A TRANSITION CALCULUS FOR BOOLEAN FUNCTIONS

by

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LIST OF SYMBOLS

$\underline{\overline{A}}$	A point with coordinates A1,...,Ai,...,An
(A1,...,Ai,...,An)	Same as $\underline{\overline{A}}$
$\underline{\overline{A}}^j$	A point with coordinates the same as those of $\underline{\overline{A}}$ except the jth coordinate of $\underline{\overline{A}}^j$ is the complement of the jth coordinate of $\underline{\overline{A}}$
(A1,...,$\overline{Aj}$,...,An)	Same as $\underline{\overline{A}}^j$
A1	The first coordinate of $\underline{\overline{A}}$
Ai	The ith coordinate of $\underline{\overline{A}}$
An	The nth coordinate of $\underline{\overline{A}}$
$\underline{\overline{B}}$	A point on the vertex of an n-cube
$\underline{\overline{C}}$	A point on the vertex of an n-cube
C0, C1, C2, C3	Constants
$\underline{\overline{D}}$	A point on the vertex of an n-cube
$D_{Xi}(F)$	The Boolean difference of F with respect to Xi
$D_{XjXi}(F)$	The Boolean difference of $D_{Xi}(F)$ with respect to Xj
$D_{\rho}(F)$	The Boolean difference of F with respect to ρ
$D^0_{Xi}(F)$	A function equal to F
$D^1_{Xi}(F)$	A function equal to $D_{Xi}(F)$
$D^{Kj}_{Xi}(F)$	A function equal to F if Kj = 0 and equal to $D_{Xi}(F)$ if Kj = 1

$D_{Xi}(F)\|_{\bar{\underline{A}}}$	The Boolean difference of F with respect to Xi evaluated at point $\bar{\underline{A}}$
$D_{Xi}F(A1,\ldots,Ai,\ldots,An)$	Same as $D_{Xi}(F)\|_{\bar{\underline{A}}}$
F	A function of $X1,\ldots,Xi,\ldots,Xn$
$F(X1,\ldots,Xi,\ldots,Xn)$	Same as F
$F(Xi)_{Xi}$	Same as F
$\bar{F}$	Not F or the complement of F
$F(1)_{Xi}$	The function F with Xi replaced by 1
$F(0)_{Xi}$	The function F with Xi replaced by 0
$F(\overline{Xi})_{Xi}$	The function F with Xi replaced by $\overline{Xi}$
$F\|_{\bar{\underline{A}}}$	The function F evaluated at point $\bar{\underline{A}}$
$F(A1,\ldots,Ai,\ldots,An)$	Same as $F\|_{\bar{\underline{A}}}$
$F(A1,\ldots,Xi,\ldots,An)$	$F(A1,\ldots,Ai,\ldots,An)$ with Ai replaced by Xi
$F(A1,\ldots,1,\ldots,An)$	$F(A1,\ldots,Ai,\ldots,An)$ with Ai replaced by 1
$F(A1,\ldots,0,\ldots,An)$	$F(A1,\ldots,Ai,\ldots,An)$ with Ai replaced by 0
F1	A function of $X1,\ldots,Xi,\ldots,Xn$
F2	A function of $X1,\ldots,Xi,\ldots,Xn$
Fj	A function of $X1,\ldots,Xi,\ldots,Xn$
Fm	Same as Fj with j set equal to m
F_{m+1}	Same as Fj with j set equal to m+1
G	A function independent of Xi
$G(1)_{Xi}$	The function G with Xi replaced by 1
$G(0)_{Xi}$	The function G with Xi replaced by 0
$\bar{\underline{K}}$	A positive integer

K1	The least significant bit of the binary representation of $\bar{\underline{K}}$
Kj	The jth bit of the binary representation of $\bar{\underline{K}}$
Km	The most significant bit of the binary representation of $\bar{\underline{K}}$
$P_{\bar{\underline{K}}}$	A constant
X1	The first independent variable
Xi	The ith independent variable
Xn	The nth independent variable
$\overline{X1}$	Not X1 or the complement of X1
$\overline{Xn}$	Not Xn or the complement of Xn
Xi^{Ki}	A term equal to 1 if Ki = 0 and equal to Xi if Ki = 1
$(Xi \oplus Ai)^{Ki}$	A term equal to 1 if Ki = 0 and equal to $(Xi \oplus Ai)$ if Ki = 1
Zi	An unspecified variable or constant
dF	The Boolean differential of F
$d\xi$	A differential expression
$d\xi_k$	One of the terms in $d\xi$
$d\zeta$	A differential expression
$d\int_k dF$	The Boolean differential of $\int_k dF$
i	An integer
j	An integer
k	An integer
k'	An integer

m	An integer
m'	An integer
n	A positive integer used to indicate the number of independent variables
x	Symbol used on Karnaugh maps to indicate a don't-care condition
ΔF	A function of time with a value of 1 at the instant at which F changes from 0 to 1, and 0 elsewhere
∇F	A function of time with a value of 1 at the instant at which F changes from 1 to 0, and 0 elsewhere
$\prod$	A symbol used to indicate multiple AND operations
$\sum$	A symbol used to indicate multiple of OR operations
$\sum_{\oplus}$	A symbol used to indicate multiple EXCLUSIVE OR operations
αi	The coefficient of dXi in the differential expression $d\xi$
βi	The coefficient of $\overline{dXi}$ in the differential expression $d\xi$
γi	The coefficient of dXi in the differential expression $d\zeta$

δi	The coefficient of $d\overline{Xi}$ in the differential expression $d\zeta$
θ	A function of $X1,\ldots,Xi,\ldots,Xn$ and $A1,\ldots,Ai,\ldots,An$
ξ	The exact integral of $d\xi$
ρ	A sequence of variables defined by equation (3-4.6)
ϕ	A function of $F1,\ldots,Fj,\ldots,Fm$
ψ	A function of $Xi,\ldots,Xi,\ldots,Xn$
$\frac{\partial F}{\partial Xi}$	The Boolean partial derivative of F with respect to Xi
$\frac{\partial F}{\partial \overline{Xi}}$	The Boolean partial derivative of F with respect to $\overline{Xi}$
$\int_1 dF$	The integral of order 1 of dF
$\int_0 dF$	The integral of order 0 of dF
$\int_k dF$	The integral of order k of dF
$\int_k dF\|_A$	$\int_k dF$ evaluated at point $\overline{\underline{A}}$
$=$	The symbol used to indicate equality
$\geq$	The symbol used to indicate "greater than or equal to"
$\leq$	The symbol used to indicate "less than or equal to"
$\supseteq$	The symbol used to indicate "covered by"

$\cdot$	The symbol used to indicate the AND operation
$+$	The symbol used to indicate either the logical OR operation or arithmetic addition
$\oplus$	The symbol used to indicate the EXCLUSIVE OR operation
$\overline{}$	The symbol used to indicate complementation
$\vert_{\underline{\bar{A}}}$	The symbol used to indicate "evaluated at $\underline{\bar{A}}$"

Chapter I

THE TRANSITION CALCULUS

1-1. Introduction

Boolean, or switching algebra is used extensively in the analysis and design of digital circuits. Boolean algebra is quite satisfactory for describing the static behavior of logic circuits when input changes are not occurring; however, difficulties arise in certain situations, notably when input changes are occurring or when portions of the circuit elements respond to transitions or edges instead of levels. A convenient method is thus needed for describing and analyzing the dynamic behavior of logic circuits. The transition calculus reported here has been developed towards fulfilling this need. It is a tool that can be used in a number of instances when Boolean algebra is inadequate.

In this thesis it is shown that a Boolean differential can be found for any switching function. This Boolean differential is analogous to the differential of a real function (i.e. in both instances the differential describes how the function is affected by changes in its variables). It is also shown that a nonconstant switching function is uniquely determined by its Boolean differential. Thus, given the differential of a function, it is possible to find that function. This is somewhat analogous to the process of integration in real functions. Several types of Boolean integrals, all of which have useful applications, are defined, and the conditions for their existence are established. Algorithms are given so that differentiation and integration can be performed using Karnaugh maps, thus making both of these processes quite simple.

1-2. Review of the Literature

This work is not the first to recognize the feasibility and desirability of establishing a mathematical system for Boolean functions analogous to ordinary calculus. Based on earlier work by Reed [1], Akers [2] in 1959 obtained the mathematical properties of the Boolean difference. With the Boolean difference Akers was able to determine the conditions under which changes in the variables of a function would cause changes in the function. This led to the discovery of a number of relationships and theorems for Boolean functions that correspond to those of finite differences and the calculus of real functions. For example, Akers gives a series expansion for Boolean functions that closely resembles the Taylor series. In 1962 Calingaert [3] made limited use of the Boolean difference; however, it was not until late in the 1960's that the Boolean difference began to receive widespread attention. In 1967 several independent papers were published in which Boolean difference was utilized. Hartman [4], employing the Boolean difference, developed a Boolean differential calculus, introducing in it a number of new concepts. Amar and Condulmari [5], and Sellers, Hsiao, and Bearnson [6] applied the Boolean difference to the problem of fault diagnosis. In recent years considerable work [7-30] has been done using the Boolean difference for fault detection and diagnosis;

moreover, Thayse, Davio, Deschamps, and Bioul [31-43] have shown that the Boolean difference is applicable to a number of areas other than fault diagnosis.

Although the work reported here is related to the Boolean difference, it is more powerful and applicable to a wider class of problems. Surprisingly very little work of a similar nature has appeared in the literature. Following is a summary of the work that has employed ideas similar to the basic concept developed here.

About the same time Akers in the United States investigated the Boolean difference, Talantsev [44] in Russia published a paper in which he defined certain special logical operators. One of Talantsev's operators, the d operator, has an important advantage over the Boolean difference. It not only gives the conditions under which a function will change due to changes in its arguments, but also describes the manner in which the function itself will change. Talantsev suggested that his d operator might be called the differential of a logical function, and in fact the Boolean differential defined in this work is essentially Talantsev's d operator with a few minor refinements. After briefly discussing differentiation, Talantsev gives an algorithm for integration which, for a function of n variables, involves solving $nx2^n$ simultaneous Boolean equations. Obviously this is a tedious process. Lazarev and Piil have used Talantsev's operators for sequential circuit synthesis [45-48], and in one short paper [49] they present a method for integration. Even though there are situations where their method of integration is easier than Talantsev's in general, it too is quite awkward.

In 1969 Brown and Young [50] introduced the Boolean variational derivative and applied it to fault diagnosis. Although they developed the Boolean variational derivative independently, it is basically the same as Talantsev's d operator. Brown and Young [51] later published a more extensive treatment and presented a number of additional properties of the variational derivative. In the area of differentiation there is very little overlap between the work reported here and that of Brown and Young. Furthermore, the type of integration considered by Brown and Young requires considerably more information about the function than just knowing its differential. In fact, their integration is nothing more than a restatement of the Shannon expansion theorem [52]. The Boolean integration developed here can be used to determine the function from its differential alone, and the ease in which this can be done makes Boolean integration applicable to practical design problems.

Chapter II

BOOLEAN DIFFERENTIATION

2-1. Introduction

This chapter is concerned with defining and developing the concept of Boolean Differentiation. It will be shown that Boolean differentiation can be used to completely describe the effect on a switching function of a change in any of its variables. Unless specifically stated to the contrary it will be assumed that multiple simultaneous changes in the arguments of a function do not occur.

Given a function, $F(X1,\ldots,Xi,\ldots,Xn)$, of n independent variables it will often be required to evaluate the function for some particular value of one or more of the variables. For convenience the function F with the variable Xi replaced by Zi will be written as $F(Zi)_{Xi}$ or occasionally as $F|_{Xi=Zi}$. In a similar way $F(Zi, Zj)_{Xi,Xj}$ indicates that in the function F, Xi has been replaced by Zi and Xj has been replaced by Zj. When considering the function F, it will occasionally be convenient to use the above notation to write F as $F(Xi)_{Xi}$.

Any Xi can assume either a value of 0 or 1. For specified values of the variables the function F will assume a value of 0 or 1. Holding all variables constant except for some Xi a change in Xi will either cause a change in F or the value of F will not change. If F changes it is said to undergo the same change as Xi

if when Xi changes from 0 to 1, F also changes from 0 to 1, and if Xi changes from 1 to 0, F also changes from 1 to 0. F is said to undergo the opposite change as Xi if when Xi changes from 0 to 1, F changes from 1 to 0, and if Xi changes from 1 to 0, F changes from 0 to 1.

2-2. The Boolean Partial Derivatives

Given a Boolean function, $F(Xi)_{Xi}$, the partial derivative of F with respect to Xi will be defined as

$$\frac{\partial F}{\partial Xi} = F(1)_{Xi}\ \overline{F(0)}_{Xi} \tag{2-2.1}$$

and the partial derivative of F with respect to $\overline{Xi}$ will be defined as

$$\frac{\partial F}{\partial \overline{Xi}} = F(0)_{Xi}\ \overline{F(1)}_{Xi} \tag{2-2.2}$$

As shown by the two theorems given below $\partial F/\partial Xi$ and $\partial F/\partial \overline{Xi}$ describe how F is affected by changes in Xi.

Theorem 2.1: The function $\partial F/\partial Xi$ will take on a value of 1 if and only if while holding the other variables constant, a change in Xi will cause the same change in F.

Proof: In order to set all variables constant except Xi, let

$$Xj = Aj \tag{2-2.3}$$

for all $j \neq i$, $1 \leq j \leq n$ and where Aj is a constant with a value of 0 or 1. If F(A1,..,Xi,...,An) changes from 0 to 1 when Xi changes from 0 to 1, and F(A1,..,Xi,...,An) changes from 1 to 0 when Xi changes from 1 to 0, then

$$F(A1,\ldots,1,\ldots,An) = 1 \qquad (2\text{-}2.4)$$

and

$$F(A1,\ldots,0,\ldots,An) = 0 \qquad (2\text{-}2.5)$$

therefore

$$\frac{\partial}{\partial \overline{Xi}} F(A1,\ldots,Xi,\ldots,An) =$$

$$F(A1,\ldots,1,\ldots,An)\ \overline{F(A1,\ldots,0,\ldots,An)} = 1 \qquad (2\text{-}2.6)$$

Thus, necessity is proved. To show sufficiency note that if equation (2-2.6) is satisfied then equations (2-2.4) and (2-2.5) are also satisfied; hence, a change in Xi from 1 to 0 causes F to change from 1 to 0 and a change in Xi from 0 to 1 causes F to change from 0 to 1.

Theorem 2.2: The function $\partial F/\partial \overline{Xi}$ will take on a value of 1 if and only if while holding the other variables constant a change in Xi will cause the opposite change in F.

Proof: The proof is identical to Theorem 2.1 with F(A1,...,1,...,An) replaced by F(A1,...,0,...,An) and F(A1,...,0,...,An) replaced by F(A1,...,1,...An).

Another obvious theorem follows immediately from the definitions given in equation (2-2.1) and (2-2.2)

Theorem 2.3: Neither $\partial F/\partial Xi$ nor $\partial F/\partial \overline{Xi}$ is a function of Xi.

Proof: $F(1)_{Xi}$ is not a function of Xi. Also $F(0)_{Xi}$ is not a function of Xi. Hence,

$$\frac{\partial F}{\partial Xi} = F(1)_{Xi}\ \overline{F(0)_{Xi}} \qquad (2\text{-}2.7)$$

is not a function of Xi, and

$$\frac{\partial F}{\partial \overline{Xi}} = F(0)_{Xi}\ \overline{F(1)_{Xi}} \qquad (2\text{-}2.8)$$

is not a function of Xi.

2-3. The Boolean Differential

The effect on an n-variable function F, of a single change in any of its arguments can be concisely expressed as a Boolean differential of F, denoted by dF and defined as shown below.

$$dF = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial Xi}\, dXi + \frac{\partial F}{\partial \overline{Xi}}\, d\overline{Xi} \right) \qquad (2\text{-}3.1)$$

From Theorems 2.1 and 2.2, equation (2-3.1) can be interpreted as follows. Whenever the function $\partial F/\partial Xi$ takes on a value of one, a change in Xi (while holding the other variables constant) will cause the same change in F. Similarly, when the function $\partial F/\partial \overline{Xi}$ takes on a value of one, a change in Xi (while holding the other variables constant) will cause the opposite change in F. Whenever both $\partial F/\partial Xi$ and $\partial F/\partial\overline{Xi}$ take on a value of zero, a change in Xi will have no effect on F. It should be noted that $\partial F/\partial Xi$ and $\partial F/\partial\overline{Xi}$ can never both be one simultaneously since from equation (2-2.1) and (2-2.2)

$$\frac{\partial F}{\partial Xi}\cdot\frac{\partial F}{\partial\overline{Xi}} = \left[F(1)_{Xi}\ \overline{F(0)_{Xi}}\right]\ \left[F(0)_{Xi}\ \overline{F(1)_{Xi}}\right] = 0 \qquad (2\text{-}3.2)$$

This is equivalent to stating that a change in Xi cannot cause F to undergo the same change as Xi, and at the same time to undergo the opposite change as Xi.

The following example illustrates the algebraic evaluation of dF.

<u>Example 2.1</u>: Given the function

$$F = X1\overline{X3} + X2X3 \qquad (2\text{-}3.3)$$

to find dF. First note that for the above function

$$\overline{F} = \overline{X1}\ \overline{X3} + \overline{X2}\ X3 \qquad (2\text{-}3.4)$$

then from the definition of the partial derivatives

$$\frac{\partial F}{\partial X1} = F(1)_{X1}\ \overline{F(0)_{X1}}$$

$$= (1\cdot\overline{X3} + X2\ X3)\ (1\cdot\overline{X3} + \overline{X2}\ X3)$$

$$= \overline{X3} \qquad (2\text{-}3.5)$$

$$\frac{\partial F}{\partial \overline{X1}} = F(0)_{X1}\ \overline{F(1)}_{X1}$$

$$= (0 \cdot X3 + X2\ X3)\ (0 \cdot \overline{X3} + \overline{X2}\ X3)$$

$$= 0 \qquad (2\text{-}3.6)$$

$$\frac{\partial F}{\partial X2} = F(1)_{X2}\ \overline{F(0)}_{X2}$$

$$= (X1\ \overline{X3} + 1 \cdot X3)\ (\overline{X1}\ \overline{X3} + 1 \cdot X3)$$

$$= X3 \qquad (2\text{-}3.7)$$

$$\frac{\partial F}{\partial \overline{X2}} = F(0)_{X2}\ \overline{F(1)}_{X2}$$

$$= (X1\ \overline{X3} + 0 \cdot X3)\ (\overline{X1}\ \overline{X3} + 0 \cdot X3)$$

$$= 0 \qquad (2\text{-}3.8)$$

$$\frac{\partial F}{\partial X3} = F(1)_{X3}\ \overline{F(0)}_{X3}$$

$$= (X1 \cdot 0 + X2 \cdot 1)\ (\overline{X1} \cdot 1 + \overline{X2} \cdot 0)$$

$$= \overline{X1}\ X2 \qquad (2\text{-}3.9)$$

$$\frac{\partial F}{\partial \overline{X3}} = F(0)_{X3}\ \overline{F(1)}_{X3}$$

$$= (X1 \cdot 1 + X2 \cdot 0)\ (\overline{X1} \cdot 0 + \overline{X2} \cdot 1)$$

$$= X1\overline{X2} \qquad (2\text{-}3.10)$$

Hence, for

$$F = X1\ \overline{X3} + X2\ X3 \qquad (2\text{-}3.11)$$

from the above equations and equation (2-3.1) it follows that

$$dF = \overline{X3}\ dX1 + X3\ dX2 + \overline{X1}\ X2\ dX3 + X1\ \overline{X2}\ d\overline{X3} \qquad (2\text{-}3.12)$$

The above differential describes the behavior of a logic circuit which has the output given by equation (2-3.11). When $X3 = 0$ a transition on the $X1$ input will cause the same transition of the output, when $X3 = 1$ a transition on the $X2$ input will cause the output to undergo the same transition, when $X1 = 0$ and $X2 = 1$ a change in $X3$ will cause the same change in F, and when $X1 = 1$ and $X2 = 0$ a change in $X3$ will cause F to change in the opposite way.

2-4. Basic Identities

In this section a number of basic identities involving the partial derivative will be obtained. These identities will often be used in the following sections.

From the definitions given in equations (2-2.1) and (2-2.2)

$$\frac{\partial \overline{F}}{\partial Xi} = \overline{F(1)_{Xi}}\ \overline{\overline{F(0)_{Xi}}}$$

$$= F(0)_{Xi}\ \overline{F(1)_{Xi}}$$

$$= \frac{\partial F}{\partial \overline{Xi}} \qquad (2\text{-}4.1)$$

which shows that if $\overline{F}$ changes in the same way Xi changed then obviously F will change in the opposite way Xi changed. Similarly

$$\frac{\partial \overline{F}}{\partial \overline{Xi}} = \overline{F(0)_{Xi}}\ \overline{\overline{F(1)_{Xi}}}$$

$$= F(1)_{Xi}\ \overline{F(0)_{Xi}}$$

$$= \frac{\partial F}{\partial Xi} \qquad (2\text{-}4.2)$$

Hence, if $\overline{F}$ changes in the opposite way Xi changed then F will change in the same way Xi changed.

There are numerous other identities that will be useful later. From the definition of $\partial F/\partial Xi$

$$\frac{\partial F}{\partial Xi} = F(1)_{Xi}\ \overline{F(0)_{Xi}}$$

$$= F(1)_{Xi}\ F(1)_{Xi}\ \overline{F(0)_{Xi}}$$

$$= F(1)_{Xi} \; \frac{\partial F}{\partial Xi} \qquad (2\text{-}4.3)$$

and also

$$\frac{\partial F}{\partial Xi} = F(1)_{Xi} \; \overline{F(0)_{Xi}} \; \overline{F(0)_{Xi}}$$

$$= \overline{F(0)_{Xi}} \; \frac{\partial F}{\partial Xi} \qquad (2\text{-}4.4)$$

Similarly

$$\frac{\partial F}{\partial \overline{Xi}} = F(0)_{Xi} \; \overline{F(1)_{Xi}}$$

$$= F(0)_{Xi} \; F(0)_{Xi} \; \overline{F(1)_{Xi}}$$

$$= F(0)_{Xi} \; \frac{\partial F}{\partial \overline{Xi}} \qquad (2\text{-}4.5)$$

and

$$\frac{\partial F}{\partial \overline{Xi}} = F(0)_{Xi} \; \overline{F(1)_{Xi}} \; \overline{F(1)_{Xi}}$$

$$= \overline{F(1)_{Xi}} \; \frac{\partial F}{\partial \overline{Xi}} \qquad (2\text{-}4.6)$$

It should also be noted that

$$F(0)_{Xi} \frac{\partial F}{\partial Xi} = F(0)_{Xi} F(1)_{Xi} \overline{F(0)}_{Xi}$$

$$= 0 \qquad (2\text{-}4.7)$$

and

$$\overline{F(1)}_{Xi} \frac{\partial F}{\partial Xi} = \overline{F(1)}_{Xi} F(1)_{Xi} \overline{F(0)}_{Xi}$$

$$= 0 \qquad (2\text{-}4.8)$$

Similarly

$$F(1)_{Xi} \frac{\partial F}{\partial \overline{Xi}} = F(1)_{Xi} F(0)_{Xi} \overline{F(1)}_{Xi}$$

$$= 0 \qquad (2\text{-}4.9)$$

and

$$\overline{F(0)}_{Xi} \frac{\partial F}{\partial \overline{Xi}} = \overline{F(0)}_{Xi} F(0)_{Xi} \overline{F(1)}_{Xi}$$

$$= 0 \qquad (2\text{-}4.10)$$

From equations (2-4.3) and (2-4.7) two identities are obtained

$$F(Xi)_{Xi} \frac{\partial F}{\partial Xi} = \left[Xi\ F(1)_{Xi} + \overline{Xi}\ F(0)_{Xi} \right] \frac{\partial F}{\partial Xi}$$

$$= Xi \frac{\partial F}{\partial Xi} \tag{2-4.11}$$

and

$$F(\overline{Xi})_{Xi} \frac{\partial F}{\partial Xi} = \left[\overline{Xi}\ F(1)_{Xi} + Xi\ F(0)_{Xi}\right] \frac{\partial F}{\partial Xi}$$

$$= \overline{Xi} \frac{\partial F}{\partial Xi} \tag{2-4.12}$$

From equations (2-4.5) and (2-4.9) the similar identities for $\partial F/\partial \overline{Xi}$ are

$$F(\overline{Xi})_{Xi} \frac{\partial F}{\partial \overline{Xi}} = \left[\overline{Xi}\ F(1)_{Xi} + Xi\ F(0)_{Xi}\right] \frac{\partial F}{\partial \overline{Xi}}$$

$$= Xi \frac{\partial F}{\partial \overline{Xi}} \tag{2-4.13}$$

and

$$F(Xi)_{Xi} \frac{\partial F}{\partial \overline{Xi}} = \left[Xi\ F(1)_{Xi} + \overline{Xi}\ F(0)_{Xi}\right] \frac{\partial F}{\partial \overline{Xi}}$$

$$= \overline{Xi} \frac{\partial F}{\partial \overline{Xi}} \tag{2-4.14}$$

From equation (2-4.13) replacing F by $\overline{F}$ gives

$$\overline{F(\overline{Xi})_{Xi}} \frac{\partial \overline{F}}{\partial \overline{Xi}} = Xi \frac{\partial \overline{F}}{\partial \overline{Xi}} \tag{2-4.15}$$

and applying equation (2-4.2) to the above results in a new identity

$$\overline{F(\overline{Xi})}\ \frac{\partial F}{\partial Xi} = Xi\ \frac{\partial F}{\partial Xi} \qquad (2\text{-}4.16)$$

Applying the same procedure to equation (2-4.14) gives the identity

$$\overline{F(Xi)}_{Xi}\ \frac{\partial F}{\partial Xi} = \overline{Xi}\ \frac{\partial F}{\partial Xi} \qquad (2\text{-}4.17)$$

