties

$$X_{av_1+bv_2}(f) = aX_{v_1}(f) + bX_{v_2}(f) \quad \text{for every } v_1, v_2 \in TG_{>0}, \text{ and } a, b \in \mathbb{R}, \tag{2.5}$$

and

(2.6)

$$X_v(af_1 + bf_2) = aX_v(f_1) + bX_v(f_2) \quad \text{for every } f_1, f_2 \in C^\infty(G_{>0}, \mathbb{R}), \text{ and } a, b \in \mathbb{R}.$$

The last property is a consequence of the linearity of the space $T_{f(x)}\mathbb{R}$ for each point $x \in G_{>0}$.

For any $u, v \colon G_{>0} \to TG_{>0}$ define a vector field $[u, v]$ by

$$X_{[u,v]}f = X_u(X_v f) - X_v(X_u f). \tag{2.7}$$

Using property (2.6) one immediatly shows that the Jacobi identity

$$[v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]] = 0 \tag{2.8}$$

is satisfied for any three vector fields $v_1, v_2, v_3 \in TG_{>0}$.

Since $(\mathbb{R}^\infty)^* \cong \mathbb{R}^\infty_0$, and we have the natural isomorphism $T_x G_{>0} \cong \mathbb{R}^\infty$, it follows that we have an isomorphism between the dual spaces $(T_x G_{>0})^* \cong (\mathbb{R}^\infty)^* \cong \mathbb{R}^\infty_0$ at each point $x \in G_{>0}$. Then every cotangent vector $\zeta \in (T_x G_{>0})^*$ is given by a finite sum

$$\zeta = \sum_{i=1}^{n} \zeta_i dx_i, \qquad \zeta_i \in C^\infty(G_{>0}),$$

for some $n \in \mathbb{N}$. Here $\{dx_i\}_{i\in\mathbb{N}}$ forms a basis of $(T_x G_{>0})^*$ dual to a basis $\{\frac{\partial}{\partial x_i}\}_{i\in\mathbb{N}}$ of $T_x G_{>0}$.

For the purpose of defining Poisson structures on $G_{>0}$ we will use the existing duality between the tangent and the cotangent spaces at each point.

Let $T_e G_{>0}$ be the tangent space at the identity of the group. Let $\varphi_{>0} : \mathbb{R}^\infty \to G_{>0}$ be a local coordinate system with $\varphi_{>0}(0) = e$. Then the multiplication $\cdot : G_{>0} \times G_{>0} \to G_{>0}$ on $G_{>0}$ (the map m_∞ restricted to $G_{>0} \times G_{>0}$) lifts to a multiplication $* : \mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}^\infty$ on the model space $\mathbb{R}^\infty$ by

$$x * y = \varphi_{>0}^{-1}(\varphi_{>0}(x) \cdot \varphi_{>0}(y)) \quad \text{for any} \quad x, y \in \mathbb{R}^\infty. \tag{2.9}$$

The first several terms of the Taylor expansion of (2.9) around 0 are given by

$$x * y = x + y + b(x, y) + \dots \tag{2.10}$$

Here $b(x, y)$ is a bilinear vector-valued form and the dots stand for terms of higher order. Later we will compute explicitly the form $b(x, y)$ for the group $G_{>0}$. Since we also have

$$x * 0 = \varphi_{>0}^{-1}(\varphi_{>0}(x) \cdot e) = x = \varphi_{>0}^{-1}(e \cdot \varphi_{>0}(x)) = 0 * x \tag{2.11}$$

it follows that the constant term in the expansion is 0. Also from

$$x * x^{-1} = \varphi_{>0}^{-1}(\varphi_{>0}(x) \cdot \varphi_{>0}^{-1}(x)) = 0 = \varphi_{>0}^{-1}(\varphi_{>0}^{-1}(x) \cdot \varphi_{>0}(x)) = x^{-1} * x \tag{2.12}$$

we have

$$0 = x * x^{-1} = x + x^{-1} + b(x, x^{-1}) + \dots .$$

Therefore

$$\begin{aligned} x^{-1} &= -x - b(x, x^{-1}) + \dots \\ &= -x + b(x, x) + \dots . \end{aligned}$$

Finally we compute

$$\begin{aligned} x * y * x^{-1} &= x * (y * x^{-1}) \\ &= x + y * x^{-1} + b(x, y * x^{-1}) + \dots \\ &= x + y + x^{-1} + b(y, x^{-1}) + \dots + b(x, y + x^{-1}) + \dots \\ &= x + y - x + b(x, x) - b(y, x) + \dots + b(x, y) - b(x, x) + \dots \\ &= y + \Big(b(x, y) - b(y, x)\Big) + \dots . \end{aligned} \tag{2.13}$$

Let $t \mapsto y(t)$ and $s \mapsto x(s)$ be two paths in $\mathbb{R}^\infty$ passing through 0, $t \mapsto \varphi_{>0}^{-1}(y(t))$ and $s \mapsto \varphi_{>0}^{-1}(x(s))$ their corresponding paths in $G_{>0}$ passing through the identity e with tangent vectors $u \in T_eG_{>0}$ and $v \in T_eG_{>0}$ respectively. From (2.13) we have

$$x(s) * y(t) * x^{-1}(s) = y(t) + \Big(b(x(s), y(t)) - b(y(t), x(s))\Big) + \dots . \tag{2.14}$$

Since we have the natural isomorphism $T_0\mathbb{R}^\infty \cong \mathbb{R}^\infty$ the derivative of the coordinate system map $\varphi_{>0} : \mathbb{R}^\infty \to G_{>0}$ evaluated at 0,

$$\Big(\varphi_{>0}\Big)_* \Big|_0 : T_0\mathbb{R}^\infty \to T_eG_{>0}, \tag{2.15}$$

supplies a natural isomorphism $T_eG_{>0} \cong \mathbb{R}^\infty$ between the vector space $T_eG_{>0}$ and the model vector space $\mathbb{R}^\infty$. Using this isomorphism (and after differentiating (2.14) w.r.t. t at $t = 0$ first and then w.r.t. s at $s = 0$) we define the adjoint action of $T_eG_{>0}$ on itself by the formula

$$ad_u(v) = b(u, v) - b(v, u). \tag{2.16}$$

Definition 2.8. The Lie algebra $\mathcal{G}_\infty$ of the group G_∞ is defined to be the topological vector space $T_eG_{>0}$ equiped with the bracket

$$[u, v] \equiv ad_u(v) = b(u, v) - b(v, u) \quad \text{for any} \quad u, v \in T_eG_{>0}. \tag{2.17}$$

The so defined bracket is continuous, bilinear and anti-symmetric, and it satisfies

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0. \tag{2.18}$$

The last identity is proved using the associativity condition

$$x * (y * z) = (x * y) * z,$$

from which we obtain

$$x + y * z + b(x, y * z) + \dots = x * y + z + b(x * y, z) + \dots .$$

After expanding the products on both sides of the above equality, canceling the linear and bilinear terms, and comparing the terms of third order we deduce that

$$b(x, b(y, z)) = b(b(x, y), z).$$

Therefore we have

$$\begin{aligned}[u,[v,w]] =& b(u,b(v,w)) - b(u,b(w,v)) - b(b(v,w),u) + b(b(w,v),u)\\ =& b(u,b(v,w)) - b(u,b(w,v)) - b(v,b(w,u)) + b(w,b(v,u)),\end{aligned}$$

and after cyclic permutation of u, v, w, (2.18) follows. In order to compute the bracket for $\mathcal{G}_\infty$ we need to find explicitly the bilinear form $b(x,y)$.

Let us consider the Taylor expansions around 0 of the first several components of the product $x * y$, where $x, y \in \mathbb{R}^\infty$:

$$\begin{aligned}(x*y)_1 =& x_1 + y_1\\ (x*y)_2 =& e^{x_1}y_2 + x_2e^{2y_1}\\ =& x_2 + y_2 + (x_1y_2 + 2x_2y_1) + \dots\\ (x*y)_3 =& e^{x_1}y_3 + 2x_2y_2e^{y_1} + x_3e^{3y_1}\\ =& x_3 + y_3 + (x_1y_3 + 2x_2y_2 + 3x_3y_1) + \dots\\ (x*y)_4 =& e^{x_1}y_4 + x_2(y_2^2 + 2e^{y_1}y_3) + 3x_3e^{2y_1}y_2 + x_4e^{4y_1}\\ =& x_4 + y_4 + (x_1y_4 + 2x_2y_3 + 3x_3y_2 + 4x_4y_1) + \dots\\ &\vdots\end{aligned}$$

The above calculation supplies us with the necessary intuition to be able to formulate and prove the following

Lemma 2.9. *For the group $G_{>0}$ the components $b_n(x,y)$ of the vector-valued bilinear form $b(x,y)$ in the expansion formula* (2.10) *are given by*

$$b_n(x,y) = \begin{cases} 0, & n = 1\\ \sum_{i=1}^n ix_iy_{n-i+1}, & n > 1.\end{cases} \tag{2.19}$$

Proof. For $n = 1$ the statement is obvious. Let $n > 1$. From (2.1) it follows that the n-th component of the product $x * y$ is given by

$$(x*y)_n = e^{x_1}y_n + e^{ny_1}x_n + \sum_{i=2}^{n-1} x_i \sum_{(\sum_{\alpha=1}^i j_\alpha)=n} y'_{j_1}\dots y'_{j_i}, \tag{2.20}$$

where

$$y'_{j_\alpha} = \begin{cases} e^{y_1}, & j_\alpha = 1\\ y_{j_\alpha}, & j_\alpha \neq 1.\end{cases} \tag{2.21}$$

Let us analyze an arbitrary term in the first sum in (2.20). It has the form

$$x_i \sum_{(\sum_{\alpha=1}^{i} j_\alpha)=n} y'_{j_1} \dots y'_{j_i}. \tag{2.22}$$

Clearly the only terms bilinear in x and y that could arise from (2.22) come from products $y'_{j_1} \dots y'_{j_i}$ with exactly $(i-1)$ multiples equal to e^{y_1} and the remaining one equal to y_{j_α} for some α, $2 \le \alpha \le i$. This fixes the value of j_α to be $j_\alpha = n - \sum_{s=1}^{i-1} 1 = n-i+1$. The sum in (2.22) contains exactly $\binom{i}{i-1} = i$ terms of this form. Therefore we conclude that the contribution from (2.22) to the bilinear form $b_n(x,y)$ is ix_iy_{n-i+1}. Thus

$$\sum_{i=2}^{n-1} x_i \sum_{(\sum_{\alpha=1}^{i} j_\alpha)=n} y'_{j_1} \dots y'_{j_i} = \sum_{i=2}^{n-1} ix_iy_{n-i+1} + \dots .$$

Expanding the exponents in the first two terms of (2.20) we obtain

$$(x * y)_n = x_n + y_n + \sum_{i=1}^{n} ix_iy_{n-i+1} + \dots .$$

Therefore we have

$$b_n(x,y) = \sum_{i=1}^{n} ix_iy_{n-i+1}. \blacksquare$$

To better understand the bracket (2.17) defined by this bilinear form let us choose a basis $\{e_n\}_{n=1}^{\infty}$ of vectors in $T_eG_{>0}$. The vectors e_n are defined by their components $(e_n)_j$, the latter being equal to

$$(e_n)_j = \delta_j^n,$$

where δ_j^n is the Kronecker symbol. Let us compute the j-th component of $[e_n, e_m]$:

$$\begin{aligned}
[e_n, e_m]_j =& b_j(e_n, e_m) - b_j(e_m, e_n) \\
=& \sum_{s=1}^{j} s(e_n)_s(e_m)_{j-s+1} - \sum_{s=1}^{j} s(e_m)_s(e_n)_{j-s+1} \\
=& \sum_{s=1}^{j} s(e_n)_s(e_m)_{j-s+1} - \sum_{s=1}^{j} (j-s+1)(e_n)_s(e_m)_{j-s+1} \\
=& \sum_{s=1}^{j} (2s-j-1)(e_n)_s(e_m)_{j-s+1} \\
=& \sum_{s=1}^{j} (2s-j-1)\delta_s^n\delta_{j-s+1}^m \neq 0 \Leftrightarrow s = n \text{ and } j-s+1 = m \Rightarrow j = m+n-1.
\end{aligned}$$

Therefore we obtain

$$[e_n, e_m]_j = \begin{cases} (n-m), & j = n+m-1 \\ 0, & j \neq n+m-1. \end{cases}$$

From this we deduce that

$$[e_n, e_m] = (n-m)e_{n+m-1} \quad \text{for every} \quad n, m \in \mathbb{Z}_+. \tag{2.23}$$

Had we chosen to enumerate the set of vectors $\{e_n\}$ with $n \in \mathbb{Z}_+$, and the components to be given by $(e_n)_j = \delta_j^{n+1}$, then $[e_n, e_m]$ would have assumed the form

$$[e_n, e_m] = (n-m)e_{n+m} \quad \text{for every} \quad n, m \in \mathbb{N}. \tag{2.24}$$

In the sequel we use both enumerations when we find the one more convenient in calculations than the other. One could easily switch between the two by shifting the indices by 1.

As a result of the above calculation we found explicitly the Lie algebra $\mathcal{G}_\infty$ of the group G_∞. The Jacobi identity (2.18) follows immediatly from (2.23) or (2.24) in this case.

Another way to find the Lie algebra structure on $\mathcal{G}_\infty$ is described by the following

Lemma 2.10. *The Lie structure on $\mathcal{G}_\infty$ is given by*

$$[e_n, e_m] = (n-m)e_{n+m}; \qquad n, m \in \mathbb{Z}_+, \tag{2.25}$$

where $\{e_n\}_{n=0}^\infty$ is the following basis for $\mathcal{G}_\infty$: if $\mathcal{C}_t(u) = u + tu^{n+1}$ is a 1-parameter path then e_n is the tangent vector to this path at the identity. In other words we make the identification $u^{n+1} \leftrightarrow e_n$, for $n \geq 0$.

Proof. Let us take two 1-parameter paths on the group passing through the identity

$$\mathcal{A}_t(u) = u + tu^n,$$

$$\mathcal{B}_t(u) = u + tu^m,$$

and let us take their inverses

$$\mathcal{A}_t^{-1}(u) = u - tu^n + nt^2u^{2n-1} + \mathrm{O}_{\mathcal{A}}(t^3),$$

$$\mathcal{B}_t^{-1}(u) = u - tu^m + mt^2u^{2m-1} + \mathrm{O}_{\mathcal{B}}(t^3).$$

From the commutator

$$\mathcal{Q}_t(u) = (\mathcal{A}_t \circ \mathcal{B}_t \circ \mathcal{A}_t^{-1} \circ \mathcal{B}_t^{-1})(u) = u + t^2(n-m)u^{n+m-1} + \mathrm{O}(t^3),$$

we deduce that

$$[e_n, e_m] = \lim_{t\to 0} \frac{1}{t^2}(\mathcal{Q}_t(e) - e) = (n-m)e_{n+m}. \blacksquare$$

The group G_∞ as a projective limit of groups of finite jets.

Still another viewpoint on the infinite-dimensional group is possible. The group G_∞ can be viewed as a projective limit of the following family of finite dimensional Lie groups. Let us consider the family of Lie groups and maps $(G_n, \pi_{n+1,n})_{n\in\mathbb{N}}$, where $G_n = \{\mathcal{X}_n(u) = \sum_{i=1}^n x_i u^i \mid x_1 \neq 0\}$ with a multiplication $m_n\colon G_n \times G_n \to G_n$ defined by

$$(\mathcal{X}_n\mathcal{Y}_n)(u) = \mathcal{X}_n(\mathcal{Y}_n(u)) \mod u^{n+1}.$$

The group G_n is topologically $\mathbb{R}^n \setminus M^n$ where $M^n = \{\,(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 = 0\,\}$. That is, the group G_n is an open subset of $\mathbb{R}^n$, and carries the structure of a finite dimensional C^∞ manifold modeled on $\mathbb{R}^n$. Clearly the maps $\pi_{n+1,n}\colon G_{n+1} \to G_n$ defined by

$$\pi_{n+1,n}(x_1, \dots, x_{n+1}) = (x_1, \dots, x_n)$$

are homomorphisms, i.e.

$$\pi_{n+1,n} \circ m_{n+1} = m_n \circ (\pi_{n+1,n} \times \pi_{n+1,n}).$$

This follows from the definition of $\pi_{n+1,n}$ and (2.1). Then the family $(G_n, \pi_{n+1,n})_{n\in\mathbb{N}}$ has a projective limit $(G_\infty, (\pi_{\infty,n})_{n\in\mathbb{N}})$, where G_∞ is an open subset of $\mathbb{R}^\infty$: $G_\infty = \{x \in \mathbb{R}^\infty \mid x_1 \neq 0\}$, and we could consider it as an infinite-dimensional manifold modeled on $\mathbb{R}^\infty$ as explained earlier in this chapter. The maps $\pi_{\infty,n}\colon G_\infty \to G_n$ are defined by

$$\pi_{\infty,n}(x_1, \dots, x_n, x_{n+1}, \dots) = (x_1, \dots, x_n).$$

Obviously, these maps satisfy $\pi_{\infty,n} = \pi_{n+1,n} \circ \pi_{\infty,n+1}$. On G_∞ a multiplication $m_\infty\colon G_\infty \times G_\infty \to G_\infty$ is defined by (2.1). Also the maps $\pi_{\infty,n}$ are homomorphisms, i.e. $\pi_{\infty,n} \circ m_\infty = m_n \circ (\pi_{\infty,n} \times \pi_{\infty,n})$, which follows from the definitions. If H_∞ is another candidate for a limit and $h_n\colon H_\infty \to G_n$ for every $n \in \mathbb{N}$, then we define $h\colon H_\infty \to G_\infty$ by $(h(x))_n = (h_n(x))_n$ where $(h(x))_n$ is the n-th component of $h(x) \in G_\infty$, $x \in H_\infty$.

Let us consider now the family of spaces and maps $(C^\infty(G_n), \iota_{n,n+1})_{n\in\mathbb{N}}$, where the space $C^\infty(G_n)$ is the space of smooth functions on G_n and the maps $\iota_{n,n+1}\colon C^\infty(G_n) \to C^\infty(G_{n+1})$ are defined as follows. For any $f \in C^\infty(G_n)$ we define $\iota_{n,n+1}(f) \in C^\infty(G_{n+1})$ by

$$(\iota_{n,n+1}(f))(x_1, \dots, x_n, x_{n+1}) = f(x_1, \dots, x_n).$$

Then the above family has an inductive limit $(C^\infty(G_\infty), \iota_{n,\infty})$, where $\iota_{n,\infty}\colon C^\infty(G_n) \to C^\infty(G_\infty)$ is defined as

$$((\iota_{n,\infty})(f))(x) = f(x_1, \dots, x_n)$$

for $x \in G_\infty$ and $f \in C^\infty(G_n)$, and $C^\infty(G_\infty)$ is the space of smooth functions on G_∞ defined as the smooth functions on $\mathbb{R}^\infty$ of finite number of variables restricted to $\mathbb{R}^\infty \setminus M$, where $M = \{x \in \mathbb{R}^\infty \mid x_1 = 0\}$. Suppose that W is another function space which is a candidate for an inductive limit of $(C^\infty(G_n), \iota_{n,n+1})_{n=1}^\infty$, and $g_{n,\infty}\colon C^\infty(G_n) \to W$. We define a map $g\colon C^\infty(G_\infty) \to W$ as follows

(i) $g(0) = 0$;

(ii) if $f \in C^\infty(G_\infty)$, let n be the maximal number of directions in G_∞ with respect to which f has non-zero directional derivatives. Then there exists a unique $\overline{f} \in C^\infty(G_n)$ such that $\iota_{n,\infty}(\overline{f}) = f$ and $g_{n,\infty} = g \circ \iota_{n,\infty}$.

One could define the Lie algebra of G_∞ in many different ways. Probably the most efficient one is as the Lie algebra of derivations (smooth vector fields) of the ring $C^\infty(G_\infty)$. These are of the form

$$X = \sum_{i=1}^{\infty} v_i \frac{\partial}{\partial x_i}, \qquad v_i \in C^\infty(G_\infty). \tag{2.26}$$

Note that if $f \in C^\infty(G_\infty)$, then

$$X(f) = \sum_{i=1}^{n} v_i \frac{\partial f}{\partial x_i}$$

is a finite sum, for some $n \in \mathbb{N}$, since f depends only upon a finite number of variables. We also have $X(f) \in C^\infty(G_\infty)$. Every automorphism $\varphi\colon G_\infty \to G_\infty$ acts on the space

of derivations by $(\varphi_* X) = (\varphi_*^{-1}) X \varphi^*$, and on $C^\infty(G_\infty)$ it acts by $(\varphi^* f)(x) = f(\varphi(x))$. Since the functions $f \in C^\infty(G_\infty)$ are functions of finite number of variables it is enough to describe the map φ_* on vector fields restricted to $C^\infty(G_n)$ for each $n \in \mathbb{N}$.

Lemma 2.11. *The set $\{X_n\}_{n\geq 1}$ of left-invariant vector fields on G_∞ is given by*

$$X_n = \sum_{i=1}^{\infty} i x_i \frac{\partial}{\partial x_{i+n-1}}. \tag{2.27}$$

Proof. From (2.1) the map $y \mapsto xy$ is given by

$$\xi_n = (xy)_n = x_1 y_n + y_1^n x_n + \sum_{i=2}^{n-1} x_i \sum_{(\sum_{\alpha=1}^{i} j_\alpha)=n} y_{j_1} \dots y_{j_i}, \quad \text{for each } n \geq 1. \tag{2.28}$$

The matrix of the tangent to the map defined by (2.28) is $\frac{\partial \xi_n}{\partial y_m}\Big|_{y=e}$. The only terms in

$$\sum_{i=2}^{n-1} x_i \sum_{(\sum_{\alpha=1}^{i} j_\alpha)=n} y_{j_1} \dots y_{j_i}, \quad \text{for each } n \geq 1,$$

that would contribute to the tangent map are the ones for which the product $y_{j_1} \dots y_{j_i}$ has exactly $(i-1)$ multiples equal to y_1 and the one remaining equal to y_{j_α} for some α, $2 \leq \alpha \leq i$. There are exactly $\binom{i}{i-1} = i$ terms of this form. Therefore we rewrite (2.28) as

$$\xi_n = \sum_{i=1}^{n} i x_i y_1^{i-1} y_{n-i+1} + \dots,$$

where the dots indicate terms that do not contribute to $\frac{\partial \xi_n}{\partial y_m}\Big|_{y=e}$. Hence,

$$\frac{\partial \xi_n}{\partial y_m}\Big|_{y=e} = \sum_{i=1}^{n} i x_i \delta_{n-i+1}^m = (n-m+1) x_{n-m+1}.$$

If $\{\frac{\partial}{\partial y_i}\}$ is a basis of vector fields at the identity, then

$$\begin{aligned}
\left(\varphi_* \frac{\partial}{\partial y_m}\right)_x &= \sum_{i=1}^{n} \frac{\partial \xi_i}{\partial y_m}\Big|_{y=e} \frac{\partial}{\partial x_i} \\
&= \sum_{i=m}^{n} (i-m+1) x_{i-m+1} \frac{\partial}{\partial x_i} \\
&= \sum_{i=1}^{n-m+1} i x_i \frac{\partial}{\partial x_{i+m-1}}.
\end{aligned}$$

Therefore for each $n \in \mathbb{N}$, the set of vector fields $\{X_k\}_{k=1}^{n}$, where

$$X_k = \sum_{i=1}^{n-k+1} ix_i \frac{\partial}{\partial x_{i+k-1}}, \quad \text{for } 1 \leq k \leq n,$$

forms a basis of left-invariant vector fields on G_n. Therefore the set $\{X_n\}_{n\geq 1}$, where

$$X_n = \sum_{i=1}^{\infty} ix_i \frac{\partial}{\partial x_{i+n-1}},$$

forms a basis of left invariant vector fields on G_∞.■

Lemma 2.12. *Every smooth vector field on G_∞ is generated by the set $\{X_n\}_{n\geq 1}$ of left-invariant vector fields* (2.27) *on G_∞.*

Proof. Let

$$Y = \sum_{i=1}^{\infty} v_i \frac{\partial}{\partial x_i}$$

be a smooth vector field on G_∞. We define inductively the following sequence of smooth vector fields. Let

$$\begin{aligned}
Y_1 =& Y - \frac{v_1}{x_1} X_1 = Y - \psi_1 X_1, \quad \text{where } \psi_1 \equiv \frac{v_1}{x_1}, \\
Y_2 =& Y_1 - \psi_2 X_2, \quad \text{where } \psi_2 \equiv \frac{1}{x_1}\Big(v_2 - 2x_2\psi_1\Big), \\
&\vdots \\
Y_n =& Y_{n-1} - \psi_n X_n, \quad \text{where } \psi_n \equiv \frac{1}{x_1}\Big(v_n - \sum_{i=2}^{n} ix_i\psi_{n-i+1}\Big), \\
&\vdots
\end{aligned} \tag{2.29}$$

Summing up the first n equalities in (2.29) we get

$$Y = \sum_{i=1}^{n} \psi_i X_i + Y_n.$$

By construction Y_n is such that $Y_n \Big|_{C^\infty(G_n)} = 0$, for any $n \in \mathbb{N}$. Hence,

$$Y = \sum_{i=1}^{\infty} \psi_i X_i.■$$

We now show that $\{X_n\}_{n\geq 1}$ forms a Lie subalgebra of the Lie algebra of vector fields on G_∞ with a Lie bracket equal to (2.23). For that we need to compute the commutator of two left-invariant vector fields $X_n = \sum_{i=1}^{\infty} ix_i \frac{\partial}{\partial x_{i+n-1}}$ and $X_m = \sum_{j=1}^{\infty} jx_j \frac{\partial}{\partial x_{j+m-1}}$. Namely,

$$\begin{aligned}
\left[X_n, X_m\right] =& \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} ix_i j\delta^j_{i+n-1}\frac{\partial}{\partial x_{j+m-1}} - \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} jx_j i\delta^i_{j+n-1}\frac{\partial}{\partial x_{i+n-1}} \\
=& \sum_{i=1}^{\infty} ix_i(i+n-1)\frac{\partial}{\partial x_{i+n+m-2}} - \sum_{j=1}^{\infty} jx_j(j+m-1)\frac{\partial}{\partial x_{j+n+m-2}} \\
=&(n-m)\sum_{i=1}^{\infty} ix_i\frac{\partial}{\partial x_{i+n+m-2}} \\
=&(n-m)X_{n+m-1}.
\end{aligned}$$

Let us assume now that G_n are endowed with Poisson-Lie structures $\{\ ,\ \}_n \colon C^\infty(G_n)\times C^\infty(G_n) \to C^\infty(G_n)$. It is natural to require that the projection maps $\pi_{n+1,n}\colon G_{n+1} \to G_n$ are Poisson, i.e. $\pi^*_{n+1,n}(\{f,g\}_n) = \{\pi^*_{n+1,n}(f), \pi^*_{n+1,n}(g)\}_{n+1}$, where, as usual, $(\pi^*_{n+1,n}(f))(x) \stackrel{def}{=} f(\pi_{n+1,n}(x))$ for every $f \in C^\infty(G_n)$ and $x \in G_{n+1}$. From the definition of $\pi^*_{n+1,n}$ it is clear that $\iota_{n,n+1} = \pi^*_{n+1,n}$. Therefore we have

$$\iota_{n,n+1}(\{f,g\}_n) = \{\iota_{n,n+1}(f), \iota_{n,n+1}(g)\}_{n+1}.$$

In other words $\{\ ,\ \}_{n+1}|_{G_n} = \{\ ,\ \}_n$.

On the other hand let $\omega \in TG_\infty \wedge TG_\infty$ be a smooth bivector field and let us define a Poisson-Lie structure on G_∞ as a map $\{\ ,\ \}_\infty \colon C^\infty(G_\infty)\times C^\infty(G_\infty) \to C^\infty(G_\infty)$ defined by $\{f,g\} =< \omega, df \wedge dg >$ for any $f,g \in C^\infty(G_\infty)$, where $<,>$ denotes the pairing between the tangent and cotangent spaces at each point. Then it is also natural to require that the maps $\pi_{\infty,n}$ are Poisson. That is, we want the condition

$$\iota_{n,\infty}(\{f,g\}_n) = \{\iota_{n,\infty}(f), \iota_{n,\infty}(g)\}_\infty$$

to be satisfied, where $f,g \in C^\infty(G_n)$. Then we could define $(G_\infty, \{\ ,\ \}_\infty, (\pi_{\infty,n})_{n\in\mathbb{N}})$ to be the inverse limit of the family of Poisson-Lie groups and maps $(G_n, \{\ ,\ \}_n, \pi_{n+1,n})_{n\in\mathbb{N}}$. Later it will become clear that for the Poisson-Lie groups studied in this text these conditions are automatically satisfied.

Are there any Poisson-Lie structures on G_∞? If such structures exist, could we classify them?

Also, since for any finite n there is a one-to-one correspondence between the Poisson-Lie structures on G_n (if they exist) and the Lie-bialgebra structures on the Lie algebra $\mathcal{G}_n$ of G_n, one is led to enquire if there are any Lie-bialgebra structures on the Lie algebra $\mathcal{G}_\infty$ of G_∞.

All these questions shall be fully answered.

Let us turn our attention to the Lie-algebraic picture first. Our first goal is to find all bialgebra structures on $\mathcal{G}_\infty$. The next chapter is devoted to this.

CHAPTER III

BIALGEBRA STRUCTURES ON $\mathcal{G}_\infty$

In this chapter we find all 1-cocycles on the Lie algebra $\mathcal{G}_\infty$ of the group G_∞. All of them turn out to be coboundaries, and they are all explicitly enumerated.

Let $\{e_i\}_{i\geq 1}$ be the previously fixed basis of the Lie algebra $\mathcal{G}_\infty$ of G_∞. The Lie algebra $\mathcal{G}_\infty$ has the form (2.23)

$$[e_k, e_m] = (k-m)e_{k+m-1}; \qquad k, m \in \mathbb{N}. \tag{3.1}$$

The cocycle condition reads (cf. (ii), Def. 1.3, Chap. I)

$$\alpha([e_k, e_m]) = e_k.\alpha(e_m) - e_m.\alpha(e_k). \tag{3.2}$$

Using (3.1), we rewrite (3.2) as

$$(k-m)\alpha(e_{k+m-1}) = e_k.\alpha(e_m) - e_m.\alpha(e_k). \tag{3.3}$$

Let $n = k+m-1$, so that $k = n-m+1 \quad (1 \leq m \leq n)$. For fixed n, (3.3) is equivalent to

$$[n-(2m-1)]\,\alpha(e_n) = e_{n-(m-1)}.\alpha(e_m) - e_m.\alpha(e_{n-(m-1)}).$$

Writing $\alpha(e_n) = \sum_{i,j=1}^{\infty} \alpha_{ij}^n e_i \wedge e_j$ we obtain

$$[n-(2m-1)]\,\alpha_{ij}^n e_i \wedge e_j = \alpha_{ij}^m\,[e_{n-m+1}, e_i] \wedge e_j + \alpha_{ij}^m e_i \wedge [e_{n-m+1}, e_j] -$$

$$- \alpha_{ij}^{n-m+1}\,[e_m, e_i] \wedge e_i - \alpha_{ij}^{n-m+1} e_i \wedge [e_m, e_j] =$$

$$= (n-m+1-i)\alpha_{ij}^{m} e_{i+n-m} \wedge e_j + (n-m+1-j)\alpha_{ij}^{m} e_i \wedge e_{j+n-m} -$$

$$-(m-i)\alpha_{ij}^{n-m+1} e_{i+m-1} \wedge e_j - (m-j)\alpha_{ij}^{n-m+1} e_i \wedge e_{j+m-1}. \tag{3.4}$$

We rewrite (3.4) (after shifting indices whenever necessary)

(3.5)

$$[n-(2m-1)]\,\alpha_{ij}^{n} = [2(n-m)+1-i]\,\alpha_{i-n+m,j}^{m} + [2(n-m)+1-j]\,\alpha_{i,j-n+m}^{m} -$$

$$-(2m-i-1)\alpha_{i-m+1,j}^{n-m+1} - (2m-j-1)\alpha_{i,j-m+1}^{n-m+1}.$$

This is the cocycle condition in its most general form in terms of α_{ij}^{n}'s.

Remark. The infinite sums $\alpha(e_n) = \sum_{i,j=1}^{\infty} \alpha_{ij}^{n} e_i \wedge e_j$ are considered as elements of the completed tensor product $\mathcal{G}_\infty \widehat{\otimes} \mathcal{G}_\infty = \prod_{n=1}^{\infty} \left(\bigoplus_{i+j=n} \mathcal{G}_i \otimes \mathcal{G}_j \right)$, where each $\mathcal{G}_i$ is a one-dimensional subspace of $\mathcal{G}_\infty$ spanned by e_i [Di,ZS]. In what follows we always look for solutions of (3.2) as elements of $\mathcal{G}_\infty \widehat{\otimes} \mathcal{G}_\infty$.

Theorem 3.1. *All solutions of (3.5) are described as follows (here $\alpha_{ij} \equiv \alpha_{ij}^{1}$). The coalgebra structure constants $\alpha_{ij}^{1} \equiv \alpha_{ij}$ are arbitrary, for all $i, j \geq 1$, while*

(i) $\alpha_{1n}^{n} = 0$, $\forall\, n \geq 2$;

(ii) $\alpha_{sn}^{n} = \frac{(n-1)}{(s-1)} \alpha_{1s}$, $\forall\, n \geq 3$ *and* $2 \leq s \leq n-1$;

(iii) $\alpha_{ij}^{n} = 0$, $\forall\, n \geq 3$ *and* $1 \leq i < j \leq n-1$;

(iv) $\alpha_{ij}^{n} = \frac{(2n-1-j)}{(n-i-j+1)} \alpha_{i,j-n+1}$ $\left(\alpha_{i,2n-1}^{n} = 0, \forall\, n \geq 2 \right)$, $\forall\, n \geq 2$, $1 \leq i \leq n-1$, $j \geq n+1$;

(v) $\alpha_{nj}^{n} = \frac{(n-1)}{(1-j)} \alpha_{1j} + \frac{(2n-1-j)}{(1-j)} \alpha_{n,j-n+1}$, $\forall\, n \geq 2$, *and* $j \geq n+1$;

(vi) $\alpha_{i,i+n-1}^{n} = 0$, $\forall\, n \geq 2$ *and* $2 \leq i \leq n-1$;

(vii) $\alpha_{ij}^{n} = \frac{(2n-1-i)}{(n-i-j+1)} \alpha_{i-n+1,j} + \frac{(2n-1-j)}{(n-i-j+1)} \alpha_{i,j-n+1}$, $\forall\, n \geq 2$ *and* $n+1 \leq i < j$.

The proof is split into eight lemmas and uses the symmetry of (3.5) together with inductive arguments.

Lemma 3.2. *(a)* $\alpha_{1n}^{n} = \alpha_{12}^{2}(n-3) + \alpha_{13}^{3}$,
(b) $\alpha_{1n}^{n} = \alpha_{1k}^{k} + \alpha_{1,n-(k-1)}^{n-(k-1)}$, $2 \le k \le \left[\frac{n+1}{2}\right]$, $\forall\, n > 3$.

Proof. (a) From (3.5) with shifting of indices whenever necessary we obtain

$$[n-(2m-1)]\,\alpha_{1n}^{n} = 2(n-m)\alpha_{m-n+1,n}^{m} + [n-(2m-1)]\,\alpha_{1m}^{m} -$$

$$-2(m-1)\alpha_{2-m,n}^{n-m+1} + [n-(2m-1)]\,\alpha_{1,n-m+1}^{n-m+1} \quad (1 \le m \le n). \tag{3.6}$$

Note that (3.6) is invariant under the transformation $m \to n-m+1$. Thus, it is enough to study only the cases when $1 \le m \le \left[\frac{n+1}{2}\right]$.

(i) Case $m = 1$, leads to identity for all n;
(ii)Case $m = 2$. From (3.6), we have

$$(n-3)\alpha_{1n}^{n} = 2(n-2)\alpha_{3-n,n}^{2} + (n-3)\alpha_{12}^{2} + (n-3)\alpha_{1,n-1}^{n-1}. \tag{3.7}$$

(ii.1) $n = 3$ gives an identity;
(ii.2) $n > 3$ leads to

$$\alpha_{1n}^{n} = \alpha_{12}^{2} + \alpha_{1,n-1}^{n-1}. \tag{3.8}$$

Therefore by induction $\alpha_{1n}^{n} = \alpha_{13}^{3} + (n-3)\alpha_{12}^{2}$, for $n > 3$.

