AIP | Journal of Mathematical Physics

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Citation: Journal of Mathematical Physics **48**, 123502 (2007); doi: 10.1063/1.2815810
View online: http://dx.doi.org/10.1063/1.2815810
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/48/12?ver=pdfcov
Published by the AIP Publishing

JOURNAL OF MATHEMATICAL PHYSICS **48**, 123502 (2007)

# Coupling constant behavior of eigenvalues of Zakharov-Shabat systems

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(Received 25 May 2007; accepted 26 October 2007; published online 5 December 2007)

We consider the eigenvalues of the non-self-adjoint Zakharov-Shabat systems as the coupling constant of the potential is varied. In particular, we are interested in eigenvalue collisions and eigenvalue trajectories in the complex plane. We identify shape features in the potential that are responsible for the occurrence of collisions and we prove asymptotic formulas for large coupling constants that tell us where eigenvalues collide or where they emerge from the continuous spectrum. Some examples are provided which show that the asymptotic methods yield results that compare well with exact numerical computations. © *2007 American Institute of Physics.* [DOI: 10.1063/1.2815810]

## I. INTRODUCTION

In this paper, we are concerned with the eigenvalues of Zakharov-Shabat systems of the form[1,16,21]

$$v_1' = -i\xi v_1 + q(t)v_2, \quad v_2' = -q(t)^* v_1 + i\xi v_2, \tag{1.1}$$

where $\xi$ is a complex-valued eigenvalue parameter, $q$ is a real function (also called the potential) of the real variable $t$ ($-\infty < t < \infty$), and the asterisk denotes the complex conjugate. Further conditions will be placed on $q$ below. An eigenvalue of (1.1) is a complex number $\xi$ for which there exists a nontrivial solution of (1.1) that is square integrable on the real line, i.e., $\int_{-\infty}^{\infty}(|v_1(t)|^2 + |v_2(t)|^2)dt < \infty$. The assumption that $q$ is real is very common in the engineering literature as it includes the conventional profiles such as Gaussians, hyperbolic secants, and rectangular (piecewise constant) shapes. As a consequence, purely imaginary eigenvalues appear as conjugate pairs and nonimaginary eigenvalues come in groups of four: $\xi$, $\xi^*$, $-\xi$, and $-\xi^*$. There are no real eigenvalues and the real axis constitutes the continuous spectrum of (1.1). Therefore, we need only be concerned with eigenvalues in $\mathbb{C}^+ = \{\xi \in \mathbb{C} : \text{Im } \xi > 0\}$. The eigenvalues of (1.1) play an important role in the solution via the inverse scattering technique of certain nonlinear evolution equations, such as the nonlinear Schrödinger equation (NLS), the sine-Gordon equation, the modified Korteweg–de Vries equation, etc. For example, in the case of the NLS, eigenvalues of (1.1) correspond to soliton solutions and thus are directly linked to applications in fiber optics (see, for example, the section on "eigenvalue communication" on p. 59 of Ref. 6). In the case of the sine-Gordon and modified Korteweg–de Vries (mKdV) equations eigenvalue pairs are associated with "breathers."[1,3] It has been observed in many situations[3,6,9] that if the potential depends on a parameter, then, as the parameter passes through certain threshold values, eigenvalues may emerge into $\mathbb{C}^+$ or reversely, eigenvalues may get absorbed into the continuous spectrum. It may also happen that, as the parameter approaches certain values, two (or more) eigenvalues coalesce into

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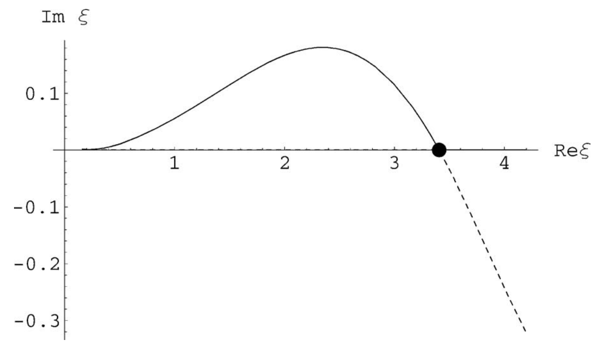


FIG. 1. Eigenvalue branch near a double root at $\xi=0$. The solid part of the curve is the eigenvalue, the dashed part is the continuation of the eigenvalue as a root of $v_1(d;\xi,\mu)=0$ into the lower half plane.

a multiple eigenvalue. Which of these possibilities are realized depends on the specific manner in which the potential depends on the parameter. In this paper, we will use as parameter the strength of the pulse, represented by a coupling constant $\mu>0$, and consider potentials of the form

$$q(t)=\mu p(t), \tag{1.2}$$

where $p(t)$ is a given real-valued function obeying suitable conditions. With a few exceptions we will assume throughout the paper that $p$ has compact support and is nonnegative, because from the viewpoint of applications in optical communication this is the most realistic situation. Note that the parameter $\mu$ also controls the area of the pulse which is known to play an important role in the study of optical pulses going back at least to Ref. 15.

Numerical studies show that, as functions of $\mu$, eigenvalues may exhibit some interesting and noteworthy behavior. For example, we find that nonimaginary eigenvalues follow the upper portions of certain meandering curves (see Figs. 1–5), a behavior that, to the best of our knowledge, has not been reported previously in the literature. In this paper, we make an effort by analytical means toward a more thorough understanding of these eigenvalue trajectories. Another goal is to study collisions of eigenvalues in order to better understand what properties of the potential might be causing them and how frequent they are. We know from prior work[9,11] that if a potential is "single lobe" (i.e., is nondecreasing on the left of some point $t_0$ and nonincreasing on the right), then all eigenvalues must be purely imaginary and simple, and hence cannot collide as $\mu$ varies. Therefore, we are forced to look at potentials with more than one peak in order to find nonimaginary eigenvalues and collisions. As we will see in the course of the paper, the case of a two-hump potential with two symmetric peaks is already quite rich in interesting phenomena. It turns out that apart from being doubly peaked, other details of the shape of the potential are factors in whether there are collisions; in particular, the peaks of a (symmetric) potential must be sufficiently pronounced (see Theorem 2.4). We should mention a technical caveat concerning the types of collisions we have observed numerically and are able to treat analytically. For potentials of the form

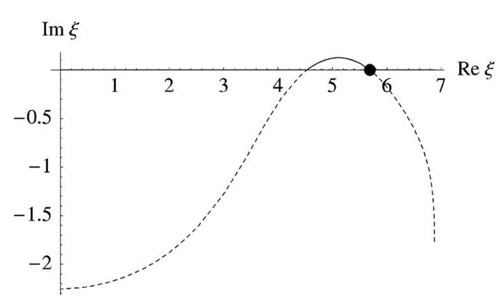


FIG. 2. Eigenvalue branch with $\check{\mu}=\check{\xi}=3\pi/(4a)=5.68$ and $\gamma=1$ in $\mathcal{S}_{-}$.

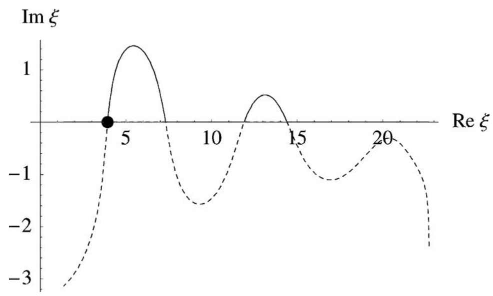


FIG. 3. Eigenvalue branch with $\check{\mu}=35.97$, $\check{\xi}=3.92$, and $\gamma=0.11$ in $\mathcal{S}_{-}$.

(1.2) (with $p$ of compact support and nonnegative), we have only observed the situation where two eigenvalues approach the imaginary axis, coalesce into a double eigenvalue (with algebraic multiplicity 2 and geometric multiplicity 1) which in turn splits into two purely imaginary eigenvalues with the lower eigenvalue dropping down to zero, where it is subsumed into the continuous spectrum. The reverse process, where two initially purely imaginary eigenvalues collide and then split into a pair symmetric about the imaginary axis, we have not seen, and it is our conjecture that this is a consequence of the potential being of one sign. However, a proof of this conjecture has so far eluded us. If the potential has both signs, then the latter type of collisions can occur; an example will be given in Sec. II.

Our motivation for studying collisions stems in part from earlier work on Zakharov-Shabat systems with chirped potentials[9] and from numerical studies on the formation of solitons and breathers for the mKdV equation[3] which raise the question as to what shape features of the potential profile may be primarily responsible for collisions. Since previous work on collisions seems to have been only numerical and for a very restricted class of potentials, we felt motivated to make a push on the analytical side to obtain a better insight into the occurrence of collisions with the aim of lending more reliable support to researchers doing numerical calculations.

The paper is organized as follows. In Sec. II we concentrate on the point $\xi=0$ as the location where eigenvalues either emerge from or get absorbed into the continuous spectrum. For reasons of symmetry, the point $\xi=0$ plays a distinguished role. We establish a connection between eigenvalue absorption at $\xi=0$ and eigenvalue collisions on the imaginary axis, and we prove several results that elucidate how certain shapes of the pulse profile $p(t)$ support collisions while others do not. For certain types of potentials, we find that, as $\mu$ increases to infinity, there are at most finitely many eigenvalue absorptions at $\xi=0$, while for others there are infinitely many. Furthermore, in some cases it is possible to obtain the asymptotic behavior as $\mu\to\infty$ of the number of absorptions at $\xi=0$; (see Theorem 2.6 for details).

In Sec. III we study eigenvalue collisions on the imaginary axis for a class of potentials having two symmetric lobes. The main result is Theorem 3.3 which contains asymptotic formulas for the points on the imaginary axis where eigenvalue collisions take place. The collision points

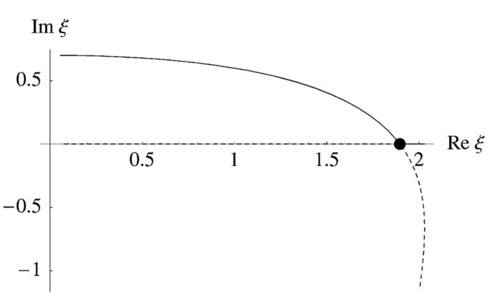


FIG. 4. Eigenvalue branch with $\check{\mu}=\check{\xi}=\pi/(4a)=1.89$ and $\gamma=1$ in $\mathcal{S}_{+}$.

TABLE I. Exact and approximate spectral singularities and coupling constants for the potential used for Fig. 5.

| $n$ | $j$ | $\check{\mu}_{j;ap}$ | $\check{\mu}_{j;ex}$ | $\check{\xi}_{j;ap}$ | $\check{\xi}_{j;ex}$ |
|---|---|---|---|---|---|
| 5 | 5 | 21.77 | 22.51 | 18.18 | 17.85 |
| 4 | 5 | 27.10 | 25.99 | 15.78 | 15.96 |
| 3 | 6 | 31.58 | 31.78 | 10.98 | 10.95 |
| 2 | 6 | 35.40 | 35.29 | 7.84 | 7.85 |
| 1 | 7 | 38.69 | 38.71 | 3.67 | 3.67 |

computed from these formulas are surprisingly accurate even for small values of $\mu$ and they provide us with excellent starting values for a numerical search. The numerical results for two cases are presented at the end of the section.

Section IV is devoted to the localization of spectral singularities for a class of two-lobe potentials. These are points on the real axis where eigenvalues appear or disappear as the coupling constant is varied. The main result (Theorem 4.8) tells us where these spectral singularities are asymptotically located as $\mu \to \infty$. Comparsion of analytical and numerical results shows good agreement (see Table I). An interesting and new result is that the spectral singularities for the family $\{\mu p : \mu > 0\}$ have an infinite number of cluster points that can be determined exactly. For piecewise constant two-lobe potentials, this can be seen by explicit calculations, but for a more general class of potentials, the rigorous treatment requires detailed estimates which is the main reason why this section is quite long. Our analysis will also provide strong evidence for the picture of meandering eigenvalue branches shown in Figs. 5 and 3. Furthermore, Theorem 4.1 shows that for certain double lobe potentials spectral singularities different from zero cannot occur. This allows us to argue that all eigenvalues must be purely imaginary and thus we obtain an extension of the single lobe theorem of Ref. 11.

An Appendix contains the proof of Theorem 3.3.

We also would like to mention that while this manuscript was in the review stage, we came across the paper of Ref. 4. It contains results on the location of complex eigenvalues for rectangular symmetric double lobe profiles and also shows how such eigenvalues affect the time evolution of optical pulses.

## II. EMERGENCE AND ABSORPTION OF EIGENVALUES AT $\xi$=0

In this section, we consider (1.1) with potential (1.2), where $p$ has compact support $[-d,d]$ $(d>0)$, is even and real, and satisfies further conditions detailed below. Our goal is to establish some results that tell us when, as $\mu$ varies, an eigenvalue either emerges from $\xi=0$ and enters $\mathbb{C}^+$ or, alternatively, approaches $\xi=0$ and is absorbed into the continuous spectrum. The condition for $\xi \in \mathbb{C}^+$ to be an eigenvalue of (1.1) with potential (1.2) is equivalent to the requirement that (1.1) have a solution $v(t;\xi,\mu)=(v_1(t;\xi,\mu),v_2(t;\xi,\mu))^T$ satisfying

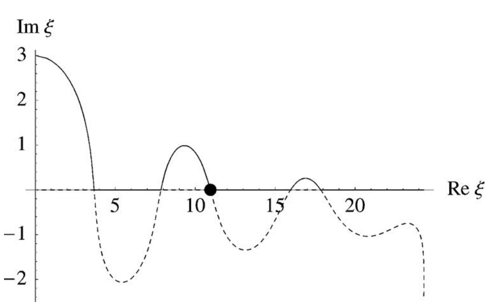


FIG. 5. Eigenvalue branch with $\check{\mu}=31.78$, $\check{\xi}=10.95$, and $\gamma=0.34$ in $\mathcal{S}_+$.

$$v_1(-d;\xi,\mu)=1, \quad v_2(-d;\xi,\mu)=0, \quad v_1(d;\xi,\mu)=0,$$

where the first equality is a normalization. It follows that the geometric multiplicity of an eigenvalue is always 1. A well known feature of system (1.1) is that for $\xi=0$, it can be solved explicitly.[5] The solution satisfying $v(-d;0,\mu)=(1,0)^T$ is given by

$$v_1(t;0,\mu)=\cos\left(\mu\int_{-d}^{t} p(\tau)d\tau\right), \tag{2.1}$$

$$v_2(t;0,\mu)=-\sin\left(\mu\int_{-d}^{t} p(\tau)d\tau\right). \tag{2.2}$$

Hence, the critical values of $\mu$ for which an eigenvalue appears at $\xi=0$ are given by the zeros of $v_1(d;0,\mu)$, that is,

$$\mu_k=\frac{(2k-1)\pi}{4\int_0^d p(t)dt}, \quad k=1,2,3,\ldots . \tag{2.3}$$

This means that for $\mu=\mu_k$, the area under the curve of $p(t)$ is an odd multiple of $\pi/2$. Actually, this is a well known fact which holds for general nonnegative $L^1$ potentials without the assumption of even symmetry.[11] Let $(u_1,u_2)^T$ denote a second solution of (1.1) so that

$$\det\begin{pmatrix} v_1 & u_1 \\ v_2 & u_2 \end{pmatrix}=1,$$

and let a subscript $\mu$ ($\xi$) indicate the partial derivative with respect to $\mu$ ($\xi$). We note that $v_1(d;\xi,\mu)$ is analytic in both $\xi$ and $\mu$. Application of the variation of parameters formula yields, at an eigenvalue $\xi$,

$$v_{1;\mu}(d;\xi,\mu)=-u_1(d;\xi,\mu)\int_{-d}^{d} p(t)[(v_1(t;\xi,\mu)^2+v_2(t;\xi,\mu)^2\ ]dt,$$

$$v_{1;\xi}(d;\xi,\mu)=2iu_1(d;\xi,\mu)\int_{-d}^{d} v_1(t;\xi,\mu)v_2(t;\xi,\mu)dt.$$

Now suppose that $\xi(\mu)\in\mathbb{C}^+$ is a (continuous) eigenvalue branch of (1.1), that is, we have $v_1(d;\xi(\mu),\mu)=0$. Thus

$$\xi'(\mu)=-\frac{v_{1;\mu}(d;\xi(\mu),\mu)}{v_{1;\xi}(d;\xi(\mu),\mu)}, \tag{2.4}$$

provided $v_{1;\xi}(d;\xi(\mu),\mu)\neq 0$; hence,

$$\xi'(\mu)=-\frac{i\int_{-d}^{d} p(t)[v_1(t;\xi(\mu),\mu)^2+v_2(t;\xi(\mu),\mu)^2]dt}{2\int_{-d}^{d} v_1(t;\xi(\mu),\mu)v_2(t;\xi(\mu),\mu)dt}. \tag{2.5}$$

If $\mu=\mu_k$, so that $\xi(\mu_k)=0$, then (2.5) also appears in Ref. 5, where it was applied analytically and numerically to some special potentials (sech-like potentials and Gaussians). Equation (2.4) [(2.5)] represents a differential equation for $\xi(\mu)$ which has proved useful for tracing eigenvalues. All of our numerical eigenvalue calculations are based on (2.4). Note that $v_1(t;\xi,\mu)$ is readily obtained from (1.1) by numerical integration and then can be fed into the nonhomogeneous systems satisfied by $v_{1;\xi}(t;\xi,\mu)$ and $v_{1;\mu}(t;\xi,\mu)$. Then (2.4) can be solved by standard numerical methods.

If $\xi$ is purely imaginary, then $v(t;\xi,\mu)$ has real components. Hence the sign of the denominator in (2.5) determines whether an eigenvalue moves up or down along the imaginary axis as $\mu$ is increased.

For the subsequent analysis, it is important to realize that by the even symmetry of $p$, eigenfunctions must belong to one of the subspaces,

$$\mathcal{S}_{\pm}=\{v \text{ bounded}: v_1(0)=\pm v_2(0)\}, \tag{2.6}$$

so that

$$v_1(t)=v_2(-t) \quad \text{on } \mathcal{S}_+,$$

$$v_1(t)=-v_2(-t) \quad \text{on } \mathcal{S}_-.$$

Henceforth we will often suppress arguments and parameters from our notation if, in our judgement, this should not cause any difficulty in understanding. Note that in (2.6) we do not require $v$ to be in $L^2$ because we also want to use these subspaces for solutions at real $\xi$. In particular, if $\xi=0$, then $v(t;0,\mu_k)$ belongs to $\mathcal{S}_+$ ($\mathcal{S}_-$) precisely when $k$ is even (odd). We will also use the notation "(1.1$\pm$)" to denote the restriction of (1.1) to $\mathcal{S}_{\pm}$, respectively. It is clear from the linear independence of (nonzero) vectors in $\mathcal{S}_+$ and $\mathcal{S}_-$ that "(1.1+)" and "(1.1−)" cannot have eigenvalues in common and hence eigenvalues associated with different subspaces cannot collide. This is also a direct consequence of the next result which is elementary but may be worth pointing out.