In similar way equations (2-4.11), (2-4.12), and (2-4.1) result in

$$\overline{F(Xi)}_{Xi}\ \frac{\partial F}{\partial \overline{Xi}} = Xi\ \frac{\partial F}{\partial \overline{Xi}} \qquad (2\text{-}4.18)$$

and

$$\overline{F(\overline{Xi})}_{Xi}\ \frac{\partial F}{\partial \overline{Xi}} = \overline{Xi}\ \frac{\partial F}{\partial \overline{Xi}} \qquad (2\text{-}4.19)$$

2-5. Relationship to the Boolean Difference

The partial derivatives defined in equations (2-2.1) and (2-2.2) are closely related to the Boolean difference of F with respect to Xi, denoted by $D_{Xi}(F)$. Two definitions of Boolean difference are commonly used. Reference [2] shows that the two are equivalent. The first definition is

$$D_{Xi}(F) = F(1)_{Xi} \oplus F(0)_{Xi} \qquad (2\text{-}5.1)$$

An equivalent definition is

$$D_{Xi}(F) = F(Xi)_{Xi} \oplus F(\overline{Xi})_{Xi} \qquad (2\text{-}5.2)$$

Using the first definition given by equation (2-5.1) and the definition of EXCLUSIVE OR

$$D_{Xi}(F) = F(1)_{Xi} \; \overline{F(0)}_{Xi} + F(0)_{Xi} \; \overline{F(1)}_{Xi}$$

$$= \frac{\partial F}{\partial Xi} + \frac{\partial F}{\partial \overline{Xi}} \qquad (2\text{-}5.3)$$

Hence, the Boolean difference can be expressed in terms of the partial derivatives. The INCLUSIVE OR in equation (2-5.3) can be replaced by the EXCLUSIVE OR since from equation (2-3.2)

$$\frac{\partial F}{\partial Xi} \cdot \frac{\partial F}{\partial \overline{Xi}} = 0 \qquad (2\text{-}5.4)$$

Therefore

$$D_{Xi}(F) = \left(\frac{\partial F}{\partial Xi} + \frac{\partial F}{\partial \overline{Xi}}\right) \overline{\frac{\partial F}{\partial Xi} \; \frac{\partial F}{\partial \overline{Xi}}}$$

$$= \left(\frac{\partial F}{\partial Xi} + \frac{\partial F}{\partial \overline{Xi}}\right) \left(\overline{\frac{\partial F}{\partial Xi}} + \overline{\frac{\partial F}{\partial \overline{Xi}}}\right)$$

$$= \frac{\partial F}{\partial Xi} \overline{\frac{\partial F}{\partial \overline{Xi}}} + \frac{\partial F}{\partial \overline{Xi}} \overline{\frac{\partial F}{\partial Xi}}$$

$$= \frac{\partial F}{\partial Xi} \oplus \frac{\partial F}{\partial \overline{Xi}} \qquad (2\text{-}5.5)$$

Again from equation (2-3.2) it follows that

$$\frac{\partial F}{\partial Xi} D_{Xi}(F) = \frac{\partial F}{\partial Xi} \left(\frac{\partial F}{\partial Xi} + \frac{\partial F}{\partial \overline{Xi}}\right)$$

$$= \frac{\partial F}{\partial Xi} \qquad (2\text{-}5.6)$$

and

$$\frac{\partial F}{\partial \overline{Xi}} D_{Xi}(F) = \frac{\partial F}{\partial \overline{Xi}} \left(\frac{\partial F}{\partial Xi} + \frac{\partial F}{\partial \overline{Xi}}\right)$$

$$= \frac{\partial F}{\partial \overline{Xi}} \qquad (2\text{-}5.7)$$

From equation (2-5.3), that defines the Boolean difference in terms of the partial derivatives, it follows immediately that

$$D_{\overline{Xi}}(F) = \frac{\partial F}{\partial \overline{Xi}} + \frac{\partial F}{\partial Xi}$$

$$= \frac{\partial F}{\partial Xi} + \frac{\partial F}{\partial \overline{Xi}}$$

$$= D_{Xi}(F) \qquad (2\text{-}5.8)$$

Again from equation (2-5.3)

$$D_{Xi}(\overline{F}) = \frac{\partial \overline{F}}{\partial Xi} + \frac{\partial \overline{F}}{\partial \overline{Xi}} \qquad (2\text{-}5.9)$$

but from equations (2-4.1) and (2-4.2)

$$\frac{\partial \overline{F}}{\partial Xi} = \frac{\partial F}{\partial \overline{Xi}} \qquad (2\text{-}5.10)$$

and

$$\frac{\partial \overline{F}}{\partial \overline{Xi}} = \frac{\partial F}{\partial Xi} \qquad (2\text{-}5.11)$$

Hence

$$D_{Xi}(\overline{F}) = D_{Xi}(F) \qquad (2\text{-}5.12)$$

The Boolean difference results given in equations (2-5.8) and (2-5.12) are well known and are repeated here only because they will be required later.

There are several ways to express $\partial F/\partial Xi$ and $\partial F/\partial \overline{Xi}$ in terms of $D_{Xi}(F)$. From equations (2-4.3), (2-4.9) and (2-5.3)

$$\frac{\partial F}{\partial Xi} = F(1)_{Xi}\,\frac{\partial F}{\partial Xi}$$

$$= F(1)_{Xi}\left[\frac{\partial F}{\partial Xi} + \frac{\partial F}{\partial \overline{Xi}}\right]$$

$$= F(1)_{Xi}\; D_{Xi}(F) \qquad (2\text{-}5.13)$$

Also, since

$$F(1)_{Xi} = Xi\;F(1)_{Xi} + \overline{Xi}\;F(1)_{Xi}$$

$$= Xi\;\left[Xi\,F(1)_{Xi} + \overline{Xi}\;F(0)_{Xi}\right] + \overline{Xi}\;\left[\overline{Xi}\;F(1)_{Xi} + Xi\;F(0)_{Xi}\right]$$

$$= Xi\;F(Xi)_{Xi} + \overline{Xi}\;F(\overline{Xi})_{Xi} \qquad (2\text{-}5.14)$$

equation (2-5.13) can be written as

$$\frac{\partial F}{\partial Xi} = \left[Xi\;F(Xi)_{Xi} + \overline{Xi}\;F(\overline{Xi})_{Xi}\right]\;D_{Xi}(F) \qquad (2\text{-}5.15)$$

Similarly from equations (2-4.5), (2-4.7) and (2-5.3)

$$\frac{\partial F}{\partial \overline{Xi}} = F(0)_{Xi}\,\frac{\partial F}{\partial \overline{Xi}}$$

$$= F(0)_{Xi} \left[\frac{\partial F}{\partial Xi} + \frac{\partial F}{\partial \overline{Xi}}\right]$$

$$= F(0)_{Xi}\, D_{Xi}(F) \qquad (2\text{-}5.16)$$

and since

$$F(0)_{Xi} = \overline{Xi}\, F(Xi)_{Xi} + Xi\, F(\overline{Xi})_{Xi} \qquad (2\text{-}5.17)$$

equation (2-5.16) can be written as

$$\frac{\partial F}{\partial \overline{Xi}} = \left[\overline{Xi}\, F(Xi)_{Xi} + Xi\, F(\overline{Xi})_{Xi}\right] D_{Xi}(F) \qquad (2\text{-}5.18)$$

It is obvious that

$$\frac{\partial F}{\partial Xi} = F\left(F\,\frac{\partial F}{\partial Xi} + \overline{F}\,\frac{\partial F}{\partial \overline{Xi}}\right) + \overline{F}\left(\overline{F}\,\frac{\partial F}{\partial Xi} + F\,\frac{\partial F}{\partial \overline{Xi}}\right) \qquad (2\text{-}5.19)$$

From equation (2-4.11)

$$F\,\frac{\partial F}{\partial Xi} = Xi\,\frac{\partial F}{\partial Xi} \qquad (2\text{-}5.20)$$

from equation (2-4.18)

$$\overline{F}\,\frac{\partial F}{\partial \overline{Xi}} = Xi\,\frac{\partial F}{\partial \overline{Xi}} \qquad (2\text{-}5.21)$$

from equation (2-4.17)

$$\overline{F}\ \frac{\partial F}{\partial Xi} = \overline{Xi}\ \frac{\partial F}{\partial Xi} \qquad (2\text{-}5.22)$$

and from equation (2-4.14)

$$F\ \frac{\partial F}{\partial \overline{Xi}} = \overline{Xi}\ \frac{\partial F}{\partial \overline{Xi}} \qquad (2\text{-}5.23)$$

Hence

$$\frac{\partial F}{\partial Xi} = F\left(Xi\ \frac{\partial F}{\partial Xi} + Xi\ \frac{\partial F}{\partial \overline{Xi}}\right) + \overline{F}\left(\overline{Xi}\ \frac{\partial F}{\partial Xi} + \overline{Xi}\ \frac{\partial F}{\partial \overline{Xi}}\right)$$

(2-5.24)

which can be rearranged to give

$$\frac{\partial F}{\partial Xi} = \left(F\ Xi + \overline{F}\ \overline{Xi}\right)\left(\frac{\partial F}{\partial Xi} + \frac{\partial F}{\partial \overline{Xi}}\right) \qquad (2\text{-}5.25)$$

Finally from equation (2-5.3) and the definition of the EXCLUSIVE OR operation

$$\frac{\partial F}{\partial Xi} = (\overline{Xi} \oplus F)\ D_{Xi}(F) \qquad (2\text{-}5.26)$$

and from equations (2-4.1), (2-5.26), and (2-5.12)

$$\frac{\partial F}{\partial \overline{Xi}} = \frac{\partial \overline{F}}{\partial Xi}$$

$$= (\overline{Xi} \oplus \overline{F})\ D_{Xi}(\overline{F})$$

$$= (Xi \oplus F)\ D_{Xi}(F) \qquad (2\text{-}5.27)$$

Other identities can also be obtained, for example from equations (2-4.2), (2-5.12), and (2-5.16)

$$\frac{\partial F}{\partial Xi} = \frac{\partial \overline{F}}{\partial \overline{Xi}}$$

$$= \overline{F(0)}_{Xi}\ D_{Xi}(F) \qquad (2\text{-}5.28)$$

and from equations (2-4.2), (2-5.12), and (2-5.18)

$$\frac{\partial F}{\partial Xi} = \frac{\partial \overline{F}}{\partial \overline{Xi}}$$

$$= \left[\overline{Xi}\ \overline{F(Xi)}_{Xi} + Xi\ \overline{F(\overline{Xi})_{Xi}}\right]\ D_{Xi}(F) \qquad (2\text{-}5.29)$$

From equations (2-4.1), (2-5.12), and (2-5.13)

$$\frac{\partial F}{\partial \overline{Xi}} = \frac{\partial \overline{F}}{\partial Xi}$$

$$= \overline{F(1)}_{Xi}\ D_{Xi}(F) \qquad (2\text{-}5.30)$$

and from equations (2-4.1), (2-5.12) and (2-5.15)

$$\frac{\partial F}{\partial \overline{Xi}} = \frac{\partial \overline{F}}{\partial Xi} = \left[Xi\ \overline{F(Xi)_{Xi}} + \overline{Xi}\ \overline{F(\overline{Xi})_{Xi}} \right] D_{Xi}(F) \qquad (2\text{-}5.31)$$

In all of the above equations or in any other form that $\partial F/\partial Xi$ or $\partial F/\partial \overline{Xi}$ might be expressed in terms of $D_{Xi}(F)$ it is necessary to have some information about the function in addition to $D_{Xi}(F)$. Hence $D_{Xi}(F)$ alone does not uniquely define the partial derivatives $\partial F/\partial Xi$ and $\partial F/\partial \overline{Xi}$.

2-6. General Expressions for the Boolean Difference

The Boolean difference of the ANDing of two functions can be found from equations (2-5.2) to be given by

$$D_{Xi}(F1 \cdot F2) = F1(Xi)_{Xi}\ F2(Xi)_{Xi} \oplus F1(\overline{Xi})_{Xi}\ F2(\overline{Xi})_{Xi} \qquad (2\text{-}6.1)$$

but by EXCLUSIVE-ORing both sides of equation (2-5.2) with $F(Xi)_{Xi}$ it follows that

$$F(\overline{Xi})_{Xi} = F(Xi)_{Xi} \oplus D_{Xi}(F) \qquad (2\text{-}6.2)$$

Therefore

$$D_{Xi}(F1\ F2) = F1\ F2 \oplus \left[F1 \oplus D_{Xi}(F1)\right] \left[F2 \oplus D_{Xi}(F2)\right]$$

$$= F1\ F2 \oplus F1\ F2 \oplus F1\ D_{Xi}(F2)$$

$$\oplus\ F2\ D_{Xi}(F1) \oplus D_{Xi}(F1)\ D_{Xi}(F2) \qquad (2\text{-}6.3)$$

However, since

$$F \oplus F = 0 \qquad (2\text{-}6.4)$$

equation (2-6.3) can be written as

$$D_{Xi}(F1\ F2) = F1\ D_{Xi}(F2) \oplus D_{Xi}(F1)\ F2 \oplus D_{Xi}(F1)\ D_{Xi}(F2) \qquad (2\text{-}6.5)$$

This result for the ANDing of two functions is reported in numerous sources [2, 4, 6, 7, 9]; however, to the author's knowledge the following result for the general case has not been reported previously. The general case will be obtained by induction, but before deriving the general formula it is convenient to introduce a new notation. Let

$$D_{Xi}^{Kj}(Fj) = \begin{cases} Fj & \text{if} \quad Kj = 0 \\ D_{Xi}(Fj) & \text{if} \quad Kj = 1 \end{cases} \qquad (2\text{-}6.6)$$

Using the above notation equation (2-6.5)

$$D_{Xi}(F1\ F2) = D_{Xi}^{o}(F1)\ D_{Xi}^{1}(F2) \oplus D_{Xi}^{1}(F1)\ D_{Xi}^{o}(F2)$$

$$\oplus \; D^{1}_{Xi}(F1)\; D^{1}_{Xi}(F2) \qquad (2\text{-}6.7)$$

Now letting

$$\underline{\bar{K}} = Km x 2^{m-1} + \ldots + Kj x 2^{j-1} + \ldots + K1 \qquad (2\text{-}6.8)$$

For

$$\underline{\bar{K}} \leq 2^{m}-1 \qquad (2\text{-}6.9)$$

where the above is an arithmetic expression and the " + " sign represents the arithmetic sum. Hence, if $\underline{\bar{K}}$ is expressed as a binary number Kj is the jth bit in that number. Equation (2-6.7) can now be written as

$$D_{Xi} \prod_{j=1}^{2} Fj = \sum_{\underline{\bar{K}}=1}^{2^{2}-1} \!\!\!\!\oplus\; \prod_{j=1}^{2} D^{Kj}_{Xi}(Fj) \qquad (2\text{-}6.10)$$

It shall now be shown that in general

$$D_{Xi}\left(\prod_{j=1}^{m} Fj \right) = \sum_{\underline{\bar{K}}=1}^{2^{m}-1} \!\!\!\!\oplus\; \prod_{j=1}^{m} D^{Kj}_{Xi}(Fj) \qquad (2\text{-}6.11)$$

<u>Proof</u>: Assume equation (2-6.11) is correct then from equation (2-6.7)

$$D_{Xi}\left(\prod_{j=1}^{m+1} Fj\right) = D_{Xi}\left[\prod_{j=1}^{m} Fj\ F_{m+1}\right]$$

$$= D^{o}_{Xi}\left(\prod_{j=1}^{m} Fj\right) D^{1}_{Xi}(F_{m+1}) \oplus D^{1}_{Xi}\left(\prod_{j=1}^{m} Fj\right) D^{o}_{Xi}(F_{m+1})$$

$$\oplus\ D^{1}_{Xi}\left(\prod_{j=1}^{m} Fj\right) D^{1}_{Xi}(F_{m+1}) \qquad (2\text{-}6.12)$$

It is obvious that

$$D^{o}_{Xi}\left(\prod_{j=1}^{m} Fj\right) = \prod_{j=1}^{m} D^{o}_{Xi}(Fj) \qquad (2\text{-}6.13)$$

then from equations (2-6.11) and (2-6.12)

$$D_{Xi}\left(\prod_{j=1}^{m+1} Fj\right) = \left(\prod_{j=1}^{m} D^{o}_{Xi}\ Fj\right) D^{1}_{Xi}(F_{m+1}) \oplus \left[\sum_{\underline{K}=1}^{2^m-1} \prod_{j=1}^{m} D^{Kj}_{Xi}(Fj)\right] D^{o}_{Xi}(F_{m+1})$$

$$\oplus\left[\sum_{\underline{K}=1}^{2^m-1} \prod_{j=1}^{m} D^{Kj}_{Xi}(Fj)\right] D^{1}_{Xi}(F_{m+1})$$

$$= \sum_{\underline{K}=1}^{2^{m+1}-1} \prod_{j=1}^{m+1} D^{Kj}_{Xi}(Fj) \qquad (2\text{-}6.14)$$

Hence, if equation (2-6.11) is valid for m it must also be valid for $m + 1$. From equation (2-6.10) it is seen that equation (2-6.11) is correct for $m = 2$; hence, it is also correct for all values of m.

Now consider the ORing of m functions. From De Morgan's theorem

$$D_{Xi}\left(\sum_{j=1}^{m} Fj\right) = D_{Xi}\left(\overline{\prod_{j=1}^{m} \overline{Fj}}\right) \qquad (2\text{-}6.15)$$

but from equation (2-5.12)

$$D_{Xi}(\overline{F}) = D_{Xi}(F) \qquad (2\text{-}6.16)$$

Hence

$$D_{Xi}\left(\sum_{j=1}^{m} Fj\right) = D_{Xi}\left(\prod_{j=1}^{m} \overline{Fj}\right) \qquad (2\text{-}6.17)$$

Now applying equation (2-6.11) to the right hand portion of equation (2-6.17) results in the general expression for the Boolean difference of the ORing of m functions. Such that

$$D_{Xi}\left(\sum_{j=1}^{m} Fj\right) = \sum_{\underline{\underline{K}}=1}^{2^m-1}{}^{\oplus} \prod_{j=1}^{m} D_{Xi}^{Kj}\ (\overline{Fj}) \qquad (2\text{-}6.18)$$

The general expression for the Boolean difference of EXCLUSIVE-ORing several functions can easily be found. From the definition of Boolean difference given by equation (2-5.2)

$$D_{Xi}\left(\sum_{j=1}^{m} Fj\right) = \sum_{j=1}^{m} Fj(Xi)_{Xi} \oplus \sum_{j=1}^{m} Fj(\overline{Xi})_{Xi} \qquad (2\text{-}6.19)$$

Rearranging the terms in equations (2-6.19) gives

$$D_{Xi}\left(\sum_{j=1}^{m} Fj\right) = \sum_{j=1}^{m}\left(Fj(Xi)_{Xi} \oplus Fj(\overline{Xi})_{Xi}\right) \qquad (2\text{-}6.20)$$

Again applying the definition of Boolean difference given by equation (2-5.2) results in the desired general expression

$$D_{Xi}\left(\sum_{j=1}^{m} Fj\right) = \sum_{j=1}^{m} D_{Xi}(Fj) \qquad (2\text{-}6.21)$$

2-7. General Expressions for the Partial Derivative

General expressions can also be obtained for the partial derivatives. If

$$\phi = \prod_{j=1}^{m} Fj(Xi)_{Xi} \qquad (2\text{-}7.1)$$

Then

$$\frac{\partial\phi}{\partial Xi} = \frac{\partial}{\partial Xi}\prod_{j=1}^{m} Fj(Xi)_{Xi} \qquad (2\text{-}7.2)$$

Which by definition can be written as

$$\frac{\partial\phi}{\partial Xi} = \prod_{j=1}^{m} Fj(1)_{Xi} \overline{\prod_{k=1}^{m} Fk(0)_{Xi}} \tag{2-7.3}$$

Applying the generalized De Morgan's theorem yields

$$\frac{\partial\phi}{\partial Xi} = \prod_{j=1}^{m} Fj(1)_{Xi} \sum_{k=1}^{m} \overline{Fk(0)}_{Xi} \tag{2-7.4}$$

which can in turn be written as

$$\frac{\partial\phi}{\partial Xi} = \prod_{j=1}^{m} Fj(1)_{Xi} \sum_{k=1}^{m} \left[Fk(1)_{Xi} \quad \overline{Fk(0)}_{Xi} \right] \tag{2-7.5}$$

Hence

$$\frac{\partial}{\partial Xi} \prod_{j=1}^{m} Fj(Xi)_{Xi} = \prod_{j=1}^{m} Fj(1)_{Xi} \sum_{k=1}^{m} \frac{\partial Fk}{\partial Xi} \tag{2-7.6}$$

It is also seen that

$$\frac{\partial\phi}{\partial \overline{Xi}} = \prod_{j=1}^{m} Fj(0)_{Xi} \overline{\prod_{k=1}^{m} Fk(1)_{Xi}} \tag{2-7.7}$$

which can be written as

$$\frac{\partial\phi}{\partial\overline{Xi}} = \prod_{j=1}^{m} Fj(0)_{Xi} \sum_{k=1}^{m} \left[Fk(0)_{Xi}\ \overline{Fk(1)_{Xi}}\right] \qquad (2\text{-}7.8)$$

Hence

$$\frac{\partial}{\partial\overline{Xi}} \prod_{j=1}^{m} Fj(Xi)_{Xi} = \prod_{j=1}^{m} Fj(0)_{Xi} \sum_{k=1}^{m} \frac{\partial Fk}{\partial\overline{Xi}} \qquad (2\text{-}7.9)$$

From De Morgan's theorem

$$\frac{\partial}{\partial Xi} \sum_{j=1}^{m} Fj(Xi)_{Xi} = \frac{\partial}{\partial Xi} \overline{\left[\prod_{j=1}^{m} \overline{Fj(Xi)_{Xi}}\right]} \qquad (2\text{-}7.10)$$

but from equation (2-4.1)

$$\frac{\partial\overline{F}}{\partial Xi} = \frac{\partial F}{\partial\overline{Xi}} \qquad (2\text{-}7.11)$$

Hence

$$\frac{\partial}{\partial Xi} \sum_{j=1}^{m} Fj(Xi)_{Xi} = \frac{\partial}{\partial\overline{Xi}} \prod_{j=1}^{m} \overline{Fj(Xi)_{Xi}} \qquad (2\text{-}7.12)$$

and from equation (2-7.9)

$$\frac{\partial}{\partial Xi} \sum_{j=1}^{m} Fj(Xi)_{Xi} = \prod_{j=1}^{m} \overline{Fj(0)_{Xi}} \sum_{k=1}^{m} \frac{\partial \overline{Fk}}{\partial \overline{Xi}} \qquad (2\text{-}7.13)$$

Finally applying De Morgan's theorem and equation (2-4.2)

$$\frac{\partial}{\partial Xi} \sum_{j=1}^{m} Fj(Xi)_{Xi} = \overline{\sum_{j=1}^{m} Fj(0)_{Xi}} \sum_{k=1}^{m} \frac{\partial Fk}{\partial Xi} \qquad (2\text{-}7.14)$$

Similarly it can easily be shown that

$$\frac{\partial}{\partial \overline{Xi}} \sum_{j=1}^{m} Fj(Xi)_{Xi} = \overline{\sum_{j=1}^{m} Fj(1)_{Xi}} \sum_{k=1}^{m} \frac{\partial Fk}{\partial \overline{Xi}} \qquad (2\text{-}7.15)$$

Given the function

$$\phi = \bigoplus_{j=1}^{m} Fj(Xi)_{Xi} \qquad (2\text{-}7.16)$$

where the summation is with respect to the EXCLUSIVE-OR. Then from equation (2-5.26)

$$\frac{\partial \phi}{\partial Xi} = (\overline{Xi} \oplus \phi)\, D_{Xi}(\phi) \qquad (2\text{-}7.17)$$

but since from equation (2-6.21)

$$D_{Xi}(\phi) = \sum_{j=1}^{m} D_{Xi}(Fj) \qquad (2\text{-}7.18)$$

Then

$$\frac{\partial}{\partial Xi} \sum_{j=1}^{m} Fj = \left(\overline{Xi} \oplus \sum_{j=1}^{m} Fj\right) \sum_{j=1}^{m} D_{Xi}(Fj) \qquad (2\text{-}7.19)$$

Similarly

$$\frac{\partial}{\partial \overline{Xi}} \sum_{j=1}^{m} Fj = \left(Xi \oplus \sum_{j=1}^{m} Fj\right) \sum_{j=1}^{m} D_{Xi}(Fj) \qquad (2\text{-}7.20)$$

All of the general partial derivative expressions obtained can be put in different forms. For example if equation (2-5.13) is used in place of equation (2-5.26), equation (2-7.19) becomes

$$\frac{\partial}{\partial Xi} \sum_{j=1}^{m} Fj = \sum_{j=1}^{m} Fj(1)_{Xi} \sum_{j=1}^{m} D_{Xi}(Fj) \qquad (2\text{-}7.21)$$

It could also be shown that

$$\frac{\partial}{\partial \overline{Xi}} \sum_{j=1}^{m} Fj = \sum_{j=1}^{m} Fj(0)_{Xi} \sum_{j=1}^{m} D_{Xi}(Fj) \qquad (2\text{-}7.22)$$

2-8. Special Case Expressions for the Partial Derivatives

There are several useful special cases that should be considered. Let the function G be independent of Xi such that

$$G = G(1)_{Xi}$$

$$= G(0)_{Xi} \qquad (2\text{-}8.1)$$

Then from equation (2-2.1)

$$\frac{\partial G}{\partial Xi} = G(1)_{Xi}\ \overline{G(0)}_{Xi}$$

$$= G\ \overline{G}$$

$$= 0 \qquad (2\text{-}8.2)$$

and from equation (2-2.2)

$$\frac{\partial G}{\partial \overline{Xi}} = G(0)_{Xi}\ \overline{G(1)}_{Xi}$$

$$= G\ \overline{G}$$

$$= 0 \qquad (2\text{-}8.3)$$

From equation (2-7.6)

$$\frac{\partial}{\partial Xi} (G\ F) = G(1)_{Xi}\ F(1)_{Xi} \left(\frac{\partial G}{\partial Xi} + \frac{\partial F}{\partial Xi} \right) \tag{2-8.4}$$