(b) In general, $\forall\, n > 3$ and for $2 \le m \le \left[\frac{n+1}{2}\right]$ we have $m-n+1 \le 0$, and $2-m \le 0$, so $\alpha_{2-m,n}^{n-m+1} = 0 = \alpha_{m-n+1,n}^{m}$. Therefore

$$[n-(2m-1)]\,(\alpha_{1n}^{n} - \alpha_{1m}^{m} - \alpha_{1,n-(m-1)}^{n-(m-1)}) = 0.$$

If $n = 2k$, then $\alpha_{1n}^{n} = \alpha_{1m}^{m} + \alpha_{1,n-(m-1)}^{n-(m-1)}$ for $2 \le m \le \left[k + \frac{1}{2}\right] = k = \frac{n}{2}$.

If $n = 2k+1$, then $\alpha_{1n}^{n} = \alpha_{1m}^{m} + \alpha_{1,n-(m-1)}^{n-(m-1)}$ for $2 \le m \le k$.■

Lemma 3.3. $\alpha^n_{sn} = \frac{(n-1)}{(s-1)}\alpha_{1s}$ *for* $2 \le s \le n-1$, $\forall\, n \ge 3$, *where* $\alpha_{1s} \equiv \alpha^1_{1s}$.

Proof. From the cocycle condition (3.5) we have

$$[n-(2m-1)]\,\alpha^n_{sn} = [2(n-m))+1-s]\,\alpha^m_{s+m-n,n} + [n-(2m-1)]\,\alpha^m_{sm}$$

$$-(2m-s-1)\alpha^{n-m+1}_{s-m+1,n} + [n-(2m-1)]\,\alpha^{n-m+1}_{s,n-m+1} \quad (1 \le m \le n). \tag{3.9}$$

Setting $m = 1$ we get

$$(n-1)\alpha^n_{sn} = (2n-s-1)\alpha_{s+1-n,n} + (n-1)\alpha_{s1} -$$

$$-(1-s)\alpha^n_{sn} + (n-1)\alpha^n_{sn}. \tag{3.10}$$

Since $2 \le s \le n-1$ we have $s+1-n \le 0, \forall\, n \ge 3$, thus $\alpha_{s+1-n,n} = 0$, and then $(s-1)\alpha^n_{sn} = (n-1)\alpha_{1s}$. So, $\alpha^n_{sn} = \frac{(n-1)}{(s-1)}\alpha_{1s}$.■

Lemma 3.4. $\alpha^n_{sk} = 0$, $\forall\, n \ge 3$ *such that* $1 \le s < k \le n-1$, *and* $n \ne s+k-1$.

Proof. From the cocycle condition (3.5) again we have

$$[n-(2m-1)]\,\alpha^n_{sk} = [2(n-m)+1-s]\,\alpha^m_{s+m-n,k} +$$

$$+ [2(n-m)+1-k]\,\alpha^m_{s,m+k-n}$$

$$-(2m-s-1)\alpha^{n-m+1}_{s-m+1,k} - (2m-k-1)\alpha^{n-m+1}_{s,k-m+1} \tag{3.11}$$

for $1 \le s < k \le n-1$, and $1 \le m \le n$, and $n \ge 3$. Setting $m = 1$, we get

$$(n-1)\alpha^n_{sk} = (2n-1-s)\alpha_{s+1-n,k} + (2n-1-k)\alpha_{s,k+1-n} -$$

$$-(1-s)\alpha^n_{sk} - (1-k)\alpha^n_{sk}. \tag{3.12}$$

Therefore

$$(n-s-k+1)\alpha^n_{sk} = (2n-1-s)\alpha_{s+1-n,k} + (2n-1-k)\alpha_{s,k+1-n}.$$

Since $1 \leq s < k \leq n-1$ we have $s+1-n < 0$ and $k+1-n \leq 0$, so $\alpha_{s+1-n,k} = 0 = \alpha_{s,k+1-n}$.

Thus $(n-s-k+1)\alpha^n_{sk} = 0 \implies \alpha^n_{sk} = 0$ if $n \neq s+k-1$ and $1 \leq s \leq n-1$.■

The case $n = k+s-1$ has to be treated separately.

For given n, suppose now that the conditions $1 \leq s < k \leq n-1$ and $n = s+k-1$ are satisfied. Then, from $k = n-s+1$ and $k \leq n-1$ it follows that $2 \leq s \leq n-s$, therefore $2 \leq s \leq \left[\frac{n}{2}\right]$. Clearly, in what follows, the arguments will be given for $n \geq 4$.

Lemma 3.5. $\alpha^n_{2,n-1} = -(n-2)\alpha^2_{12} - \alpha^3_{13}$, $\forall\, n \geq 4$ *and* $\alpha^n_{s,n-s+1} = 0$, $\forall\, s \geq 3 \forall\, n \geq 2s$.

Proof. From (3.5) we have

$$[n-(2m-1)]\,\alpha^n_{s,n-s+1} = [2(n-m)+1-s]\,\alpha^m_{s+m-n,n-s+1} +$$
$$+(n-2m+s)\alpha^m_{s,m-s+1} -$$
$$-(2m-s-1)\alpha^{n-m+1}_{s-m+1,n-s+1} - (2m-n+s-2)\alpha^{n-m+1}_{s,n-m-s+2} \tag{3.13}$$

Setting $m = 2$ we get

$$(n-3)\alpha^n_{s,n-s+1} = (2n-3-s)\alpha^2_{s+2-n,n-s+1} + (n+s-4)\alpha^2_{s,3-s} -$$
$$-(3-s)\alpha^{n-1}_{s-1,n-s+1} - (2-n+s)\alpha^{n-1}_{s,n-s}.$$

Since $2 \leq s \leq \left[\frac{n}{2}\right]$ and $n \geq 4$ we have $s+2-n \leq 0$, so $\alpha^2_{s+2-n,n-s+1} = 0$. Therefore

$$(n-3)\alpha^n_{s,n-s+1} = (n+s-4)\alpha^2_{s,3-s} - (3-s)\alpha^{n-1}_{s-1,n-s+1} - (2-n+s)\alpha^{n-1}_{s,n-s}. \tag{3.14}$$

We will break the remaining part of the proof of Lemma 3.5 into three steps.

Claim 1. $\alpha^n_{2,n-1} = -(n-2)\alpha^2_{12} - \alpha^3_{13}$, $\forall\, n \geq 4$.

Proof. Setting $s = 2$ in (3.13) we obtain

$$(n-3)\alpha^n_{2,n-1} = (n-2)\alpha^2_{21} - \alpha^{n-1}_{1,n-1} + (n-4)\alpha^{n-1}_{2,n-2} =$$

$$(3.15) \qquad = -(n-2)\alpha_{12}^2 - (n-4)\alpha_{12}^2 - \alpha_{13}^3 + (n-4)\alpha_{2,n-2}^{n-1} \quad \text{[using Lemma 3.2]}$$

$$= -(2n-6)\alpha_{12}^2 - \alpha_{13}^3 + (n-4)\alpha_{2,n-2}^{n-1}.$$

For $n = 4$, $\alpha_{23}^4 = -2\alpha_{12}^2 - \alpha_{13}^3$, the statement is true. Assume now, that it is true for some $n > 4$, i.e. $\alpha_{2,n-1}^n = -(n-2)\alpha_{12}^2 - \alpha_{13}^3$, and show that it is true for $n+1$.

From (3.15) we have

$$(n-2)\alpha_{2,n}^{n+1} = -(n-1)\alpha_{12}^2 - \alpha_{1n}^n + (n-3)\alpha_{2,n-1}^n$$

$$= -(n-1)\alpha_{12}^2 - (n-3)\alpha_{12}^2 - \alpha_{13}^3 - (n-3)(n-2)\alpha_{12}^2 - (n-3)\alpha_{13}^3$$

$$= -\left[(n-1) + (n-3) + (n-3)(n-2)\right]\alpha_{12}^2 - (n-2)\alpha_{13}^3$$

$$= -(n-1)(n-2)\alpha_{12}^2 - (n-2)\alpha_{13}^3$$

$$\Longrightarrow \alpha_{2,n}^{n+1} = -(n-1)\alpha_{12}^2 - \alpha_{13}^3.$$

Claim 2. $\alpha_{3,n-2}^n = 0$, $\forall\, n \geq 6$.

Proof. From (3.14) with $s = 3$ we have

$$(n-3)\alpha_{3,n-2}^n = (n-5)\alpha_{3,n-3}^{n-1} \quad \Longrightarrow \quad \alpha_{3,n-2}^n = \frac{(n-5)}{(n-3)}\alpha_{3,n-3}^{n-1}.$$

Since $\alpha_{34}^6 = 0$, we have $\alpha_{3,n-2}^n = 0\ \forall\, n \geq 6$.

Claim 3. *For* $3 \leq s \leq \left[\frac{n}{2}\right]$, *if* $\alpha_{s-1,n-(s-1)+1}^n = 0$, $\forall\, n \geq 2(s-1)$, *then* $\alpha_{s,n-s+1}^n = 0$, $\forall\, n \geq 2s$. *Therefore, using Claim 2, we have* $\alpha_{s,n-s+1}^n = 0$, $\forall\, s$ *such that* $3 \leq s \leq \left[\frac{n}{2}\right]$ *and* $\forall\, n \geq 2s$.

Proof. Fix $s \geq 3$. Assume $n \geq 2s$. Then, $2(s-1) < 2s-1 \leq n-1$, and $\alpha_{s-1,n-1-(s-1)+1}^{n-1} = 0$ by assumption. Then from (3.14) it follows that

$$(n-3)\alpha_{s,n-s+1}^n = (n-2-s)\alpha_{s,n-s}^{n-1}, \qquad \forall\, n \geq 2s.$$

But for $n = 2s$ we have $(n-3)\alpha_{s,s+1}^{2s} = 0$, since $\alpha_{s,s}^{2s-1} = 0$. Therefore $\alpha_{s,n-s+1}^n = 0$, $\forall\, n > 2s$.■

Lemma 3.6. $\alpha^2_{12} = 0 = \alpha^3_{13}$.

Proof. Assume $n \geq 5$. Then, since $1 \leq m \leq \left[\frac{n+1}{2}\right]$ we could set $m = 3$ in (3.13) and get

$$(n-5)\alpha^n_{s,n-s+1} = (2n-5-s)\alpha^3_{s+3-n,n-s+1} + (n-6+s)\alpha^3_{s,4-s}$$

$$-(5-s)\alpha^{n-2}_{s-2,n-s+1} - (4-n+s)\alpha^{n-2}_{s,n-4-s} \tag{3.16}$$

Since $2 \leq s \leq \left[\frac{n}{2}\right]$ and $n \geq 5$, let us explore (3.16) for $s = 2$, obtaining

$$(n-5)\alpha^n_{2,n-1} = (2n-7)\alpha^3_{5-n,n-1} + (n-6)\alpha^{n-2}_{2,n-6}.$$

Since $n \geq 5$, $\alpha^3_{5-n,n-1} = 0$, thus $(n-5)\alpha^n_{2,n-1} = (n-6)\alpha^{n-2}_{2,n-6}$.

For $n = 6$ we get (by Lemma 3.5, Claim 1) $\alpha^6_{25} = -4\alpha^2_{12} - \alpha^3_{13} = 0 \implies \alpha^3_{13} = -4\alpha^2_{12}$.

For $n = 7$ we get (using Lemmas 3.4 and 3.5 (Claim 1)) $2\alpha^7_{26} = \alpha^5_{21} = -2(5\alpha^2_{12}+\alpha^3_{13}) = 0 \implies \alpha^3_{13} = -5\alpha^2_{12}$.

Therefore $\alpha^2_{12} = 0 = \alpha^3_{13}$.■

Corollary. $\alpha^n_{1n} = 0 = \alpha^n_{2,n-1}$, $\forall\ n \geq 2$.

Lemma 3.7. *For* $\forall\ n \geq 2$, $1 \leq i \leq n$, $j \geq n+1$ *we have*

(a) $\alpha^n_{nj} = \frac{(n-1)}{(1-j)}\alpha_{1j} + \frac{(2n-1-j)}{(1-j)}\alpha_{n,j-n+1}$;

(b) $\alpha^n_{ij} = \frac{(2n-1-j)}{(n-i-j+1)}\alpha_{i,j-n+1}$, *for* $1 \leq i \leq n-1$, ($\alpha^n_{i,2n-1} = 0$, $\forall\ n \geq 2$).

Proof. From (3.5) with $m = 1$ we have

$$(n-1)\alpha^n_{ij} = (2n-1-i)\alpha_{i-n+1,j} + (2n-1-j)\alpha_{i,j-n+1} +$$

$$+(i-1)\alpha^n_{ij} + (j-1)\alpha^n_{ij}, \tag{3.17}$$

which is equivalent to

$$(n-i-j+1)\alpha^n_{ij} = (2n-1-i)\alpha_{i-n+1,j} + (2n-1-j)\alpha_{i,j-n+1}. \tag{3.18}$$

(a) Set $i = n$. Then

$$(1-j)\alpha^n_{nj} = (n-1)\alpha_{1j} + (2n-1-j)\alpha_{n,j-n+1}.$$

(b) Now assume that $1 \leq i \leq n-1$. Then $i-n+1 \leq 0$, so $\alpha_{i-n+1,j} = 0$. Therefore from (2.18) we obtain

$$(n-i-j+1)\alpha^n_{ij} = (2n-1-j)\alpha_{i,j-n+1}.$$

Thus, $\alpha^n_{ij} = \frac{(2n-1-j)}{(n-i-j+1)}\alpha_{i,j-n+1}$. Clearly, if $j = 2n-1$ then $\alpha^n_{i,2n-1} = 0, \forall\, n \geq 2$.■

Lemma 3.8. $\alpha^n_{ij} = \frac{(2n-1-i)}{(n-i-j+1)}\alpha_{i-n+1,j} + \frac{(2n-1-j)}{(n-i-j+1)}\alpha_{i,j-n+1}$, $\forall\, n \geq 2$ *and* $n+1 \leq i < j$.

Proof. Since $n+1 \leq i < j \implies i-n+1 \geq 2$ and $j-n+1 > 2$. Also $n-i-j+1 \neq 0$. From (3.18) the lemma follows.■

In order to finish the proof of Theorem 3.1 we have to check that the so obtained solution satisfies identically the cocycle equation. This we are going to show by using the following important observation.

Lemma 3.9. *The general solution of the cocycle equation given in Theorem 3.1 is a coboundary.*

Proof. Let us introduce $\lambda_{ij} := \frac{1}{(2-i-j)}\alpha_{ij}$ for $1 \leq i \neq j$. Let $\alpha^0 = \lambda_{ij} e_i \wedge e_j$ be a 0-cochain. Then the coboundary 1-cochain obtained from it will be

$$\begin{aligned}
\alpha(e_n) &= e_n.\alpha^0 \\
&= \lambda_{ij}([e_n, e_i] \wedge e_j + e_i \wedge [e_n, e_j]) \\
&= \lambda_{ij}(C^{ni}_k e_k \wedge e_j + C^{nj}_k e_i \wedge e_k) \\
&= \lambda_{lj} C^{nl}_i e_i \wedge e_j + \lambda_{il} C^{nl}_j e_i \wedge e_j \\
&= (\lambda_{lj} C^{nl}_i + \lambda_{il} C^{nl}_j) e_i \wedge e_j
\end{aligned}$$

Thus

$$\alpha^n_{ij} = \lambda_{lj} C^{nl}_i + \lambda_{il} C^{nl}_j,$$

where C^{ij}_k are the structure constants of the Lie algebra. For $\mathcal{G}_\infty$ we have $C^{ij}_k = (i-j)\delta^{i+j-1}_k$. Therefore

$$\alpha_{ij}^n = (2n-i-1)\lambda_{i-n+1,j} + (2n-j-1)\lambda_{i,j-n+1}.$$

Now, let us investigate the above formula for the different ranges of (i,j), as described in the statement of Theorem 3.1. In the arguments below we implicitly assume that $\lambda_{ij} = 0$ ($\alpha_{ij} = 0$) whenever $i < 1$ or $j < 1$.

(i) Assume that $n \geq 2$. Then

$$\alpha_{1n}^n = (n-1)\lambda_{-n,n} + (n-1)\lambda_{11} = 0.$$

(ii) Assume that $n \geq 3$ and $2 \leq s \leq n-1$. Then

$$\alpha_{sn}^n = (2n-s-1)\lambda_{s-n+1,n} + (n-1)\lambda_{s1} = -(n-1)\lambda_{1s} = \frac{(n-1)}{(s-1)}\alpha_{1s}.$$

(iii) Assume that $n \geq 3$ and $1 \leq i < j \leq n-1$. Then

$$\alpha_{ij}^n = 0,$$

since $i-n+1 < 0$ and $j-n+1 \leq 0$.

(iv) Assume that $n \geq 2$, $1 \leq i \leq n-1$, and $j \geq n+1$. Then

$$\begin{aligned}\alpha_{ij}^n = (2n-i-1)\lambda_{i-n+1,j} + (2n-j-1)\lambda_{i,j-n+1} &= (2n-j-1)\lambda_{i,j-n+1} \\ &= \frac{(2n-j-1)}{(n-i-j+1)}\alpha_{i,j-n+1}.\end{aligned}$$

(v) Assume that $n \geq 2$ and $j \geq n+1$. Then

$$\begin{aligned}\alpha_{nj}^n &= (n-1)\lambda_{1j} + (2n-j-1)\lambda_{n,j-n+1} \\ &= \frac{(n-1)}{(1-j)}\alpha_{1j} + \frac{(2n-j-1)}{(1-j)}\alpha_{n,j-n+1}.\end{aligned}$$

(vi) Assume that $n \geq 2$ and $2 \leq i \leq n-1$. Then

$$\alpha_{i,i+n-1}^n = (2n-i-1)\lambda_{i-n+1,i+n-1} + (n-i)\lambda_{i,i} = 0,$$

since $i-n+1 \leq 0$.

(vii) Assume that $n \geq 2$ and $n+1 \leq i < j$. Then

$$\begin{aligned}\alpha_{ij}^{n} &= (2n-i-1)\lambda_{i-n+1,j} + (2n-j-1)\lambda_{i,j-n+1} \\ &= \frac{(2n-i-1)}{(n-i-j+1)}\alpha_{i-n+1,j} + \frac{(2n-j-1)}{(n-i-j+1)}\alpha_{i,j-n+1}.\end{aligned}$$

This shows that for $\mathcal{G}_\infty$ all 1-cocycles are coboundaries.■

CHAPTER IV

POISSON-LIE STRUCTURES ON G_∞

In this chapter we study Poisson-Lie structures on the group G_∞. It turns out that there exists a large class of such structures, which can be described explicitly. In the next chapter we prove that in fact this class exhausts all Poisson-Lie structures on G_∞.

We recall the definition of a Poisson-Lie structure on the group G_∞ given in Chapter II. It is defined as a skew-symmetric map $\{\ ,\ \}\colon C^\infty(G_\infty) \times C^\infty(G_\infty) \to C^\infty(G_\infty)$ which is multiplicative, is a derivation in both arguments, and satisfies the Jacobi identity. The derivation property implies that there is a bi-vector field $\omega \in \wedge^2 TG_\infty$ such that $\{f,g\} =< \omega, df \wedge dg >$ for any $f,g \in C^\infty(G_\infty)$. In local coordinates the bi-vector field $\omega_x \in \wedge^2 T_x G_\infty$ is defined as

$$\omega_x = \omega_{ij}(x)\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j},$$

where $\omega_{ij} \in C^\infty(G_\infty)$ are smooth function on G_∞. Then for every $f,g \in C^\infty(G_\infty)$ we have

$$\{f,g\}(x) = \omega_{ij}(x)\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}.$$

Note that on the right hand side we have in effect a finite sum since the functions f and g depend only upon finite number of arguments. Similarly, the 1-cocycle equation (1.6) for ω_{ij} is given by

$$\omega_{ij}(xy) = \omega_{kl}(x)\frac{\partial \xi_i}{\partial x_k}\frac{\partial \xi_j}{\partial x_l} + \omega_{kl}(y)\frac{\partial \xi_i}{\partial y_k}\frac{\partial \xi_j}{\partial y_l}. \tag{4.0}$$

Here again the sums on the right-hand-side are finite, since for every $n \in \mathbb{N}$ we have $\xi_n = \xi_n(x_1, \dots, x_n; y_1, \dots, y_n)$. The same is true for the sums in the Jacobi identity (1.1) for the functions ω_{ij}.

Let us define

$$(*) \qquad \Omega(u,v;\mathcal{X}) := \sum_{i,j=1}^{\infty} \omega_{ij}(x)u^i v^j, \quad \text{for } \mathcal{X} = \sum_{i=1}^{\infty} x_i u^i.$$

Thus $\Omega(u,v;\mathcal{X})$ is a generating series for the brackets $\omega_{ij}(x)$.

Lemma 4.0. *In terms of Ω the cocycle condition (4.0) has the form*

(4.1)
$$\Omega(u,v;\mathcal{Z}) = \Omega(\mathcal{Y}(u),\mathcal{Y}(v);\mathcal{X}) + \Omega(u,v;\mathcal{Y})\mathcal{X}'(\mathcal{Y}(u))\mathcal{X}'(\mathcal{Y}(v)), \qquad \mathcal{Z}(u) = \mathcal{X}(\mathcal{Y}(u)).$$

Proof. Recall that $\mathcal{X}(u) = \sum_{i=1}^{\infty} x_i u^i$ (cf. Chap. II) and $\mathcal{Z}(u) = \mathcal{X}(\mathcal{Y}(u)) = \sum_{i=1}^{\infty} x_i[\mathcal{Y}(u)]^i = \sum_{i=1}^{\infty} \xi_i u^i$, where $\xi_i = (xy)_i$. From the last formula we obtain that

$$\frac{\partial \mathcal{Z}}{\partial x_k} = [\mathcal{Y}(u)]^k \quad \left(= \sum_{i=1}^{\infty} \frac{\partial \xi_i}{\partial x_k} u^i\right),$$

$$\begin{aligned}\frac{\partial \mathcal{Z}}{\partial y_k} &= \sum_{i=1}^{\infty} i x_i u^k [\mathcal{Y}(u)]^{i-1} \\ &= u^k \sum_{i=1}^{\infty} i x_i [\mathcal{Y}(u)]^{i-1} \\ &= u^k \mathcal{X}'(\mathcal{Y}(u)) \quad \left(= \sum_{i=1}^{\infty} \frac{\partial \xi_i}{\partial y_k} u^i\right).\end{aligned}$$

If we multiply both sides of equation (4.0) by $u^i v^j$ and sum over i and j we obtain

$$\begin{aligned}\sum_{i,j=1}^{\infty} \omega_{ij}(\xi)u^i v^j &= \sum_{k,l=1}^{\infty} \omega_{kl}(x) \sum_{i=1}^{\infty} \frac{\partial \xi_i}{\partial x_k} u^i \sum_{j=1}^{\infty} \frac{\partial \xi_j}{\partial x_l} v^j + \sum_{k,l=1}^{\infty} \omega_{kl}(y) \sum_{i=1}^{\infty} \frac{\partial \xi_i}{\partial y_k} u^i \sum_{j=1}^{\infty} \frac{\partial \xi_j}{\partial y_l} v^j \\ &= \sum_{k,l=1}^{\infty} \omega_{kl}(x)[\mathcal{Y}(u)]^k [\mathcal{Y}(v)]^l + \mathcal{X}'(\mathcal{Y}(u))\mathcal{X}'(\mathcal{Y}(v)) \sum_{k,l=1}^{\infty} \omega_{kl}(y) u^k v^l.\end{aligned}$$

Now, using the definition of Ω we finally obtain that

$$\Omega(u,v;\mathcal{Z}) = \Omega(\mathcal{Y}(u),\mathcal{Y}(v);\mathcal{X}) + \Omega(u,v;\mathcal{Y})\mathcal{X}'(\mathcal{Y}(u))\mathcal{X}'(\mathcal{Y}(v)).$$

Notice also that both sides of the above equation are divisible by uv.■

Equation (4.1) has a large class of solutions of the type (*). Namely, we have the following theorem.

Theorem 4.1. *For any function $\varphi = \varphi(u,v)$ with the properties*
(i) $\varphi(u,v)$ is divisible by u and v;
(ii)$\varphi(u,v) = -\varphi(v,u)$,
we have the following solution of (4.1):

$$\Omega(u,v;\mathcal{X}) = \varphi(u,v)\mathcal{X}'(u)\mathcal{X}'(v) - \varphi(\mathcal{X}(u),\mathcal{X}(v)). \tag{4.2}$$

Proof. The proof consists of a simple check. The left hand side of the equation (4.1) in terms of (4.2) reads

$$\begin{aligned}\Omega(u,v;\mathcal{Z}) &= \varphi(u,v)\mathcal{Z}'(u)\mathcal{Z}'(v) - \varphi(\mathcal{Z}(u),\mathcal{Z}(v))\\ &= \varphi(u,v)\mathcal{X}'(\mathcal{Y}(u))\mathcal{Y}'(u)\mathcal{X}'(\mathcal{Y}(v))\mathcal{Y}'(v) - \varphi(\mathcal{Z}(u),\mathcal{Z}(v)).\end{aligned}$$

The right hand side of (4.1) gives

$$\begin{aligned}&\Omega(\mathcal{Y}(u),\mathcal{Y}(v);\mathcal{X}) + \Omega(u,v;\mathcal{Y})\mathcal{X}'(\mathcal{Y}(u))\mathcal{X}'(\mathcal{Y}(v)) =\\ &\quad + \varphi(\mathcal{Y}(u),\mathcal{Y}(v))\mathcal{X}'(\mathcal{Y}(u))\mathcal{X}'(\mathcal{Y}(v)) - \varphi(\mathcal{Z}(u),\mathcal{Z}(v)) +\\ &\quad + \varphi(u,v)\mathcal{X}'(\mathcal{Y}(u))\mathcal{Y}'(u)\mathcal{X}'(\mathcal{Y}(v))\mathcal{Y}'(v) - \varphi(\mathcal{Y}(u),\mathcal{Y}(v))\mathcal{X}'(\mathcal{Y}(u))\mathcal{X}'(\mathcal{Y}(v)).\end{aligned}$$

Compairing both sides we obtain an identity.

Condition (ii) is equivalent to $\Omega(u,v;\mathcal{X}) = -\Omega(v,u;\mathcal{X})$ which on the other hand is equivalent to the skew-symmetry of the ω_{ij}'s.

The condition (i) is needed since as noticed above $\Omega(u,v;\mathcal{X})$ is divisible by uv. This requires that the r.h.s. of (4.2) is divisible by uv. From definition of $\mathcal{X}(u)$ it is clear that $\mathcal{X}'(u)\mathcal{X}'(v)$ is not divisible by uv. It begins with a term $x_1^2 + 2x_1x_2(u+v) + \dots$. Suppose that $\varphi(u,v)$ is not divisible by uv. Then $\varphi(\mathcal{X}(u),\mathcal{X}(v))$ is also not divisible by uv, and so is the difference $\varphi(u,v)\mathcal{Z}'(u)\mathcal{Z}'(v) - \varphi(\mathcal{Z}(u),\mathcal{Z}(v))$. The last becomes clear if we consider the formal expansions of $\varphi(u,v)$ and $\varphi(\mathcal{X}(u),\mathcal{X}(v))$ around $(0,0)$:

$$\begin{aligned}\varphi(u,v) &= \varphi_{10}u + \varphi_{01}v + \varphi_{12}uv^2 + \varphi_{21}u^2v + \dots\\ &= \varphi_{01}(v-u) + \varphi_{12}uv(v-u) + \dots,\end{aligned}$$

$$\begin{aligned}\varphi(\mathcal{X}(u),\mathcal{X}(v)) &= \varphi_{10}\mathcal{X}(u) + \varphi_{01}\mathcal{X}(v) + \varphi_{12}\mathcal{X}(u)\mathcal{X}(v)^2 + \varphi_{21}\mathcal{X}(u)^2\mathcal{X}(v) + \dots\\ &= \varphi_{10}x_1u + \varphi_{01}x_1v + \varphi_{12}x_1^3uv^2 + \varphi_{21}x_1^3u^2v + \dots\\ &= \varphi_{01}x_1(v-u) + \varphi_{12}x_1^3uv(v-u) + \dots.\end{aligned}$$

Also we have

$$\begin{aligned}\varphi(u,v)\mathcal{X}'(u)\mathcal{X}'(v) &= [\varphi_{01}(v-u)+\varphi_{12}uv(v-u)+\dots]\times \\ &\qquad \times [x_1^2+2x_1x_2(u+v)+3x_1x_3(u^2+v^2)+\dots] \\ &= \varphi_{01}x_1^2(v-u)+2\varphi_{01}x_1x_2(v^2-u^2)+\varphi_{12}x_1^2uv(v-u)+\dots .\end{aligned}$$

Here we have used that $\varphi_{ij}=-\varphi_{ji}$ which follows from $\varphi(u,v)=-\varphi(v,u)$. Clearly the r.h.s. of (4.2) would be divisible by uv only if $\mathcal{X}(u)=u$, i.e. when $\mathcal{X}=e$. But then $\Omega(u,v;e)=0$. Therefore we conclude that $\varphi(u,v)$ must be divisible by uv. From this it follows that this is also true for $\varphi(\mathcal{X}(u),\mathcal{X}(v))$, and thus for the r.h.s. of (4.2).■

Next, we would like to find out for which classes of φ's the Jacobi identity is satisfied. This will be an important step in the solution of the problem of classifying all possible Lie-Poisson structures on G_∞.

Let $\mathcal{U}=\{u_i\}_{i\in\mathbb{Z}_+}$ be a countable set of indeterminates. Consider the ring of formal power series $C^\infty(G_\infty)[[\mathcal{U}]]$ in $\mathcal{U}$ over the ring $C^\infty(G_\infty)$. We shall use the following definition

$$\{\mathcal{X}(u),\mathcal{X}(v)\}\equiv\Omega(u,v;\mathcal{X})=\sum_{i,j=1}^{\infty}\{x_i,x_j\}u^iv^j,$$

where $u\equiv u_i$ and $v\equiv u_j$ for any $u_i,u_j\in\mathcal{U}$. The above equality defines a map $\{\ ,\ \}\colon C^\infty(G_\infty)[[\mathcal{U}]]\times C^\infty(G_\infty)[[\mathcal{U}]]\to C^\infty(G_\infty)[[\mathcal{U}]]$ induced by the map $\{\ ,\ \}\colon C^\infty(G_\infty)\times C^\infty(G_\infty)\to C^\infty(G_\infty)$. Then the Jacobi identity in terms of generating series reads

$$\{\mathcal{X}(w),\{\mathcal{X}(u),\mathcal{X}(v)\}\}+c.p.=0, \tag{4.3}$$

for any $u,v,w\in\mathcal{U}$. Indeed,

$$\begin{aligned}\{\mathcal{X}(w),\{\mathcal{X}(u),\mathcal{X}(v)\}\}+c.p. &= \sum_{i,j,k=1}^{\infty}\{x_i,\{x_j,x_k\}\}u^iv^jw^k+c.p. \\ &= \sum_{i,j,k=1}^{\infty}\Big[\{x_i,\{x_j,x_k\}\}+\mathrm{cycl}(i,j,k)\Big]u^iv^jw^k \\ &= \sum_{i,j,k=1}^{\infty}\Big[\{x_i,\omega_{jk}\}+\mathrm{cycl}(i,j,k)\Big]u^iv^jw^k \\ &= \sum_{i,j,k,l=1}^{\infty}\Big[\{x_i,x_l\}\frac{\partial\omega_{jk}}{\partial x_l}+\mathrm{cycl}(i,j,k)\Big]u^iv^jw^k \\ &= \sum_{i,j,k,l=1}^{\infty}\Big[\omega_{il}\frac{\partial\omega_{jk}}{\partial x_l}+\mathrm{cycl}(i,j,k)\Big]u^iv^jw^k,\end{aligned}$$

where we used the derivation property of $\{\ ,\ \}$ and the fact that $\omega_{ij}(x) = \{x_i, x_j\}$. On the other hand we have

$$\{\mathcal{X}(w), \{\mathcal{X}(u), \mathcal{X}(v)\}\} = \{\mathcal{X}(w), \varphi(u,v)\mathcal{X}'(u)\mathcal{X}'(v) - \varphi(\mathcal{X}(u), \mathcal{X}(v)\}$$

$$= \varphi(u,v)\Big[\{\mathcal{X}(w), \mathcal{X}'(u)\}\mathcal{X}'(v) + \{\mathcal{X}(w), \mathcal{X}'(v)\}\mathcal{X}'(u)\Big] -$$

$$- \partial_1\varphi(\mathcal{X}(u), \mathcal{X}(v))\{\mathcal{X}(w), \mathcal{X}(u)\} - \partial_2\varphi(\mathcal{X}(u), \mathcal{X}(v))\{\mathcal{X}(w), \mathcal{X}(v)\},$$

where ∂_1 denotes the derivative with respect to the first argument and ∂_2 is the derivative with respect to the second argument. Also

$$\begin{aligned}\{\mathcal{X}(w), \mathcal{X}'(u)\} =& \partial_u\{\mathcal{X}(w), \mathcal{X}(u)\}\\ =& \partial_u\varphi(w,u)\mathcal{X}'(w)\mathcal{X}'(u) +\\ &+ \varphi(w,u)\mathcal{X}'(w)\mathcal{X}''(u) + \partial_2\varphi(\mathcal{X}(w), \mathcal{X}(u))\mathcal{X}'(u),\end{aligned}$$

and we have similar formulae when considering the remaining two terms in (4.3) with w, u, v permuted.

