Spin and Rotations in Galois Field Quantum Mechanics

Lay Nam Chang, Zachary Lewis, Djordje Minic and Tatsu Takeuchi

Department of Physics, Virginia Tech, Blacksburg, VA 24061, USA

E-mail: `laynam@vt.edu`,`zlewis@vt.edu`,`dminic@vt.edu`,`takeuchi@vt.edu`

Abstract. We discuss the properties of Galois Field Quantum Mechanics constructed on a vector space over the finite Galois field $GF(q)$. In particular, we look at 2-level systems analogous to spin, and discuss how $SO(3)$ rotations could be embodied in such a system. We also consider two-particle 'spin' correlations and show that the Clauser-Horne-Shimony-Holt (CHSH) inequality is nonetheless not violated in this model.

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1. Introduction

Correlations of physical properties between states provide the cleanest differentiation between quantum descriptions and classical ones. By quantum descriptions we mean those in which entangled states have correlations that cannot be reproduced by ascribing to the physical properties distinctive and unchanging values. By classical descriptions we mean those wherein such assignments are explicit, or through hidden variables. The best known of this distinction is encoded in the super-classical descriptions of spins in canonical quantum mechanics (QM), as exemplified by the celebrated Bell's inequalities [1].

In the formulation by Clauser, Horne, Shimony, and Holt (CHSH) [2], the observables A_α and B_β, associated with the spins of two spacelike separated particles and the subscripts indicating detector settings, are assumed to yield either one of the two numbers ± 1 upon measurement. The combination of correlators given by

$$\langle A_1, A_2; B_1, B_2 \rangle \equiv \langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle - \langle A_2 B_2 \rangle \,, \tag{1}$$

can be shown to have its absolute value bounded by 2 for classical hidden variable theory, that is:

$$\left| \langle A_1, A_2; B_1, B_2 \rangle \right| \leq 2 \,, \tag{2}$$

for arbitrary pairs of detector settings. This inequality is violated by correlations in canonical QM in which the upper bound is replaced by $2\sqrt{2}$ [3, 4]. In the present context, by canonical QM we mean a description based upon a Hilbert space defined over the complex number field. Observables have correspondence with hermitian operators on the space, which do not necessarily commute with each other.

However, we demonstrate in [5] that canonical QM is not a unique quantum description. There, variants of canonical QM are constructed on vector spaces over the finite Galois field $GF(q)$. Both canonical QM and our variant quantum description make crucial use of the underlying linear properties, and the CHSH bound for both is predicated upon properties of the vector space. In [4] we point out the importance of the inner product in deriving the conventional CHSH bound. The vector spaces on which our variants are constructed do not support an inner product because of the cyclicity of Galois fields. Consequently, the CHSH bound of our variant differs from the canonical value: we find that it is 2, the classical hidden variable value. Nevertheless, we show that the resultant range of correlations also cannot be reproduced in any classical context with hidden variables [5]. The system is therefore quantum in the sense described above, despite the CHSH bound remaining at 2.

The use of Galois fields in physics is not extensive [6]. In this paper, we elaborate on our model presented in [5] to better provide an intuitive feel for how the properties of Galois fields are utilized. Details of implementation using concrete examples are presented, as well as the full derivation of the CHSH bound for 'spin' systems.

Our model system for 'spin' has much similarity with canonical spin in conventional 3D space. We describe the extent to which our 'spin' can be mapped to ordinary spin, and the associated finite projective geometry and projective group to 3D space rotations. This relationship is demonstrated via transformations on the associated polyhedron. The emergent geometry provides a useful framework to visualize the underlying physics of the approach. In general, the role of $SU(N)$ familiar in descriptions of spin-like N-level systems in conventional QM is now replaced by

$PGL(N, q)$, the projective linear group acting on the vector space over Galois fields of order q.

This paper is organized as follows. In section II, we review our construction of Galois field quantum mechanics (GQM). Section III presents a detailed analysis of the $GF(2)$ case. We pay particular attention to the CHSH bound, and the impossibility of hidden variables to account for this result. We also present the analogs of singlet and triplet states under $SU(2)$ in conventional QM in the present formulation, and how these account for the resultant bound. Section IV is devoted to the discussion of the geometric structure of 'rotations' in the contexts of $GF(3)$, $GF(4)$ and $GF(5)$ fields. We explicitly construct and identify some of the transformations with those of some polyhedral group, for the case of $N = 2$ and $q = 2$, 3, 4, and 5, thereby justifying the spin-like context for these levels. In the concluding section, we comment on how these results can impact upon more general foundational questions in quantum theory.

Before we proceed, we emphasize that our model is distinct from 'Galois quantum systems' discussed in the literature [8, 9]. These papers consider a phase space which is assumed to be $GF(q)\times GF(q)$; that is, the position and momentum of particles take values in $GF(q)$. In this paper, it is the wave-functions that take values in $[GF(q)]^N$, while the outcomes of measurements take on values in $\mathbb{R}$, the real number line.

2. Galois Field Quantum Mechanics

The key variation introduced in [5] is the replacement of the Hilbert space of an N-level quantum system, $\mathcal{H}_{\mathbb{C}} = \mathbb{C}^N$, with a discrete vector space over a finite field: $\mathcal{H}_q = \mathbb{Z}_q^N$ [7, 10]. Here, $\mathbb{Z}_q$ denotes the Galois field $GF(q)$, where $q = p^n$ with p prime and $n \in \mathbb{N}$. For the $n = 1$ case, $GF(p)$ is simply $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$. The states of the system are represented by vectors $|\psi\rangle \in \mathcal{H}_q$, and outcomes of measurements by dual-vectors $\langle x| \in \mathcal{H}_q^*$. Observables are associated with a choice of basis of $\mathcal{H}_q^*$, each dual-vector in it representing a different outcome. The probability of obtaining the outcome represented by $\langle x|$ when a measurement of the observable is performed on the state represented by $|\psi\rangle$ is given by the canonical form

$$P(x|\psi) \;=\; \frac{\big|\langle x|\psi\rangle\big|^2}{\sum_y \big|\langle y|\psi\rangle\big|^2}\;, \tag{3}$$

where the bracket $\langle x|\psi\rangle \in \mathbb{Z}_q$ is converted into a non-negative real number $|\langle x|\psi\rangle| \in \mathbb{R}$ via the absolute value function:

$$|\,\underline{k}\,| \;=\; \begin{cases} 0 & \text{if } \underline{k} = \underline{0}\;, \\ 1 & \text{if } \underline{k} \neq \underline{0}\;. \end{cases} \tag{4}$$

Here, underlined numbers and symbols represent elements of $\mathbb{Z}_q$, to distinguish them from elements of $\mathbb{R}$ or $\mathbb{C}$. Note that all non-zero elements of $\mathbb{Z}_q$ are assigned an absolute value of 1, effectively making them all 'phases.'

Definition (4) is not arbitrary. In order that the absolute value function generate probability amplitudes for multi-particle states, we must ensure, for consistency, that it satisfies the critical factorizability criterion for product states. Since $\mathbb{Z}_q\backslash\{\underline{0}\}$ is a cyclic multiplicative group, this definition of absolute value is the only one consistent with the requirement

$$|\underline{k}\underline{l}| \;=\; |\underline{k}||\underline{l}|\;, \tag{5}$$

which is necessary for the probabilities of product states to factorize.

Since the multiplication of $|\psi\rangle$ with a non-zero element of $\mathbb{Z}_q$ will not affect the probability as defined above, vectors that differ by a non-zero multiplicative constant are identified as representing the same physical state, and the state space is endowed with the finite projective geometry [11, 12, 13]

$$PG(N-1,q) \;=\; (\,\mathbb{Z}_q^N\backslash\{\underline{\mathbf{0}}\}\,)\,/\,(\,\mathbb{Z}_q\backslash\{\underline{0}\}\,)\;. \tag{6}$$

The group of all possible basis transformations in this space is the projective group $PGL(N,q)$:

$$PGL(N,q) \;=\; GL(N,q)\,/\,Z(N,q)\;. \tag{7}$$

Here, $GL(N,q)$ is the general linear group of $\mathbb{Z}_q^N$, and $Z(N,q)$ is its center, which consists of $N\times N$ unit matrices multiplied by a 'phase' in $\mathbb{Z}_q\backslash\{\underline{0}\}$. Effectively, the elements of $PGL(N,q)$ lead to permutations of the states in $PG(N-1,q)$, and is thus a subgroup of the symmetric group of all possible state permutations. Physically, basis transformations should correspond to the change of 'detector setting,' *e.g.* the rotation of the polarizaton axis of a spin-measuring device.

Let us denote the GQM model resulting from this procedure as $GQM(N,q)$. Spin-like systems with two possible outcomes ±1 can be constructed on the space $V_q\equiv\mathbb{Z}_q^2$ as $GQM(2,q)$, and two-particle spin-like systems on $V_q\otimes V_q=\mathbb{Z}_q^2\otimes\mathbb{Z}_q^2=\mathbb{Z}_q^4$ as $GQM(4,q)$. In the following, we will consider the cases $q=2$, 3, 4, and 5 as concrete examples of this procedure.