Applying equations (2-8.1) and (2-8.2), equation (2-8.4) reduces to

$$\frac{\partial}{\partial Xi} (G\ F) = G\ F(1)_{Xi}\ \frac{\partial F}{\partial Xi} \tag{2-8.5}$$

but from equation (2-4.3)

$$F(1)_{Xi}\ \frac{\partial F}{\partial Xi} = \frac{\partial F}{\partial Xi} \tag{2-8.6}$$

Hence

$$\frac{\partial}{\partial Xi} (G\ F) = G\ \frac{\partial F}{\partial Xi} \tag{2-8.7}$$

In a similar way it can be shown that

$$\frac{\partial}{\partial \overline{Xi}} (G\ F) = G\ \frac{\partial F}{\partial \overline{Xi}} \tag{2-8.8}$$

From De Morgan's theorem

$$\frac{\partial}{\partial Xi}(G + F) = \frac{\partial}{\partial Xi}(\overline{\overline{G}\ \overline{F}}) \tag{2-8.9}$$

but since as shown by equation (2-4.1)

$$\frac{\partial \overline{F}}{\partial Xi} = \frac{\partial F}{\partial \overline{Xi}} \tag{2-8.10}$$

then

$$\frac{\partial}{\partial Xi}(G + F) = \frac{\partial}{\partial \overline{Xi}}(\overline{G}\ \overline{F}) \tag{2-8.11}$$

From equation (2-8.8)

$$\frac{\partial}{\partial Xi}(G + F) = \overline{G}\,\frac{\partial \overline{F}}{\partial \overline{Xi}} \tag{2-8.12}$$

but as given by equation (2-4.2)

$$\frac{\partial \overline{F}}{\partial \overline{Xi}} = \frac{\partial F}{\partial Xi} \tag{2-8.13}$$

Hence

$$\frac{\partial}{\partial Xi}(G + F) = \overline{G}\,\frac{\partial F}{\partial Xi} \tag{2-8.14}$$

Similarly it can be shown that

$$\frac{\partial}{\partial \overline{Xi}} (G + F) = \overline{G} \frac{\partial F}{\partial \overline{Xi}} \tag{2-8.15}$$

From equation (2-7.21)

$$\frac{\partial}{\partial Xi}(G \oplus F) = \left[G \oplus F(1)_{Xi}\right] \left(\frac{\partial F}{\partial Xi} + \frac{\partial F}{\partial \overline{Xi}}\right) \tag{2-8.16}$$

since

$$D_{Xi}(G) = \frac{\partial G}{\partial Xi} + \frac{\partial G}{\partial \overline{Xi}} = 0 \tag{2-8.17}$$

Equation (2-8.16) may be written as

$$\frac{\partial}{\partial Xi}(G \oplus F) = \left[G \frac{\partial F}{\partial Xi} \oplus F(1)_{Xi} \frac{\partial F}{\partial Xi}\right] + \left[G \frac{\partial F}{\partial \overline{Xi}} \oplus F(1)_{Xi} \frac{\partial F}{\partial \overline{Xi}}\right] \tag{2-8.18}$$

Since equation (2-4.3) states that

$$F(1)_{Xi} \frac{\partial F}{\partial Xi} = \frac{\partial F}{\partial Xi} \tag{2-8.19}$$

and from equation (2-4.9)

$$F(1)_{Xi} \frac{\partial F}{\partial \overline{Xi}} = 0 \qquad (2\text{-}8.20)$$

equation (2-8.18) reduces to

$$\frac{\partial}{\partial Xi}(G \oplus F) = \overline{G} \frac{\partial F}{\partial Xi} + G \frac{\partial F}{\partial \overline{Xi}} \qquad (2\text{-}8.21)$$

In a similar way it can be shown that

$$\frac{\partial}{\partial \overline{Xi}}(G \oplus F) = \overline{G} \frac{\partial F}{\partial \overline{Xi}} + G \frac{\partial F}{\partial Xi} \qquad (2\text{-}8.22)$$

Other basic relations are

$$\frac{\partial Xi}{\partial Xi} = 1 \cdot \overline{0}$$

$$= 1 \qquad (2\text{-}8.23)$$

$$\frac{\partial \overline{Xi}}{\partial Xi} = 0 \cdot \overline{1}$$

$$= 0 \qquad (2\text{-}8.24)$$

$$\frac{\partial Xi}{\partial \overline{Xi}} = 0 \cdot \overline{1}$$

$$= 0 \qquad (2\text{-}8.25)$$

and

$$\frac{\partial \overline{Xi}}{\partial \overline{Xi}} = 1 \cdot \overline{0}$$

$$= 1 \qquad (2\text{-}8.26)$$

2-9. Operations Involving Differentials

In the next chapter it will be necessary to perform the ANDing and the ORing of differentials. The rules for accomplishing these operations are given in this section. It must be emphasized that the definitions given here are intended for use with the restriction that only one variable changes at a time.

A differential expression $d\xi$ is an expression of the form

$$d\xi = \sum_{i=1}^{n} (\alpha i \; dXi + \beta i \; d\overline{Xi}) \qquad (2\text{-}9.1)$$

where in general αi and βi are functions of X1 through Xn. Obviously from the above definition all differentials are differential expressions; however, the converse is not true. For a differential expression to be a differential, there must exist a function, with a differential which is the given differential expression. In equation (2-9.1) if the function ξ actually exists then $d\xi$ is a differential. When this is the case then naturally

$$\alpha i = \frac{\partial \xi}{\partial Xi} \tag{2-9.2}$$

and

$$\beta i = \frac{\partial \xi}{\partial \overline{Xi}} \tag{2-9.3}$$

for all i, $1 \leq i \leq n$. Given a second differential expression $d\zeta$ of the form

$$d\zeta = \sum_{i=1}^{n} (\gamma i \; dXi + \delta i \; d\overline{Xi}) \tag{2-9.4}$$

where in general γi and δi are functions of $X1$ through Xn. Then <u>$d\xi$ and $d\zeta$ are equal</u>, that is

$$d\xi = d\zeta \tag{2-9.5}$$

if and only if

$$\alpha i = \gamma i \tag{2-9.6}$$

and

$$\beta i = \delta i \tag{2-9.7}$$

for all i, $1 \leq i \leq n$. dξ ANDed with dζ is defined as

$$d\xi \cdot d\zeta = \sum_{i=1}^{n} (\alpha i \ \gamma i \ dXi + \beta i \ \delta i \ d\overline{Xi}) \qquad (2\text{-}9.8)$$

Similarly, dξ ORed with dζ is defined as

$$d\xi + d\zeta = \sum_{i=1}^{n} \left[(\alpha i + \gamma i)\ dXi + (\beta i + \delta i)\ d\overline{Xi}\right] \qquad (2\text{-}9.9)$$

2-10. The Oriented Difference Operators

In this section the oriented difference operators, Δ and ∇, will be defined and their close relationship to the Boolean differential established. Smith and Routh [53, 54] have shown that these operators can be used to describe the behavior of differential mode sequential circuits. Furthermore, Talantsev's [44] d operator is the same as the Δ operator.

By definition, ΔF is a function such that $\Delta F = 1$ only at the instant of time when F is changing from 0 to 1. Otherwise $\Delta F = 0$. Similarly, ∇F is a function such that $\nabla F = 1$ only at the instant of time when F is changing from 1 to 0. Otherwise $\nabla F = 0$. Figure 1 shows ΔF and ∇F for changes in a typical F as a function of time.

Certain identities are obvious from the above definitions. For example

$$\Delta F = \nabla\overline{F} \qquad (2\text{-}10.1)$$

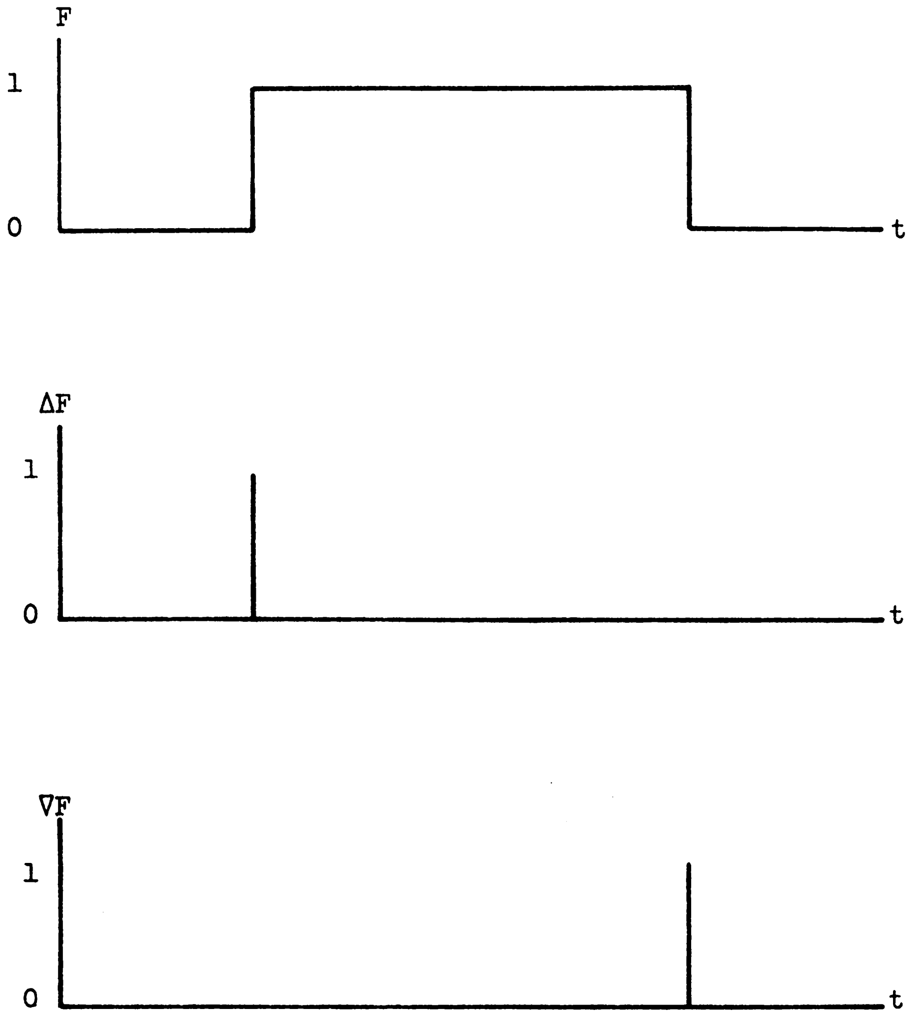


Figure 1. - ΔF and ∇F for changes in F as a function of time.

since when F changes from 0 to 1, $\overline{F}$ changes from 1 to 0. Likewise

$$\nabla F = \Delta \overline{F} \tag{2-10.2}$$

Although other identities could be obtained it is not the intent to develop the properties of the oriented difference operators here, but merely to establish their relationship to the Boolean differential. As shown by the following theorem, ΔF and ∇F can be expressed in terms of the partial derivatives of F.

<u>Theorem 2.4</u>: Given a function F where

$$F = F\ (X1, \ldots, Xi, \ldots, Xn) \tag{2-10.3}$$

and $X1, \ldots, Xi, \ldots, Xn$ are all independent variables. Then

$$\Delta F = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial Xi}\ \Delta Xi + \frac{\partial F}{\partial \overline{Xi}}\ \nabla Xi \right) \tag{2-10.4}$$

and

$$\nabla F = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial Xi}\ \nabla Xi + \frac{\partial F}{\partial \overline{Xi}}\ \Delta Xi \right) \tag{2-10.5}$$

<u>Proof</u>: If Xi changes all other variables will remain constant, since it is assumed that only one variable changes at a time. By

Theorem 2.1, F changes from 0 to 1 when Xi changes from 0 to 1 provided $\partial F/\partial Xi$ has a value of 1. Likewise by Theorem 2.2, F changes from 0 to 1 when Xi changes from 1 to 0 provided $\partial F/\partial \overline{Xi}$ has a value of 1. Thus ΔF is given by equation (2-10.4). A similar argument shows that ∇F is given by equation (2-10.5).

The above theorem shows that ΔF and ∇F are expressible in the same basic form as dF; thus given any one of these, the other two can readily be determined. The close relationship between ΔF, ∇F, and dF is illustrated by the following examples.

Example 2.2: As shown in Example 2.1, for the function

$$F = X1\ \overline{X3} + X2\ X3 \tag{2-10.6}$$

the differential is given by

$$dF = \overline{X3}\ dX1 + X3\ dX2 + \overline{X1}\ X2\ dX3 + X1\ \overline{X2}\ d\overline{X3} \tag{2-10.7}$$

From the above equation ΔF and ∇F can be written by inspection as

$$\Delta F = \overline{X3}\ \Delta X1 + X3\ \Delta X2 + \overline{X1}\ X2\ \Delta X3 + X1\ \overline{X2}\ \nabla X3 \tag{2-10.8}$$

and

$$\nabla F = \overline{X3}\ \nabla X1 + X3\ \nabla X2 + \overline{X1}\ X2\ \nabla X3 + X1\ \overline{X2}\ \Delta X3 \tag{2-10.9}$$

It should also be noted that from equation (2-10.2)

$$\nabla X3 = \Delta\overline{X3} \tag{2-10.10}$$

Thus equation (2-10.8) can be written as

$$\Delta F = \overline{X3}\ \Delta X1 + X3\ \Delta X2 + \overline{X1}\ X2\ \Delta X3 + X1\ \overline{X2}\ \Delta\overline{X3} \tag{2-10.11}$$

and similarly $\Delta X3$ can be replaced by $\nabla\overline{X3}$ in equation (2-10.9).

<u>Example 2.3</u>: Given that for some function F

$$\Delta F = (X2 + X3)\ \Delta X1 + \overline{X1}\ \overline{X3}\ \nabla X2 + \overline{X1}\ \overline{X2}\ \nabla X3 \tag{2-10.12}$$

It follows that

$$\nabla F = (X2 + X3)\ \nabla X1 + \overline{X1}\ \overline{X3}\ \Delta X2 + \overline{X1}\ \overline{X2}\ \Delta X3 \tag{2-10.13}$$

and

$$dF = (X2 + X3)\ dX1 + \overline{X1}\ \overline{X3}\ d\overline{X2} + \overline{X1}\ \overline{X2}\ d\overline{X3} \tag{2-10.14}$$

Chapter III

BOOLEAN INTEGRATION

3-1. Introduction

In the previous chapter it was shown how to obtain the differential of a switching function. In this chapter techniques will be developed for obtaining a function whose differential is known. This process will be called finding the exact integral of a Boolean differential. It can not always be performed since, for a given differential expression, there may not exist a function such that its differential is equal to the given differential expression. Other types of Boolean integrals are defined which, for a given differential expression, can be found even though an exact integral does not exist.

3-2. Definition of the Integral Operators

Before examining the possibility of integrating Boolean differentials, it is desirable to introduce certain operators that act on differential expressions. Consider a differential of the function F given by

$$dF = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial Xi} dXi + \frac{\partial F}{\partial \overline{Xi}} d\overline{Xi} \right) \tag{3-2.1}$$

The zeroth order integral of dF, written as $\int_0 dF$, is defined by

$$\int_0 dF = \sum_{i=1}^{n} \left(\overline{Xi} \frac{\partial F}{\partial Xi} + Xi \frac{\partial F}{\partial \overline{Xi}} \right) \tag{3-2.2}$$

<u>The first order integral of dF</u>, written as $\int_1 dF$, is defined by

$$\int_1 dF = \sum_{i=1}^{n} \left(Xi \frac{\partial F}{\partial Xi} + \overline{Xi} \frac{\partial F}{\partial \overline{Xi}} \right) \tag{3-2.3}$$

In general for $k > 1$, <u>the kth order integral of dF</u>, written as $\int_k dF$, is defined by

$$\int_k dF = \overline{\int_{k-2} dF} \int_0 d \int_{k-1} dF \quad , \quad k > 1 \tag{3-2.4}$$

and for $k < 0$, by

$$\int_k dF = \overline{\int_{k+2} dF} \int_0 d \int_{k+1} dF \quad , \quad k < 0 \tag{3-2.5}$$

The significance of these definitions and their use in integrating Boolean differential expressions is not immediately apparent. Before the above definitions can be interpreted and applied to the integration of Boolean differentials, several theorems and a number of basic identities are required; however, it is constructive at this point to compute the integrals of various orders for a particular differential expression.

<u>Example 3.1</u>:

Consider the function

$$F = X2 + \overline{X1}\, X3 \tag{3-2.6}$$

Since,

$$\overline{F} = X1\ \overline{X2} + \overline{X2}\ \overline{X3} \tag{3-2.7}$$

and from equations (2-2.1), (2-2.2), and (2-3.1) the differential of F is given by

$$dF = \overline{X2}\ X3\ d\overline{X1} + (X1 + \overline{X3})\ dX2 + \overline{X1}\ \overline{X2}\ dX3 \tag{3-2.8}$$

From equation (3-2.2)

$$\int_0 dF = \sum_{i=1}^{3} \left(\overline{Xi} \frac{\partial F}{\partial Xi} + Xi \frac{\partial F}{\partial \overline{Xi}} \right)$$

$$= \overline{X1} \cdot 0 + X1\ \overline{X2}\ X3 + \overline{X2}\ (X1 + \overline{X3}) + X2 \cdot 0$$

$$+ \overline{X3}\ \overline{X1}\ \overline{X2} + X3 \cdot 0$$

$$= X1\ \overline{X2} + \overline{X2}\ \overline{X3} \tag{3-2.9}$$

From equation (3-2.3)

$$\int_1 dF = \sum_{i=1}^{3} \left(Xi \frac{\partial F}{\partial Xi} + \overline{Xi} \frac{\partial F}{\partial \overline{Xi}} \right)$$

$$= X1 \cdot 0 + \overline{X1}\ \overline{X2}\ X3 + X2(X1 + \overline{X3}) + \overline{X2} \cdot 0 + X3\ \overline{X1}\ \overline{X2} + \overline{X3} \cdot 0$$

$$= X1\ X2 + X2\ \overline{X3} + \overline{X1}\ \overline{X2}\ X3 \tag{3-2.10}$$

In order to compute

$$\int_2 dF = \overline{\int_0 dF} \int_0 d \int_1 dF \qquad (3\text{-}2.11)$$

it is first necessary to find

$$d \int_1 dF = d[X1\ X2 + X2\ \overline{X3} + \overline{X1}\ \overline{X2}\ X3] \qquad (3\text{-}2.12)$$

which, using the techniques of the previous chapter, can be shown to be given by

$$d \int_1 dF = X2\ X3\ dX1 + \overline{X2}\ X3\ d\overline{X1} + (X1 + \overline{X3})dX2$$

$$+ \overline{X1}\ X3\ d\overline{X2} + \overline{X1}\ \overline{X2}\ dX3 + \overline{X1}\ X2\ d\overline{X3} \qquad (3\text{-}2.13)$$

Thus, from equations (3-2.2) and (3-2.13)

$$\int_0 d \int_1 dF = \overline{X1}\ X2\ X3 + X1\ \overline{X2}\ X3 + \overline{X2}\ (X1 + \overline{X3})$$

$$+ X2\ \overline{X1}\ X3 + \overline{X3}\ \overline{X1}\ \overline{X2} + X3\ \overline{X1}\ X2$$

$$= X1\ \overline{X2} + \overline{X2}\ \overline{X3} + \overline{X1}\ X2\ X3 \qquad (3\text{-}2.14)$$

From the above and equations (3-2.9) and (3-2.11)

$$\int_2 dF = \overline{(X1\ \overline{X2} + \overline{X2}\ \overline{X3})}\ (X1\ \overline{X2} + \overline{X2}\ \overline{X3} + \overline{X1}\ X2\ X3)$$

$$= \overline{(X1\ \overline{X2} + \overline{X2}\ \overline{X3})}\ (X1\ \overline{X2} + \overline{X2}\ \overline{X3}) + \overline{(X1\ \overline{X2} + \overline{X2}\ X3)}\ \overline{X1}\ X2\ X3$$

$$= \overline{X1}\ X2\ X3 \qquad (3\text{-}2.15)$$

In a similar way it can be shown that for the differential expression given by equation (3-2.8)

$$\int_k dF = 0 \qquad (3\text{-}2.16)$$

for all $k > 2$. It can be shown that for $k = -1$

$$\int_{-1} dF = \overline{\int_1 dF}\int_0 d\int_0 dF$$

$$= 0 \qquad (3\text{-}2.17)$$

and

$$\int_k dF = 0 \qquad (3\text{-}2.18)$$

for all $k < -1$.

3-3. Basic Identities

By the definition given in equation (3-2.2)

$$\int_0 dF = \sum_{i=1}^{n} \left(\overline{Xi} \frac{\partial F}{\partial Xi} + Xi \frac{\partial F}{\partial \overline{Xi}} \right) \tag{3-3.1}$$

from equation (2-4.1) and (2-4.2) the above equation can be written as

$$\int_0 dF = \sum_{i=1}^{n} \left(\overline{Xi} \frac{\partial \overline{F}}{\partial \overline{Xi}} + Xi \frac{\partial \overline{F}}{\partial Xi} \right) \tag{3-3.2}$$

Thus, from the above and equation (3-2.3)

$$\int_0 dF = \int_1 d\overline{F} \tag{3-3.3}$$

Equation (3-3.3) is very useful and will be generalized in a later section. From the definition given by equation (3-2.3)

$$\int_1 dF = \sum_{i=1}^{n} \left(Xi \frac{\partial F}{\partial Xi} + \overline{Xi} \frac{\partial F}{\partial \overline{Xi}} \right) \tag{3-3.4}$$

Using equations (2-4.11) and (2-4.14) the above equation may be written as

$$\int_1 dF = \sum_{i=1}^{n} \left(F \frac{\partial F}{\partial Xi} + F \frac{\partial F}{\partial \overline{Xi}} \right)$$

$$= F \sum_{i=1}^{n} \left(\frac{\partial F}{\partial Xi} + \frac{\partial F}{\partial \overline{Xi}} \right) \qquad (3\text{-}3.5)$$

From equation (2-5.3) equation (3-3.5) may be written as

$$\int_1 dF = F \sum_{i=1}^{n} D_{Xi}(F) \qquad (3\text{-}3.6)$$

Equation (3-3.6) relates the first order integral of dF to the Boolean differences of F.