*Lemma 2.1: Suppose that p is even and has compact support. Let, for a given* $\xi\in\overline{\mathbb{C}^+}$ *and* $\mu\geqslant 0$, $v^+(t)=(v_1^+(t),v_2^+(t))^T$ *denote the solution of (1.1+) satisfying* $v^+(0)=(1,1)^T$ *and let* $v^-(t)=(v_1^-(t),v_2^-(t))^T$ *denote the solution of (1.1−) satisfying* $v^-(0)=(1,-1)^T$. *Let* $v(t)=(v_1(t),v_2(t))^T$ *be the solution of (1.1) obeying* $v(-d)=(1,0)^T$. *Then,*

$$v_1(d;\xi,\mu)=v_1^+(d;\xi,\mu)v_1^-(d;\xi,\mu).$$

*Proof:* Write $v$ as a linear combination of $v^+$ and $v^-$ and use the symmetries of $v^+$ and $v^-$, together with the fact that the Wronskian of $v^+$ and $v^-$ is constant (equal to −2). □

As we will see below, it is perfectly possible that eigenvalues associated with the same subspace may collide.

We now turn our attention to those eigenvalue branches that approach zero as $\mu$ approaches one of the values $\mu_k$. Assuming $v_{1;\xi}(d;0,\mu_k)\neq 0$, the implicit function theorem gives us a unique function $\xi_k(\mu)$ defined for $\mu$ near $\mu_k$, such that $v_1(d;\xi_k(\mu),\mu)=0$. If $\operatorname{Im}\xi_k'(\mu_k)>0$, then there is an $\epsilon>0$ such that for all $\mu\in(\mu_k,\mu_k+\epsilon)$ the eigenvalue $\xi_k(\mu)$ lies on the positive imaginary axis and converges to zero as $\mu\downarrow\mu_k$. If $\mu$ is slightly less than $\mu_k$, then $\xi_k(\mu)$ lies on the negative imaginary axis; it is a root of the equation $v_1(d;\xi,\mu)=0$ but does not represent a genuine eigenvalue, because the eigenfunction for an eigenvalue in $\mathbb{C}^-$ would have to satisfy $v_1(-d;\xi,\mu)=0$. If $\operatorname{Im}\xi_k'(\mu_k)<0$, then the situation is reversed in an obvious way. If $v_{1;\xi}(d;0,\mu_k)=0$, in addition to $v_1(d;0,\mu_k)=0$, then we have a special situation which will be addressed later in this section. Our reason for monitoring eigenvalues as they pass through zero is that there is a connection between eigenvalue absorptions and eigenvalue collisions on the imaginary axis. In the following, a superscript $c$ stands for "collision," so $\mu^c$ is a coupling constant at which a collision takes place. By a "simple" eigenvalue we mean an eigenvalue of algebraic multiplicity one. Hence, "nonsimple" eigenvalues have algebraic multiplicity at least 2 (but always geometric multiplicity 1) and hence may split into two or more eigenvalue branches as $\mu$ varies. For the results of the next theorem it suffices that $p$ be real, in $L^1$, and have compact support; symmetry is not required and, in fact, the assumption of compact support could also be dropped [since (2.5) holds for general $L^1$ potentials when $\operatorname{Im}\xi(\mu)>0$].

*Lemma 2.2: Suppose that* $\operatorname{Im} \xi_k'(\mu_k) < 0$ *for some* $k$, *where* $\mu_k$ *is given by (2.3) such that* $\xi_k(\mu_k)=0$. *Then there exists a positive number* $\mu_k^c < \mu_k$ *and a function* $\xi_k(\mu)$ *defined for* $\mu_k^c \leqslant \mu \leqslant \mu_k$ *such that* $\xi_k(\mu)$ *is a simple purely imaginary eigenvalue of (1.1) for* $\mu_k^c < \mu < \mu_k$ *and a nonsimple eigenvalue for* $\mu = \mu_k^c$.

*Proof:* This follows from the basic existence theory for solutions of differential equations. The largest open $\mu$-interval with right endpoint $\mu_k$ on which a unique solution $\xi_k(\mu)$ of (2.4) satisfying $\xi_k(\mu_k)=0$ exists is the interval $(\mu_k^c, \mu_k)$. For $\mu=\mu_k^c$ and $\xi=\xi_k^c$ $(=\xi_k(\mu_k^c))$, the denominator on the right-hand side of (2.4) becomes zero, which means that the eigenvalue is nonsimple (see Ref. 11). □

Lemma 2.2 says that an eigenvalue which is absorbed into the continuous spectrum at zero must originate from a collision of (at least) two eigenvalues. As already mentioned in the Introduction, the only scenario we have observed is that when two eigenvalues approach (as $\mu$ increases) a point on the imaginary axis from the left and right half planes, respectively, coalesce into a double eigenvalue and then splits into a pair of purely imaginary eigenvalues. The lower one of the eigenvalues then drops down to zero as described in Lemma 2.2. The other scenario, where two initially imaginary eigenvalues collide and split into a pair of nonimaginary eigenvalues (as $\mu$ increases), we have so far observed only if the potential has both signs. Here is an example: Suppose that $p(t)=8$ for $-1 \leqslant t \leqslant 0$, $p(t)=-4$ for $0 < t \leqslant 1$, and $p(t)=0$ otherwise. Then, as $\mu$ increases, two nonimaginary eigenvalues collide when $\mu=2.401$ at $\xi=5.783i$. The upper eigenvalue of the resulting imaginary pair collides again with an eigenvalue coming from above when $\mu=2.409$ at $\xi=6.816i$, producing two nonimaginary eigenvalues. The upper eigenvalue in this second collision is actually an eigenvalue that made a "U-turn" at $\mu=2.381$ and $\xi=7.167i$. Even though, as we will see later, such a reversal of direction is not possible when $p$ is of one sign, it does not allow us to conclude that the second scenario cannot occur for potentials of one sign. As observed in p. 389 of Ref. 3, the type of collision exhibited in this example also occurs when the width of a potential (having both signs) is varied instead of the coupling constant.

To see what happens analytically when a collision occurs, we assume that two or more eigenvalues meet at a collision point $\xi^c$ on the imaginary axis when $\mu=\mu^c$. Expanding $v_1(d;\xi,\mu)$ near $(\xi^c,\mu^c)$, we have that

$$v_1(d;\xi,\mu) = \sum_{s,j=0}^{\infty} \alpha_{sj}(\xi-\xi^c)^s(\mu-\mu^c)^j,$$

where $\alpha_{00}=\alpha_{10}=0$, $\alpha_{01}\neq 0$, and $\alpha_{sj}=(-1)^s\overline{\alpha_{sj}}$, since $v_1(d;\xi,\mu)$ is real when $\xi$ is purely imaginary. In the standard situation (first scenario), we have that $\alpha_{20}\neq 0$ and $\alpha_{01}/\alpha_{20}>0$ so that $\xi-\xi^c \sim \pm i(\alpha_{01}/\alpha_{20})^{1/2}(\mu-\mu^c)^{1/2}$ as $\mu\to\mu^c$. We note that the leading term stated here is in fact the leading term of a convergent series (the Puiseux series[8]) in powers of $(\mu-\mu^c)^{1/2}$. The second scenario occurs if $\alpha_{01}/\alpha_{20}<0$. Of course, it could also happen that $\alpha_{20}=0$, or more generally, that $\alpha_{s0}=0$ for $s=0,1,\ldots,m-1$ $(m\geqslant 3)$ and $\alpha_{m0}\neq 0$; note that $a_{m0}=0$ for all $m$ is not possible, since $v_1(d;\xi,\mu^c)$ is analytic in $\xi$ and not identically zero. Then we have $m$ distinct simple eigenvalues colliding at $\xi^c$ according to $\xi-\xi^c\sim(-\alpha_{01}/\alpha_{m0})^{1/m}(\mu-\mu^c)^{1/m}$, with $m$ possible values for the roots. These higher-order collisions will not be studied here.

At $\xi=0$ and for $\mu=\mu_k$, (2.1), (2.2), and (2.5) reduce to

$$\xi_k'(\mu_k) = \frac{i(-1)^{k+1}A}{\int_0^d \cos\left(2\mu_k\int_0^t p(\tau)d\tau\right)dt}, \tag{2.7}$$

where

$$A = \int_0^d p(t)dt.$$

In deriving (2.7) we have also used the fact that $p(t)$ and $v_1(t;0,\mu_k)v_2(t;0,\mu_k)$ are even functions of $t$. For some of the proofs below, it will be convenient to set

$$\tilde{p}(t) = p(d-t) \tag{2.8}$$

and to use, in place of (2.7), the formula

$$\xi_k'(\mu_k) = \frac{iA}{\int_0^d \sin\left(2\mu_k \int_0^t \tilde{p}(\tau)d\tau\right)dt}, \tag{2.9}$$

which easily follows from (2.7) on using (2.3) and (2.8).

The next two theorems give us information about the sign of Im $\xi_k'(\mu_k)$ and tell us whether an eigenvalue is emerging from $\xi=0$ or is getting absorbed there.

**Theorem 2.3:** *Suppose that $p$ is even, absolutely continuous on $[0,d]$, and positive on $(0,d)$. If either*

(i) *$p(0)>0$ and $p(d)>0$, or*

(ii) *$p(0)>0$, $p(d)=0$, and there exists an $\epsilon_0\in(0,d)$ such that on $(d-\epsilon_0,d)$, $p'(t)/p(t)^3$ is negative and decreasing,*
*then, for sufficiently large $k$,*

$$\int_0^d \sin\left(2\mu_k \int_0^t \tilde{p}(\tau)d\tau\right)dt > 0, \tag{2.10}$$

*and thus* Im $\xi_k'(\mu_k)>0$.

(iii) *Suppose $p(d)>0$, $p(0)=0$, and, as $t\to 0$, $p(t)\sim ct^\beta$ ($c>0$, $\beta>0$). Then* Im $\xi_k'(\mu_k)>0$ *for all large enough odd $k$ and* Im $\xi_k'(\mu_k)<0$ *for all large enough even $k$.*

*Remarks:* In part (ii) the assumption concerning $p'(t)/p(t)^3$ is equivalent to the demand that $1/p(t)^2$ be convex near $d$. Included are potentials that behave like $(d-t)^\gamma$ $(0<\gamma<\infty)$ or $e^{-1/(d-t)^\gamma}$ $(\gamma>0)$ as $t\to d$. The reader may have noticed that the vanishing conditions in (ii) near $d$ and those in (iii) near 0 do not mirror each other. In fact, if in (iii) we were to replace the condition near zero by the requirement that $1/p(t)^2$ be convex (near zero), then the conclusion of (iii) would no longer be true in general. The following example illustrates this point. Set $w(t)=\int_0^t p(\tau)d\tau$ and define $w(t)$ through the implicit equation $w-w\ln w=t$ for $0<t\le d$, where $d\in(0,1)$ is the right endpoint of the support of $p(\tau)$, and set $w(0)=0$. Then $w(t)\sim -t/\ln t$ as $t\downarrow 0$ and hence $p(t)=w'(t)\sim -1/\ln t$. Also, $p(t)$ is continuous, increasing on $[0,d]$, and $1/p(t)^2$ is convex, since $p'(t)/p(t)^3=1/w(t)$. For the denominator in (2.7), we obtain [since $dt=(-\ln w)dw$]

$$\int_0^d \cos\left(2\mu_k \int_0^t p(\tau)d\tau\right)dt = \int_0^{w(d)} \cos(2\mu_k w)(-\ln w)dw = \frac{(-1)^{k+1}}{2\mu_k}(-\ln w(d)) + \frac{\pi}{4\mu_k} + O\left(\frac{1}{\mu_k^2}\right)$$

as $k\to\infty$. We see that if $-\ln w(d)>\pi/2$ [note that $w(d)<1$], then Im $\xi_k'(\mu_k)>0$ for sufficiently large $k$. This happens when $d$ is chosen sufficiently small. In this case, the conclusion of part (iii) is not true. However, if $-\ln w(d)<\pi/2$, which holds if $d$ is close to 1, then Im $\xi_k'(\mu_k)>0$, provided $k$ is large enough and odd, while Im $\xi_k'(\mu_k)<0$ if $k$ is large enough and even. This is the same conclusion as that in (iii).

*Proof:* (i) follows directly from (2.10) via an integration by parts which shows that the integral in (2.10) is equal to $(2\mu_k p(d))^{-1}+o(\mu_k^{-1})$ for large $k$. To prove (ii), we first make a change of variable. We set

$$u = \frac{1}{A}\int_0^t \tilde{p}(\tau)d\tau, \tag{2.11}$$

where $A$ is given in (2.7), thereby defining a one-to-one function $t(u)$ satisfying $t(0)=0$ and $t(1)=d$. Define

$$I(\lambda) = \int_0^1 f(u)\sin(\lambda u)du, \tag{2.12}$$

with

$$f(u) = \frac{1}{\tilde{p}(t(u))}, \tag{2.13}$$

so that by (2.3) and (2.11), the integral in (2.10) is equal to $AI(\lambda_k)$, where

$$\lambda_k = (2k-1)\pi/2.$$

Hence we need to show that under the given assumptions $I(\lambda_k)>0$ for sufficiently large $k$. We note that, by (2.11),

$$\tilde{p}(t(u))t'(u) = A,$$

and hence

$$f'(u) = -A\frac{\tilde{p}'(t(u))}{\tilde{p}(t(u))^3}. \tag{2.14}$$

Let $u_{\epsilon_0}$ be such that $t(u_{\epsilon_0})=\epsilon_0$ and let $k$ be so large that $\pi/(2\lambda_k)<u_{\epsilon_0}/2$. Also, define $\tilde{u}_{\lambda_k}$ as the unique point in $[u_{\epsilon_0}-2\pi/\lambda_k, u_{\epsilon_0}]$, where $\sin(\lambda_k\tilde{u}_{\lambda_k})=1$.

Now write $I(\lambda_k)$ as a sum of integrals over $[0,\pi/(2\lambda_k)]$, $[\pi/(2\lambda_k),\tilde{u}_{\lambda_k}]$, $[\tilde{u}_{\lambda_k},1]$, and denote these integrals by $I_1(\lambda_k)$, $I_2(\lambda_k)$, and $I_3(\lambda_k)$, respectively. The assumption about the behavior of $p(t)$ near $t=d$ translates into the statement that for $0<u<u_{\epsilon_0}$, $f'(u)$ is negative and increasing. Also, $f(u)$ is decreasing on $[0,\pi/(2\lambda_k)]$. These observations imply the next two estimates,

$$I_1(\lambda_k) > f(\pi/(2\lambda_k))\lambda_k^{-1}, \tag{2.15}$$

$$I_2(\lambda_k) = \int_{\pi/(2\lambda_k)}^{\tilde{u}_{\lambda_k}} f'(u)\frac{\cos(\lambda_k u)}{\lambda_k}du > 0, \tag{2.16}$$

where in deriving (2.16), we used an integration by parts and the fact that $\cos(\lambda_k u)$ vanishes at the endpoints of the integration interval. A similar integration by parts yields

$$|I_3(\lambda_k)| \le C\lambda_k^{-1}, \tag{2.17}$$

where $C=\sup_{u\in[u_{\epsilon_0}/2,1]}(|f(u)|+|f'(u)|)$. Since $f(\pi/(2\lambda_k))\to\infty$ as $k\to\infty$, (2.10) follows by combining (2.15)–(2.17). This proves (ii). From the assumptions of (iii), it follows that

$$f(u) \sim a_\beta(1-u)^{-\beta/(\beta+1)}, \quad u\to 1,$$

where $a_\beta=[A(\beta+1)]^{-\beta/(\beta+1)}c^{-1/(\beta+1)}$. Then the stationary phase method (see, Sec. 6.1 of Ref. 2) yields

$$I(\lambda_k) \sim d_\beta(-1)^{k+1}\lambda_k^{-1/(\beta+1)}, \quad k\to\infty, \tag{2.18}$$

where

$$d_\beta = a_\beta \int_0^\infty w^{-\beta/(\beta+1)} \cos w \, dw = a_\beta \Gamma\left(\frac{1}{\beta+1}\right) \sin\left(\frac{\pi\beta}{2(\beta+1)}\right).$$

The assertions of (iii) are now obvious. □

We would like to comment that integrals of type (2.12) have been studied long ago by Pólya[17] in connection with the distribution of zeros of entire functions. More recently, Sedletskii[19] has relaxed some of Pólya's assumptions and a result equivalent to $I(\lambda_k)>0$ for all $k\geqslant 2$ was proved in Ref. 19 under suitable conditions on $f(u)$.

Theorem 2.3 in essence tells us that an even, smooth potential of compact support that is positive (or negative) everywhere on its support exhibits at most a finite number of eigenvalue absorptions at $\xi=0$ as $\mu$ increases [case (i)]. This still holds true if the potential vanishes at the endpoints of the support [case (ii)] but need not be true any longer if it vanishes at an interior point [case (iii)].

Under some more stringent assumptions, we can obtain results that do not require $k$ to be sufficiently large.

**Theorem 2.4:** *Suppose that $p$ is even, absolutely continuous on $[0,d]$, and positive on $(0,d)$.*

(i) *If $p'(t)\leqslant 0$ for all $t\in(0,d)$, then $\operatorname{Im}\xi_k'(\mu_k)>0$ for all $k$.*

(ii) *If $p(t)$ has a single maximum at the point $0<t_0\leqslant d$, and $p(t_0)<2p(0)$, then $\operatorname{Im}\xi_k'(\mu_k)>0$ for all $k$. The same conclusion holds if $p(t)$ is only absolutely continuous on $[0,d]\setminus\{t_0\}$, increasing on $(0,t_0)$, decreasing on $(t_0,d)$, and such that $\max\{p(t_0-),p(t_0+)\}<2p(0)$.*

(iii) *If $p'(t)\geqslant 0$ $(p'(t)\equiv 0)$ and $p'(t)/p(t)^3$ is decreasing for all $t\in(0,d)$, then $\operatorname{Im}\xi_k'(\mu_k)>0$ for all odd $k$ (i.e., on $\mathcal{S}_-$).*

We remark that under the assumptions in (i), $p$ is single lobe, and hence the conclusion follows from earlier results.[11] Moreover, it is clear that the conditions of (i) or (ii) imply that $p(0)>0$ so that $p(t)>0$ on $[0,d]$. However, in (iii), $p(0)=0$ is possible.