Lemma 4.2. *The solution (4.2) satisfies the Jacobi identity (4.3) iff $\varphi(u,v)$ satisfies the following functional (partial) differential equation*

$$\varphi(u,v)[\partial_u\varphi(w,u) + \partial_v\varphi(w,v)] + c.p. = 0. \tag{4.4}$$

Proof. After substituting (4.2) into (4.3), using the formulae derived above, and collecting terms we obtain

$$\begin{aligned}(*)\quad &\Big(\varphi(u,v)\,[\partial_u\varphi(w,v) + \partial_v\varphi(w,v)] + c.p.\Big)\mathcal{X}'(w)\mathcal{X}'(u)\mathcal{X}'(v) +\\ &+ \Big(\varphi(\mathcal{X}(v), \mathcal{X}(u))[\partial_2\varphi(\mathcal{X}(w), \mathcal{X}(v)) + \partial_2\varphi(\mathcal{X}(w), \mathcal{X}(u))] + c.p.\Big) = 0.\end{aligned}$$

Let us define the function $\Phi(w,u,v)$ by

$$\Phi(w,u,v) \equiv \varphi(u,v)[\partial_u\varphi(w,u) + \partial_v\varphi(w,v)] + c.p.$$

It is easily verified that the function $\Phi(w,u,v)$ is antisymmetric with respect to each pair of its arguments. For example $\Phi(w,u,v) = -\Phi(u,w,v)$. Then (*) becomes

$$\Phi(\mathcal{X}(w), \mathcal{X}(u), \mathcal{X}(v)) = \mathcal{X}'(w)\mathcal{X}'(u)\mathcal{X}'(v)\Phi(w,u,v).$$

This equation is satisfied for every function $\mathcal{X}(u)$. In particular, it is true for the function $\mathcal{X}(u) = \lambda u$, where $\lambda \neq 0$. In this case the above equation is equivalent to

$$\Phi(\lambda w, \lambda u, \lambda v) = \lambda^3 \Phi(w, u, v).$$

In other words Φ is homogeneous of degree 3, and satisfies the Euler equation

$$w\partial_w \Phi(w, u, v) + u\partial_u \Phi(w, u, v) + v\partial_v \Phi(w, u, v) = 3\Phi(w, u, v).$$

But the only homogeneous function $\Phi(w, u, v)$ of degree 3 which is also antisymmetric with respect to each pair of its arguments is $\Phi(w, u, v) = 0$. Therefore the statement of the Lemma follows.■

Theorem 4.3. *The map $g \mapsto g^{-1}$ is an anti-Poisson map.*

Proof. Let $\overline{\mathcal{X}}(u)$ denotes the inverse of $\mathcal{X}(u)$. Then we have the identities

$$\overline{\mathcal{X}}(\mathcal{X}(u)) = u, \qquad \text{and} \qquad \mathcal{X}(\overline{\mathcal{X}}(u)) = u,$$

as well as (following from them)

$$\overline{\mathcal{X}}'(\mathcal{X}(u))\mathcal{X}'(u) = 1, \qquad \text{and} \qquad \mathcal{X}'(\overline{\mathcal{X}}(u))\overline{\mathcal{X}}'(u) = 1.$$

On the other hand we have

$$\begin{aligned} 0 &= \{u, \mathcal{X}(v)\} \\ &= \{\overline{\mathcal{X}}(\mathcal{X}(u)), \mathcal{X}(v)\} \\ &= \{\overline{\mathcal{X}}(w), \mathcal{X}(v)\}|_{w=\mathcal{X}(u)} + \overline{\mathcal{X}}'(w)|_{w=\mathcal{X}(u)}\{\mathcal{X}(u), \mathcal{X}(v)\}. \end{aligned}$$

Therefore,

$$\{\mathcal{X}(v), \overline{\mathcal{X}}(w)\}|_{w=\mathcal{X}(u)} = \overline{\mathcal{X}}'(w)|_{w=\mathcal{X}(u)}\{\mathcal{X}(u), \mathcal{X}(v)\}. \tag{4.5}$$

Also, we have the following chain of identities

$$\begin{aligned} 0 &= \{v, \overline{\mathcal{X}}(w)\}|_{w=\mathcal{X}(u)} \\ &= \{\overline{\mathcal{X}}(\mathcal{X}(v)), \overline{\mathcal{X}}(w)\}|_{w=\mathcal{X}(u)} \\ &= \{\overline{\mathcal{X}}(z), \overline{\mathcal{X}}(w)\}|_{z=\mathcal{X}(v), w=\mathcal{X}(u)} + \overline{\mathcal{X}}'(z)|_{z=\mathcal{X}(v)}\{\mathcal{X}(v), \overline{\mathcal{X}}(w)\}|_{w=\mathcal{X}(u)}. \end{aligned}$$

Using (4.2) and (4.5), the last identity can be rewritten as

$$0 = \varphi(\mathcal{X}(v), \mathcal{X}(u))\overline{\mathcal{X}}'(\mathcal{X}(v))\overline{\mathcal{X}}'(\mathcal{X}(u)) - \varphi(v,u)$$

$$+ \overline{\mathcal{X}}'(\mathcal{X}(v))\overline{\mathcal{X}}'(\mathcal{X}(u))\left[\varphi(u,v)\mathcal{X}'(u)\mathcal{X}'(v) - \varphi(\mathcal{X}(u), \mathcal{X}(v))\right]$$

$$= \{\overline{\mathcal{X}}(z), \overline{\mathcal{X}}(w)\}|_{z=\mathcal{X}(v), w=\mathcal{X}(u)} + \varphi(u,v) - \overline{\mathcal{X}}'(w)\overline{\mathcal{X}}'(z)\varphi(w,z).$$

Thus,

$$\{\overline{\mathcal{X}}(w), \overline{\mathcal{X}}(z)\} = -\left[\overline{\mathcal{X}}'(w)\overline{\mathcal{X}}'(z)\varphi(w,z) - \varphi(\overline{\mathcal{X}}(w), \overline{\mathcal{X}}(z))\right],$$

and the proof is finished.■

Theorem 4.4. *For every $d \in \mathbb{Z}_+$, the function $\varphi(u,v) = uv(u^d - v^d)$ solves (4.4), thus giving rise to a family of Lie-Poisson structures on the group G_∞.*

Proof. Substituting $\varphi(u,v) = uv(u^d - v^d)$ into the equation (4.4) we obtain

$$[u^{d+1}v - uv^{d+1}][w^{d+1} - (d+1)wu^d + w^{d+1} - (d+1)wv^d] + c.p. \stackrel{?}{=} 0.$$

But this is an identity, since

$$2u^{d+1}w^{d+1}v - 2uv^{d+1}w^{d+1} + c.p. = 0,$$

$$(d+1)uwv^{2d+1} - (d+1)u^{2d+1}vw + c.p = 0,$$

and

$$(d+1)u^{d+1}v^{d+1}w - (d+1)wu^{d+1}v^{d+1} + c.p. = 0.■$$

Writing formula (4.2) in components, with $\varphi(u,v) = uv(u^d - v^d)$, we obtain

$$\omega_{ij}(x) = (i-d)jx_jx_{i-d} - i(j-d)x_ix_{j-d}$$

$$+ x_i \sum_{\sum_{k=1}^{d+1} s_k = j} x_{s_1} \dots x_{s_{d+1}} - x_j \sum_{\sum_{k=1}^{d+1} s_k = i} x_{s_1} \dots x_{s_{d+1}}.$$

These formulae describe a countable family of Poisson-Lie structures on G_∞, thus answering the question of existence of such.

In order to classify *all* Poisson-Lie structures on G_∞ one has, in particular, to classify all solutions of the functional (partial) differential equation (4.4). The main result of this section is formulated below.

Theorem 4.5. *For each $d \in \mathbb{N}$, there is a solution of (4.4) given by the function*

$$\varphi_d(u,v) = \frac{1}{\lambda_{1,d+1}}\Big[f_d(u)g_d(v) - f_d(v)g_d(u)\Big],$$

where the functions $f_d(u), g_d(u)$ are such that $f_d'(u)g_d(u) - f_d(u)g_d'(u) = -d\lambda_{1,d+1}f_d(u)$, where $\lambda_{1,d+1} \neq 0$ is an arbitrary parameter, and f_d has a zero of order $d+1$ at $u = 0$. The set of all solutions of (4.4) is described in this way.

First, we prove a helpful lemma.

Lemma 4.6. *For any two functions $f(u)$, $g(u)$ satisfying the relation $f'(u)g(u) - f(u)g'(u) = \alpha f(u) + \beta g(u)$, where α, β are arbitrary constants, (4.4) has a solution in the form $\varphi(u,v) = f(u)g(v) - f(v)g(u)$.*

Proof. After substituting $\varphi(u,v) = f(u)g(v) - f(v)g(u)$ into (4.4) and collecting terms we obtain

$$[f(u)g'(u) - f'(u)g(u)]\, f(w)g(v) - [f(u)g'(u) - f'(u)g(u)]\, f(v)g(w) + c.p. = 0.$$

Using the relation

$$f'(u)g(u) - f(u)g'(u) = \alpha f(u) + \beta g(u)$$

the above equality transforms to

$$-[\alpha f(u) + \beta g(u)]f(w)g(v) + [\alpha f(u) + \beta g(u)]f(v)g(w) + c.p. = 0.$$

The last equality is equivalent to

$$\alpha[f(u)f(v)g(w) - f(u)f(w)g(v) + c.p.] + \beta[f(v)g(u)g(w) - f(w)g(u)g(v) + c.p.] = 0.$$

But the expressions in the square brackets are identically zero as one can easily check. Thus we obtain an identity.■

In fact, we will show that all solutions of the functional differential equation (4.4) with the additional assumption that $\varphi(u,v)$ is divisible by uv are of the above form, with $\beta = 0$.

We will seek the general solution of (4.4) as a formal power series $\varphi(u,v) = \sum_{n,m=1}^{\infty} \lambda_{nm} u^n v^m$. Here, the antisymmetry of $\varphi(u,v)$ implies the antisymmetry of λ_{nm}, namely $\lambda_{nm} = -\lambda_{mn}$.

Substituting into (4.4) we obtain

$$\sum_{k,n,r} \sum_{s} s[\lambda_{k-s+1,n}\lambda_{rs} + \lambda_{k,n-s+1}\lambda_{rs} + \lambda_{n-s+1,r}\lambda_{ks} + \lambda_{n,r-s+1}\lambda_{ks} + \lambda_{r-s+1,k}\lambda_{ns} + \\ + \lambda_{r,k-s+1}\lambda_{ns}]u^k v^n w^r = 0,$$

or

$$\sum_{s=1}^{max(k,n,r)} s[(\lambda_{k-s+1,n} + \lambda_{k,n-s+1})\lambda_{rs} + (\lambda_{n-s+1,r} + \lambda_{n,r-s+1})\lambda_{ks} + (\lambda_{r-s+1,k} + \\ + \lambda_{r,k-s+1})\lambda_{ns}] = 0. \tag{4.6}$$

We may assume $k < n < r$, since if at least two of the indices k, n, r are equal then (4.6) is identically satisfied. Then $\max(k,n,r) = r$. Let $k = 1$ and $n < r$, then we have

$$\sum_{s=1}^{r} s\,[\lambda_{2-s,n}\lambda_{rs} + \lambda_{1,n-s+1}\lambda_{rs} + \lambda_{n-s+1,r}\lambda_{1s} + \lambda_{n,r-s+1}\lambda_{1s} + \lambda_{r-s+1,1}\lambda_{ns} + \lambda_{r,2-s}\lambda_{ns}] = 0,$$

which is equivalent to

$$\lambda_{1n}\lambda_{r1} + \sum_{s=1}^{n} s\lambda_{1,n-s+1}\lambda_{rs} + \sum_{s=1}^{n} s\lambda_{n-s+1,r}\lambda_{1s} + \sum_{s=1}^{r} s\lambda_{n,r-s+1}\lambda_{1s} + \\ + \sum_{s=1}^{r} s\lambda_{r-s+1,1}\lambda_{ns} + \lambda_{r1}\lambda_{n1} = 0.$$

The first and the last terms in the above equation cancel each other. We make the change of variables $s \mapsto n-s+1$, and $s \mapsto r-s+1$ in the third and the forth terms respectively to obtain

$$\sum_{s=1}^{n} s\lambda_{1,n-s+1}\lambda_{rs} + \sum_{s=1}^{n}(n-s+1)\lambda_{sr}\lambda_{1,n-s+1} + \sum_{s=1}^{r}(r-s+1)\lambda_{ns}\lambda_{1,r-s+1} + \\ + \sum_{s=1}^{r} s\lambda_{r-s+1,1}\lambda_{ns} = 0,$$

which is equivalent to

$$\sum_{s=1}^{n}(n-2s+1)\lambda_{1,n-s+1}\lambda_{sr} = \sum_{s=1}^{r}(r-2s+1)\lambda_{1,r-s+1}\lambda_{sn}. \tag{4.7}$$

A close look at the first several equations of (4.6)

$$\begin{aligned}
&\lambda_{12}\lambda_{13} = 0\\
&\lambda_{12}(2\lambda_{14}+\lambda_{23}) = 0\\
&\lambda_{13}(\lambda_{14}+\lambda_{23}) = 0\\
&3\lambda_{14}\lambda_{23} - (\lambda_{23})^2 - 4\lambda_{13}\lambda_{24} + 5\lambda_{12}\lambda_{34} = 0\\
&\lambda_{12}(3\lambda_{15}+2\lambda_{24}) = 0\\
&\lambda_{13}\lambda_{15} + \lambda_{14}\lambda_{23} + \lambda_{12}\lambda_{34} = 0\\
&3\lambda_{15}\lambda_{23} - 2\lambda_{23}\lambda_{24} - 5\lambda_{13}\lambda_{25} + 6\lambda_{12}\lambda_{35} = 0\\
&-\lambda_{14}\lambda_{15} - 2\lambda_{14}\lambda_{24} + \lambda_{13}\lambda_{25} - \lambda_{12}\lambda_{35} = 0\\
&4\lambda_{15}\lambda_{24} - 2(\lambda_{24})^2 - 5\lambda_{14}\lambda_{25} + \lambda_{23}\lambda_{25} + 7\lambda_{12}\lambda_{45} = 0\\
&5\lambda_{15}\lambda_{34} - 2\lambda_{24}\lambda_{34} - 6\lambda_{14}\lambda_{35} + \lambda_{23}\lambda_{35} + 7\lambda_{13}\lambda_{45} = 0\\
&\qquad\qquad\vdots
\end{aligned}$$

shows that the solutions of this infinite system of quadrics fall into three main classes. Namely those with $\lambda_{13} = 0$, these with $\lambda_{12} = 0$, and these with $\lambda_{12} = 0 = \lambda_{13}$, the last one being an intersection of the first two. We now proceed with describing these classes.

(i) Let $\lambda_{12} \neq 0$ and $\lambda_{13} = 0$. Then, from (4.7) with $n = 2$ we obtain

$$-\lambda_{12}\lambda_{1r} = \sum_{s=1}^{r}(r-2s+1)\lambda_{1,r-s+1}\lambda_{2s}.$$

After multiplying both sides of the above equation by u^r, and summing over r we obtain

$$-\lambda_{12}\sum_{r=1}^{\infty}\lambda_{1r}u^r = \sum_{r=1}^{\infty}\sum_{s=1}^{r}(r-2s+1)\lambda_{1,r-s+1}\lambda_{2s}u^r$$

$$\Updownarrow\ r \mapsto m+s-1$$

$$-\lambda_{12}\sum_{r=1}^{\infty}\lambda_{1r}u^r = \sum_{m=1}^{\infty}\sum_{s=1}^{\infty}(m-s)\lambda_{1m}\lambda_{2s}u^{m+s-1}$$

$$= \sum_{m=1}^{\infty}\sum_{s=1}^{\infty} m\lambda_{1m}u^{m-1}\lambda_{2s}u^s - \sum_{m=1}^{\infty}\sum_{s=1}^{\infty} s\lambda_{2s}u^{s-1}\lambda_{1m}u^m$$

Now, if we define $f_1(u) = \sum_{r=1}^{\infty}\lambda_{1r}u^r$, and $g_1(u) = \sum_{s=1}^{\infty}\lambda_{2s}u^s$, the above equation becomes

$$-\lambda_{12}f_1(u) = f_1'(u)g_1(u) - f_1(u)g_1'(u).$$

(ii) Let $\lambda_{12} = 0$ and $\lambda_{13} \neq 0$. Then, from (4.7) with $n = 3$ we have

$$-2\lambda_{13}\lambda_{1r} = \sum_{s=1}^{r}(r-2s+1)\lambda_{1,r-s+1}\lambda_{3s}.$$

Define $f_2(u)$, and $g_2(u)$ to be $f_2(u) = \sum_{r=1}^{\infty}\lambda_{1r}u^r$, and $g_1(u) = \sum_{s=1}^{\infty}\lambda_{3s}u^s$. Then, performing the same manipulations as in the case (i) above we obtain

$$-2\lambda_{13}f_2(u) = f_2'(u)g_2(u) - f_2(u)g_2'(u).$$

(iii) For $\lambda_{12} = 0 = \lambda_{13}$, let us assume temporarily that $\lambda_{14} \neq 0$. Then from (4.7) with $n = 4$ we have

$$-3\lambda_{14}\lambda_{1r} = \sum_{s=1}^{r}(r-2s+1)\lambda_{1,r-s+1}\lambda_{4s},$$

and therefore

$$-3\lambda_{14}f_3(u) = f_3'(u)g_3(u) - f_3(u)g_3'(u).$$

Where $f_3(u) = \sum_{s=1}^{\infty}\lambda_{1s}u^s$, $g_3(u) = \sum_{s=1}^{\infty}\lambda_{4s}u^s$.

The above considerations suggest the following argument. If the first non-zero element of the set $\{\lambda_{1n}\}_{n\geq 2}$ is $\lambda_{1,d+1}$ $(d \geq 1)$, then from (4.7) we deduce that

$$-d\lambda_{1,d+1}\lambda_{1r} = \sum_{s=1}^{r}(r-2s+1)\lambda_{1,r-s+1}\lambda_{d+1,s},$$

and the above equation is equivalent to

$$-d\lambda_{1,d+1}f_d(u) = f_d'(u)g_d(u) - f_d(u)g_d'(u).$$

Here f_d, g_d are defined as $f_d(u) = \sum_{s=1}^{\infty} \lambda_{1s} u^s$, and $g_d(u) = \sum_{s=1}^{\infty} \lambda_{d+1,s} u^s$. (Certainly, the first d terms in the definition of f_d are zero.)

Thus, we will parametrize all classes of solutions of (4.4) by $d \in \mathbb{N}$ such that $\lambda_{12} = \ldots = \lambda_{1d} = 0$, and $\lambda_{1,d+1} \neq 0$. In what follows, we will show that for each $d \in \mathbb{N}$ with the above property $\varphi(u,v)$ is given by $\varphi_d(u,v) = \frac{1}{\lambda_{1,d+1}} [f_d(u)g_d(v) - f_d(v)g_d(u)]$, and therefore is a solution of (4.4), according to Lemma 4.6. This we will show by proving that for each $d \in \mathbb{N}$ we have

$$\lambda_{nm} = \frac{1}{\lambda_{1,d+1}} \Big[\lambda_{1n}\lambda_{d+1,m} - \lambda_{1m}\lambda_{d+1,n} \Big], \qquad \forall\, n,m \geq 1. \tag{4.8}$$

Lemma 4.7. *For any fixed $d \in \mathbb{N}$ such that $\lambda_{1n} = 0$ for $1 \leq n \leq d$, $\lambda_{1,d+1} \neq 0$, it follows that $\lambda_{sn} = 0$ for $1 \leq s \leq d-1$, $1 \leq n \leq d$.*

Proof. Since $1 \leq n \leq d$, it follows from (4.7) that (the l.h.s. is zero)

$$\sum_{s=1}^{r-d} (r - 2s + 1)\lambda_{1,r-s+1}\lambda_{sn} = 0. \tag{4.9}$$

Since $n < r$, if
(i) $r = d+1$, then $d\lambda_{1,d+1}\lambda_{1n} = 0$, which is an identity;
(ii) $r = d+2$, then $(d+1)\lambda_{1,d+2}\lambda_{1n} + (d-1)\lambda_{1,d+1}\lambda_{2n} = 0 \implies (d-1)\lambda_{1,d+1}\lambda_{2n} = 0 \implies \lambda_{2n} = 0$.
Assume now that $\lambda_{sn} = 0$ for $1 \leq s \leq m < d-1$, $1 \leq n \leq d$. We would like to show that this implies $\lambda_{m+1,n} = 0$. But from (4.9) with $r = d+m+1$ it follows that

$$(d-m)\lambda_{1,d+1}\lambda_{m+1,n} = 0 \implies \lambda_{m+1,n} = 0.$$

Therefore, letting r run in the interval $d+1 \leq r \leq 2d-1$ finishes the proof.■

Remark. For $r = 2d$ and $r = 2d+1$ we obtain identities. For $r > 2d+1$ we obtain relations which are particular cases of (4.8). For example, when $r = 2d+2$ we have

$$\lambda_{1,d+2}\lambda_{d+1,n} - \lambda_{1,d+1}\lambda_{d+2,n} = 0 \implies \lambda_{n,d+2} = \frac{\lambda_{1,d+2}}{\lambda_{1,d+1}}\lambda_{n,d+1}.$$

Lemma 4.8. *For each fixed $d \in \mathbb{N}$, such that $\lambda_{1m} = 0$ for $1 \leq m \leq d$, $\lambda_{1,d+1} \neq 0$, it follows that $\lambda_{d+1,n}$ is a rational function of $\lambda_{1,d+1}, \dots, \lambda_{1,d+n}$, $\forall\, n \geq 1$.*

Proof. Let us consider (4.7) with $r = n + d$

$$\sum_{s=1}^{d+1}(d-2s+1)\lambda_{1,d+2-s}\lambda_{s,n+d} = \sum_{s=1}^{n}(n+d-2s+1)\lambda_{1,n+d-s+1}\lambda_{s,d+1} \quad (n \geq 1), \tag{4.10}$$

which is equivalent to

$$d\lambda_{1,d+1}\lambda_{1,n+d} = \sum_{s=1}^{n-1}(n+d-2s+1)\lambda_{1,n+d-s+1}\lambda_{s,d+1} + (d-n+1)\lambda_{1,d+1}\lambda_{n,d+1}$$

since only the first term on the l.h.s is non-zero. Therefore, solving for $\lambda_{d+1,n}$ we obtain

(4.11)

$$\lambda_{d+1,n} = -\frac{1}{(d-n+1)\lambda_{1,d+1}}\left[d\lambda_{1,d+1}\lambda_{1,n+d} - \sum_{s=1}^{n-1}(n+d-2s+1)\lambda_{1,n+d-s+1}\lambda_{s,d+1}\right].$$

The above formula gives a recursive relation for $\lambda_{n,d+1}$'s, whenever $n \neq d+1$. To finish the proof of the lemma we write the first three relations. For $n = 1$ (4.10) is an identity. For $n = 2$ we have

$$d\lambda_{1,d+1}\lambda_{1,2+d} = (d+1)\lambda_{1,2+d}\lambda_{1,d+1} + (d-1)\lambda_{1,d+1}\lambda_{2,d+1},$$

from which it follows that

$$\lambda_{2,d+1} = -\frac{1}{d-1}\lambda_{1,d+2}. \tag{4.12}$$

For $n = 3$ we have

$$\begin{aligned}\lambda_{d+1,3} &= -\frac{1}{(d-2)\lambda_{1,d+1}}\left[d\lambda_{1,d+1}\lambda_{1,d+3} - (d+2)\lambda_{1,d+3}\lambda_{1,d+1} - d\lambda_{1,d+2}\lambda_{1,d+1}\right] \\ &= -\frac{1}{(d-2)\lambda_{1,d+1}}\left[-2\lambda_{1,d+1}\lambda_{1,d+3} + \frac{d}{d-1}(\lambda_{1,d+2})^2\right],\end{aligned}$$

where we have used (4.11). Finally for $n = 4$ we have

$$\begin{aligned}\lambda_{d+1,4} = -\frac{1}{(d-3)\lambda_{1,d+1}}\Big[&d\lambda_{1,d+1}\lambda_{1,d+4} - (d+3)\lambda_{1,d+4}\lambda_{1,d+1} - (d+1)\lambda_{1,d+3}\lambda_{2,d+1} - \\ &- (d-1)\lambda_{1,d+2}\lambda_{3,d+1}\Big] \\ = -\frac{1}{(d-3)\lambda_{1,d+1}}\Big[&-3\lambda_{1,d+1}\lambda_{1,d+4} + \frac{(3d-5)d}{(d-1)(d-2)}\lambda_{1,d+2}\lambda_{1,d+3} - \frac{d}{d-2}\frac{(\lambda_{1,d+2})^3}{\lambda_{1,d+1}}\Big].\end{aligned}$$

Therefore, repeating this process n times we obtain that $\lambda_{d+1,n}$ is a rational function of the type as stated, since all $\lambda_{d+1,s}$ with $1 \leq s \leq n-1$ are rational functions. This completes the proof of the lemma.■

Remark. Lemma 4.8 shows that for each $d \in \mathbb{N}$ the functions f_d and g_d are defined only in terms of $\{\lambda_{1n}\}_{n \geq d+1}$. That is, for each $d \in \mathbb{N}$ we will have a solution of (4.4) determined by this infinite set of parameters. It turns out that these parameters are not completely independent.

Namely, we have the following lemma.

Lemma 4.9. *For each $d \in \mathbb{N}$ there exists the following single relation between λ_{1n}'s (with $d+1 \leq n \leq 2d+1$)*

$$\lambda_{1,2d+1} = -\frac{1}{d\lambda_{1,d+1}} \sum_{s=2}^{d} 2(d+1-s)\lambda_{1,2d+2-s}\lambda_{s,d+1}.$$

Here $\lambda_{s,d+1} = \lambda_{s,d+1}(\lambda_{1,d+1}, \ldots, \lambda_{1,d+s})$, $2 \leq s \leq d$, are rational functions according to the previous lemma.

Proof. From (4.11) we have

$$-(d-n+1)\lambda_{1,d+1}\lambda_{d+1,n} = d\lambda_{1,d+1}\lambda_{1,n+d} - \sum_{s=1}^{n-1}(n+d-2s+1)\lambda_{1,n+d-s+1}\lambda_{s,d+1}.$$

If $n = d+1$ the l.h.s. is zero, and we obtain

$$\begin{aligned} d\lambda_{1,d+1}\lambda_{1,2d+1} &= \sum_{s=1}^{d}(2d+2-2s)\lambda_{1,2d+2-s}\lambda_{s,d+1} \\ &= 2d\lambda_{1,2d+1}\lambda_{1,d+1} + \sum_{s=2}^{d} 2(d+1-s)\lambda_{1,2d+2-s}\lambda_{s,d+1}. \end{aligned}$$

From this, the statement follows. This means that for each $d \in \mathbb{N}$ we can solve for λ_{2d+1} in terms of $\lambda_{1,d+1}, \ldots, \lambda_{1,2d}$, and $\lambda_{1,2d+1}$ is a rational function of these variables. This is the only relation between λ_{1n}'s. This is easily seen from (4.11). Multiplying both sides of (4.10) by $(d+1-n)$ we see that the l.h.s. of the equality so obtained vanishes if and only if $n = d+1$. From this we obtain exactly one relation between λ_{1n}'s for $d+1 \leq n \leq 2d+1$. Thus for $d = 1$ we obtain $\lambda_{13} = 0$, for $d = 2$ we have

$\lambda_{15} = \frac{(\lambda_{14})^2}{\lambda_{13}}$, for $d = 3$ we have

$$\lambda_{17} = 2\frac{\lambda_{15}\lambda_{16}}{\lambda_{14}} - \frac{(\lambda_{15})^3}{(\lambda_{14})^2},$$

and so on.■

Summarizing, each $d \in \mathbb{N}$ specifies a branch in the space of solutions of (4.6) for which the set of parameters $\{\lambda_{1n}\}_{n \geq d+1, n \neq 2d+1}$ forms a basis. Here $\lambda_{1,2d+1}$ is a rational function of $\lambda_{1,d+1}, \dots, \lambda_{1,2d}$.

Lemma 4.10. *For each fixed $d \in \mathbb{N}$ and every $n, m \geq 1$ the following formula is valid*

$$\lambda_{mn} = \frac{1}{\lambda_{1,d+1}}\Big[\lambda_{1m}\lambda_{d+1,n} - \lambda_{1n}\lambda_{d+1,m}\Big]. \tag{4.13}$$

Proof. The plan of the proof is as follows
(i) First we prove (4.13) for $1 \leq m \leq d$ and $n \geq d+1$. For $n < d+1$ there is nothing to prove, because all λ_{mn}'s are zero according to Lemma 4.7;
(ii) Second, we prove (4.13) for $m \leq 2d$ and $n > m$, using (i). Here, we prove it first for $\lambda_{d+2,d+3}$ (for $\lambda_{d+1,d+2}$ the statement is trivial). Then, we prove it for $\lambda_{d+2,n}$, and every $n \geq d+3$, using an inductive argument. Next, we prove that if the statement is true for $\lambda_{m-1,m}$, then it is true for $\lambda_{m,m+1}$, for some $m \leq 2d$. Last, we fix $m \leq 2d$, and use again an inductive argument to prove it for λ_{mn}, and every $n > m$;
(iii) Our third step is to apply induction to the argument m. Namely, assuming that the statement is true for λ_{mn}, where $m \leq (k-1)d$, and every $n > m$, we prove it for $m \leq kd$, and every $n > m$.

The proof is technical, but not difficult, and uses extensively formula (4.11), which we now write as

$$\sum_{s=1}^{n-1}(n+d-2s+1)\lambda_{1,n+d-s+1}\lambda_{s,d+1} = (d-n+1)\lambda_{1,d+1}\lambda_{d+1,n} + d\lambda_{1,d+1}\lambda_{1,n+d}. \tag{4.14}$$

(i) Let $1 \leq m \leq d$, and $N \geq d+1$. Fix m, and assume that the statement is true for all $d+1 \leq n \leq N-1$. We want to prove it for λ_{mN}. But from (4.7) we have

$$\sum_{s=1}^{m-1}(m-2s+1)\lambda_{1,m-s+1}\lambda_{s,N+d} = -\sum_{s=d+1}^{N}(N+d-2s+1)\lambda_{1,N+d-s+1}\lambda_{ms}.$$

But the l.h.s. of this equation is zero, since $m \leq d$. Therefore

$$
\begin{aligned}
0 = & \sum_{s=d+1}^{N-1} (N+d-2s+1)\lambda_{1,N+d-s+1}\lambda_{ms} + (d-N+1)\lambda_{1,d+1}\lambda_{mN} \\
= & \frac{1}{\lambda_{1,d+1}} \sum_{s=d+1}^{N-1} (N+d-2s+1)\lambda_{1,N+d-s+1} \left[\lambda_{1m}\lambda_{d+1,s} - \lambda_{1s}\lambda_{d+1,m}\right] + \\
& + (d-N+1)\lambda_{1,d+1}\lambda_{mN} \\
= & -\frac{\lambda_{d+1,m}}{\lambda_{1,d+1}} \sum_{s=d+1}^{N-1} (N+d-2s+1)\lambda_{1,N+d-s+1}\lambda_{1s} + (d-N+1)\lambda_{1,d+1}\lambda_{mN} \\
= & (d-N+1)\lambda_{d+1,m}\lambda_{1N} + (d-N+1)\lambda_{1,d+1}\lambda_{mN},
\end{aligned}
$$

and we obtain

$$\lambda_{mN} = -\frac{\lambda_{1N}}{\lambda_{d+1,m}}\lambda_{1,d+1}.$$

This is exactly (4.13) with $\lambda_{1m} = 0$, which is a consequence of the assumption on m.

(ii.a) The statement is true for $\lambda_{d+1,d+2}$. Let us now assume that $m = d+2$, $n = d+3$. Then from (4.7) we have

$$\sum_{s=1}^{d+1}(d-2s+3)\lambda_{1,d-s+3}\lambda_{s,2d+3} = \sum_{s=1}^{d+1}(2d-2s+3)\lambda_{1,2d-s+3}\lambda_{s,d+2} - 2\lambda_{1,d+1}\lambda_{d+3,d+2}.$$

Now, we use (i) in both sides of the above equation to obtain

$$
\begin{aligned}
& \frac{1}{\lambda_{1,d+1}} \sum_{s=1}^{d+1}(d-2s+3)\lambda_{1,d-s+3}\left[\lambda_{1s}\lambda_{d+1,2d+3} - \lambda_{1,2d+3}\lambda_{d+1,s}\right] = \\
& = \frac{1}{\lambda_{1,d+1}} \sum_{s=1}^{d+1}(2d-2s+3)\lambda_{1,2d-s+3}\left[\lambda_{1s}\lambda_{d+1,d+2} - \lambda_{1,d+2}\lambda_{d+1,s}\right] - 2\lambda_{1,d+1}\lambda_{d+3,d+2},
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
& \frac{\lambda_{d+1,2d+3}}{\lambda_{1,d+1}} \sum_{s=1}^{d+1}(d-2s+3)\lambda_{1,d-s+3}\lambda_{1s} - \frac{\lambda_{1,2d+3}}{\lambda_{1,d+1}} \sum_{s=1}^{d+1}(d-2s+3)\lambda_{1,d-s+3}\lambda_{d+1,s} = \\
& \quad = -2\lambda_{1,d+1}\lambda_{d+3,d+2} + \frac{\lambda_{1,2d+3}}{\lambda_{1,d+1}} \sum_{s=1}^{d+1}(2d-2s+4)\lambda_{1,2d-s+4}\lambda_{1s} - \\
& \qquad - \frac{\lambda_{1,d+2}}{\lambda_{1,d+1}} \sum_{s=1}^{d+1}(2d-2s+4)\lambda_{1,2d-s+4}\lambda_{d+1,s}.
\end{aligned}
$$

The first term on the l.h.s. is zero, and the second term on the r.h.s. contains only one summand thus giving

$$-\frac{\lambda_{1,2d+3}}{\lambda_{1,d+1}}\sum_{s=1}^{d+1}(d-2s+3)\lambda_{1,d-s+3}\lambda_{d+1,s}=$$

$$=-2\lambda_{1,d+1}\lambda_{d+3,d+2}+2\lambda_{1,d+2}\lambda_{d+1,d+2}-\frac{\lambda_{1,d+2}}{\lambda_{1,d+1}}\sum_{s=1}^{d+1}(2d-2s+3)\lambda_{1,2d-s+3}\lambda_{d+1,s}.$$

The term on the l.h.s. has only two summands. The third term on the r.h.s. we transform using (4.14). Thus,

$$\frac{\lambda_{1,2d+3}}{\lambda_{1,d+1}}\left[(d+1)\lambda_{1,d+2}\lambda_{1,d+1}+(d-1)\lambda_{1,d+1}\lambda_{2,d+1}\right]=$$

$$=-2\lambda_{1,d+1}\lambda_{d+3,d+2}+2\lambda_{1,d+3}\lambda_{d+1,d+3}+\frac{\lambda_{1,d+2}}{\lambda_{1,d+1}}\left[-2\lambda_{1,d+1}\lambda_{d+1,d+3}+d\lambda_{1,d+1}\lambda_{2d+3}\right].$$

Collecting all the terms we obtain

$$\lambda_{d+2,d+3}=\frac{1}{\lambda_{1,d+1}}\left[\lambda_{1,d+2}\lambda_{d+1,d+3}-\lambda_{1,d+3}\lambda_{d+1,d+3}\right].$$

(ii.b) Let us assume now that the statement is true for $\lambda_{d+2,k}$, $d+3\le k\le n-1$. We would like to prove it for $k=n$. From (4.7) we have

$$\sum_{s=1}^{d+1}(d-2s+3)\lambda_{1,d-s+3}\lambda_{s,n+d}=\sum_{s=1}^{n-1}(n+d-2s+1)\lambda_{1,n+d-s+1}\lambda_{s,d+2}+(d-n+1)\lambda_{1,d+1}\lambda_{n,d+2}.$$

The l.h.s. has only two summands. Also, applying the inductive hypothesis we obtain

$$(d+1)\lambda_{1,d+2}\lambda_{1,n+d}+(d-1)\lambda_{1,d+1}\lambda_{2,n+d}=$$

$$=\frac{1}{\lambda_{1,d+1}}\sum_{s=1}^{n-1}(n+d-2s+1)\lambda_{1,n+d-s+1}\left[\lambda_{1s}\lambda_{d+1,d+2}-\lambda_{1,d+2}\lambda_{d+1,s}\right]+$$

$$+(d-n+1)\lambda_{1,d+1}\lambda_{n,d+2}$$

$$=\frac{\lambda_{d+1,d+2}}{\lambda_{1,d+1}}\sum_{d+1}^{n-1}(n+d-2s+1)\lambda_{1,n+d-s+1}\lambda_{1s}-$$

$$-\frac{\lambda_{1,d+2}}{\lambda_{1,d+1}}\sum_{s=1}^{n-1}(n+d-2s+1)\lambda_{1,n+d-s+1}\lambda_{d+1,s}+(d-n+1)\lambda_{1,d+1}\lambda_{n,d+2}$$

$$=(n-d-1)\lambda_{d+1,d+2}\lambda_{1n}+\frac{\lambda_{1,d+2}}{\lambda_{1,d+1}}\left[(d-n+1)\lambda_{1,d+1}\lambda_{d+1,n}+d\lambda_{1,d+1}\lambda_{1,n+d}\right]+$$

$$+(d-n+1)\lambda_{1,d+1}\lambda_{n,d+2}.$$

The second term in the above expression has been obtained by using formula (4.14). Applying the inductive argument to $\lambda_{2,n+d}$, namely $\lambda_{2,n+d} = -\frac{1}{\lambda_{1,d+1}}\left[-\lambda_{1,n+d}\lambda_{d+1,2}\right]$, and collecting terms we obtain

$$\lambda_{d+2,n} = \frac{1}{\lambda_{1,d+1}}\Big[\lambda_{1,d+2}\lambda_{d+1,n} - \lambda_{1n}\lambda_{d+1,d+2}\Big].$$

(ii.c) Suppose that the statement is true for $\lambda_{m-1,n}$, $\forall\ n \geq m-1$. We are going to prove it for $\lambda_{m,m+1}$. From (4.7) we have

$$\sum_{s=1}^{m-1}(m-2s+1)\lambda_{1,m-s+1}\lambda_{s,m+d+1} = \sum_{s=1}^{m}(m+d-2s+2)\lambda_{1,m+d-s+2}\lambda_{sm}+(d-m)\lambda_{1,d+1}\lambda_{m+1,m}.$$

Using the induction hypothesis the above equation transforms to

$$\frac{1}{\lambda_{1,d+1}}\sum_{s=1}^{m-1}(m-2s+1)\lambda_{1,m-s+1}\left[\lambda_{1s}\lambda_{d+1,m+d+1} - \lambda_{1,m+d+1}\lambda_{d+1,s}\right] =$$

$$= (d-m)\lambda_{1,d+1}\lambda_{m+1,m}+\frac{1}{\lambda_{1,d+1}}\sum_{s=1}^{m-1}(m+d-2s+2)\lambda_{1,m+d-s+2}\left[\lambda_{1s}\lambda_{d+1,m} - \lambda_{1m}\lambda_{d+1,s}\right].$$

Expanding, we have

$$\frac{\lambda_{d+1,m+d+1}}{\lambda_{1,d+1}}\sum_{s=d+1}^{m-d}(m-2s+1)\lambda_{1,m-s+1}\lambda_{1s}-\frac{\lambda_{1,m+d+1}}{\lambda_{1,d+1}}\sum_{s=1}^{m-d}(m-2s+1)\lambda_{1,m-s+1}\lambda_{d+1,s} =$$

$$= (d-m)\lambda_{1,d+1}\lambda_{m+1,m} + \frac{\lambda_{d+1,m}}{\lambda_{1,d+1}}\sum_{s=d+1}^{m}(m+d-2s+2)\lambda_{1,m+d-s+2}\lambda_{1s}-$$

$$-\frac{\lambda_{1m}}{\lambda_{1,d+1}}\sum_{s=1}^{(m+1)-1}(m+d-2s+2)\lambda_{1,m+d-s+2}\lambda_{d+1,s}.$$

The first term on the l.h.s. is zero. From the others we obtain

$$-(2d-m+1)\lambda_{1,m+d+1}\lambda_{d+1,m-d}+\frac{\lambda_{1,m+d+1}}{\lambda_{1,d+1}}\Big[(2d-m+1)\lambda_{1,d+1}\lambda_{d+1,m-d}+d\lambda_{1,d+1}\lambda_{1m}\Big] =$$

$$= (d-m)\lambda_{1,d+1}\lambda_{m+1,m} + (m-d)\lambda_{1,m+1}\lambda_{d+1,m}+$$

$$+\frac{\lambda_{1m}}{\lambda_{1,d+1}}\Big[(d-m)\lambda_{1,d+1}\lambda_{d+1,m+1} + d\lambda_{1,d+1}\lambda_{1,m+d+1}\Big].$$

Collecting terms we arrive at

$$\lambda_{m,m+1} = \frac{1}{\lambda_{1,d+1}}\Big[\lambda_{1,m}\lambda_{d+1,m+1} - \lambda_{1,m+1}\lambda_{d+1,m}\Big].$$

(ii.d) Let us assume now, that the statement is true for λ_{mk}, where $m \leq 2d$, and all $k \geq m+1$ up to $k = n-1$. We will prove it for $k = n$. Again, from (4.7) we have

$$\sum_{s=1}^{m-1}(m-2s+1)\lambda_{1,m-s+1}\lambda_{s,n+d} = \sum_{s=1}^{n+1}(n+d-2s+1)\lambda_{1,n+d-s+1}\lambda_{sm}+(d-n+1)\lambda_{1,d+1}\lambda_{nm}.$$

Next, we apply the induction hypothesis to obtain

$$\frac{1}{\lambda_{1,d+1}}\sum_{s=1}^{m-1}(m-2s+1)\lambda_{1,m-s+1}\left[\lambda_{1s}\lambda_{d+1,n+d} - \lambda_{1,n+d}\lambda_{d+1,s}\right] =$$

$$= \frac{1}{\lambda_{1,d+1}}\sum_{s=1}^{n-1}(n+d-2s+1)\lambda_{1,n+d-s+1}\left[\lambda_{1s}\lambda_{d+1,m} - \lambda_{1m}\lambda_{d+1,s}\right] + (d-n+1)\lambda_{1,d+1}\lambda_{nm},$$

which leads to

(*)

$$\frac{\lambda_{d+1,n+d}}{\lambda_{1,d+1}}\sum_{s=d+1}^{m-1}(m-2s+1)\lambda_{1,m-s+1}\lambda_{1s} - \frac{\lambda_{1,n+d}}{\lambda_{1,d+1}}\sum_{s=1}^{m-1}(m-2s+1)\lambda_{1,m-s+1}\lambda_{d+1,s} =$$

$$= (d-n+1)\lambda_{1,d+1}\lambda_{nm} + \frac{\lambda_{d+1,m}}{\lambda_{1,d+1}}\sum_{s=d+1}^{n-1}(n+d-2s+1)\lambda_{1,n+d-s+1}\lambda_{1s} -$$

$$- \frac{\lambda_{1m}}{\lambda_{1,d+1}}\sum_{s=1}^{n-1}(n+d-2s+1)\lambda_{1,n+d-s+1}\lambda_{d+1,s}.$$

The first term on the l.h.s. is zero. The second one we represent as (notice that the sum is up to $m-d$)

$$-(2d-m+1)\lambda_{1,n+d}\lambda_{d+1,n-d} + \sum_{s=1}^{m-(d+1)}(m-2s+1)\lambda_{1,m-s+1}\lambda_{s,d+1}.$$

For the second term in the expression above we apply (4.14), namely

$$\sum_{s=1}^{m-(d+1)}(m-2s+1)\lambda_{1,m-s+1}\lambda_{s,d+1} = (2d-m+1)\lambda_{1,d+1}\lambda_{d+1,m-d} + d\lambda_{1,d+1}\lambda_{1m}.$$

We do the same for the last term on the r.h.s. of (*) too. Thus finally, we come to the equality

$$-(2d-m+1)\lambda_{1,n+d}\lambda_{d+1,m-d} + (2d-m+1)\lambda_{1,n+d}\lambda_{d+1,m-d} + d\lambda_{1,n+d}\lambda_{1m} =$$

$$= (d-n+1)\lambda_{1,d+1}\lambda_{nm} + (n-d-1)\lambda_{d+1,m}\lambda_{1n} + (d-n+1)\lambda_{d+1,n}\lambda_{1m} + d\lambda_{1,n+d}\lambda_{1m},$$

and after cancellation we get

$$\lambda_{mn} = \frac{1}{\lambda_{1,d+1}}\Big[\lambda_{1m}\lambda_{d+1,n} - \lambda_{1n}\lambda_{d+1,m}\Big].$$

(iii) In this case the arguments repeat vis-à-vis the arguments presented in case (ii). Thus, we omit them.