Replacing F with $\overline{F}$ in equation (3-3.6) and applying the results given in equation (3-3.3) produces another new identity. That is

$$\int_0 dF = \int_1 d\overline{F}$$

$$= \overline{F} \sum_{i=1}^{n} D_{Xi}(\overline{F})$$

$$= \overline{F} \sum_{i=1}^{n} D_{Xi}(F) \qquad (3\text{-}3.7)$$

The last step in the above equation follows from equation (2-5.12). Equation (3-3.6) and the definition of Boolean difference given by equation (2-5.2) may be used to obtain the relation

$$\int_1 dF = \sum_{i=1}^{n} F\,[F \oplus F(\overline{Xi})_{Xi}]$$

$$= \sum_{i=1}^{n} F\,[1 \oplus F(\overline{Xi})_{Xi}]$$

$$= F \sum_{i=1}^{n} \overline{F(\overline{Xi})_{Xi}} \qquad (3\text{-}3.8)$$

Replacing F with $\overline{F}$ in the above equation and using equation (3-3.3) yields

$$\int_0 dF = \overline{F} \sum_{i=1}^{n} F(\overline{Xi})_{Xi} \qquad (3\text{-}3.9)$$

Equations (3-3.6) and (3-3.7) can also be used to obtain the following useful identity.

$$\int_0 dF + \int_1 dF = \overline{F} \sum_{i=1}^{n} D_{Xi}(F) + F \sum_{i=1}^{n} D_{Xi}(F)$$

$$= \sum_{i=1}^{n} D_{Xi}(F) \qquad (3\text{-}3.10)$$

Given two functions F1 and F2, <u>F1 covers F2</u> written as F1 $\supseteq$ F2 if and only if

$$F1 \cdot F2 = F2 \qquad (3\text{-}3.11)$$

It is thus apparent that

$$F \supseteq \int_1 dF \qquad (3\text{-}3.12)$$

since from equation (3-3.6)

$$F \cdot \int_1 dF = F \cdot F \sum_{i=1}^{n} D_{Xi}(F)$$

$$= F \sum_{i=1}^{n} D_{Xi}(F)$$

$$= \int_1 dF \qquad (3\text{-}3.13)$$

Similarly

$$\overline{F} \supseteq \int_0 dF \qquad (3\text{-}3.14)$$

since from equation (3-3.7)

$$\overline{F} \cdot \int_0 dF = \overline{F} \cdot \overline{F} \sum_{i=1}^{n} D_{Xi}(F)$$

$$= \overline{F} \sum_{i=1}^{n} D_{Xi}(F)$$

$$= \int_0 dF \qquad (3\text{-}3.15)$$

3.4 Taylor Series Expansion

Any switching function can be expanded in series about a point in the n-space over which the function is defined. Letting A_i, $1 \leq i \leq n$, be a constant with value 0 or 1, $F(X1,\ldots,Xi,\ldots,Xn)$ can be expanded around an arbitrary point $\overline{\underline{A}}$ as a generalized Reed-Muller expansion [1,55] of the form

$$F = \sum_{\underline{K}=0}^{2^n-1} \prod_{i=1}^{n} (Xi \oplus Ai)^{Ki} P_{\underline{K}} \qquad (3\text{-}4.1)$$

where

$$\overline{\underline{A}} = (A1,\ldots,Ai,\ldots,An) \qquad (3\text{-}4.2)$$

and as in Chapter II

$$\overline{\underline{K}} = Knx2^{n-1} + \ldots + Kix2^{i-1} + \ldots + K1 \qquad (3\text{-}4.3)$$

Equation (3-4.3) is an arithmetic expression with the "+" sign representing the arithmetic sum. The term $(Xi \oplus Ai)^{Ki}$ is defined as

$$(Xi \oplus Ai)^{Ki} = \begin{cases} 1 & \text{if } Ki = 0 \\ (Xi \oplus Ai) & \text{if } Ki = 1 \end{cases} \qquad (3\text{-}4.4)$$

and $P_{\underline{K}}$ is a constant which must be determined. The various $P_{\underline{K}}$'s in equation (3-4.1) can be evaluated by taking Boolean differences of both the sides of equation (3-4.1) and evaluating the results at point $\underline{\bar{A}}$. The technique is similar to that used in ordinary calculus to obtain the Taylor series for a function of several variables. It can be shown [2, 32, 33, 42] that equation (3-4.1) can be expressed as

$$F(X1,\ldots,Xi,\ldots,Xn) = F(A1,\ldots,Ai,\ldots,An)$$

$$\oplus \sum_{\underline{\bar{K}}=1}^{2^n-1} \prod_{i=1}^{n} (Xi \oplus Ai)^{Ki} \, D_\rho F(A1,\ldots,Ai,\ldots An) \qquad (3\text{-}4.5)$$

where

$$\rho = \prod_{i=1}^{n} Xi^{Ki} \qquad (3\text{-}4.6)$$

and

$$Xi^{Ki} = \begin{cases} 1 & \text{if } Ki = 0 \\ Ki & \text{if } Ki = 1 \end{cases} \qquad (3\text{-}4.7)$$

In equation (3-2.5) it is understood that

$$D_\rho F(A1,\ldots,Ai,\ldots,An) = D_\rho F(X1,\ldots,Xi,\ldots,Xn)\big|_{\underline{A}} \qquad (3\text{-}4.8)$$

$D_\rho F$ may be a first order Boolean difference, but in general it is of higher order than the first. For example, if

$$\rho = X1\ X3\ X4 \qquad (3\text{-}4.9)$$

then

$$D_\rho F = D_{X1\ X3\ X4}\ F$$

$$= D_{X1}\left[D_{X3}(D_{X4}\ F)\right] \qquad (3\text{-}4.10)$$

It should be noted from equation (3-4.5) that any function can be expressed as

$$F(X1,\ldots,Xi,\ldots,Xn) = F(A1,\ldots,Ai,\ldots,An) \oplus \theta(X1,\ldots,Xi,\ldots,Xn) \qquad (3\text{-}4.11)$$

where θ is uniquely determined by the Boolean differences of F. In order to determine F, knowledge of the Boolean differences alone is not sufficient, but additional information is required to evaluate the function at a particular point so that $F(A1,\ldots,Ai,\ldots,An)$ can be determined. Thus the Boolean differences alone are not sufficient to distinguish between F and $\overline{F}$. This is not at all surprising since from equation (2-5.12)

$$D_{Xi}(F) = D_{Xi}(\overline{F}) \tag{3-4.12}$$

It will be shown that if the first order partial derivatives of F are known no other information is required to uniquely determine F provided F is not a constant.

3-5. Uniqueness

An important theorem can now be proven. This theorem will be used frequently in the development of Boolean integration and it forms the basis of a Karnaugh map method for finding F when the differential of F is given.

Theorem 3.1: At any point $\underline{\overline{A}}$, where

$$\underline{\overline{A}} = (A1,\ldots,Ai,\ldots,An) \tag{3-5.1}$$

is a vertex of the n-cube [56] on which the function $F(X1,\ldots,Xi,\ldots,Xn)$ is defined:

(a) If

$$\int_1 dF|_{\underline{\overline{A}}} = 1 \tag{3-5.2}$$

then

$$F|_{\underline{\bar{A}}} = 1 \tag{3-5.3}$$

(b) If

$$\int_0 dF|_{\underline{\bar{A}}} = 1 \tag{3-5.4}$$

then

$$F|_{\underline{\bar{A}}} = 0 \tag{3-5.5}$$

(c) If

$$\int_1 dF|_{\underline{\bar{A}}} = 0 \tag{3-5.6}$$

and

$$\int_0 dF|_{\underline{\bar{A}}} = 0 \tag{3-5.7}$$

then F evaluated at $\underline{\bar{A}}$ has the same value as F evaluated at any point adjacent to $\underline{\bar{A}}$. That is

$$F(A1,\ldots,Ai,\ldots,An) = F(A1,\ldots,\overline{Ai},\ldots,An) \qquad (3\text{-}5.8)$$

for all i, $1 \leq i \leq n$.

<u>Proof</u>: From equation (3-3.11)

$$F \supseteq \int_1 dF \qquad (3\text{-}5.9)$$

Therefore, at any point $\overline{\underline{A}}$

$$\left[F \cdot \int_1 dF \right] \Big|_{\overline{\underline{A}}} = \int_1 dF \Big|_{\overline{\underline{A}}} \qquad (3\text{-}5.10)$$

If

$$\int_1 dF \Big|_{\overline{\underline{A}}} = 1 \qquad (3\text{-}5.11)$$

then from equation (3-5.10)

$$F \Big|_{\overline{\underline{A}}} = 1 \qquad (3\text{-}5.12)$$

which proves (a).

If

$$\int_0 dF\Big|_{\overline{\underline{A}}} = 1 \tag{3-5.13}$$

then from the identity given by equation (3-2.3)

$$\int_1 d\overline{F}\Big|_{\overline{\underline{A}}} = 1 \tag{3-5.14}$$

Applying (a) to the above equation gives

$$\overline{F}\Big|_{\overline{\underline{A}}} = 1 \tag{3-5.15}$$

or

$$F\Big|_{\overline{\underline{A}}} = 0 \tag{3-5.16}$$

thus, proving (b).

To prove (c) note that if

$$\int_0 dF\Big|_{\overline{\underline{A}}} = 0 \tag{3-5.17}$$

and

$$\int_{1} dF|_{\underline{A}} = 0 \tag{3-5.18}$$

Then

$$\left[\int_{0} dF + \int_{1} dF\right]|_{\underline{A}} = 0 \tag{3-5.19}$$

From the above equation and the identity given by equation (3-3.10)

$$\sum_{i=1}^{n} D_{Xi}(F)|_{\underline{A}} = 0 \tag{3-5.20}$$

Hence

$$D_{Xi}(F)|_{\underline{A}} = D_{Xi}F(A1,\ldots,Ai,\ldots,An)$$

$$= 0 \tag{3-5.21}$$

for all i, $1 \leq i \leq n$.

By the definition of the Boolean difference given in equation (2-5.2)

$$D_{Xi}F(X1,\ldots,Xi,\ldots,Xn)$$

$$= F(X1,\ldots,Xi,\ldots,Xn) \oplus F(X1,\ldots,\overline{Xi},\ldots,Xn) \tag{3-5.22}$$

Taking the EXCLUSIVE-OR of both sides of equation (3-5.22) with $F(X1,\ldots,\overline{Xi},\ldots,Xn)$ gives

$$F(X1,\ldots,Xi,\ldots,Xn)$$

$$= F(X1,\ldots,\overline{Xi},\ldots,Xn) \oplus D_{Xi}F(X1,\ldots,Xi,\ldots,Xn) \qquad (3\text{-}5.23)$$

Evaluating the above equation at point $\underline{\bar{A}}$ yields

$$F(A1,\ldots,Ai,\ldots,An)$$

$$= F(A1,\ldots,\overline{Ai},\ldots,An) \oplus D_{Xi}F(A1,\ldots,Ai,\ldots,An) \qquad (3\text{-}5.24)$$

but from equation (3-5.21)

$$D_{Xi}(A1,\ldots,Ai,\ldots,An) = 0$$

Therefore

$$F(A1,\ldots,Ai,\ldots,An) = F(A1,\ldots,\overline{Ai},\ldots,An) \qquad (3\text{-}5.25)$$

which proves (c).

From the above theorem and equation (3-4.5) it is seen that P_0 can readily be determined if the expansion given in equation (3-4.1) is about a point $\underline{\bar{A}}$, such that

$$\int_1 dF|_{\underline{\bar{A}}} = 1 \qquad (3\text{-}5.26)$$

or

$$\int_0 dF|_{\underline{\bar{A}}} = 1 \qquad (3\text{-}5.27)$$

If equation (3-5.26) is satisfied

$$P_0 = 1 \qquad (3\text{-}5.28)$$

If equation (3-5.27) is satisfied

$$P_0 = 0 \qquad (3\text{-}5.29)$$

The remaining $P_{\underline{\bar{K}}}$'s in equation (3-4.1) can be determined from the Boolean differences as shown by equation (3-4.5). In fact only the first order Boolean differences are required, since the higher order differences can be obtained from the first order differences, but as shown by equation (2-5.3) the first order Boolean differences can be determined from the partial derivatives of F. Therefore, F, provided it exists, is uniquely determined by the partial derivatives of F, if there exists at least one point $\underline{\bar{A}}$ so that either equation

(3-5.26) or equation (3-5.27) is satisfied. It will now be shown that, provided F is not a constant, there is at least one point such that equation (3-5.26) is satisfied and at least one point such that equation (3-5.27) is satisfied.

Lemma 3.1: A necessary and sufficient condition for F to be a constant is that

$$\frac{\partial F}{\partial Xi} = 0 \qquad (3\text{-}5.30)$$

and

$$\frac{\partial F}{\partial \overline{Xi}} = 0 \qquad (3\text{-}5.31)$$

for all i, $1 \leq i \leq n$.

Proof: If F is a constant it is independent of Xi for all i, then from equation (2-8.2)

$$\frac{\partial F}{\partial Xi} = 0 \qquad (3\text{-}5.32)$$

and from equation (2-8.3)

$$\frac{\partial F}{\partial \overline{Xi}} = 0 \qquad (3\text{-}5.33)$$

for all $i, 1 \leq i \leq n$.

To prove sufficiency note that if for all $i, 1 \leq i \leq n$

$$\frac{\partial F}{\partial Xi} = 0 \tag{3-5.34}$$

and

$$\frac{\partial F}{\partial \overline{Xi}} = 0 \tag{3-5.35}$$

then from equation (2-5.3)

$$D_{Xi}(F) = \frac{\partial F}{\partial Xi} + \frac{\partial F}{\partial \overline{Xi}} = 0 \tag{3-5.36}$$

for all i. Theorem 1 in reference [2] shows that a necessary and sufficient condition for a function $F(Xi)_{Xi}$ to be independent of Xi is that

$$D_{Xi}(F) = 0 \tag{3-5.37}$$

Thus it follows that F is independent of Xi for all i; therefore, F must be a constant.

Theorem 3.2: Given a function F,

$$\int_1 dF = 0 \tag{3-5.38}$$

if and only if F is a constant.

Proof: If F is a constant, from Lemma 3.1

$$\frac{\partial F}{\partial Xi} = 0 \qquad (3\text{-}5.39)$$

and

$$\frac{\partial F}{\partial \overline{Xi}} = 0 \qquad (3\text{-}5.40)$$

for all i, $1 \leq i \leq n$; therefore,

$$\int_1 dF = \sum_{i=1}^{n} \left(Xi \frac{\partial F}{\partial Xi} + \overline{Xi} \frac{\partial F}{\partial \overline{Xi}} \right) = 0 \qquad (3\text{-}5.41)$$

To show sufficiency, note that if the above equation holds then for all i

$$Xi \frac{\partial F}{\partial Xi} + \overline{Xi} \frac{\partial F}{\partial \overline{Xi}} = 0 \qquad (3\text{-}5.42)$$

Hence

$$Xi \frac{\partial F}{\partial Xi} = 0 \qquad (3\text{-}5.43)$$

and

$$\overline{Xi}\,\frac{\partial F}{\partial \overline{Xi}} = 0 \qquad (3\text{-}5.44)$$

From Theorem 2.3, $\partial F/\partial Xi$ is not a function of Xi; therefore, in equation (3-5.43) Xi may be set equal to 1 without affecting $\partial F/\partial Xi$. Hence

$$\frac{\partial F}{\partial Xi} = 0 \qquad (3\text{-}5.45)$$

Similarly from equation (3-5.44)

$$\frac{\partial F}{\partial \overline{Xi}} = 0 \qquad (3\text{-}5.46)$$

for all i, $1 \leq i \leq n$. Thus, from Lemma 3.1, F is a constant, and Theorem 3.2 is proved.

<u>Corollary 3.1</u>: If a function F is not a constant, there exists at least one point $\underline{\overline{A}}$ on the n-cube over which the function is defined such that

$$\int_1 dF\Big|_{\underline{\overline{A}}} = 1 \qquad (3\text{-}5.47)$$

Proof: From Theorem 3.2 if F is not a constant $\int_1 dF$ is not identically equal to 0 ; therefore, there must be at least one point $\underline{\bar{A}}$ that satisfies equation (3-5.47).

Theorem 3.3: Given a function F

$$\int_0 dF = 0 \tag{3-5.48}$$

if and only if F is a constant.

Proof: The proof is essentially the same as that for Theorem 3.2.

Corollary 3.2: If a function F is not a constant there exists at least one point $\underline{\bar{A}}$, on the n-cube over which the function is defined, such that

$$\int_0 dF\Big|_{\underline{\bar{A}}} = 1 \tag{3-5.49}$$

Proof: The proof is similar to that of Corollary 3.1.

Theorem 3.4: Given a function F

$$dF = 0 \tag{3-5.50}$$

if and only if F is a constant.

Proof: If

$$dF = 0 \tag{3-5.51}$$

then

$$\sum_{i=1}^{n} \left(\frac{\partial F}{\partial Xi} dXi + \frac{\partial F}{\partial \overline{Xi}} d\overline{Xi} \right) = 0 \qquad (3\text{-}5.52)$$

Therefore

$$\frac{\partial F}{\partial Xi} = 0 \qquad (3\text{-}5.53)$$

and

$$\frac{\partial F}{\partial \overline{Xi}} = 0 \qquad (3\text{-}5.54)$$

for all i, $1 \leq i \leq n$. Hence, from Lemma 3.1, F is a constant. Thus, equation (3-5.51) is a necessary condition for F to be a constant. If F is a constant, from Lemma 3.1

$$\frac{\partial F}{\partial Xi} = 0 \qquad (3\text{-}5.55)$$

and

$$\frac{\partial F}{\partial \overline{Xi}} = 0 \qquad (3\text{-}5.56)$$

for all i; therefore,

$$dF = 0 \qquad (3\text{-}5.57)$$

Thus equation (3-5.51) is also a sufficient condition for F to be a constant.

From Corollary 3.1 or Corollary 3.2 it is seen that, if F is not a constant, there always exists a point $\bar{\underline{A}}$, such that P_0 in equation (3-4.2) can be evaluated. The function F can be obtained in the form of a Taylor-like expansion around this point provided the partial derivatives of F are known, but the partial derivatives of F are known if the differential of F is given. From Theorem 3.4, F is a constant if and only if

$$dF = 0 \qquad (3\text{-}5.58)$$

Hence, F, provided it exists, is uniquely determined by a non-zero dF. As shown by the following theorem, two different nonconstant functions cannot have the same differential, and two distinct differentials cannot be obtained from the same function.

<u>Theorem 3.5</u>: Given two nonconstant function F1 and F2,

$$dF1 = dF2 \qquad (3\text{-}5.59)$$

if and only if

$$F1 \;=\; F2 \qquad (3\text{-}5.60)$$

Proof: If

$$dF1 \;=\; dF2 \qquad (3\text{-}5.61)$$

then from the definition of Boolean differentials,

$$\sum_{i=1}^{n}\left(\frac{\partial F}{\partial Xi}\,dXi + \frac{\partial F1}{\partial \overline{Xi}}\,d\overline{Xi}\right) \;=\; \sum_{i=1}^{n}\left(\frac{\partial F2}{\partial Xi}\,dXi + \frac{\partial F2}{\partial \overline{Xi}}\,d\overline{Xi}\right) \qquad (3\text{-}5.62)$$

but equation (3-5.62) can be true only if

$$\frac{\partial F1}{\partial Xi} \;=\; \frac{\partial F2}{\partial Xi} \qquad (3\text{-}5.63)$$

and

$$\frac{\partial F1}{\partial \overline{Xi}} \;=\; \frac{\partial F2}{\partial \overline{Xi}} \qquad (3\text{-}5.64)$$

for all i, $1 \leq i \leq n$. From equations (3-5.63) and (3-5.64), it follows that

$$\sum_{i=1}^{n} \left(Xi \frac{\partial F1}{\partial Xi} + \overline{Xi} \frac{\partial F1}{\partial \overline{Xi}} \right) = \sum_{i=1}^{n} \left(Xi \frac{\partial F2}{\partial Xi} + \overline{Xi} \frac{\partial F2}{\partial \overline{Xi}} \right) \qquad (3\text{-}5.65)$$

Hence

$$\int_1 dF1 = \int_1 dF2 \qquad (3\text{-}5.66)$$

By Theorem 3.2 there exists at least one point $\underline{\overline{A}}$ such that

$$\int_1 dF1 \Big|_{\underline{\overline{A}}} = 1 \qquad (3\text{-}5.67)$$

and

$$\int_1 dF2 \Big|_{\underline{\overline{A}}} = 1 \qquad (3\text{-}5.68)$$

Hence, from Theorem 3.1(a)

$$F1 \Big|_{\underline{\overline{A}}} = 1 \qquad (3\text{-}5.69)$$

and

$$F2 \Big|_{\underline{\overline{A}}} = 1 \qquad (3\text{-}5.70)$$

Therefore

$$F1\big|_{\bar{\underline{A}}} = F2\big|_{\bar{\underline{A}}} \qquad (3\text{-}5.71)$$

From equations (2-5.3), (3-5.63), and (3-5.64)

$$D_{Xi}(F1) = D_{Xi}(F2) \qquad (3\text{-}5.72)$$

for all i, $1 \leq i \leq n$. Boolean differences may be taken of both the sides of equation (3-5.72) to establish that any order Boolean difference of F1 is equal to the corresponding Boolean difference of F2. Therefore, from equations (3-5.71) and (3-5.72), if the functions F1 and F2 are both expanded about the point $\bar{\underline{A}}$ as in equation (3-4.5), each term in the expansion of F1 is equal to the corresponding term in the expansion of F2. Thus

$$F1 = F2 \qquad (3\text{-}5.73)$$

To prove sufficiency, note that if

$$F1 = F2 \qquad (3\text{-}5.74)$$

then

$$F1(1)_{Xi} = F2(1)_{Xi} \tag{3-5.75}$$

and

$$F1(0)_{Xi} = F2(0)_{Xi} \tag{3-5.76}$$

Therefore, from the definition of $\partial F/\partial Xi$ given by equation (2-2.1)

$$\frac{\partial F1}{\partial Xi} = \frac{\partial F2}{\partial Xi} \tag{3-5.77}$$

and from the definition of $\partial F/\partial \overline{Xi}$ given by equation (2-2.2)

$$\frac{\partial F1}{\partial \overline{Xi}} = \frac{\partial F2}{\partial \overline{Xi}} \tag{3-5.78}$$

Since equations (3-5.78) and (3-5.77) hold for all i, $1 \leq i \leq n$

$$\sum_{i=1}^{n} \left(\frac{\partial F1}{\partial Xi}\, dXi + \frac{\partial F1}{\partial \overline{Xi}}\, d\overline{Xi}\right) = \sum_{i=1}^{n} \left(\frac{\partial F2}{\partial Xi}\, dXi + \frac{\partial F2}{\partial \overline{Xi}}\, d\overline{Xi}\right) \tag{3-5.79}$$

Hence

$$dF1 = dF2 \tag{3-5.80}$$

3-6. Relationship Between Integrals of Order k and Order 1-k

The identity given by equation (3-3.3) can be generalized with the following important theorem.

Theorem 3.6:

$$\int_{k} dF = \int_{1-k} d\bar{F} \qquad (3\text{-}6.1)$$

for all k.

Proof: From equation (3-3.3) equation (3-6.1) is valid for $k = 0$. Replacing F with $\bar{F}$ in equation (3-3.3) shows that equation (3-6.1) is correct for $k = 1$. Assume the theorem is valid for all k such that, $0 \leq k \leq k'$, when $k' \geq 1$, from equation (3-2.4)

$$\int_{k'+1} dF = \overline{\int_{k'-1} dF} \int_{0} d \int_{k'} dF \quad , \quad k' \geq 1 \qquad (3\text{-}6.2)$$

Applying equation (3-6.1) to the terms on the right hand side of the above equation gives

$$\int_{k'+1} dF = \overline{\int_{2-k'} d\bar{F}} \int_{0} d \int_{1-k'} d\bar{F} \quad , \quad k' \geq 1 \qquad (3\text{-}6.3)$$

From equation (3-2.5)

$$\int_{-k'} dF = \overline{\int_{2-k'} d\bar{F}} \int_{0} d \int_{1-k'} d\bar{F} \quad , \quad k' \geq 1 \qquad (3\text{-}6.4)$$

Hence

$$\int_{k'+1} dF = \int_{-k'} d\overline{F}$$

$$= \int_{1-(k'+1)} d\overline{F} \quad , \quad k' \geq 1 \tag{3-6.5}$$

Thus, if equation (3-6.1) is valid for $k=k'$ when $k' \geq 1$ it is also valid for $k=k'+1$. Since equation (3-6.1) is valid for $k'=1$ it must be valid for all $k \geq 1$. Hence

$$\int_{k} dF = \int_{1-k} d\overline{F} \quad , \quad k \geq 1 \tag{3-6.6}$$

To show that the above equation holds for all k, replace k with 1-k to obtain

$$\int_{1-k} dF = \int_{k} d\overline{F} \quad , \quad 1-k \geq 1 \tag{3-6.7}$$

If $1-k \geq 1$, then $k \leq 0$; therefore, equation (3-6.7) can be written as

$$\int_{k} d\overline{F} = \int_{1-k} dF \quad , \quad k \leq 0 \tag{3-6.8}$$

Finally replacing $\overline{F}$ with F in the above equation gives

$$\int_k dF = \int_{1-k} d\overline{F} \qquad , \qquad k \leq 0 \qquad (3\text{-}6.9)$$

Thus, Theorem 3.6 is valid for all k.

3-7. Useful Concepts

The point $\underline{\overline{A}}$ is called a zero of the function F if and only if

$$F|_{\underline{\overline{A}}} = 0 \qquad (3\text{-}7.1)$$

The point $\underline{\overline{A}}$ is called a one of the function F if and only if

$$F|_{\underline{\overline{A}}} = 1 \qquad (3\text{-}7.2)$$

The following example illustrates the concept of ones and zeros of a function.

Example 3.2: Given the function

$$F(X1,\ X2) = X1 + \overline{X2} \qquad (3\text{-}7.3)$$

Since

$$F(X1,\ X2)|_{(1,0)} = 1 \qquad (3\text{-}7.4)$$

The point (1,0) is a one of the function F given by equation (3-7.3). Other ones of this function are located at the points (1,1) and (0,0). The only zero of the given function is located at the (0,1).

Two basic lemmas will occasionally be required.

Lemma 3.2: A zero of the function F is a one of the function $\overline{F}$.

Proof: Let the point $\underline{\overline{A}}$ be any zero of the function F then

$$F|_{\underline{\overline{A}}} = 0 \tag{3-7.5}$$

The above is equivalent to

$$\overline{F}|_{\underline{\overline{A}}} = 1 \tag{3-7.6}$$

Hence, $\underline{\overline{A}}$ must be a one of the function $\overline{F}$.

Lemma 3.3: A one of the function F is a zero of the function $\overline{F}$.

Proof: The proof is similar to the proof of Lemma 3.2.

The distance between two points on an n-cube is the number of coordinates in which the binary representation of the two points differ. If two points are adjacent, that is their binary representations differ in only one coordinate, then the distance between the two points is one. In general if two points on an n-cube are separated by a distance k, $0 \leq k \leq n$, then their binary representations will differ in k coordinates and they will be the same in n-k coordinates. Two intuitively obvious lemmas will now be proven. These lemmas will be used in the next section to prove two important theorems.

Lemma 3.4: If the point $\bar{A}$ is distance k from the zero of the function F nearest point $\bar{A}$, then there exists no zero of F with distance less than k-1 from point $\bar{A}^j$, where $\bar{A}^j$ is any point unit distance from $\bar{A}$.

Proof: Assume there is a zero of F with distance less than k-1 from $\bar{A}^j$. Since $\bar{A}$ is unit distance from $\bar{A}^j$ it must be a distance less than k from that zero of F, but this contradicts the hypothesis.

Lemma 3.5: If the point $\bar{A}$ is distance k from the zero of the function F nearest point $\bar{A}$, then there exists a zero of F less than or equal to distance k+1 from point $\bar{A}^j$, where $\bar{A}^j$ is any point unit distance from $\bar{A}$.

Proof: The distance between $\bar{A}$ and $\bar{A}^j$ is one. Between $\bar{A}$ and the zero of F nearest $\bar{A}$ the distance is k. Hence, $\bar{A}^j$ cannot be further removed from that zero of F than a distance of k+1.

3-8. The Distance Interpretation of the kth Order Integral

In this section it will be shown how the kth order integral of the differential of a function can be interpreted in terms of the distance to the ones or zeros of the function. The results for the special cases when k=1 and k=0 are given in Lemma 3.8 and Lemma 3.9 respectively. The results for the general case are obtained in

Theorem 3.7 and Theorem 3.8. Besides providing an insight into what the kth order integrals represent, these two theorems will greatly simplify the proof of several other important theorems that will be obtained in the next section.

The two lemmas given below will be used to facilitate the proof of Theorem 3.7.