*Proof:* Let $t(u)$ be defined as in (2.11) and $f(u)$ as in (2.13). If $p'(t)\leqslant 0$, then $\tilde{p}(t)\geqslant 0$ and $f(u)$ is decreasing. This immediately shows that $I(\lambda)>0$ if $\lambda>0$, proving (i). We consider the situation described in the second half of (ii). Let $u_0$ be such that $t(u_0)=d-t_0$, where $t(u_0)$ is defined by (2.11). From the assumptions, we conclude that $f(u)$ is decreasing on $[0,u_0)$ and increasing on $(u_0,1]$. However, note that, since $p(d)$ may be zero, $f(u)$ may diverge as $u\to 0$. If $k=1$, then $\lambda_1=\pi/2$ and so $I(\lambda_1)>0$ because $f(u)$ is positive. If $k\geqslant 2$, then $\pi/(2\lambda_k)\in(0,1/3]$ and we first assume that $u_0\leqslant\pi/(2\lambda_k)$. By using an integration by parts, we obtain

$$I(\lambda_k)=\int_0^{u_0} f(u)\sin(\lambda_k u)du+\int_{u_0}^1 f(u)\sin(\lambda_k u)du \geqslant \frac{f(u_0-)}{\lambda_k}(1-\cos(\lambda_k u_0))+\frac{f(u_0+)}{\lambda_k}\cos(\lambda_k u_0) + \int_{u_0}^1 f'(u)\frac{\cos(\lambda_k u)}{\lambda_k}du,$$

since $f(u)\geqslant f(u_0-)$ on $(0,u_0)$. Since $f(u)$ is increasing on $(u_0,1)$, the last integral is bounded below by $-\lambda_k^{-1}(f(1)-f(u_0+))$. It follows that

$$I(\lambda_k)\geqslant\frac{f(u_0-)}{\lambda_k}(1-\cos(\lambda_k u_0))+\frac{f(u_0+)}{\lambda_k}(1+\cos(\lambda_k u_0))-\frac{f(1)}{\lambda_k}\geqslant\frac{2}{\lambda_k}\min\{f(u_0-),f(u_0+)\} - \frac{f(1)}{\lambda_k}. \tag{2.19}$$

By the assumptions and in view of (2.8) and (2.13), this expression is positive. Now we assume that $k\geqslant 2$ and $u_0>\pi/(2\lambda_k)$. In this case, we divide [0,1] into three pieces: $[0,\pi/(2\lambda_k)]$, $[\pi/(2\lambda_k),u_0]$, and $[u_0,1]$. On the first interval, we use $f(u)\geqslant f(\pi/(2\lambda_k))$. On the second and third interval, we use an integration by parts. Then, from $[\pi/(2\lambda_k),u_0]$, we obtain

$$\int_{\pi/(2\lambda_k)}^{u_0} \frac{\cos(\lambda_k u)}{\lambda_k} f'(u)du \geqslant \lambda_k^{-1}[f(u_0-) - f(\pi/(2\lambda_k))],$$

which leads to a cancellation of the terms involving the point $\pi/(2\lambda_k)$. As a result, we end up with the same lower bound as in (2.19); so part (ii) is proved. From the assumptions in (iii), we conclude that $f'(u)$ is increasing and so $f(u)$ is convex. However, if $p(0)=0$, then $f(u)$ is singular at $u=1$ and $f'(u)$ is in general not integrable there, so we cannot directly integrate by parts. To fix this problem we first observe that on $(1-\pi/(2\lambda_k),1]$, $\sin \lambda_k u > 0$, since $k$ is odd. On this interval we now replace $f(u)$ by its linear approximation at $1-\pi/(2\lambda_k)$ and denote the function $f$ thus modified by $\tilde{f}(u)$ $(0 \leqslant u \leqslant 1)$ and the integral corresponding to $I(\lambda)$ by $\tilde{I}(\lambda)$. Note that $f(u) \geqslant \tilde{f}(u)$ on $(1-\pi/(2\lambda_k),1)$ because $f$ is convex. Since $\sin \lambda_k u > 0$ on $(1-\pi/(2\lambda_k),1]$, we have that $I(\lambda_k) \geqslant \tilde{I}(\lambda_k)$. The interval $[0,\pi/(2\lambda_k)]$ clearly gives a positive contribution to $\tilde{I}(\lambda_k)$ and on the interval $[\pi/(2\lambda_k),1]$ we can now use an integration by parts (the boundary terms vanish). The fact that $\tilde{f}(u)$ is increasing guarantees that this contribution is non-negative. □

We take a brief look at the case when $p$ has infinite support.

**Theorem 2.5:** *Suppose that $p$ is locally absolutely continuous, even, $tp(t) \in L^1(0,\infty)$, strictly positive and such that $p'(t)/p(t)^3$ is negative and decreasing as $t\to\infty$. Then* Im $\xi_k'(\mu_k) > 0$ *for sufficiently large $k$.*

*Proof:* The proof that (2.7) still holds for potentials which satisfy $tp(t) \in L^1(0,\infty)$ will not be given here. We write the denominator in (2.7) as

$$\mathcal{J}(\mu_k) = \int_0^\infty \cos\left(2\mu_k \int_0^t p(\tau)d\tau\right)dt = (-1)^{k+1}\int_0^\infty \sin\left(2\mu_k \int_t^\infty p(\tau)d\tau\right)dt,$$

where we have used (2.3) with $d=\infty$. Note that the second integral is finite if and only if $tp(t) \in L^1(0,\infty)$, since $p(t)$ is positive. Now we make the substitution $u=(1/A)\int_t^\infty p(\tau)d\tau$ and then proceed exactly as in the proof of part (ii) of Theorem 2.3. The function $f(u)=1/p(t(u))$ is convex, decreasing near $u=0$, and diverges as $u\to 0$. Thus the interval $[0,\pi/(2\lambda_k)]$ gives the dominant contribution to $\mathcal{J}(\mu_k)$. Estimates similar to those in (2.15)–(2.17) yield the desired result. □

Smooth positive potentials going to zero like $t^{-\beta}$, with $\beta > 2$, potentials with exponential decay, or Gaussian and super-Gaussian potentials $[\exp(-(t/t_0)^{2m}), m > 2]$ satisfy the hypotheses of Theorem 2.5. The following examples augment and extend some of the results of Theorems 2.3–2.5.

*Example 1:* If $p(t) \sim c_0 t^{-\beta}$ as $t\to\infty$ with $\beta > 2$ and $c_0 > 0$, then $p(t)$ is strictly positive for large enough $t$, but it may also have zeros as long as they are confined to a bounded interval. It can easily be seen that the leading behavior of $\mathcal{J}(\mu_k)$ as $k\to\infty$ is $c_1(-1)^{k+1}\mu_k^{1/(\beta-1)}$ with $c_1 > 0$ [so $\mathcal{J}(\mu_k)$ is actually growing]. It follows from (2.7) that Im $\xi_k'(\mu_k) > 0$ for all sufficiently large $k$.

*Example 2:* If $p(t) \sim ce^{-\alpha t}$ as $t\to\infty$, then $\mathcal{J}(\mu_k) \sim (-1)^{k+1}\pi/(2\alpha)$ and hence, if $k$ is large enough, we only see eigenvalues emerging from $\xi=0$. The purpose of this example is to make the point that $k$ indeed needs to be sufficiently large. A class of potentials of this form that has been discussed in the literature (pp. 57 and 154 of Ref. 6) is

$$p(t) = \operatorname{sech}(t-t_0) + \operatorname{sech}(t+t_0).$$

As we glean from the comments made there such potentials appear to have only purely imaginary eigenvalues. However, this is not the case. In this example, we have $\mu_k=(2k-1)/4$ (independently of $t_0$) and if $t_0=5$, then numerical calculations show that Im $\xi_k'(\mu_k) < 0$ if $k$ is even and $\leqslant 8$, while Im $\xi_k'(\mu_k) > 0$ if $k$ is either odd or $k$ is even and $\geqslant 10$. Hence, eigenvalue collisions must occur on $\mathcal{S}_+$. For example, one such collision takes place at $\xi=0.08i$ when $\mu=0.72$ (approximately). The eigenvalue that drops down to zero is the one corresponding to $k=2$; it gets absorbed at $\mu_2=3/4$.

*Example 3:* Suppose that $p$ vanishes at $d$ according to $p(t) \sim \tilde{c}(d-t)^{\sigma}$ $(\tilde{c},\sigma > 0)$ and assume that this relation is differentiable. Suppose also that $p(t) > 0$ on $[0,d)$. Then we are in case (ii) of Theorem 2.3 and, similarly to (2.18), we obtain

$$I(\lambda_k) \sim \tilde{d}_\sigma \lambda_k^{-1/(\sigma+1)}, \quad k \to \infty, \tag{2.20}$$

where

$$\tilde{d}_\sigma = [A(\sigma+1)]^{-\sigma/(\sigma+1)} \tilde{c}^{-1/(\sigma+1)} \Gamma(1/(\sigma+1)) \cos(\pi\sigma/(2(\sigma+1))),$$

and thus Im $\xi_k'(\mu_k) > 0$ for sufficiently large $k$. If $p$ vanishes both at 0 and $d$ (and nowhere else), then the leading contribution to the large-$k$ asymptotics of $I(\lambda_k)$ comes from that endpoint at which the order of vanishing of $p$ is largest. If $p$ vanishes to the same order at both endpoints, then the leading contributions from the two endpoints may cancel each other out. As we can see from (2.19) and (2.20), this can happen only when $k$ is even. An example illustrating this behavior is the following.

*Example 4:* Define

$$p(t) = \begin{cases} t, & 0 \leq t \leq d/2 \\ d-t, & d/2 < t \leq d. \end{cases}$$

Let $k$ be even. Then

$$I(\lambda_k) \sim \frac{8\sqrt{2}(-1)^{k/2}}{(2k-1)^2 \pi^2 d}, \quad k \to \infty.$$

Thus, if $k=4m$ $(k=4m+2)$ an eigenvalue emerges (is absorbed), provided $m$ is sufficiently large. In the case of absorption, by Lemma 2.2, a collision must have preceded the absorption.

All the potentials considered thus far, except for the ones covered by the second part of Theorem 2.4 (ii), are continuous on $[0,d]$. If we allow discontinuities within the interval $(0,d)$, we can get behaviors not encountered thus far. We consider in some more detail the case of one discontinuity and make the following hypothesis.

$(H)$: Suppose that $p$ has support $[-d,d]$, is positive and even, and has the property that there exists an $a \in (0,d)$ such that $p$ is absolutely continuous on each of the subintervals $(0,a)$ and $(a,d)$.

As a consequence of the assumptions, the left and right-hand limits $p(a-)$ and $p(a+)$ exist, so $p(t)$ is piecewise continuous with one possible discontinuity at $t=a$.

To state the next theorem, we need some more notation. Let

$$N_{\text{tot}}(\mu) = \#\{k: 0 < \mu_k \leq \mu\},$$

$$N_{\text{up}}(\mu) = \#\{k: 0 < \mu_k \leq \mu \text{ and Im } \xi_k'(\mu_k) > 0\},$$

$$N_{\text{down}}(\mu) = \#\{k: 0 < \mu_k \leq \mu \text{ and Im } \xi_k'(\mu_k) < 0\},$$

and note that from (2.3) we have

$$N_{\text{tot}}(\mu) = \left\lfloor \frac{1}{2} + \frac{\mu \int_{-d}^{d} p(\tau) d\tau}{\pi} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x$. $N_{\text{up}}(\mu)$ represents the number of strict upward crossings [crossings of roots of $v_1(d;\xi,\mu)$ from the lower into the upper half plane with strictly positive derivative] as the coupling constant increases from 0 to $\mu$. Similarly, $N_{\text{down}}(\mu)$ represents the number of strict downward crossings. The sum of $N_{\text{up}}(\mu)$ and $N_{\text{down}}(\mu)$ may be strictly smaller than $N_{\mathrm{tot}}(\mu)$ because $N_{\mathrm{tot}}(\mu)$ also includes those crossings where $\xi_k'(\mu_k)$ is not defined (i.e., is infinite) because $v_{1;\xi}(d;0,\mu_k)=0$. We also need the following quantities:

$$\omega = \frac{\int_0^a p(t)dt}{2\int_0^d p(t)dt},$$

so that

$$2\mu_k \int_0^a p(t)dt = (2k-1)\pi\omega.$$

Also, let

$$\rho = \frac{p(a+)p(a-)}{p(d)(p(a+)-p(a-))},$$

and assume henceforth that $p(a+)\neq p(a-)$; note that $\rho\neq 0$ is guaranteed by assumption (H). A special case corresponding to $\rho=0$ will be analyzed in detail in Secs. III and IV. Now $\omega$ may be rational or irrational. If $\omega\in\mathbb{Q}$, with $\omega=r/s$ in lowest terms, we set $\mathcal{A}=\{1,2,\ldots,s\}$ and define

$$\nu_1 = \#\{k\in\mathcal{A}:(-1)^k\sin((2k-1)\pi\omega) > \rho\},$$

$$\nu_2 = \#\{k\in\mathcal{A}:(-1)^k\sin((2k-1)\pi\omega) < \rho\}.$$

Note that $s\geqslant 3$, since $\omega<1/2$; $\omega=1/2$ implies $p(t)=0$ on $(a,d]$, contradicting the positivity of $p$. We also set

$$\mathcal{A}' = \{k\in\mathcal{A}:(-1)^k\sin((2k-1)\pi\omega)\neq\rho\}.$$

**Theorem 2.6:** *Assume hypothesis* (*H*) *holds. Then the following are true.*

(i) *If* $|\rho|>1$, *then* $\mathrm{Im}\,\xi_k'(\mu_k)>0$ *for all sufficiently large* $k$.

(ii) *If* $|\rho|\leqslant 1$ *and* $\omega\notin\mathbb{Q}$, *then*

$$\lim_{\mu\to\infty}\frac{N_{\mathrm{down}}(\mu)}{N_{\mathrm{tot}}(\mu)} = \frac{1}{2} - \frac{\arcsin|\rho|}{\pi}, \tag{2.21}$$

$$\lim_{\mu\to\infty}\frac{N_{\mathrm{up}}(\mu)}{N_{\mathrm{tot}}(\mu)} = \frac{1}{2} + \frac{\arcsin|\rho|}{\pi}. \tag{2.22}$$

(iii) *If* $|\rho|\leqslant 1$ *and* $\omega\in\mathbb{Q}$, *with* $\omega=r/s$ *in lowest terms, then*

$$\liminf_{\mu\to\infty}\frac{N_{\mathrm{down}}(\mu)}{N_{\mathrm{tot}}(\mu)} \geqslant \begin{cases} \dfrac{\nu_1}{s} & \text{if } p(a+)>p(a-) \\ \dfrac{\nu_2}{s} & \text{if } p(a+)<p(a-), \end{cases} \tag{2.23}$$

$$\limsup_{\mu\to\infty}\frac{N_{\mathrm{down}}(\mu)}{N_{\mathrm{tot}}(\mu)} \leqslant \begin{cases} \dfrac{s-\nu_2}{s} & \text{if } p(a+)>p(a-) \\ \dfrac{s-\nu_1}{s} & \text{if } p(a+)<p(a-), \end{cases} \tag{2.24}$$

$$\liminf_{\mu\to\infty}\frac{N_{\rm up}(\mu)}{N_{\rm tot}(\mu)} \geqslant \begin{cases}\dfrac{\nu_2}{s} & \text{if } p(a+)>p(a-)\\[2mm] \dfrac{\nu_1}{s} & \text{if } p(a+)<p(a-),\end{cases}$$

$$\limsup_{\mu\to\infty}\frac{N_{\rm up}(\mu)}{N_{\rm tot}(\mu)} \leqslant \begin{cases}\dfrac{s-\nu_1}{s} & \text{if } p(a+)>p(a-)\\[2mm] \dfrac{s-\nu_2}{s} & \text{if } p(a+)<p(a-).\end{cases}$$

*If* $\mathcal{A}'=\mathcal{A}$, *then* $\nu_1+\nu_2=s$ *and the limits exist in all the relations of part* (iii).

*Proof:* (i) In view of the assumptions, upon integrating by parts, we obtain

$$\mathcal{J}(\mu_k)=\frac{(-1)^{k+1}}{2\mu_k}\left\{\frac{1}{p(d)}+\frac{(-1)^{k+1}(p(a+)-p(a-))}{p(a+)p(a-)}\sin\left(2\mu_k\int_0^a p(\tau)d\tau\right)\right\}+r_k=\frac{(-1)^{k+1}}{2\mu_k p(d)\rho}\{\rho+\delta_k - (-1)^k\sin((2k-1)\pi\omega)\}, \tag{2.25}$$

where $r_k=o(1/\mu_k)$ and thus $\delta_k=2(-1)^{k+1}p(d)\rho\mu_k r_k=o(1)$ as $k\to\infty$; $\mathcal{J}(\mu_k)$ is the denominator in (2.7).

If $\rho>1$ $(\rho<-1)$, then for large $k$ the sum between braces in (2.25) is positive (negative) which, by (2.7), proves the claim. To prove (ii) it suffices to consider the case $0<\rho\leqslant 1$; the case $-1\leqslant\rho<0$ can be dealt with similarly. Assuming $0<\rho<1$, pick $\epsilon>0$ so that $0<\rho-\epsilon<\rho+\epsilon<1$, and then pick $k_0$ so large that

$$\rho-\epsilon<\rho+\delta_k<\rho+\epsilon \tag{2.26}$$

for $k>k_0$. For convenience also assume that $k_0$ is an integer multiple of $s$. Let $m$ be a positive integer. A glance at (2.25) and (2.26) shows that

$$a(m,\epsilon)\leqslant N_{\rm down}(\mu_{k_0+m})\leqslant b(m,\epsilon),$$

where

$$a(m,\epsilon)=\#\{k\in(k_0,k_0+m]:\rho+\epsilon<(-1)^k\sin((2k-1)\pi\omega)\},$$

$$b(m,\epsilon)=k_0+\#\{k\in(k_0,k_0+m]:\rho-\epsilon<(-1)^k\sin((2k-1)\pi\omega)\}.$$

Considering the lower bound, assuming (without loss) that $k_0$ is a even multiple of $s$, we write

$$\frac{a(m,\epsilon)}{k_0+m}=\frac{\lfloor m/2\rfloor}{k_0+m}\left(\frac{\#\{k \text{ even}:k\in(k_0,k_0+m],\rho+\epsilon<\sin((2k-1)\pi\omega)\}}{\lfloor m/2\rfloor}\right) + \frac{\lfloor (m+1)/2\rfloor}{k_0+m}\left(\frac{\#\{k \text{ odd}:k\in(k_0,k_0+m],\sin((2k-1)\pi\omega)<-\rho-\epsilon\}}{\lfloor (m+1)/2\rfloor}\right).$$

From results about the value distribution of $(2k-1)\pi\omega \pmod{2\pi}$ for $k$ even and $k$ odd[7,20] and, in particular, the fact that these values are uniformly distributed in $[0,2\pi]$, we conclude that

$$\liminf_{\mu\to\infty}\frac{N_{\rm down}(\mu)}{N_{\rm tot}(\mu)}\geqslant\lim_{m\to\infty}\frac{a(m,\epsilon)}{k_0+m}=\frac{1}{2}\frac{|\{x\in[0,2\pi]:|\sin x|>\rho+\epsilon\}|}{2\pi}=\frac{1}{2}-\frac{\arcsin(\rho+\epsilon)}{\pi}.$$

Here the vertical bars denote Lebesgue measure. In a similar manner, we find that

$$\limsup_{\mu\to\infty}\frac{N_{\mathrm{down}}(\mu)}{N_{\mathrm{tot}}(\mu)}\leqslant\frac{1}{2}-\frac{\arcsin(\rho-\epsilon)}{\pi}.$$

Taking $\epsilon\to 0$ proves (2.21) for $0<\rho<1$. If $\rho=1$, then we have $a(m,\epsilon)=0$ and for the lim sup we get $1/2-\arcsin(1-\epsilon)/\pi$. So (2.21) also holds in this case. The limit in (2.22) is proved similarly.