3. $\mathbb{Z}_2$ Quantum Mechanics

3.1. One-Particle Spin

We begin our discussion with $q=2$. The spin-like system is $GQM(2,2)$, perhaps the simplest quantum system imaginable, which is constructed on the finite field consisting of only two elements, $GF(2)=\mathbb{Z}_2=\mathbb{Z}/2\mathbb{Z}=\{\underline{0},\underline{1}\}$, with the following addition and multiplication tables:

$+$	$\underline{0}$	$\underline{1}$
$\underline{0}$	$\underline{0}$	$\underline{1}$
$\underline{1}$	$\underline{1}$	$\underline{0}$

$\times$	$\underline{0}$	$\underline{1}$
$\underline{0}$	$\underline{0}$	$\underline{0}$
$\underline{1}$	$\underline{0}$	$\underline{1}$

As stated above, we use numbers with underlines to indicate elements of $\mathbb{Z}_2$ to distinguish them from elements of $\mathbb{R}$ or $\mathbb{C}$. Since $\mathbb{Z}_2\backslash\{\underline{0}\}=\{\underline{1}\}$ is trivial, there are no phases and each physical state is represented by a unique non-zero vector in $V_2=\mathbb{Z}_2^2$.

There exist only $2^2-1=3$ non-zero vectors in V_2, which we denote:

$$|a\rangle \;=\; \begin{bmatrix}\underline{1}\\ \underline{0}\end{bmatrix}\;,\quad |b\rangle \;=\; \begin{bmatrix}\underline{0}\\ \underline{1}\end{bmatrix}\;,\quad |c\rangle \;=\; \begin{bmatrix}\underline{1}\\ \underline{1}\end{bmatrix}\;. \tag{8}$$

Thus, there are 3 possible states of the system. Since

$$|a\rangle \;=\; |b\rangle+|c\rangle\;,\quad |b\rangle \;=\; |c\rangle+|a\rangle\;,\quad |c\rangle \;=\; |a\rangle+|b\rangle\;, \tag{9}$$

the three vectors are completely equivalent; any pair of them can be used as a basis for V_2. A change of basis in V_2 would permute the above column representations among the three vectors, or equivalently, permute the vector labels a, b, and c on the three column vectors. Since all permutations of the three vectors are possible, the group of basis transformations on V_2 is $S_3\cong PGL(2,2)$.

The dual vector space V_2^* is the space of all linear maps from V_2 to $\mathbb{Z}_2$. There are $2^2 - 1 = 3$ non-zero dual vectors in V_2^*, and following [5], we denote them as:

$$\langle\bar{a}| \;=\; \left[\; \underline{0}\;\; \underline{1} \;\right] , \quad \langle\bar{b}| \;=\; \left[\; \underline{1}\;\; \underline{0} \;\right] , \quad \langle\bar{c}| \;=\; \left[\; \underline{1}\;\; \underline{1} \;\right] . \tag{10}$$

This labeling allows us to write:

$$\langle\bar{r}|s\rangle \;=\; \begin{cases} \underline{0} & \text{if } r = s, \\ \underline{1} & \text{if } r \neq s, \end{cases} \tag{11}$$

and consequently,

$$|\langle\bar{r}|s\rangle| \;=\; 1 - \delta_{rs} \;. \tag{12}$$

This relation pairs up vectors with dual-vectors in a somewhat non-standard way, and leads to the various consequence of GQM. To maintain it, we assume that a relabeling of vectors in V_2 is always accompanied by a corresponding relabeling of dual-vectors in V_2^*.

Observables are associated with *a choice of basis* in V_2^*. There are six possible choices:

$$A_{rs} \;=\; \{\, \langle\bar{r}|,\, \langle\bar{s}|\, \} \;, \tag{13}$$

with $rs = ab$, ba, bc, cb, ca, and ac. Each of the dual vectors in each basis represents an *outcome*‡ which could occur as the result of a measurement of the observable represented by that basis, that is, the measurement of the observable A_{rs} would result in one of the two outcomes represented by $\langle\bar{r}|$ and $\langle\bar{s}|$.

Though elements of V_2^* map elements of V_2 onto $\mathbb{Z}_2$, the outcomes they represent need not be elements of $\mathbb{Z}_2$ themselves. We assign the numerical values ± 1 to the two outcomes represented by the pair of dual vectors in each basis: $+1$ to the first dual vector, and -1 to the second dual vector. Thus, $\langle\bar{r}|$ represents the outcome $+1$ when A_{rs} is measured, while $\langle\bar{s}|$ represents the outcome -1 when A_{rs} is measured. If the ordering of the dual vectors is reversed to $\{\, \langle\bar{s}|,\, \langle\bar{r}|\, \}$, then this pair corresponds to $-A_{rs}$. Thus $A_{sr} = -A_{rs}$, and we can consider A_{rs} and A_{sr} to be essentially the same observable. So the number of observables in our system is three: A_{ab}, A_{bc}, and A_{ca}.

The above assignment of outcomes allows us to view A_{rs} as representing observables akin to 'spin,' with the indices rs representing the 'direction' of the spin. [7] labels them as:

$$X \;=\; A_{bc} \;, \qquad Y \;=\; A_{ca} \;, \qquad Z \;=\; A_{ab} \;. \tag{14}$$

Indeed, if we look at their transformation properties under the S_3 group of basis transformations, we find:

$$\begin{array}{lll} (ab)X \;=\; -Y\,, & (ab)Y \;=\; -X\,, & (ab)Z \;=\; -Z\,, \\ (bc)X \;=\; -X\,, & (bc)Y \;=\; -Z\,, & (bc)Z \;=\; -Y\,, \\ (ca)X \;=\; -Z\,, & (ca)Y \;=\; -Y\,, & (ca)Z \;=\; -X\,, \\ (abc)X \;=\; +Y\,, & (abc)Y \;=\; +Z\,, & (abc)Z \;=\; +X\,, \\ (acb)X \;=\; +Z\,, & (acb)Y \;=\; +X\,, & (acb)Z \;=\; +Y\,, \end{array} \tag{15}$$

which can all be considered $SO(3)$ rotations of the mutually orthogonal X, Y, and Z axes as shown in figure 1. The six basis transformations in S_3 can be mapped onto the six rotations of the dihedral group D_3 which keep the equilateral triangle abc in figure 1 invariant.

‡ Schumacher and Westmoreland call it an *effect* in [7]

Thus, our 'spins' transform in an analogous way to canonical spin under 'rotations.' However, a significant difference also exists. With our 'spin,' the same dual vector can represent different outcomes of different observables: in addition to the outcomes of A_{ab}, $\langle\bar{a}|$ represents the outcome -1 when A_{ca} is measured, while $\langle\bar{b}|$ represents the outcome $+1$ when A_{bc} is measured. What each dual vector represents depends on the observable under consideration.

The probabilities of outcomes are calculated with (3). For the measurement of the observable $Z = A_{ab}$, for instance, we find:

$$\begin{aligned}
P(A_{ab}\,;+\,|\,a) &= \frac{\left|\langle\bar{a}|a\rangle\right|^2}{\left|\langle\bar{a}|a\rangle\right|^2+\left|\langle\bar{b}|a\rangle\right|^2} \;=\; 0\,,\\
P(A_{ab};-\,|\,a) &= \frac{\left|\langle\bar{b}|a\rangle\right|^2}{\left|\langle\bar{a}|a\rangle\right|^2+\left|\langle\bar{b}|a\rangle\right|^2} \;=\; 1\,,\\
P(A_{ab};+\,|\,b) &= \frac{\left|\langle\bar{a}|b\rangle\right|^2}{\left|\langle\bar{a}|b\rangle\right|^2+\left|\langle\bar{b}|b\rangle\right|^2} \;=\; 1\,,\\
P(A_{ab};-\,|\,b) &= \frac{\left|\langle\bar{b}|b\rangle\right|^2}{\left|\langle\bar{a}|b\rangle\right|^2+\left|\langle\bar{b}|b\rangle\right|^2} \;=\; 0\,,\\
P(A_{ab};+\,|\,c) &= \frac{\left|\langle\bar{a}|c\rangle\right|^2}{\left|\langle\bar{a}|c\rangle\right|^2+\left|\langle\bar{b}|c\rangle\right|^2} \;=\; \frac{1}{2}\,,\\
P(A_{ab};-\,|\,c) &= \frac{\left|\langle\bar{b}|c\rangle\right|^2}{\left|\langle\bar{a}|c\rangle\right|^2+\left|\langle\bar{b}|c\rangle\right|^2} \;=\; \frac{1}{2}\,.
\end{aligned} \tag{16}$$

The expectation values of $Z = A_{ab}$ on the three states are therefore:

$$\begin{aligned}
\langle A_{ab}\rangle_a &= (+1)\times 0 \;+(-1)\times 1 \;=\; -1\,,\\
\langle A_{ab}\rangle_b &= (+1)\times 1 \;+(-1)\times 0 \;=\; +1\,,\\
\langle A_{ab}\rangle_c &= (+1)\times \tfrac{1}{2} \;+(-1)\times \tfrac{1}{2} \;=\; 0\,.
\end{aligned} \tag{17}$$

Thus, $|a\rangle$ and $|b\rangle$ take on the role of the 'eigenstates' of A_{ab}, while $|c\rangle$ is the superposition of the two with a 50-50 chance of obtaining either $+1$ or -1. The

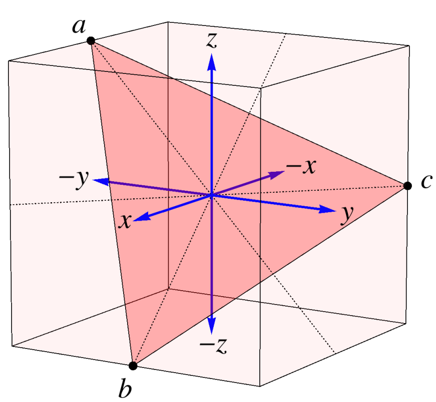


Figure 1. The 6 'spin' directions of $GQM(2,2)$. Allowed $SO(3)$ rotations are those that rotate the equilateral triangle abc onto itself.