Lemma 3.6: If at point $\underline{\bar{A}}$

$$\int_0 dF|_{\underline{\bar{A}}} = 1 \qquad (3\text{-}8.1)$$

then there exists a point $\underline{\bar{A}}^j$, adjacent to $\underline{\bar{A}}$, such that

$$\int_1 dF|_{\underline{\bar{A}}^j} = 1 \qquad (3\text{-}8.2)$$

Proof: Let

$$\underline{\bar{A}} = (A1,\ldots,Aj,\ldots,An) \qquad (3\text{-}8.3)$$

and

$$\underline{\bar{A}}^j = (A1,\ldots,\overline{Aj},\ldots,An) \qquad (3\text{-}8.4)$$

If

$$\int_0 dF|_{\underline{A}} = 1 \qquad (3\text{-}8.5)$$

then from equation (3-2.2)

$$\sum_{i=1}^{n} \left(\overline{Xi} \frac{\partial F}{\partial Xi} + Xi \frac{\partial F}{\partial \overline{Xi}} \right) |_{\underline{A}} = 1 \qquad (3\text{-}8.6)$$

From the above equation there exists at least one j, $1 \leq j \leq n$, such that

$$\left(\overline{Xj} \frac{\partial F}{\partial Xj} + Xj \frac{\partial F}{\partial \overline{Xj}} \right) |_{\underline{A}} = 1 \qquad (3\text{-}8.7)$$

Aj is either 0 or 1. If

$$Aj = 0 \qquad (3\text{-}8.8)$$

then from equation (3-8.7)

$$\frac{\partial F}{\partial Xj}|_{\underline{A}} = 1 \qquad (3\text{-}8.9)$$

Since, from Theorem 2.3, $\partial F/\partial Xj$ is not a function of Xj

$$\frac{\partial F}{\partial Xj}|_{\underline{A}^j} = 1 \qquad (3\text{-}8.10)$$

From the above equation and equation (3-8.8)

$$\left(Xj \frac{\partial F}{\partial Xj}\right)\Big|_{\underline{A}^j} = 1 \qquad (3\text{-}8.11)$$

In a similar way it can be shown that if

$$Aj = 1 \qquad (3\text{-}8.12)$$

then

$$\left(\overline{Xj} \frac{\partial F}{\partial \overline{Xj}}\right)\Big|_{\underline{A}^j} = 1 \qquad (3\text{-}8.13)$$

In either case

$$\left(Xj \frac{\partial F}{\partial Xj} + \overline{Xj} \frac{\partial F}{\partial \overline{Xj}}\right)\Big|_{\underline{A}^j} = 1 \qquad (3\text{-}8.14)$$

for some j, $1 \leq j \leq n$. Hence,

$$\sum_{i=1}^{n} \left(Xi \frac{\partial F}{\partial Xi} + \overline{Xi} \frac{\partial F}{\partial \overline{Xi}}\right)\Big|_{\underline{A}^j} = 1 \qquad (3\text{-}8.15)$$

From the above equation and equation (3-2.3)

$$\int_1 dF\Big|_{\underline{A}^j} = 1 \qquad (3\text{-}8.16)$$

Thus, Lemma 3.6 is proven.

Lemma 3.7: If at point $\bar{\underline{A}}$

$$\int_1 dF|_{\bar{\underline{A}}} = 1 \tag{3-8.17}$$

then there exists a point $\bar{\underline{A}}^j$, adjacent to $\bar{\underline{A}}$, such that

$$\int_0 dF|_{\bar{\underline{A}}^j} = 1 \tag{3-8.18}$$

Proof: In Lemma 3.6 replace F with $\bar{F}$ then apply the identity given by equation (3-3.3).

The next two lemmas give an interpretation of $\int_k dF$ for the special case when k=1 and k=0. These results are used to obtain Theorem 3.7 and Theorem 3.8 which are applicable for any value of k.

Lemma 3.8: The zero of the function F nearest point $\bar{\underline{A}}$ is unit distance from $\bar{\underline{A}}$ if and only if

$$\int_1 dF|_{\bar{\underline{A}}} = 1 \tag{3-8.19}$$

Proof: If the zero of F nearest $\bar{\underline{A}}$ is unit distance from $\bar{\underline{A}}$ then the point $\bar{\underline{A}}$ cannot be a zero of F (since the distance from $\bar{\underline{A}}$ to $\bar{\underline{A}}$ is zero); hence

$$F|_{\underline{\bar{A}}} = 1 \qquad (3\text{-}8.20)$$

Let the point $\underline{\bar{A}}^j$, where

$$\underline{\bar{A}}^j = (A1,\ldots,\bar{A}j,\ldots,An) \qquad (3\text{-}8.21)$$

be the zero of F unit distance from $\underline{\bar{A}}$, where

$$\underline{\bar{A}} = (A1,\ldots,Aj,\ldots,An) \qquad (3\text{-}8.22)$$

then

$$F|_{\underline{\bar{A}}^j} = 0 \qquad (3\text{-}8.23)$$

Aj must be either 0 or 1. If

$$Aj = 0 \qquad (3\text{-}8.24)$$

then

$$F(0)_{Xj}|_{\underline{\bar{A}}} = F|_{\underline{\bar{A}}} = 1 \qquad (3\text{-}8.25)$$

and

$$F(1)_{Xj}\Big|_{\underline{\bar{A}}} = F(1)_{Xj}\Big|_{\underline{\bar{A}}^j}$$

$$= F\big|_{\underline{\bar{A}}^j}$$

$$= 0 \qquad (3\text{-}8.26)$$

Hence

$$\left(\overline{Xj}\,\frac{\partial F}{\partial \overline{Xj}}\right)\Bigg|_{\underline{\bar{A}}} = \left(\overline{Aj}\,\frac{\partial F}{\partial \overline{Xj}}\right)\Bigg|_{\underline{\bar{A}}}$$

$$= \frac{\partial F}{\partial \overline{Xj}}\Bigg|_{\underline{\bar{A}}}$$

$$= [F(0)_{Xj}\ \overline{F(1)}_{Xj}]\Big|_{\underline{\bar{A}}}$$

$$= 1 \qquad (3\text{-}8.27)$$

In a similar way it can be shown that if

$$Aj = 1 \qquad (3\text{-}8.28)$$

then

$$Xj \left. \frac{\partial F}{\partial Xj} \right|_{\underline{\bar{A}}} = 1 \tag{3-8.29}$$

Thus, for either value of Aj

$$\left. \left(Xj \frac{\partial F}{\partial Xj} + \overline{Xj} \frac{\partial F}{\partial \overline{Xj}} \right) \right|_{\underline{\bar{A}}} = 1 \tag{3-8.30}$$

Hence

$$\int_1 dF|_{\underline{\bar{A}}} = \sum_{i=1}^{n} \left(Xi \frac{\partial F}{\partial Xi} + \overline{Xi} \frac{\partial F}{\partial \overline{Xi}} \right)$$

$$= 1 \tag{3-8.31}$$

To prove sufficiency let

$$\int_1 dF|_{\underline{\bar{A}}} = 1 \tag{3-8.32}$$

then from equation (3-2.3) there exists at least one j, $1 \leq j \leq n$, such that

$$\left. \left(X_j \frac{\partial F}{\partial Xj} + \overline{Xj} \frac{\partial F}{\partial \overline{Xj}} \right) \right|_{\underline{\bar{A}}} = 1 \tag{3-8.33}$$

Again Aj must be either 0 or 1. If

$$Aj = 0 \tag{3-8.34}$$

then from equation (3-8.33)

$$\left.\frac{\partial F}{\partial \overline{Xj}}\right|_{\underline{\overline{A}}} = 1 \tag{3-8.35}$$

From Theorem 2.3, $\partial F/\partial \overline{Xj}$ is not a function of Xj; hence, from the above equation

$$\left.\frac{\partial F}{\partial \overline{Xj}}\right|_{\underline{\overline{A}}^j} = 1 \tag{3-8.36}$$

where as before

$$\underline{\overline{A}}^j = (A1,\ldots,\overline{Aj},\ldots,An) \tag{3-8.37}$$

Thus

$$\left.\left(X_j \frac{\partial F}{\partial \overline{Xj}}\right)\right|_{\underline{\overline{A}}^j} = \overline{Aj} \left.\frac{\partial F}{\partial \overline{Xj}}\right|_{\underline{\overline{A}}^j}$$

$$= 1 \tag{3-8.38}$$

In a similar way it can also be shown that if

$$Aj = 1 \tag{3-8.39}$$

then

$$\left(\overline{Xj}\,\frac{\partial F}{\partial Xj}\right)\Bigg|_{\underline{\overline{A}}^j} = 1 \tag{3-8.40}$$

Thus, no matter what the value of Aj

$$\left(\overline{Xj}\,\frac{\partial F}{\partial Xj} + Xj\,\frac{\partial F}{\partial \overline{Xj}}\right)\Bigg|_{\underline{\overline{A}}^j} = 1 \tag{3-8.41}$$

From which it follows that

$$\int_0 dF\Big|_{\underline{\overline{A}}^j} = \sum_{i=1}^{n}\left(\overline{Xi}\,\frac{\partial F}{\partial Xi} + Xi\,\frac{\partial F}{\partial \overline{Xi}}\right)\Bigg|_{\underline{\overline{A}}^j}$$

$$= 1 \tag{3-8.42}$$

and by Theorem 3.1(b)

$$F\Big|_{\underline{\overline{A}}^j} = 0 \tag{3-8.43}$$

Hence, $\overline{\underline{A}}^j$ is a zero of the function F. The distance between $\overline{\underline{A}}$ and $\overline{\underline{A}}^j$ is one. The only point closer to $\overline{\underline{A}}$ than $\overline{\underline{A}}^j$ is $\overline{\underline{A}}$ itself, but $\overline{\underline{A}}$ is not a zero of F since from Theorem 3.1(a) and equation (3-8.32)

$$F|_{\overline{\underline{A}}} = 1 \tag{3-8.44}$$

Thus, the zero of F nearest $\overline{\underline{A}}$ is unit distance from $\overline{\underline{A}}$.

<u>Lemma 3.9</u>: The one of the function F nearest point $\overline{\underline{A}}$ is unit distance from $\overline{\underline{A}}$ if and only if

$$\int_0 dF|_{\overline{\underline{A}}} = 1 \tag{3-8.45}$$

<u>Proof</u>: Replace F with $\overline{F}$ in Lemma 3.8. Then apply Lemma 3.2 and the identity given by equation (3-3.3).

Lemma 3.8 is extended to cover all values of k greater than or equal to one by Theorem 3.7.

<u>Theorem 3.7</u>: The zero of the function F nearest point $\overline{\underline{A}}$ is a distance of k, where $k \geq 1$, from $\overline{\underline{A}}$ if and only if

$$\int_k dF|_{\overline{\underline{A}}} = 1 \quad , \quad k \geq 1 \tag{3-8.46}$$

<u>Proof</u>: Assume the above theorem holds for $1 \leq k \leq k'$, where $k' \geq 1$. From equation (3-2.4)

$$\int_{k'+1} dF = \overline{\int_{k'-1} dF} \int_0 d \int_{k'} dF \tag{3-8.47}$$

If

$$\int_{k'+1} dF|_{\underline{\bar{A}}} = 1 \tag{3-8.48}$$

then from equation (3-8.47)

$$\int_{k'-1} dF|_{\underline{\bar{A}}} = 0 \tag{3-8.49}$$

and

$$\int_0 d \int_{k'} dF|_{\underline{\bar{A}}} = 1 \tag{3-8.50}$$

From the above equation and Theorem 3.1(b)

$$\int_{k'} dF|_{\underline{\bar{A}}} = 0 \tag{3-8.51}$$

Equation (3-8.50) and Lemma 3.6 also implies that there exists a point $\underline{\bar{A}}^j$ adjacent to $\underline{\bar{A}}$ such that

$$\int_{1} d \int_{k'} dF|_{\underline{\bar{A}}^j} = 1 \tag{3-8.52}$$

From the above equation and Theorem 3.1(a)

$$\int_{k'} dF|_{\underline{\bar{A}}^j} = 1 \tag{3-8.53}$$

Since it was assumed that Theorem 3.7 is valid for all k, $1 \leq k \leq k'$, the zero of F nearest the point $\underline{\bar{A}}^j$ is distance k' from $\underline{\bar{A}}^j$. By Lemma 3.4, there does not exist a zero of F with distance less than $k'-1$ from point $\underline{\bar{A}}$; however, by Lemma 3.5 there does exist a zero of F less than or equal to distance $k'+1$ from point $\underline{\bar{A}}$. Hence, the zero of F nearest $\underline{\bar{A}}$ is either distance $k'-1$, k, or $k'+1$ from $\underline{\bar{A}}$. To show that the distance is not $k'-1$, two cases must be considered. First if $k'=1$, it must be shown that $\underline{\bar{A}}$ itself is not a zero of F. In order to do this assume $\underline{\bar{A}}$ is a zero of F. From equation (3-8.53) when $k'=1$

$$\int_{1} dF|_{\underline{\bar{A}}^j} = 1 \tag{3-8.54}$$

Hence, from Theorem 3.1(a)

$$F|_{\underline{\bar{A}}^j} = 1 \tag{3-8.55}$$

and by definition $\underline{\bar{A}}^j$ is a one of the function F. $\underline{\bar{A}}^j$ is unit distance from $\underline{\bar{A}}$; therefore, the one of F nearest point $\underline{\bar{A}}$ is unit distance from $\underline{\bar{A}}$. Then by Lemma 3.9

$$\int_0 dF|_{\underline{\bar{A}}} = 1 \tag{3-8.56}$$

Equation (3-8.56) contradicts equation (3-8.49) thus it is concluded that if k'=1, $\underline{\bar{A}}$ is not a zero of F. If k' > 1, it can also be shown that the zero of F nearest $\underline{\bar{A}}$ is not distance k'-1 from $\underline{\bar{A}}$. It has been assumed that Theorem 3.7 holds for k=k'-1 if k' > 1 and from equation (3-8.49)

$$\int_{k'-1} dF|_{\underline{\bar{A}}} \neq 1 \tag{3-8.57}$$

then the zero of F nearest $\underline{\bar{A}}$ is not at a distance of k'-1 from $\underline{\bar{A}}$. From equation (3-8.51) it is also apparent that provided Theorem 3.7 holds for k=k', the zero of F nearest $\underline{\bar{A}}$ is not a distance k' from $\underline{\bar{A}}$. Thus for all $k' \geq 1$ the zero of F nearest $\underline{\bar{A}}$ must be a distance k'+1 from $\underline{\bar{A}}$ if

$$\int_{k'+1} dF|_{\underline{\bar{A}}} = 1 \tag{3-8.58}$$

To show sufficiency, let the zero of F nearest $\bar{\underline{A}}$ be a distance of $k'+1$ from $\bar{\underline{A}}$. It then must be shown that

$$\int_{k'+1} dF|_{\bar{\underline{A}}} = 1 \tag{3-8.59}$$

If Theorem 3.7 holds for all k, $1 \leq k \leq k'$, then

$$\int_{k'} dF|_{\bar{\underline{A}}} = 0 \tag{3-8.60}$$

and

$$\int_{k'-1} dF|_{\bar{\underline{A}}} = 0 \quad , \quad k' \geq 2 \tag{3-8.61}$$

If $k'=1$, then from Theorem 3.1(b)

$$\int_{k'-1} dF|_{\bar{\underline{A}}} = 0 \quad , \quad k'=1 \tag{3-8.62}$$

since by hypothesis

$$F|_{\bar{\underline{A}}} = 1 \tag{3-8.63}$$

Thus from equations (3-8.61) and (3-8.62)

$$\overline{\int_{k'-1} dF|_{\underline{\bar{A}}}} = 1 \qquad (3\text{-}8.64)$$

for $k' \geq 1$.

It must now be shown that

$$\int_0 d \int_{k'} dF|_{\underline{\bar{A}}} = 1 \qquad (3\text{-}8.65)$$

From Lemma 3.9 the above equation is satisfied if and only if the one of the function $\int_{k'} dF$ nearest point $\underline{\bar{A}}$ is unit distance from $\underline{\bar{A}}$. It is easily shown that such is the case. First note that from equation (3-8.60) $\underline{\bar{A}}$ is not a one of $\int_{k'} dF$; therefore, if there exists a one of $\int_{k'} dF$ unit distance from $\underline{\bar{A}}$, it is the one of $\int_{k'} dF$ nearest point $\underline{\bar{A}}$. By hypothesis $\underline{\bar{A}}$ is a distance of $k'+1$ from the zero of F nearest $\underline{\bar{A}}$. Obviously there must be a point $\underline{\bar{A}}^j$ adjacent to $\underline{\bar{A}}$ such that the distance from the zero of F nearest $\underline{\bar{A}}^j$ is a distance of k' from $\underline{\bar{A}}^j$. If as has been assumed, Theorem 3.7 is valid for $k=k'$ then

$$\int_{k'} dF|_{\underline{\bar{A}}^j} = 1 \qquad (3\text{-}8.66)$$

Hence, $\underline{\bar{A}}^j$ is the one of the function $\int_{k'} dF$ nearest point $\underline{\bar{A}}$. Thus, by Lemma 3.9 equation (3-8.65) is valid. From equations (3-8.47), (3-8.64) and (3-8.65)

$$\int_{k'+1} dF|_{\underline{\bar{A}}} = 1 \qquad (3\text{-}8.67)$$

if the zero of F nearest $\underline{\bar{A}}$ is a distance of k'+1 from $\underline{\bar{A}}$. It has thus been shown that if Theorem 3.7 is valid for k=k' it must also be valid for k=k'+1. It is only left to observe that from Lemma 3.8, Theorem 3.7 is valid for k=1; therefore, it must be valid for all $k \geq 1$.

The Theorem given below is similar to Theorem 3.7 except it applies when $k \leq 0$.

<u>Theorem 3.8</u>: The one of the function F nearest point $\underline{\bar{A}}$ is a distance of 1-k, where $k \leq 0$, from $\underline{\bar{A}}$ if and only if

$$\int_{k} dF|_{\underline{\bar{A}}} = 1 \quad , \quad k \leq 0 \qquad (3\text{-}8.68)$$

<u>Proof</u>: Replace F with $\bar{F}$ and k with 1-k in Theorem 3.7. Then apply Lemma 3.2 and Theorem 3.6.

From Theorem 3.7 and Theorem 3.8 it is possible to interpret $\int_k dF$ so that an intuitive fell can be obtained for its significance. From Theorem 3.7, the function $\int_k dF$, $k \geq 1$, has a value of one at those points and only those points a distance of k from the nearest zero of the function F. From Theorem 3.8, the function $\int_k dF$, $k \leq 0$, has a value of one at those points and only those points a distance of 1-k from the nearest one of the function F.

3-9. Results of the Distance Interpretation of the kth Order Integrals

In this section a number of results will be given that follow from the two theorems given in the previous section. Theorem 3.7 leads to a generalization of Lemma 3.7.

Theorem 3.9: For all points $\underline{\bar{A}}$ such that

$$\int_k dF|_{\underline{\bar{A}}} = 1, \quad k \geq 1 \tag{3-9.1}$$

there exists a point $\underline{\bar{A}}^j$, adjacent to $\underline{\bar{A}}$ such that

$$\int_{k-1} dF|_{\underline{\bar{A}}^j} = 1 \tag{3-9.2}$$

Proof: From Theorem 3.7, $\underline{\bar{A}}$ is a distance of k from the zero of F nearest $\underline{\bar{A}}$. If $k \geq 1$ there must exist another point $\underline{\bar{A}}^j$, adjacent to $\underline{\bar{A}}$, with a distance of $k-1$ from the zero of F nearest $\underline{\bar{A}}$. Thus, $\underline{\bar{A}}^j$ is a distance of $k-1$ from the zero of F nearest $\underline{\bar{A}}^j$. If $k=1$ by Theorem 3.8, $\underline{\bar{A}}^j$ satisfies equation (3-9.2) and if $k > 1$ by Theorem 3.7, $\underline{\bar{A}}^j$ satisfies equation (3-9.2).

Lemma 3.6 can be generalized by the following theorem.

Theorem 3.10: For all points $\underline{\bar{A}}$ such that

$$\int_k dF|_{\underline{\bar{A}}} = 1, \quad k \leq 0 \tag{3-9.3}$$

then there exists a point $\underline{\bar{A}}^j$, adjacent to $\underline{\bar{A}}$ such that

$$\int_{k+1} dF|_{\underline{\bar{A}}^j} = 1 \qquad (3\text{-}9.4)$$

Proof: Replace F with $\bar{F}$ and k with 1-k in Theorem 3.9. Then apply Theorem 3.6.

Before developing additional theorems it is convenient to introduce two lemmas that follow from the definition of covering given by equation (3-3.11).

Lemma 3.10:

$$F1 \supseteq F2 \qquad (3\text{-}9.5)$$

if and only if for all points $\underline{\bar{A}}$ such that

$$F2|_{\underline{\bar{A}}} = 1 \qquad (3\text{-}9.6)$$

then

$$F1|_{\underline{\bar{A}}} = 1 \qquad (3\text{-}9.7)$$

Proof: If F1 covers F2 then by equation (3-3.11)

$$F1 \cdot F2 = F2 \tag{3-9.8}$$

therefore, at all points satisfying equation (3-9.6) equation (3-9.7) is also satisfied. For all points $\underline{\bar{A}}$ such that

$$F2|_{\underline{\bar{A}}} = 0 \tag{3-9.9}$$

then

$$F1 \cdot F2|_{\underline{\bar{A}}} = 0 \tag{3-9.10}$$

and if for all points $\underline{\bar{A}}$ satisfying equation (3-9.6), equation (3-9.7) is satisfied, it follows that for all points

$$F1 \cdot F2 = F2 \tag{3-9.11}$$

Thus, from the definition of covering

$$F1 \supseteq F2 \tag{3-9.12}$$

Lemma 3.11:

$$F1 = F2 \tag{3-9.13}$$

if and only if

$$F1 \supseteq F2 \qquad (3\text{-}9.14)$$

and

$$F2 \supseteq F1 \qquad (3\text{-}9.15)$$

<u>Proof</u>: If equation (3-9.13) is satisfied

$$F1 \cdot F2 = F2 \qquad (3\text{-}9.16)$$

and

$$F2 \cdot F1 = F1 \qquad (3\text{-}9.17)$$

Thus, from the definition of covering, equations (3-9.14) and (3-9.15) are satisfied. To show sufficiency note that if equations (3-9.14) and (3-9.15) are satisfied from the definition of covering equations (3-9.16) and (3-9.17) are also satisfied; therefore

$$F1 = F2 \qquad (3\text{-}9.18)$$

Theorem 3.7 can now be used to obtain the following generalization of the identity given by equation (3-3.12).

<u>Theorem 3.11</u>: For all $k \geq 1$

$$F \supseteq \int_k dF, \qquad k \geq 1 \tag{3-9.19}$$

<u>Proof</u>: From Theorem 4.7 for all points $\underline{\bar{A}}$ such that

$$\int_k dF\Big|_{\underline{\bar{A}}} = 1, \qquad k \geq 1 \tag{3-9.20}$$

$\underline{\bar{A}}$ is a distance of k, $k \geq 1$, from the zero of F nearest $\underline{\bar{A}}$; therefore, $\underline{\bar{A}}$ cannot be a zero of F. Thus

$$F\Big|_{\underline{\bar{A}}} = 1 \tag{3-9.21}$$

Therefore, from Lemma 3.10

$$F \supseteq \int_k dF, \qquad k \geq 1 \tag{3-9.22}$$

Equation (3-3.14) is generalized by the following theorem.

<u>Theorem 3.12</u>: For all $k \leq 0$

$$\bar{F} \supseteq \int_k dF, \qquad k \leq 0 \tag{3-9.23}$$

Proof: The proof is similar to Theorem 3.11.

Theorem 3.13: For any n-variable function F

$$\int_k dF = 0 \qquad (3\text{-}9.24)$$

for all $k > n$ and for all $k < 1 - n$.

Proof: Assume the above is not true for $k > n$. Then there must exist at least one point $\underline{\bar{A}}$ such that

$$\int_k dF|_{\underline{\bar{A}}} = 1 \quad , \quad k > n \qquad (3\text{-}9.25)$$

From Theorem 3.7 the point $\underline{\bar{A}}$ must be a distance of k from the zero of F nearest $\underline{\bar{A}}$; however, this is impossible if $k > n$ since no two points on an n-cube can be separated by a distance greater than n. Theorem 3.8 can be used to show that there is a similar contradiction if it is assumed equation (3-9.24) is not true when $k < 1 - n$.

Lemma 3.12: If for any function F

$$\int_k dF = 0 \quad , \quad k \geq 1 \qquad (3\text{-}9.26)$$

then

$$\int_{k+1} dF = 0 \qquad (3\text{-}9.27)$$

Proof: Assume that when $k \geq 1$ that there exists a point $\underline{\bar{A}}$ such that

$$\int_{k+1} dF|_{\underline{\bar{A}}} = 1 \qquad (3\text{-}9.28)$$

then by Theorem 3.9 there must exist another point $\underline{\bar{A}}^j$ such that

$$\int_{k} dF|_{\underline{\bar{A}}^j} = 1 \qquad (3\text{-}9.29)$$

however, this violates equation (3-9.26). It is thus concluded that there cannot exist a point $\underline{\bar{A}}$ satisfying equation (3-9.28). Therefore, equation (3-9.27) follows from equation (3-9.26).

Theorem 3.14: If for any function F

$$\int_{k} dF = 0 \qquad (3\text{-}9.30)$$

then

$$\int_{m} dF = 0 \qquad (3\text{-}9.31)$$

for all $m \geq k$ if $k \geq 1$ and for all $m \leq k$ if $k \leq 0$.