This concludes the proof Lemma 4.10 and of Theorem 4.5.■

Using the above results we are going to describe now a one-parameter deformation of the solution $\varphi(u,v) = uv(u^d - v^d)$ obtained in Theorem 4.5, which is of course a particular case of the general infinite-parameter solution.

Lemma 4.11. *For each $d \in \mathbb{N}\setminus\{1\}$, if $\lambda_{1n} = \frac{(\lambda_{1,d+2})^{n-d-1}}{(\lambda_{1,d+1})^{n-d-2}}$, $\forall\, n \geq d+1$, it follows that $\lambda_{n,d+1} = 0$, $\forall\, n \geq d+1$, and $\lambda_{n,d+1} = -\frac{1}{d-1}\frac{(\lambda_{1,d+2})^{n-1}}{(\lambda_{1,d+1})^{n-2}}$ for $2 \leq n \leq d$.*

Proof. From (4.10) we obtain

(4.15)

$$\lambda_{n,d+1} = \frac{1}{(d-n+1)\lambda_{1,d+1}}\left[d\frac{(\lambda_{1,d+1})^{n-1}}{(\lambda_{1,d+1})^{n-3}} - \sum_{s=1}^{n-1}(n+d-2s+1)\frac{(\lambda_{1,d+2})^{n-s}}{(\lambda_{1,d+1})^{n-s-1}}\lambda_{s,d+1}\right].$$

If $n = 2$ we have

$$\lambda_{2,d+1} = \frac{1}{(d-n+1)\lambda_{1,d+1}}\Big[d\lambda_{1,d+1}\lambda_{1,d+2} - (d+1)\lambda_{1,d+2}\lambda_{1,d+1}\Big] = -\frac{1}{d-1}\lambda_{1,d+2}.$$

Let as assume now that $\lambda_{k,d+1} = -\frac{1}{d-1}\frac{(\lambda_{1,d+2})^{k-1}}{(\lambda_{1,d+1})^{k-2}}$ for all k, such that $2 \leq k \leq n-1 < d$.

We will now prove that this relation is true for $k = n \leq d$. From (4.14) we obtain

$$\begin{aligned}
\lambda_{n,d+1} =& \frac{1}{(d-n+1)\lambda_{1,d+1}} \left[d\frac{(\lambda_{1,d+2})^{n-1}}{(\lambda_{1,d+1})^{n-3}} + \frac{1}{d-1}\sum_{s=1}^{n-1}(n+d-2s+1)\frac{(\lambda_{1,d+2})^{n-s}}{(\lambda_{1,d+1})^{n-s-1}}\lambda_{s,d+1} \right] \\
=& \frac{1}{(d-n+1)\lambda_{1,d+1}} \left[d\frac{(\lambda_{1,d+2})^{n-1}}{(\lambda_{1,d+1})^{n-3}} + \frac{1}{d-1}\frac{(\lambda_{1,d+2})^{n-1}}{(\lambda_{1,d+1})^{n-3}}\sum_{s=2}^{n-1}(n+d-2s+1) \right] - \\
& - \frac{(\lambda_{1,d+2})^{n-1}}{(\lambda_{1,d+1})^{n-2}} \\
=& \frac{1}{(d-n+1)\lambda_{1,d+1}} \left[-(n-1) + \frac{1}{d-1}(n-2)(n+d+1) - \right. \\
& \left. - \frac{2}{d-1}\left(\frac{n(n-1)}{2} - 1\right) \right] \frac{(\lambda_{1,d+2})^{n-1}}{(\lambda_{1,d+1})^{n-3}} \\
=& \frac{1}{(d-n+1)(d-1)} \left[-(n-1)(d-1) + (n-2)(n+d+1) - \right. \\
& \left. - n(n-1) + 2 \right] \frac{(\lambda_{1,d+2})^{n-1}}{(\lambda_{1,d+1})^{n-2}} \\
=& \frac{1}{(d-n+1)(d-1)} \left[-d+n-1 \right] \frac{(\lambda_{1,d+2})^{n-1}}{(\lambda_{1,d+1})^{n-2}} \\
=& -\frac{1}{d-1}\frac{(\lambda_{1,d+2})^{n-1}}{(\lambda_{1,d+1})^{n-2}}.
\end{aligned}$$

This proves the second part of the lemma. If $n = d+1$, then the l.h.s. of (4.15) is zero. We check for consistency whether the r.h.s. is zero. The expression in the brackets reads

$$\begin{aligned}
-d + \frac{1}{d-1}\sum_{s=2}^{d} 2(d-s+1) &= \frac{1}{d-1}\left[-d^2 + d + 2(d+1)(d-1) - 2\left(\frac{d(d+1)}{2} - 1\right) \right] \\
&= 0.
\end{aligned}$$

Now, we check whether the statement is true for $\lambda_{d+2,d+1}$. From (4.15)

$$\begin{aligned}
\lambda_{d+2,d+1} &= -\left[-(d+1) + \frac{1}{d-1}\sum_{s=2}^{d}(2d+3-2s) \right] \frac{(\lambda_{1,d+2})^{d+1}}{(\lambda_{1,d+1})^{d}} \\
&= -\frac{1}{d-1}\left[-d^2 + 1 + (d-1)(2d+3) - 2\left(\frac{d(d+1)}{2} - 1\right) \right] \frac{(\lambda_{1,d+2})^{d+1}}{(\lambda_{1,d+1})^{d}} \\
&= 0.
\end{aligned}$$

Thus $\lambda_{d+2,d+1} = 0$. We use again an inductive argument. We assume that $\lambda_{k,d+1} = 0$ for all k such that $d+1 \leq k \leq n-1$, and we would like to show that this implies $\lambda_{n,d+1} = 0$. From (4.15) we have

$$\begin{aligned}
\lambda_{n,d+1} =& \frac{1}{(d-n+)}\left[d\frac{(\lambda_{1,d+2})^{n-1}}{(\lambda_{1,d+1})^{n-2}} - \sum_{s=1}^{d}(n+d-2s+1)\frac{(\lambda_{1,d+2})^{n-s}}{(\lambda_{1,d+1})^{n-2}}\lambda_{s,d+1}\right] \\
=& \frac{1}{(d-n+1)}\left[-(n-1)+\frac{1}{d-1}\sum_{s=2}^{d}(n+d-2s+1)\right]\frac{(\lambda_{1,d+2})^{n-1}}{(\lambda_{1,d+1})^{n-2}} \\
=& \frac{1}{(d-n+1)(d-1)}\Bigg[-(n-1)(d-1)+(d-1)(n+d+1)- \\
& \qquad -2\left(\frac{d(d+1)}{2}-1\right)\Bigg]\frac{(\lambda_{1,d+2})^{n-1}}{(\lambda_{1,d+1})^{n-2}} \\
=& 0.
\end{aligned}$$

This concludes the proof of the lemma.■

Theorem 4.12. *For every $d \geq 2$ and $\lambda \equiv \frac{\lambda_{1,d+2}}{\lambda_{1,d+1}}$ the function*

(4.16)
$$\varphi_{d,\lambda}(u,v) = \frac{1}{(d-1)(1-\lambda u)(1-\lambda v)}\left\{(d-1)uv(v^d-u^d)+\lambda d u^2v^2(u^{d-1}-v^{d-1})\right\}$$

is a solution of (4.4). It is a one-parameter deformation of the solution $\varphi(u,v) = uv(v^d-u^d)$ $\forall\ d \geq 2$, which we obtain back from (4.16) by setting $\lambda = 0$.

Remark. One can obtain the solution $\varphi(u,v) = uv(v-u)$, which gives the Poisson-Lie structure with $d=1$ in Theorem 4.4 from (4.16) in the following way. Rewriting (4.16) as

$$\varphi_{d,\lambda}(u,v) = \frac{uv(v^d-u^d)}{(1-\lambda u)(1-\lambda v)} + \frac{\lambda d u^2v^2}{(1-\lambda u)(1-\lambda v)}\frac{u^{d-1}-v^{d-1}}{d-1},$$

we pass to the limit $d \to 1$ and then set $\lambda = 0$.

Proof. With the assumptions of Lemma 4.11 we have

$$f_d(u) = \sum_{n=d+1}^{\infty} \lambda_{1n} u^n = \sum_{n=d+1}^{\infty} \frac{(\lambda_{1,d+2})^{n-d-1}}{(\lambda_{1,d+1})^{n-d-2}} u^n$$
$$= \lambda_{1,d+1} u^{d+1} \sum_{n=0}^{\infty} \left(\frac{\lambda_{1,d+2}}{\lambda_{1,d+1}} u \right)^n$$
$$= \frac{\lambda_{1,d+1}}{1-\lambda u} u^{d+1}$$

$$g_d(u,v) = -\sum_{n=1}^{d} \lambda_{n,d+1} u^n = -\lambda_{1,d+1} u + \frac{1}{d-1} \sum_{n=2}^{d} \frac{(\lambda_{1,d+2})^{n-1}}{(\lambda_{1,d+1})^{n-2}} u^n$$
$$= -\lambda_{1,d+1} u + \frac{\lambda_{1,d+2} u^2}{d-1} \sum_{n=2}^{d} \left(\frac{\lambda_{1,d+2}}{\lambda_{1,d+1}} u \right)^{n-2}$$
$$= -\lambda_{1,d+1} u + \frac{1}{d-1} \lambda_{1,d+2} u^2 \left[\frac{1-(\lambda u)^{d-1}}{1-\lambda u} \right]$$
$$= \frac{\lambda_{1,d+1}}{(d-1)(1-\lambda u)} \left\{ -(d-1)(1-\lambda u)u + \lambda u^2 \left[1-(\lambda u)^{d-1} \right] \right\}$$
$$= \frac{\lambda_{1,d+1}}{(d-1)(1-\lambda u)} \left[d\lambda u^2 - u(d-1+(\lambda u)^d) \right].$$

By Theorem 4.5 we obtain

$$\varphi_{d,\lambda}(u,v) = \frac{1}{\lambda_{1,d+1}} \left[f_d(u) g_d(v) - f_d(v) g_d(u) \right]$$
$$= \frac{1}{(d-1)(1-\lambda u)(1-\lambda v)} \left\{ v^{d+1} \left[-\lambda d u^2 + u(d-1+(\lambda u)^d) \right] \right.$$
$$\left. - u^{d+1} \left[-\lambda d v^2 + v(d-1+(\lambda v)^d) \right] \right\}.$$

After simplification we finally have

$$\varphi_{d,\lambda}(u,v) = \frac{1}{(d-1)(1-\lambda u)(1-\lambda v)} \left\{ (d-1)uv(v^d - u^d) + \lambda d u^2 v^2 (u^{d-1} - v^{d-1}) \right\}. \blacksquare$$

Remark. One can independently check that (4.16) solves the functional (partial) differential equation by directly substituting (4.16) into the equation.

Summarizing, we showed existence of an infinite parameter family of Poisson-Lie structures on the group G_∞. In the next chapter we will show that this family exhausts all such possible structures on G_∞.

CHAPTER V

THE GROUP G_∞ AND THE r-MATRIX

We find here all Lie-bialgebra structures on the Lie algebra $\mathcal{G}_\infty$ of the group G_∞. Then we show that there is a one-to-one correspondence between these Lie-bialgebra structures and the Poisson-Lie structures found in the previous chapter.

In this chapter we will describe the correspondence between the solution

$$\Omega(u,v;\mathcal{X}) = \varphi(u,v)\mathcal{X}'(u)\mathcal{X}'(v) - \varphi(\mathcal{X}(u),\mathcal{X}(v)) \tag{5.1}$$

of the cocycle equation and the classical r-matrix on $\mathcal{G}_\infty$. It turns out that there is a one-to-one correspondence between the Poisson-Lie structures on G_∞ given by (5.1) and r-matrices on $\mathcal{G}_\infty$. Namely, if we write $\varphi(u,v) = \sum_{m,n=1}^{\infty} \lambda_{mn} u^m v^n$, where $\lambda_{mn} = -\lambda_{nm}$, and $r = r_{ij} e_i \wedge e_j$, where $r_{ij} = -r_{ji}$ and $\{e_i\}_{i\geq 0}$ form a basis of the Lie algebra $\mathcal{G}_\infty$, one shows that $\lambda_{i+1,j+1} = r_{ij}$, $\forall_{i,j\geq 0}$. This we will prove by demonstrating that the λ_{ij}'s and the r_{ij}'s satisfy the same infinite system of algebraic equations. In what follows, $\{x_i\}_{i\geq 1}$ will be again a set of local coordinates of a point of the group G_∞. We recall also that a 1-cochain $\alpha\colon \mathcal{G}_\infty \to \mathcal{G}_\infty \wedge \mathcal{G}_\infty$ acting on a basis element $e_n \in \mathcal{G}_\infty$ is written as $\alpha(e_n) = \alpha_{ij}^n e_i \wedge e_j$, where summation is understood over the repeated indices. If α is 1-cocycle, then

$$(\delta\alpha)(e_l, e_m) = e_l.\alpha(e_m) - e_m.\alpha(e_l) - \alpha([e_l, e_m]) = 0. \tag{5.2}$$

where δ is the coboundary operator in the Chevalley-Eilenberg cohomology of Lie algebras (see formula (*) in Ch.I).

Let us assume for a momment that $\mathcal{G}$ is a finite-dimensional Lie algebra. Let $r = r_{ij} e_i \wedge e_j \in \wedge^2\mathcal{G}$ be a 0-cochain, and let $\alpha\colon \mathcal{G} \to \mathcal{G} \wedge \mathcal{G}$ be defined as $\alpha = \delta r$. Let us also define $< r, r > \in \otimes^3\mathcal{G}$ as

$$< r, r > \equiv \left[r^{12}, r^{13}\right] + \left[r^{12}, r^{23}\right] + \left[r^{13}, r^{23}\right], \tag{5.3}$$

where we have

$$\left[r^{12}, r^{13}\right] \equiv r_{ij} r_{kl} \left[e_i, e_k\right] \wedge e_j \wedge e_l = r_{ij} r_{kl} C_n^{ik} e_n \wedge e_j \wedge e_l,$$

$$[r^{12}, r^{23}] \equiv r_{ni} r_{kl} e_n \wedge [e_i, e_k] \wedge e_l = r_{ni} r_{kl} C_j^{ik} e_n \wedge e_j \wedge e_l,$$
$$[r^{13}, r^{23}] \equiv r_{ni} r_{jk} e_n \wedge e_j \wedge [e_i, e_k] = r_{ni} r_{jk} C_l^{ik} e_n \wedge e_j \wedge e_l.$$

In the above expressions we used $[e_i, e_k] = C_n^{ik} e_n$, where C_n^{ki} are the structure constants of the Lie algebra $\mathcal{G}$. Now, we can rewrite (5.3) in tensor notation as

$$< r, r >= [C_n^{ik} r_{ij} r_{kl} + C_j^{ik} r_{il} r_{kn} + C_l^{ik} r_{in} r_{kj}] e_n \wedge e_j \wedge e_l. \tag{5.4}$$

We have the following:

Lemma 5.1 [Drinfel'd]. *The coboundary α satisfies the co-Jacobi identity, i.e. α defines a Lie-bialgebra structure on $\mathcal{G}$, if and only if $< r, r >$ is $\mathcal{G}$-invariant with respect to the adjoint action of $\mathcal{G}$ on itself.*

Proof. Let $\alpha(e_n) = \alpha_{ij}^n e_i \wedge e_j$. Then α satisfies the co-Jacobi identity if and only if

$$\alpha_{ij}^n \alpha_{sp}^j + \alpha_{pj}^n \alpha_{is}^j + \alpha_{sj}^n \alpha_{pi}^j = 0. \tag{5.5}$$

Using the fact that α is a coboundary $\alpha(e_n) = \delta r(e_n) = (r_{is} C_j^{ns} + r_{sj} C_i^{ns}) e_i \wedge e_j$ we rewrite (5.5) as

$$\begin{aligned}
&(r_{is} C_j^{ns} + r_{sj} C_i^{ns})(r_{kp} C_l^{jp} + r_{pl} C_k^{jp}) + \\
&+ (r_{ks} C_j^{ns} + r_{sj} C_k^{ns})(r_{lp} C_i^{jp} + r_{pi} C_l^{jp}) + \\
&+ (r_{ls} C_j^{ns} + r_{sj} C_l^{ns})(r_{ip} C_k^{jp} + r_{pk} C_i^{jp}) = 0.
\end{aligned}$$

This system of equations is equivalent to

$$\begin{aligned}
&r_{is} C_j^{ns} r_{kp} C_l^{jp} + r_{sj} C_i^{ns} r_{kp} C_l^{jp} + r_{is} C_j^{ns} r_{pl} C_k^{jp} + r_{sj} C_i^{ns} r_{pl} C_k^{jp} \\
&+ r_{ks} C_j^{ns} r_{lp} C_i^{jp} + r_{sj} C_k^{ns} r_{lp} C_i^{jp} + r_{ks} C_j^{ns} r_{pi} C_l^{jp} + r_{sj} C_k^{ns} r_{pi} C_l^{jp} \\
&+ r_{ls} C_j^{ns} r_{ip} C_k^{jp} + r_{sj} C_l^{ns} r_{ip} C_k^{jp} + r_{ls} C_j^{ns} r_{pk} C_i^{jp} + r_{sj} C_l^{ns} r_{pk} C_i^{jp} = 0.
\end{aligned}$$

Next we perform some algebraic manipulations using the Jacobi identity for the Lie algebra structure constants of $\mathcal{G}$

$$C_m^{ij} C_n^{mk} + C_m^{jk} C_n^{mi} + C_m^{ki} C_n^{mj} = 0.$$

We observe that

$$\begin{aligned}
r_{is} C_j^{ns} r_{kp} C_l^{jp} + r_{ks} C_j^{ns} r_{pi} C_l^{jp} &= r_{is} C_j^{ns} r_{kp} C_l^{jp} + r_{kp} C_j^{np} r_{si} C_l^{js} \\
&= r_{is} r_{kp} (C_j^{ns} C_l^{jp} - C_j^{np} C_l^{js}) \\
&= -r_{is} r_{kp} (C_j^{sn} C_l^{jp} + C_j^{np} C_l^{js}) \\
&= r_{is} r_{kp} C_j^{ps} C_l^{jn}.
\end{aligned}$$

Similarly we have

$$r_{is}C_j^{ns}r_{pl}C_k^{jp} + r_{ls}C_j^{ns}r_{ip}C_k^{jp} = r_{is}r_{pl}C_j^{ps}C_k^{jn},$$

and

$$r_{ks}C_j^{ns}r_{lp}C_i^{jp} + r_{ls}C_j^{ns}r_{pk}C_i^{jp} = r_{ks}r_{lp}C_j^{ps}C_i^{jn}.$$

Therefore the co-Jacobi identity is equivalent to

$$\begin{aligned} &r_{is}r_{kp}C_j^{ps}C_l^{jn} + r_{sj}C_i^{ns}r_{kp}C_l^{jp} + r_{sj}C_i^{ns}r_{pl}C_k^{jp} \\ &+r_{sj}C_k^{ns}r_{lp}C_i^{jp} + r_{sj}C_k^{ns}r_{pi}C_l^{jp} + r_{is}r_{pl}C_j^{ps}C_k^{jn} \\ &+r_{ks}r_{lp}C_j^{ps}C_i^{jn} + r_{sj}C_l^{ns}r_{ip}C_k^{jp} + r_{sj}C_l^{ns}r_{pk}C_i^{jp} = 0. \end{aligned} \tag{5.6}$$

Let us now consider the system of equations $e_m. < r, r >= 0$ for every $m \geq 0$. That is,

$$[C_n^{ik}r_{ij}r_{kl} + C_j^{ik}r_{ni}r_{kl} + C_l^{ik}r_{ni}r_{jk}]e_m.(e_n \wedge e_j \wedge e_l) = 0. \tag{5.7}$$

Calculating

$$\begin{aligned} e_m.(e_n \wedge e_j \wedge e_l) &= [e_m, e_n] \wedge e_j \wedge e_l + e_n \wedge [e_m, e_j] \wedge e_l + e_n \wedge e_j \wedge [e_m, e_l] \\ &= C_s^{mn}e_s \wedge e_j \wedge e_l + C_s^{mj}e_n \wedge e_s \wedge e_l + C_s^{ml}e_n \wedge e_j \wedge e_s, \end{aligned}$$

and renaming indices when necessary we obtain that (5.7) is equivalent to

$$\begin{aligned} &r_{ij}r_{kl}C_s^{ik}C_n^{ms} + r_{is}C_n^{ik}r_{kl}C_j^{ms} + r_{ij}C_n^{ik}r_{ks}C_l^{ms} \\ &+r_{si}r_{kl}C_j^{ik}C_n^{ms} + r_{ni}C_s^{ik}r_{kl}C_j^{ms} + r_{ni}C_j^{ik}r_{ks}C_l^{ms} \\ &+r_{si}r_{jk}C_l^{ik}C_n^{ms} + r_{ni}C_l^{ik}r_{sk}C_j^{ms} + r_{ni}C_s^{ik}r_{jk}C_l^{ms} = 0. \end{aligned} \tag{5.8}$$

Again after renaming indices we conclude that the above system of equations is identical to (5.6). This concludes the proof.■

A subclass of coboundary Lie-bialgebra structures is obtained when $< r, r >= 0$; written explicitly, this condition has the form

$$C_n^{ik}r_{ij}r_{kl} + C_j^{ik}r_{il}r_{kn} + C_l^{ik}r_{in}r_{kj} = 0. \tag{5.9}$$

This is the so-called Classical Yang-Baxter Equation (CYBE).

For the Lie algebra $\mathcal{G}_\infty$ the structure constants are $C_k^{ij} = (i-j)\delta_k^{i+j}$, where $i, j, k \geq 1$. One easily sees that the arguments given above apply in this case, since

the presence of the Kronecker symbol in the formula for the structure constants as well as the fact that $r_{ij} = 0$ whenever $i < 0,\ j < 0$ make all sums finite. Therefore (5.9) becomes

$$(i-k)\delta_n^{i+k} r_{ij} r_{kl} + (i-k)\delta_j^{i+k} r_{il} r_{kn} + (i-k)\delta_l^{i+k} r_{in} r_{kj} = 0,$$

or

$$\sum_{k=0}^{\max(n,j,l)} [(n-2k) r_{n-k,j} r_{kl} + (j-2k) r_{j-k,l} r_{kn} + (l-2k) r_{l-k,n} r_{kj}] = 0. \tag{5.10}$$

Using the fact that

$$\sum_{k=0}^{n} (n-k) r_{n-k,j} r_{kl} = \sum_{k=0}^{n} k r_{kj} r_{n-k,l},$$

we have

$$\begin{aligned}\sum_{k=0}^{n} (n-2k) r_{n-k,j} r_{kl} &= \sum_{k=0}^{n} [(n-k) r_{n-k,j} r_{kl} - k r_{n-k,j} r_{kl}] \\ &= \sum_{k=0}^{n} k\,[r_{kj} r_{n-k,l} - r_{n-k,j} r_{kl}] \\ &= \sum_{k=0}^{n} k\,[r_{kj} r_{n-k,l} + r_{j,n-k} r_{kl}],\end{aligned}$$

and similarly for the other two terms in (5.10). Thus, (5.10) assumes the form

$$\sum_{k=0}^{\max(n,j,l)} k\,[(r_{n-k,l} + r_{n,l-k}) r_{kj} + (r_{j,n-k} + r_{j-k,n}) r_{kl} + (r_{l,j-k} + r_{l-k,j}) r_{kn}] = 0. \tag{5.11}$$

(In the above formulae we implicitly assume that $r_{ij} = 0$ whenever $i < 0$ or $j < 0$.) This is the CYBE for $\mathcal{G}_\infty$.

We now proceed with the proof of the following important result.

Lemma 5.2. *For the Lie algebra $\mathcal{G}_\infty$ the system of equations $e_n . < r, r >= 0$, for every $n \in \mathbb{Z}_+$, implies $< r, r >= 0$.*

Remark. This means that all coboundary Lie-bialgebra structures on $\mathcal{G}_\infty$ are given by the solutions of CYBE.

Proof. For $\mathcal{G}_\infty$ the equation $e_m . < r, r >= 0$, for any $m \in \mathbb{Z}_+$, becomes

$$(A) + (B) + (C) = 0,$$

where

$$\begin{aligned}(A) =&(i-k)(m-s)\delta_s^{i+k}\delta_n^{m+s}r_{ij}r_{kl}+(i-k)(m-s)\delta_n^{i+k}\delta_j^{m+s}r_{is}r_{kl}+\\ &+(i-k)(m-s)\delta_n^{i+k}\delta_l^{m+s}r_{ij}r_{ks}\\ =&(i-k)(m-i-k)\delta_n^{m+i+k}r_{ij}r_{kl}+(2i-n)(2m-j)r_{i,j-m}r_{n-i,l}+\\ &+(2i-n)(2m-j)r_{ij}r_{n-i,l-m}\\ =&(2i+m-n)(2m-n)r_{ij}r_{n-m-i,l}+(2i-n)(2m-j)r_{i,j-m}r_{n-i,l}+\\ &+(2i-n)(2m-j)r_{ij}r_{n-i,l-m},\end{aligned}$$

$$(B)=(2i-j)(2m-n)r_{n-m,i}r_{j-i,l}+(2i-j+m)(2m-j)r_{ni}r_{j-m-i,l}+(2i-j)(2m-l)r_{ni}r_{j-i,l-m},$$

and

$$(C)=(2i-l)(2m-n)r_{n-m,i}r_{j,l-i}+(2i-l)(2m-j)r_{ni}r_{j-m,l-i}+(2i-l+m)(2m-l)r_{ni}r_{j,l-m-i}.$$

Therefore the system of equations $e_m.<r,r>=0$ is equivalent to the following system of equations

$$\begin{aligned}&\sum_{i=0}^{n-m}(2i+m-n)(2m-n)r_{ij}r_{n-m-i,l}+\sum_{i=0}^{j-m}(2i-j+m)(2m-j)r_{ni}r_{j-m-i,l}+\\ &+\sum_{i=0}^{l-m}(2i-l+m)(2m-l)r_{ni}r_{j,l-m-i}\\ &+\sum_{i=0}^{n}[(2i-n)(2m-j)r_{i,j-m}r_{n-i,l}+(2i-n)(2m-j)r_{ij}r_{n-i,l-m}]\\ &+\sum_{i=0}^{j}[(2i-j)(2m-n)r_{n-m,i}r_{j-i,l}+(2i-j)(2m-l)r_{ni}r_{j-i,l-m}]\\ &+\sum_{i=0}^{l}[(2i-l)(2m-n)r_{n-m,i}r_{j,l-i}+(2i-l)(2m-j)r_{ni}r_{j-m,l-i}]=0.\end{aligned}$$

In order to prove that $e_m.<r,r>=0$ implies $<r,r>=0$ it is enough to prove this implication for $m=0$, i.e. that $e_0.<r,r>=0$ implies $<r,r>=0$. Let $m=0$

in the above system of equations. Then we have

$$
\begin{aligned}
&-\sum_{i=0}^{n}(2i-n)nr_{ij}r_{n-i,l}-\sum_{i=0}^{j}(2i-j)jr_{ni}r_{j-i,l}-\sum_{i=0}^{l}(2i-l)lr_{ni}r_{j,l-i}\\
&+\sum_{i=0}^{n}[-(2i-n)jr_{ij}r_{n-i,l}-(2i-n)lr_{ij}r_{n-i,l}]\\
&+\sum_{i=0}^{j}[-(2i-j)nr_{ni}r_{j-i,l}-(2i-j)lr_{ni}r_{j-i,l}]\\
&+\sum_{i=0}^{l}[-(2i-l)nr_{ni}r_{j,l-i}-(2i-l)jr_{ni}r_{j,l-i}]=0,
\end{aligned}
$$

which is finally equivalent to

$$
(n+j+l)\left\{\sum_{i=0}^{n}(2i-n)r_{ij}r_{n-i,l}+\sum_{i=0}^{j}(2i-j)r_{ni}r_{j-i,l}+\sum_{i=0}^{l}(2i-l)r_{ni}r_{j,l-i}\right\}=0.
$$

Now, using the identities

$$
\begin{aligned}
\sum_{i=0}^{n}(2i-n)r_{ij}r_{n-i,l}&=\sum_{i=0}^{n}ir_{ij}r_{n-i,l}+\sum_{i=0}^{n}(i-n)r_{ij}r_{n-i,l}\\
&=\sum_{i=0}^{n}i(r_{ij}r_{n-i,l}-r_{n-i,j}r_{il})\\
&=\sum_{i=0}^{n}i(r_{ij}r_{n-i,l}+r_{j,n-i}r_{il}),
\end{aligned}
$$

and similarly for

$$
\sum_{i=0}^{j}(2i-j)r_{ni}r_{j-i,l}=\sum_{i=0}^{j}i(r_{ni}r_{j-i,l}+r_{j-i,n}r_{il}),
$$

and for

$$
\sum_{i=0}^{l}(2i-l)r_{ni}r_{j,l-i}=\sum_{i=0}^{l}i(r_{ni}r_{j,l-i}+r_{n,l-i}r_{ij}),
$$

we conclude that $e_0.<r,r>=0$ is equivalent to

$$
(n+j+l)\left\{\sum_{i=0}^{\max(n,j,l)}i\left[(r_{n-i,l}+r_{n,l-i})r_{ij}+(r_{j,n-i}+r_{j-i,n})r_{il}+(r_{l,j-i}+r_{l-i,j})r_{in}\right]\right\}=0.
$$

Since $n+j+l=0$ if and only if $n=j=l=0$, and in this case $e_m. < r,r >$ is identically zero for any $m \in \mathbb{Z}_+$, the above system of equations implies that the expression in the curly brackets vanishes, and by formula (5.11) this is exactly $< r,r >= 0$. ■

As we have seen, if α is a coboundary of r then α has the form

$$\begin{aligned}\alpha(e_n) = (\delta r)(e_n) &= e_n.r \\ &= r_{ij}([e_n, e_i] \wedge e_j + e_i \wedge [e_n, e_j]) \\ &= (r_{lj}C_i^{nl} + r_{il}C_j^{nl})e_i \wedge e_j.\end{aligned}$$

Therefore,

$$\begin{aligned}\alpha_{ij}^n &= r_{lj}C_i^{nl} + r_{il}C_j^{nl} \\ &= (2n-i)r_{i-n,j} + (2n-j)r_{i,j-n}.\end{aligned} \tag{5.12}$$

Now, we turn our attention to the Poisson bracket on the group, written in local coordinates, by writing $\Omega(u,v;\mathcal{X})$ in components. In order to do this we will need the formula

$$\begin{aligned}\mathcal{X}(u)^n &= \sum_{s_1=1}^{\infty} x_{s_1}u^{s_1} \dots \sum_{s_n=1}^{\infty} x_{s_n}u^{s_n} \\ &= \sum_{s_1=1}^{\infty} \dots \sum_{s_n=1}^{\infty} x_{s_1} \dots x_{s_n} u^{s_1+\dots+s_n} \\ &= \sum_{i=n}^{\infty} \left(\sum_{(\sum_{k=1}^n s_k)=i} x_{s_1} \dots x_{s_n} \right) u^i.\end{aligned}$$