To prove the first inequality of (iii) when $p(a+)>p(a-)$, we can again assume that $0<\rho<1$. We first observe that the values of $(-1)^k\sin((2k-1)\pi r/s)$ as $k$ ranges over $\mathcal{A}$ comprise the entire range of this function (when $s$ is even this is already true when $k$ ranges only over half of $\mathcal{A}$). Let $(\omega=r/s)$ and set

$$\epsilon^{*}=\min\{|(-1)^k\sin((2k-1)\pi\omega)-\rho|:k\in\mathcal{A}'\}.$$

Choose $\epsilon$ and $k_0$ as in the proof of (ii) and suppose also that $\epsilon<\epsilon^{*}$. Then, for all $k>k_0$, if $(-1)^k\sin((2k-1)\pi\omega)>\rho$, then

$$(-1)^k\sin((2k-1)\pi\omega)-\rho\geqslant\epsilon^{*}>\epsilon>\delta_k.$$

This implies that the term between braces in (2.25) is negative. By the definition of $\nu_1$, if $m\geqslant 1$, then the interval $(k_0,k_0+m]$ contains at least $\lfloor m/s\rfloor\nu_1$ values of $k$ that contribute to $N_{\mathrm{down}}(\mu_{k_0+m})$. Since $\lfloor m/s\rfloor\nu_1/(k_0+m)\to\nu_1/s$ as $m\to\infty$, the first equation of (2.23) follows. If $\rho=1$, then $\nu_1=0$ and (2.23) is trivial. To prove the second relation in (2.23), we must show that the term in braces is positive because now $-1<\rho<0$. Using the definition of $\nu_2$, the proof proceeds like in the first case. To prove the first equation of (2.24), we note that if $k>k_0$ and $\rho+\delta_k-(-1)^k\sin((2k-1)\pi\omega)<0$, then

$$\rho-(-1)^k\sin((2k-1)\pi\omega)<|\delta_k|<\epsilon<\epsilon^{*}.$$

Hence, by the definitions of $\mathcal{A}'$ and $\epsilon^{*}$, $\rho-(-1)^k\sin((2k-1)\pi\omega)\leqslant 0$. There are $s-\nu_2$ values of $k$ with this property in each interval $(k_0+ns,k_0+(n+1)s]$ $(n=0,1,2,\ldots)$. From this, inequality (2.24) follows immediately. The other limits in part (iii) are proved similarly. The last assertion is obvious from the definitions. □

We add a few remarks about situations not covered by Theorem 2.6. If $\mathcal{A}\neq\mathcal{A}'$, then $\sin((2k-1)\pi\omega)=\rho$ for an infinite sequence of $k$-values. Then it depends on the remainders $\delta_k$ in (2.25) whether or not the $\mu_k$ corresponding to such a $k$-value contributes to $N_{\mathrm{down}}(\mu)$ or $N_{\mathrm{up}}(\mu)$ and whether or not $N_{\mathrm{down}}(\mu)/N_{\mathrm{tot}}(\mu)$ has a limit. We may also encounter the case where both $(-1)^k\sin((2k-1)\pi\omega)=\rho$ and $\delta_k=0$ for an infinite sequence of $k$'s. Then we have a multiple zero of $v_1(d;\xi,\mu_k)$ at $\xi=0$. These $\mu_k$ are not counted in $N_{\mathrm{down}}$ or $N_{\mathrm{up}}$. We also note that if $p$ satisfies the assumptions of Theorem 2.4(ii), then $|\rho|>1$ and we are in case (i).

The simplest potentials satisfying hypothesis (H) are those of the form

$$p(t)=\begin{cases}h_1, & |t|<a\\ h_2, & a\leqslant|t|\leqslant d\\ 0, & |t|>d,\end{cases}$$

where $h_1>0$, $h_2>0$, and $0<a<d$. For completeness, we also include the limiting cases $h_1=0$ and $h_2=0$ in our discussion. Then (2.7) becomes

$$\xi_k'(\mu_k)=\begin{cases}\dfrac{i(-1)^k h_1h_2(2k-1)\pi}{2((-1)^k h_1+(h_1-h_2)\sin[(2k-1)\pi\omega])}, & h_1>0,\ h_2>0\\[2ex] \dfrac{i(-1)^k h_2(d-a)(2k-1)\pi}{2(-1)^k(d-a)-a(2k-1)\pi}, & h_1=0,\ h_2>0\\[2ex] \dfrac{ih_1(2k-1)\pi}{2}, & h_1>0,\ h_2=0,\end{cases}\tag{2.27}$$

where

$$\omega = \frac{ah_1}{2A}, \quad A = ah_1 + (d-a)h_2.$$

The explicit expressions in (2.27) allow us to make a number of observations.

If $h_2=0$, then eigenvalues can only emerge at $\xi=0$. This is consistent with the fact that $p$ is single lobe.[11] Also, in the context of Theorem 2.6, we have (in the limit as $h_2 \to 0$) $\rho=h_1/(h_2-h_1)\to -1$ and $\omega \to 1/2$. Since $\rho < -1$ when $h_2>0$, this case also has $\nu_1=s$ and $\nu_2=0$ as limiting values. We see that (i) and (iii) [with $p(a+)=0<h_1=p(a-)$] of Theorem 2.6 are in agreement.

The property that Im $\xi_k'(\mu_k)>0$ for all $k$ persists as long as $h_2<2h_1$, and this is in agreement with Theorem 2.4 (ii) and Theorem 2.6 since $|\rho|>1$. If $h_2 \geqslant 2h_1$, then it may happen that Im $\xi_k'(\mu_k)<0$ or that $\xi_k'(\mu_k)$ does not exist. These cases are covered by Theorem 2.6 [(ii) and (iii)] since $|\rho| \leqslant 1$. Moreover, for these particular potentials the $\delta_k$ terms in (2.25) are all zero. It follows that the $k$-values for which $(-1)^k \sin((2k-1)\pi\omega)=\rho$ correspond to multiple roots of $v_1(d;\xi,\mu_k)$ and hence are not picked up by either $N_{\text{down}}$ or $N_{\text{up}}$. Consequently, if $\omega \in \mathbb{Q}$, then $\lim N_{\text{down}}(\mu)/N_{\text{tot}}(\mu)=\nu_1/s$ and $\lim N_{\text{up}}(\mu)/N_{\text{tot}}(\mu)=\nu_2/s$; that is, the limits exist.

If $h_1=0$, then for odd $k$ an eigenvalue always emerges from $\xi=0$ while for sufficiently large even $k$ eigenvalues can only get absorbed at $\xi=0$.

We have also analyzed what happens when the denominator in one of the first two relations in (2.27) becomes zero for certain values of $k$, which means that $v_1(d;0,\mu_k)=v_{1;\xi}(d;0,\mu_k)=0$; note that $h_2 \geqslant 2h_1$ is a necessary condition for this to happen. We confine ourselves to briefly discussing two results. These were obtained by means of the expansion

$$v_1(d;\xi,\mu) = a_1\xi^2 + a_2\xi^3 + a_3(\mu-\mu_k) + a_4(\mu-\mu_k)\xi + \cdots, \tag{2.28}$$

where the displayed terms are those needed for our purposes. The initial conditions are $v(0;\xi,\mu)^T=(1,\pm 1)^T$ on $\mathcal{S}_\pm$. The explicit expressions for the coefficients $a_1$, $a_2$, and $a_4$ are lengthy, so we do not write them down here; however, $a_3=\sqrt{2}Ai^{k+1}$ if $k$ is odd and $a_3=\sqrt{2}Ai^k$ if $k$ is even. Also, $a_1 \neq 0$ and $a_3/a_1>0$, so that on equating the right-hand side of (2.28) to zero, we get

$$\xi_k(\mu) = \pm\sqrt{\frac{a_3}{a_1}}\sqrt{\mu_k-\mu} + \frac{a_1a_4-a_2a_3}{2a_1^2}(\mu_k-\mu) + o((\mu_k-\mu)^{3/2}). \tag{2.29}$$

A computation also shows that $\text{Re}(a_1a_4-a_2a_3)=0$.

*Case 1:* If $h_1=0$, then the denominator in (2.27) can vanish only if $k$ is even and

$$\frac{a}{d-a} = \frac{2}{(2k-1)\pi}.$$

Then (2.29) becomes

$$\xi_k(\mu) = \pm\sqrt{\frac{(2k-1)\pi h_2}{a(4+(2k-1)\pi)}}\sqrt{\mu_k-\mu} \times on - \frac{8h_2(3+(2k-1)\pi)i}{3(4+(2k-1)\pi)^2}(\mu_k-\mu) + o((\mu_k-\mu)^{3/2}).$$

This tells us that for $\mu$ slightly less than $\mu_k$, there are two roots $\xi_k(\mu)$ of $v_1(d;\xi,\mu)$ situated in the lower half plane and symmetric with respect to the imaginary axis. These roots move toward the origin as $\mu \uparrow \mu_k$ along two parabolical arcs. For $\mu$ slightly larger than $\mu_k$, there is a pair of roots on the imaginary axis with the upper root corresponding to an eigenvalue of (1.1).

*Case 2:* This example complements the first by exhibiting a pair of eigenvalues converging to zero from the upper half plane. It is illustrated in Fig. 1. The parameters are

$$a=1,\quad d=\frac{3}{4}+\frac{19\pi}{8(2\pi+\arcsin(1/3))},\quad h_1=1,\quad h_2=4.$$

The solid dot marks the point $\xi=3.40$ which corresponds to a root of $v_1(d;\cdot,\mu)$ when $\mu=2.85$. The parameter $k=10$ and the eigenvalue is absorbed at $\xi=0$ when $\mu_{10}=(1/2)(2\pi+\arcsin(1/3))=3.31$.

## III. NONSIMPLE PURELY IMAGINARY EIGENVALUES

In this section, we prove some general results on the existence and location of eigenvalue collisions on the imaginary axis. For a special class of potentials, we derive asymptotic formulas, valid for large $\mu$, for the points on the imaginary axis where collisions take place. As is often the case with asymptotic formulas, they give quite satisfactory results even for relatively small values of $\mu$.

**Theorem 3.1:** *Let $\xi(\mu)=i\, s(\mu)$ denote a purely imaginary eigenvalue branch of (1.1) for the potentials $q=\mu p$. Then the following is true.*

(i) *If $p$ is even, absolutely continuous and positive on $[0,d]$, zero for $t>d$, and*

$$s(\mu)\geqslant \operatorname*{ess\,sup}_{0\leqslant t\leqslant d}\frac{p'(t)}{2p(t)},\tag{3.1}$$

*then $\xi(\mu)$ is a simple eigenvalue and $s'(\mu)>0$.*

(ii) *If there is an interval $[0,a)$ on which $p=0$ and $p$ is positive and absolutely continuous on $[a,d]$, then any purely imaginary eigenvalue for (1.1−) with*

$$s(\mu)\geqslant \operatorname*{ess\,sup}_{a\leqslant t\leqslant d}\frac{p'(t)}{2p(t)}\tag{3.2}$$

*is simple and $s'(\mu)>0$.*

(iii) *If $p$ is non-negative and absolutely integrable on $[0,d]$, and there exists a point $a\in(0,d)$ such that $p$ is decreasing for $t>a$, then all purely imaginary eigenvalues of (1.1−) are simple and $s'(\mu)>0$ so long as $\mu\leqslant\pi(4\int_0^a p(t)dt)^{-1}$.*

Note that for the right-hand side of (3.1) or (3.2) to be positive, a smooth potential $p$ must be increasing on a subinterval of $[0,d]$. Part (i) is essentially contained in Ref. 11 but we have included a proof here because we will need it to prove (ii). As we will see below, (ii) and (iii) are not true on $\mathcal{S}_+$. Of course, if $p=0$ on $[0,a)$, then the conclusion of (iii) holds for any $\mu$.

*Proof:*

(i) We recall from (Appendix of Ref. 11) or Sec. II that an eigenvalue is simple if and only if

$$\int_0^d v_1(t)v_2(t)dt\neq 0,\tag{3.3}$$

where $v=(v_1,v_2)^T$ is the corresponding eigenfunction. Since we are dealing with purely imaginary eigenvalues, $v(t;\xi,\mu)$ is real. From the first equation in (1.1), we obtain

$$\int_0^d v_1(t)v_2(t)dt=\int_0^d\frac{v_1(t)v_1'(t)}{\mu p(t)}dt-s\int_0^d\frac{v_1(t)^2}{\mu p(t)}dt=-\frac{v_1(0)^2}{2\mu p(0)}+\int_0^d\frac{v_1(t)^2p'(t)}{2\mu p(t)^2}-s\int_0^d\frac{v_1(t)^2}{\mu p(t)}dt.$$

Looking at the two integral terms on the right, we see that their combined value is non-positive if (3.1) is satisfied. Since $v_1(0)\neq 0$ on either $\mathcal{S}_+$ or $\mathcal{S}_-$, the integral in (3.3) is strictly negative. In view of (2.5), (i) is proved.

(ii) On $\mathcal{S}_-$, assuming $v_1(0)=-v_2(0)=1$, we have that $v_1(t)=e^{st}$ and $v_2=-e^{-st}$ on $[0,a)$. Thus

$$\int_0^d v_1(t)v_2(t)dt = -a - \frac{v_1(a)^2}{2\mu p(a)} + \int_a^d \frac{v_1(t)^2 p'(t)}{2\mu p(t)^2} - s\int_a^d \frac{v_1(t)^2}{\mu p(t)}dt.$$

and the result follows as in (i).

(iii) First, by integrating as in (i) or (ii), we see that the contribution from $[a,d]$ to the integral in (3.3) is nonpositive. We are going to show that the assumption in (iii) forces the integral over $[0,a)$ to be strictly negative. To see this we employ a Prüfer transformation (as in Ref. 11) of the form

$$v_1(t) = \rho(t)\cos\theta(t), \quad v_2(t) = \rho(t)\sin\theta(t), \quad \rho(t)^2 = v_1(t)^2 + v_2(t)^2.$$

Then one verifies that

$$\theta' = -\mu p(t) - s\sin 2\theta, \quad \theta(0) = -\frac{\pi}{4}, \tag{3.4}$$

since we are in $\mathcal{S}_-$. A look at (3.4) tells us that $\theta(t)$ can never become positive. It follows that $\theta(t)$ must first reach the value $-\pi/2$ before it can reach $-\pi$ which corresponds to the first zero of $v_2(t)$. In other words, if $\tilde{t}$ denotes the first positive zero of $v_1(t)$, then $\theta(\tilde{t}) = -\pi/2$ and $\tilde{t}$ is strictly less than the first positive zero of $v_2(t)$. By (3.4)

$$\theta(t) \geqslant -\frac{\pi}{4} - \mu\int_0^t p(\tau)d\tau$$

for $0 < t \leqslant \tilde{t}$ and hence,

$$\mu\int_0^{\tilde{t}} p(\tau)d\tau \geqslant \frac{\pi}{4}.$$

Therefore, by the assumption on $\mu$, $\tilde{t} \geqslant a$ and hence $v_1(t) > 0$, $v_2(t) < 0$ on $[0,a)$. Thus

$$\int_0^a v_1(t)v_2(t)dt < 0,$$

proving (iii).

Considering again the situation at $\xi=0$, if we assume that $p=0$ on $[0,a)$ and $p>0$ on $[a,d]$ but do not require $p$ to be decreasing on $[a,d]$, then we have that

$$\int_0^d \cos\left(2\mu_k\int_0^t p(\tau)d\tau\right)dt = a + \frac{(-1)^{k+1}}{2\mu_k p(d)} + o(1/\mu_k). \tag{3.5}$$

Thus, by (2.7), Im $\xi_k'(\mu_k) < 0(>0)$ if $k$ is even (odd) and sufficiently large. This means that we may restrict ourselves to the subspace $\mathcal{S}_+$ to study the asymptotics of collision points.

Our first result gives an estimate for an interval on the imaginary axis where nonsimple eigenvalues *cannot* occur. This result is an improvement over Theorem 3.1(ii) and it will be needed for the proof of Theorem 3.3.

**Theorem 3.2:** *Suppose that p is even, $p=0$ on $[0,a)$, and that, on $[a,d]$, p is positive and has an essentially bounded derivative. Then there exist positive constants $c_1$ and $c_2$ such that if $\xi(\mu)$ is a nonsimple purely imaginary eigenvalue of* (1.1+), *then $\xi(\mu) \in (ic_1\ln\mu, ic_2\ln\mu)$ provided $\mu$ is sufficiently large. As $\mu$ increases, eigenvalues in $(0, ic_1\ln\mu]$ move downward and eigenvalues in $[ic_2\ln\mu, i\infty)$ move upward along the imaginary axis.*

*Proof:* We will use the inequality

$$(v_1^2+v_2^2)' = 2s(v_1^2-v_2^2) \leqslant 2s(v_1^2+v_2^2),$$

which easily follows from (1.1). Here $v=(v_1,v_2)^T$ is the real-valued eigenfunction for a putative purely imaginary eigenvalue $\xi=is$ $(s>0)$. Integrating the inequality from $a$ to $t$ and using the fact that for $t\leqslant a$, $v_1(t)=e^{st}$, and $v_2(t)=e^{-st}$ (in $\mathcal{S}_+$), we conclude that

$$v_1^2+v_2^2 \leqslant 2\cosh(2as)e^{2s(t-a)} \leqslant 2e^{2st}, \quad t \geqslant a.$$

Note that these bounds do not involve $q(t)$. Setting $q(t)=\mu p(t)$, we obtain as above

$$\int_0^d v_1(t)v_2(t)dt = a - \frac{v_1(a)^2}{2\mu p(a)} + \int_a^d \frac{v_1(t)^2 p'(t)}{2\mu p(t)^2}dt - s\int_a^d \frac{v_1(t)^2}{\mu p(t)}dt \geqslant a - \frac{e^{2sa}}{2\mu p(a)} - \frac{2Cd}{\mu}e^{2sd}(1+s), \tag{3.6}$$

where

$$C = \operatorname*{ess\,sup}_{a\leqslant t\leqslant d}\left(\frac{|p'(t)|}{2p(t)^2}+\frac{1}{p(t)}\right),$$

and we have used the crude estimate $v_1(t)^2 \leqslant 2e^{2sd}$. Now let $c_1$ be any positive number strictly less than $1/(2d)$. Then, for $s\leqslant c_1 \ln \mu$, we have that $e^{2sd}\leqslant \mu^{2dc_1}=o(\mu)$ as $\mu\to\infty$, so that when $\mu$ is large the right-hand side of (3.6) is positive. Thus there can be no nonsimple eigenvalues in $(0, ic_1 \ln \mu]$. If $\mu$ is so large that $c_1 \ln \mu \geqslant \sup_{[a,d]} p'(t)/(2p(t))$ and $s \geqslant c_1 \ln \mu$, then

$$\int_0^d v_1(t)v_2(t)dt \leqslant a - \frac{e^{2sa}}{2\mu p(a)}, \tag{3.7}$$

since the two integrals on the right-hand side of the first equality in (3.6) together are nonpositive. Thus if we choose $c_2 > 1/(2a)$ and $s \geqslant c_2 \ln \mu$, then for $\mu$ large enough the right-hand side of (3.7) is negative. The remaining assertions follow from (2.5). □

In the next theorem, we will show that there exist unbounded sequences $\{\mu_j^c\}$ and $\{\xi_j^c\}$ such that $\xi_j^c$ is a purely imaginary nonsimple eigenvalue for the potential $\mu_j^c p$; in other words (at least) two eigenvalues collide at $\xi=\xi_j^c$ when $\mu=\mu_j^c$. Since the essential ingredients needed for the proof are similar to those used in the proof of Theorem 4.8, we defer the proof of Theorem 3.3 to the Appendix.