Table 1. The probabilities of the two outcomes + and − for all combinations of observables and states in $GQM(2,2)$.

observable	state	$P(+)$	$P(-)$	Expectation Value
A_{ab}	a	0	1	−1
	b	1	0	+1
	c	$\frac{1}{2}$	$\frac{1}{2}$	0
A_{bc}	a	$\frac{1}{2}$	$\frac{1}{2}$	0
	b	0	1	−1
	c	1	0	+1
A_{ca}	a	1	0	+1
	b	$\frac{1}{2}$	$\frac{1}{2}$	0
	c	0	1	−1

probabilities and expectation values of all other observables can be calculated in a similar fashion and we obtain the results listed in table 1.

The probabilities for $X = A_{bc}$ and $Y = A_{ca}$ can also be calculated by 'rotating' the results for $Z = A_{ab}$ via (15). For instance, since $(abc)A_{ab} = A_{bc}$, we can conclude that

$$\begin{aligned} P(A_{bc};+\,|\,b) = (abc)\,P(A_{ab};+\,|\,a) \;&=\; 0\;, \\ P(A_{bc};-\,|\,b) = (abc)\,P(A_{ab};-\,|\,a) \;&=\; 1\;. \end{aligned} \tag{18}$$

Note that, due to our construction, states $|r\rangle$ and $|s\rangle$ are 'eigenstates' of A_{rs} for all rs. However, in our approach, observables are not linear hermitian maps from V_2 to V_2 as in canonical QM, and thus do not have eigenstates in the usual sense of the term.

Note, also, that each state is an 'eigenstate' of two observables at a time: $|a\rangle$ of A_{ab} and A_{ca}, $|b\rangle$ of A_{bc} and A_{ab}, and $|c\rangle$ of A_{ca} and A_{bc}. This is due to each of the three dual vectors appearing in two observables. Thus, despite the resemblance to spins in the x, y, and z directions in canonical QM, this quantum system is quite distinct.

3.2. Two Particle States

Two particle states are expressed as vectors in the tensor product space $V_2 \otimes V_2 = \mathbb{Z}_2^4$, and the system becomes $GQM(4,2)$. There are $2^4 - 1 = 15$ non-zero vectors in this space, of which nine are product states and six are entangled states. The nine product states are:

$$\begin{aligned}
|aa\rangle &= |a\rangle \otimes |a\rangle &= \begin{bmatrix} \underline{1} & \underline{0} & \underline{0} & \underline{0} \end{bmatrix}^{\mathrm{T}} , \\
|ab\rangle &= |a\rangle \otimes |b\rangle &= \begin{bmatrix} \underline{0} & \underline{1} & \underline{0} & \underline{0} \end{bmatrix}^{\mathrm{T}} , \\
|ac\rangle &= |a\rangle \otimes |c\rangle &= \begin{bmatrix} \underline{1} & \underline{1} & \underline{0} & \underline{0} \end{bmatrix}^{\mathrm{T}} , \\
|ba\rangle &= |b\rangle \otimes |a\rangle &= \begin{bmatrix} \underline{0} & \underline{0} & \underline{1} & \underline{0} \end{bmatrix}^{\mathrm{T}} , \\
|bb\rangle &= |b\rangle \otimes |b\rangle &= \begin{bmatrix} \underline{0} & \underline{0} & \underline{0} & \underline{1} \end{bmatrix}^{\mathrm{T}} , \\
|bc\rangle &= |b\rangle \otimes |c\rangle &= \begin{bmatrix} \underline{0} & \underline{0} & \underline{1} & \underline{1} \end{bmatrix}^{\mathrm{T}} , \\
|ca\rangle &= |c\rangle \otimes |a\rangle &= \begin{bmatrix} \underline{1} & \underline{0} & \underline{1} & \underline{0} \end{bmatrix}^{\mathrm{T}} , \\
|cb\rangle &= |c\rangle \otimes |b\rangle &= \begin{bmatrix} \underline{0} & \underline{1} & \underline{0} & \underline{1} \end{bmatrix}^{\mathrm{T}} , \\
|cc\rangle &= |c\rangle \otimes |c\rangle &= \begin{bmatrix} \underline{1} & \underline{1} & \underline{1} & \underline{1} \end{bmatrix}^{\mathrm{T}} .
\end{aligned} \tag{19}$$

The six entangled states can be classified according to their transformation properties under global S_3 'rotations.'

The first is a singlet which transforms into itself under all permutations of S_3:

$$\big|S\big\rangle \;=\; |aa\rangle + |bb\rangle + |cc\rangle \;=\; \begin{bmatrix} \underline{0} & \underline{1} & \underline{1} & \underline{0} \end{bmatrix}^{\mathrm{T}} . \tag{20}$$

This state is the closest analog to the spin singlet state $|0,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$ in canonical QM, as can be seen from the fact that $|S\rangle$ can also be written as

$$\big|S\big\rangle \;=\; |ab\rangle + |ba\rangle \;=\; |bc\rangle + |cb\rangle \;=\; |ca\rangle + |ac\rangle \,. \tag{21}$$

This state is symmetric, however, under the interchange of the two particles since there is no analog of -1 in $\mathbb{Z}_2$.

Three more states are symmetric, and transform as a triplet:

$$\begin{aligned}
\big|(ab)\big\rangle &= |ab\rangle + |ba\rangle + |cc\rangle &= \begin{bmatrix} \underline{1} & \underline{0} & \underline{0} & \underline{1} \end{bmatrix}^{\mathrm{T}} , \\
\big|(bc)\big\rangle &= |aa\rangle + |bc\rangle + |cb\rangle &= \begin{bmatrix} \underline{1} & \underline{1} & \underline{1} & \underline{0} \end{bmatrix}^{\mathrm{T}} , \\
\big|(ca)\big\rangle &= |ac\rangle + |bb\rangle + |ca\rangle &= \begin{bmatrix} \underline{0} & \underline{1} & \underline{1} & \underline{1} \end{bmatrix}^{\mathrm{T}} .
\end{aligned} \tag{22}$$

These would be the analog of the spin-one triplet in canonical QM.

The remaining two states are asymmetric under the interchange of the two particles:

$$\begin{aligned}
\big|(abc)\big\rangle &= |ab\rangle + |bc\rangle + |ca\rangle &= \begin{bmatrix} \underline{1} & \underline{1} & \underline{0} & \underline{1} \end{bmatrix}^{\mathrm{T}} , \\
\big|(acb)\big\rangle &= |ac\rangle + |cb\rangle + |ba\rangle &= \begin{bmatrix} \underline{1} & \underline{0} & \underline{1} & \underline{1} \end{bmatrix}^{\mathrm{T}} .
\end{aligned} \tag{23}$$

These transform as a doublet: they are invariant under even permutations, but transform into each other under odd permtations. There is a one-to-one correspondence between these states and the elements of S_3, as well as a correspondence between the state multiplets and the conjugate classes of S_3.

3.3. Geometric Characterization

As discussed in section II, the space of the fifteen state vectors in $GQM(4,2)$ possesses the projective geometry $PG(3,2)$. In this geometry, three 'points' $|r\rangle$, $|s\rangle$, $|t\rangle$ are on a 'line' if they add up to the zero vector. The 15 'points' lie on 35 'lines,' with 7 'lines' crossing at each 'point.' The 35 'lines' are contained in 15 'planes,' with 3 'planes' intersecting at each 'line.'

In the current context, the nine product states are 'points' that lie on six 'lines,' no three of which are in the same 'plane,' forming a 'non-planar' grid:

$$\begin{array}{ccccc} |aa\rangle & = & |ab\rangle & = & |ac\rangle \\ \| & & \| & & \| \\ |ba\rangle & = & |bb\rangle & = & |bc\rangle \\ \| & & \| & & \| \\ |ca\rangle & = & |cb\rangle & = & |cc\rangle \end{array} \qquad (24)$$

The six entangled states are each a sum of three product state 'points,' no two of which lie on the same 'line' of this grid, that is, no two states in the sum share the same 'row' or 'column.'