Then we have

$$\Omega(u,v;\mathcal{X}) = \varphi(u,v)\mathcal{X}'(u)\mathcal{X}'(v) - \varphi(\mathcal{X}(u),\mathcal{X}(v)) =$$

$$\begin{aligned}= \sum_{p,q=1}^{\infty} \lambda_{pq}u^pv^q \sum_{i=1}^{\infty} ix_iu^{i-1} \sum_{j=1}^{\infty} jx_jv^{j-1} - \\ - \sum_{p,q=1}^{\infty} \lambda_{pq} \sum_{i=p}^{\infty} \left(\sum_{(\sum_{k=1}^p r_k)=i} x_{r_1} \dots x_{r_p} \right) u^i \sum_{j=q}^{\infty} \left(\sum_{(\sum_{l=1}^q s_l)=j} x_{s_1} \dots x_{s_q} \right) v^j\end{aligned}$$

$$= \sum_{p,q=1}^{\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \lambda_{pq} i x_i j x_j u^{p+i-1} v^{q+j-1} -$$

$$- \sum_{p,q=1}^{\infty}\sum_{i=p}^{\infty}\sum_{j=q}^{\infty} \lambda_{pq} \left(\sum_{(\sum_{k=1}^{p} r_k)=i} x_{r_1} \dots x_{r_p} \sum_{(\sum_{l=1}^{q} s_l)=j} x_{s_1} \dots x_{s_q} \right) u^i v^j$$

$$= \sum_{p,q=1}^{\infty}\sum_{i=p}^{\infty}\sum_{j=q}^{\infty} \left(\lambda_{i-p+1,j-q+1} p x_p q x_q - \lambda_{pq} \sum_{(\sum_{k=1}^{p} r_k)=i} x_{r_1} \dots x_{r_p} \sum_{(\sum_{l=1}^{q} s_l)=j} x_{s_1} \dots x_{s_q} \right) u^i v^j$$

$$= \sum_{i,j=1}^{\infty} \left[\sum_{p=1}^{i}\sum_{q=1}^{j} \left(\lambda_{i-p+1,j-q+1} p x_p q x_q - \lambda_{pq} \sum_{(\sum_{k=1}^{p} r_k)=i} x_{r_1} \dots x_{r_p} \sum_{(\sum_{l=1}^{q} s_l)=j} x_{s_1} \dots x_{s_q} \right) \right] u^i v^j.$$

Therefore for $\{x_i, x_j\} = \omega_{ij}(x)$ we obtain

(5.13) $\omega_{ij}(x) =$

$$= \sum_{p=1}^{i}\sum_{q=1}^{j} p x_p q x_q \lambda_{i-p+1,j-q+1} - \sum_{p=1}^{i}\sum_{q=1}^{j} \lambda_{pq} \left(\sum_{(\sum_{k=1}^{p} r_k)=i} x_{r_1} \dots x_{r_p} \sum_{(\sum_{l=1}^{q} s_l)=j} x_{s_1} \dots x_{s_q} \right).$$

Before we continue further, let us deduce the following useful formulae

$$\frac{\partial}{\partial x_n} \sum_{(\sum_{k=1}^{p} r_k)=i} x_{r_1} \dots x_{r_p} = \sum_{(\sum_{k=1}^{p} r_k)=i} \left(\sum_{l=1}^{p} x_{r_1} \dots \delta^n_{r_l} \dots x_{r_p} \right)$$

$$= \sum_{l=1}^{p} \left(\sum_{(\sum_{k=1}^{p} r_k)=i} x_{r_1} \dots \delta^n_{r_l} \dots x_{r_p} \right)$$

$$= p \sum_{(\sum_{k=1}^{p-1} r_k)=i-n} x_{r_1} \dots x_{r_{p-1}},$$

as well as

$$\sum_{(\sum_{k=1}^{p-1} r_k)=i-n} x_{r_1} \dots x_{r_{p-1}} \Big|_e = \sum_{(\sum_{k=1}^{p-1} r_k)=i-n} \delta^1_{r_1} \dots \delta^1_{r_{p-1}} = \delta^{p-1}_{i-n},$$

$$\left(\frac{\partial}{\partial x_n} \sum_{(\sum_{k=1}^{p} r_k)=i} x_{r_1} \dots x_{r_p} \right) \Bigg|_e = p \delta^{p-1}_{i-n}.$$

Differentiating (5.13) with respect to x_n we obtain

$$\begin{aligned}\frac{\partial\omega_{ij}(x)}{\partial x_n} &= \sum_{p=1}^{i}\sum_{q=1}^{j}(p\delta_p^n q x_q \lambda_{i-p+1,j-q+1} + p x_p \delta_q^n q \lambda_{i-p+1,j-q+1}) \\ &- \sum_{p=1}^{i}\sum_{q=1}^{j}\left[p \sum_{(\sum_{k=1}^{p-1} r_k)=i-n} x_{r_1}\dots x_{r_{p-1}} \sum_{(\sum_{l=1}^{q} s_l)=j} x_{s_1}\dots x_{s_q}\right]\lambda_{pq} \\ &- \sum_{p=1}^{i}\sum_{q=1}^{j}\left[q \sum_{(\sum_{k=1}^{p} r_k)=i} x_{r_1}\dots x_{r_p} \sum_{(\sum_{l=1}^{q-1} s_k)=j-n} x_{s_1}\dots x_{s_{q-1}}\right]\lambda_{pq}.\end{aligned}$$

From the above formula we have (keeping in mind that $x_p|_e = \delta_p^1$)

$$\begin{aligned}\beta_{ij}^n \equiv \left.\frac{\partial\omega_{ij}(x)}{\partial x_n}\right|_e &= \sum_{p=1}^{i} p\delta_p^n \lambda_{i-p+1,j} + \sum_{q=1}^{j} q\delta_q^n \lambda_{i,j-q+1} - \sum_{p=1}^{i}\sum_{q=1}^{j}\lambda_{pq}\left[p\delta_{i-n+1}^p\delta_j^q + q\delta_i^p\delta_{j-n+1}^q\right] \\ &= \sum_{p=1}^{i}(p\delta_p^n\lambda_{i-p+1,j} - p\delta_{i-n+1}^p\lambda_{pj}) + \sum_{q=1}^{j}(q\delta_q^n\lambda_{i,j-q+1} - q\delta_{j-n+1}^q\lambda_{iq}) \\ &= n\lambda_{i-n+1,j} - (i-n+1)\lambda_{i-n+1,j} + n\lambda_{i,j-n+1} - (j-n+1)\lambda_{i,j-n+1} \\ &= (2n-i-1)\lambda_{i-n+1,j} + (2n-j-1)\lambda_{i,j-n+1}.\end{aligned}$$

Thus, we finally obtain that

$$\beta_{ij}^n = (2n-i-1)\lambda_{i-n+1,j} + (2n-j-1)\lambda_{i,j-n+1}, \tag{5.14}$$

which is the same as (5.12) after we make the identification $r_{i,j-1} = \lambda_{i+1,j}$ $\forall i \geq 0, j \geq 1$, which is equivalent to $r_{ij} = \lambda_{i+1,j+1}$ $\forall i,j \geq 0$, since r_{ij} and λ_{ij} are antisymmetric.

On the other hand the system of equations (5.11) is exactly the same as the system of equations (4.6) which λ_{ij} satisfy. This allows us to conclude that the elements of the r-matrix are given by λ_{ij}, since λ_{ij} and r_{ij} satisfy the same system of equations.

The results of the calculations made in this chapter may be summarized in the following.

Theorem 5.3. *There is a one-to-one correspondence between the coboundary Lie bialgebra structures on $\mathcal{G}_\infty$ given by r and the Poisson-Lie structures of the type (5.1) on G_∞. Since all Lie bialgebra structures on $\mathcal{G}_\infty$ are given by r (cf. Theorem*

3.1), Theorem 4.5 gives a classification of all solutions of the classical Yang-Baxter equation for $\mathcal{G}_\infty$.

Proof. We give here an alternative proof. Recall that ω_{mn} satisfy the infinite system of functional equations (1.6)

$$\omega_{mn}(\xi) = \omega_{kl}(x)\frac{\partial \xi_m}{\partial x_k}\frac{\partial \xi_n}{\partial x_l} + \omega_{kl}(y)\frac{\partial \xi_m}{\partial y_k}\frac{\partial \xi_n}{\partial y_l}, \quad \text{where } x, y \in G_\infty, \tag{5.15}$$

and $\xi_n = \xi_n(x, y)$ is given by formula (2.1)

$$\xi_n = \sum_{i=1}^{n} x_i \sum_{(\sum_{\alpha=1}^{i} j_\alpha)=n} y_{j_1} \cdots y_{j_i}. \tag{5.16}$$

From (5.16) it follows that

$$\left.\frac{\partial \xi_i}{\partial y_k}\right|_{y=e} = (i-k+1)x_{i-k+1} \quad \text{and} \quad \left.\frac{\partial \xi_i}{\partial x_k}\right|_{y=e} = \delta_i^k. \tag{5.17}$$

Let us fix $n \in \mathbb{N}$ and consider a subsystem of the system of equations (5.15) for all ω_{ij} with $1 \le i < j \le n$. After differentiating (5.15) with respect to y_j, for each j such that $1 \le j \le n$, and setting $y = e$ we deduce that ω_{mn} satisfy the following inhomogeneous system of linear partial differential equations

(5.18)
$$\sum_{i=j}^{n}(i+1-j)x_{i+1-j}\frac{\partial \omega_{mn}}{\partial x_i} = \omega_{m+1-j,n}(x)(m+1-j) + \omega_{m,n+1-j}(x)(n+1-j) + \\ + \sum_{k=1}^{m}\sum_{l=1}^{n}\beta_{kl}^{j}(m+1-k)(n+1-l)x_{m+1-k}x_{n+1-l},$$

for $1 \le j \le n$, and where $\beta_{kl}^{j} = \left.\frac{\partial \omega_{kl}(y)}{\partial y_j}\right|_{y=e}$.

The idea of the proof is as follows. Let β_{kl}^{j} be given by (5.14). For each $n \in \mathbb{N}$ the general solution of (5.18) is a linear combination of the general solution of the homogeneous system of equations

$$\sum_{i=j}^{n}(i+1-j)x_{i+1-j}\frac{\partial \omega_{mn}}{\partial x_i} = \omega_{m+1-j,n}(x)(m+1-j) + \omega_{m,n+1-j}(x)(n+1-j) \tag{5.19}$$

and a particular solution of the inhomogeneous system (5.18). We now show that for each $n \in \mathbb{N}$ and $1 \leq m < n$ the system (5.18) has a unique solution by demonstrating that the only solution of the homogeneous system is the zero solution. Therefore, since every solution of the system of functional equations (5.15) is a solution of the system of partial differential equations (5.18) it follows that the class of solutions of (5.15) found in Chapter IV exhausts all possible solutions of (5.15). We will prove that the only solution of (5.19) is the zero solution by induction applied in several steps. Recall that

$$\omega_{mn}(e) = 0, \quad \text{for every } n, m \in \mathbb{N}. \tag{5.20}$$

In the following arguments we implicitly assume that $\omega_{mn} = 0$ whenever $n < 1$ or $m < 1$.

(i) If $n = 1$ there is nothing to prove. Let $n = 2$. Then from (5.19) we obtain

$$x_1 \frac{\partial \omega_{12}}{\partial x_1} = 3\omega_{12}. \tag{5.21}$$

The most general solution of this equation is

$$\omega_{12}(x) = Cx_1^3,$$

where C is an arbitrary constant. From (5.20) it follows that $C = 0$. Therefore $\omega_{12}(x) = 0$ is the only solution of (5.21). Let $n = 3$. Then from (5.19) we obtain

$$x_1 \frac{\partial \omega_{13}}{\partial x_1} + 2x_2 \frac{\partial \omega_{13}}{\partial x_2} = 4\omega_{13} \tag{5.22}$$

$$x_1 \frac{\partial \omega_{13}}{\partial x_2} = 0. \tag{5.23}$$

From (5.23) it follows that $\omega_{13}(x) = \omega_{13}(x_1)$. From (5.22) we deduce that $\omega_{13}(x_1)$ satisfies the equation

$$x_1 \frac{\partial \omega_{13}}{\partial x_1} = 4\omega_{13}. \tag{5.24}$$

Therefore $\omega_{13}(x) = Cx_1^5$, and from (5.20) it follows that $C = 0$, and $\omega_{13}(x) = 0$. Let us assume now that $\omega_{1k}(x) = 0$ for $2 \leq k \leq n-1$. From (5.19) we have

(5.25)

$$\sum_{i=j}^{n} (i+1-j) x_{i+1-j} \frac{\partial \omega_{1n}}{\partial x_i} = \omega_{2-j,n}(x)(2-j) + \omega_{1,n+1-j}(x)(n+1-j), \quad \text{for } 1 \leq j \leq n,$$

which implies

$$\sum_{i=1}^{n} i x_i \frac{\partial \omega_{1n}}{\partial x_i} = (n+1)\omega_{1n}(x), \tag{5.26}$$

$$\sum_{i=j}^{n} (i+1-j) x_{i+1-j} \frac{\partial \omega_{1n}}{\partial x_i} = 0, \quad \text{for } 2 \le j \le n. \tag{5.27}$$

We used above the induction hypothesis: $\omega_{1k}(x) = 0$ for $2 \le k \le n-1$ from which follows that the r.h.s of (5.27) is zero. From (5.27) it follows that $\omega_{1n}(x) = \omega_{1n}(x_1)$. Then from (5.26) it follows that $\omega_{1n}(x_1)$ satisfies

$$x_1 \frac{\partial \omega_{1n}}{\partial x_1} = (n+1)\omega_{1n}. \tag{5.28}$$

From (5.28) we have that $\omega_{1n}(x) = C x_1^{n+1}$ for an arbitrary constant C. Applying again (5.20) we conclude that $\omega_{1n}(x) = 0$. Therefore $\omega_{1n}(x) = 0$ for every $n \in \mathbb{N}$.

(ii) Let $m = 2$ and $n = 3$. Then from (5.19) we have the following homogeneous system of partial differential equations for ω_{23}:

$$\begin{aligned} x_1 \frac{\partial \omega_{23}}{\partial x_1} + 2x_2 \frac{\partial \omega_{23}}{\partial x_2} + 3x_3 \frac{\partial \omega_{23}}{\partial x_3} &= 5\omega_{23} \\ x_1 \frac{\partial \omega_{23}}{\partial x_2} + 2x_2 \frac{\partial \omega_{23}}{\partial x_3} &= \omega_{13} = 0 \\ x_1 \frac{\partial \omega_{23}}{\partial x_3} &= -\omega_{12} = 0. \end{aligned}$$

Arguing in a similar manner as above we obtain that $\omega_{23}(x) = 0$. Let us assume that $\omega_{2k}(x) = 0$ for all k such that $3 \le k \le n-1$. We now prove that $\omega_{2n}(x) = 0$. From (5.19) we have

(5.29)

$$\sum_{i=j}^{n} (i+1-j) x_{i+1-j} \frac{\partial \omega_{2n}}{\partial x_i} = \omega_{3-j,n}(x)(3-j) + \omega_{2,n+1-j}(x)(n+1-j), \quad \text{for } 1 \le j \le n.$$

After using the induction hypothesis and the already proved fact that $\omega_{1n} = 0$, $\forall n \in \mathbb{N}$, (5.29) yields

$$\sum_{i=1}^{n} i x_i \frac{\partial \omega_{2n}}{\partial x_i} = (n+2)\omega_{2n}(x), \tag{5.30}$$

$$\sum_{i=j}^{n} (i+1-j) x_{i+1-j} \frac{\partial \omega_{2n}}{\partial x_i} = 0, \quad \text{for } 2 \le j \le n. \tag{5.31}$$

Therefore from (5.31) and (5.30) it follows that $\omega_{2n}(x) = \omega_{2n}(x_1) = Cx_1^{n+2}$, and imposing (5.20) again we obtain that $\omega_{2n}(x_1) = 0$. Thus $\omega_{2n}(x) = 0$ for every $n \in \mathbb{N}$.

(iii) Let us assume that $\omega_{sn} = 0$ for all s such that $1 \leq s \leq m-1$, for some $m \geq 2$ and all $n > s$. We will prove that $\omega_{mn} = 0$ for all $n \geq m$. Let $n = m+1$. From (5.19) we have

(5.32)
$$\sum_{i=j}^{m+1}(i+1-j)x_{i+1-j}\frac{\partial\omega_{m,m+1}}{\partial x_i} = \omega_{m+1-j,n}(x)(m+1-j) + \omega_{m,m+2-j}(x)(m+2-j), \quad \text{for } 1 \leq j \leq m+1.$$

We apply now the induction hypothesis and deduce from (5.32) the following system of equations

$$\sum_{i=1}^{m+1} ix_i\frac{\partial\omega_{m,m+1}}{\partial x_i} = (2m+1)\omega_{m,m+1}(x), \tag{5.33}$$

$$\sum_{i=j}^{m+1}(i+1-j)x_{i+1-j}\frac{\partial\omega_{m,m+1}}{\partial x_i} = 0, \quad \text{for } 2 \leq j \leq m+1. \tag{5.34}$$

From (5.34) it follows that $\omega_{m,m+1}(x) = \omega_{m,m+1}(x_1)$, and from (5.33) we deduce that $\omega_{m,m+1}(x)$ must satisfy

$$x_1\frac{\partial\omega_{m,m+1}}{\partial x_1} = (2m+1)\omega_{m,m+1}. \tag{5.35}$$

The solution of the above equation is $\omega_{m,m+1}(x) = Cx_1^{2m+1}$, where C is an arbitrary constant. Then from $\omega_{m,m+1}(e) = C = 0$ we obtain that $\omega_{m,m+1}(x) = 0$. Finally, we assume that $\omega_{mk} = 0$ for all k such that $m+1 \leq k \leq n-1$, and we prove it for $k = n$. Indeed, from (5.19), after applying the induction hypothesis, we obtain

$$\sum_{i=1}^{n} ix_i\frac{\partial\omega_{mn}}{\partial x_i} = (m+n)\omega_{mn}(x), \tag{5.36}$$

$$\sum_{i=j}^{n}(i+1-j)x_{i+1-j}\frac{\partial\omega_{mn}}{\partial x_i} = 0, \quad \text{for } 2 \leq j \leq n. \tag{5.37}$$

Again, from (5.37) it follows that $\omega_{mn}(x) = \omega_{mn}(x_1)$, and that $\omega_{mn}(x)$ must satisfy

$$x_1\frac{\partial\omega_{mn}}{\partial x_1} = (m+n)\omega_{mn}. \tag{5.38}$$

From here we conclude that $\omega_{mn}(x) = Cx_1^{m+n}$ for an arbitrary constant C. But the requirement $\omega_{mn}(e) = 0$ fixes the value of this constant to be $C = 0$. Therefore $\omega_{mn}(x) = 0$.

Thus, we showed that for every $m, n \in \mathbb{N}$ the only solution of (5.19) is the zero solution. Therefore the system of partial differential equations (5.18) has a unique solution. The existence of the solution follows from the existence of the solution of the system of functional equations (5.15) of which (5.18) is a consequence. Thus, the structure constants β_{kl}^j of the Lie-bialgebra $\mathcal{G}_\infty$, as given by (5.14), determine uniquely all Poisson-Lie structures on the group G_∞. The proof of Theorem 5.3 is completed. ■

We conclude this chapter by writing an explicit formula for the family of Lie-bialgebra structures arising from the family of Poisson-Lie structures obtained in Theorem 4.4 as well as the more general one-parameter family of which it is a particular case for $d \geq 2$. An elegant way to do this is by deriving a global formula for the Lie-bialgebra structures on $\mathcal{G}_\infty$ in terms of generating series and solutions of

$$\varphi(u,v)[\partial_u\varphi(w,u) + \partial_v\varphi(w,v)] + c.p. = 0.$$

Let us define $\mathcal{A}_n(u,v)$ as

$$\mathcal{A}_n(u,v) = \frac{\partial}{\partial x_n}\Omega(u,v;\mathcal{X})\Big|_e = \sum_{i,j=1}^{\infty} \alpha_{ij}^n u^i v^j.$$

Then we have the following lemma.

Lemma 5.4. *The generating series $\mathcal{A}_n(u,v)$ is given by*

$$\mathcal{A}_n(u,v) = n\varphi(u,v)(u^{n-1} + v^{n-1}) - [u^n\partial_u\varphi(u,v) + v^n\partial_v\varphi(u,v)].$$

Proof. We use formula (5.1) and the following facts. If $\mathcal{X}(u) = \sum_{i=1}^{\infty} x_i u^i$ then

$$\mathcal{X}'(u) = \sum_{i=1}^{\infty} i x_i u^{i-1} \implies \mathcal{X}'(u)\,|_e = 1,$$

and also

$$\frac{\partial}{\partial x_n}\mathcal{X}'(u) = \sum_{i=1}^{\infty} i\delta_i^n u^{i-1} = nu^{n-1}.$$

From this it follows that

$$\frac{\partial}{\partial x_n}\left[\mathcal{X}'(u)\mathcal{X}'(v)\right] = nu^{n-1}\mathcal{X}'(v) + nv^{n-1}\mathcal{X}'(u) \implies$$

$$\implies \frac{\partial}{\partial x_n}\left[\mathcal{X}'(u)\mathcal{X}'(v)\right]\Big|_e = n(u^{n-1}+v^{n-1}).$$

Finally we have

$$\frac{\partial}{\partial x_n}\varphi(\mathcal{X}(u),\mathcal{X}(v))\Big|_e = \frac{\partial\mathcal{X}(u)}{\partial x_n}\partial_1\varphi(\mathcal{X}(u),\mathcal{X}(v))\Big|_e + \frac{\partial\mathcal{X}(v)}{\partial x_n}\partial_2\varphi(\mathcal{X}(u),\mathcal{X}(v))\Big|_e$$

$$= u^n\partial_u\varphi(u,v) + v^n\partial_v\varphi(u,v).$$

In the above equality we used that $\frac{\partial\mathcal{X}(u)}{\partial x_n}\Big|_e = u^n$. ■

Proposition 5.5 [Taft]. *For each $d \in \mathbb{N}$ the family of Poisson-Lie structures (5.1) given by $\varphi_d(u,v) = uv(v^d - u^d)$ gives rise to the following family of Lie-bialgebra structures on $\mathcal{G}_\infty$:*

$$\alpha(e_n) = 2ne_d \wedge e_n - 2(n-d)e_0 \wedge e_{d+n}, \quad (n \geq 0) \tag{5.39}$$

where $\{e_n\}_{n\in\mathbb{Z}_+}$ is a basis for $\mathcal{G}_\infty$.

Proof. The generating series $\mathcal{A}_{n,d}$ in this case is

$$\begin{aligned}\mathcal{A}_{n,d}(u,v) =& n[uv^{d+1} - vu^{d+1}](u^{n-1}+v^{n-1}) - \\ &- \left\{u^n[v^{d+1} - (d+1)vu^d] + v^n[(d+1)uv^d - u^{d+1}]\right\} \\ =&(n-1)u^nv^{d+1} - (n-1)v^nu^{d+1} + (n-d-1)uv^{n+d} - (n-d-1)vu^{n+d} \\ =&\left\{(n-1)[\delta_i^n\delta_j^{d+1} - \delta_j^n\delta_i^{d+1}] + (n-d-1)[\delta_i^1\delta_j^{d+n} - \delta_j^1\delta_i^{d+n}]\right\}u^iv^j \\ =& \sum_{i,j=1}^{\infty}\alpha_{ij|d}^n u^iv^j,\end{aligned}$$

where

$$\alpha_{ij|d}^n = \left\{(n-1)[\delta_i^n\delta_j^{d+1} - \delta_j^n\delta_i^{d+1}] + (n-d-1)[\delta_i^1\delta_j^{d+n} - \delta_j^1\delta_i^{d+n}]\right\}.$$

Therefore

$$\begin{aligned}\alpha_d(e_n) &= \alpha_{ij|d}^n e_i \wedge e_j \\ &= (n-1)[e_n \wedge e_{d+1} - e_{d+1} \wedge e_n] + (n-d-1)[e_1 \wedge e_{d+n} - e_{d+n} \wedge e_1] \\ &= 2(n-1)e_n \wedge e_{d+1} + 2(n-d-1)e_1 \wedge e_{n+d},\end{aligned}$$

and after shifting indices by 1 we obtain

$$\alpha_d(e_n) = -2ne_d \wedge e_n + 2(n-d)e_0 \wedge e_{n+d} \quad \text{for every} \quad n \in \mathbb{Z}_+.$$

A second way to derive the above formula is by using Theorem 5.3. The r-matrix is given in this case by $r_{ij} = \delta^1_{i+1}\delta^{d+1}_{j+1} - \delta^{d+1}_{i+1}\delta^1_{j+1} = \lambda_{i+1,j+1}$. Therefore, using (5.12), we have

$$\begin{aligned}
\alpha(e_n) &= \alpha^n_{ij} e_i \wedge e_j \\
&= [(2n-i)r_{i-n,j} + (2n-j)r_{i,j-n}]\, e_i \wedge e_j \\
&= -(2n-i)\delta^{d+1}_{i-n+1}\delta^1_{j+1} e_i \wedge e_j + (2n-i)\delta^1_{i-n+1}\delta^{d+1}_{j+1} e_i \wedge e_j \\
&\quad - (2n-j)\delta^{d+1}_{i+1}\delta^1_{j-n+1} e_i \wedge e_j + (2n-j)\delta^1_{i+1}\delta^{d+1}_{j-n+1} e_i \wedge e_j \\
&= -(n-d)e_{d+n} \wedge e_0 + ne_n \wedge e_{d+1} - ne_{d+1} \wedge e_n + (n-d)e_0 \wedge e_{d+n} \\
&= -2ne_d \wedge e_n + 2(n-d)e_0 \wedge e_{d+n}.
\end{aligned}$$

This concludes the proof.■

Proposition 5.6. *For every $d \geq 2$ the family of Poisson-Lie structures described by (4.15) gives rise to the following family of Lie-bialgebra structures on $\mathcal{G}_\infty$:*

$$\begin{aligned}
\alpha_{d,\lambda}(e_n) =& 2\sum_{i=d+n}^{\infty}(2n-i)\lambda^{i-(n+d)} e_0 \wedge e_i - 2n\sum_{i=d}^{\infty}\lambda^{i-d} e_i \wedge e_n \\
&+ \frac{2}{d-1}\sum_{i=d+n}^{\infty}\sum_{j=1}^{d-1}(2n-i)\lambda^{i+j-(n+d)} e_i \wedge e_j \\
&+ \frac{2}{d-1}\sum_{i=d}^{\infty}\sum_{j=n+1}^{d+n-1}(2n-j)\lambda^{i+j-(n+d)} e_i \wedge e_j,
\end{aligned} \tag{5.40}$$

for every $n \in \mathbb{Z}_+$.

Proof. Let $\alpha_{d,\lambda}(e_n) = \alpha^n_{ij|d} e_i \wedge e_j$, where $\{e_n\}_{n\geq 1}$ is a basis of $\mathcal{G}_\infty$, and

$$\alpha^n_{ij|d} = (2n-i-1)\lambda_{i-n+1,j} + (2n-j-1)\lambda_{i,j-n+1}, \quad \text{for} \quad n,i,j \in \mathbb{N}.$$

With the assumptions of Lemma 4.11 and Theorem 4.12 we have

$$\lambda_{ij} = \frac{1}{\lambda_{1,d+1}}\Big[\lambda_{1i}\lambda_{d+1,j} - \lambda_{1j}\lambda_{d+1,i}\Big],$$

where

$$
\begin{aligned}
&\lambda_{n,d+1} = 0, \quad \text{for every} \quad n \geq d+1, \\
(5.41)\quad &\lambda_{n,d+1} = -\frac{1}{d-1}\frac{(\lambda_{1,d+2})^{n-1}}{(\lambda_{1,d+1})^{n-2}} = -\frac{1}{d-1}\lambda_{1,d+1}\lambda^{n-1}, \quad \text{where} \quad 2 \leq n \leq d, \\
&\lambda_{1n} = \frac{(\lambda_{1,d+2})^{n-d-1}}{(\lambda_{1,d+1})^{n-d-2}} = \lambda_{1,d+1}\lambda^{n-d-1}, \quad \text{for every} \quad n \geq d+1,
\end{aligned}
$$

and where we have introduced $\lambda \equiv \frac{\lambda_{1,d+2}}{\lambda_{1,d+1}}$. Then

$$
\begin{aligned}
\alpha_{d,\lambda}(e_n) =& \Big[(2n-i-1)\lambda_{i-n+1,j} + (2n-j-1)\lambda_{i,j-n+1}\Big] e_i \wedge e_j \\
=& \frac{1}{\lambda_{1,d+1}}\Big\{(2n-i-1)\Big[\lambda_{1,i-n+1}\lambda_{d+1,j} - \lambda_{1j}\lambda_{d+1,i-n+1}\Big] + \\
&+ (2n-j-1)\Big[\lambda_{1i}\lambda_{d+1,j-n+1} - \lambda_{1,j-n+1}\lambda_{d+1,i}\Big]\Big\} e_i \wedge e_j \\
=& \frac{1}{\lambda_{1,d+1}}\Big\{\sum_{i=d+n}^{\infty}\sum_{j=1}^{d}(2n-i-1)\lambda_{1,i-n+1}\lambda_{d+1,j} e_i \wedge e_j - \\
&- \sum_{i=n}^{d+n-1}\sum_{j=d+1}^{\infty}(2n-i-1)\lambda_{1j}\lambda_{d+1,i-n+1} e_i \wedge e_j \\
&+ \sum_{i=d+1}^{\infty}\sum_{j=n}^{d+n-1}(2n-j-1)\lambda_{1i}\lambda_{d+1,j-n+1} e_i \wedge e_j - \\
&- \sum_{i=1}^{d}\sum_{j=d+n}^{\infty}(2n-j-1)\lambda_{1,j-n+1}\lambda_{d+1,i} e_i \wedge e_j\Big\} \\
=& \frac{1}{\lambda_{1,d+1}}\Big\{\sum_{i=d+n}^{\infty}\sum_{j=2}^{d}(2n-i-1)\lambda_{1,i-n+1}\lambda_{d+1,j} e_i \wedge e_j + \\
&+ \lambda_{d+1,1}\sum_{i=d+n}^{\infty}(2n-i-1)\lambda_{1,i-n+1} e_i \wedge e_1 \\
&- \sum_{i=n+1}^{d+n-1}\sum_{j=d+1}^{\infty}(2n-i-1)\lambda_{1j}\lambda_{d+1,i-n+1} e_i \wedge e_j - \lambda_{d+1,1}\sum_{j=d+1}^{\infty}(n-1)\lambda_{1j} e_n \wedge e_j \\
&+ \sum_{i=d+1}^{\infty}\sum_{j=n+1}^{d+n-1}(2n-j-1)\lambda_{1i}\lambda_{d+1,j-n+1} e_i \wedge e_j + \lambda_{d+1,1}\sum_{i=d+1}^{\infty}(n-1)\lambda_{1i} e_i \wedge e_n \\
&- \sum_{i=2}^{d}\sum_{j=d+n}^{\infty}(2n-j-1)\lambda_{1,j-n+1}\lambda_{d+1,i} e_i \wedge e_j -
\end{aligned}
$$

$$
\begin{aligned}
&- \lambda_{d+1,1} \sum_{j=d+n}^{\infty} (2n-j-1)\lambda_{1,j-n+1} e_1 \wedge e_j \Big\} \\
=&\frac{1}{\lambda_{1,d+1}} \Big\{ 2 \sum_{i=d+n}^{\infty} \sum_{j=2}^{d} (2n-i-1)\lambda_{1,i-n+1}\lambda_{d+1,j} e_i \wedge e_j + \\
&+ 2 \sum_{i=d+1}^{\infty} \sum_{j=n+1}^{d+n-1} (2n-j-1)\lambda_{1i}\lambda_{d+1,j-n+1} e_i \wedge e_j \\
&+ 2\lambda_{1,d+1} \sum_{i=d+n}^{\infty} (2n-i-1)\lambda_{1,i-n+1} e_1 \wedge e_i - 2\lambda_{1,d+1} \sum_{i=d+1}^{\infty} (n-1)\lambda_{1i} e_i \wedge e_n \Big\} \\
=&\lambda_{1,d+1} \Big\{ \frac{2}{d-1} \sum_{i=d+n}^{\infty} \sum_{j=2}^{d} (2n-i-1)\lambda^{i+j-(n+d+1)} e_i \wedge e_j + \\
&+ \frac{2}{d-1} \sum_{i=d+1}^{\infty} \sum_{j=n+1}^{d+n-1} (2n-j-1)\lambda^{i+j-(n+d+1)} e_i \wedge e_j \\
&+ 2 \sum_{i=d+n}^{\infty} (2n-i-1)\lambda^{i-(n+d)} e_1 \wedge e_i - 2 \sum_{i=d+1}^{\infty} (n-1)\lambda^{i-(d+1)} e_i \wedge e_n \Big\},
\end{aligned}
$$

where we used formulae (5.41) to obtain the last equality. Hence, after normalizing by the factor $\lambda_{1,d+1} \neq 0$ and shifting indices by 1 we obtain (5.40).■

Remark. One could show directly that $\alpha_{d,\lambda}$ satisfies the co-Jacobi identity. The r.h.s. of (5.40) is understood as an element of the completed tensor product $\mathcal{G}_\infty \widehat{\otimes} \mathcal{G}_\infty$ [Di,ZS].

CHAPTER VI

THE GROUP $G_{0\infty}$ AND POISSON-LIE STRUCTURES ON IT

In this chapter we study the group $G_{0\infty}$ of which G_∞ is a subgroup. We classify all Poisson-Lie structures on $G_{0\infty}$ corresponding to coboundary Lie-bialgebra structures on the Lie algebra $\mathcal{G}_{0\infty}$ of $G_{0\infty}$.

Let $X = \{x_i\}_{i\in\mathbb{Z}_+}$ be a countable set of indeterminates. Let $k[[X]]$ be the ring of formal power series over X whitout constant term with the standard multiplication. Here k is a commutative field assumed to be of characteristic zero. Let $Y = \{y_i\}_{i\in\mathbb{Z}_+}$ be a second set of indeterminates, and $k[[Y]]$ be the corresponding ring of formal power series over Y. Consider the formal group $G_{0\infty}$ defined by a formal group law $F = (F_i)_{i\in\mathbb{Z}_+}$ [Se,Di] in infinite number of variables, where $F_i \in k[[X,Y]]$ for every $i \in \mathbb{Z}_+$, induced by a substitution of formal power series in one variable. Let $\mathcal{X}(u) \in k[[X]][[u]]$ and $\mathcal{Y}(u) \in k[[Y]][[u]]$ be elements in the rings of formal power series with a constant term in the variable u over the rings $k[[X]]$ and $k[[Y]]$ respectively. The multiplication of formal power series in the variable u is defined again as the substitution:

$$
\begin{aligned}
(\mathcal{X}\mathcal{Y})(u) = \mathcal{X}(\mathcal{Y}(u)) &= \sum_{i=0}^{\infty} x_i(\mathcal{Y}(u))^i \\
&= \sum_{i=0}^{\infty} x_i \left[y_0^i + \sum_{j=1}^{\infty} \Big(\sum_{(\sum_{\alpha=1}^{i} s_\alpha)=j} y_{s_1}\cdots y_{s_i} \Big) u^j \right] \\
&= \sum_{i=0}^{\infty} x_i y_0^i + \sum_{i=1}^{\infty} x_i \sum_{j=1}^{\infty} \left(\sum_{(\sum_{\alpha=1}^{i} s_\alpha)=j} y_{s_1}\cdots y_{s_i} \right) u^j \\
&= \sum_{i=0}^{\infty} x_i y_0^i + \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} x_i \sum_{(\sum_{\alpha=1}^{i} s_\alpha)=j} y_{s_1}\cdots y_{s_i} \right) u^j .
\end{aligned}
\tag{6.1}
$$

Therefore from (6.1) we obtain

$$F_0(X,Y) = \sum_{i=0}^{\infty} x_i y_0^i,$$

$$F_j(X,Y) = \sum_{i=1}^{\infty} x_i \sum_{(\sum_{\alpha=1}^{i} s_\alpha)=j} y_{s_1} \dots y_{s_i}, \quad \text{for every } \ j \geq 1.$$

This is a model of the group of diffeomorphisms of $\mathbb{R}^1$ not necessarily leaving the point $u = 0$ fixed. The identity here is $e = (0,1,0,0,\dots)$. Clearly G_∞, if viewed as a formal group, will be a subgroup of $G_{0\infty}$. We define a Poisson structure $\omega(x) = \omega_{ij}(x)\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$ on the group $G_{0\infty}$ as a derivation $\omega(x)\colon k[[X]] \otimes k[[X]] \to k[[X]]$, where $\omega_{ij}(x) \in k[[X]]$, satisfying the Jacobi identity. The methods developed in analyzing the Poisson-Lie structures on G_∞ apply without major changes to the case of $G_{0\infty}$, but with two important differences. Namely, Theorem 4.1 still holds with $\Omega(u,v;\mathcal{X})$ defined as

$$\Omega(u,v;\mathcal{X}) := \sum_{i,j=0}^{\infty} \omega_{ij}(x) u^i v^j,$$

but in the solution of the cocycle equation

$$\Omega(u,v;\mathcal{X}) = \varphi(u,v)\mathcal{X}'(u)\mathcal{X}'(v) - \varphi(\mathcal{X}(u),\mathcal{X}(v)) \tag{6.2}$$

$\varphi(u,v)$ does not have to be divisible by uv. Thus, this condition is dropped. This change affects the analysis of the equation

$$\varphi(u,v)\left[\partial_u \varphi(w,u) + \partial_v \varphi(w,v)\right] + c.p. = 0, \tag{6.3}$$

as well as the structure of its solutions.