Proof: This theorem follows from Lemma 3.12 by induction. Assume that for $k \geq 1$ the theorem is valid for all m, $k \leq m \leq m'$. Thus given equation (3-9.30) then

$$\int_{m'} dF = 0 \qquad (3\text{-}9.32)$$

From Lemma 3.12 the above equation implies that

$$\int_{m'+1} dF = 0 \qquad (3\text{-}9.33)$$

Thus if the theorem is correct for $m = m'$ it must also be valid for $m = m' + 1$. Obviously since the theorem is correct when $m = k$. It must also be correct for $m = k + 1$, and thus for all values of $m \geq k$. To show that the theorem is correct for $k \leq 0$ replace F with $\overline{F}$ and k with $1-k$. Then apply Theorem 3.6.

Theorem 3.15:

$$\int_j dF \cdot \int_k dF = 0 \qquad (3\text{-}9.34)$$

for all j and k, $j \neq k$

Proof: If equation (3-9.34) is not correct there must be at least one point $\bar{\underline{A}}$ such that

$$\int_j dF \cdot \int_k dF|_{\underline{\bar{A}}} = 1 \tag{3-9.35}$$

or

$$\int_j dF|_{\underline{\bar{A}}} = 1 \tag{3-9.36}$$

and

$$\int_k dF|_{\underline{\bar{A}}} = 1 \tag{3-9.37}$$

To show that equations (3-9.36) and (3-9.37) cannot be true if $j \neq k$, first assume $j \geq 1$ and $k \geq 1$. Then from equation (3-9.36) and Theorem 3.7, $\underline{\bar{A}}$ is a distance of j from the zero of F nearest $\underline{\bar{A}}$, but from equation (3-9.37) and Theorem 3.7, $\underline{\bar{A}}$ is a distance of k from the zero of F nearest $\underline{\bar{A}}$. Thus equation (3-9.34) cannot be satisfied unless $j = k$ when $j \geq 1$ and $k \geq 1$. Now suppose $j \geq 1$ and $k \leq 0$. Again by Theorem 3.7 and equation (3-9.36), $\underline{\bar{A}}$ is a distance of j from the zero of F nearest $\underline{\bar{A}}$, but from Theorem 3.8 and equation (3-9.37) $\underline{\bar{A}}$ is a distance of $1-k$ from the one of F nearest $\underline{\bar{A}}$. Thus, since $\underline{\bar{A}}$ cannot be both a one and a zero of F, equation (3-9.34) cannot be satisfied for $j \geq 1$ and $k \leq 0$. It has now been shown that

$$\int_j dF \cdot \int_k dF = 0 \tag{3-9.38}$$

for all j, $j \geq 1$ and for all k, $k \neq j$. To show that the above equation is also valid for $j \leq 0$, replace j with $1-j$, k with $1-k$, and F with $\overline{F}$. Then

$$\int_{1-j} d\overline{F} \cdot \int_{1-k} d\overline{F} = 0 \tag{3-9.39}$$

where $1 - j \geq 1$ and $1 - k \neq 1 - j$, or $j \leq 0$ and $j \neq k$. From Theorem 3.6, the above equation can be written as

$$\int_j dF \cdot \int_k dF = 0 \tag{3-9.40}$$

where $j \leq 0$ and $j \neq k$. Thus equation (3-9.34) is correct for all $j \neq k$.

3-10. Obtaining a Function from its Differential

In this section it will be shown that F can be expressed in terms of $\int_k dF$. Since $\int_k dF$ is determined by the partial derivatives of F which in turn are known if the differential of F is specified, the equation given in the theorem below provides a means of finding F from its differential.

<u>Theorem 3.16</u>: Any nonconstant switching function of n-variables can be expressed as

$$F = \sum_{k=1}^{n} \int_k dF \qquad (3\text{-}10.1)$$

<u>Proof</u>: For all points $\underline{\bar{A}}$ such that

$$\sum_{k=1}^{\infty} \int_k dF|_{\underline{\bar{A}}} = 1 \qquad (3\text{-}10.2)$$

there is some $k \geq 1$ such that

$$\int_k dF|_{\underline{\bar{A}}} = 1 \qquad (3\text{-}10.3)$$

Thus, from Theorem 3.7, $\underline{\bar{A}}$ is a distance of k from the zero of F nearest $\underline{\bar{A}}$. Since $k \geq 1$, $\underline{\bar{A}}$ must be a one of F; therefore,

$$F|_{\underline{\bar{A}}} = 1 \qquad (3\text{-}10.4)$$

From Lemma 3.10 and equations (3-10.2) and (3-10.4).

$$F \supseteq \sum_{i=1}^{n} \int_k dF \qquad (3\text{-}10.5)$$

For all $\underline{\bar{A}}$ such that

$$F|_{\underline{\bar{A}}} = 1 \tag{3-10.6}$$

If F is not a constant then point $\underline{\bar{A}}$ is some distance $k \geq 1$ from zero of F nearest $\underline{\bar{A}}$. Thus from Theorem 3.7

$$\int_k dF|_{\underline{\bar{A}}} = 1 \tag{3-10.7}$$

or

$$\sum_{k=1}^{\infty} \int_k dF|_{\underline{\bar{A}}} = 1 \tag{3-10.8}$$

Therefore, from Lemma 3.10 and equations (3-10.6) and (3-10.8)

$$\sum_{k=1}^{\infty} \int_k dF \supseteq F \tag{3-10.9}$$

From Lemma 3.11 and equations (3-10.5) and (3-10.9) it follows that

$$F = \sum_{k=1}^{\infty} \int_k dF \tag{3-10.10}$$

but from Theorem 3.13

$$\int_{k} dF = 0 \qquad (3\text{-}10.11)$$

for all $k > n$. Thus,

$$F = \sum_{k=1}^{n} \int_{k} dF \qquad (3\text{-}10.12)$$

3-11. Exact and Compatible Integrals

As has been mentioned, there may not always exist a function from which an arbitrary differential expression can be obtained. If such a function does exist, for a given differential expression, the function will be referred to as an exact integral. Even when an exact integral does not exist there may exist a compatible integral whose differential is closely related to the original differential expression. In this section the properties of exact and compatible integrals will be presented.

Consider a differential expression $d\xi$ of the form

$$d\xi = \sum_{i=1}^{n} (\alpha i \; dXi + \beta i \; d\overline{Xi}) \qquad (3\text{-}11.1)$$

If a function F exists such that

$$dF = d\xi \qquad (3\text{-}11.2)$$

Then F will be referred to as <u>the exact integral of $d\xi$</u>. Obviously from Theorem 3.5 if F is an exact integral of $d\xi$ then the function ξ exists since

$$\xi = F \tag{3-11.3}$$

As shown by the following theorem a Boolean differential possesses at most one exact integral.

<u>Theorem 3.17</u>: If an exact integral of a Boolean differential exists then the exact integral is unique.

<u>Proof</u>: Assume that a Boolean differential $d\xi$ possesses two exact integrals $F1$ and $F2$. Then by definition of exact integral

$$dF1 = d\xi \tag{3-11.4}$$

and

$$dF2 = d\xi \tag{3-11.5}$$

Hence

$$dF1 = dF2 \tag{3-11.6}$$

and it follows immediately from Theorem 3.5 that

$$F1 = F2 \tag{3-11.7}$$

Thus, the exact integral is unique.

It should be recalled that given two functions F1 and F2, F1 is said to cover F2 if and only if

$$F1 \cdot F2 = F2 \tag{3-11.8}$$

Similarly <u>dF1 covers dF2</u> written as

$$dF1 \supseteq dF2 \tag{3-11.9}$$

if and only if

$$dF1 \cdot dF2 = dF2 \tag{3-11.10}$$

Since only the case of one variable changing at a time is being considered

$$dF1 \cdot dF2$$

$$= \sum_{i=1}^{n} \left(\frac{\partial F1}{\partial Xi} dXi + \frac{\partial F1}{\partial \overline{Xi}} d\overline{Xi} \right) \cdot \sum_{i=1}^{n} \left(\frac{\partial F2}{\partial Xi} dXi + \frac{\partial F2}{\partial \overline{Xi}} d\overline{Xi} \right)$$

$$= \sum_{i=1}^{n} \left(\frac{\partial F1}{\partial Xi} \frac{\partial F2}{\partial Xi} dXi + \frac{\partial F1}{\partial \overline{Xi}} \frac{\partial F2}{\partial \overline{Xi}} d\overline{Xi} \right) \qquad (3\text{-}11.11)$$

Equating each term in equation (3-11.11) with the corresponding term of dF2 given by

$$dF2 = \sum_{i=1}^{n} \left(\frac{\partial F2}{\partial Xi} dXi + \frac{\partial F2}{\partial \overline{Xi}} d\overline{Xi} \right) \qquad (3\text{-}11.12)$$

it is seen that equation (3-11.10) is equivalent to stating that

$$\frac{\partial F1}{\partial Xi} \quad \frac{\partial F2}{\partial Xi} = \frac{\partial F2}{\partial Xi} \qquad (3\text{-}11.13)$$

and

$$\frac{\partial F1}{\partial \overline{Xi}} \quad \frac{\partial F2}{\partial \overline{Xi}} = \frac{\partial F2}{\partial \overline{Xi}} \qquad (3\text{-}11.14)$$

for all i, $1 \leq i \leq n$.

<u>Theorem 3.18</u>: Given switching functions F1 and F2 with the property that

$$dF1 \supseteq dF2 \qquad (3\text{-}11.15)$$

if F2 undergoes a change F1 will undergo the same change.

Proof: By Theorem 2.1 if F2 changes the same way Xi changes then $\partial F2/\partial Xi$ will take on a value of 1. If

$$dF1 \supseteq dF2 \tag{3-11.16}$$

then from equation (3-11.13)

$$\frac{\partial F2}{\partial Xi} = \frac{\partial F1}{\partial Xi}\frac{\partial F2}{\partial Xi} \tag{3-11.17}$$

therefore, $\partial F1/\partial Xi$ also takes on a value of 1. It thus follows from Theorem 2.1 that F1 must change the same way Xi changes whenever F2 changes the same way Xi changes. It can also be shown, by using equation (3-11.14) and Theorem 2.2, that F1 changes the opposite way Xi changes whenever F2 changes the opposite way Xi changes. If F2 changes, it must change because some Xi, $1 \leq i \leq n$, changed, in which case as has been shown F1 undergoes the same change as F2.

Given a differential expression $d\xi$ if a function F exists such that

$$dF \supseteq d\xi \tag{3-11.18}$$

F will be referred to as a compatible integral of $d\xi$. If $d\xi$ possesses an exact integral F1, then by definition of exact integral

$$dF1 = d\xi \tag{3-11.19}$$

It thus follows that F1 is also a compatible integral since

$$dF1 \cdot d\xi = d\xi \tag{3-11.20}$$

which by definition of covering is equivalent to

$$dF1 \supseteq d\xi \tag{3-11.21}$$

Even though all exact integrals are compatible integrals, it is not necessary for a compatible integral to be an exact integral. In fact the definition of compatible integral does not require an exact integral to exist. Thus, the existence of a compatible integral is a weaker condition than is the existence of an exact integral. The difference between exact and compatible integrals is illustrated in the following examples.

Example 3.3: Consider the differential expression

$$d\xi = (X2 \oplus X3)dX1 + (X1 \oplus X3)\, dX2 + (X1 \oplus X2)\, dX3 \tag{3-11.22}$$

The function

$$F1 = X1\ X2 + X2\ X3 + X1\ X3 \qquad (3\text{-}11.23)$$

is an exact integral of $d\xi$ since

$$dF1 = (X2 \oplus X3)\ dX1 + (X1 \oplus X3)\ dX2 + (X1 \oplus X2)dX3 \qquad (3\text{-}11.24)$$

Equation (3-11.24) shows that F1 undergoes all transitions specified by $d\xi$ and only those transitions specified by $d\xi$. Now consider the function

$$F2 = \overline{X1}\ X2\ X3 + X1\ \overline{X2}\ X3 + X1\ X2\ \overline{X3} \qquad (3\text{-}11.25)$$

It may easily be shown that

$$dF2 = (X2 \oplus X3)\ dX1 + (X1 \oplus X3)\ dX2 + (X1 \oplus X2)dX3$$

$$+ X2\ X3\ d\overline{X1} + X1\ X3\ d\overline{X2} + X1\ X2\ d\overline{X3} \qquad (3\text{-}11.26)$$

Hence, F2 is a compatible integral since

$$dF2 \cdot d\xi = d\xi \qquad (3\text{-}11.27)$$

It is apparent from equations (3-11.22) and (3-11.26) that F2 undergoes all transitions specified by $d\xi$ but F2 also undergoes additional transitions not specified by $d\xi$.

Example 3.4: Consider the differential expression

$$d\xi = X2\ dX1 \qquad (3\text{-}11.28)$$

The above $d\xi$ does not possess an exact integral, but there are several compatible integrals. One compatible integral is

$$F1 = X1 \qquad (3\text{-}11.29)$$

Since

$$dF1 = dX1$$

$$\supseteq d\xi \qquad (3\text{-}11.30)$$

Another compatible integral is

$$F2 = X1\ X2 \qquad (3\text{-}11.31)$$

since

$$dF2 = X2\ dX1 + X1\ dX2$$

$$\supseteq d\xi \qquad (3\text{-}11.32)$$

Also

$$F3 = \overline{X1 \oplus X2} \qquad (3\text{-}11.33)$$

is a compatible integral since

$$dF3 = X2\ dX1 + \overline{X2}\ d\overline{X1} + X1\ dX2 + \overline{X1}\ d\overline{X2}$$

$$\supseteq d\xi \qquad (3\text{-}11.34)$$

If a switching function is desired that will change in the same way X1 changes whenever X2 = 1 and it does not matter how the function behaves under other circumstances, then either F1, F2, or F3 can be used.

3-12. Necessary and Sufficient Conditions for the Existence of a Compatible Integral

Necessary and sufficient conditions for the existence of an exact integral have not been obtained; however, a very simple necessary and sufficient condition for the existence of a compatible integral is given in this section. Since, as was shown in the previous section, all exact integrals are also compatible integrals, the necessary and sufficient condition for the existence of a compatible integral is also a necessary condition for the existence of an exact integral. For applications compatible integrals are probably more useful than exact integrals; therefore, the results obtained in this section are very important.

Given a $d\xi$ of the form

$$d\xi = \sum_{i=1}^{n} (\alpha i \; Xi + \beta i \; \overline{Xi}) \qquad (3\text{-}12.1)$$

The terms αi and βi cannot be completely arbitrary, but they must be restricted so that neither αi nor βi is a function of Xi. If this restriction is not imposed on αi and βi then $d\xi$ does not have meaning in the same sense as the differentials discussed so far. From Theorem 1 reference [2] the restriction that neither αi nor βi is a function of Xi is equivalent to requiring

$$D_{Xi}(\alpha i) = D_{Xi}(\beta i) = 0 \qquad (3\text{-}12.2)$$

With this restriction on αi and βi, the necessary and sufficient condition for the existence of a compatible integral can be given in the following theorem.

<u>Theorem 4.19</u>: Given a differential expression

$$d\xi = \sum_{i=1}^{n} (\alpha i \; dXi + \beta i \; d\overline{Xi}) \qquad (3\text{-}12.3)$$

where

$$D_{Xi}(\alpha i) = D_{Xi}(\beta i) = 0 \qquad (3\text{-}12.4)$$

for all i, $1 \leq i \leq n$. A necessary and sufficient condition for a compatible integral of $d\xi$ to exist is for

$$\int_1 d\xi \cdot \int_0 d\xi = 0 \qquad (3\text{-}12.5)$$

Proof: To show necessity, note that if there exists a compatible integral F of $d\xi$, then by definition of compatible integral

$$dF \supseteq d\xi \qquad (3\text{-}12.6)$$

or

$$dF \cdot d\xi = d\xi \qquad (3\text{-}12.7)$$

It will be shown that

$$\int_1 d\xi \cdot \int_0 d\xi = 0 \qquad (3\text{-}12.8)$$

by assuming

$$\int_1 d\xi \cdot \int_0 d\xi \neq 0 \tag{3-12.9}$$

and showing a contradiction results. If equation (3-12.9) is the case there must exist at least one point $\underline{\bar{A}}$ such that

$$\int_1 d\xi \cdot \int_0 d\xi \Big|_{\underline{\bar{A}}} = 1 \tag{3-12.10}$$

Hence,

$$\int_1 d\xi \Big|_{\underline{\bar{A}}} = \sum_{i=1}^{n} (\alpha i \ Xi + \beta i \ \overline{Xi}) \Big|_{\underline{\bar{A}}}$$

$$= 1 \tag{3-12.11}$$

and

$$\int_0 d\xi \Big|_{\underline{\bar{A}}} = \sum_{i=1}^{n} (\alpha i \ \overline{Xi} + \beta i \ Xi) \Big|_{\underline{\bar{A}}}$$

$$= 1 \tag{3-12.12}$$

From equation (3-12.11) there must exist at least one j, $1 \leq j \leq n$ such that

$$(\alpha j\ Xj + \beta j\ \overline{Xj})\Big|_{\underline{\overline{A}}} = 1 \qquad (3\text{-}12.13)$$

Similarly, from equation (3-12.12), there must exist at least one k, $1 \leq k \leq n$ such that

$$(\alpha k\ \overline{Xk} + \beta k\ Xk)\Big|_{\underline{\overline{A}}} = 1 \qquad (3\text{-}12.14)$$

From equations (3-12.6), (3-11.13), and (3-11.14) it follows that

$$\alpha i = \alpha i \frac{\partial F}{\partial Xi} \qquad (3\text{-}12.15)$$

and

$$\beta i = \beta i \frac{\partial F}{\partial \overline{Xi}} \qquad (3\text{-}12.16)$$

for all i, $1 \leq i \leq n$. Applying the above equations to equation (3-12.13) for $i = j$ gives

$$\left(\alpha j \frac{\partial F}{\partial Xj} Xj + \beta j \frac{\partial F}{\partial \overline{Xj}} \overline{Xj}\right)\Big|_{\underline{\overline{A}}} = 1 \qquad (3\text{-}12.17)$$

Similarly applying equations (3-12.15) and (3-12.16) to equation (3-12.14) for $i = k$ gives

$$\left(\alpha k \frac{\partial F}{\partial Xk} \overline{Xk} + \beta k \frac{\partial F}{\partial \overline{Xk}}\right) \Big|_{\underline{\overline{A}}} = 1 \qquad (3\text{-}12.18)$$

From equation (3-12.17) either

$$\frac{\partial F}{\partial Xj} Xj \Big|_{\underline{\overline{A}}} = 1 \qquad (3\text{-}12.19)$$

or

$$\frac{\partial F}{\partial \overline{Xj}} \overline{Xj} \Big|_{\underline{\overline{A}}} = 1 \qquad (3\text{-}12.20)$$

Hence,

$$\left(\frac{\partial F}{\partial Xj} Xj + \frac{\partial F}{\partial \overline{Xj}} \overline{Xj}\right) \Big|_{\underline{\overline{A}}} = 1 \qquad (3\text{-}12.21)$$

and since $1 \leq j \leq n$

$$\int_1 dF \Big|_{\underline{\overline{A}}} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial Xi} Xi + \frac{\partial F}{\partial \overline{Xi}} \overline{Xi}\right) \Big|_{\underline{\overline{A}}}$$

$$= 1 \qquad (3\text{-}12.22)$$

It thus follows from Theorem 3.1(a) that

$$F|_{\underline{\bar{A}}} = 1 \tag{3-12.23}$$

In a similar way it can be shown from equation (3-12.18) that

$$\int_0 dF|_{\underline{\bar{A}}} = 1 \tag{3-12.24}$$

From Theorem 4.1(b), the above equation can be correct only if

$$F|_{\underline{\bar{A}}} = 0 \tag{3-12.25}$$

but this contradicts equation (3-12.23). Thus, it is concluded for a function F to exist such that

$$dF \supseteq d\xi \tag{3-12.26}$$

it is necessary for

$$\int_1 d\xi \cdot \int_0 d\xi = 0 \tag{3-12.27}$$

The procedure to show sufficiency is also somewhat involved. First note that

$$\int_1 d\xi \cdot \int_0 d\xi = \sum_{j=1}^{n} (\alpha j\ Xj + \beta j\ \overline{Xj}) \cdot \sum_{i=1}^{n} (\alpha i\ \overline{Xi} + \beta i\ Xi)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} (\alpha i\ \alpha j\ \overline{Xi}\ Xj + \beta i\ \alpha j\ Xi\ Xj + \alpha i\ \beta j\ \overline{Xi}\ \overline{Xj}$$

$$+ \beta i\ \beta j\ Xi\ \overline{Xj}) \qquad (3\text{-}12.28)$$

Thus, if

$$\int_1 d\xi \cdot \int_0 d\xi = 0 \qquad (3\text{-}12.29)$$

from equation (3-12.28)

$$\alpha i\ \alpha j\ \overline{Xi}\ Xj + \beta i\ \alpha j\ Xi\ Xj + \alpha i\ \beta j\ \overline{Xi}\ \overline{Xj} + \beta i\ \beta j\ Xi\ \overline{Xj} = 0 \qquad (3\text{-}12.30)$$

for all i, $1 \leq i \leq n$, and for all j, $1 \leq j \leq n$. It is seen that

$$\alpha i \int_1 d\xi = \alpha i \sum_{j=1}^{n} (\alpha j\ Xj + \beta j\ \overline{Xj})$$

$$= \alpha i\ Xi + \alpha i\ \beta i\ \overline{Xi} + \sum_{\substack{j=1 \\ j \neq i}}^{n} (\alpha i\ \alpha j\ Xj + \alpha i\ \beta j\ \overline{Xj})$$

$$= \alpha i \ Xi + \alpha i \ \beta i \ \overline{Xi}$$

$$+ \sum_{\substack{j=1 \\ j \neq i}}^{n} (\alpha i \ \alpha j \ Xi \ Xj + \alpha i \ \alpha j \ \overline{Xi} \ Xj + \alpha i \ \beta j \ Xi \ \overline{Xj}$$

$$+ \alpha i \ \beta j \ \overline{Xi} \ \overline{Xj}) \qquad (3\text{-}12.31)$$

From equation (3-12.30)

$$\alpha i \ \alpha j \ \overline{Xi} \ Xj = 0 \qquad (3\text{-}12.32)$$

and

$$\alpha i \ \beta j \ \overline{Xi} \ \overline{Xj} = 0 \qquad (3\text{-}12.33)$$

By setting $j = i$ in equation (3-12.30) it is also obvious that

$$\alpha i \ \beta i = 0 \qquad (3\text{-}12.34)$$

Substituting the above relations into equation (3-12.31) and simplifying gives

$$\alpha i \int_{1} d\xi = \alpha i \ Xi + \sum_{\substack{j=1 \\ j \neq i}}^{n} (\alpha i \ \alpha j \ Xi \ Xj + \alpha i \ \beta j \ Xi \ \overline{Xj})$$

$$= \alpha i \; Xi \; [1 + \sum_{\substack{j=1 \\ j \neq i}}^{n} (\alpha j \; Xj + \beta j \; \overline{Xj})]$$

$$= \alpha i \; Xi \qquad (3\text{-}12.35)$$

By interchanging i and j in equation (3-12.30) it can likewise be shown that

$$\alpha i \int_0 d\xi = \alpha i \; \overline{Xi} \qquad (3\text{-}12.36)$$

From the above equation

$$\alpha i \; \overline{\int_0 d\xi} = \alpha i (1 \oplus \int_0 d\xi)$$

$$= \alpha i \oplus \alpha i \int_0 d\xi$$

$$= \alpha i \oplus \alpha i \; \overline{Xi}$$

$$= \alpha i \; (1 \oplus \overline{Xi})$$

$$= \alpha i \; Xi \qquad (3\text{-}12.37)$$

It can now be shown that a compatible integral of $d\xi$ is the function F where

$$F = \int_1 d\xi \; + \; \psi \overline{\int_0 d\xi} \qquad (3\text{-}12.38)$$

and ψ is any arbitrary function. From equations (3-11.13) and (3-11.14), F is a compatible integral of $d\xi$ provided

$$\alpha i \, \frac{\partial F}{\partial Xi} = \alpha i \qquad (3\text{-}12.39)$$

and

$$\beta i \, \frac{\partial F}{\partial \overline{Xi}} = \beta i \qquad (3\text{-}12.40)$$

for all i, $1 \leq i \leq n$. To show that the function given by equation (3-12.38) satisfies equation (3-12.39), notice that from equations (3-12.35) and (3-12.37)

$$\alpha i \; F = \alpha i \int_1 d\xi + \psi \; \alpha i \; \overline{\int_0 d\xi}$$

$$= \alpha i \; Xi + \psi \; \alpha i \; Xi$$

$$= \alpha i \; Xi \qquad (3\text{-}12.41)$$

Recalling that αi is not a function of Xi the identities given by equations (2-8.7) and (2-8.23) may be used with the above equation to obtain

$$\alpha i \frac{\partial F}{\partial Xi} = \frac{\partial}{\partial Xi} (\alpha i \ F)$$

$$= \frac{\partial}{\partial Xi} (\alpha i \ Xi)$$

$$= \alpha i \frac{\partial Xi}{\partial Xi}$$

$$= \alpha i \qquad (3\text{-}12.42)$$

To verify that F also satisfies equation (3-12.40), it may be shown that

$$\beta i \int_1 dF = \beta i \ \overline{Xi} \qquad (3\text{-}12.43)$$

and

$$\beta i \int_0 dF = \beta i \ Xi \qquad (3\text{-}12.44)$$

from which it follows that

$$\beta i \frac{\partial F}{\partial \overline{Xi}} = \beta i \qquad (3\text{-}12.45)$$

Thus, F is a compatible integral of $d\xi$ provided

$$\int_1 d\xi \cdot \int_0 d\xi = 0 \qquad (3\text{-}12.46)$$

It will now be shown that all compatible integrals are of the form given by equation (3-12.38), and all functions given by equation (3-12.38) are compatible integrals of $d\xi$, provided $d\xi$ possesses a compatible integral.

Theorem 3.20: If the differential expression $d\xi$ possesses a compatible integral, then F is a compatible integral if and only if F is of the form

$$F = \int_1 d\xi + \psi \overline{\int_0 d\xi} \qquad (3\text{-}12.47)$$

where ψ is any arbitrary function.