Let

$$\hat{\mu}_j = \frac{(2j-1)\pi}{2\int_a^d p(t)dt}, \quad j=1,2,3\ldots, \tag{3.8}$$

and let

$$\omega_1 = \int_a^d p(t)dt, \quad \omega_2 = \frac{1}{2}\int_a^d \frac{dt}{p(t)}. \tag{3.9}$$

Of course, $\omega_1 = A$ but we prefer using $\omega_1$ in this section and in the Appendix.

**Theorem 3.3:** *Suppose that $p$ is even, $p=0$ on $[0,a)$, positive and absolutely continuous on $[a,d]$, and zero outside $[-d,d]$. Then for large $j$, every interval $[\mu_{2j-2},\mu_{2j}]$ contains exactly one $\mu_j^c$ for which two eigenvalues of (1.1+) collide on the imaginary axis at $\xi_j^c = i\, s_j^c$, where, asymptotically,*

$$\mu_j^c = \hat{\mu}_j + \frac{\omega_2}{4a^2\omega_1}\frac{(\ln(2a\hat{\mu}_j))^2}{\hat{\mu}_j} + O\left(\frac{(\ln\hat{\mu}_j)\ln\ln\hat{\mu}_j}{\hat{\mu}_j}\right), \tag{3.10}$$

$$s_j^c = \frac{\ln(2a\hat{\mu}_j)}{2a} - \frac{1}{2a}\ln\left(\frac{\omega_2}{a}\ln(2a\hat{\mu}_j)\right) + O\left(\frac{\ln\ln\hat{\mu}_j}{\ln\hat{\mu}_j}\right). \tag{3.11}$$

*The downward moving eigenvalue produced by the collision vanishes into the continuous spectrum at* $\mu=\mu_{2j}$.

*Moreover, if* $p$ *is constant on* $[a,d]$, *then for sufficiently large* $j$, $\mu_j^c$ *and* $s_j^c$ *are given by absolutely convergent series of the form*

$$\sum_{n,m,k\geq 0} c(n,m,k)\tau_1^n\tau_2^m\tau_3^k, \tag{3.12}$$

*where*

$$\tau_1 = \frac{(\ln(2a\hat{\mu}_j))^2}{2a\hat{\mu}_j}, \quad \tau_2 = \frac{1}{\ln(2a\hat{\mu}_j)}, \quad \tau_3 = \frac{\ln\ln(2a\hat{\mu}_j)}{\ln(2a\hat{\mu}_j)}. \tag{3.13}$$

*Proof:* See the Appendix. We mention that the proof uses some notation and results from Sec. IV. □

We mention that the idea for (3.12) came from earlier work on the coupling constant threshold problem for Schrödinger operators (Appendix of Ref. 12), where expansions of a similar kind were proved.

For large $k$ ($k$ even) and $j$, the numbers $\mu_k$ and $\mu_j^c$ are interlaced in the following manner: Put $j=1+k/2$ with $k\geq 2$ even, then $\mu_k<\hat{\mu}_j<\mu_{k+2}<\hat{\mu}_{j+1}<\cdots$ by (2.3) and (3.8). Since $\hat{\mu}_j-\mu_k=3\pi/(4A)$, $\mu_{k+2}-\hat{\mu}_j=\pi/(4A)$, and $\mu_j^c-\hat{\mu}_j\to 0$ as $j\to\infty$ by (3.10), we have that $\mu_k<\mu_j^c<\mu_{k+2}<\mu_{j+1}^c<\cdots$, provided $j$ is large enough. If Im $\xi_2'(\mu_2)<0$, then there is also a collision at some $\mu_1^c<\mu_2$ which produces the eigenvalue that is absorbed at zero when $\mu=\mu_2$. We see that the movement of eigenvalues on the imaginary axis must be "well-orchestrated" as a brief chronology of events shown. At some $\mu=\mu_1$, the first eigenvalue (in the subspace $\mathcal{S}_-$) appears and starts to move up the imaginary axis. At some $\mu=\mu_1^c>\mu_1$ typically—but not always—the first collision of two eigenvalues in $\mathcal{S}_+$ occurs. One eigenvalue drops down to zero and arrives there at $\mu=\mu_2$; the other eigenvalue moves up the imaginary axis. At $\mu=\mu_3$, the second eigenvalue in the subspace $\mathcal{S}_-$ appears and moves up the imaginary axis without collision. At $\mu=\mu_2^c$, the second collision (in $\mathcal{S}_+$) occurs, the falling eigenvalue arrives at zero when $\mu=\mu_4$, and so on. It is obvious that the eigenvalues on $\mathcal{S}_-$ must rise fast enough in order to avoid collisions with the falling eigenvalues that are created right after they are launched.

In (3.10) and (3.11) not all the terms we have actually computed are displayed. Indeed, the proof yields an expansion for $\mu_j^c$ with an error of only $o(1/\hat{\mu}_j)$ [see (A14)]. Using this expansion for $\mu_j^c$ and (3.11) for $s_j^c$, we have checked some examples that satisfy the conditions of Theorem 3.3. We write $\mu_{j;ex}^c$ and $s_{j;ex}^c$ for the exact values and $\mu_{j;ap}^c$ and $s_{j;ap}^c$ for their approximations based on (A14) and (3.11), respectively. Some results for the following two examples are shown in Table II.

*Example 1:* An even box profile given by $p(t)=0$ for $0\leq t<1$, $p(t)=1$ for $1\leq t\leq 2$, and $p(t)=0$ for $t>2$.

*Example 2:* An even oscillatory potential given by $p(t)=0$ for $0\leq t<1$, $p(t)=1+(1/2)\sin(100t)$ for $1\leq t\leq 2$, and $p(t)=0$ for $t>2$.

The results corresponding to $j=1$, 3, and 5 have been skipped in the table. However, we mention that the first collision occurs at $s_{1;ex}^c=0.45$ when $\mu_{1;ex}^c=2.17<\mu_2=2.35$.

Example 2 was chosen to see how oscillations affect the predictions based on (A14) and (3.11). Note that in Example 1, $\omega_1=1$, while in Example 2, $\omega_1=1.002$, so that we are comparing potentials of nearly equal areas. It is interesting to see that in Example 2 the agreement between exact and approximate values for the coupling constants and eigenvalues is almost as good as in

TABLE II. Exact and approximate coupling constants and imaginary parts of collision points for Examples 1 and 2.

| | Example 1 (box) | | | | Example 2 (oscillatory) | | | |
|---|---|---|---|---|---|---|---|---|
| $j$ | $\mu^c_{j;ap}$ | $\mu^c_{j;ex}$ | $s^c_{j;ap}$ | $s^c_{j;ex}$ | $\mu^c_{j;ap}$ | $\mu^c_{j;ex}$ | $s^c_{j;ap}$ | $s^c_{j;ex}$ |
| 2 | 5.27 | 5.12 | 1.06 | 0.82 | 5.43 | 5.11 | 0.99 | 0.82 |
| 4 | 11.30 | 11.25 | 1.32 | 1.15 | 11.37 | 11.23 | 1.25 | 1.16 |
| 6 | 17.50 | 17.47 | 1.48 | 1.34 | 17.53 | 17.43 | 1.41 | 1.36 |

Example 1. However, in view of (A14) and (3.11) this is not surprising, since $p$ enters through the quantities $\omega_1$, $\omega_2$, and $\omega_3$, which means that oscillations are averaged out.

## IV. SPECTRAL SINGULARITIES

In this section, we study the question of where, on the real axis, eigenvalues either enter or leave $\mathbb{C}^+$ as $\mu$ is increased. In other words, we are interested in *real* values of $\xi$ for which there is a $\mu>0$ such that $v_1(d;\xi,\mu)=0$. Such a $\xi$ will be referred to as a spectral singularity for the potential $\mu p$. It is easy to see that a spectral singularity corresponds to a real pole of the transmission coefficient and this establishes the connection with the definition used by other authors (e.g., Ref. 3). We already know that the point $\xi=0$ is a spectral singularity if and only if $\mu=\mu_k$ for some $k$.

In order to get an idea of what to expect, we first consider the piecewise constant potentials,

$$p(t)=\begin{cases}0, & |t|<a\\ 1, & a\leqslant |t|\leqslant d\\ 0, & |t|>d\end{cases} \tag{4.1}$$

which already appeared in Sec. III. We set $b=d-a$ and $\gamma=\xi/\mu$. Then the zeros of $v_1(d;\xi,\mu)$ in $(\xi,\mu)$-space, that is, the spectral singularities and the corresponding coupling constants, are determined by the following equations:

(a) On $\mathcal{S}_+$, where $v_1(0)=v_2(0)$, the equation $v_1(d;\xi,\mu)=0$ becomes

$$\cot(\mu b\sqrt{1+\gamma^2})=-\frac{e^{2ia\xi}}{\sqrt{1+\gamma^2}}+\frac{i\gamma}{\sqrt{1+\gamma^2}},$$

or, after separating real and imaginary parts,

$$\cot(\mu b\sqrt{1+\gamma^2})=-\frac{\cos(2a\xi)}{\sqrt{1+\gamma^2}},\quad \sin(2a\xi)=\gamma. \tag{4.2}$$

From the second equation of (4.2), we see that

$$\xi=\frac{n\pi}{2a}\pm\frac{1}{2a}\arcsin\gamma,\quad n\geqslant 0, \tag{4.3}$$

with the upper or lower sign being taken according as $n$ is even or odd. Note that solutions (i.e., pairs $(\xi,\mu)$) of (4.2) and (4.3) only exist if $\gamma\leqslant 1$; that is, if $\xi\leqslant\mu$. When $n$ is even (odd), we have that $\cos(2a\xi)\geqslant 0$ ($\leqslant 0$). The first equation of (4.2) then becomes

$$\sqrt{1+\gamma^2}\cot\left(\frac{b\sqrt{1+\gamma^2}(n\pi\pm\arcsin\gamma)}{2a\gamma}\right)\pm\sqrt{1-\gamma^2}=0, \tag{4.4}$$

with the same specification regarding ± as in (4.3).

(b) On $\mathcal{S}_-$, where $v_1(0)=-v_2(0)$, we have

$$\cot(\mu b\sqrt{1+\gamma^2}) = \frac{e^{2ia\xi}}{\sqrt{1+\gamma^2}} + \frac{i\gamma}{\sqrt{1+\gamma^2}},$$

or equivalently,

$$\cot(\mu b\sqrt{1+\gamma^2}) = \frac{\cos(2a\xi)}{\sqrt{1+\gamma^2}}, \quad \sin(2a\xi) = -\gamma. \tag{4.5}$$

Thus,

$$\xi = \frac{n\pi}{2a} \mp \frac{1}{2a}\arcsin\gamma, \quad n \geqslant 1, \tag{4.6}$$

where $\gamma \in (0,1]$ is a root of

$$\sqrt{1+\gamma^2}\cot\left(\frac{b\sqrt{1+\gamma^2}(n\pi \mp \arcsin\gamma)}{2a\gamma}\right) \mp \sqrt{1-\gamma^2} = 0 \tag{4.7}$$

(with the upper/lower sign corresponding to $n$ even/odd).

To obtain the spectral singularities and corresponding coupling constants, we fix $n \geqslant 0$, solve (4.4) or (4.7) for $\gamma$, then compute $\xi$ from (4.3) or (4.6), and finally set $\mu=\xi/\gamma$. For a given $n \geqslant 1$, either one of (4.4) or (4.7) has infinitely many roots $\gamma$ converging to zero; if $n=0$, then there are at most finitely many roots. This implies that every point $n\pi(2a)^{-1}$ $(n=1,2,3,\ldots)$ is a "two-sided" accumulation point of spectral singularities for the potential family $\{\mu p:\mu>0\}$ in the following sense. Suppose that $n \geqslant 1$ is odd. Then there is a sequence $\{\check{\xi}_k^{\uparrow}\}$ of $\xi$-values converging to $n\pi(2a)^{-1}$ from below and a corresponding sequence $\{\check{\mu}_k\}$ of $\mu$-values converging to infinity such that $\check{\xi}_k^{\uparrow}$ is a spectral singularity for $\check{\mu}_k p$. Similarly, there is a sequence $\{\check{\xi}_k^{\downarrow}\}$ converging to $n\pi(2a)^{-1}$ from above and a corresponding sequence $\{\check{\mu}_k\}$ converging to infinity such that $\check{\xi}_k^{\downarrow}$ is a spectral singularity for $\check{\mu}_k p$. Moreover, the singularities $\check{\xi}_k^{\uparrow}$ occur in $\mathcal{S}_+$ and the singularities $\check{\xi}_k^{\downarrow}$ occur in $\mathcal{S}_-$. If $n \geqslant 2$ is even, then there are two similar sequences, but now $\{\check{\xi}_k^{\uparrow}\}$ is associated with $\mathcal{S}_-$ and $\{\check{\xi}_k^{\downarrow}\}$ is associated with $\mathcal{S}_+$. Setting $n=0$ in (4.3) and (4.4), we see that $\xi=0$ is not an accumulation point of spectral singularities.

A special situation occurs when $\gamma=1$ is a root of (4.4) or (4.7) for some $n$. This happens if and only if $b/(\sqrt{2}a)=p/q$ such that $p-q$ is even, provided $p$ and $q$ are coprime. Then there are (infinitely many) non-negative integers $s$ and $m$ such that $p/q=(1+2s)/(1+4m)$, and the points $\check{\xi}_m=(\pi/(4a))(1+4m)$ are spectral singularities for (1.1+) with potentials $\check{\mu}_m p$, where $\check{\mu}_m=\check{\xi}_m$. Similarly, there are (infinitely many) non-negative integers $s$ and $m$ such that $p/q=(1+2s)/(3+4m)$, and the points $\check{\xi}_m=(\pi/(4a))(3+4m)$ are spectral singularities for (1.1−) with potentials $\check{\mu}_m p$, where $\check{\mu}_m=\check{\xi}_m$.

Figures 2–5 show some typical trajectories of roots $\xi(\mu)$ of $v_1(d;\xi,\mu)=0$. As $\mu$ increases the roots move toward the imaginary axis where they collide with a symmetrically located eigenvalue or root, depending on whether the collision takes place in the upper or lower half plane. The dots mark the starting points chosen for the numerical calculations which were done by integrating (2.4). The parameters specifying the trajectories are indicated in each figure caption.

The potential $p$ is of the form (4.1) with $a=1/(1+\sqrt{2})$ and $d=1$, so $b/(\sqrt{2}a)=1$.

We now turn our attention to more general potentials. The next result in essence says that if we can rule out nonzero spectral singularities, then eigenvalues must be purely imaginary.

**Theorem 4.1:** *Suppose that p is real, absolutely integrable, and of compact support* $[-d,d]$. *Then the following holds.*

(i) *If for all* $\mu \in [0,1]$, (1.1) *with potential* $\mu p$ *has no nonzero spectral singularities, then any eigenvalue of* (1.1) *with potential p must be purely imaginary.*

(ii) *Suppose that p satisfies the assumptions of part (ii) of Theorem* 2.4. *Then for any* $\mu > 0$, (1.1) *with potential* $\mu p$ *has no spectral singularities different from zero and hence has only purely imaginary eigenvalues.*

Note that in (i), $p$ is not required to be symmetric. All we need is the fact that the spectrum is symmetric about the imaginary axis and this follows from the realness of $p$. Part (ii) represents an extension of the "single lobe theorem" of Ref. 11.