Still, we will show that the Poisson-Lie structures on $G_{0\infty}$ fall in two main classes. Also, as we found in the previous chapter, the solution (6.2) corresponds to a cocycle that is a coboundary in the Lie algebra $\mathcal{G}_\infty$. This result carries over to the case of $\mathcal{G}_{0\infty}$ without change. We showed that all cocycles on $\mathcal{G}_\infty$ are coboundaries. This allowed us to completely classify them. We do not know whether the same fact is true for $\mathcal{G}_{0\infty}$. We have not been able to prove that all 1-cocycles on $\mathcal{G}_{0\infty}$ are coboundaries. Thus, the class of Poisson-Lie structures on $G_{0\infty}$ that we will describe is the one that corresponds to coboundaries on $\mathcal{G}_{0\infty}$. That is, we will give a classification of all r-matrices on $\mathcal{G}_{0\infty}$.

We start with two, in some sense exceptional, solutions of (6.3). Exceptional here means that these two solutions are only two "points" in the otherwise infinite-parameter space of solutions of (6.3).

Theorem 6.1. *The functions*
(i) $\varphi(u,v) = u - v$, *and*
(ii) $\varphi(u,v) = e^{\lambda u} - e^{\lambda v}$, *where* λ *is an arbitrary parameter,*
are solutons of (6.3), thus giving rise to two Poisson-Lie structures on $G_{0\infty}$.

Remark. These are the only solutions of (6.3) of the form $\varphi(u,v) = a(u) - a(v)$.

The formal proof, following from more general considerations, will be postponed until later, even though one could check the claim by directly substituting (i) or (ii) into (6.3). Here, we only write the Poisson brackets in coordinates.

For the case (i) we have

$$\omega_{ij}(x) = i(j+1)x_i x_{j+1} - (i+1)j x_{i+1} x_j - x_i \delta_j^0 + x_j \delta_i^0, \quad i,j \in \mathbb{Z}_+.$$

Notice, that there are no terms higher than quadratic in the right hand side. For the case (ii) we obtain

$$\omega_{ij}(x) = (j+1)x_{j+1} \sum_{p=0}^{i+1} \frac{p x_p}{(i-p+1)!} - (i+1)x_{i+1} \sum_{q=0}^{j+1} \frac{q x_q}{(j-q+1)!}$$
$$- \delta_j^0 \sum_{p=0}^{i} \frac{1}{p!} \sum_{r_1+\ldots+r_p=i} x_{r_1} \ldots x_{r_p} + \delta_i^0 \sum_{q=0}^{j} \frac{1}{q!} \sum_{r_1+\ldots+r_q=j} x_{r_1} \ldots x_{r_q}.$$

Now, we proceed with the main result of this chapter.

Theorem 6.2. *All solutions of (6.3) fall into the following two classes*
(a) The first class is given by Theorem 4.4.
(b) The second class is given by

$$\varphi(u,v) = \frac{1}{\lambda_{01}} \Big[f(u)g(v) - f(v)g(u) \Big]$$

where $\lambda_{01} \neq 0$, *and* $f(u)$ *and* $g(u)$ *are arbitrary functions satisfying the relation*

$$f'(u)g(u) - f(u)g'(u) = \lambda_{01} g(u) - 2\lambda_{02} f(u).$$

Here, λ_{01} *and* λ_{02} *are arbitrary parameters with* λ_{01} *being subject to the above restriction:* $\lambda_{01} \neq 0$.

Proof. We look again for a solution of (6.3) in a form of a formal power series $\varphi(u,v) = \sum_{m,n=0}^{\infty} \lambda_{mn} u^m v^n$, where $\lambda_{mn} = -\lambda_{nm}$. Substituting into (6.3) we obtain (in a similar way as in the proof of Theorem 4.5)

(6.4)
$$\sum_{s=0}^{k+1}(k-2s+1)\lambda_{n,k-s+1}\lambda_{rs} - \sum_{s=0}^{n+1}(n-2s+1)\lambda_{k,n-s+1}\lambda_{rs} - \sum_{s=0}^{r+1}(r-2s+1)\lambda_{n,r-s+1}\lambda_{ks} = 0$$

where $k < n < r$.

Let $k = 0$. Then (6.4) becomes

$$(6.4^*)\quad \lambda_{n1}\lambda_{r0} - \lambda_{n0}\lambda_{r1} - \sum_{s=0}^{n+1}(n-2s+1)\lambda_{0,n-s+1}\lambda_{rs} - \sum_{s=0}^{r+1}(r-2s+1)\lambda_{n,r-s+1}\lambda_{0s} = 0.$$

Notice that the summation in the sums above is to n and r respectively. If we now let $n = 1$, we finally obtain

$$\begin{aligned} -\lambda_{01}\lambda_{1r} + 2\lambda_{02}\lambda_{0r} &= \sum_{s=0}^{r}(r-2s+1)\lambda_{1,r-s+1}\lambda_{0s} \\ &= -\sum_{s=0}^{r}(r-2s+1)\lambda_{0,r-s+1}\lambda_{1s}. \end{aligned}$$

Therefore,

$$(6.5)\qquad \sum_{s=0}^{r}(r-s+1)\lambda_{0,r-s+1}\lambda_{1s} - \sum_{s=0}^{r} s\lambda_{0,r-s+1}\lambda_{1s} = \lambda_{01}\lambda_{1r} - 2\lambda_{02}\lambda_{0r}.$$

Let us define the functions $f(u) = \sum_{n=0}^{\infty}\lambda_{0n}u^n$ and $g(u) = \sum_{n=0}^{\infty}\lambda_{1n}u^n$. Then, after multiplying both sides of (6.5) by u^r, and summing over r we obtain

$$\sum_{r=0}^{\infty}\sum_{s=0}^{r}(r-s+1)\lambda_{0,r-s+1}\lambda_{1s}u^r - \sum_{r=0}^{\infty}\sum_{s=0}^{r} s\lambda_{0,r-s+1}\lambda_{1s}u^r = \lambda_{01}g(u) - 2\lambda_{02}f(u),$$

which, after the change of variables $r = q + s - 1$, is equivalent to

$$\sum_{s=0}^{\infty}\sum_{q=0}^{\infty} q\lambda_{0q}\lambda_{1s}u^{q+s-1} - \sum_{s=0}^{\infty}\sum_{q=0}^{\infty} s\lambda_{0q}\lambda_{1s}u^{q+s-1} = \lambda_{01}g(u) - 2\lambda_{02}f(u).$$

Thus, the functions $f(u)$, and $g(u)$ satisfy

$$f'(u)g(u) - f(u)g'(u) = \lambda_{01}g(u) - 2\lambda_{02}f(u).$$

Next, we split the remaining part of the proof into five lemmas.

Lemma 6.3. *For every $n \geq 1$ the parameters λ_{1n} are rational functions of λ_{0n}, provided that $\lambda_{01} \neq 0$.*

Remark. The case of $\lambda_{01} = 0$ will be treated in Lemmas 6.5, 6.6, and 6.7.

Proof. From (6.5) we have

$$\sum_{s=0}^{r-1}(r-2s+1)\lambda_{0,r-s+1}\lambda_{1s} - (r-1)\lambda_{01}\lambda_{1r} = \lambda_{01}\lambda_{1r} - 2\lambda_{02}\lambda_{0r},$$

thus

$$\lambda_{1r} = \frac{1}{r\lambda_{01}}\left[\sum_{s=0}^{r-1}(r-2s+1)\lambda_{0,r-s+1}\lambda_{1s} + 2\lambda_{02}\lambda_{0r}\right].$$

This gives us a recursive relation for λ_{1r}. Here are the first several λ_{1r}'s

$$\begin{aligned}
\lambda_{12} &= \frac{1}{\lambda_{01}}\left[3\lambda_{03}\lambda_{10} + 2(\lambda_{02})^2\right] = \frac{1}{2}\left[2\frac{(\lambda_{02})^2}{\lambda_{01}} - 3\lambda_{03}\right] \\
\lambda_{13} &= \frac{1}{3}\left[2\frac{\lambda_{02}\lambda_{03}}{\lambda_{01}} - 4\lambda_{04}\right] \\
\lambda_{14} &= \frac{1}{24}\left[2\frac{(\lambda_{02})^2\lambda_{03}}{(\lambda_{01})^2} - 9\frac{(\lambda_{03})^2}{\lambda_{01}} + 20\frac{\lambda_{02}\lambda_{04}}{\lambda_{01}} - 30\lambda_{05}\right] \\
&\vdots
\end{aligned}$$

Therefore, the claim is proved by induction.■

Lemma 6.4. *If $\lambda_{01} \neq 0$, then we have the following formula:*

$$\lambda_{nr} = \frac{1}{\lambda_{01}}\left[\lambda_{0n}\lambda_{1r} - \lambda_{1n}\lambda_{0r}\right], \quad \forall\, n, r \geq 0.$$

Proof. From (6.4*) with $n = 2$ we obtain

$$\lambda_{12}\lambda_{0r} - \lambda_{01}\lambda_{2r} + 3\lambda_{03}\lambda_{0r} - \sum_{s=0}^{r+1}(r-2s+1)\lambda_{2,r-s+1}\lambda_{0s} = 0. \tag{6.6}$$

If we now let $r = 3$, we obtain a formula for λ_{23}:

$$\lambda_{12}\lambda_{03} - \lambda_{01}\lambda_{23} + 3(\lambda_{03})^2 - 2\lambda_{23}\lambda_{01} - 2\lambda_{12}\lambda_{03} - 4\lambda_{02}\lambda_{04} = 0,$$

from which

$$\begin{aligned}\lambda_{23} &= \frac{1}{\lambda_{01}}\left[-\frac{1}{3}\frac{(\lambda_{02})^2}{\lambda_{01}}\lambda_{03} + \frac{2}{3}(\lambda_{03})^2 - \frac{4}{3}\lambda_{02}\lambda_{04}\right] \\ &= \frac{1}{\lambda_{01}}\left[\frac{1}{3}\left(2\frac{\lambda_{02}\lambda_{03}}{\lambda_{01}} - 4\lambda_{04}\right)\lambda_{02} - \frac{1}{2}\left(2\frac{(\lambda_{02})^2}{\lambda_{01}} - 3\lambda_{03}\right)\lambda_{03}\right] \\ &= \frac{1}{\lambda_{01}}\left[\lambda_{02}\lambda_{13} - \lambda_{12}\lambda_{03}\right].\end{aligned}$$

Now, let us assume that $\lambda_{2k} = \frac{1}{\lambda_{01}}[\lambda_{02}\lambda_{1k} - \lambda_{12}\lambda_{0k}]$ for $1 \le k \le r-1$. We need to prove that it is true for $k = r$. From (6.6) we have

$$\lambda_{12}\lambda_{0r} - \lambda_{01}\lambda_{2r} + 3\lambda_{03}\lambda_{0r} + (1-r)\lambda_{01}\lambda_{2r} + \sum_{s=0}^{r-1}(r-2s+1)\lambda_{0,r-s+1}\lambda_{2s} = 0.$$

Using the induction hypothesis we transform the above equation into

$$\lambda_{12}\lambda_{0r} + 3\lambda_{03}\lambda_{0r} + \frac{1}{\lambda_{01}}\sum_{s=0}^{r-1}(r-2s+1)\lambda_{0,r-s+1}\left[\lambda_{02}\lambda_{1s} - \lambda_{12}\lambda_{0s}\right] = r\lambda_{01}\lambda_{2r}.$$

Next, using the fact that $\sum_{s=2}^{r-1}(r-2s+1)\lambda_{0,r-s+1}\lambda_{0s} = 0$ we transform the above equality further into

$$(2-r)\lambda_{12}\lambda_{0r} + 3\lambda_{03}\lambda_{0r} + \frac{\lambda_{02}}{\lambda_{01}}\sum_{s=0}^{r-1}(r-2s+1)\lambda_{0,r-s+1}\lambda_{1s} = r\lambda_{01}\lambda_{2r}.$$

The final step is to use (6.5) and the formula for λ_{12} obtained in the proof of Lemma 6.3. This leads to

$$\left(2\frac{(\lambda_{02})^2}{\lambda_{01}} - 3\lambda_{03}\right)\lambda_{0r} - r\lambda_{12}\lambda_{0r} + 3\lambda_{03}\lambda_{0r} + \frac{\lambda_{02}}{\lambda_{01}}\left[r\lambda_{01}\lambda_{1r} - 2\lambda_{02}\lambda_{0r}\right] = r\lambda_{01}\lambda_{2r}.$$

After collecting all terms we obtain

$$\lambda_{2r} = \frac{1}{\lambda_{01}}\left[\lambda_{02}\lambda_{1r} - \lambda_{12}\lambda_{0r}\right].$$

Finally, we assume that $\lambda_{nk} = \frac{1}{\lambda_{01}}[\lambda_{0n}\lambda_{1k} - \lambda_{1n}\lambda_{0k}]$ for $1 \le k \le r-1$ and each $n \ge 1$. Then, we show that it is true for $k = r$. The steps are essentially the same as in the previous calculation. From (6.4*) we have

$$\lambda_{1n}\lambda_{0r} - \lambda_{0n}\lambda_{1r} + \sum_{s=0}^{n-1}(n-2s+1)\lambda_{0,n-s+1}\lambda_{sr} + \sum_{s=0}^{r-1}(r-2s+1)\lambda_{0,r-s+1}\lambda_{ns} = (r+n-2)\lambda_{01}\lambda_{nr}.$$

Applying the induction hypothesis, the above equation transforms to

$$(2-r)\lambda_{1n}\lambda_{0r}+(n-2)\lambda_{1r}\lambda_{0n}-\frac{\lambda_{0r}}{\lambda_{01}}\sum_{s=0}^{n-1}(n-2s+1)\lambda_{0,n-s+1}\lambda_{1s}+\frac{\lambda_{0n}}{\lambda_{01}}\sum_{s=0}^{r-1}(r-2s+1)\lambda_{0,r-s+1}\lambda_{1s}=$$
$$=(r+n-2)\lambda_{01}\lambda_{nr}.$$

Now, we use (6.5) to obtain

$$(2-r)\lambda_{1n}\lambda_{0r}+(n-2)\lambda_{1r}\lambda_{0n}-\frac{\lambda_{0r}}{\lambda_{01}}\Big[n\lambda_{01}\lambda_{1n}-2\lambda_{02}\lambda_{0n}\Big]+\frac{\lambda_{0n}}{\lambda_{01}}\Big[r\lambda_{01}\lambda_{1r}-2\lambda_{02}\lambda_{0r}\Big]=$$
$$=(r+n-2)\lambda_{01}\lambda_{nr},$$

and collecting terms we have

$$\lambda_{nr}=\frac{1}{\lambda_{01}}\Big[\lambda_{0n}\lambda_{1r}-\lambda_{1n}\lambda_{0r}\Big].$$

This completes the proof of the lemma.■

As a consequence,

$$\varphi(u,v)=\sum_{n,m=0}^{\infty}\lambda_{nm}u^nv^m=\frac{1}{\lambda_{01}}\Big[f(u)g(v)-f(v)g(u)\Big],$$

where $f(u)=\sum_{s=0}^{\infty}\lambda_{0s}u^s$ and $g(u)=\sum_{s=0}^{\infty}\lambda_{1s}u^s$.

Lemma 6.5. *If $\lambda_{01}=0$, then $\lambda_{02}=0$. If in addition we assume that $\lambda_{12}\neq 0$, then it follows that $\lambda_{0n}=0$, $\forall\ n\geq 2$.*

Proof. Again, from (6.5) with $\lambda_{01}=0$ we obtain

$$\sum_{s=1}^{r-1}(r-2s+1)\lambda_{0,r-s+1}\lambda_{1s}+2\lambda_{02}\lambda_{0r}=0,\quad \forall\ r\geq 2.$$

For $r=2$ we have $(\lambda_{02})^2=0$. Using this fact, the above equation reduces to

$$\sum_{s=1}^{r-1}(r-2s+1)\lambda_{0,r-s+1}\lambda_{1s}=0,\quad \forall\ r\geq 3. \tag{6.7}$$

For $r=3$ we have an identity. For $r=4$ we obtain

$$\lambda_{03}\lambda_{12}-\lambda_{02}\lambda_{13}=\lambda_{03}\lambda_{12}=0\implies\lambda_{03}=0.$$

Therefore, if we assume that $0=\lambda_{01}=\lambda_{02}=\ldots=\lambda_{0,r-2}$, equation (6.7) gives

$$(r-3)\lambda_{0,r-1}\lambda_{12}=0\implies\lambda_{0,r-1}=0,\forall\ r>3.$$

As a result the system of equations (6.4) reduces to the system (4.6). But for $\lambda_{12}\neq 0$ the solution of that system of equations is the one corresponding to $d=1$ as described by Theorem 4.5.■

Lemma 6.6. *Suppose that* $\lambda_{01} = 0 = \lambda_{02}$. *If* $\lambda_{12} = \ldots = \lambda_{1n} = 0$ *for some* $n \geq 2$, *then* $\lambda_{03} = \ldots = \lambda_{0,n+1} = 0$.

Proof. From (6.4) with $k = 0$ and $r = n + 1$ we obtain

$$\sum_{s=1}^{n}(n-2s+1)\lambda_{n+1,n-s+1}\lambda_{0s} + \sum_{s=1}^{n+1}(n-2s+2)\lambda_{0,n-s+2}\lambda_{ns} =$$

$$-\lambda_{0,n+1}\lambda_{1n} + \lambda_{0n}\lambda_{1,n+1} - (n+1)(\lambda_{0,n+1})^2 + (n+2)\lambda_{0,n+2}\lambda_{0n} \tag{6.8}$$

For $n = 2$ the above equation gives $3(\lambda_{03})^2 = \lambda_{03}\lambda_{12}$. Thus if $\lambda_{12} = 0$ then $\lambda_{03} = 0$.

Assume that $\lambda_{12} = \ldots = \lambda_{1,n-1} = 0$ and $\lambda_{03} = \ldots = \lambda_{0n} = 0$. Then from (6.8) we obtain

$$(n+1)(\lambda_{0,n+1})^2 = -(n-1)\lambda_{0,n+1}\lambda_{1n}.$$

Therefore if $\lambda_{1n} = 0$, then $\lambda_{0,n+1} = 0$.■

Lemma 6.7. *Suppose that* $\lambda_{01} = 0$, *and* $\lambda_{12} = \ldots = \lambda_{1d} = 0$, *and* $\lambda_{1,d+1} \neq 0$ *for some* $d \in \mathbb{N}$. *Then* $\lambda_{0n} = 0$ *for every* $n \geq 2$.

Proof. From Lemma 6.6 it follows that $\lambda_{02} = \ldots = \lambda_{0,d+1} = 0$. Further, from (6.7) we have

$$\sum_{s=d+1}^{n}(n-2s+1)\lambda_{0,n-s+1}\lambda_{1s} = 0. \tag{6.9}$$

Let $n = 2d + 2$. Then

$$\sum_{s=d+1}^{2d+2}(2d-2s+3)\lambda_{0,2d-s+3}\lambda_{1s} = \lambda_{0,d+2}\lambda_{1,d+1} - \lambda_{0,d+1}\lambda_{1,d+2} = 0.$$

But $\lambda_{0,d+1} = 0$ according to Lemma 6.5. Therefore $\lambda_{0,d+2}\lambda_{1,d+1} = 0$. Since $\lambda_{1,d+1} \neq 0$ then $\lambda_{0,d+2} = 0$. Now, from (6.9) with $n = 2d + 3$ we obtain

$$\sum_{s=d+1}^{2d+3}(2d-2s+4)\lambda_{0,2d-s+4}\lambda_{1s} = 2\lambda_{0,d+3}\lambda_{1,d+1} = 0,$$

therefore $\lambda_{0,d+3} = 0$. By induction the statetment follows.■

The above arguments show that if $\lambda_{01} = 0$ the classification of the solutions of (6.3) reduces to the classification of all those solutions $\varphi(u,v)$ of (6.3) which are divisible by uv. But all such solutions have been given by Theorem 4.5, and the proof of Theorem 6.2 is finished.■

Now, we describe how the solutions given in Theorem 6.1 are obtained as particular cases of the general solution.

If we assume that $\lambda_{nm} = 0, \forall\, n, m \geq 1$, then $\varphi(u,v) = \sum_{n=0}^{\infty} \lambda_{0n}(u^n - v^n)$. From (6.4) with $k = 0$ we have

$$(n+1)\lambda_{0,n+1}\lambda_{0r} - (r+1)\lambda_{0n}\lambda_{0,r+1} = 0, \qquad \forall\, 1 \leq n < r.$$

Therefore

$$\lambda_{0,n+1} = (r+1)\frac{\lambda_{0,r+1}}{\lambda_{0r}}\left(\frac{\lambda_{0n}}{n+1}\right).$$

Let us introduce $\alpha := 2\frac{\lambda_{02}}{\lambda_{01}}$, then $\lambda_{0,n+1} = \alpha(\frac{\lambda_{0n}}{n+1})$, and

$$\begin{aligned}
\lambda_{02} &= \alpha\frac{\lambda_{01}}{2} \\
\lambda_{03} &= \alpha\frac{\lambda_{02}}{3} = \alpha^2\frac{\lambda_{01}}{3!} \\
&\vdots \\
\lambda_{0n} &= \alpha^{n-1}\frac{\lambda_{01}}{n!} \\
&\vdots
\end{aligned}$$

Thus,

$$\varphi(u,v) = \sum_{n=0}^{\infty} \lambda_{0n}(u^n - v^n) = \frac{\lambda_{01}}{\alpha}\left[e^{\alpha u} - e^{\alpha v}\right].$$

If $\alpha = 0$, then $\lambda_{0r} = 0\ \forall\, r \geq 2$, and $\varphi(u,v) = \lambda_{01}(u - v)$.

Next, we discuss an interesting particular result which follows from the general formulae. First we write the Lie bialgebra structures on $\mathcal{G}_{0\infty}$ that correspond to the Poisson-Lie structures (i) and (ii) of Theorem 6.1. Namely, if $e_n \in \mathcal{G}_{0\infty}$ is a basis element we have

$$\begin{aligned}
\alpha(e_n) &= \alpha_{ij}^n e_i \wedge e_j \\
&= \left[(\delta_{i-n+1}^1\delta_{j+1}^0 - \delta_{i-n+1}^0\delta_{j+1}^0)(2n-i) + (\delta_{i+1}^1\delta_{j-n+1}^0 - \delta_{i+1}^0\delta_{j-n+1}^1)(2n-j)\right] e_i \wedge e_j\,. \\
&= -2ne_{-1} \wedge e_n + 2(n+1)e_0 \wedge e_{n-1}
\end{aligned}$$

Thus, for the case (i) we obtain

$$\alpha(e_n) = -2ne_{-1} \wedge e_n + 2(n+1)e_0 \wedge e_{n-1}. \tag{6.10}$$

Similarly, for the case (ii)

$$\begin{aligned}\alpha(e_n) =& \left[\frac{1}{(i-n+1)!}\delta^0_{j+1} - \frac{1}{(j+1)!}\delta^0_{i-n+1}\right](2n-i)e_i \wedge e_j \\ &+ \left[\frac{1}{(i+1)!}\delta^0_{j-n+1} - \frac{1}{(j-n+1)!}\delta^0_{i+1}\right](2n-j)e_i \wedge e_j \\ =& -2\sum_{j=n-1}^{\infty}\frac{(2n-j)}{(j-n+1)!}e_{-1} \wedge e_j + 2(n+1)\sum_{i=-1}^{\infty}\frac{1}{(i+1)!}e_i \wedge e_{n-1}.\end{aligned}$$

The commutator for $\mathcal{G}_{0\infty}$ has the standard form

$$[e_n, e_m] = (n-m)e_{n+m}, \qquad (n, m \geq -1).$$

That is, $\mathcal{G}_{0\infty}$ is the Witt algebra, $\mathcal{W}$.

It is well known that $\mathcal{W}$ contains sl_2 as a Lie subalgebra. In our notation the defining relations are given by

$$\begin{aligned}[e_1, e_{-1}] &= 2e_0 \\ [e_1, e_0] &= e_1 \\ [e_0, e_{-1}] &= e_{-1}.\end{aligned}$$

If we now turn to the formula (6.10)

$$\alpha(e_n) = -2ne_{-1} \wedge e_n + 2(n+1)e_0 \wedge e_{n-1},$$

it gives us the following Lie bialgebra structure on sl_2:

$$\begin{aligned}\alpha(e_{-1}) &= 0 \\ \alpha(e_0) &= 2e_0 \wedge e_{-1} \\ \alpha(e_1) &= -2e_{-1} \wedge e_1.\end{aligned}$$

On the other hand if we take formula (5.15)

$$\alpha(e_n) = -2ne_d \wedge e_n + 2(n-d)e_0 \wedge e_{d+n},$$

and consider it for $d = 1$, that is

$$\alpha(e_n) = -2ne_1 \wedge e_n + 2(n-1)e_0 \wedge e_{n+1},$$

it gives a second Lie bialgebra structure on sl_2, namely

$$\begin{aligned}\alpha(e_{-1}) &= 2e_1 \wedge e_{-1}\\ \alpha(e_0) &= -2e_0 \wedge e_1\\ \alpha(e_1) &= 0.\end{aligned}$$

These two Lie bialgebra structures lead to all (up to isomorphism) Poisson-Lie structures on the group SL_2 [Ku4].

CHAPTER VII

ELEMENTS OF REPRESENTATION THEORY

In this chapter we study the Poisson action of the Poisson-Lie group G_∞ on the space of λ-densities V_λ.

Let G be a Poisson-Lie group and $\overline{\omega}$ be a Poisson-Lie structure on G. Let V be a space on which G acts, i.e. there is a map $G \times V \to V$. Such a space is called a G-space. Assume that V is equipped with a Poisson structure ω. Recall the following definition.

Definition 7.1. The action of G on V is called Poisson if the map $G \times V \to V$ is Poisson. Here $G \times V$ is equipped with the product Poisson structure.

In this chapter we study the following problem. Suppose that we are given the Poisson-Lie group G_∞. Consider the space $V_\lambda = \{\mathcal{X}(u)(du)^\lambda \mid \mathcal{X}(u) = \sum_{i=0}^\infty x_i u^i\}$, $\lambda \in \mathbb{R}$. The space V_λ is sometimes referred to as the space of λ-densities (Jacobians) over the real line. The group G_∞ acts naturally on V_λ. Let $\mathcal{Y} \in G_\infty$ and $\mathcal{X}(u)(du)^\lambda \in V_\lambda$. Then the action of G_∞ on V is defined by

$$\mathcal{X}(u)(du)^\lambda \mapsto \mathcal{X}\left(\mathcal{Y}(u)\right)\left(\mathcal{Y}'(u)\right)^\lambda (du)^\lambda,$$

where $\mathcal{Y}(u) = \sum_{i=1}^\infty y_i u^i$, and

$$\begin{aligned}
\left(\mathcal{Y}'(u)\right)^\lambda &= \left(\sum_{i=1}^\infty i y_i u^{i-1}\right)^\lambda \\
&= \left(y_1 + \sum_{i=2}^\infty i y_i u^{i-1}\right)^\lambda \\
&= y_1^\lambda \left(1 + \sum_{i=2}^\infty i \frac{y_i}{y_1} u^{i-1}\right)^\lambda \\
&= y_1^\lambda \left[1 + \frac{\lambda}{1!}\sum_{i=2}^\infty i \frac{y_i}{y_1} u^{i-1} + \frac{\lambda(\lambda-1)}{2!}\left(\sum_{i=2}^\infty i \frac{y_i}{y_1} u^{i-1}\right)^2 + \dots\right].
\end{aligned}$$

We consider the problem: are there Poisson structures on the space V_λ such that the above action of G_∞ on V_λ is a Poisson action? In other words, is there a Poisson strucuture ω on V_λ such that the map $G_\infty \times V_\lambda \to V_\lambda$ is a Poisson map? Here again $G_\infty \times V_\lambda$ is equipped with the product Poisson structure:

$$\overline{\omega} \times \omega = 1 \times \omega + \overline{\omega} \times 1.$$

Let $\mathcal{Y}(u) = \sum_{i=1}^{\infty} y_i u^i \in G_\infty$, and $\mathcal{X}(u)(du)^\lambda \in V_\lambda$. Let us define

$$\mathcal{Z}_\lambda(u) = \mathcal{X}(\mathcal{Y}(u))[\mathcal{Y}'(u)]^\lambda = \sum_{i=0}^{\infty} \xi_i u^i,$$

where $\xi_i = \xi_i(x, y; \lambda)$ are the coordinate functions of $\mathcal{Z}_\lambda$. If we also introduce the notation $J(u) \equiv \mathcal{Y}'(u) = \sum_{i=1}^{\infty} i y_i u^{i-1}$, we have

$$\begin{aligned}
\mathcal{X}(u)(du)^\lambda \to \mathcal{X}(\mathcal{Y}(u))(\mathcal{Y}'(u))^\lambda (du)^\lambda &= \mathcal{X}(\mathcal{Y}(u))(J(u))^\lambda (du)^\lambda \\
&= \mathcal{Z}_\lambda(u)(du)^\lambda \\
&= \sum_{i=0}^{\infty} x_i(\mathcal{Y}(u))^i \left(\sum_{i=1}^{\infty} i y_i u^{i-1} \right)^\lambda (du)^\lambda.
\end{aligned}$$

Defining $\mathcal{Z}(u) = \mathcal{X}(\mathcal{Y}(u))$ and using the definition $\mathcal{Z}_\lambda(u) = \mathcal{X}(\mathcal{Y}(u))[J(u)]^\lambda$ we deduce that

$$\mathcal{Z}_\lambda'(u) = \mathcal{Z}'(u)(J(u))^\lambda + \mathcal{Z}(u)\left[(J(u))^\lambda\right]',$$

where $'$ stands for the derivative with respect to u.

An argument analogous to the argument given in Chapter I implies that the map $G_\infty \times V_\lambda \to V_\lambda$ is Poisson if and only if

$$\omega_{ij}(\xi) = \omega_{kl}(x)\frac{\partial \xi_i}{\partial x_k}\frac{\partial \xi_j}{\partial x_l} + \overline{\omega}_{kl}(y)\frac{\partial \xi_i}{\partial y_k}\frac{\partial \xi_j}{\partial y_l}. \tag{7.2}$$

Here $\omega_{ij}(x) = \{x_i, x_j\}$ and $\overline{\omega}_{ij}(y) = \{y_i, y_j\}$ where $\{x_i\}_{i\in\mathbb{Z}_+}$ and $\{y_i\}_{i\in\mathbb{Z}_+}$ are the coordinate functions on V_λ and G_∞ respectively. Also, let us introduce in a manner similar to the one used in Capter IV a generating series for the Poisson strucutures on V_λ as

$$\Omega(u, v; \mathcal{X}) \equiv \sum_{i,j=0}^{\infty} \omega_{ij}(x) u^i v^j.$$

Lemma 7.2. *The multiplicativity condition (7.2) is equivalent to the following functional equation*

(7.3)

$$\begin{aligned}\Omega(u,v;\mathcal{Z}_\lambda) =&\Omega(\mathcal{Y}(u),\mathcal{Y}(v);\mathcal{X})(J(u))^\lambda(J(v))^\lambda + \overline{\Omega}(u,v;\mathcal{Y})\mathcal{Z}'(u)(J(u))^{\lambda-1}\mathcal{Z}'(v)(J(v))^{\lambda-1}+\\ &+\lambda\partial_u\overline{\Omega}(u,v;\mathcal{Y})\mathcal{Z}(u)(J(u))^{\lambda-1}\mathcal{Z}'(v)(J(v))^{\lambda-1}+\\ &+\lambda\partial_v\overline{\Omega}(u,v;\mathcal{Y})\mathcal{Z}'(u)(J(u))^{\lambda-1}\mathcal{Z}(v)(J(v))^{\lambda-1}+\\ &+\lambda^2\partial^2_{u,v}\overline{\Omega}(u,v;\mathcal{Y})\mathcal{Z}(u)(J(u))^{\lambda-1}\mathcal{Z}(v)(J(v))^{\lambda-1}.\end{aligned}$$

Here $\overline{\Omega}$ stands for the generating series of the Poisson-Lie strucutures on G_∞.

Proof. Multiplying both sides of equation (7.2) by u^iv^j, summing over i and j, and using the definition of Ω we obtain

$$\Omega(u,v;\mathcal{Z}_\lambda) = \omega_{kl}(x)\frac{\partial\mathcal{Z}_\lambda}{\partial x_k}\frac{\partial\mathcal{Z}_\lambda}{\partial x_l} + \overline{\omega}_{kl}(y)\frac{\partial\mathcal{Z}_\lambda}{\partial y_k}\frac{\partial\mathcal{Z}_\lambda}{\partial y_l}. \tag{7.4}$$

On the other hand, the following formulae are valid

$$\frac{\partial\mathcal{Z}_\lambda}{\partial x_k} = \left(\sum_{j=1}^\infty y_ju^j\right)^k\left(\sum_{i=1}^\infty iy_iu^{i-1}\right)^\lambda = [\mathcal{Y}(u)]^k[J(u)]^\lambda,$$

$$\begin{aligned}\frac{\partial\mathcal{Z}_\lambda}{\partial y_k} =& \left[\sum_{i=0}^\infty ix_i\left(\sum_{j=1}^\infty y_ju^j\right)^{i-1}u^k\right]\left(\sum_{i=1}^\infty iy_iu^{i-1}\right)^\lambda +\\ &+\left[\sum_{i=0}^\infty x_i\left(\sum_{j=1}^\infty y_ju^j\right)^i\right]\lambda\left(\sum_{i=1}^\infty iy_iu^{i-1}\right)^{\lambda-1}ku^{k-1}\\ =&\mathcal{X}'(\mathcal{Y}(u))(J(u))^\lambda u^k + \lambda\mathcal{X}(\mathcal{Y}(u))[J(u)]^{\lambda-1}ku^{k-1}\\ =&\mathcal{Z}'(u)[J(u)]^{\lambda-1}u^k + \lambda\mathcal{Z}_\lambda(u)[J(u)]^{-1}ku^{k-1}.\end{aligned}$$

Therefore equation (7.4) takes the form

$$\begin{aligned}
\Omega(u,v;\mathcal{Z}_\lambda) =&\Omega(\mathcal{Y}(u),\mathcal{Y}(v);\mathcal{X})(J(u))^\lambda(J(v))^\lambda+\\
&+\overline{\omega}_{kl}(y)\left[\mathcal{Z}'(u)(J(u))^{\lambda-1}u^k+\lambda\mathcal{Z}(u)(J(u))^{\lambda-1}ku^{k-1}\right]\times\\
&\times\left[\mathcal{Z}'(v)(J(v))^{\lambda-1}v^l+\lambda\mathcal{Z}(v)(J(v))^{\lambda-1}lv^{l-1}\right]\\
=&\Omega(\mathcal{Y}(u),\mathcal{Y}(v);\mathcal{X})(J(u))^\lambda(J(v))^\lambda+\overline{\omega}_{kl}(y)\mathcal{Z}'(u)(J(u))^{\lambda-1}u^k\mathcal{Z}'(v)(J(v))^{\lambda-1}v^l+\\
&+\overline{\omega}_{kl}(y)\lambda\mathcal{Z}(u)(J(u))^{\lambda-1}ku^{k-1}\mathcal{Z}'(v)(J(v))^{\lambda-1}v^l+\\
&+\overline{\omega}_{kl}(y)\mathcal{Z}'(u)(J(u))^{\lambda-1}u^k\lambda\mathcal{Z}(v)(J(v))^{\lambda-1}lv^{l-1}+\\
&+\lambda^2\overline{\omega}_{kl}(y)\mathcal{Z}(u)(J(u))^{\lambda-1}ku^{k-1}\mathcal{Z}(v)(J(v))^{\lambda-1}lv^{l-1}\\
=&\Omega(\mathcal{Y}(u),\mathcal{Y}(v);\mathcal{X})(J(u))^\lambda(J(v))^\lambda+\overline{\Omega}(u,v;\mathcal{Y})\mathcal{Z}'(u)(J(u))^{\lambda-1}\mathcal{Z}'(v)(J(v))^{\lambda-1}+\\
&+\lambda\partial_u\overline{\Omega}(u,v;\mathcal{Y})\mathcal{Z}(u)(J(u))^{\lambda-1}\mathcal{Z}'(v)(J(v))^{\lambda-1}+\\
&+\lambda\partial_v\overline{\Omega}(u,v;\mathcal{Y})\mathcal{Z}'(u)(J(u))^{\lambda-1}\mathcal{Z}(v)(J(v))^{\lambda-1}+\\
&+\lambda^2\partial^2_{u,v}\overline{\Omega}(u,v;\mathcal{Y})\mathcal{Z}(u)(J(u))^{\lambda-1}\mathcal{Z}(v)(J(v))^{\lambda-1}.
\end{aligned}$$

This concludes the proof of the Lemma.■

In the above formulae $\overline{\Omega}$ is given by (cf. Ch. VI)

$$\overline{\Omega}(u,v;\mathcal{Y}) = \varphi(u,v)\mathcal{Y}'(u)\mathcal{Y}'(v) - \varphi(\mathcal{Y}(u),\mathcal{Y}(v)), \tag{7.5}$$

where the function $\varphi(u,v)$ satisfies the equation

$$\varphi(u,v)\left[\partial_u\varphi(w,u)+\partial_v\varphi(w,v)\right]+c.p.=0. \tag{7.6}$$

In other words $\varphi(u,v)$ is given by (cf. Ch. VI)

$$\varphi(u,v) = f(u)g(v) - f(v)g(u),$$

where the functions f and g satisfy the relation

(7.7)

$$f'(u)g(u) - f(u)g'(u) = \alpha f'(u) + \beta g'(u) \quad \Big(\Longrightarrow \; f''(u)g(u) - f(u)g''(u) = \alpha f''(u) + \beta g''(u) \Big).$$

Here, α and β are constants.