Beyond this, we have been unsuccessful in finding a geometric characterization, or differentiation, of product and entangled states. It is unclear whether a similar characterization is possible in cases other than $q = 2$. The discovery of a geometrical understanding applicable to the generic $GQM(4, q)$ case with $PG(3, q)$ geometry could be enlightening.

3.4. Local Rotations

It will be useful to see how the six entangled states listed above transform into each other under 'rotations' of either the first, or the second particle only. We find:

$$\begin{array}{rcrcl} (ab)_1|S\rangle & = & (ab)_2|S\rangle & = & |(ab)\rangle\;, \\ (bc)_1|S\rangle & = & (bc)_2|S\rangle & = & |(bc)\rangle\;, \\ (ca)_1|S\rangle & = & (ca)_2|S\rangle & = & |(ca)\rangle\;, \\ (acb)_1|S\rangle & = & (abc)_2|S\rangle & = & |(abc)\rangle\;, \\ (abc)_1|S\rangle & = & (acb)_2|S\rangle & = & |(acb)\rangle\;. \end{array} \qquad (25)$$

The transformation properties of the other states can be obtained from these relations, for instance:

$$(ab)_1|(bc)\rangle \;=\; (ab)_1(bc)_1|S\rangle \;=\; (abc)_1|S\rangle \;=\; |(acb)\rangle\;. \qquad (26)$$

The fact that all six entangled states transform into each other this way means that they are all equivalent and equally entangled, since the transformations considered here amount to simple relabelings of the states in the two vector spaces that are tensored.

3.5. Two Particle Observables

There are fifteen non-zero dual vectors in $V_2^* \otimes V_2^*$, but we will only be looking at the nine product observables constructed from the nine product dual vectors which are of the form

$$A_{rs}A_{tu} \;=\; \{\, \langle\bar{r}|\otimes\langle\bar{t}|\,,\,\langle\bar{r}|\otimes\langle\bar{u}|\,,\,\langle\bar{s}|\otimes\langle\bar{t}|\,,\,\langle\bar{s}|\otimes\langle\bar{u}|\,\}\;, \qquad (27)$$

where the indices rs and tu are ab, bc, or ca. The four tensored dual vectors in this expression respectively represent the outcomes, $++$, $+-$, $-+$, and $--$ when $A_{rs}A_{tu}$ is measured. The row vector representations of all nine tensor products are given by

$$\begin{aligned} \langle\bar{a}|\otimes\langle\bar{a}| &= \begin{bmatrix} \underline{0} & \underline{0} & \underline{0} & \underline{1} \end{bmatrix}\;, \\ \langle\bar{a}|\otimes\langle\bar{b}| &= \begin{bmatrix} \underline{0} & \underline{0} & \underline{1} & \underline{0} \end{bmatrix}\;, \\ \langle\bar{a}|\otimes\langle\bar{c}| &= \begin{bmatrix} \underline{0} & \underline{0} & \underline{1} & \underline{1} \end{bmatrix}\;, \\ \langle\bar{b}|\otimes\langle\bar{a}| &= \begin{bmatrix} \underline{0} & \underline{1} & \underline{0} & \underline{0} \end{bmatrix}\;, \end{aligned}$$

$$\begin{aligned}
\langle\bar{b}|\otimes\langle\bar{b}| &= \left[\,\underline{1}\;\underline{0}\;\underline{0}\;\underline{0}\,\right] , \\
\langle\bar{b}|\otimes\langle\bar{c}| &= \left[\,\underline{1}\;\underline{1}\;\underline{0}\;\underline{0}\,\right] , \\
\langle\bar{c}|\otimes\langle\bar{a}| &= \left[\,\underline{0}\;\underline{1}\;\underline{0}\;\underline{1}\,\right] , \\
\langle\bar{c}|\otimes\langle\bar{b}| &= \left[\,\underline{1}\;\underline{0}\;\underline{1}\;\underline{0}\,\right] , \\
\langle\bar{c}|\otimes\langle\bar{c}| &= \left[\,\underline{1}\;\underline{1}\;\underline{1}\;\underline{1}\,\right] .
\end{aligned} \tag{28}$$

3.6. Probabilities and Correlations

Applying (3) to product observables, the probability of obtaining an outcome (x, y) represented by the product dual vector $\langle xy| = \langle x|\otimes\langle y|$ when observable O_1O_2 is measured on state $|\psi\rangle$ is given by:

$$P(O_1O_2\,;xy\,|\,\psi) \;=\; \frac{|\langle xy|\psi\rangle|^2}{\sum_{zw}|\langle zw|\psi\rangle|^2} \;. \tag{29}$$

For a product state $|\psi\rangle = |r\rangle\otimes|s\rangle \equiv |rs\rangle$, the brackets factorize:

$$\langle xy|rs\rangle \;=\; \big(\langle x|\otimes\langle y|\,\big)\big(\,|r\rangle\otimes|s\rangle\big) \;=\; \langle x|r\rangle\langle y|s\rangle \;, \tag{30}$$

and due to the condition (5) we imposed on the absolute values, we have

$$\big|\langle xy|rs\rangle\big| \;=\; \big|\langle x|r\rangle\big|\,\big|\langle y|s\rangle\big| \;. \tag{31}$$

Consequently, the probability also factorizes as

$$\begin{aligned}
P(O_1O_2\,;xy\,|\,rs) &= \frac{|\langle xy|rs\rangle|^2}{\sum_{zw}|\langle zw|rs\rangle|^2} \\
&= \frac{|\langle x|r\rangle|^2|\langle y|s\rangle|^2}{\sum_z\sum_w|\langle z|r\rangle|^2|\langle w|s\rangle|^2} \\
&= \left[\frac{|\langle x|r\rangle|^2}{\sum_z|\langle z|r\rangle|^2}\right]\left[\frac{\langle y|s\rangle|^2}{\sum_w|\langle w|s\rangle|^2}\right] \\
&= P(O_1\,;x\,|\,r)\,P(O_2\,;y\,|\,s) \;,
\end{aligned} \tag{32}$$

which is a property we would like to preserve for unentangled states. Otherwise, no isolated particle would be possible. Note the importance of (5) for this factorization to occur. The expectation value of the product observable O_1O_2, *i.e.* the correlation between O_1 and O_2, will be

$$\langle O_1O_2\rangle_\psi \;=\; \sum_{xy} xy\;P(O_1O_2\,;xy\,|\,\psi) \;=\; \frac{\sum_{xy}xy\,|\langle xy|\psi\rangle|^2}{\sum_{zw}|\langle zw|\psi\rangle|^2} \;, \tag{33}$$

which for product states factorizes as

$$\langle O_1O_2\rangle_{rs} \;=\; \langle O_1\rangle_r\langle O_2\rangle_s \;. \tag{34}$$

Thus, the correlations for the nine product states will simply be products of those listed in table 1.

The probabilities and expectation values of the nine product observables for all six entangled states are shown in table 2, where we have used the notation $X = A_{bc}$, $Y = A_{ca}$, $Z = A_{ab}$. The entries can be 'rotated' into each other via (15) and (25). For instance, since $(ab)_1X_1X_2 \;=\; -Y_1X_2$ and $(ab)_1|(ab)\rangle = |S\rangle$, we have

$$\begin{aligned}
P(Y_1X_2;++,S) &= P(-Y_1X_2;-+,S) \\
&= (ab)_1P(X_1X_2;-+,(ab)) \;=\; \frac{1}{3} \;,
\end{aligned} \tag{35}$$

and so on.