Proof: Sufficiency was shown in the proof of the previous theorem. To show necessity let F be a compatible integral of $d\xi$. Then,

$$dF \supseteq d\xi \qquad (3\text{-}12.48)$$

which from equations (3-11.13) and (3-11.14) is equivalent to requiring that

$$\alpha i \frac{\partial F}{\partial Xi} = \alpha i \qquad (3\text{-}12.49)$$

and

$$\beta i \frac{\partial F}{\partial \overline{Xi}} = \beta i \qquad (3\text{-}12.50)$$

for all i, $1 \leq i \leq n$. For any point $\underline{\overline{A}}$ such that

$$\int_1 d\xi \big|_{\underline{\overline{A}}} = \sum_{i=1}^{n} (\alpha i \; Xi + \beta i \; \overline{Xi}) \big|_{\underline{\overline{A}}}$$

$$= 1 \qquad (3\text{-}12.51)$$

then substituting into the above from equations (3-12.49) and (3-12.50) gives

$$\sum_{i=1}^{n} \left(\alpha i \frac{\partial F}{\partial Xi} Xi + \beta i \frac{\partial F}{\partial \overline{Xi}} \overline{Xi}\right) \big|_{\underline{\overline{A}}} = 1 \qquad (3\text{-}12.52)$$

Therefore

$$\int_1 dF \big|_{\underline{\overline{A}}} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial Xi} Xi + \frac{\partial F}{\partial \overline{Xi}} \overline{Xi}\right) \big|_{\underline{\overline{A}}} = 1 \qquad (3\text{-}12.53)$$

From the above equation and Theorem 3.1(a)

$$F\Big|_{\underline{\bar{A}}} = 1 \qquad (3\text{-}12.54)$$

It can likewise be shown that at any point $\underline{\bar{A}}$ where

$$\int_0 d\xi\Big|_{\underline{\bar{A}}} = 1 \qquad (3\text{-}12.55)$$

then

$$F\Big|_{\underline{\bar{A}}} = 0 \qquad (3\text{-}12.56)$$

Since equation (3-12.54) follows from equation (3-12.51)

$$F = \int_1 d\xi + F \qquad (3\text{-}12.57)$$

and since equation (3-12.56) follows from equation (3-12.55)

$$\bar{F} = \int_0 d\xi + \bar{F} \qquad (3\text{-}12.58)$$

The function F must simultaneously satisfy the two equations given above with the constraint imposed by Theorem 3.19 that

$$\int_1 d\xi \cdot \int_0 d\xi = 0 \tag{3-12.59}$$

Solving the two equations simultaneously by using one of the available techniques such as Tapia's method [57] shows that F must be expressible as

$$F = \int_1 d\xi + \psi \overline{\int_0 d\xi} \tag{3-12.60}$$

where ψ is arbitrary.

Several examples will now be given that illustrate the use of the results obtained in this section.

Example 3.5: Consider the differential expression

$$d\xi = dX1 + dX2 \tag{3-12.61}$$

From equation (3-2.3) it is seen that

$$\int_1 d\xi = X1 + X2 \tag{3-12.62}$$

and from equation (3-2.2)

$$\int_0 d\xi = \overline{X1} + \overline{X2} \tag{3-12.63}$$

Therefore,

$$\int_1 d\xi \cdot \int_0 d\xi = (X1 + X2)\ (\overline{X1} + \overline{X2})$$

$$= X1 \oplus X2$$

$$\neq 0 \qquad (3\text{-}12.64)$$

and from Theorem 4.19 there does not exist a compatible integral of $d\xi$.

<u>Example 3.6</u>: In Example 3.4 several compatible integrals were given for the differential expression

$$d\xi = X2\ dX1 \qquad (3\text{-}12.65)$$

for which

$$\int_1 d\xi = X2\ X1 \qquad (3\text{-}12.66)$$

and

$$\int_0 d\xi = X2\ \overline{X1} \qquad (3\text{-}12.67)$$

As expected

$$\int_1 d\xi \cdot \int_0 d\xi = X2\ X1\ X2\ \overline{X1}$$

$$= 0 \qquad (3\text{-}12.68)$$

To obtain the compatible integrals of $d\xi$ it is necessary to find functions that can be expressed as

$$F = \int_1 d\xi + \psi\ \overline{\int_0 d\xi}$$

$$= X2\ X1 + \psi\ \overline{X2\ \overline{X1}}$$

$$= X2\ X1 + \psi\ (\overline{X2} + X1) \qquad (3\text{-}12.69)$$

If the arbitrary function ψ is restricted to be a function of only X1 and X2 then in general ψ can be expressed as

$$\psi\ (X1,X2) = C0\ \overline{X1}\ \overline{X2} + C1\ \overline{X1}\ X2 + C2\ X1\ \overline{X2} + C3\ X1\ X2 \qquad (3\text{-}12.70)$$

where C0, C1, C2, and C3 are constants with values of 0 or 1. Thus, the compatible integrals are given by

$$F = X2\ X1 + (C0\ \overline{X1}\ \overline{X2} + C1\ \overline{X1}\ X2 + C2\ X1\ \overline{X2} + C3\ X1\ X2)\ (\overline{X2} + X1)$$

$$= X1\ X2\ (1 + C3) + C0\ \overline{X1}\ \overline{X2} + C2\ X1\ \overline{X2}$$

$$= X1\ X2 + C0\ \overline{X1}\ \overline{X2} + C2\ X1\ \overline{X2} \tag{3-12.71}$$

If C0 = 0 and C2 = 1

$$F1 = X1\ X2 + X1\ \overline{X2}$$

$$= X1 \tag{3-12.72}$$

If C0 = 0 and C2 = 0

$$F2 = X1\ X2 \tag{3-12.73}$$

If C0 = 1 and C2 = 0

$$F3 = X1\ X2 + \overline{X1}\ \overline{X2}$$

$$= \overline{X1 \oplus X2} \tag{3-12.74}$$

If C0 = 1 and C2 = 1

$$F4 = X1\ X2 + \overline{X1}\ \overline{X2} + X1\ \overline{X2}$$

$$= X1 + \overline{X2} \tag{3-12.75}$$

In Example 3.4 it was verified that the first three functions are compatible integrals of $d\xi$. The differential of F4 is

$$dF4 = X2\ dX1 + \overline{X1}\ d\overline{X2} \tag{3-12.76}$$

and it is seen that F4 is not an exact integral since its differential contains the term $\overline{X1}\ d\overline{X2}$ which is not contained in $d\xi$.

<u>Example 3.7</u>: Given the differential expression

$$d\xi = \overline{X1}\ dX2 + \overline{X2}\ dX1 \tag{3-12.77}$$

then

$$\int_1 d\xi = \overline{X1}\ X2 + \overline{X2}\ X1 \tag{3-12.78}$$

and

$$\int_0 d\xi = \overline{X1}\ \overline{X2} \tag{3-12.79}$$

Obviously

$$\int_1 d\xi \cdot \int_0 d\xi = 0 \tag{3-12.80}$$

All compatible integrals are of the form

$$F = \int_1 d\xi + \psi \overline{\int_0 d\xi}$$

$$= \overline{X1}\ X2 + X1\ \overline{X2} + \psi\ (X1 + X2) \qquad (3\text{-}12.81)$$

If ψ is restricted to be a function of only X1 and X2 such that

$$\psi = C0\ \overline{X1}\ \overline{X2} + C1\ \overline{X1}\ X2 + C2\ X1\ \overline{X2} + C3\ X1\ X2 \qquad (3\text{-}12.82)$$

it can be shown that

$$F = \overline{X1}\ X2 + X1\ \overline{X2} + C3\ X1\ X2 \qquad (3\text{-}12.83)$$

Letting C3 = 0 gives the first compatible integral,

$$F1 = \overline{X1}\ X2 + X1\ \overline{X2} \qquad (3\text{-}12.84)$$

and letting C3 = 1 gives the second compatible integral,

$$F2 = \overline{X1}\ X2 + X1\ \overline{X2} + X1\ X2$$

$$= X1 + X2 \qquad (3\text{-}12.85)$$

Determining if either of these is an exact integral is accomplished by finding the differential of both functions and comparing the results with $d\xi$. Having done this, it is seen that

$$dF1 = \overline{X2}\ dX1 + \overline{X1}\ dX2 + X2\ d\overline{X1} + X1\ d\overline{X2}$$

$$= d\xi + X2\ d\overline{X1} + X1\ d\overline{X2} \qquad (3\text{-}12.86)$$

and

$$dF2 = \overline{X2}\ dX1 + \overline{X1}\ dX2$$

$$= d\xi \qquad (3\text{-}12.87)$$

Thus F2 is an exact integral of $d\xi$.

3-13. Integration by parts

In this section the concept of Boolean integration by parts will be introduced. It will be shown that all differential expressions of the type that have been previously considered are integrable by parts.

Given a differential expression

$$d\xi = \sum_{i=1}^{n} (\alpha i\ dXi + \beta i\ d\overline{Xi}) \qquad (3\text{-}13.1)$$

where

$$D_{Xi}\ (\alpha i) = D_{Xi}\ (\beta i) = 0 \qquad (3\text{-}13.2)$$

for all i, $1 \leq i \leq n$. Then $d\xi$ is <u>integrable by parts</u> if $d\xi$ can be written as

$$d\xi = \sum_{k=1}^{m} d\xi_k \;, \; m \geq 1 \qquad (3\text{-}13.3)$$

where each of the differential expressions, $d\xi_k$, possesses a compatible integral. Any compatible integral of $d\xi_k$, $1 \leq k \leq m$, will be referred to as a <u>partial integral of $d\xi$</u>. A <u>complete set of partial integrals</u> is a set of functions, {F1,...,Fk,...,Fm}, where for each $d\xi_k$ in equation (3-13.3) there is a function Fk in the set such that

$$dFk \supseteq d\xi_k \qquad (3\text{-}13.4)$$

It will now be shown that all differential expressions given by equation (3-13.1), where neither αi nor βi are functions of Xi, are integrable by parts.

<u>Theorem 3.21</u>: Any differential expression

$$d\xi = \sum_{i=1}^{n} (\alpha i \; dXi + \beta i \; d\overline{Xi}) \qquad (3\text{-}13.5)$$

where for all i, $1 \leq i \leq n$,

$$D_{Xi}\ (\alpha i) = D_{Xi}\ (\beta i) = 0 \qquad (3\text{-}13.6)$$

is integrable by parts.

Proof: First note that

$$\int_1 \alpha i\ dXi \cdot \int_0 \alpha i\ dXi = (\alpha i\ Xi) \cdot (\alpha i\ \overline{Xi})$$

$$= 0 \qquad (3\text{-}13.7)$$

and

$$\int_1 \beta i\ d\overline{Xi} \cdot \int_0 \beta i\ d\overline{Xi} = (\beta i\ \overline{Xi}) \cdot (\beta i\ Xi)$$

$$= 0 \qquad (3\text{-}13.8)$$

Thus by Theorem 3.19 $\alpha i\ dXi$ and $\beta i\ d\overline{Xi}$ possess compatible integrals. Since each term in equation (3-13.5) possesses a compatible integral, $d\xi$ is always integrable by parts.

Integration by parts is illustrated by the following example.

Example 3.8: Consider the differential expression

$$d\xi = \overline{X3}\ dX1 + \overline{X3}\ dX2 + \overline{X2}\ dX3 \qquad (3\text{-}13.9)$$

For this differential expression

$$\int_1 d\xi = \overline{X3}\ X1 + \overline{X3}\ X2 + \overline{X2}\ X3 \tag{3-13.10}$$

and

$$\int_0 d\xi = \overline{X3}\ \overline{X1} + \overline{X3}\ \overline{X2} + \overline{X2}\ \overline{X3}$$

$$= \overline{X1}\ \overline{X3} + \overline{X2}\ \overline{X3} \tag{3-13.11}$$

It can easily be shown that

$$\int_1 d\xi \cdot \int_0 d\xi = X1\ \overline{X2}\ \overline{X3} + \overline{X1}\ X2\ \overline{X3}$$

$$\neq 0 \tag{3-13.12}$$

Thus by Theorem 3.19, $d\xi$ does not possess a compatible integral; however, $d\xi$ is integrable by parts. Numerous complete sets of partial integrals could be obtained, but only two will be given here. First let

$$d\xi = d\xi_1 + d\xi_2 + d\xi_3 \tag{3-13.13}$$

where

$$d\xi_1 = \overline{X3}\ dX1 \tag{3-13.14}$$

$$d\xi_2 = \overline{X3}\ dX2 \tag{3-13.15}$$

and

$$d\xi_3 = \overline{X2}\ dX3 \tag{3-13.16}$$

Thus {F1, F2, F3} is a complete set of partial integrals if F1 is a compatible integral of $d\xi_1$, F2 is a compatible integral of $d\xi_2$, and F3 is a compatible integral of ξ_3. Using the results of the previous section it is seen that one way of satisfying the above conditions is by letting

$$F1 = X1 \tag{3-13.17}$$

$$F2 = X2 \tag{3-13.18}$$

and

$$F3 = X3 \tag{3-13.19}$$

since

$$dX1 \supseteq d\xi_1 \tag{3-13.20}$$

$$dX2 \supseteq d\xi_2 \tag{3-13.21}$$

and

$$dX3 \supseteq d\xi_3 \tag{3-13.22}$$

Therefore, the set $\{X1, X2, X3\}$ is a complete set of partial integrals of $d\xi$. It is also possible to find a complete set of partial integrals using only two partial integrals. Let

$$d\xi = d\xi_1 + d\xi_2 \tag{3-13.23}$$

where

$$d\xi_1 = \overline{X3}\ dX1 \tag{3-13.24}$$

and

$$d\xi_2 = \overline{X3}\ dX2 + \overline{X2}\ dX3 \tag{3-13.25}$$

The simplest compatible integral of $d\xi_1$ is X1, and the simplest compatible integral of $d\xi_2$ is $X2 + X3$ which happens to be an exact integral. Therefore, $\{X1, X2 + X3\}$ is also a complete set of partial integrals of $d\xi$.

Chapter IV

MAP METHODS

4-1. Introduction

In this chapter algorithms will be given for differentiation and integration using Karnaugh maps. When dealing with not more than five or six variables the algorithms are very simple to apply and with practice the process of Boolean differentiation and integration becomes almost as easy and natural as the Karnaugh map method of simplifying switching functions.

It will of course be assumed that all differential expressions considered in this chapter are of the form

$$d\xi = \sum_{i=1}^{n} (\alpha i\ dXi + \beta i\ d\overline{Xi}) \qquad (4\text{-}1.1)$$

where neither αi nor βi is a function of Xi. Even though only completely specified functions are considered, no serious difficulty should be encountered in extending the Karnaugh map techniques to incompletely specified functions.

4-2. Differentiation

In this section a Karnaugh map algorithm will be presented for finding the partial derivatives of a completely specified function. The algorithm is given for obtaining $\partial F/\partial Xi$; the modifications necessary to obtain $\partial F/\partial \overline{Xi}$ are given in parentheses.

Algorithm 4.1: Given F, a completely specified function of n variables, $\partial F/\partial Xi$ (or $\partial F/\partial \overline{Xi}$) may be found as follows:

1. Construct two n-variable Karnaugh maps. Plot the function F on one of the maps. The function $\partial F/\partial Xi$ (or $\partial F/\partial \overline{Xi}$) will be plotted on the other map.

2. On the map of F, if a cell where $Xi = 1$ (or 0) and $F = 1$ is adjacent to another cell where $Xi = 0$ (or 1) and $F = 0$, group these two cells.

3. On the map for $\partial F/\partial Xi$ (or $\partial F/\partial \overline{Xi}$) enter 1's in the cells corresponding to the pair of cells grouped in step 2.

4. Repeat steps 2 and 3 until no new groups can be formed.

5. On the map for $\partial F/\partial Xi$ (or $\partial F/\partial \overline{Xi}$) enter 0's in all cells that do not contain a 1. This map now represents the function $\partial F/\partial Xi$ (or $\partial F/\partial \overline{Xi}$).

Proof: In step 2 the only pairs of cells grouped are those for which a change in Xi will cause the same (or opposite) change in F, and since the two cells in a grouped pair must be adjacent, all variables except Xi are held constant, when change in Xi is constrained within the pair of cells. By Theorem 2.1 (or 2.2) $\partial F/\partial Xi$ (or $\partial F/\partial \overline{Xi}$) will take on a value of 1 at those cells and only those cells that can be grouped in step 2; therefore, in step 3 on the map for $\partial F/\partial Xi$ (or $\partial F/\partial \overline{Xi}$) 1's are entered in the cells corresponding to the pair of cells grouped in step 2. Step 4 insures that on the map for $\partial F/\partial Xi$ (or $\partial F/\partial \overline{Xi}$) 1's are entered in all appropriate cells. After completing step 4 only those cells for which a change in Xi does not cause the same (or opposite) change in F are left blank. Thus by Thoerem 2.1 (or 2.2), in step 5 0's are entered in all cells that do not contain a 1.

The application of Algorithm 4.1 is illustrated by the following example

Example 4.1: Given the function

$$F = \overline{X1}\ X3 + X1\ \overline{X4} + X1\ X2\ \overline{X3} \qquad (4\text{-}2.1)$$

Algorithm 4.1 can be used to find the various partial derivatives. Figure 2 shows the various steps in obtaining $\partial F/\partial X1$, and Figure 3 shows the results of applying the algorithm to find $\partial F/\partial X2$, $\partial F/\partial X3$, and $\partial F/\partial X4$. The various steps for obtaining $\partial F/\partial \overline{X1}$ are shown in Figure 4, and Figure 5 shows the results for obtaining $\partial F/\partial \overline{X2}$, $\partial F/\partial \overline{X3}$, and $\partial F/\partial \overline{X4}$. From Figure 2 through Figure 5 it follows that

$$\begin{aligned} dF = {} & (X2\ \overline{X3} + \overline{X3}\ \overline{X4})\ dX1 + X3\ X4\ d\overline{X1} \\ & + X1\ \overline{X3}\ X4\ dX2 + \overline{X1}\ dX3 + X1\ X2\ X4\ d\overline{X3} \\ & + (X1\ \overline{X2} + X1\ X3)\ d\overline{X4} \end{aligned} \qquad (4\text{-}2.2)$$

4-3. Exact Integrals

An algorithm will now be given for obtaining the exact integral of a differential expression. This algorithm will always give the

F

X3 X4 \ X1 X2	00	01	11	10
00	0	0	1	1
01	0	0	1	0
11	1	1	0	0
10	1	1	1	1

∂F/∂X1

X3 X4 \ X1 X2	00	01	11	10
00				
01				
11				
10				

(a) Result after applying step 1.

F

X3 X4 \ X1 X2	00	01	11	10
00	0	0	1	1
01	0	0	1	0
11	1	1	0	0
10	1	1	1	1

∂F/∂X1

X3 X4 \ X1 X2	00	01	11	10
00	1	1	1	1
01		1	1	
11				
10				

(b) Result after applying steps 2, 3, and 4.

F

X3 X4 \ X1 X2	00	01	11	10
00	0	0	1	1
01	0	0	1	0
11	1	1	0	0
10	1	1	1	1

∂F/∂X1

X3 X4 \ X1 X2	00	01	11	10
00	1	1	1	1
01	0	1	1	0
11	0	0	0	0
10	0	0	0	0

(c) Final result showing that $\partial F/\partial X1 = X2\ \overline{X3} + \overline{X3}\ \overline{X4}$

Figure 2. - Maps illustrating the application of Algorithm 4.1 to find $\partial F/\partial X1$ for $F = \overline{X1}\ X3 + X1\ \overline{X4} + X1\ X2\ \overline{X3}$

F

X3 X4 \ X1 X2	00	01	11	10
00	0	0	1	1
01	0	0	1	0
11	1	1	0	0
10	1	1	1	1

∂F/∂X2

X3 X4 \ X1 X2	00	01	11	10
00	0	0	0	0
01	0	0	1	1
11	0	0	0	0
10	0	0	0	0

(a) Result showing that $\partial F/\partial X2 = X1\ \overline{X3}\ X4$

F

X3 X4 \ X1 X2	00	01	11	10
00	0	0	1	1
01	0	0	1	0
11	1	1	0	0
10	1	1	1	1

∂F/∂X3

X3 X4 \ X1 X2	00	01	11	10
00	1	1	0	0
01	1	1	0	0
11	1	1	0	0
10	1	1	0	0

(b) Result showing that $\partial F/\partial X3 = \overline{X1}$

F

X3 X4 \ X1 X2	00	01	11	10
00	0	0	1	1
01	0	0	1	0
11	1	1	0	0
10	1	1	1	1

∂F/∂X4

X3 X4 \ X1 X2	00	01	11	10
00	0	0	0	0
01	0	0	0	0
11	0	0	0	0
10	0	0	0	0

(c) Result showing that $\partial F/\partial X4 = 0$

Figure 3. - Maps illustrating the application of Algorithm 4.1 to find $\partial F/\partial X2$, $\partial F/\partial X3$, and $\partial F/\partial X4$ for $F = \overline{X1}\ X3 + X1\ \overline{X4} + X1\ X2\ \overline{X3}$

F

X3 X4 \ X1 X2	00	01	11	10
00	0	0	1	1
01	0	0	1	0
11	1	1	0	0
10	1	1	1	1

$\partial F/\partial \overline{X1}$

X3 X4 \ X1 X2	00	01	11	10
00				
01				
11				
10				

(a) Result after applying step 1.

F

X3 X4 \ X1 X2	00	01	11	10
00	0	0	1	1
01	0	0	1	0
11	1	1	0	0
10	1	1	1	1

$\partial F/\partial \overline{X1}$

X3 X4 \ X1 X2	00	01	11	10
00				
01				
11	1	1	1	1
10				

(b) Result after applying steps 2, 3, and 4.

F

X3 X4 \ X1 X2	00	01	11	10
00	0	0	1	1
01	0	0	1	0
11	1	1	0	0
10	1	1	1	1

$\partial F/\partial \overline{X1}$

X3 X4 \ X1 X2	00	01	11	10
00	0	0	0	0
01	0	0	0	0
11	1	1	1	1
10	0	0	0	0

(c) Final result showing that $\partial F/\partial \overline{X1} = X3\ X4$

Figure 4. - Maps illustrating the application of Algorithm 4.1 to find $\partial F/\partial \overline{X1}$ for $F = \overline{X1}\ X3 + X1\ \overline{X4} + X1\ X2\ \overline{X3}$

F

X3 X4 \ X1 X2	00	01	11	10
00	0	0	1	1
01	0	0	1	0
11	1	1	0	0
10	1	1	1	1

$\partial F/\partial \overline{X2}$

X3 X4 \ X1 X2	00	01	11	10
00	0	0	0	0
01	0	0	0	0
11	0	0	0	0
10	0	0	0	0

(a) Result showing that $\partial F/\partial \overline{X2} = 0$

F

X3 X4 \ X1 X2	00	01	11	10
00	0	0	1	1
01	0	0	1	0
11	1	1	0	0
10	1	1	1	1

$\partial F/\partial \overline{X3}$

X3 X4 \ X1 X2	00	01	11	10
00	0	0	0	0
01	0	0	1	0
11	0	0	1	0
10	0	0	0	0

(b) Result showing that $\partial F/\partial \overline{X3} = X1\ X2\ X4$

F

X3 X4 \ X1 X2	00	01	11	10
00	0	0	1	1
01	0	0	1	0
11	1	1	0	0
10	1	1	1	1

$\partial F/\partial \overline{X4}$

X3 X4 \ X1 X2	00	01	11	10
00	0	0	0	1
01	0	0	0	1
11	0	0	1	1
10	0	0	1	1

(c) Result showing that $\partial F/\partial \overline{X4} = X1\ \overline{X2} + X1\ X3$

Figure 5. - Maps illustrating the application of Algorithm 4.1 to find $\partial F/\partial \overline{X2}$, $\partial F/\partial \overline{X3}$, and $\partial F/\partial \overline{X4}$ for $F = \overline{X1}\ X3 + X1\ \overline{X4} + X1\ X2\ \overline{X3}$

exact integral provided it exists. If an exact integral does not exist a contradiction may or may not be encountered when applying the algorithm; therefore, it is important to verify that the function obtained is indeed an exact integral. This can be done by comparing the differential of the function with the original differential expression. If they are the same then the function is an exact integral.

Algorithm 4.2: Given a nonzero differential expression $d\xi$ with no constraints the exact integral, provided it exists, can be found as follows:

1. Construct a Karnaugh map on which the exact integral will be plotted.

2. Enter 1's in all cells whose coordinates cause the function $\int_1 d\xi$ to take on a value of 1.

3. Enter 0's in all cells whose coordinates cause the function $\int_0 d\xi$ to take on a value of 1. If this requires that both a 1 and a 0 be entered in the same cell an exact integral does not exist.

4. Enter 1's in all blank cells adjacent to a cell already containing a 1. Enter 0's in all blank cells adjacent to a cell already containing a 0. If this requires that both a 1 and a 0 be entered in the same cell an exact integral does not exist.

5. Repeat step 4 until there are no blank cells on the map. The function obtained is an exact integral provided an exact integral exists.

Proof: This algorithm follows directly from Theorem 3.1. Let F be the exact integral of $d\xi$ then be definition of exact integral

$$d\xi = dF \qquad (4\text{-}3.1)$$

Thus from Theorem 3.1(a), F has a value of 1 whenever $\int_1 dF$ takes on a value of 1; therefore, step 2 is justified. Likewise, from Theorem 3.1(b) F has a value of 0 whenever $\int_0 dF$ takes on a value of 1; thus, step 3 is justified. Steps 4 and 5 follow from Theorem 3.1(c).

Several examples will now be given to illustrate the application of the above algorithm.