*Proof:* We must show that nonimaginary eigenvalues cannot "come in from infinity" or be produced by a collision of two purely imaginary eigenvalues, or appear because zero might be a multiple root of $v_1(d;\xi,\mu_k)$ (for some $k$) that splits. Let $\mathcal{B}=\{\mu \in [0,1]:(1.1)$ with potential $\mu p$ has a nonimaginary eigenvalue$\}$ and define $\tilde{\mu}=\inf \mathcal{B}$. Suppose that $\mathcal{B}\neq\emptyset$. Then $\tilde{\mu}<1$ by the continuity of eigenvalues in $\mu$. Also, $\tilde{\mu}\geq\mu_1>0$ by a result of Ref. 10 (if $\mu_1\geq 1$, then $\mathcal{B}=\emptyset$), so we can assume that $0<\tilde{\mu}<1$ and that $\tilde{\mu}p$ has no nonimaginary eigenvalues. As was shown in Ref. 13, there exists $R>0$ such that the closed semidisk $D_R=\{\xi \in \overline{\mathbb{C}^+}:|\xi|\leq R\}$ contains all the eigenvalues of $\mu p$, $0\leq\mu\leq 1$. Thus there exist sequences $\{\mu_n\}$ and $\{\xi_n\}$ ($\xi_n\in D_R$ with Re $\xi_n>0$) such that $\mu_n\downarrow\tilde{\mu}$ and $\xi_n$ is an eigenvalue for $\mu_n p$. Hence we may assume that $\xi_n\to\tilde{\xi}\in D_R$. Then $v_1(d;\tilde{\xi},\tilde{\mu})=0$ and therefore $\tilde{\xi}$ is either an eigenvalue or a spectral singularity for $\tilde{\mu}p$. If it is an eigenvalue, it must be purely imaginary (by continuity and the definition of $\tilde{\mu}$), and hence we also have that $-\xi_n^*\to\tilde{\xi}$. This implies that $\tilde{\xi}$ is a nonsimple imaginary eigenvalue. If $\tilde{\xi}$ is a spectral singularity then, by the assumptions, it must be zero. First, suppose that $\tilde{\xi}$ is an eigenvalue. Then, since there are no nonimaginary eigenvalues when $\mu$ is slightly less than $\tilde{\mu}$, we have, near $(\tilde{\xi},\tilde{\mu})$, the expansion

$$v_1(d;\xi,\mu)=\alpha_{01}(\mu-\tilde{\mu})+\alpha_{20}(\xi-\tilde{\xi})^2+\cdots,$$

with $\alpha_{01}/\alpha_{20}<0$, so that $\xi-\tilde{\xi}\sim(-\alpha_{01}/\alpha_{20})^{1/2}(\mu-\tilde{\mu})^{1/2}$; this is the second scenario described below Lemma 2.2. Note that higher order branch points invariably involve nonimaginary eigenvalues when $\mu<\tilde{\mu}$. So there is one eigenvalue that approaches $\tilde{\xi}$ from above as $\mu\uparrow\tilde{\mu}$. By the argument used in the proof of Lemma 2.2, this eigenvalue must be the offspring of a collision. Since every collision involves nonimaginary eigenvalues, this contradicts the definition of $\tilde{\mu}$. Now suppose that $\tilde{\xi}=0$ is a spectral singularity, so $\tilde{\mu}=\mu_k$ for some $k$. Then there are two possibilities. Either $v_1(d;\xi,\mu)$ has the same behavior near $(\tilde{\xi},\tilde{\mu})$ as above and we have a contradiction, or we have that

$$v_1(d;\xi,\mu)=\alpha_{01}(\mu-\tilde{\mu})+\alpha_{30}\xi^3+\cdots,$$

where $\alpha_{01}/\alpha_{30}$ is purely imaginary with negative imaginary part, so that $\xi\sim(-\alpha_{01}/\alpha_{30})^{1/3}(\mu-\tilde{\mu})^{1/3}$. The latter case is the only branch point situation where no nonimaginary eigenvalues exist near zero for $\mu$ slightly less than $\tilde{\mu}$. However, now, as before, there is again a purely imaginary eigenvalue converging to zero as $\mu\uparrow\tilde{\mu}$; in addition, there are two roots (not eigenvalues) of $v_1(d;\xi,\mu)$ approaching 0 along rays making angles of $-\pi/6$ and $-5\pi/6$ with the Re $\xi$-axis. As in the first part of the proof, this contradicts the definition of $\tilde{\mu}$. In other words, $\mathcal{B}$ must be empty, which proves part (i).

To prove (ii) we suppose that $\xi_0>0$ is a spectral singularity for $\mu_0 p(t)$ and let $v_k=v_k(t;\xi_0,\mu_0)$, $k=1,2$, be normalized so that $v_1(0)=\pm v_2(0)=1$ on $\mathcal{S}_\pm$, respectively. Following Refs. 11 and 18 we derive from (1.1) the identity

$$i\int_0^d (v_1'v_2^*-v_2'v_1^*)dt-i\mu_0\int_0^d p(t)(|v_1|^2+|v_2|^2)dt=\xi_0\int_0^d (v_1v_2^*+v_2v_1^*)dt. \tag{4.8}$$

The first integral in (4.8) is real, as an integration by parts shows. Thus the left-hand side of (4.8) is purely imaginary while the right-hand side is real. Therefore, we conclude that

$$\int_0^d (v_1 v_2^* + v_2 v_1^*)dt = 0. \tag{4.9}$$

Rewriting this integral with the help of (1.1) as in Ref. 11 we obtain [since $\xi_0 \in \mathbb{R}$, $v_1(0)=1$]

$$\int_0^d (v_1 v_2^* + v_2 v_1^*)dt = -\frac{1}{\mu_0 p(0)} + \frac{|v_1(t_0)|^2}{\mu_0}\left[\frac{1}{p(t_0-)} - \frac{1}{p(t_0+)}\right] + \int_0^{t_0} \frac{|v_1(t)|^2 p'(t)}{\mu_0 p(t)^2}dt + \int_{t_0}^d \frac{|v_1(t)|^2 p'(t)}{\mu_0 p(t)^2}dt.$$

Since $\xi_0$ is real, we have that $|v_1(t)|^2+|v_2(t)|^2=2$ for all $t$. This follows from (1.1) which implies that[13]

$$(|v_1(t)|^2 + |v_2(t)|^2)' = 2(\mathrm{Im}\ \xi)(|v_1(t)|^2 - |v_2(t)|^2)$$

for any solution. Thus, since $p$ is increasing on $(0,t_0)$, decreasing on $(t_0,d)$, and strictly positive on $[0,d]$ by the assumptions, we conclude that

$$\int_0^d (v_1 v_2^* + v_2 v_1^*)dt \leqslant -\frac{1}{\mu_0 p(0)} + \frac{|v_1(t_0)|^2}{\mu_0}\left[\frac{1}{p(t_0-)} - \frac{1}{p(t_0+)}\right] + 2\int_0^{t_0} \frac{p'(t)}{\mu_0 p(t)^2}dt = \frac{1}{\mu_0 p(0)} + \frac{|v_1(t_0)|^2}{\mu_0}\left[\frac{1}{p(t_0-)} - \frac{1}{p(t_0+)}\right] - \frac{2}{\mu_0 p(t_0-)} \leqslant \frac{1}{\mu_0 p(0)} - \frac{2}{\mu_0 \max\{p(t_0-),p(t_0+)\}} < 0,$$

where the second inequality follows by using $0 \leqslant |v_1(t_0)|^2 \leqslant 2$ and the last inequality is an immediate consequence of the assumptions. This contradicts (4.9), and (ii) is proved. □

Our next goal is to analyze in more detail the location of spectral singularities for certain kinds of potentials. As a preparation we make in (1.1) the substitution

$$w = S^{-1}v, \tag{4.10}$$

where

$$S = \begin{pmatrix} (i(\xi+\sigma))/\mu p & (i(\xi-\sigma))/\mu p \\ 1 & 1 \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} -i\mu p/2\sigma & (\sigma-\xi)/2\sigma \\ i\mu p/2\sigma & (\xi+\sigma)/2\sigma \end{pmatrix}, \tag{4.11}$$

and $\sigma=\sigma(t)=\sqrt{\xi^2+\mu^2 p(t)^2}$. Hence,

$$w' = -i\sigma J w + (S^{-1})' S w, \tag{4.12}$$

where

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.13}$$

Substituting (4.11) into (4.12) yields

$$w' = \begin{pmatrix} -i\sigma + \alpha_0' & -\beta_0' \\ -\alpha_0' & i\sigma + \beta_0' \end{pmatrix} w, \tag{4.14}$$

where

$$\alpha_0(t) = \int_0^t \frac{\xi(\xi+\sigma)p'}{2p\sigma^2}d\tau, \quad \beta_0(t) = \int_0^t \frac{\xi(\xi-\sigma)p'}{2p\sigma^2}d\tau \tag{4.15}$$

[we leave the integrals unevaluated since we only need to evaluate their difference, see (4.19)]. We define

$$\Sigma_0(t) = \int_0^t \sigma(\tau)d\tau, \tag{4.16}$$

$$g_1(t) = e^{i\Sigma_0(t)-\alpha_0(t)}w_1(t), \tag{4.17}$$

$$g_2(t) = e^{-i\Sigma_0(t)-\beta_0(t)}w_2(t), \tag{4.18}$$

$$\eta_0(t) = e^{\beta_0(t)-\alpha_0(t)} = \frac{p(0)(\xi+\sigma(t))}{p(t)(\xi+\sigma(0))}, \tag{4.19}$$

where the second equality in (4.19) follows from (4.15) by doing the necessary integration. Converting (4.14) to a system of integral equations, we obtain

$$g_1(t) - g_1(0) = -\int_0^t e^{2i\Sigma_0(\tau)}\eta_0(\tau)\beta_0'(\tau)g_2(0)d\tau - \int_0^t e^{2i\Sigma_0(\tau)}\eta_0(\tau)\beta_0'(\tau)(g_2(\tau)-g_2(0))d\tau, \tag{4.20}$$

$$g_2(t) - g_2(0) = -\int_0^t e^{-2i\Sigma_0(\tau)}(\alpha_0'(\tau)/\eta_0(\tau))g_1(0)d\tau - \int_0^t e^{-2i\Sigma_0(\tau)}(\alpha_0'(\tau)/\eta_0(\tau))(g_1(\tau)-g_1(0))d\tau. \tag{4.21}$$

The bounds contained in the next lemma will be needed to control various quantities associated with (4.20) and (4.21).

*Lemma 4.2: Let* $\gamma=\xi/\mu$. *Suppose that on* $[0,d]$, $p>0$ *and that* $p$ *is absolutely continuous. Then for* $0\leq t\leq d$, *we have*

$$\left|\frac{\eta_0(t)\beta_0'(t)}{\sigma(t)}\right| \leq \frac{C\gamma|p'(t)|}{\mu(\gamma+1)^4}, \quad \left|\frac{\alpha_0'(t)}{\eta_0(t)\sigma(t)}\right| \leq \frac{C\gamma|p'(t)|}{\mu(\gamma+1)^2}. \tag{4.22}$$

*Suppose that* $p'$ *is absolutely continuous on* $[0,d]$. *Then for* $0\leq t\leq d$, *we have*

$$\left|\left(\frac{\eta_0(t)\beta_0'(t)}{\sigma(t)}\right)'\right| \leq \frac{C\gamma(1+|p''(t)|)}{\mu(\gamma+1)^4}, \quad \left|\left(\frac{\alpha_0'(t)}{\eta_0(t)\sigma(t)}\right)'\right| \leq \frac{C\gamma(1+|p''(t)|)}{\mu(\gamma+1)^2}. \tag{4.23}$$

*The constant* $C$ *depends on* $p(t)$ *and* $p'(t)$ *via their sup norms, but not on* $\gamma$ *and* $\mu$.

*Proof:* These estimates follow from (4.15) and (4.19) on substituting $\xi=\gamma\mu$. □

The next lemma provides estimates for those solutions of (4.20) and (4.21) that are associated with (1.1+) or (1.1−). We assume that these solutions are normalized such that $v(0)=(1,\pm 1)^T$. Setting

$$\tilde{\sigma}(t) = \sqrt{\gamma^2 + p(t)^2}, \tag{4.24}$$

so that $\sigma(t)=\mu\tilde{\sigma}(t)$, and using (4.10), (4.11), (4.17), and (4.18), we obtain

$$g_1(0) = w_1(0) = \frac{\pm(\tilde{\sigma}(0)-\gamma) - ip(0)}{2\tilde{\sigma}(0)}, \tag{4.25}$$

$$g_2(0) = w_2(0) = \frac{\pm(\gamma + \tilde{\sigma}(0)) + ip(0)}{2\tilde{\sigma}(0)}, \tag{4.26}$$

where $\pm$ refers to $\mathcal{S}_{\pm}$.

*Lemma 4.3: Suppose that p is even, $p=0$ for $|t|>d$, $p>0$ for $|t|\leq d$, and that $p'$ is absolutely continuous on $[-d,d]$. Let $g(t)$ be a vector solution of* (4.20) and (4.21) *obeying* (4.25) and (4.26) *for either choice of $\pm$. Then,*

$$g_1(t) = g_1(0)(1 + \delta_1(t)), \tag{4.27}$$

$$g_2(t) = g_2(0)(1 + \delta_2(t)), \tag{4.28}$$

*where $\delta_1(t)$ and $\delta_2(t)$ satisfy*

$$|\delta_k(t)| \leq \frac{C\gamma}{\mu(\gamma+1)^3}, \quad k = 1,2. \tag{4.29}$$

*Proof:* Our goal is to apply Gronwall's inequality to (4.20) and (4.21). To do so, we must first estimate the first terms on the right-hand side of either equation. By means of an integration by parts, we obtain

$$\int_0^t e^{2i\Sigma_0(\tau)}\eta_0(\tau)\beta_0'(\tau)d\tau = e^{2i\Sigma_0(t)}\frac{\eta_0(t)\beta_0'(t)}{2i\sigma(t)} - \frac{\eta_0(0)\beta_0'(0)}{2i\sigma(0)} - \int_0^t e^{2i\Sigma_0(\tau)}\left(\frac{\eta_0(\tau)\beta_0'(\tau)}{2i\sigma(\tau)}\right)' d\tau. \tag{4.30}$$

Using (4.22) and (4.23), we obtain for $t \in [0,d]$

$$\left|\int_0^t e^{2i\Sigma_0(\tau)}\eta_0(\tau)\beta_0'(\tau)d\tau\right| \leq \frac{C\gamma}{\mu(\gamma+1)^4}. \tag{4.31}$$

Here the continuity of $p'(t)$ was used and the integral involving $|p''(t)|$ was absorbed in $C$. Proceeding similarly with the first term on the right-hand side of (4.21), we get that

$$\left|\int_0^t e^{-2i\Sigma_0(\tau)}(\alpha_0'(\tau)/\eta_0(\tau))d\tau\right| \leq \frac{C\gamma}{\mu(\gamma+1)^2}. \tag{4.32}$$

From (4.25) and (4.26), the following bounds are obvious (for either choice of $\pm$),

$$C_1(1+\gamma)^{-1} \leq |g_1(0)| \leq C_2(1+\gamma)^{-1}, \quad C_3 \leq |g_2(0)| \leq C_4, \tag{4.33}$$

with suitable positive constants $C_j$, $j=1,\ldots,4$. Let $\Delta_k(t)=g_k(t)-g_k(0)$ for $k=1,2$. Using (4.31)–(4.33) and the estimates of Lemma 4.2 in (4.20) and (4.21), we obtain

$$|\Delta_1(t)| \leq \frac{K_1\gamma}{\mu(\gamma+1)^4} + \frac{K_2\gamma}{(\gamma+1)^3}\int_0^t |p'(\tau)||\Delta_2(\tau)|d\tau, \tag{4.34}$$

$$|\Delta_2(t)| \leq \frac{K_3\gamma}{\mu(\gamma+1)^3} + \frac{K_4\gamma}{\gamma+1}\int_0^t |p'(\tau)||\Delta_1(\tau)|d\tau. \tag{4.35}$$

Consequently, after substituting (4.34) and (4.35) and vice versa, Gronwall's inequality gives

$$|\Delta_k(t)| \leq \frac{K_5\gamma}{\mu(\gamma+1)^{5-k}}, \quad t \in [0,d], \quad k = 1,2. \tag{4.36}$$

Solving (4.27) and (4.28) for $\delta_1(t)$ and $\delta_2(t)$ and using (4.33) and (4.36), we obtain (4.29). □

**Theorem 4.4:** *Suppose that $p$ satisfies the conditions of Lemma 4.3. Then there exists $\mu_0 > 0$ such that if $\xi > 0$ and $\mu > \mu_0$, then $v_1(d;\xi,\mu) \neq 0$. In other words, if $\mu > \mu_0$, then there are no nonzero spectral singularities.*

*Proof:* From (4.10) and (4.11), we have

$$v_1(t) = \frac{i(\xi + \sigma(t))}{\mu p(t)} w_1(t) + \frac{i(\xi - \sigma(t))}{\mu p(t)} w_2(t),$$

where

$$w_1(t) = e^{-i\Sigma_0(t)+\alpha_0(t)} w_1(0)(1 + \delta_1(t)),$$

$$w_2(t) = e^{i\Sigma_0(t)+\beta_0(t)} w_2(0)(1 + \delta_2(t)).$$

Setting $v_1(d)=0$ gives

$$1 + e^{2i\Sigma_0(d)} \eta_0(d) \frac{\gamma - \tilde{\sigma}(d)}{\gamma + \tilde{\sigma}(d)} \frac{w_2(0)}{w_1(0)} \frac{1 + \delta_2(d)}{1 + \delta_1(d)} = 0, \tag{4.37}$$

where $\tilde{\sigma}(t)$ is defined in (4.24). After some calculations using

$$p(0)^2 + (\gamma \pm \tilde{\sigma}(0))^2 = 2\tilde{\sigma}(0)(\tilde{\sigma}(0) \pm \gamma),$$

where the upper sign corresponds to $\mathcal{S}_+$, the lower sign to $\mathcal{S}_-$, we find that

$$\eta_0(d) \frac{\tilde{\sigma}(d) - \gamma}{\gamma + \tilde{\sigma}(d)} \frac{|w_2(0)|}{|w_1(0)|} = \frac{p(d)}{\gamma + \tilde{\sigma}(d)}. \tag{4.38}$$

Now

$$\left| \frac{1 + \delta_2(d)}{1 + \delta_1(d)} \right| \leqslant 1 + \frac{C\gamma}{\mu(\gamma+1)^3}, \tag{4.39}$$

on account of (4.29) and hence,

$$\frac{p(d)}{\gamma + \tilde{\sigma}(d)} \left| \frac{1 + \delta_2(d)}{1 + \delta_1(d)} \right| \leqslant \frac{p(d)}{\gamma + p(d)} \left| \frac{1 + \delta_2(d)}{1 + \delta_1(d)} \right| \leqslant 1 - \frac{\gamma}{\gamma + p(d)} \left( 1 - \frac{Cp(d)}{\mu(\gamma+1)^3} \right). \tag{4.40}$$

The right-hand side is less than 1 provided that $\gamma \neq 0$ and $\mu > Cp(d)$. On setting $\mu_0 = Cp(d)$ and using (4.38) and (4.40), we conclude that (4.37) cannot hold. Hence $v_1(d;\xi,\mu) \neq 0$ whenever $\xi > 0$ and $\mu > \mu_0$. Since $v_1(d;-\xi,\mu) = v_1(d;\xi,\mu)^*$ when $\xi$ is real, the result also holds when $\xi < 0$.□

Theorem 4.4 actually holds under weaker conditions, in particular, without the symmetry assumption. We state this result as a corollary.