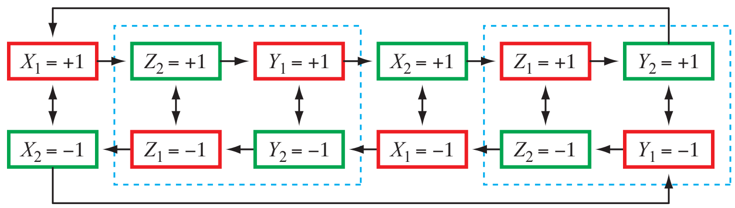
Table 2. Correlations of observables for the six entangled states.

	state	$++$	$+-$	$-+$	$--$	E.V.
X_1X_2	S	0	$\frac{1}{2}$	$\frac{1}{2}$	0	-1
	(ab)	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$
	(bc)	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$+1$
	(ca)	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$
	(abc)	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	$+\frac{1}{3}$
	(acb)	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	$+\frac{1}{3}$
X_1Y_2	S	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	$+\frac{1}{3}$
	(ab)	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$+1$
	(bc)	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$
	(ca)	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$
	(abc)	0	$\frac{1}{2}$	$\frac{1}{2}$	0	-1
	(acb)	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	$+\frac{1}{3}$
X_1Z_2	S	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	$+\frac{1}{3}$
	(ab)	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$
	(bc)	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$
	(ca)	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$+1$
	(abc)	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	$+\frac{1}{3}$
	(acb)	0	$\frac{1}{2}$	$\frac{1}{2}$	0	-1

	state	$++$	$+-$	$-+$	$--$	E.V.
Y_1X_2	S	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	$+\frac{1}{3}$
	(ab)	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$+1$
	(bc)	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$
	(ca)	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$
	(abc)	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	$+\frac{1}{3}$
	(acb)	0	$\frac{1}{2}$	$\frac{1}{2}$	0	-1
Y_1Y_2	S	0	$\frac{1}{2}$	$\frac{1}{2}$	0	-1
	(ab)	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$
	(bc)	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$
	(ca)	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$+1$
	(abc)	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	$+\frac{1}{3}$
	(acb)	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	$+\frac{1}{3}$
Y_1Z_2	S	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	$+\frac{1}{3}$
	(ab)	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$
	(bc)	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$+1$
	(ca)	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$
	(abc)	0	$\frac{1}{2}$	$\frac{1}{2}$	0	-1
	(acb)	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	$+\frac{1}{3}$

	state	$++$	$+-$	$-+$	$--$	E.V.
Z_1X_2	S	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	$+\frac{1}{3}$
	(ab)	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$
	(bc)	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$
	(ca)	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$+1$
	(abc)	0	$\frac{1}{2}$	$\frac{1}{2}$	0	-1
	(acb)	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	$+\frac{1}{3}$
Z_1Y_2	S	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	$+\frac{1}{3}$
	(ab)	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$
	(bc)	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$+1$
	(ca)	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$
	(abc)	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	$+\frac{1}{3}$
	(acb)	0	$\frac{1}{2}$	$\frac{1}{2}$	0	-1
Z_1Z_2	S	0	$\frac{1}{2}$	$\frac{1}{2}$	0	-1
	(ab)	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$+1$
	(bc)	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$
	(ca)	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$
	(abc)	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	$+\frac{1}{3}$
	(acb)	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	$+\frac{1}{3}$

Figure 2. The implication chart for the state $|S\rangle$. Arrows point from the condition toward the implication. By tracing the arrows, it is easy to see that no classical configurations, and thus no hidden variable theory, can satisfy all of these requirements. If we ignore the observable X and look at only Y and Z, then the assignments within the dashed boxes are possible classical configurations. However, neither allow for the pairs $(Y_1 Z_2)$ and $(Z_1 Y_2)$ to be anti-correlated, which occurs with probability 1/3 in our QM.



3.7. Entanglement and the Impossibility of Hidden Variables

We now demonstrate that hidden variables cannot reproduce the probabilities and correlations predicted by GQM for entangled states. The argument is analogous to those of Greenberger, Horne, Shimony, and Zeilinger [14], and of Hardy [15] for canonical QM.

Since all six entangled states are equivalent, it suffices to consider only one, for which we will use the state $|S\rangle$. From table 2, we see that

$$\begin{aligned} P(X_1X_2;{+}{+},S) &= P(X_1X_2;{-}{-},S) &= 0\,, \\ P(Y_1Y_2;{+}{+},S) &= P(Y_1Y_2;{-}{-},S) &= 0\,, \\ P(Z_1Z_2;{+}{+},S) &= P(Z_1Z_2;{-}{-},S) &= 0\,, \end{aligned} \tag{36}$$

which means that the pairs (X_1, X_2), (Y_1, Y_2), and (Z_1, Z_2) are all completely anti-correlated. We next note that

$$P(X_1Z_2;{+}{-},S) \;=\; 0\,, \tag{37}$$

which means that $X_1 = +1$ necessarily implies $Z_2 = +1$, while $Z_2 = -1$ necessarily implies $X_1 = -1$. Similarly,

$$P(Y_1Z_2;{-}{+},S) \;=\; 0 \tag{38}$$

means that $Z_2 = +1$ necessarily implies $Y_1 = +1$, while $Y_1 = -1$ necessarily implies $Z_2 = -1$. Going through table 2 in this fashion, we obtain the implication diagram shown in figure 2. As is clear from the diagram, no classical configuration exists which would be compatible with all of these constraints. For instance, $X_1 = +1$ implies $Z_2 = +1$, which implies $Y_1 = +1$, which implies $X_2 = +1$, which contradicts the requirement that X_1 and X_2 are anti-correlated.

It should be noted that even if we limit our attention to only two of the three observables available for each particle, hidden variables still cannot reproduce the quantum probabilities. For instance, consider only Y and Z for both particles. Then, the selection of values within the dashed boxes on figure 2 give possible classical configurations. However, the combinations $(Y_1Z_2) = ({+}{-})$ and $(Z_1Y_2) = ({-}{+})$ cannot occur even though they are possible quantum mechanically. Thus, the entangled states in our model are truly 'quantum' in the sense discussed in the introduction, and entangled.

3.8. The CHSH Bound

Let us now find the CHSH bound of our model, *i.e.* the upper bound of the absolute value of the CHSH correlator defined in (1). Owing to the equivalence of all entangled states, we only need to look at the correlations for one state for all possible observable combinations. That is, using (15) and (25), we can convert the correlations for any entangled state into those for the state $|S\rangle$. For instance:

$$\begin{aligned}\langle X_1, Y_1; X_2, Y_2\rangle_{(ab)} &= -\langle Y_1, X_1; X_2, Y_2\rangle_S \\ &= -\langle X_1, Y_1; Y_2, X_2\rangle_S \;. \end{aligned} \tag{39}$$

We also need not consider the negatives of the observables as long as all possible choices for A_1, A_2, B_1, and B_2 are considered since

$$\begin{aligned}&\langle A_1, A_2\,; B_1, B_2\rangle \\ &= \langle A_1, -A_2\,; B_2, B_1\rangle \;=\; -\langle -A_1, A_2\,; B_2, B_1\rangle \\ &= \langle A_2, A_1\,; B_1, -B_2\rangle \;=\; -\langle A_2, A_1\,; -B_1, B_2\rangle \;. \end{aligned} \tag{40}$$

Then, from simple inspection of table 2, we can see that the maximum absolute value of the CHSH correlator is achieved for

$$\begin{aligned}\langle X, Y; Y, X\rangle_S &= -2 \;, \\ \langle X, Z; Y, Z\rangle_S &= +2 \;, \end{aligned} \tag{41}$$

with arbitrary permutations of the three observables leading to the same values. All the other correlators yield $\pm\frac{2}{3}$. Thus, the CHSH bound for our model is 2, the classical value, despite the fact that it is 'quantum' and does not allow for any hidden variables.

That this bound is different from the Cirel'son bound of $2\sqrt{2}$ for spin systems in canonical QM should not be a surprise [3]. The underlying vector space in the present 'spin' system has no inner product, a key element in the derivation of the Cirel'son bound [4], and the symmetry group in this context is not equivalent to $SU(2)$. The entangled states have parallels to those of $SU(2)$, but have an independent existence and structure. These features conspire to lower the bound from the canonical value. While its agreement with the classical value seems to be a happenstance, it demonstrates that the CHSH bound by itself does not necessarily distinguish between quantum and classical descriptions.

4. 'Rotations' and Rotations

We now consider the cases $q = 3$, 4, and 5. Since the details of how $GQM(2, q)$ is implemented for these cases, and the calculation of the CHSH bound are not that different from the $q = 2$ case, we will not go into detail about these cases here and refer the reader to the argument presented in [5] which applies to all values of q. In this section, we will mostly be concerned with how the 'spin directions' in $GQM(2, q)$ can be mapped to actual directions in 3D space, and how the elements of $PGL(2, q)$ can be identified with $SO(3)$ rotations.

4.1. $\mathbb{Z}_3$ case

$GQM(2, 3)$ is constructed on the field consisting of three elements, $GF(3) = \mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z} = \{\underline{0}, \underline{1}, \underline{2}\}$, with addition and multiplication tables given by

$+$	$\underline{0}$	$\underline{1}$	$\underline{2}$
$\underline{0}$	$\underline{0}$	$\underline{1}$	$\underline{2}$
$\underline{1}$	$\underline{1}$	$\underline{2}$	$\underline{0}$
$\underline{2}$	$\underline{2}$	$\underline{0}$	$\underline{1}$

$\times$	$\underline{0}$	$\underline{1}$	$\underline{2}$
$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$
$\underline{1}$	$\underline{0}$	$\underline{1}$	$\underline{2}$
$\underline{2}$	$\underline{0}$	$\underline{2}$	$\underline{1}$

In the following, we will write $\underline{2}$ as $-\underline{1}$. Since $\mathbb{Z}_3\backslash\{\underline{0}\} = \{\underline{1}, -\underline{1}\}$, each physical state will be represented by two vectors in $V_3 = \mathbb{Z}_3^2$ which differ by the multiplicative 'phase' $-\underline{1}$.