Now we will describe a class of solutions of equation (7.3).

Theorem 7.3. *If $\varphi(u,v)$ is defined by the equation (7.6) and $\overline{\Omega}$ is defined by (7.5) one has the following solution of (7.3):*

$$\begin{aligned} \Omega(u,v;\mathcal{X}) =& \varphi(u,v)\mathcal{X}'(u)\mathcal{X}'(v) + \lambda\partial_u\varphi(u,v)\mathcal{X}(u)\mathcal{X}'(v) + \\ &+ \lambda\partial_v\varphi(u,v)\mathcal{X}'(u)\mathcal{X}(v) + \lambda^2\partial^2_{u,v}\varphi(u,v)\mathcal{X}(u)\mathcal{X}(v). \end{aligned} \tag{7.8}$$

Proof. Using formula (7.5) in the r.h.s. of equation (7.4) we obtain

$$\begin{aligned} r.h.s. = \Omega(\mathcal{Y}(u),\mathcal{Y}(v);\mathcal{X})(J(u))^\lambda(J(v))^\lambda + \varphi(u,v)\mathcal{Z}'(u)\mathcal{Z}'(v)(J(u))^\lambda(J(v))^\lambda - \\ -\varphi(\mathcal{Y}(u),\mathcal{Y}(v))\mathcal{Z}'(u)\mathcal{Z}'(v)(J(u))^{\lambda-1}(J(v))^{\lambda-1} + +\lambda\partial_u\varphi(u,v)\mathcal{Z}_\lambda(u)\mathcal{Z}'(v)(J(v))^\lambda + \\ +\varphi(u,v)\mathcal{Z}(u)\mathcal{Z}'(v)[(J(u))^\lambda]'(J(v))^\lambda - \lambda\partial_1\varphi(\mathcal{Y}(u),\mathcal{Y}(v))\mathcal{Z}_\lambda(u)\mathcal{Z}'(v)(J(v))^{\lambda-1} + \\ +\lambda\partial_v\varphi(u,v)\mathcal{Z}_\lambda(v)\mathcal{Z}'(u)(J(u))^\lambda + \varphi(u,v)\mathcal{Z}'(u)\mathcal{Z}(v)[(J(v))^\lambda]'(J(u))^\lambda - \\ -\lambda\partial_2\varphi(\mathcal{Y}(u),\mathcal{Y}(v))\mathcal{Z}_\lambda(v)\mathcal{Z}'(u)(J(u))^{\lambda-1} + \lambda^2\partial^2_{u,v}\varphi(u,v)\mathcal{Z}_\lambda(u)\mathcal{Z}_\lambda(v) + \\ +\lambda\partial_u\varphi(u,v)\mathcal{Z}_\lambda(u)\mathcal{Z}(v)[(J(v))^\lambda]' + \lambda\partial_v\varphi(u,v)\mathcal{Z}_\lambda(v)\mathcal{Z}(u)[(J(u))^\lambda]' + \\ +\varphi(u,v)\mathcal{Z}(u)\mathcal{Z}(v)[(J(u))^\lambda]'[(J(v))^\lambda]' - \lambda^2\partial^2_{1,2}\varphi(\mathcal{Y}(u),\mathcal{Y}(v))\mathcal{Z}_\lambda(u)\mathcal{Z}_\lambda(v). \end{aligned}$$

We used above the formulae

$$\partial_u\overline{\Omega}(u,v;\mathcal{Y}) = \partial_u\varphi(u,v)J(u)J(v) + \varphi(u,v)J'(u)J(v) - \partial_1\varphi(\mathcal{Y}(u),\mathcal{Y}(v))J(u),$$

$$\partial_v\overline{\Omega}(u,v;\mathcal{Y}) = \partial_v\varphi(u,v)J(u)J(v) + \varphi(u,v)J(u)J'(v) - \partial_2\varphi(\mathcal{Y}(u),\mathcal{Y}(v))J(v).$$

For the l.h.s. of equation (7.4) we have

$$\begin{aligned}
l.h.s. =& \varphi(u,v)\mathcal{Z}'_\lambda(u)\mathcal{Z}'_\lambda(v) + \lambda\partial_u\varphi(u,v)\mathcal{Z}_\lambda(u)\mathcal{Z}'_\lambda(v) + \\
&+ \lambda\partial_v\varphi(u,v)\mathcal{Z}'_\lambda(u)\mathcal{Z}_\lambda(v) + \lambda^2\partial^2_{u,v}\varphi(u,v)\mathcal{Z}_\lambda(u)\mathcal{Z}_\lambda(v) \\
=& \varphi(u,v)\Big\{\mathcal{Z}'(u)(J(u))^\lambda + \mathcal{Z}(u)\left[(J(u))^\lambda\right]'\Big\}\Big\{\mathcal{Z}'(v)(J(v))^\lambda + \mathcal{Z}(v)\left[(J(v))^\lambda\right]'\Big\} + \\
&+ \text{remaining terms} \\
=& \varphi(u,v)\mathcal{Z}'(u)\mathcal{Z}'(v)(J(u))^\lambda(J(v))^\lambda + \varphi(u,v)\mathcal{Z}'(u)\mathcal{Z}(v)(J(u))^\lambda\left[(J(v))^\lambda\right]' + \\
&+ \varphi(u,v)\mathcal{Z}(u)\mathcal{Z}'(v)(J(v))^\lambda\left[(J(u))^\lambda\right]' + \varphi(u,v)\mathcal{Z}(u)\mathcal{Z}(v)\left[(J(u))^\lambda\right]'\left[(J(v))^\lambda\right]' + \\
&+ \lambda\partial_u\varphi(u,v)\mathcal{Z}_\lambda(u)\mathcal{Z}'(v)(J(v))^\lambda + \lambda\partial_u\varphi(u,v)\mathcal{Z}_\lambda(u)\mathcal{Z}(v)\left[(J(v))^\lambda\right]' + \\
&+ \lambda\partial_v\varphi(u,v)\mathcal{Z}_\lambda(v)\mathcal{Z}'(u)(J(u))^\lambda + \lambda\partial_v\varphi(u,v)\mathcal{Z}_\lambda(v)\mathcal{Z}(u)\left[(J(u))^\lambda\right]' + \\
&+ \lambda^2\partial^2_{u,v}\varphi(u,v)\mathcal{Z}_\lambda(u)\mathcal{Z}_\lambda(v).
\end{aligned}$$

On the other hand we also have

$$\begin{aligned}
\Omega(\mathcal{Y}(u),\mathcal{Y}(v);\mathcal{X})(J(u))^\lambda(J(v))^\lambda =& \varphi(\mathcal{Y}(u),\mathcal{Y}(v))\mathcal{Z}'(u)\mathcal{Z}'(v)(J(u))^{\lambda-1}(J(v))^{\lambda-1} + \\
&+ \lambda\partial_1\varphi(\mathcal{Y}(u),\mathcal{Y}(v))\mathcal{Z}_\lambda(u)\mathcal{Z}'(v)(J(v))^{\lambda-1} + \\
&+ \lambda\partial_2\varphi(\mathcal{Y}(u),\mathcal{Y}(v))\mathcal{Z}_\lambda(v)\mathcal{Z}'(u)(J(u))^{\lambda-1} + \\
&+ \lambda^2\partial^2_{1,2}\varphi(\mathcal{Y}(u),\mathcal{Y}(v))\mathcal{Z}_\lambda(u)\mathcal{Z}_\lambda(v).
\end{aligned}$$

After comparing the terms on the l.h.s. and the r.h.s. of equation (7.4) we obtain an identity. This concludes the proof of the Theorem.■

Remark. Notice that for $\lambda \neq 0$ we can not have an inhomogeneous term of the form $\overline{\varphi}(\mathcal{X}(u), \mathcal{X}(v))$ in (7.8). Had it been the case it would impose on $\overline{\varphi}(u,v)$ the condition of being a homogeneous function of degree 1 in both arguments. In order to satisfy the equation (7.4) $\overline{\varphi}(u,v)$ must have the property $\overline{\varphi}(\mathcal{Z}_\lambda(u), \mathcal{Z}_\lambda(v)) = \overline{\varphi}(\mathcal{Z}(u)(J(u))^\lambda, \mathcal{Z}(v)(J(v))^\lambda) = \overline{\varphi}(\mathcal{Z}(u), \mathcal{Z}(v))(J(u))^\lambda(J(v))^\lambda)$. But since $\overline{\varphi}(u,v)$ must be also antisymmetric it follows that the only function with these properties is $\overline{\varphi} = 0$. On the contrary, for $\lambda = 0$ we have a solution of (7.4) of the form

$$\Omega(u,v;\mathcal{X}) = \varphi(u,v)\mathcal{X}'(u)\mathcal{X}'(v) - \overline{\varphi}(\mathcal{X}(u), \mathcal{X}(v)),$$

where the function $\overline{\varphi}$ has the property $\overline{\varphi}(u,v) = -\overline{\varphi}(v,u)$. The Jacobi identity for Ω then implies that $\overline{\varphi}$ must satisfy (7.6).

Next we come to the following remarkable fact.

Theorem 7.4. *If $\varphi(u,v)$ is defined by (7.6) then the solution (7.8) satisfies the Jacobi identity, thus defining a class of Poisson structures on V_λ for which the action of G_∞ on V_λ is a Poisson action.*

Proof. We will use again the definition

$$\{\mathcal{X}(u), \mathcal{X}(v)\} \equiv \Omega(u,v;\mathcal{X}).$$

Then we have

$$\begin{aligned}\{\mathcal{X}(u), \mathcal{X}(v)\} =& \varphi(u,v)\mathcal{X}'(u)\mathcal{X}'(v) + \lambda\partial_u\varphi(u,v)\mathcal{X}(u)\mathcal{X}'(v) + \\ &+ \lambda\partial_v\varphi(u,v)\mathcal{X}'(u)\mathcal{X}(v) + \lambda^2\partial^2_{u,v}\varphi(u,v)\mathcal{X}(u)\mathcal{X}(v),\end{aligned} \tag{7.9}$$

as well as

$$\begin{aligned}\partial_u\{\mathcal{X}(u), \mathcal{X}(v)\} =& \partial_v\varphi(u,v)\mathcal{X}'(u)\mathcal{X}'(v) + \varphi(u,v)\mathcal{X}'(u)\mathcal{X}''(v) + \\ &+ \lambda^2\partial^2_{u,v}\varphi(u,v)\mathcal{X}(u)\mathcal{X}'(v) + \lambda\partial_u\varphi(u,v)\mathcal{X}(u)\mathcal{X}''(v) + \\ &+ \lambda\partial^2_v\varphi(u,v)\mathcal{X}'(u)\mathcal{X}(v) + \lambda\partial_v\varphi(u,v)\mathcal{X}'(u)\mathcal{X}'(v) + \\ &+ \lambda^2\partial^3_{u,v^2}\varphi(u,v)\mathcal{X}(u)\mathcal{X}(v) + \lambda^2\partial^2_{u,v}\varphi(u,v)\mathcal{X}(u)\mathcal{X}'(v).\end{aligned}$$

The Jacobi identity

$$\{\mathcal{X}(w), \{\mathcal{X}(u), \mathcal{X}(v)\}\} + c.p. = 0$$

is equivalent to the following equation:

$$(\mathrm{a}) + (\mathrm{b}) + (\mathrm{c}) + (\mathrm{d}) = 0.$$

Here,

$$\begin{aligned}
(\mathrm{a}) =& \varphi(u,v)\{\mathcal{X}(w), \mathcal{X}'(u)\mathcal{X}'(v)\} + c.p. \\
=& \varphi(u,v)\Big[\partial_u\{\mathcal{X}(w), \mathcal{X}(u)\}\mathcal{X}'(v) + \partial_v\{\mathcal{X}(w), \mathcal{X}(v)\}\mathcal{X}'(u)\Big] + c.p. \\
(\mathrm{b}) =& \lambda\partial_u\varphi(u,v)\{\mathcal{X}(w), \mathcal{X}(u)\mathcal{X}'(v)\} + c.p. \\
=& \lambda\partial_u\varphi(u,v)\Big[\{\mathcal{X}(w), \mathcal{X}(u)\}\mathcal{X}(v) + \partial_v\{\mathcal{X}(w), \mathcal{X}(v)\}\mathcal{X}(u)\Big] + c.p. \\
(\mathrm{c}) =& \lambda\partial_v\varphi(u,v)\{\mathcal{X}(w), \mathcal{X}'(u)\mathcal{X}(v)\} + c.p. \\
=& \lambda\partial_v\varphi(u,v)\Big[\partial_u\{\mathcal{X}(w), \mathcal{X}(u)\}\mathcal{X}(v) + \{\mathcal{X}(w), \mathcal{X}(v)\}\mathcal{X}'(u)\Big] + c.p. \\
(\mathrm{d}) =& \lambda^2\partial^2_{u,v}\varphi(u,v)\{\mathcal{X}(w), \mathcal{X}(u)\mathcal{X}(v)\} + c.p. \\
=& \lambda^2\partial^2_{u,v}\varphi(u,v)\Big[\{\mathcal{X}(w), \mathcal{X}(u)\}\mathcal{X}(v) + \{\mathcal{X}(w), \mathcal{X}(v)\}\mathcal{X}(u)\Big] + c.p.
\end{aligned}$$

For the expressions in the square brackets for each term we therefore obtain

$$\begin{aligned}
(\mathrm{a}') =& \partial_u\varphi(w,u)\mathcal{X}'(w)\mathcal{X}'(u)\mathcal{X}'(v) + \varphi(w,u)\mathcal{X}'(w)\mathcal{X}''(u)\mathcal{X}'(v) + \\
&+ \lambda\partial^2_{w,u}\varphi(w,u)\mathcal{X}(w)\mathcal{X}'(u)\mathcal{X}'(v) + \lambda\partial_w\varphi(w,u)\mathcal{X}(w)\mathcal{X}''(u)\mathcal{X}'(v) + \\
&+ \lambda\partial^2_u\varphi(w,u)\mathcal{X}'(w)\mathcal{X}(u)\mathcal{X}'(v) + \lambda\partial_u\varphi(w,u)\mathcal{X}'(w)\mathcal{X}'(u)\mathcal{X}'(v) +
\end{aligned}$$

$$+ \lambda^2 \partial^3_{w,u^2}\varphi(w,u)\mathcal{X}(w)\mathcal{X}(u)\mathcal{X}'(v) + \lambda^2 \partial^2_{w,u}\varphi(w,u)\mathcal{X}(w)\mathcal{X}'(u)\mathcal{X}'(v)+$$

$$+ \partial_v\varphi(w,v)\mathcal{X}'(w)\mathcal{X}'(u)\mathcal{X}'(v) + \varphi(w,v)\mathcal{X}'(w)\mathcal{X}'(u)\mathcal{X}''(v)+$$

$$+ \lambda\partial^2_{w,v}\varphi(w,v)\mathcal{X}(w)\mathcal{X}'(u)\mathcal{X}'(v) + \lambda\partial_w\varphi(w,v)\mathcal{X}(w)\mathcal{X}'(u)\mathcal{X}''(v)+$$

$$+ \lambda\partial^2_v\varphi(w,v)\mathcal{X}'(w)\mathcal{X}'(u)\mathcal{X}(v) + \lambda\partial_v\varphi(w,v)\mathcal{X}'(w)\mathcal{X}'(u)\mathcal{X}'(v)+$$

$$+ \lambda^2 \partial^3_{w,v^2}\varphi(w,v)\mathcal{X}(w)\mathcal{X}'(u)\mathcal{X}(v) + \lambda^2 \partial^2_{w,v}\varphi(w,v)\mathcal{X}(w)\mathcal{X}'(u)\mathcal{X}'(v),$$

$$\text{(b')} = \varphi(w,u)\mathcal{X}'(w)\mathcal{X}'(u)\mathcal{X}'(v) + \lambda\partial_w\varphi(w,u)\mathcal{X}(w)\mathcal{X}'(u)\mathcal{X}'(v)+$$

$$+ \lambda\partial_u\varphi(w,u)\mathcal{X}'(w)\mathcal{X}(u)\mathcal{X}'(v) + \lambda^2 \partial^2_{w,u}\varphi(w,u)\mathcal{X}(w)\mathcal{X}(u)\mathcal{X}'(v)+$$

$$+ \partial_v\varphi(w,v)\mathcal{X}'(w)\mathcal{X}(u)\mathcal{X}'(v) + \varphi(w,v)\mathcal{X}'(w)\mathcal{X}(u)\mathcal{X}''(v)+$$

$$+ \lambda\partial^2_{w,v}\varphi(w,v)\mathcal{X}(w)\mathcal{X}(u)\mathcal{X}'(v) + \lambda\partial_w\varphi(w,v)\mathcal{X}(w)\mathcal{X}(u)\mathcal{X}''(v)+$$

$$+ \lambda\partial^2_v\varphi(w,v)\mathcal{X}'(w)\mathcal{X}(u)\mathcal{X}(v) + \lambda\partial_v\varphi(w,v)\mathcal{X}'(w)\mathcal{X}(u)\mathcal{X}'(v)+$$

$$+ \lambda^2 \partial^3_{w,v^2}\varphi(w,v)\mathcal{X}(w)\mathcal{X}(u)\mathcal{X}(v) + \lambda^2 \partial^2_{w,v}\varphi(w,v)\mathcal{X}(w)\mathcal{X}(u)\mathcal{X}'(v),$$

$$\text{(c')} = \partial_u\varphi(w,u)\mathcal{X}'(w)\mathcal{X}'(u)\mathcal{X}(v) + \varphi(w,u)\mathcal{X}'(w)\mathcal{X}''(u)\mathcal{X}(v)+$$

$$+ \lambda\partial^2_{w,u}\varphi(w,u)\mathcal{X}(w)\mathcal{X}'(u)\mathcal{X}(v) + \lambda\partial_w\varphi(w,u)\mathcal{X}(w)\mathcal{X}''(u)\mathcal{X}(v)+$$

$$+ \lambda\partial^2_u\varphi(w,u)\mathcal{X}'(w)\mathcal{X}(u)\mathcal{X}(v) + \lambda\partial_u\varphi(w,u)\mathcal{X}'(w)\mathcal{X}'(u)\mathcal{X}(v)+$$

$$+ \lambda^2 \partial^3_{w,u^2}\varphi(w,u)\mathcal{X}(w)\mathcal{X}(u)\mathcal{X}(v) + \lambda^2 \partial^2_{w,u}\varphi(w,u)\mathcal{X}(w)\mathcal{X}'(u)\mathcal{X}(v)+$$

$$+\varphi(w,v)\mathcal{X}'(w)\mathcal{X}'(u)\mathcal{X}'(v)+\lambda\partial_w\varphi(w,v)\mathcal{X}(w)\mathcal{X}'(u)\mathcal{X}'(v)+$$

$$+\lambda\partial^2_{w,v}\varphi(w,v)\mathcal{X}(w)\mathcal{X}'(u)\mathcal{X}(v)+\lambda\partial_v\varphi(w,v)\mathcal{X}'(w)\mathcal{X}'(u)\mathcal{X}(v),$$

$$\text{(d')}=\varphi(w,u)\mathcal{X}'(w)\mathcal{X}'(u)\mathcal{X}(v)+\lambda\partial_w\varphi(w,u)\mathcal{X}(w)\mathcal{X}'(u)\mathcal{X}(v)+$$

$$+\lambda\partial_u\varphi(w,u)\mathcal{X}'(w)\mathcal{X}(u)\mathcal{X}(v)+\lambda^2\partial^2_{w,u}\varphi(w,u)\mathcal{X}(w)\mathcal{X}(u)\mathcal{X}(v)+$$

$$+\varphi(w,v)\mathcal{X}'(w)\mathcal{X}(u)\mathcal{X}'(v)+\lambda\partial_w\varphi(w,v)\mathcal{X}(w)\mathcal{X}(u)\mathcal{X}'(v)+$$

$$+\lambda\partial_v\varphi(w,v)\mathcal{X}'(w)\mathcal{X}(u)\mathcal{X}(v)+\lambda^2\partial^2_{w,v}\varphi(w,u)\mathcal{X}(w)\mathcal{X}(u)\mathcal{X}(v).$$

As a result we will study the four terms in the Jacobi identity written as

$$\begin{aligned}
\text{(a)} &= \varphi(u,v)*\text{(a')}+c.p.\\
\text{(b)} &= \lambda\partial_u\varphi(u,v)*\text{(b')}+c.p.\\
\text{(c)} &= \lambda\partial_v\varphi(u,v)*\text{(c')}+c.p.\\
\text{(d)} &= \lambda^2\partial^2_{u,v}\varphi(u,v)*\text{(d')}+c.p.
\end{aligned}$$

We split the analysis of the Jacobi identity into seven steps (A)-(G). Our analysis starts with

(A) terms of the form $\mathcal{X}'\mathcal{X}'\mathcal{X}'$. From them we obtain (after cyclicly permuting the arguments of some of them):

$$\left\{\varphi(u,v)[\partial_u\varphi(w,u)+\partial_v\varphi(w,v)]+c.p.\right\}\mathcal{X}'(w)\mathcal{X}'(u)\mathcal{X}'(v).$$

But from (7.6) it follows that the above term is zero.

(B) Terms of the form $\mathcal{X}\mathcal{X}''\mathcal{X}'$ cancel each other out after cyclic permutation.

(C) Terms of the form $\mathcal{X}\mathcal{X}''\mathcal{X}$ cancel each other out after cyclic permutation.

(D) Terms of the form $\mathcal{X}\mathcal{X}\mathcal{X}$ give

$$\lambda^3\left[\partial_u\varphi(u,v)\partial^3_{w,v^2}\varphi(w,v)+\partial_v\varphi(u,v)\partial^3_{w,u^2}\varphi(w,u)+c.p.\right]\mathcal{X}(w)\mathcal{X}(u)\mathcal{X}(v)+$$
$$+\lambda^4\left[\partial^2_{u,v}\varphi(u,v)\left(\partial^2_{w,u}\varphi(w,u)+\partial^2_{w,v}\varphi(w,v)\right)+c.p.\right]\mathcal{X}(w)\mathcal{X}(u)\mathcal{X}(v).$$

Since $\varphi(u,v)$ is a solution of (7.6) the results obtained in Ch. VI showed that $\varphi(u,v) = f(u)g(v) - f(v)g(u)$, where f and g satisfy (7.7). Therefore the λ^3-term becomes

$$\lambda^3\Big\{\Big[f'(u)g(v) - f(v)g'(u)\Big]\Big[f'(w)g''(v) - f''(v)g'(w)\Big] + \\ + \Big[f(u)g'(v) - f'(v)g(u)\Big]\Big[f'(w)g''(u) - f''(u)g'(w)\Big] + c.p.\Big\}\mathcal{X}(w)\mathcal{X}(u)\mathcal{X}(v),$$

and the λ^4-term assumes the form

$$\lambda^4\Big\{\Big[f'(u)g'(v) - f'(v)g'(u)\Big]\times \\ \times\Big[f'(w)g'(u) - f'(u)g'(w) + f'(w)g'(v) - f'(v)g'(w)\Big] + c.p.\Big\}\mathcal{X}(w)\mathcal{X}(u)\mathcal{X}(v).$$

Using the identities $f'(u)g(u) - f(u)g'(u) = \alpha f'(u) + \beta g'(u)$ $\Big(\Longrightarrow$ $f''(u)g(u) - f(u)g''(u) = \alpha f''(u) + \beta g''(u)\Big)$ one easily shows that both terms proportional to λ^3 and λ^4 respectively are identically zero.

(E) Terms of the form $\mathcal{X}\mathcal{X}\mathcal{X}'$ give

$$\lambda^2\Big[\varphi(u,v)\partial^3_{w,u^2}\varphi(w,u) + \varphi(v,w)\partial^3_{u,w^2}\varphi(u,w) + \\ + \partial_u\varphi(u,v)\partial^2_{w,v}\varphi(w,v) + \partial_w\varphi(w,u)\partial^2_u\varphi(v,u) \\ + \partial_u\varphi(w,u)\partial^2_w\varphi(v,w) + \partial_w\varphi(v,w)\partial^2_{u,v}\varphi(u,v)\Big]\mathcal{X}(w)\mathcal{X}(u)\mathcal{X}'(v) + c.p.$$

The expression in the square brackets becomes identically zero after using (7.7) in a similar way as in (D). Also we have a term proportional to λ^3 which reads

$$\lambda^3\Big[\partial_u\varphi(u,v)\partial^2_{w,u}\varphi(w,u) + \partial_u\varphi(u,v)\partial^2_{w,v}\varphi(w,v) + \\ + \partial_w\varphi(v,w)\partial^2_{u,v}\varphi(u,v) + \partial_w\varphi(v,w)\partial^2_{u,w}\varphi(u,w) + \\ + \partial_w\varphi(v,w)\partial^2_{w,u}\varphi(w,u) + \partial_u\varphi(u,v)\partial^2_{v,w}\varphi(v,w) + \\ + \partial_w\varphi(w,v)\partial^2_{u,v}\varphi(u,v) + \partial_u\varphi(v,u)\partial^2_{w,u}\varphi(w,u)\Big]\mathcal{X}(w)\mathcal{X}(u)\mathcal{X}'(v) + c.p.,$$

which is identically zero.

(F) Terms of the form $\mathcal{X}\mathcal{X}'\mathcal{X}'$ give

$$\lambda\Big[\varphi(u,v)\partial^2_{w,u}\varphi(w,u) + \varphi(w,u)\partial^2_w\varphi(v,w) + \\ + \varphi(u,v)\partial^2_{w,v}\varphi(w,v) + \varphi(v,w)\partial^2_w\varphi(u,w) + \\ + \partial_w\varphi(w,u)\partial_u\varphi(v,u) + \partial_w\varphi(v,w)\partial_v\varphi(u,v)\Big]\mathcal{X}(w)\mathcal{X}'(u)\mathcal{X}'(v) + c.p.$$

The expression in the square brackets can be shown to be identically zero after using (7.7) in a similar way as in (D) and (E).

The other two terms of the same form are

$$\lambda^2\Big[\varphi(u,v)\partial^2_{w,u}\varphi(w,u)+\varphi(u,v)\partial^2_{w,v}\varphi(w,v)+ \\ +\varphi(v,u)\partial^2_{w,u}\varphi(w,u)+\varphi(u,v)\partial^2_{w,v}\varphi(v,w)\Big]\mathcal{X}(w)\mathcal{X}'(u)\mathcal{X}'(v)+c.p.,$$

which is identically zero, and

$$\lambda^2\Big[\partial_w\varphi(w,u)\partial_u\varphi(u,v)+\partial_w\varphi(w,u)\partial_w\varphi(v,w)+ \\ +\partial_w\varphi(w,u)\partial_u\varphi(v,u)+\partial_w\varphi(v,w)\partial_v\varphi(u,v)+ \\ +\partial_v\varphi(u,v)\partial_w\varphi(w,v)+\partial_w\varphi(v,w)\partial_w\varphi(u,w)\Big]\mathcal{X}(w)\mathcal{X}'(u)\mathcal{X}'(v)+c.p.,$$

which is again identically zero.

(F) Terms of the form $\mathcal{X}'\mathcal{X}''\mathcal{X}'$ cancel each other.

Thus all terms have been covered and the proof of Theorem 7.4 is completed.■

The consequence of Theorem 7.3 and Theorem 7.4 is that for each Poisson-Lie structure on G_∞ defined by a function φ satisfying the equation (7.6) there exists a Poisson structure on V_λ for which the action of G_∞ is Poisson. Thus we obtain a series of representations $V_{\varphi,\lambda}$ of G_∞ on V_λ.

CHAPTER VIII

QUANTIZATION

This chapter is devoted to the quantization of some of the Poisson-Lie structures on the group G_∞ restricted to the finite dimensionsional factor-groups G_n.

In this chapter we shall construct explicitly families of finite dimensional quantum (semi)groups. Their quasi-classical limits are the finite-dimensional Poisson-Lie groups endowed with Poisson-Lie structures which are restrictions of the Poisson-Lie structures on the group G_∞ belonging to the countable family obtained in Theorem 4.4. This means that we shall consider factor-groups $G_n = G_\infty \mathrm{mod}\{u^{n+1}\}$, for $n \geq 5$ (cf. Chapter II). In our aproach to quantization we shall start from the quasi-classical limits, i.e. the corresponding Poisson-Lie groups, and reconstruct from this data their quantum counterparts. Our quantization procedure is a procedure of deformation of the Poisson algebra of C^∞ functions on the corresponding finite-dimensional Poisson-Lie groups turning it into a "quantized algebra" of functions. We shall describe this algebra on the set of generators of $C^\infty(G_\infty)$, i.e. the coordinate functions on the corresponding (non-commutative) quantum group space.

(i) Let $X = \{x_i\}_{i\in\mathbb{N}}$ be the set of coordinate functions on G_∞. Let us introduce a grading on the space $k[X]$, where k denotes the ground field (assumed to be of characteristic zero), by assigning a degree (denoted $|\ |$) to each of the generators x_i of $k[X]$ by the following definition:

$$|x_i| = i - 1, \quad \text{for every } i \in \mathbb{N}, \tag{8.1}$$

and

$$|AB| = |A| + |B|, \quad \text{for every two monomials } A, B.$$

As we have mentioned above, in this chapter we shall address the quantization problem for the countable family of Poisson-Lie structures on the group G_∞, the

formulae for which we now recall:

(8.2)

$$\{x_i, x_j\} = (i-d)jx_jx_{i-d} - i(j-d)x_ix_{j-d}$$

$$+ x_i \sum_{(\sum_{k=1}^{d+1} s_k)=j} x_{s_1} \dots x_{s_{d+1}} - x_j \sum_{(\sum_{k=1}^{d+1} s_k)=i} x_{s_1} \dots x_{s_{d+1}}, \quad \forall\, d \in \mathbb{N}.$$

It is clear from the right hand side of (8.2) that for each $d \in \mathbb{N}$ the degree of the bracket $\{x_i, x_j\}$ is given by

$$|\{x_i, x_j\}| = |x_i| + |x_j| - d = i + j - d - 2. \tag{8.3}$$

(ii) Let $X = \{x_i\}_{i\in\mathbb{N}}$ be a set of generators and let $\langle X \rangle$ be a free associative semigroup with identity on X. Consider the set of relations

$$\mathcal{R}_h = \{x_ix_j - x_jx_i = f_{ij|h}(x) \mid i, j \in \mathbb{N}\} \tag{8.4}$$

which depend continuously on the parameter h and $f_{ij|h}(x)$ are polynomials in x_i, where $i \in \mathbb{N}$, such that $f_{ij|0} = 0$.

One of our postulates of quantization is the following. As explained earlier we require that

$$[x_i, x_j] = h\{x_i, x_j\} + \mathrm{O}(h^2). \tag{8.5}$$

Here $[x_i, x_j] = x_ix_j - x_jx_i$. In other words we would like to recover the Poisson-Lie bracket on G_∞ (or the factor groups $G_n = G_\infty \mathrm{mod}\ u^{n+1}$) in the quasi-classical limit $h \to 0$. This also means that for $h \to 0$ we should have $f_{ij|h}(x) = h\{x_i, x_j\} + \mathrm{O}(h^2)$ or

$$x_ix_j - x_jx_i = h\{x_i, x_j\} + \mathrm{O}(h^2). \tag{8.6}$$

After computing the degree of the right hand side of the above equality

$$|h\{x_i, x_j\}| = |h| + |\{x_i, x_j\}| = |h| + i + j - d - 2,$$

we deduce that for each $d \in \mathbb{N}$ the parameter h must have degree $|h| = d$, since

$$|[x_i, x_j]| = i + j - 2.$$

(iii) Consider the semigroup algebra $k[[h]]\langle X\rangle$ of $\langle X\rangle$ over the field $k[[h]]$ of formal power series in the parameter h. Here k is assumed to be a field of characteristic zero. For each $d \in \mathbb{N}$ consider the set of relations

$$\mathcal{R}_h^d = \{x_i x_j = x_j x_i + f_{ij|h}^d(x) \mid i < j \text{ for } i, j \in \mathbb{N}\} \tag{8.7}$$

where $f_{ij|h}^d(x) \in k[[h]]\langle X\rangle$ are linear combinations of monomials $h^n x_{i_1}^{n_1} \dots x_{i_k}^{n_k}$ such that $i_1 > \dots > i_k$ and $nd + \sum_{s=1}^k n_s(i_s - 1) = i + j - d - 2$. We will call these monomials in canonical form. Thus $f_{ij|h}^d(x)$ are linear combinations of monomials in canonical form. Recall the following definition.

Definition 8.1. The semigroup algebra $k[[h]]\langle X\rangle$ has the Poincaré-Birkhoff-Witt (PBW) property if every monomial $A \in \langle X\rangle$ can be reduced to a unique expression as a linear combination of monomials in canonical form using the set of relations (8.7) independently of the choice of a reduction procedure.

For example, if we are given the monomial $x_i x_j x_k$ then the two reduction procedures starting with

$$x_i x_j x_k \to (x_j x_i + f_{ij|h}^d(x)) x_k \to \dots ,$$

and

$$x_i x_j x_k \to x_i(x_j x_k + f_{jk|h}^d(x)) \to \dots$$

should lead to the same unique canonical expression. The above definition is equivalent to the assertion that the monomials in canonical form comprise a basis for $k[[h]]\langle X\rangle$ as a vector space over $k[[h]]$. We shall use a result known as the Diamond Lemma [Be]. Let us define a total order $\leq$ on the set X by $x_1 \leq x_2 \leq \dots \leq x_n \leq \dots$. Define a misordering index [Be] of an element $x_{i_1} \dots x_{i_n} \in \langle X\rangle$ as the number of pairs (p, q) such that $p < q$ but $x_{i_p} > x_{i_q}$. For example, if $x_{i_1} \leq x_{i_2} \leq \dots \leq x_{i_n}$ the misordering index is 0, if $x_{i_1} > x_{i_2} > \dots > x_{i_n}$ then the misordering index is $\frac{n(n-1)}{2}$. One defines a partial order $<$ on $\langle X\rangle$ by setting $A < B$ if A has a smaller length than B, or if A is a permutation of the terms of B but has a smaller misordering index. Then the l.h.s. of each of the relations from the set $\mathcal{R}_h^d$ is smaller than every monomial on the r.h.s. Also it is easy to verify that every reduction procedure using the set of relations $\mathcal{R}_h^d$ terminates. Under these conditions the Diamond Lemma asserts that the semigroup algebra $k[[h]]\langle X\rangle$ has the PBW property.