Example 4.2: Consider the differential obtained in example 4.1. Since

$$dF = (X2\ \overline{X3} + \overline{X3}\ \overline{X4})\ dX1 + X3\ X4\ d\overline{X1}$$
$$+ X1\ \overline{X3}\ X4\ dX2 + \overline{X1}\ dX3 + X1\ X2\ X4\ d\overline{X3}$$
$$+ (X1\ \overline{X2} + X1\ X3)\ d\overline{X4} \qquad (4\text{-}3.2)$$

it follows that

$$\int_1 dF = (X2\ \overline{X3} + \overline{X3}\ \overline{X4})\ X1 + X3\ X4\ \overline{X1}$$
$$+ X1\ \overline{X3}\ X4\ X2 + \overline{X1}\ X3 + X1\ X2\ X4\ \overline{X3}$$
$$+ (X1\ \overline{X2} + X1\ X3)\ \overline{X4}$$
$$= \overline{X1}\ X3 + X1\ \overline{X4} + X1\ X2\ \overline{X3} \qquad (4\text{-}3.3)$$

and

$$\int_0 dF = (X2\ \overline{X3} + \overline{X3}\ \overline{X4})\ \overline{X1} + X3\ X4\ X1$$

$$+ X1\ \overline{X3}\ X4\ \overline{X2} + \overline{X1}\ \overline{X3} + X1\ X2\ X4\ X3$$

$$+ (X1\ \overline{X2} + X1\ X3)\ X4$$

$$= \overline{X1}\ \overline{X3} + X1\ X3\ X4 + X1\ \overline{X2}\ X4 \qquad (4\text{-}3.4)$$

The results of applying steps 1, 2, and 3 of Algorithm 4.2 are shown in Figure 6. There are no blank cells on the map for F; therefore, steps 4 and 5 can be omitted. It is seen that the exact integral of dF is the original function given in example 4.1.

Example 4.3: Consider the differential expression

$$d\xi = (X3\ \overline{X4} + \overline{X2}\ X3)\ dX1 + \overline{X1}\ X3\ X4\ dX2$$

$$+ (\overline{X1}\ \overline{X2} + \overline{X1}\ \overline{X4})\ d\overline{X3} + \overline{X1}\ X2\ X3\ dX4 \qquad (4\text{-}3.5)$$

It is seen that

$$\int_1 d\xi = (X3\ \overline{X4} + \overline{X2}\ X3)\ X1 + \overline{X1}\ X3\ X4\ X2$$

$$+ (\overline{X1}\ \overline{X2} + \overline{X1}\ \overline{X4})\ \overline{X3} + \overline{X1}\ X2\ X3\ X4$$

$$= X1\ X3\ \overline{X4} + \overline{X1}\ \overline{X2}\ \overline{X3} + \overline{X1}\ \overline{X3}\ \overline{X4}$$

$$+ \overline{X1}\ X2\ X3\ X4 + X1\ \overline{X2}\ X3 \qquad (4\text{-}3.6)$$

X3 X4 \ X1 X2	00	01	11	10
00			1	1
01			1	
11	1	1		
10	1	1	1	1

F

(a) Result after applying steps 1 and 2.

X3 X4 \ X1 X2	00	01	11	10
00	0	0	1	1
01	0	0	1	0
11	1	1	0	0
10	1	1	1	1

F

(b) Result after applying step 3 showing that the exact integral is $F = \overline{X1}\ X3 + X1\ \overline{X4} + X1\ X2\ \overline{X3}$

Figure 6. - Maps illustrating the application of Algorithm 4.2 to the differential expression given by equation (4-3.2).

and

$$\int_0 d\xi = (X3\ \overline{X4} + \overline{X2}\ X3)\ \overline{X1} + \overline{X1}\ X3\ X4\ \overline{X2}$$

$$+ (\overline{X1}\ \overline{X2} + \overline{X1}\ \overline{X4})\ X3 + \overline{X1}\ X2\ X3\ \overline{X4}$$

$$= \overline{X1}\ \overline{X2}\ X3 + \overline{X1}\ X3\ \overline{X4} \qquad (4\text{-}3.7)$$

Figure 7 shows the use of Algorithm 4.2 to find a function F such that

$$dF = d\xi \qquad (4\text{-}3.8)$$

The results of applying steps 1, 2, and 3 are shown in Figure 7(a). It is seen that several cells are blank. The result of the first application of step 4 is shown in Figure 7(b) where 1's have been entered in all cells adjacent to a 1. There is only one blank cell in Figure 7(b) and it is adjacent to other cells all of which contain a 1; therefore, a 1 is entered in this cell as shown in Figure 7(c) and it is seen that

$$F = X1 + \overline{X3} + X2\ X4 \qquad (4\text{-}3.9)$$

If an exact integral of $d\xi$ exists it is given by equation (4-3.9) To verify that F is an exact integral of $d\xi$, Algorithm 4.1 can be used to obtain dF. If this is done it is seen that

F

X3 X4 \ X1 X2	00	01	11	10
00	1	1		
01	1			
11	0	1		1
10	0	0	1	1

(a) Result after applying steps 1, 2, and 3.

F

X3 X4 \ X1 X2	00	01	11	10
00	1	1	1	1
01	1	1		1
11	0	1	1	1
10	0	0	1	1

(b) Result after the first application of step 4.

F

X3 X4 \ X1 X2	00	01	11	10
00	1	1	1	1
01	1	1	1	1
11	0	1	1	1
10	0	0	1	1

(c) Final result showing that the exact integral is
$F = X1 + \overline{X3} + X2\ X4$

Figure 7. - Maps illustrating the application of Algorithm 4.2 to the differential expression given by equation (4-3.5)

$$dF = d\xi \tag{4-3.10}$$

therefore, the function given by equation (4-3.9) is indeed an exact integral of $d\xi$.

For both of the above examples, an exact integral actually does exist. In the next three examples this is not the case. In Examples 4.4 and 4.5 the algorithm leads to a clear contradiction, and it is apparent that an exact integral does not exist. In Example 4.6 no contradiction occurs, yet an exact integral does not exist.

Example 4.4: Consider the differential expression

$$d\xi = X2\ dX1 + \overline{X1}\ d\overline{X2} + \overline{X1}\ \overline{X2}\ dX3 \tag{4-3.11}$$

It is seen that

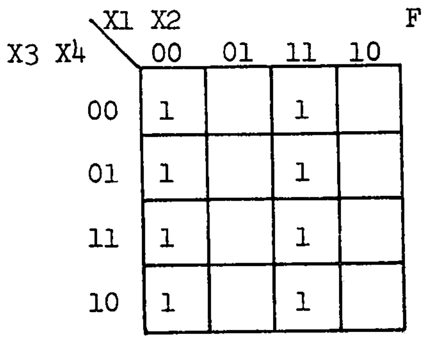
$$\int_1 d\xi = X2\ X1 + \overline{X1}\ \overline{X2} + \overline{X1}\ \overline{X2}\ X3$$

$$= X1\ X2 + \overline{X1}\ \overline{X2} \tag{4-3.12}$$

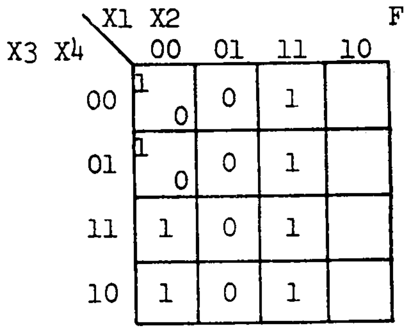
and

$$\int_0 d\xi = X2\ \overline{X1} + \overline{X1}\ X2 + \overline{X1}\ \overline{X2}\ \overline{X3}$$

$$= \overline{X1}\ X2 + \overline{X1}\ \overline{X2}\ \overline{X3} \tag{4-3.13}$$



(a) Result after applying steps 1 and 2.



(b) Result after applying step 3 showing the contradiction in the cells where X1 = 0, X2 = 0, and X3 = 0.

Figure 8. - Maps illustrating the application of Algorithm 4.2 to the differential expression given by equation (4-3.11)

As shown in Figure 8 a contradiction results since step 2 requires that the cells where X1 = 0, X2 = 0, and X3 = 0 contain a 1 while step 3 requires that the same cells contain a 0. This contradiction arises because

$$\int_1 d\xi \cdot \int_0 d\xi \neq 0 \qquad (4\text{-}3.14)$$

thus $d\xi$ does not even posses a compatible integral.

Example 4.5: Consider the differential expression

$$d\xi = X2\ X3\ dX1 + X1\ X3\ dX2 + \overline{X1}\ \overline{X3}\ d\overline{X2} + \overline{X2}\ d\overline{X3} \qquad (4\text{-}3.15)$$

It is seen that

$$\int_1 d\xi = X2\ X3\ X1 + X1\ X3\ X2 + \overline{X1}\ \overline{X3}\ \overline{X2} + \overline{X2}\ \overline{X3}$$

$$= \overline{X2}\ \overline{X3} + X1\ X2\ X3 \qquad (4\text{-}3.16)$$

and

$$\int_0 d\xi = X2\ X3\ \overline{X1} + X1\ X3\ \overline{X2} + \overline{X1}\ \overline{X3}\ X2 + \overline{X2}\ X3$$

$$= \overline{X1}\ X2 + \overline{X2}\ X3 \qquad (4\text{-}3.17)$$

F

X3 \ X1 X2	00	01	11	10
0	1	0		1
1	0	0	1	0

(a) Result after applying steps 1 and 2.

F

X3 \ X1 X2	00	01	11	10
0	1	0	1 0	1
1	0	0	1	0

(b) Result after applying step 4 showing that a contradiction occurs in the cell where X1 = 1, X2 = 1, and X3 = 0.

Figure 9. - Maps illustrating the application of Algorithm 4.2 to the differential expression given by equation (4-3.15)

As shown in Figure 9 no contradiction is encountered in applying steps 2 and 3; however, difficulty arises when attempting to apply step 4 since the only blank cell is adjacent to both a 1 and a 0. It is thus concluded that an exact integral does not exist for the differential expression given by equation (4-3.15).

Example 4.6: Consider the differential expression

$$d\xi = X2\ X3\ dX1 + X1\ X3\ dX2 + \overline{X3}\ d\overline{X2} + \overline{X2}\ d\overline{X3} \qquad (4\text{-}3.18)$$

It is seen that

$$\int_1 d\xi = X2\ X3\ X1 + X1\ X3\ X2 + \overline{X3}\ \overline{X2} + \overline{X2}\ \overline{X3}$$

$$= \overline{X2}\ \overline{X3} + X1\ X2\ X3 \qquad (4\text{-}3.19)$$

and

$$\int_0 d\xi = X2\ X3\ \overline{X1} + X1\ X3\ \overline{X2} + \overline{X3}\ X2 + \overline{X2}\ X3$$

$$= \overline{X1}\ X3 + \overline{X2}\ X3 + X2\ \overline{X3} \qquad (4\text{-}3.20)$$

As shown in Figure 10 no contradiction is encountered in applying Algorithm 4.2; therefore, if an exact integral exists it is given by

$$F = \overline{X2}\ \overline{X3} + X1\ X2\ X3 \qquad (4\text{-}3.21)$$

X3 \ X1 X2	00	01	11	10
0	1			1
1			1	

F

(a) Result after applying steps 1 and 2.

X3 \ X1 X2	00	01	11	10
0	1	0	0	1
1	0	0	1	0

F

(b) Result after applying step 3 showing that if an exact integral exists it is given by $F = \overline{X2}\ \overline{X3} + X1\ X2\ X3$

Figure 10. - Maps illustrating the application of Algorithm 4.2 to the differential expression given by equation (4-3.18)

However, the differential of the above function is

$$dF = X2\ X3\ dX1 + X1\ X3\ dX2 + \overline{X3}\ d\overline{X2}$$

$$+ X1\ X2\ dX3 + \overline{X2}\ d\overline{X3}$$

$$= d\xi + X1\ X2\ dX3$$

$$\neq d\xi \qquad (4\text{-}3.22)$$

Since dF includes the term X1 X2 dX3 not contained in $d\xi$, the function F is a compatible integral but not an exact integral. It is thus concluded that the differential expression given by equation (4-3.18) does not possess an exact integral.

4-4. Compatible Integrals

In this section an algorithm is given for obtaining compatible integrals of a differential expression. The algorithm is very similar to Algorithm 4.2, but it has the advantage that if a compatible integral does not exist a contradiction always occurs.

Algorithm 4.3: Given a differential expression $d\xi$ with no constraints, the compatible integrals may be found as follows:

1. Construct a Karnaugh map on which the compatible integral will be plotted.

2. Enter 1's in all cells whose coordinates cause the function $\int_1 d\xi$ to take on a value of 1.

3. Enter 0's in all cells whose coordinates cause the function $\int_0 d\xi$ to take on a value of 1. If there is at least one cell where the value of $\int_1 d\xi$ and $\int_0 d\xi$ are 1, steps 2 and 3 lead to a contradictory assignment of value in that cell and a compatible integral does not exist. If there is no such cell, proceed to step 4.

4. Any function that can be selected by treating each blank cell as a don't-care condition is a compatible integral.

Proof: From Theorem 3.20, F is a compatible integral of $d\xi$ if and only if

$$F = \int_1 d\xi + \psi \overline{\int_0 d\xi} \qquad (4\text{-}4.1)$$

where ψ is an arbitrary function. Obviously at any point $\underline{\bar{A}}$ such that

$$\int_1 d\xi \Big|_{\underline{\bar{A}}} = 1 \qquad (4\text{-}4.2)$$

then

$$F\Big|_{\underline{\bar{A}}} = 1 \qquad (4\text{-}4.3)$$

Thus, justifying step 2. At any point $\underline{\bar{B}}$ such that

$$\int_0 d\xi \Big|_{\underline{\bar{B}}} = 1 \qquad (4\text{-}4.4)$$

then from the above equation and equation (4-4.1)

$$F\Big|_{\underline{\bar{B}}} = \int_1 d\xi \Big|_{\underline{\bar{B}}} \qquad (4\text{-}4.5)$$

However, from Theorem 3.19 if a compatible integral exists then

$$\int_1 d\xi \cdot \int_0 d\xi = 0 \qquad (4\text{-}4.6)$$

Thus if a compatible integral exists, for all points $\underline{\bar{B}}$ satisfying equation (4-4.4)

$$\int_1 d\xi \Big|_{\underline{\bar{B}}} = 0 \qquad (4\text{-}4.7)$$

and from equation (4-4.5)

$$F\Big|_{\underline{\bar{B}}} = 0 \qquad (4\text{-}4.8)$$

Therefore, step 3 is valid provided a compatible integral exists. If there exists a cell $\underline{\bar{C}}$ where

$$\int_1 d\xi \Big|_{\overline{\underline{C}}} = 1 \tag{4-4.9}$$

and

$$\int_0 d\xi \Big|_{\overline{\underline{C}}} = 1 \tag{4-4.10}$$

then

$$\int_1 d\xi \cdot \int_0 d\xi \neq 0 \tag{4-4.11}$$

and by Theorem 3.19 a compatible integral does not exist. However, if no such cell $\overline{\underline{C}}$ exists, then

$$\int_1 d\xi \cdot \int_0 d\xi = 0 \tag{4-4.12}$$

Therefore, by Theorem 3.19 a compatible integral does exist. Step 4 follows immediately from equation (4-4.1). Observe that cells $\overline{\underline{D}}$ where

$$\int_1 d\xi \Big|_{\overline{\underline{D}}} = 0 \tag{4-4.13}$$

and

$$\int_0 d\xi|_{\overline{\underline{D}}} = 0 \qquad (4\text{-}4.14)$$

are left blank in the map of F. Hence, from equation (4-4.1)

$$F|_{\overline{\underline{D}}} = \psi|_{\overline{\underline{D}}} \qquad (4\text{-}4.15)$$

However, ψ is completely arbitrary so that F may have a value of either 0 or 1 at point $\overline{\underline{D}}$ and still be a compatible integral.

The following examples illustrate the application of Algorithm 4.3.

<u>Example 4.7</u>: Consider the differential expression

$$d\xi = X2\ X3\ dX1 + X1\ X3\ dX2 + X1\ X2\ dX3 \qquad (4\text{-}4.16)$$

It is seen that

$$\int_1 d\xi = X1\ X2\ X3 \qquad (4\text{-}4.17)$$

and

$$\int_0 d\xi = X2\ X3\ \overline{X1} + X1\ X3\ \overline{X2} + X1\ X2\ \overline{X3} \qquad (4\text{-}4.18)$$

X3 \ X1 X2	00	01	11	10
0				
1			1	

F

(a) Result after applying steps 1 and 2.

X3 \ X1 X2	00	01	11	10
0			0	
1		0	1	0

F

(b) Result after applying step 3.

X3 \ X1 X2	00	01	11	10
0	x	x	0	x
1	x	0	1	0

F

(c) Result after applying step 4 showing the don't-care conditions.

Figure 11. - Maps illustrating the application of Algorithm 4.3 to the differential expression given by equation (4-4.16)

The application of Algorithm 4.3 to the above differential expression is shown in Figure 11. It is seen that no contradiction occurs in applying steps 2 and 3. Thus, a compatible integral exists and as shown in Figure 11(c) there are several don't-care conditions indicated by X's on the map of F. From Figure 11(c) it is seen that the simplest choice for F is

$$F = X1\ X2\ X3 \tag{4-4.19}$$

which is an exact integral.

<u>Example 4.8</u>: Consider the differential expression

$$d\xi = \overline{X3}\ \overline{X4}\ dX1 + X3\ X4\ dX2 \tag{4-4.20}$$

Then

$$\int_1 d\xi = \overline{X3}\ \overline{X4}\ X1 + X3\ X4\ X2 \tag{4-4.21}$$

and

$$\int_0 d\xi = \overline{X3}\ \overline{X4}\ \overline{X1} + X3\ X4\ \overline{X2} \tag{4-4.22}$$

As shown in Figure 12(a), no difficulty is encountered in applying steps 1, 2, and 3 of Algorithm 4.3. As shown in Figure 12(b), there

F

X3 X4 \ X1 X2	00	01	11	10
00	0	0	1	1
01				
11	0	1	1	0
10				

(a) Result after applying steps 1, 2, and 3.

F

X3 X4 \ X1 X2	00	01	11	10
00	0	0	1	1
01	x	x	x	x
11	0	1	1	0
10	x	x	x	x

(b) Result after applying step 4.

Figure 12. - Maps illustrating the application of Algorithm 4.3 to the differential expression given by equation (4-4.20)

are several don't-care conditions, and no simplest compatible integral. For example

$$F1 = X1\ \overline{X3} + X2\ X4 \qquad (4\text{-}4.23)$$

$$F2 = X1\ \overline{X3} + X2\ X3 \qquad (4\text{-}4.24)$$

$$F3 = X1\ \overline{X4} + X2\ X3 \qquad (4\text{-}4.25)$$

and

$$F4 = X1\ \overline{X4} + X2\ X4 \qquad (4\text{-}4.26)$$

are all compatible integrals of equal complexity; however, depending on the application one might be preferred over the others.

Example 4.9: For the differential expression

$$d\xi = dX1 + dX2 + dX3 + dX4 \qquad (4\text{-}4.27)$$

it follows that

$$\int_1 d\xi = X1 + X2 + X3 + X4 \qquad (4\text{-}4.28)$$

and

$$\int_0 d\xi = \overline{X1} + \overline{X2} + \overline{X3} + \overline{X4} \qquad (4\text{-}4.29)$$

If Algorithm 4.3 is applied to $d\xi$ it is found that step 3 requires 0's to be entered in several cells that already contain a 1. Thus, a compatible integral does not exist for the differential expression given by equation (4-4.27).

Chapter V

SUMMARY AND CONCLUSION

5-1. Summary

In this thesis the mathematical properties of a transition calculus for switching functions are developed under the assumption that only one variable changes at a time. Both differentiation and integration are considered.

Chapter II is concerned with the differential portion of the transition calculus. Boolean partial derivatives are introduced for determining the conditions under which a change in a variable will cause the same or opposite change in a function. The partial derivatives are used to define the Boolean differential. The Boolean differential provides a convenient and concise method for describing the effect on a function of changes in its variables. After deriving a number of basic identities involving the partial derivatives, the relationship between the partial derivatives and the Boolean difference, which has been used extensively for fault diagnosis, is established. It is shown that the Boolean difference can be easily found from the partial derivatives. General expressions for the partial derivatives are determined, which show that the partial derivatives do not distribute over the AND, OR, or EXCLUSIVE OR operations. This property unfortunately complicates the direct algebraic evaluation of the partial derivatives; however, simple expressions are obtained for the partial derivatives in a number of special cases. It is shown that the Boolean differential can be related to the oriented difference operators in such a way that the methodology developed in this thesis provides a tool for applications using these operators.

Chapter III is concerned with Boolean integration. In order to facilitate the development of Boolean integration, certain integral operators are introduced, and their properties established. It is shown that any nonconstant switching function is uniquely determined by its differential. Given the differential of a function, it is possible to find that function; however, given an arbitrary differential expression, a function with that differential may not exist. If such a function does exist, it is called the exact integral of the differential expression. Even when an exact integral does not exist, it is often possible to find a compatible integral, which is a function that undergoes all transitions specified by the given differential expression. Even though a differential expression possesses at most one exact integral, it may possess several compatible integrals. A simple necessary and sufficient condition for the existence of a compatible integral is obtained, and methods for computing both exact and compatible integrals are given. It is also shown that any differential expression of the proper form can always be broken into parts such that each of these parts possesses a compatible integral. This process is referred to as integration by parts.

In Chapter IV Karnaugh map algorithms are developed for both differentiation and integration. The methods are given only for completely specified functions, but can be extended to include incompletely specified ones. It is shown that Karnaugh maps can be used to find the partial derivatives with which the Boolean differential is obtained. The algorithm for finding an exact integral assumes that an exact integral exists. It is thus necessary to differentiate the function obtained with this algorithm to verify that it is, indeed, an exact integral of the given differential expression. If the differential of the function is not the same as the original differential expression, then no exact integral exists. The algorithm for obtaining compatible integrals produces all compatible integrals that a given differential expression possesses. If the differential expression does not possess a compatible integral, this is indicated by the algorithm.

5-2. Applications

In this section a brief general discussion of applications for the transition calculus will be given. From the simple relationship expressed by equation (2-5.3) the partial derivatives could obviously be used for all applications for which the Boolean difference is used. However, the transition calculus should not be considered a technique which competes with the Boolean difference for the same class of applications. Rather the methodology developed in this thesis and the work that has been done in the area of Boolean difference closely complement each other. In fact, considerable use has been made of the Boolean difference in the development of the transition calculus. However, it must be noted that the partial derivatives cannot be obtained from the Boolean difference alone. Additional information about the function must be provided. For this reason there are applications for the transition calculus that cannot be adequately handled by the Boolean difference.

For applications such as fault diagnosis and hazard detection where the function is known, either the transition calculus or Boolean difference can be used. If the direction in which the function changes is of no consequence (such as in the case of stuck at 1 and stuck at 0 type errors), the Boolean difference is entirely adequate. However, if the direction in which the function changes is important (such as in the case of inversion type errors), it is probably more appropriate to use the partial derivatives.

For applications in which a function with known transitions is to be determined, the integration techniques developed in this thesis can be used. One of the most outstanding examples of such an application is the synthesis of asynchronous sequential networks using edge-sensitive flip-flops. It is desirable to synthesize such circuits using standard, commercially available flip-flops which have only a single clock input. The procedure for accomplishing this involves writing a differential expression describing the transitions each clock input must undergo, and then finding an appropriate compatible integral for each of these differential expressions.

5-3. Areas for Further Development

There are a number of areas in which further research and development of the transition calculus is desirable. Additional work is needed to investigate multiple simultaneous transitions, and general expressions are also needed for obtaining the differential of a function of functions. Circuits with input constraints and sequential circuits are important applications and extension of the transition calculus in these areas should prove to be very fruitful. Obtaining simple necessary and sufficient conditions for the existence of an exact integral is a challenging problem. The development of a computationally simpler algebraic method for exact integration is also an interesting problem.

5-4. Conclusion

The transition calculus is a powerful tool that is useful when dealing with input changes and their effect on logic circuits. The Boolean differential, the oriented difference operators, and the Boolean difference can all be defined in terms of the Boolean partial derivatives introduced in this thesis. Thus, all of these concepts are fundamentally related and can be used in conjunction with one another. The Boolean differential provides an effective method for describing how a switching function is affected by changes in its variables, and it is shown that a nonconstant function is uniquely determined by its Boolean differential. The concept of the exact integral provides a method for finding a function when its differential is known. Even when a function does not exist with a differential equal to a given differential expression, it may still be possible to find a compatible integral. The compatible integral is very important for applications such as the synthesis of sequential circuits using edge sensitive flip flops. The necessary and sufficient condition for determining if a differential expression possesses a compatible integral is given. A differential expression that does not possess either an exact integral or a compatible integral can still be integrated by parts and this concept also has potential applications. The Karnaugh map algorithms provide simple and easy methods for differentiation and integration and with practice both of these processes become quite natural.

The transition calculus has been developed with the restriction that only one variable changes at a time and that all functions are completely specified. A similar development could obviously be made allowing multiple simultaneous changes in the variables; however, there would naturally be a corresponding increase in the complexity. Incompletely specified circuits can be handled with slight modifications of the Karnaugh map methods that have been presented in Chapter IV.

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A TRANSITION CALCULUS FOR
BOOLEAN FUNCTIONS

BY

JERRY H. TUCKER

(ABSTRACT)

A transition calculus is developed for describing and analyzing the dynamic behavior of logic circuits. Boolean partial derivatives are introduced that are more powerful and applicable to a wider class of problems than the Boolean difference. The partial derivatives are used to define a Boolean differential which provides a concise method for describing the effect on a switching function of changes in its variables. It is shown that a nonconstant function is uniquely determined by its differential, and integration techniques are developed for finding a function when its differential is known. The useful concepts of exact integrals, compatible integrals, and integration by parts are introduced and the conditions for their existence are established. Algorithms for both differentiation and integration are simply implemented using Karnaugh maps.