Thus, of the $3^2 - 1 = 8$ non-zero vectors in V_3, there are pairs of vectors that are equivalent, and the inequivalent ones can be taken to be:

$$|a\rangle = \begin{bmatrix} \underline{1} \\ \underline{0} \end{bmatrix}, \quad |b\rangle = \begin{bmatrix} \underline{0} \\ \underline{1} \end{bmatrix}, \quad |c\rangle = \begin{bmatrix} -\underline{1} \\ \underline{1} \end{bmatrix}, \quad |d\rangle = \begin{bmatrix} \underline{1} \\ \underline{1} \end{bmatrix}. \tag{42}$$

Note that any pair of these states can be written as the sum and difference of the other two up to phases, *e.g.*

$$|c\rangle \;=\; -|a\rangle + |b\rangle\;, \qquad |d\rangle \;=\; |a\rangle + |b\rangle\;. \tag{43}$$

Thus, a basis transformation which interchanges a pair of states would leave the other two unaffected. In the above example, interchanging $|a\rangle$ and $|b\rangle$ would leave $|d\rangle$ unchanged, while $|c\rangle$ only acquires an unphysical phase $-\underline{1}$. Thus, single transpositions of the vector labels are possible, and the group generated by those transpositions would be $S_4 \cong PGL(2,3)$. That is, all permutations of the vector labels are possible under basis transformations. This is the group of 'rotations' for $GQM(2,3)$.

The inequivalent dual-vectors of V_3^* can be taken to be

$$\begin{aligned} \langle\bar{a}| &= \begin{bmatrix} \underline{0} & -\underline{1} \end{bmatrix}, \quad & \langle\bar{b}| &= \begin{bmatrix} \underline{1} & \underline{0} \end{bmatrix}, \\ \langle\bar{c}| &= \begin{bmatrix} \underline{1} & \underline{1} \end{bmatrix}, \quad & \langle\bar{d}| &= \begin{bmatrix} \underline{1} & -\underline{1} \end{bmatrix}. \end{aligned} \tag{44}$$

The actions of these dual-vectors on the vectors are given by:

	$\vert a\rangle$	$\vert b\rangle$	$\vert c\rangle$	$\vert d\rangle$
$\langle\bar{a}\vert$	$\underline{0}$	$-\underline{1}$	$-\underline{1}$	$-\underline{1}$
$\langle\bar{b}\vert$	$\underline{1}$	$\underline{0}$	$-\underline{1}$	$\underline{1}$
$\langle\bar{c}\vert$	$\underline{1}$	$\underline{1}$	$\underline{0}$	$-\underline{1}$
$\langle\bar{d}\vert$	$\underline{1}$	$-\underline{1}$	$\underline{1}$	$\underline{0}$

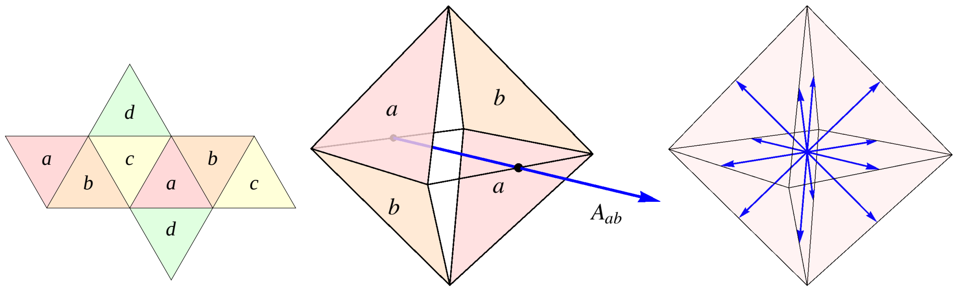
Thus,

$$\begin{aligned} \langle\bar{r}|s\rangle &= \underline{0} \quad \text{if } r = s, \\ &\neq \underline{0} \quad \text{if } r \neq s, \end{aligned} \tag{45}$$

and the relation $|\langle\bar{r}|s\rangle| = 1 - \delta_{rs}$ is obtained in this case also. Maintaining this relation would require relabeling the dual vectors in the same way as the vectors under basis transformations.

From the set of four dual-vectors, we can define $4 \times 3 = 12$ observables with outcomes ± 1, or 6 if we count the 'spins' pointing in opposite directions as the same observable. These 'spins' can be associated with actual directions in 3D, and their $PGL(2,3) \cong S_4$ transformations with rotations in $SO(3)$ as shown in figure 3: First, label the faces of an octahedron with four letters *abcd*, each letter appearing twice, on opposing faces as shown in figure 3(a). The octahedral group O [16] which rotates the octahedron onto itself consists of 24 elements. Each of these elements will permute the four letters on the faces of the octahedron. Thus, there exists a one-to-one correspondence between elements of the octahedral group and the 24 permutations of S_4. The 'direction' of the 'spin' observable A_{ab} can be associated with the direction

Figure 3. To map 'rotations' in $PGL(2,3) \cong S_4$ to rotations in $SO(3)$: (a) label the faces of an octahedron with four symbols as shown. Then, every permutation of the four labels *abcd* will correspond a rotation of the octahedral group. (b) The 'spin' observable A_{ab} in $GQM(2,3)$ can be mapped onto a direction in 3D space as shown. (c) The urchin diagram showing all 12 'spin' directions $GQM(2,3)$.



(a) (b) (c)

of the arrow shown in figure 3(b). All 12 'spin' directions can be mapped this way, and figure 3(c) shows the resulting urchin of spin-directions. These 12 'spins' transform into each other under $S_4 \cong O$ rotations.

4.2. $\mathbb{Z}_2[\underline{\omega}]$ *case*

$GQM(2,4)$ is constructed on the field consisting of four elements, $GF(4) = \mathbb{Z}_2[\underline{\omega}] = \{\underline{0}, \underline{1}, \underline{\omega}, \underline{\omega}^2\}$, which is the Galois extension of $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ with solutions to the equation

$$\underline{x}^2 + \underline{x} + \underline{1} \;=\; \underline{0}\,, \tag{46}$$

which we denote $\underline{\omega}$ and $\underline{\omega}^2 = \underline{1} + \underline{\omega}$. The addition and multiplication tables of this field are given by

$+$	$\underline{0}$	$\underline{1}$	$\underline{\omega}$	$\underline{\omega}^2$
$\underline{0}$	$\underline{0}$	$\underline{1}$	$\underline{\omega}$	$\underline{\omega}^2$
$\underline{1}$	$\underline{1}$	$\underline{0}$	$\underline{\omega}^2$	$\underline{\omega}$
$\underline{\omega}$	$\underline{\omega}$	$\underline{\omega}^2$	$\underline{0}$	$\underline{1}$
$\underline{\omega}^2$	$\underline{\omega}^2$	$\underline{\omega}$	$\underline{1}$	$\underline{0}$

$\times$	$\underline{0}$	$\underline{1}$	$\underline{\omega}$	$\underline{\omega}^2$
$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$
$\underline{1}$	$\underline{0}$	$\underline{1}$	$\underline{\omega}$	$\underline{\omega}^2$
$\underline{\omega}$	$\underline{0}$	$\underline{\omega}$	$\underline{\omega}^2$	$\underline{1}$
$\underline{\omega}^2$	$\underline{0}$	$\underline{\omega}^2$	$\underline{1}$	$\underline{\omega}$

Since $\mathbb{Z}_2[\underline{\omega}]\backslash\{\underline{0}\} = \{\underline{1}, \underline{\omega}, \underline{\omega}^2\}$, each physical state will be represented by three vectors in $V_4 = \{\,\mathbb{Z}_2[\underline{\omega}]\,\}^2$ which differ by multiplicative 'phases' $\underline{\omega}$ or $\underline{\omega}^2$.

Thus, of the $4^2 - 1 = 15$ non-zero vectors in V_4, every three of them are equivalent, and the $15/3 = 5$ inequivalent ones can be taken to be:

$$\begin{array}{lll} |a\rangle = \begin{bmatrix} \underline{1} \\ \underline{0} \end{bmatrix}, & |b\rangle = \begin{bmatrix} \underline{0} \\ \underline{1} \end{bmatrix}, & |c\rangle = \begin{bmatrix} \underline{\omega} \\ \underline{1} \end{bmatrix}, \\ |d\rangle = \begin{bmatrix} \underline{\omega}^2 \\ \underline{1} \end{bmatrix}, & |e\rangle = \begin{bmatrix} \underline{1} \\ \underline{1} \end{bmatrix}, & \end{array} \tag{47}$$

Let us choose a pair of vectors as a basis and express the other three as linear combinations of those two, *e.g.*

$$|c\rangle \;=\; \underline{\omega}\,|a\rangle + |b\rangle\,, \quad |d\rangle \;=\; \underline{\omega}^2|a\rangle + |b\rangle\,, \quad |e\rangle \;=\; |a\rangle + |b\rangle\,. \tag{48}$$

Now consider a basis transformation that would interchange $|a\rangle$ and $|b\rangle$. This would leave $|e\rangle$ unchanged, but $|c\rangle$ and $|d\rangle$ would transform into each other:

$$\begin{aligned} |c\rangle &\to |a\rangle + \underline{\omega}\,|b\rangle \;\cong\; \underline{\omega}^2|a\rangle + |b\rangle \;=\; |d\rangle\;, \\ |d\rangle &\to |a\rangle + \underline{\omega}^2|b\rangle \;\cong\; \underline{\omega}\,|a\rangle + |b\rangle \;=\; |c\rangle\;. \end{aligned} \tag{49}$$

Thus, single transpositions of the vector labels are impossible. Transpositions must always come in pairs, and these would generate the alternating group $A_5 \cong PGL(2,4)$, the group of all even permutations of the five labels *abcde*. This is the group of 'rotations' for $GQM(2,4)$.