Definition 8.2. For each $d \in \mathbb{N}$ a quantum semigroup $G_{\infty|h}^d$ is defined as follows. As a quantum space $G_{\infty|h}^d$ is defined by its factor semigroup algebra of coordinate

functions $k[[h]]\langle X\rangle/\mathcal{I}_h^d \equiv \mathrm{Fun}(G^d_{\infty|h})$, where $\mathcal{I}_h^d \subset k[[h]]\langle X\rangle$ is the ideal generated by the set of relations (8.7), and we require that in the quasi-classical limit

$$[x_i, x_j] = h\{x_i, x_j\} + \mathrm{O}(h^2)$$

one obtains the Poisson algebra of functions on the Poisson-Lie group G^d_∞ defined by (8.2). The multiplication of formal power series in one variable under the operation of substitution

$$(\mathcal{X}\mathcal{Y})(u) = \mathcal{X}(\mathcal{Y}(u)),$$

where $\mathcal{X}(u) = \sum_{i=1}^\infty x_i u^i$, and $\mathcal{Y}(u) = \sum_{i=1}^\infty y_i u^i$, gives rise to a comultiplication map

$$\Delta\colon \mathrm{Fun}(G^d_{\infty|h}) \to \mathrm{Fun}(G^d_{\infty|h}) \otimes \mathrm{Fun}(G^d_{\infty|h}),$$

which is defined on the generators by

$$\Delta(x_k) = \sum_{i=1}^k x_i \otimes \sum_{\sum_{\alpha=1}^i j_\alpha = k} x_{j_1} \dots x_{j_i}, \qquad k \in \mathbb{N}.$$

Also one defines a counit map $c\colon \mathrm{Fun}(G^d_{\infty|h}) \to k[[h]]$ by

$$c(x_i) = \delta_i^1, \qquad i \in \mathbb{N}.$$

This endows $\mathrm{Fun}(G^d_{\infty|h})$ with a structure of a bialgebra and the quantum semigroup $G^d_{\infty|h}$ is defined to be the bialgebra $\mathrm{Fun}(G^d_{\infty|h})$. In order to do computations we will introduce the notion of multiplication of commuting points of $G^d_{\infty|h}$. Let A be a non-commutative algebra over $k[[h]]$. Define the set $\mathrm{Hom}_{k[[h]]-alg}(\mathrm{Fun}(G^d_{\infty|h}), A)$ to be the set (functor of points [Ma2]) of points of $G^d_{\infty|h}$. Two points $f, f' \in \mathrm{Hom}_{k[[h]]-alg}(\mathrm{Fun}(G^d_{\infty|h}), A)$ commute if $f(x)f'(x') = f'(x')f(x)$, for all $x, x' \in \mathrm{Fun}(G^d_{\infty|h})$. A product between two commuting points f, f' is defined [Ma1] as the homomorphism

$$ff'\colon \mathrm{Fun}(G^d_{\infty|h}) \xrightarrow{\Delta} \mathrm{Fun}(G^d_{\infty|h}) \otimes \mathrm{Fun}(G^d_{\infty|h}) \xrightarrow{f\otimes f'} A \otimes A \xrightarrow{m_A} A.$$

Let $u_i = f(x_i)$, $v_i = f'(x_i)$, and $w_i = ff'(x_i)$, for all $i \in \mathbb{N}$, where x_i's are the generators of $\mathrm{Fun}(G^d_{\infty|h})$. Then we obtain that

$$w_k = \sum_{i=1}^k u_i \sum_{\sum_{\alpha=1}^i j_\alpha = k} v_{j_1} \dots v_{j_i}, \qquad k \in \mathbb{N},$$

where

$$u_i v_j = v_j u_i, \qquad \text{for every } i, j \in \mathbb{N}.$$

Remark. Another way to think of the multiplication map is the following. Let $X = \{x_i\}_{i\in\mathbb{N}}$ and $Y = \{y_i\}_{i\in\mathbb{N}}$ be two sets of generators. We make the identification

$$k[[h]]\langle X\rangle/\mathcal{I}_h^d(X) \otimes k[[h]]\langle Y\rangle/\mathcal{I}_h^d(Y) \simeq k[[h]]\langle X,Y\rangle/\{\mathcal{I}_h^d(X),\mathcal{I}_h^d(Y),\mathcal{J}(X,Y)\},$$

and require that

$$k[[h]]\langle X,Y\rangle/\{\mathcal{I}_h^d(X),\mathcal{I}_h^d(Y),\mathcal{J}(X,Y)\} \equiv k[[h]]\langle Z\rangle/\mathcal{I}_h^d(Z),$$

where Z is the set $Z = \{z_k\}_{k\in\mathbb{N}}$ with z_k's given by

$$z_k = \sum_{i=1}^{k} x_i \sum_{\sum_{\alpha=1}^{i} j_\alpha = k} y_{j_1}\cdots y_{j_i}, \qquad k \in \mathbb{N}, \tag{8.8}$$

and where $\mathcal{J}(X,Y)$ is the ideal in $k[[h]]\langle X,Y\rangle$ generated by the set of relations

$$\{x_i y_j - y_j x_i = 0 \mid \text{for every } i,j \in \mathbb{N}\}. \tag{8.9}$$

Does such an object exist? We do not know yet. However, if we consider the Poisson-Lie factor groups $G_n^d = G_\infty^d \mathrm{mod}\{u^{n+1}\}$, for $n \geq 5$, then there exist quantum objects that satisfy the above definition for $d \leq 3$ and whose quasi-classical limits are the Poisson-Lie groups G_n^d. The definition of the finite dimensional quantum groups $G_{n|h}^d$ is the same as the definition above with $\mathcal{I}_h^d$ being an ideal generated by a finite set of relations $\mathcal{R}_h^d$. Their defining semigroup algebras of functions turn out to have other interesting properties.

We will describe in more detail the construction of the quantum semigroup $G_{5|h}^2$, while omitting parts of the construction that consist of lengthy and tedious calculations, and we will state the results for the quantum semigroups $G_{4|h}^1$ and $G_{5|h}^3$ without entering into the details of the calculations. The construction of the last two mimics exactly the construction of $G_{5|h}^2$.

Let $G_5^2 = G_\infty^2 \mathrm{mod}\{u^{n+1}\}$, for $n \geq 5$, be the finite dimensional (dim=5, $d = 2$) Poisson-Lie group with a Poisson-Lie structure defined by

$$\begin{aligned}\{x_i, x_j\} =& (i-2)j x_j x_{i-2} - i(j-2) x_i x_{j-2} \\ &+ x_i \sum_{s_1+s_2+s_3=j} x_{s_1} x_{s_2} x_{s_3} - x_j \sum_{s_1+s_2+s_3=i} x_{s_1} x_{s_2} x_{s_3}.\end{aligned} \tag{8.10}$$

The above formulae are obtained from (8.2) with $d = 2$ and we adopt the convention that $x_i = 0$ whenever $i < 1$. In more detail we have

$$\{x_1, x_2\} = 0$$

$$\{x_1, x_3\} = -x_1^2 + x_1^4$$

$$\{x_2, x_3\} = -2x_2x_1 + x_2x_1^3$$

$$\{x_1, x_4\} = -2x_2x_1 + 3x_2x_1^3$$

$$\{x_2, x_4\} = -4x_2^2 + 3x_2^2x_1^2$$

$$\{x_3, x_4\} = 4x_4x_1 - x_4x_1^3 - 6x_3x_2 + 3x_3x_2x_1^2 \tag{8.11}$$

$$\{x_1, x_5\} = -3x_3x_1 + 3x_2^2x_1^2 + 3x_3x_1^3$$

$$\{x_2, x_5\} = -6x_3x_2 + 3x_2^3x_1 + 3x_3x_2x_1^2$$

$$\{x_3, x_5\} = 5x_5x_1 - 9x_3^2 + 3x_3x_2^2x_1 + 3x_3^2x_1^2 - x_5x_1^3$$

$$\{x_4, x_5\} = 10x_5x_2 - 12x_4x_3 + 3x_4x_2^2x_1 + 3x_4x_3x_1^2 - 3x_5x_2x_1^2.$$

Theorem 8.3. *Let $X = \{x_i\}_{1 \le i \le 5}$ be a set and let $\langle X \rangle$ be the associative semigroup with identity generated by X. Consider an ideal $\mathcal{I}_h^2$ generated by the set of relations $\mathcal{R}_h^2$ in $k[[h]]\langle X \rangle$*

$$x_1x_2 = x_2x_1$$

$$x_1x_3 = x_3x_1 + h(-x_1^2 + x_1^4)$$

$$x_2x_3 = x_3x_2 + h(-2x_2x_1 + x_2x_1^3)$$

$$x_1x_4 = x_4x_1 + h(-2x_2x_1 + 3x_2x_1^3)$$

$$x_2x_4 = x_4x_2 + h(-4x_2^2 + 3x_2^2x_1^2)$$

$$\begin{aligned} x_3x_4 = x_4x_3 &+ h(4x_4x_1 - x_4x_1^3 - 6x_3x_2 + 3x_3x_2x_1^2) \\ &+ 2h^2x_2x_1 \end{aligned} \tag{8.12}$$

$$\begin{aligned} x_1x_5 = x_5x_1 &+ h(-3x_3x_1 + 3x_2^2x_1^2 + 3x_3x_1^3) \\ &+ h^2(-6x_1^4 + \frac{9}{2}x_1^6 + \frac{3}{2}x_1^2) \end{aligned}$$

$$\begin{aligned} x_2x_5 = x_5x_2 &+ h(-6x_3x_2 + 3x_2^3x_1 + 3x_3x_2x_1^2) \\ &+ h^2x_2(6x_1 - 9x_1^3 + \frac{9}{2}x_1^5) \end{aligned}$$

$$\begin{aligned} x_3x_5 = x_5x_3 &+ h(5x_5x_1 - 9x_3^2 + 3x_3x_2^2x_1 + 3x_3^2x_1^2 - x_5x_1^3) \\ &+ h^2x_3(-\frac{15}{2}x_1 + 6x_1^3 + \frac{3}{2}x_1^5) \\ &+ h^3C(x_1^8 - x_1^2) \end{aligned}$$

$$\begin{aligned} x_4x_5 = x_5x_4 &+ h(10x_5x_2 - 12x_4x_3 + 3x_4x_2^2x_1 + 3x_4x_3x_1^2 - 3x_5x_2x_1^2) \\ &+ h^2[x_4(-24x_1 + 9x_1^3 + \frac{3}{2}x_1^5) + 6x_3x_2] \\ &+ h^3x_2[-(6+2C)x_1 + 3Cx_1^7], \end{aligned}$$

where $C \in k$ is an arbitrary parameter. Then the semigroup factor algebra $k[[h]]\langle X\rangle/\mathcal{I}_h^2$ defines a quantum semigroup $G^2_{5|h}$ in the sense of Definition 8.2 *with a multiplication defined by* (8.8). *Namely,*

$$z_1 = x_1y_1$$

$$z_2 = x_1y_2 + x_2y_1^2$$

$$z_3 = x_1y_3 + x_2y_1y_2 + x_2y_2y_1 + x_3y_1^3$$

$$z_4 = x_1y_4 + x_2y_1y_3 + x_2y_2^2 + x_2y_3y_1 + x_3y_1^2y_2 + x_3y_1y_2y_1 + x_3y_2y_1^2 + x_4y_1^4$$

$$
\begin{aligned}
z_5 =& x_1y_5 + x_2y_1y_4 + x_2y_2y_3 + x_2y_3y_2 + x_2y_4y_1 + x_3y_1^2y_3 \\
&+ x_3y_1y_2^2 + x_3y_1y_3y_1 + x_3y_2y_1y_2 + x_3y_2^2y_1 + x_3y_3y_1^2 + x_4y_1^3y_2 \\
&+ x_4y_1^2y_2y_1 + x_4y_1y_2y_1^2 + x_4y_2y_1^3 + x_5y_1^4.
\end{aligned} \tag{8.13}
$$

Moreover the semigroup algebra $k[[h]]\langle X\rangle$ has the Poincaré-Birkhoff-Witt property.

Proof. The proof is constructive. We look for a set of relations $\mathcal{R}_h^2$ in $k[[h]]\langle X\rangle$ in the following form

$$x_1x_2 = x_2x_1$$

$$x_1x_3 = x_3x_1 + h(-x_1^2 + x_1^4)$$

$$x_2x_3 = x_3x_2 + h(-2x_2x_1 + x_2x_1^3)$$

$$x_1x_4 = x_4x_1 + h(-2x_2x_1 + 3x_2x_1^3)$$

$$x_2x_4 = x_4x_2 + h(-4x_2^2 + 3x_2^2x_1^2) + h^2f_1(x_1)$$

$$
\begin{aligned}
x_3x_4 = x_4x_3 &+ h(4x_4x_1 - x_4x_1^3 - 6x_3x_2 + 3x_3x_2x_1^2) \\
&+ 2h^2x_2f_2(x_1)
\end{aligned}
$$

$$
\begin{aligned}
x_1x_5 = x_5x_1 &+ h(-3x_3x_1 + 3x_2^2x_1^2 + 3x_3x_1^3) \\
&+ h^2f_3(x_1)
\end{aligned}
$$

$$
\begin{aligned}
x_2x_5 = x_5x_2 &+ h(-6x_3x_2 + 3x_2^3x_1 + 3x_3x_2x_1^2) \\
&+ h^2x_2f_4(x_1)
\end{aligned}
$$

$$
\begin{aligned}
x_3x_5 = x_5x_3 &+ h(5x_5x_1 - 9x_3^2 + 3x_3x_2^2x_1 + 3x_3^2x_1^2 - x_5x_1^3) \\
&+ h^2[x_3f_5(x_1) + x_2^2f_6(x_1)] \\
&+ h^3f_7(x_1)
\end{aligned}
$$

$$
\begin{aligned}
x_4x_5 = x_5x_4 &+ h(10x_5x_2 - 12x_4x_3 + 3x_4x_2^2x_1 + 3x_4x_3x_1^2 - 3x_5x_2x_1^2) \\
&+ h^2[x_4f_8(x_1) + x_3x_2f_9(x_1) + x_2^3f_{10}(x_1)] \\
&+ h^3x_2f_{11}(x_1),
\end{aligned} \tag{8.14}
$$

where $\{f_i\}_{1\leq i\leq 11}$ is a set of arbitrary functions. Since the degree of h is $|h| = 2$ and the degree of x_1 is $|x_1| = 0$ this is the most general form of the set of relations that one can have such that their quasi-classical limit gives the Poisson-Lie structure on G_5^2.

If $z = xy \in G^2_{5|h}$ is the product of two commuting points $x, y \in G^2_{5|h}$ then its coordinate functions z_i, $1 \leq i \leq 5$ should satisfy the set of relations (8.14). This leads to some restrictions on the functions $f_i(x)$. Using the multiplication formulae (8.13) one can see that the coordinate functions of z satisfy the first four relations of (8.14) identically. This is done by substituting the formulae for the functions z_i into the relations and reducing both sides of the corresponding relation to a canonical form which yields an identity. In a similar way one obtains functional equations for the functions $f_i(x)$ from the remaining six relations.

We will analyse first the equations that arise from terms of order h^2.

(a) After reducing both sides of the relation

$$z_2z_4 = z_4z_2 + h(-4z_2^2 + 3z_2^2z_1^2) + h^2f_1(z_1)$$

to a canonical form using the multiplication formulae (8.13) we obtain the following linear functional equation for the function $f_1(x)$:

$$-f_1(x_1y_1) + x_1^2f_1(y_1) + y_1^6f_1(x_1) = 0.$$

From now on since all the equations for the functions f_i will depend only on the variables x_1, y_1 we will use the notation $x \equiv x_1$ and $y \equiv y_1$. Therefore we have

$$-f_1(xy) + x^2f_1(y) + y^6f_1(x) = 0. \tag{8.15}$$

The most general solution of the above equation is

$$f_1(x) = C_1(x^6 - x^2), \tag{8.16}$$

where $C_1 \in k$ is an arbitrary constant. There are no terms of higher order in h that arise in the analysis of this relation. We move on to the next.

(b) Again after reducing to a canonical form both sides of

$$z_3z_4 = z_4z_3 + h(4z_4z_1 - z_4z_1^3 - 6z_3z_2 + 3z_3z_2z_1^2) + 2h^2z_2f_2(z_1)$$

we obtain two equations. One of them arises from a term proportional to x_2, i.e. we have a term of the form

$$x_2[-y_1^2f_2(x_1y_1) + (2-2C_1)x_1y_1^3 + y_1^7f_2(x_1) + (-2+2C_1)x_1y_1^7].$$

It leads to the equation

$$-y^2f_2(xy) + (2-2C_1)xy^3 + y^7f_2(x) + (-2+2C_1)xy^7 = 0. \tag{8.17}$$

The term proportional to y_2 leads to

$$-xf_2(xy) + x^2f_2(y) - 2C_1x^2y^5 + 2C_1x^6y^5 = 0. \tag{8.18}$$

Solving (8.17) and (8.18) together we obtain

$$f_2(x) = (2-2C_1)x + 2C_1x^5. \tag{8.19}$$

There are no terms of higher order in h arising from this relation.

At this stage of the calculation we check whether the PBW property is satisfied in the subalgebra of $k[[h]]\langle X\rangle$ generated by the set $\{x_1, x_2, x_3, x_4\}$ and subject to the first six relations of (8.14). By direct calculation, using the Diamond Lemma, one shows that the monomial $x_2x_3x_4$ can be reduced to a unique canonical form if and only if $C_1 = 0$. The other possible monomials of three variables have a unique canonical form. Therefore we obtain that

$$f_1(x) = 0 \quad \text{and} \quad f_2(x) = 2x. \tag{8.20}$$

(c) Then the next relation

$$z_1z_5 = z_5z_1 + h(-3z_3z_1 + 3z_2^2z_1^2 + 3z_3z_1^3) + h^2f_3(z_1),$$

after a reduction to a canonical form leads to the equation

$$-f_3(xy) + x^2f_3(y) + 6x^2y^4 + y^6f_3(x) - 6x^4y^4 - 6x^2y^6 + 6x^4y^6 = 0. \tag{8.21}$$

The most general solution of the above equation is given by

$$f_3(x) = 6x^2 - 6x^4 + C_2(x^6 - x^2), \tag{8.22}$$

where $C_2 \in k$ is an arbitrary constant. There are no terms of order h^3 or higher that arise after the reduction to a canonical form of the above relation.

(d) The analysis of the next relation

$$z_2 z_5 = z_5 z_2 + h(-6z_3 z_2 + 3z_2^3 z_1 + 3z_3 z_2 z_1^2) + h^2 z_2 f_4(z_1)$$

leads to two equations. One of them arises from a term of the form

$$h^2 x_2[-y_1^2 f_4(x_1 y_1) + (15 - 2C_2)x_1 y_1^3 + y_1^7 f_4(x_1) - 9x_1^3 y_1^5 + (-15 + 2C_2)x_1 y_1^7 + 9x_1^3 y_1^7].$$

Setting it to zero gives

$$(8.23) \quad -y^2 f_4(xy) + (15 - 2C_2)xy^3 + y^7 f_4(x) - 9x^3 y^5 + (-15 + 2C_2)xy^7 + 9x^3 y^7 = 0.$$

The second equation comes from a term proportional to $h^2 y_2$ and reads

$$(8.24) \qquad -x f_4(xy) + x^2 f_4(y) + 9x^2 y^3 - 9x^4 y^3 - C_2 x^2 y^5 + C_2 x^6 y^5 = 0.$$

Solving together (8.23) and (8.24) one obtains

$$(8.25) \qquad f_4(x) = (15 - 2C_2)x - 9x^3 + C_2 x^5.$$

There are no terms of higher order in h that do not cancel after the reduction to a canonical form.

(e) We move on to the next relation which gives

$$\begin{aligned} z_3 z_5 = z_5 z_3 &+ h(5z_5 z_1 - 9z_3^2 + 3z_3 z_2^2 z_1 + 3z_3^2 z_1^2 - z_5 z_1^3) \\ &+ h^2\Big[z_3 f_5(z_1) + z_2^2 f_6(z_1)\Big] + h^3 f_7(z_1). \end{aligned}$$

Terms of order h^2 give rise to five functional equations which we now describe.
(i) A term proportional to x_3 gives rise to

$$(8.26) \quad -y^3 f_5(xy) + (6 - 3C_2)xy^4 + y^8 f_5(x) + 6x^3 y^6 - 6x^3 y^8 + (-6 + 3C_2)xy^8 = 0,$$

and a term prortional to y_3 gives rise to the equation

$$(8.27) \quad -x f_5(xy) + x^2 f_5(y) - 6x^2 y^3 + 6x^4 y^3 + (3 - C_2)x^2 y^5 + (-3 + C_2)x^6 y^5 = 0.$$

Solving (8.26) and (8.27) together we obtain for f_5

$$(8.28) \qquad f_5(x) = (6 - 3C_2)x + 6x^3 + (-3 + C_2)x^5.$$

(ii) Terms proportional to x_2^2 and y_2^2 give rise to another two functional equations:

$$-y^4 f_6(xy) + y^8 f_6(x) = 0, \tag{8.29}$$

and

$$-x^2 f_6(xy) + x^2 f_6(y) = 0 \tag{8.30}$$

respectively. The only solution that satisfies both (8.28) and (8.29) is

$$f_6(x) = 0. \tag{8.31}$$

(iii) The last term from the terms of order h^2 is a term proportional to $x_2 y_2$ which gives rise to the following equation for f_5 and f_6:

$$-2y f_5(xy) + (12 - 6C_2)xy^2 - 2xy^2 f_6(xy) + 12x^3y^4 + (-6 + 2C_2)x^5y^6 = 0. \tag{8.32}$$

After substituting the solutions (8.25) and (8.31) into (8.32) we obtain that it is satisfied identically. There is one term of order h^3 that arises which gives rise to

$$x^2 f_7(y) + y^8 f_7(x) - f_7(xy) = 0. \tag{8.33}$$

The most general solution of (8.33) is given by

$$f_7(x) = C_3(x^8 - x^2), \tag{8.34}$$

where $C_3 \in k$ is an arbitrary constant. No terms of higher order in h arise.

(f) The last relation to be analyzed is

$$\begin{aligned} z_4 z_5 = z_5 z_4 &+ h(10 z_5 z_2 - 12 z_4 z_3 + 3 z_4 z_2^2 z_1 + 3 z_4 z_3 z_1^2 - 3 z_5 z_2 z_1^2) \\ &+ h^2\Big[z_4 f_8(z_1) + z_3 z_2 f_9(z_1) + z_2^3 f_{10}(z_1)\Big] + h^3 z_2 f_{11}(z_1). \end{aligned}$$

After reducing to a canonical form both sides of the above relation we obtain ten terms of order h^2 that do not cancel and two terms of order h^3. We analyze first the terms of order h^2.

(i) Two terms proportional to x_4 and y_4 give rise to the following two equations

$$-y^4 f_8(xy) + (-6 - 4C_2)xy^5 + y^9 f_8(x) + 9x^3y^7 + (6 + 4C_2)xy^9 - 9x^3y^9 = 0, \tag{8.35}$$

and

$$-x f_8(xy) + x^2 f_8(y) - 9x^2y^3 + 9x^4y^3 + (3 - C_2)x^2y^5 + (-3 + C_2)x^6y^5 = 0 \tag{8.36}$$

respectively. The most general solution of (8.35) and (8.36) is

$$f_8(x) = (-6 - 4C_2)x + 9x^3 + (-3 + C_2)x^5. \tag{8.37}$$

(ii) Terms proportional to x_3x_2 and y_3y_2 give rise to the equations

$$6y^5 - y^5 f_9(xy) - 6y^9 + y^9 f_9(x) = 0, \tag{8.38}$$

and

$$-x^2 f_9(xy) + x^2 f_9(y) = 0. \tag{8.39}$$

The solution of the system (8.38) and (8.39) is

$$f_9(x) = 6. \tag{8.40}$$

(iii) A term proportional to x_3y_2 gives rise to

$$-3y^2 f_8(xy) + (-12 - 12C_2)xy^3 - xy^3 f_9(xy) + 27x^3y^5 + (-9 + 3C_2)x^5y^7 = 0. \tag{8.41}$$

After substituting (8.37) and (8.40) into (8.41) it yields an identity. Similarly the term proportional to x_2y_3 leads to an identity.
(iv) Two terms proportional to x_2^3 and y_2^3 lead to the equations

$$-y^6 f_{10}(xy) + y^9 f_{10}(x) = 0, \tag{8.42}$$

and

$$x^2 f_{10}(y) - x^3 f_{10}(xy) = 0. \tag{8.43}$$

The only solution of (8.42) and (8.43) solved together is

$$f_{10} = 0. \tag{8.44}$$

(v) The terms proportional to $x_2y_2^2$ and $x_2^2y_2$ give rise to

$$-f_8(xy) + (6 - 4C_2)xy - 2xy f_9(xy) - 3x^2y^2 f_{10}(xy) + 9x^3y^3 + (-3 + C_2)x^5y^5 = 0, \tag{8.45}$$

and

$$12y^3 - 2y^3 f_9(xy) - 3xy^4 f_{10}(xy) = 0 \tag{8.46}$$

which are identically satisfied. This becomes obvious after substituting (8.37), (8.40) and (8.44) into (8.45) and (8.46).
The two terms of order h^3 are proportional to x_2 and y_2 and give rise to

$$-y^2 f_{11}(xy) + (3 - 2C_2 - 2C_3)xy^3 + y^3 f_{11}(x) + (-3 + 2C_2 + 2C_3)xy^9 = 0, \tag{8.47}$$

and

$$-x f_{11}(xy) + x^2 f_{11}(y) - 3C_3 x^2 y^7 + 3C_3 x^8 y^7 = 0 \tag{8.48}$$

respectively. The most general solution of (8.47) and (8.48) is

$$f_{11}(x) = (3 - 2C_2 - 2C_3)x + 3C_3 x^7. \tag{8.49}$$

We need one last step in order to complete the construction. We would like to find whether the so obtained set of relations define an algebra with the PBW property. After lengthy and tedious calculation one shows that the requirement that the monomials $x_1x_3x_5$, $x_1x_4x_5$, $x_2x_4x_5$, and $x_3x_4x_5$ can be reduced to a unique canonical form imposes the following single equation on the arbitrary constant C_2:

$$-9 + 2C_2 = 0. \tag{8.50}$$

The monomial $x_1x_2x_5$ is reducible to a canonical form without imposing any conditions. Thus $C_2 = \frac{9}{2}$. If we introduce $C \equiv C_3$ we obtain the statement of the Theorem. This concludes the proof.■

Remark. Notice that our construction yields a one-parameter family of quantum semigroups $G^{2|C}_{5|h}$ parametrized by the parameter C! The following theorem will describe a family of quantum semigroups parametrized by even more parameters. This phenomenon seems quite intriguing.

Let $G^1_4 = G^1_\infty \mathrm{mod}\{u^{n+1}\}$, for $n \geq 4$, be the finite dimensional (dim=4, $d = 1$) Poisson-Lie group with a Poisson-Lie structure defined by

$$\begin{aligned}\{x_i, x_j\} =& (i-1)jx_jx_{i-1} - i(j-1)x_ix_{j-1} \\ &+ x_i \sum_{s_1+s_2=j} x_{s_1}x_{s_2} - x_j \sum_{s_1+s_2=i} x_{s_1}x_{s_2}.\end{aligned} \tag{8.51}$$

The above formulae are obtained from (8.2) with $d = 1$. Writing them explicitly we have

$$
\begin{aligned}
\{x_1, x_2\} &= x_1^3 - x_1^2 \\
\{x_1, x_3\} &= 2x_2(x_1^2 - x_1) \\
\{x_2, x_3\} &= (3x_1 - x_1^2)x_3 + x_2^2(2x_1 - 4) \\
\{x_1, x_4\} &= x_3(2x_1^2 - 3x_1) + x_2^2 x_1 \\
\{x_2, x_4\} &= x_4(4x_1 - x_1^2) + x_3 x_2(2x_1 - 6) \\
\{x_3, x_4\} &= x_4 x_2(8 - 2x_1) + x_3 x_2^2 - 9x_3^2 + 2x_3^2 x_1.
\end{aligned}
\tag{8.52}
$$

Theorem 8.4. *Let $X = \{x_i\}_{1 \le i \le 4}$ be a set and let $\langle X \rangle$ be the associative semigroup with identity generated by X. Consider an ideal $\mathcal{I}_h^1$ generated by the set of relations $\mathcal{R}_h^1$ in $k[[h]]\langle X \rangle$*

$$
\begin{aligned}
x_1 x_2 &= x_2 x_1 + h(x_1^3 - x_1^2) \\
x_1 x_3 &= x_3 x_1 + h(2x_2 x_1^2 - 2x_2 x_1) \\
&\quad + h^2(2x_1^4 - 3x_1^3 + x_1^2) \\
x_2 x_3 &= x_3 x_2 + h(3x_3 x_1 - x_3 x_1^2 + 2x_2^2 x_1 - 4x_2^2) \\
&\quad + h^2(3x_2 x_1^2 - 3x_2 x_1) \\
&\quad + h^3(2 - 2C_3)(x_1^5 - x_1^2) \\
x_1 x_4 &= x_4 x_1 + h(-3x_3 x_1 + 2x_3 x_1^2 + x_2^2 x_1) \\
&\quad + h^2 x_2(3x_1 - 8x_1^2 + 5x_1^3) \\
&\quad + h^3\Big[(5x_1^5 - 12x_1^4 + 7x_1^3) + C_3(x_1^5 - x_1^2)\Big]
\end{aligned}
\tag{8.53}
$$

$$
\begin{aligned}
x_2x_4 = x_4x_2 &+ h(4x_4x_1 - x_4x_1^2 + 2x_3x_2x_1 - 6x_3x_2 + x_2^3) \\
&+ h^2(3x_2^2x_1^2 - 10x_2^2x_1 + 12x_2^2 + 12x_3x_1^2 - 2x_3x_1^3 - 15x_3x_1) \\
&+ h^3x_2\Big[(9+2C_3)x_1 - 17x_1^2 + 6x_1^3 + (5-5C_3)x_1^4\Big] \\
&+ h^4\Big[(22-22C_3)x_1^2 + (-4+4C_3)x_1^3 + (-18+18C_3)x_1^5 + C_4(x_1^6 - x_1^2)\Big]
\end{aligned}
$$

$$
\begin{aligned}
x_3x_4 = x_4x_3 &+ h(8x_4x_2 - 2x_4x_2x_1 + x_3x_2^2 - 9x_3^2 + 2x_3^2x_1) \\
&+ h^2\Big[x_3x_2(-x_1^2 + 16x_1 - 24) + x_4(-7x_1^2 + 16x_1)\Big] \\
&+ h^3\Big[(10-10C_3)x_2^2x_1^3 + (-9+8C_3)x_2^2\Big] \\
&+ h^3\Big[(-5+5C_3)x_3x_1^4 + 16x_3x_1^2 - (6+9C_3)x_3x_1\Big] \\
&+ h^4\Big[(8-9C_3-2C_4)x_2x_1 + (-8C_3+9)x_2x_1^2 + (10-10C_3)x_2x_1^4\Big] \\
&+ h^4(2C_4 - 18 + 18C_3)x_2x_1^5 \\
&+ h^5\Big[(C_4+2C_3-2)x_1^2 + (4-4C_3-C_4)x_1^6\Big] \\
&+ h^5\Big[(2C_3-2)x_1^5 + C_5(x_1^7 - x_1^2)\Big].
\end{aligned}
$$

where $C_3, C_4, C_5 \in k$ are arbitrary parameters. Then the semigroup factor algebra $k[[h]]\langle X\rangle/\mathcal{I}_h^1$ defines a quantum semigroup $G^1_{4|h}$ in the sense of Definition 8.2 *with a multiplication defined by* (8.13). *Also the semigroup algebra $k[[h]]\langle X\rangle$ has the Poincaré-Birkhoff-Witt property.*

Remark. The proof of Theorem 8.4 goes along the same lines as the proof of Theorem 8.3, i.e. it is constructive. In the course of the construction five arbitrary constants C_1, C_2, C_3, C_4, C_5, appear in solving the corresponding functional equations. The requirement that the PBW property be satisfied fixes two of them. Namely, $C_1 = 1$ and $C_2 = 2 - 2C_3$. Thus we obtain a 3-parameter family of quantum semigroups $G^{1|C_3,C_4,C_5}_{4|h}$.

Finally we describe a third quantum semigroup arising after the quantization of the Poisson algebra of functions on the finite dimensional (dim=5, $d = 3$) Poisson-Lie group G^3_5. The Poisson-Lie structure on G^3_5 is given by

$$
\begin{aligned}
\{x_i, x_j\} =& (i-3)jx_jx_{i-3} - i(j-3)x_ix_{j-3} \\
&+ x_i \sum_{s_1+s_2+s_3+s_4=j} x_{s_1}x_{s_2}x_{s_3}x_{s_4} - x_j \sum_{s_1+s_2+s_3+s_4=i} x_{s_1}x_{s_2}x_{s_3}x_{s_4}.
\end{aligned} \tag{8.54}
$$

The above formulae are obtained again from (8.2) with $d = 3$. Writing them explicitly we have

$$
\begin{aligned}
\{x_1, x_2\} &= 0 \\
\{x_1, x_3\} &= 0 \\
\{x_2, x_3\} &= 0 \\
\{x_1, x_4\} &= x_1^5 - x_1^2 \\
\{x_2, x_4\} &= x_2(x_1^4 - 2x_1) \\
\{x_3, x_4\} &= x_3(x_1^4 - 3x_1) \qquad (8.55) \\
\{x_1, x_5\} &= x_2(4x_1^4 - 2x_1) \\
\{x_2, x_5\} &= x_2^2(4x_1^4 - 4) \\
\{x_3, x_5\} &= x_3x_2(4x_1^3 - 6) \\
\{x_4, x_5\} &= x_4x_2(4x_1^3 - 8) + x_5(5x_1 - x_1^4).
\end{aligned}
$$

Then we have our last theorem.

Theorem 8.5. *Let $X = \{x_i\}_{1 \le i \le 5}$ be a set and let $\langle X \rangle$ be the associative semigroup with identity generated by X. Consider an ideal $\mathcal{I}_h^3$ generated by the set of relations $\mathcal{R}_h^3$ in $k[[h]]\langle X \rangle$:*

$$x_1x_2 = x_2x_1$$

$$x_1x_3 = x_3x_1$$

$$x_2x_3 = x_3x_2$$

$$
\begin{aligned}
x_1x_4 &= x_4x_1 + h(x_1^5 - x_1^2) \\
x_2x_4 &= x_4x_2 + hx_2(x_1^4 - 2x_1) \\
x_3x_4 &= x_4x_3 + hx_3(x_1^4 - 3x_1) \\
x_1x_5 &= x_5x_1 + hx_2(4x_1^4 - 2x_1) \\
x_2x_5 &= x_5x_2 + hx_2^2(4x_1^4 - 4) \\
x_3x_5 &= x_5x_3 + hx_3x_2(4x_1^3 - 6) \\
x_4x_5 &= x_5x_4 + hx_4x_2(4x_1^3 - 8) + x_5(5x_1 - x_1^4) + h^2 3x_2x_1.
\end{aligned}
\tag{8.56}
$$

Then the semigroup factor algebra $k[[h]]\langle X\rangle/\mathcal{I}_h^3$ *defines a quantum semigroup* $G^3_{5|h}$ *in the sense of Definition* 8.2 *with a multiplication defined by* (8.13), *and the semigroup algebra* $k[[h]]\langle X\rangle$ *has the Poincaré-Birkhoff-Witt property.*

Remark. The proof is again constructive. Note that no arbitrary parameters arise in dimension 5 for $d = 3$. Arbitrary parameters arise in higher dimensions though. There is one arbitrary parameter in dimension 6 for $d = 3$ but we refrain from describing the corresponding 6-dimensional one-parameter family of quantum semigroups $G^3_{6|h}$ here.

OPEN PROBLEMS

In this last chapter we would like to discuss briefly some open problems. There are several questions that have not been answered in this dissertation.

Are all Lie-bialgebra structures on the Lie algebra $\mathcal{G}_{0\infty}$ of coboundary type? If the answer of the above question is negative then we formulate

Problem 1. *Give a complete classification of all Lie-bialgebra structures on the Lie algebra $\mathcal{G}_{0\infty}$, and all corresponding Poison-Lie structures on the group $G_{0\infty}$.*

The next problem is related to the quantization.

Problem 2. *Find a quantum (semi)group $G^d_{\infty|h}$, for each $d \in \mathbb{N}$, of which G^d_∞ is the quasi-classical limit. More generally, what are the quantum counterparts to the each of the infinite-parameter families of Poisson-Lie groups found on G_∞?*

Problem 3. *The same as Problem 2 but for the group $G_{0\infty}$.*

There is a related problem:

Problem 4. *If $\mathcal{G}_\infty$ is the Lie bialgebra corresponding to G_∞, is it possible to quantize the Universal Enveloping Algebra $\mathcal{U}\mathcal{G}_\infty$ of $\mathcal{G}_\infty$? (cf. [D2])*

Problem 5. *The same as Problem 4 but for $\mathcal{U}\mathcal{G}_{0\infty}$.*

Problem 6. *Consider the action of $G_{0,\infty}$ on the space $\{V_\lambda$ tensored with the space of formal Laurant series$\}$. What does the representation theory of $G_{0\infty}$ on this space look like?*

Similarly we pose the following question:

Problem 7. *If $G_{0\infty|h}$ exists, what does its representation theory look like?*

Suppose that $G_{\infty|N}$ is the N-super extension of G_∞ (by the Grassmann algebra $\Lambda(N)$).

Problem 8. *Classify all Poisson-Lie structures on $G_{\infty|N}$.*

Is there a representation theory for this group?

Problem 9. *Extend all this (Poisson and Quantum pictures) for the Virasoro group/algebra, its central extension, and to the theory of highest weight representations.*

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