*Corollary 4.5: Suppose that $p > 0$ on $[0,d]$, $p=0$ for $t<0$ and $t>d$. In addition, suppose that on $[0,d]$, $p$ is absolutely continuous. Then there exists $\mu_0 > 0$ such that if $\xi \neq 0$ and $\mu > \mu_0$, then $v_1(d;\xi,\mu) \neq 0$.*

*Proof:* It suffices to consider $\xi > 0$. As initial condition, we choose $v(0) = (1,0)^T$ so that, by (4.17) and (4.18), $g(0) = w(0) = ip(0)/(2\tilde{\sigma}(0))(-1,1)^T$. Thus

$$C_1(\gamma+1)^{-1} \leqslant |g_k(0)| \leqslant C_2(\gamma+1)^{-1}, \quad k = 1,2. \tag{4.41}$$

By the absolute continuity of $p$, we have, in place of (4.31) and (4.32),

$$\left| \int_0^t e^{2i\Sigma_0(\tau)} \eta_0(\tau) \beta_0'(\tau) d\tau \right| \leqslant \frac{\gamma\delta(\mu)}{(\gamma+1)^3}, \tag{4.42}$$

$$\left| \int_0^t e^{-2i\Sigma_0(\tau)}(\alpha_0'(\tau)/\eta_0(\tau))d\tau \right| \leqslant \frac{\gamma\delta(\mu)}{\gamma+1}, \quad (4.43)$$

where $\delta(\mu)\to 0$ as $\mu\to\infty$. To see this, consider (4.42) and note that

$$\eta_0(\tau)\beta_0'(\tau) = -\frac{\gamma p(0)}{2(\gamma+\tilde{\sigma}(0))}\frac{p'(\tau)}{\tilde{\sigma}(\tau)^2},$$

so that it suffices to show that

$$\left| \int_0^t e^{2i\Sigma_0(\tau)}\frac{p'(\tau)}{\tilde{\sigma}(\tau)^2}d\tau \right| \leqslant \frac{\tilde{\delta}(\mu)}{(\gamma+1)^2}, \quad (4.44)$$

with $\tilde{\delta}(\mu)\to 0$ as $\mu\to\infty$. To prove (4.44), we choose a sequence $\{p_n\}$ of infinitely differentiable functions such that on $[0,d]$, $p_n(t)\to p(t)$ uniformly and $p_n'(t)\to p'(t)$ in the $L^1$ norm as $n\to\infty$. Using this sequence to approximate the $p'(\tau)$ in the numerator [the $p(\tau)$'s contained in $\Sigma_0(\tau)$ and $\tilde{\sigma}(\tau)$ need not be approximated], we get

$$\int_0^t e^{2i\Sigma_0(\tau)}\frac{p'(\tau)}{\tilde{\sigma}(\tau)^2}d\tau = \int_0^t e^{2i\Sigma_0(\tau)}\frac{p'(\tau)-p_n'(\tau)}{\tilde{\sigma}(\tau)^2}d\tau + \int_0^t e^{2i\Sigma_0(\tau)}\frac{p_n'(\tau)}{\tilde{\sigma}(\tau)^2}d\tau.$$

For any given $\epsilon>0$, the first integral is less than $\epsilon/(\gamma+1)^2$ if $n$ is large enough. Choosing $n_\epsilon$ such that for $n\geqslant n_\epsilon$ this is the case and using an integration by parts, we see that

$$\left| \int_0^t e^{2i\Sigma_0(\tau)}\frac{p_{n_0}'(\tau)}{\tilde{\sigma}(\tau)^2}d\tau \right| \leqslant \frac{C_{n_\epsilon}}{\mu(\gamma+1)^3}.$$

Thus

$$\tilde{\delta}(\mu) := \sup_{\gamma\geqslant 0}\left( (\gamma+1)^2 \left| \int_0^t e^{2i\Sigma_0(\tau)}\frac{p'(\tau)}{\tilde{\sigma}(\tau)^2}d\tau \right| \right) \leqslant \epsilon + \frac{C_{n_\epsilon}}{\mu},$$

and hence $\tilde{\delta}(\mu)\to 0$ as $\mu\to\infty$. Thus (4.44) and then (4.42) follow. A similar argument works for (4.43). This leads, via (4.20) and (4.21), to

$$|\Delta_k(t)| \leqslant \frac{C\gamma\delta(\mu)}{(\gamma+1)^{3-(-1)^k}}, \quad t\in[0,d], \quad k=1,2,$$

which in turn gives, by (4.27), (4.28), and (4.41),

$$|\delta_k(d)| \leqslant \frac{C\gamma\delta(\mu)}{(\gamma+1)^{2-(-1)^k}}, \quad k=1,2.$$

Hence, in place of (4.39), we now have

$$\left| \frac{1+\delta_2(d)}{1+\delta_1(d)} \right| \leqslant 1 + \frac{C\gamma\delta(\mu)}{\gamma+1}.$$

Proceeding as in the step leading from (4.39) and (4.40), we see that it suffices to choose $\mu_0$ so large that $Cp(d)\delta(\mu)<1$ if $\mu>\mu_0$. Since this is possible, the conclusion of the theorem holds.□

The results of Theorem 4.4 and Corollary 4.5 are in sharp contrast to what one finds when the requirement that $p$ be strictly positive on $[-d,d]$ is relaxed. For example, looking back to the example at the beginning of this section, we recall that there are unbounded sequences of coupling constants for which we find spectral singularities, and that the latter accumulate at certain specific points on the real axis. It is a natural question to ask whether this is just a peculiar property of piecewise constant potentials or if it holds for a larger class of potentials. It turns out that the latter is true. To answer this question, we now consider potentials of the following form.

There is a number $a$, $0<a<d$, such that

$$p(t) > 0 \quad \text{for } a \leqslant |t| \leqslant d,$$

$$p(t) = 0 \quad \text{otherwise},$$

$$p(t) = p(-t) \quad \text{for all } t \in \mathbb{R}. \tag{4.45}$$

Smoothness conditions on $p$ will be added when needed. To reduce the number of cases, we will only work out the details for (1.1+) and then merely state the results for (1.1−). From the normalization $v(0)=(1,1)^T$ on $\mathcal{S}_+$, it follows that $v(a)=(e^{-i\xi a},e^{i\xi a})^T$. Then, by (4.10),

$$w(a) = \begin{pmatrix} -(ip(a)/2\tilde{\sigma}(a))e^{-i\xi a} + (\tilde{\sigma}(a) - \gamma/2\tilde{\sigma}(a))e^{i\xi a} \\ (ip(a)/2\tilde{\sigma}(a))e^{-i\xi a} + [(\tilde{\sigma}(a) + \gamma)/2\tilde{\sigma}(a)]e^{i\xi a} \end{pmatrix}.$$

We decompose $w(a)$ as follows:

$$w(a) = w^{(-)}(a)e^{-i\xi a} + w^{(+)}(a)e^{i\xi a},$$

where

$$w^{(-)}(a) = \begin{pmatrix} -ip(a)/2\tilde{\sigma}(a) \\ ip(a)/2\tilde{\sigma}(a) \end{pmatrix}, \quad w^{(+)}(a) = \begin{pmatrix} (\tilde{\sigma}(a) - \gamma)/2\tilde{\sigma}(a) \\ (\tilde{\sigma}(a) + \gamma)/2\tilde{\sigma}(a) \end{pmatrix}. \tag{4.46}$$

Notice that here the superscript $\pm$ indicates that the vector belongs to the solution containing the factor $e^{\pm i\xi a}$. The solutions of (4.12) satisfying $w(a)=w^{(\pm)}(a)$ will be denoted by $w^{(\pm)}(t)$. In analogy to (4.15), (4.16), and (4.19), we let

$$\Sigma_a(t) = \int_a^t \sqrt{\xi^2 + \mu^2 p(\tau)^2}\, d\tau = \mu \int_a^t \tilde{\sigma}(\tau)\, d\tau,$$

$$\alpha_a(t) = \int_a^t \frac{\gamma(\gamma + \tilde{\sigma})p'}{2p\tilde{\sigma}^2}\, d\tau, \quad \beta_a(t) = \int_a^t \frac{\gamma(\gamma - \tilde{\sigma})p'}{2p\tilde{\sigma}^2}\, d\tau,$$

$$\eta_a(t) = \frac{p(a)(\gamma + \tilde{\sigma}(t))}{p(t)(\gamma + \tilde{\sigma}(a))}, \tag{4.47}$$

and introduce [compare (4.17) and (4.18)]

$$g_1^{(\pm)}(t) = e^{i\Sigma_a(t) - \alpha_a(t)} w_1^{(\pm)}(t),$$

$$g_2^{(\pm)}(t) = e^{-i\Sigma_a(t) - \beta_a(t)} w_2^{(\pm)}(t),$$

so that the integral equations (4.20) and (4.21) now take the form

$$g_1^{(\pm)}(t) - g_1^{(\pm)}(a) = -\int_a^t e^{2i\Sigma_a(\tau)} \eta_a(\tau)\beta_a'(\tau) g_2^{(\pm)}(a)\, d\tau - \int_a^t e^{2i\Sigma_a(\tau)} \eta_a(\tau)\beta_a'(\tau)(g_2^{(\pm)}(\tau) - g_2^{(\pm)}(a))\, d\tau, \tag{4.48}$$

$$g_2^{(\pm)}(t) - g_2^{(\pm)}(a) = - \int_a^t e^{-2i\Sigma_a(\tau)}(\alpha_a'(\tau)/\eta_a(\tau))g_1^{(\pm)}(a)d\tau - \int_a^t e^{-2i\Sigma_a(\tau)}(\alpha_a'(\tau)/\eta_a(\tau))(g_1^{(\pm)}(\tau) - g_1^{(\pm)} \times (a))d\tau. \tag{4.49}$$

From (4.48) and (4.49), we obtain the following estimates.

*Lemma 4.6: Suppose that p is as in (4.45) and that p′ is absolutely continuous on* $[a,d]$. *Then the following holds.*

(i) *For* $t \in [a,d]$, *we have*

$$|g_k^{(+)}(t) - g_k^{(+)}(a)| \leq \frac{C\gamma}{\mu(\gamma+1)^4}, \quad k = 1,2, \tag{4.50}$$

$$|g_1^{(-)}(t) - g_1^{(-)}(a)| \leq \frac{C\gamma}{\mu(\gamma+1)^5}, \quad |g_2^{(-)}(t) - g_2^{(-)}(a)| \leq \frac{C\gamma}{\mu(\gamma+1)^3}. \tag{4.51}$$

(ii) *For the partial derivative with respect to* $\mu$, *we have*

$$|g_{k;\mu}^{(+)}(t)| \leq \frac{C\gamma}{\mu(\gamma+1)^3}, \quad k = 1,2. \tag{4.52}$$

$$|g_{1;\mu}^{(-)}(t)| \leq \frac{C\gamma}{\mu(\gamma+1)^4}, \quad |g_{2;\mu}^{(-)}(t)| \leq \frac{C\gamma}{\mu(\gamma+1)^2}. \tag{4.53}$$

(iii) *For the partial derivative with respect to* $\gamma$, *we have*

$$|g_{k;\gamma}^{(+)}(t) - g_{k;\gamma}^{(+)}(a)| \leq \frac{C}{(\gamma+1)^4}\left(\gamma + \frac{1}{\mu}\right), \quad k = 1,2, \tag{4.54}$$

$$|g_{1;\gamma}^{(-)}(t) - g_{1;\gamma}^{(-)}(a)| \leq \frac{C}{(\gamma+1)^5}\left(\gamma + \frac{1}{\mu}\right), \tag{4.55}$$

$$|g_{2;\gamma}^{(-)}(t) - g_{2;\gamma}^{(-)}(a)| \leq \frac{C}{(\gamma+1)^3}\left(\gamma + \frac{1}{\mu}\right). \tag{4.56}$$

*Proof:* (i) The proofs of (4.50) and (4.51) are similar to those of (4.36). The only difference is that here, in place of (4.33), we have the inequalities

$$\frac{C_1}{(\gamma+1)^2} \leq |g_1^{+}(a)| \leq \frac{C_2}{(\gamma+1)^2}, \quad C_3 \leq |g_2^{(+)}(a)| \leq C_4, \tag{4.57}$$

$$\frac{C_5}{\gamma+1} \leq |g_k^{-}(a)| \leq \frac{C_6}{\gamma+1}, \quad k = 1,2. \tag{4.58}$$

Further details are therefore omitted. The proofs of (4.52) and (4.53) are based on the integral equations,

$$g_{1;\mu}^{(\pm)}(t) = -2ig_2^{(\pm)}(a)\int_a^t I(\tau)e^{2i\Sigma_a(\tau)}\eta_a(\tau)\beta_a'(\tau)d\tau - 2i\int_a^t I(\tau)e^{2i\Sigma_a(\tau)}\eta_a(\tau)\beta_a'(\tau)(g_2^{(\pm)}(\tau) - g_2^{(\pm)}(a))d\tau - \int_a^t e^{2i\Sigma_a(\tau)}\eta_a(\tau)\beta_a'(\tau)g_{2;\mu}^{(\pm)}(\tau)d\tau, \tag{4.59}$$

$$g_{2;\mu}^{(\pm)}(t) = 2ig_1^{(\pm)}(a)\int_a^t I(\tau)e^{-2i\Sigma_a(\tau)}[\alpha_a'(\tau)/\eta_a(\tau)]d\tau + 2i\int_a^t I(\tau)e^{-2i\Sigma_a(\tau)}[\alpha_a'(\tau)/\eta_a(\tau)](g_1^{(\pm)}(\tau) - g_1^{(\pm)}$$

$$\times(a))d\tau - \int_a^t e^{-2i\Sigma_a(\tau)}[\alpha_a'(\tau)/\eta_a(\tau)]g_{1;\mu}^{(\pm)}(\tau)d\tau, \tag{4.60}$$

where $I(\tau)=\int_a^\tau\tilde{\sigma}(u)du$. The first terms on the right-hand sides of (4.59) and (4.60) are handled by an integration by parts and then estimated using Lemma 4.2 together with the obvious inequalities $0\leq I(\tau)\leq C(\gamma+1)$ and $0\leq I'(\tau)\leq C(\gamma+1)$. The middle terms are estimated using (4.50) and (4.51), and Lemma 4.2. This leads to (for the + sign)

$$|g_{1;\mu}^{(+)}(t)| \leq \frac{K_1\gamma}{\mu(\gamma+1)^3} + \frac{K_2\gamma^2}{\mu(\gamma+1)^6} + \frac{K_3\gamma}{(\gamma+1)^3}\int_a^t |p'(\tau)||g_{2;\mu}^{(+)}(\tau)|d\tau,$$

$$|g_{2;\mu}^{(+)}(t)| \leq \frac{K_4\gamma}{\mu(\gamma+1)^3} + \frac{K_5\gamma^2}{\mu(\gamma+1)^4} + \frac{K_6\gamma}{\gamma+1}\int_a^t |p'(\tau)||g_{1;\mu}^{(+)}(\tau)|d\tau,$$

from which (4.52) follows by an appeal to Gronwall's lemma. The proof of (4.53) is similar. To prove (4.54)–(4.56), we start from the pair of integral equations satisfied by $g_{k;\gamma}^{(\pm)}(t)-g_{k;\gamma}^{(\pm)}(a)$ of which the first one reads (for +)

$$g_{1;\gamma}^{(+)}(t) - g_{1;\gamma}^{(+)}(a) = A_1 + A_2 + A_3 + A_4,$$

where

$$A_1 = -g_2^{(+)}(a)\int_0^t 2i\mu e^{2i\Sigma_a(\tau)}I_1(\tau)\eta_a(\tau)\beta_a'(\tau)d\tau, \tag{4.61}$$

$$A_2 = -\int_0^t e^{2i\Sigma_a(\tau)}(\eta_a(\tau)\beta_a'(\tau)g_2^{(+)}(a))_\gamma\, d\tau,$$

$$A_3 = -\int_0^t (e^{2i\Sigma_a(\tau)}\eta_a(\tau)\beta_a'(\tau))_\gamma(g_2^{(+)}(t) - g_2^{(+)}(a))d\tau,$$

$$A_4 = -\int_0^t e^{2i\Sigma_a(\tau)}\eta_a(\tau)\beta_a'(\tau)(g_{2;\gamma}^{(+)}(t) - g_{2;\gamma}^{(+)}(a))d\tau,$$

and where, in (4.61), $I_1(\tau)=\int_a^\tau\gamma/\sqrt{\gamma^2+p(s)^2}ds$. The difference $g_{2;\gamma}^{(+)}(\tau)-g_{2;\gamma}^{(+)}(a)$ is broken up in a similar way; we need not write it out. We only add some remarks about how the terms $A_1$ through $A_4$ are being treated. Using an integration by parts, we see that

$$|A_1| \leq \frac{C\gamma^2}{(\gamma+1)^5}.$$

Note that $0\leq I_1(\tau)\leq C\gamma/(\gamma+1)$ and $0\leq I_1'(\tau)\leq C\gamma/(\gamma+1)$. In $A_2$ we again use an integration by parts and obtain the bound

$$|A_2| \leq \frac{C}{\mu(\gamma+1)^4}.$$

Working out the partial derivative $(e^{2i\Sigma_a(\tau)}\eta_a(\tau)\beta_a'(\tau))_\gamma$ and using (4.22) we find that

$$|A_3| \leqslant \frac{C\gamma}{(\gamma+1)^7}\left(\gamma+\frac{1}{\mu}\right).$$

Collecting these estimates together gives

$$|A_1|+|A_2|+|A_3| \leqslant \frac{C}{(\gamma+1)^4}\left(\gamma+\frac{1}{\mu}\right).$$

Furthermore, by (4.22), we have that

$$|A_4| \leqslant \frac{C\gamma}{(\gamma+1)^3}\int_0^t |p'(\tau)||g_{2;\gamma}^{(+)}(\tau)-g_{2;\gamma}^{(+)}(a)|d\tau.$$

Estimating $g_{2;\gamma}^{(+)}(\tau)-g_{2;\gamma}^{(+)}(a)$ in a similar way and applying Gronwall's lemma, we obtain (4.54). The reasoning for (4.55) and (4.56) is similar. □

We set

$$w_1^{(\pm)}(t)=e^{-i\Sigma_a(t)+\alpha_a(t)}w_1^{(\pm)}(a)(1+r_1^{(\pm)}(t;\gamma,\mu)), \tag{4.62}$$

$$w_2^{(\pm)}(t)=e^{i\Sigma_a(t)+\beta_a(t)}w_2^{(\pm)}(a)(1+r_2^{(\pm)}(t;\gamma,\mu)), \tag{4.63}$$

and introduce the quantities

$$\hat{r}_1^{(+)}(t;\gamma,\mu)=-\frac{i\gamma(\gamma+\tilde{\sigma}(a))}{4\mu p(a)}\left(e^{2i\mu\int_a^t\tilde{\sigma}(\tau)d\tau}\frac{p'(t)}{\tilde{\sigma}(t)^3}-\frac{p'(a)}{\tilde{\sigma}(a)^3}\right),$$

$$\hat{r}_2^{(+)}(t;\gamma,\mu)=\frac{\gamma-\tilde{\sigma}(a)}{\gamma+\tilde{\sigma}(a)}\hat{r}_1^{(+)}(t;\gamma,\mu)^*,$$

$$\hat{r}_1^{(-)}(t;\gamma,\mu)=\hat{r}_2^{(+)}(t;\gamma,\mu)^*, \quad \hat{r}_2^{(-)}(t;\gamma,\mu)=\hat{r}_1^{(+)}(t;\gamma,\mu)^*. \tag{4.64}$$

Finally, we set

$$r_k^{(\pm)}(t;\gamma,\mu)=\hat{r}_k^{(\pm)}(t;\gamma,\mu)+\epsilon_k^{(\pm)}(t;\gamma,\mu). \tag{4.65}$$

Bounds on $r_k^{(\pm)}(t;\gamma,\mu)$, $\epsilon_k^{(\pm)}(t;\gamma,\mu)$, and other related quantities are summarized in the following.