The inequivalent dual-vectors of V_4^* can be taken to be

$$\begin{aligned} \langle\bar{a}| &= \begin{bmatrix} \underline{0} & \underline{1} \end{bmatrix}, & \langle\bar{b}| &= \begin{bmatrix} \underline{1} & \underline{0} \end{bmatrix}, & \langle\bar{c}| &= \begin{bmatrix} \underline{1} & \underline{\omega} \end{bmatrix}, \\ \langle\bar{d}| &= \begin{bmatrix} \underline{1} & \underline{\omega}^2 \end{bmatrix}, & \langle\bar{e}| &= \begin{bmatrix} \underline{1} & \underline{1} \end{bmatrix}. \end{aligned} \tag{50}$$

The actions of these dual-vectors on the vectors are:

	$\lvert a\rangle$	$\lvert b\rangle$	$\lvert c\rangle$	$\lvert d\rangle$	$\lvert e\rangle$
$\langle\bar{a}\rvert$	$\underline{0}$	$\underline{1}$	$\underline{1}$	$\underline{1}$	$\underline{1}$
$\langle\bar{b}\rvert$	$\underline{1}$	$\underline{0}$	$\underline{\omega}$	$\underline{\omega}^2$	$\underline{1}$
$\langle\bar{c}\rvert$	$\underline{1}$	$\underline{\omega}$	$\underline{0}$	$\underline{1}$	$\underline{\omega}^2$
$\langle\bar{d}\rvert$	$\underline{1}$	$\underline{\omega}^2$	$\underline{1}$	$\underline{0}$	$\underline{\omega}$
$\langle\bar{e}\rvert$	$\underline{1}$	$\underline{1}$	$\underline{\omega}^2$	$\underline{\omega}$	$\underline{0}$

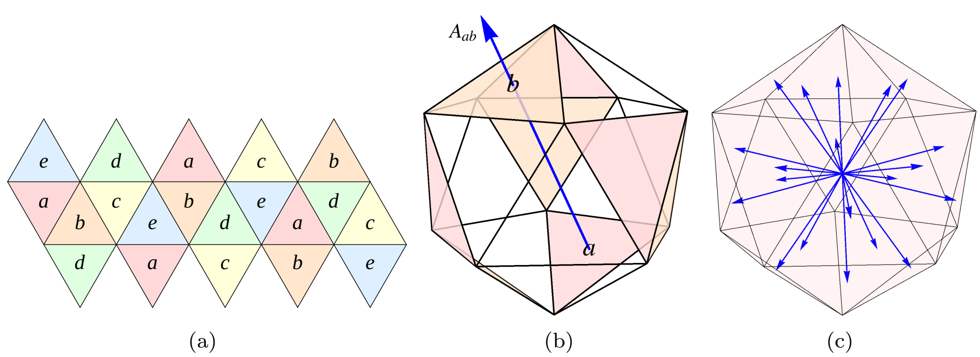
Thus,

$$\begin{aligned} \langle\bar{r}|s\rangle &= \underline{0} \quad \text{if } r = s, \\ &\neq \underline{0} \quad \text{if } r \neq s, \end{aligned} \tag{51}$$

and the relation $|\langle\bar{r}|s\rangle| = 1 - \delta_{rs}$ is obtained as before. Maintaining this relation would require relabeling the dual vectors in the same way as the vectors under basis transformations.

From the set of five dual-vectors, we can define $5 \times 4 = 20$ observables with outcomes ± 1, or 10 if we count the 'spins' pointing in opposite directions as the same observable. These 'spins' can be associated with actual directions in 3D, and their

Figure 4. To map 'rotations' in $PGL(2,4) \cong A_5$ to rotations in $SO(3)$: (a) label the faces of an icosahedron with five symbols as shown above left. Then, to every even permutation of the five labels *abcde* will correspond a rotation belonging to the icosahedral group. (b) The 'spin' observable A_{ab} in $GQM(2,4)$ can be mapped onto a direction in 3D space as shown. (c) The urchin diagram showing all 20 'spin' directions of $GQM(2,4)$.



$PGL(2,4) \cong A_5$ transformations with rotations in $SO(3)$ as shown in figure 4: First, label the faces of an icosahedron with five letters *abcde*, each letter appearing four time, as shown in figure 4(a). The centers of the four faces with the same letter are positioned at the vertices of a tetrahedron. The icosahedral group Y [16] which rotates the icosahedron onto itself consists of 60 elements. Each of these elements will lead to an even permutation of the five letters on the faces of the icosahedron. Thus, there exists a one-to-one correspondence between elements of the icosahedral group and the 60 permutations of A_5. The 'direction' of the 'spin' observable A_{ab} can be associated with the direction of the arrow shown in figure 4(b). All 20 'spin' directions can be mapped this way, and figure 4(c) shows the resulting urchin of spin-directions. These 20 'spins' transform into each other under $A_5 \cong Y$ rotations.

4.3. $\mathbb{Z}_5$ case

$GQM(2,5)$ is constructed on the field consisting of five elements, $GF(5) = \mathbb{Z}_5 = \mathbb{Z}/5\mathbb{Z} = \{\underline{0},\underline{1},\underline{2},\underline{3},\underline{4}\}$. The addition and multiplication tables of this field is given by

$+$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$
$\underline{0}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$
$\underline{1}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{0}$
$\underline{2}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{0}$	$\underline{1}$
$\underline{3}$	$\underline{3}$	$\underline{4}$	$\underline{0}$	$\underline{1}$	$\underline{2}$
$\underline{4}$	$\underline{4}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$

$\times$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$
$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$
$\underline{1}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$
$\underline{2}$	$\underline{0}$	$\underline{2}$	$\underline{4}$	$\underline{1}$	$\underline{3}$
$\underline{3}$	$\underline{0}$	$\underline{3}$	$\underline{1}$	$\underline{4}$	$\underline{2}$
$\underline{4}$	$\underline{0}$	$\underline{4}$	$\underline{3}$	$\underline{2}$	$\underline{1}$

We will denote $\underline{4} = -\underline{1}$ and $\underline{3} = -\underline{2}$ in the following. Since $\mathbb{Z}_5 \backslash \{\underline{0}\} = \{\pm\underline{1}, \pm\underline{2}\}$, each physical state will be represented by four vectors in $V_5 = \mathbb{Z}_5^2$ which differ by the multiplicative 'phases' $-\underline{1}$ or $\pm\underline{2}$.

Thus, of the $5^2 - 1 = 24$ non-zero vectors in V_5, every four of them are equivalent, and the $24/4 = 6$ inequivalent ones can be taken to be:

$$
\begin{array}{lll}
|a\rangle = \begin{bmatrix} \underline{1} \\ \underline{0} \end{bmatrix}, & |b\rangle = \begin{bmatrix} \underline{0} \\ \underline{1} \end{bmatrix}, & |c\rangle = \begin{bmatrix} \underline{2} \\ \underline{1} \end{bmatrix}, \\
|d\rangle = \begin{bmatrix} -\underline{1} \\ \underline{1} \end{bmatrix}, & |e\rangle = \begin{bmatrix} -\underline{2} \\ \underline{1} \end{bmatrix}, & |f\rangle = \begin{bmatrix} \underline{1} \\ \underline{1} \end{bmatrix}.
\end{array} \tag{52}
$$

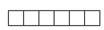
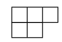
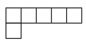
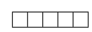
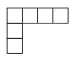
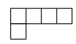
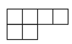
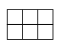
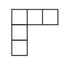
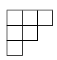
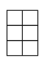
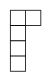
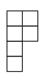
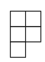
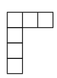
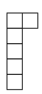
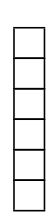
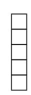
The group of basis transformations of this space is a subgroup of S_6, the group of permutations of the six vector labels. It consists of both odd and even permutations, and the distribution of its elements among the 11 conjugate classes of S_6 are shown in table 3. There are $120 = 5!$ elements in total in 7 conjugate classes. These numbers match those of S_5 exactly, and in fact, there is an isomorphism between the two, *i.e.* $PGL(2,5) \cong S_5$. Of the 120 elements of $PGL(2,5)$, a subgroup of 60 elements consisting of the even permutations, and isomorphic to A_5, can be mapped onto $SO(3)$ rotations in the icosahedral group Y as we will see below.

The inequivalent dual-vectors of V_5^* can be taken to be

$$
\begin{array}{lll}
\langle\bar{a}| = \begin{bmatrix} \underline{0} & -\underline{1} \end{bmatrix}, & \langle\bar{b}| = \begin{bmatrix} \underline{1} & \underline{0} \end{bmatrix}, & \langle\bar{c}| = \begin{bmatrix} \underline{1} & -\underline{2} \end{bmatrix}, \\
\langle\bar{d}| = \begin{bmatrix} \underline{1} & \underline{1} \end{bmatrix}, & \langle\bar{e}| = \begin{bmatrix} \underline{1} & \underline{2} \end{bmatrix}, & \langle\bar{f}| = \begin{bmatrix} \underline{1} & -\underline{1} \end{bmatrix}.
\end{array} \tag{53}
$$

The actions of these dual-vectors on the vectors are:

Table 3. $PGL(2,5)$ is a subgroup of S_6, which is isomorphic to S_5. This table shows how many elements in each conjugate class of S_6 are in $PGL(2,5)$, and the conjugate class in S_5 that they correspond to. The signs adjacent to the Young tableaux indicate the signature of the permutations in each class. Only the even permutations in $PGL(2,5)$, which form an invariant subgroup of order 60 isomorphic to A_5, can be mapped to elements in $SO(3)$.

Conjugate Classes of S_6		S_6	$PGL(2,5)$	Conjugate Classes of S_5		A_5
	$-$	120	20		$-$	
	$+$	144	24		$+$	$\surd$
	$-$	90	30		$-$	
	$+$	90	0			
	$+$	40	20		$+$	$\surd$
	$-$	120	0			
	$-$	15	10		$-$	
	$+$	45	15		$+$	$\surd$
	$+$	40	0			
	$-$	15	0			
	$+$	1	1		$+$	$\surd$
total		720	120			

	$\|a\rangle$	$\|b\rangle$	$\|c\rangle$	$\|d\rangle$	$\|e\rangle$	$\|f\rangle$
$\langle\bar{a}\|$	$\underline{0}$	$-\underline{1}$	$\underline{1}$	$-\underline{1}$	$-\underline{1}$	$-\underline{1}$
$\langle\bar{b}\|$	$\underline{1}$	$\underline{0}$	$\underline{2}$	$-\underline{1}$	$-\underline{2}$	$\underline{1}$
$\langle\bar{c}\|$	$\underline{1}$	$-\underline{2}$	$\underline{0}$	$\underline{2}$	$\underline{1}$	$-\underline{1}$
$\langle\bar{d}\|$	$\underline{1}$	$\underline{1}$	$-\underline{2}$	$\underline{0}$	$-\underline{1}$	$\underline{2}$
$\langle\bar{e}\|$	$\underline{1}$	$\underline{2}$	$-\underline{1}$	$\underline{1}$	$\underline{0}$	$-\underline{2}$
$\langle\bar{f}\|$	$\underline{1}$	$-\underline{1}$	$\underline{1}$	$-\underline{2}$	$\underline{2}$	$\underline{0}$

Thus,

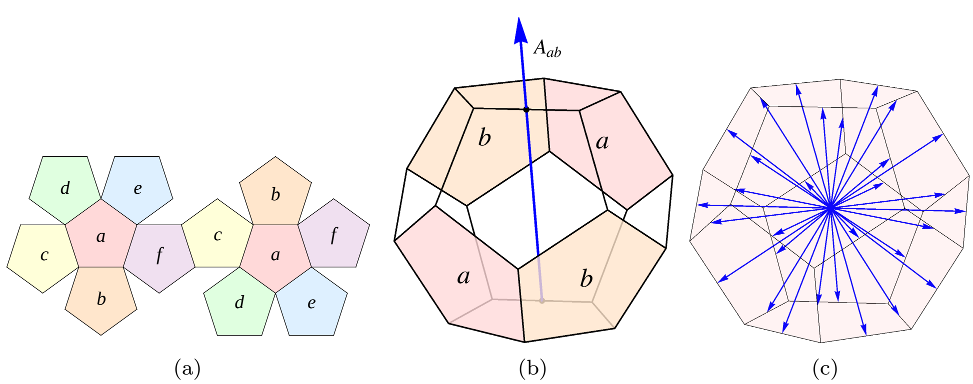
$$\begin{aligned} \langle \bar{r}|s\rangle &= \underline{0} \quad \text{if } r = s, \\ &\neq \underline{0} \quad \text{if } r \neq s, \end{aligned} \tag{54}$$

and the relation $|\langle \bar{r}|s\rangle| = 1 - \delta_{rs}$ is obtained as before. Maintaining this relation would require relabeling the dual vectors in the same way as the vectors under basis transformations.

From the set of six dual-vectors, we can define $6 \times 5 = 30$ observables with outcomes ± 1, or 15 if we count the 'spins' pointing in opposite directions as the same observable. These 'spins' can be associated with actual directions in 3D, and their transformations under the subgroup of $PGL(2,5)$ mentioned above with rotations in $SO(3)$ as shown in figure 5: First, label the 12 faces of a dodecahedron with 6 letters $abcdef$, with each letter appearing twice on faces that oppose each other, a shown in figure 5(a). Note that the dodecahedron is dual to the icosahedron, under the interchange of vertices and faces, so its symmetry group under rotations is the icosahedral group Y, which was shown to be isomorphic to A_5 in the $q = 4$ case we discussed above. Each of the rotations of the icosahedral group will lead to an even permutation of the six letters on the faces of the dodecahedron. These will generate the subgroup of $PGL(2,5)$ consisting of even permutations only. The 'direction' of the 'spin' observable A_{ab} can be associated with the direction of the arrow shown in figure 5(b). All 30 'spin' directions can be mapped this way, and figure 5(c) shows the resulting urchin of spin-directions. These 30 'spins' transform into each other under $A_5 \cong Y$ rotations.

Unfortunately, there are 60 more elements of $PGL(2,5)$ unaccounted for, and these do not seem to be representable as rotations or reflections of the dodecahedron. Thus, in a sense, the 'rotations' in the state space of $GQM(2,5)$ are much richer than a finite group of $SO(3)$ rotations.

Figure 5. Not all 'rotations' in $PGL(2,5) \cong S_5$ can be mapped to rotations in $SO(3)$. The even permutations, isomorphic to A_5, can be mapped as follows: (a) label the faces of a dodecahedron with six symbols as shown above left. Then, to every even permutation of the six labels $abcdef$ will correspond a rotation belonging to the icosahedral group. (b) The 'spin' observable A_{ab} in $GQM(2,5)$ can be mapped onto the direction in 3D space as shown. (c) The urchin diagram showing all 30 'spin' directions of $GQM(2,5)$.



4.4. $\mathbb{Z}_7$ and Beyond

As we have seen, for the $q = 2$, 3, 4, and 5 cases, the group $PGL(2, q)$ itself, or its invariant subgroup, is isomorphic to some polyhedral group, allowing for the identification of those $PGL(2, q)$ group elements with $SO(3)$ rotations. Therefore, our 'spins' can be considered objects that transform like canonical spin under 'rotations' for these cases.

Whether a similar pattern emerges for $GQM(2, 7)$ and beyond remains to be explored. In general, the $PGL(2, q)$ group is a subgroup of S_{q+1} of order $q(q^2 - 1)$. Constructing a correspondence via the method we employed in this paper would require the labeling of the faces of some polyhedron with $q + 1$ symbols. Given that only 5 Platonic solids and 13 Archimedean solids are at our disposal, it is not at all clear that such a correspondence exists for generic q.

5. Summary

In this paper, we have elaborated on Galois field quantum mechanics (GQM) introduced in our previous letter [5]. In particular we have examined in detail the cases of $GQM(2, 2)$, $GQM(2, 3)$, $GQM(2, 4)$ and $GQM(2, 5)$. The fascinating geometric structure that underlies GQM as encapsulated by the finite projective geometry [11, 12, 13] is interesting in itself, and it also represents the simplest theoretical playground for understanding some outstanding issues in the foundations of quantum theory, quantum information and quantum computation [17] (see also [10]).

It is not clear to us if GQM has an immediate realization in any physical system. Most dynamical systems exist in continuous spaces, with continuous group operations. However, there could exist parameter spaces of complex systems, such as those in quantum computation and information theory, which can be considered as finite fields, and in those cases, the results obtained here could have relevance. The pictures we present here could provide a context to better understand what is happening in these systems.

As already emphasized in [5], GQM should also prove useful in understanding the still mysterious super-quantum limit [4, 18] and its possible relation to quantum gravity [4]. That the Galois fields should be of relevance in going beyond quantum field theory has been conjectured a long time ago [6], and our effort should be understood as a natural realization of that prescient old intuition.

Features of GQM should shed light on questions raised in the geometric formulation of canonical quantum theory [19] and in the natural generalization of the geometric quantum theory [20] argued to be relevant to quantum gravity [21]. These questions lie outside of the scope of this paper, and will be taken up elsewhere [22].

Acknowledgments

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