*Lemma 4.7: Under the assumptions of Lemma 4.6, there is a constant $C$ such that for $k=1,2$,*

$$|r_k^{(\pm)}(t;\gamma,\mu)| \leqslant \frac{C\gamma}{\mu(\gamma+1)^{3\pm(-1)^k}}, \tag{4.66}$$

$$|\epsilon_k^{(\pm)}(t;\gamma,\mu)| \leqslant \frac{C\gamma}{\mu(\gamma+1)^{3\pm(-1)^k}}(\gamma+o(1)), \tag{4.67}$$

$$|\epsilon_{k;\mu}^{(\pm)}(t;\gamma,\mu)| \leqslant \frac{C\gamma}{\mu(\gamma+1)^{2\pm(-1)^k}}(\gamma+o(1)), \tag{4.68}$$

*where $o(1)$ designates a term that goes to zero as $\mu\to\infty$. Moreover,*

$$|r_{k;\gamma}^{(\pm)}(t;\gamma,\mu)| \leqslant \frac{C}{(\gamma+1)^{3\pm(-1)^k}}\left(\gamma+\frac{1}{\mu}\right). \tag{4.69}$$

*Proof:* Since $w^{(\pm)}(a)=g^{(\pm)}(a)$, we have that

$$r_k^{(\pm)}(t;\gamma,\mu)=\frac{g_k^{(\pm)}(t)-g_k^{(\pm)}(a)}{g_k^{(\pm)}(a)}. \tag{4.70}$$

Hence (4.66) is an immediate consequence of Lemma 4.6, (4.57) and (4.58). In proving (4.67), we restrict ourselves to the case of $\epsilon_1^{(+)}$; the other cases are dealt with similarly. From (4.48), (4.64), (4.65), and (4.70), along with an integration by parts, we derive

$$\epsilon_1^{(+)}(t;\gamma,\mu)=-(1/g_1^{(+)}(a))\int_a^t e^{2i\Sigma_a(\tau)}\eta_a(\tau)\beta_a'(\tau)(g_2^{(+)}(\tau)-g_2^{(+)}(a))d\tau+(g_2^{(+)}(a)/g_1^{(+)}(a))\int_a^t e^{2i\Sigma_a(\tau)} \times\left(\frac{\eta_a(\tau)\beta_a'(\tau)}{2i\mu\tilde{\sigma}(\tau)}\right)'d\tau. \tag{4.71}$$

The first term on the right-hand side of (4.71) is bounded by $C\mu^{-1}\gamma^2(\gamma+1)^{-5}$ on account of (4.22), (4.49), and (4.50). The second term is equal to

$$\frac{i\gamma g_2^{(+)}(a)p(a)}{4\mu(\gamma+\tilde{\sigma}(a))g_1^{(+)}(a)}\int_a^t e^{2i\Sigma_a(\tau)}\left(\frac{p'(\tau)}{\tilde{\sigma}(\tau)^3}\right)'d\tau.$$

First we note that the coefficient in front of the integral is bounded by $C\gamma(\gamma+1)/\mu$. The integral can be estimated by an approximation argument similar to that used to prove (4.44), except that here we choose a sequence $\{p_n(t)\}$ such that $p_n(t)\to p(t)$ and $p_n'(t)\to p'(t)$ uniformly on $[a,d]$, and $p_n''(t)\to p''(t)$ in the $L^1$ norm as $n\to\infty$. It follows that

$$\sup_{\gamma\geqslant 0}\left((\gamma+1)^3\left|\int_a^t e^{2i\Sigma_a(\tau)}\left(\frac{p'(\tau)}{\tilde{\sigma}(\tau)^3}\right)'d\tau\right|\right)\to 0 \quad \text{as } \mu\to\infty.$$

Combining this result with the bound on the first term in (4.71) yields (4.67) (for $\epsilon_1^{(+)}$). Differentiation of (4.71) with respect to $\mu$ and use of (4.52) and (4.53), in addition to the other results used to prove (4.67), we obtain (4.68). Finally, (4.69) follows on differentiating (4.70) with respect to $\gamma$ and using (4.46), (4.50), (4.51), and (4.54)–(4.58). □

The next theorem is the main result of this section. It provides asymptotic formulas for the location of the spectral singularities and for the corresponding coupling constants. Let

$$\beta=\frac{p'(a)}{p(a)^3}. \tag{4.72}$$

**Theorem 4.8:** *Suppose p satisfies the assumptions of Lemma 4.6.*

(i) *Suppose $0<\xi_0<\pi/(2a)$. Then there exists $\mu_0>0$ such that if $\mu>\mu_0$, then $(0,\xi_0]$ contains no spectral singularities of $\mu p(t)$.*

(ii) *Let $n\geqslant 2$ be even. Then there are two sequences, $\{\check{\mu}_j\}$ and $\{\check{\xi}_j\}$, such that each $\check{\xi}_j$ is a spectral singularity for (1.1+) with potential $\check{\mu}_j p(t)$ and the following asymptotics hold:*

$$\check{\xi}_j=\frac{n\pi}{2a}+\frac{\rho_1}{j+3/4}+\frac{\rho_2}{(j+3/4)^2}+O(1/j^3), \tag{4.73}$$

$$\check{\mu}_j=\frac{(j+3/4)\pi}{A}-\frac{n^2\pi\left(\int_a^d\frac{d\tau}{p(\tau)}\right)}{8a^2(j+3/4)}+O(1/j^2), \tag{4.74}$$

*as $j\to\infty$, where*

$$\rho_1 = \frac{nA}{4a^2 p(d)}, \quad \rho_2 = \frac{nA^2}{8\pi a^2}\left(-\beta + \frac{1}{ap(d)^2}\right). \tag{4.75}$$

*Moreover*, Im $\check{\xi}_j'(\check{\mu}_j) < 0$ *for sufficiently large* $j$.

(iii) *Let* $n \geq 1$ *be odd. Then there are two sequences*, $\{\check{\mu}_j\}$ *and* $\{\check{\xi}_j\}$, *such that each* $\check{\xi}_j$ *is a spectral singularity for (1.1+) with potential* $\check{\mu}_j p(t)$ *and*

$$\check{\xi}_j = \frac{n\pi}{2a} - \frac{\rho_1}{j+1/4} + \frac{\rho_2}{(j+1/4)^2} + O(1/j^3), \tag{4.76}$$

$$\check{\mu}_j = \frac{(j+1/4)\pi}{A} - \frac{n^2\pi\left(\int_a^d \frac{d\tau}{p(\tau)}\right)}{8a^2(j+1/4)} + O(1/j^2), \tag{4.77}$$

*as* $j \to \infty$. *Moreover*, Im $\check{\xi}_j'(\check{\mu}_j) > 0$ *for sufficiently large* $j$.

(iv) *Let* $n \geq 1$ *be odd. Then there are two sequences*, $\{\check{\mu}_j\}$ *and* $\{\check{\xi}_j\}$, *such that each* $\check{\xi}_j$ *is a spectral singularity for (1.1−) with potential* $\check{\mu}_j p(t)$, $\{\check{\xi}_j\}$ *obeys* (4.73), *and* $\{\check{\mu}_j\}$ *obeys* (4.74). *Moreover*, Im $\check{\xi}_j'(\check{\mu}_j) < 0$ *for sufficiently large* $j$.

(v) *Let* $n \geq 2$ *be even. Then there are two sequences*, $\{\check{\mu}_j\}$ *and* $\{\check{\xi}_j\}$, *such that each* $\check{\xi}_j$ *is a spectral singularity for (1.1−) with potential* $\check{\mu}_j p(t)$, $\{\check{\xi}_j\}$ *obeys* (4.76), *and* $\{\check{\mu}_j\}$ *obeys* (4.77). *Moreover*, Im $\check{\xi}_j'(\check{\mu}_j) > 0$ *for sufficiently large* $j$.

In preparation of the proof, we bring the equation $v_1(d;\xi,\mu)=0$ into a suitable form. Since the proofs for (1.1+) and (1.1−) are virtually the same, we will give the details only for the former.

From (4.10), (4.11), (4.62), and (4.63), we have

$$e^{-\alpha_a(d)}v_1(d;\xi,\mu) = \frac{i(\gamma+\tilde{\sigma}(d))}{p(d)}e^{-i\Sigma_a(d)}[w_1^{(-)}(a)e^{-i\xi a}(1+r_1^{(-)}(d)) + w_1^{(+)}(a)e^{i\xi a}(1+r_1^{(+)}(d))]$$
$$+ \frac{i(\gamma-\tilde{\sigma}(d))}{p(d)}e^{i\Sigma_a(d)}\eta_a(d)[w_2^{(-)}(a)e^{-i\xi a}(1+r_2^{(-)}(d)) + w_2^{(+)}(a)e^{i\xi a}(1+r_2^{(+)}(d))], \tag{4.78}$$

where $r_k^{(\pm)}(d) = r_k^{(\pm)}(d;\gamma,\mu)$ and $\eta_a(d)=\eta_a(d;\gamma)$ (with $\gamma=\xi/\mu$). We set

$$\psi_1 = \sin(2a\xi), \quad \psi_2 = \cot(\Sigma_a(d)),$$

$$\hat{r}_1^{(+)}(d;\gamma,\mu) = \hat{A}e^{2i\Sigma_a(d)} + \hat{B}, \quad \hat{r}_2^{(+)}(d;\gamma,\mu) = -\frac{\gamma-\tilde{\sigma}(a)}{\gamma+\tilde{\sigma}(a)}(\hat{A}e^{-2i\Sigma_a(d)} + \hat{B}),$$

$$\hat{r}_1^{(-)}(d;\gamma,\mu) = \frac{\gamma-\tilde{\sigma}(a)}{\gamma+\tilde{\sigma}(a)}(\hat{A}e^{2i\Sigma_a(d)} + \hat{B}), \quad \hat{r}_2^{(-)}(d;\gamma,\mu) = -\hat{A}e^{-2i\Sigma_a(d)} - \hat{B}, \tag{4.79}$$

where, from (4.64),

$$\hat{A} = -\frac{i\gamma(\gamma+\tilde{\sigma}(a))p'(d)}{4\mu p(a)\tilde{\sigma}(d)^3}, \quad \hat{B} = \frac{i\gamma(\gamma+\tilde{\sigma}(a))p'(a)}{4\mu p(a)\tilde{\sigma}(a)^3}. \tag{4.80}$$

Then (4.78) can be written as

$$-ie^{-\alpha_a(d)}e^{i\xi a}\frac{v_1(d;\xi,\mu)}{\sin(\Sigma_a(d))}=\mathcal{U}_1+\mathcal{U}_2+\mathcal{U}_3, \tag{4.81}$$

where $\mathcal{U}_k=\mathcal{U}_k(\psi_1,\psi_2;\gamma,\mu)$, $k=1,2,3$, are given by

$$\mathcal{U}_1(\psi_1,\psi_2;\gamma,\mu)=\frac{(\gamma+\tilde{\sigma}(d))}{p(d)}(\psi_2-i)[w_1^{(-)}(a)+w_1^{(+)}(a)(\pm\sqrt{1-\psi_1^2}+i\psi_1)]+\frac{(\gamma-\tilde{\sigma}(d))}{p(d)}(\psi_2+i)\eta_a(d)\times[w_2^{(-)}(a)+w_2^{(+)}(a)(\pm\sqrt{1-\psi_1^2}+i\psi_1)], \tag{4.82}$$

$$\mathcal{U}_2(\psi_1,\psi_2;\gamma,\mu)=\frac{(\gamma+\tilde{\sigma}(d))}{p(d)}(\hat{A}(\psi_2+i)+\hat{B}(\psi_2-i))\times\left[w_1^{(-)}(a)\frac{\gamma-\tilde{\sigma}(a)}{\gamma+\tilde{\sigma}(a)}+w_1^{(+)}(a)(\pm\sqrt{1-\psi_1^2}+i\psi_1)\right]-\frac{(\gamma-\tilde{\sigma}(d))}{p(d)}\eta_a(d)(\hat{A}(\psi_2-i)+\hat{B}(\psi_2+i))\times\left[w_2^{(-)}(a)+w_2^{(+)}(a)(\pm\sqrt{1-\psi_1^2}+i\psi_1)\frac{\gamma-\tilde{\sigma}(a)}{\gamma+\tilde{\sigma}(a)}\right], \tag{4.83}$$

$$\mathcal{U}_3(\psi_1,\psi_2;\gamma,\mu)=\frac{(\gamma+\tilde{\sigma}(d))}{p(d)}(\psi_2-i)\times[w_1^{(-)}(a)\epsilon_1^{(-)}(d)+w_1^{(+)}(a)(\pm\sqrt{1-\psi_1^2}+i\psi_1)\epsilon_1^{(+)}(d)]+\frac{(\gamma-\tilde{\sigma}(d))}{p(d)}\eta_a(d)(\psi_2+i)\times[w_2^{(-)}(a)\epsilon_2^{(-)}(d)+w_2^{(+)}(a)(\pm\sqrt{1-\psi_1^2}+i\psi_1)\epsilon_2^{(+)}(d)]. \tag{4.84}$$

In (4.82)–(4.84) the upper or lower sign is to be taken according as $\cos(2a\xi)>0$ or $\cos(2a\xi)<0$. Note that (4.65) was used to replace $r_k^{(\pm)}$ in (4.78). Division by $\sin(\Sigma_a(d))$ in (4.81) is permitted, since we are only interested in the parameter range where $\gamma$ is small, so that

$$v_1(d;\xi,\mu)=e^{-ia\xi}\cos(\Sigma_a(d))+e^{ia\xi}\sin(\Sigma_a(d))+O(\gamma).$$

Clearly, if $v_1(d;\xi,\mu)$ is zero, then $\sin(\Sigma_a(d))$ cannot be zero. More specifically, since

$$|e^{-ia\xi}\cos(\Sigma_a(d))+e^{ia\xi}\sin(\Sigma_a(d))|^2=1+\cos(2a\xi)\sin(2\Sigma_a(d)),$$

we see that for small $\gamma$ the zeros of $v_1(d;\xi,\mu)$ are located in regions of $(\xi,\mu)$-space where either $\xi\approx n\pi/(2a)$, with $n$ even and $\Sigma_a(d)\approx 3\pi/4+k\pi$, or $\xi\approx n\pi/(2a)$, with $n$ odd and $\Sigma_a(d)\approx\pi/4+k\pi$ [remember that $\Sigma_a(d)$ depends on $\mu$ and $\xi$]. To shorten the notation, we set $\psi=(\psi_1,\psi_2)$ and define $F_1=F_1(\psi;\gamma,\mu)$ and $F_2=F_2(\psi;\gamma,\mu)$ by

$$F_1=\mathrm{Re}(\mathcal{U}_1+\mathcal{U}_2+\mathcal{U}_3),\quad F_2=\mathrm{Im}(\mathcal{U}_1+\mathcal{U}_2+\mathcal{U}_3),$$

so that the equation $v_1(d;\xi,\mu)=0$ is equivalent to the two simultaneous equations,

$$F_1(\psi;\gamma,\mu)=0,\quad F_2(\psi;\gamma,\mu)=0. \tag{4.85}$$

For latter use, we note the following.

$$F_1(\psi;\gamma,\mu)=A_0+A_1\psi_1+A_2\psi_2\pm A_3\sqrt{1-\psi_1^2}+A_4\psi_1\psi_2\pm A_5\psi_2\sqrt{1-\psi_1^2}, \tag{4.86}$$

$$F_2(\psi;\gamma,\mu)=B_0+B_1\psi_1+B_2\psi_2\pm B_3\sqrt{1-\psi_1^2}+B_4\psi_1\psi_2\pm B_5\psi_2\sqrt{1-\psi_1^2}, \tag{4.87}$$

where the coefficients $A_0,\ldots,A_5$ and $B_0,\ldots,B_5$ depend on $\gamma$ and $\mu$ and are easily identified from (4.82)–(4.84) (although writing them out is very tedious). The following relations and bounds will be needed in the proof of Theorem 4.8. A detailed proof of these estimates is omitted because it basically involves just computations. We have used MATHEMATICA to do some of the necessary series expansions.

*Lemma 4.9: Suppose that* $\xi$ *is restricted to a compact subinterval of* $(0,\infty)$*. Then, as* $\mu\to\infty$*, we have*

$$A_0 = -\left(\frac{1}{p(a)} + \frac{1}{p(d)}\right)\frac{\gamma}{2} + O(1/\mu^2),$$

$$A_1 = 1 - \left(\frac{1}{p(a)} - \frac{1}{p(d)}\right)\frac{\gamma}{2} + O(1/\mu^2),$$

$$A_2 = O(1/\mu^2), \quad A_3 = O(1/\mu^2), \quad A_4 = O(1/\mu^2),$$

$$A_5 = -\left(\frac{1}{p(a)} - \frac{1}{p(d)}\right)\frac{\gamma}{2} + O(1/\mu^2),$$

$$B_0 = O(1/\mu^2), \quad B_1 = O(1/\mu^2),$$

$$B_2 = -1 + \left(\frac{1}{p(a)} - \frac{1}{p(d)}\right)\frac{\gamma}{2} + O(1/\mu^2),$$

$$B_3 = -1 + \left(\frac{1}{p(a)} - \frac{1}{p(d)}\right)\frac{\gamma}{2} + O(1/\mu^2),$$

$$B_4 = -\left(\frac{1}{p(a)} - \frac{1}{p(d)}\right)\frac{\gamma}{2} + O(1/\mu^2),$$

$$B_5 = O(1/\mu^2).$$

*Moreover, the following estimates hold.*

$$|A_0| \leqslant \frac{C\gamma}{\gamma+1}, \quad |A_1| \leqslant \frac{C}{\gamma+1}, \quad |A_2| \leqslant \frac{C\gamma}{\mu(\gamma+1)^2},$$

$$|A_3| \leqslant \frac{C\gamma}{\mu(\gamma+1)^2}, \quad |A_4| \leqslant \frac{C\gamma}{\mu(\gamma+1)^2}, \quad |A_5| \leqslant \frac{C\gamma}{(\gamma+1)^2},$$

$$|B_0| \leqslant \frac{C\gamma}{\mu(\gamma+1)^2}, \quad |B_1| \leqslant \frac{C\gamma}{\mu(\gamma+1)^2}, \quad |B_2| \leqslant C,$$

$$|B_3| \leqslant \frac{C}{\gamma+1}, \quad |B_4| \leqslant \frac{C\gamma}{(\gamma+1)^2}, \quad |B_5| \leqslant \frac{C\gamma}{\mu(\gamma+1)^2}.$$

From now on we give the details only for the case when $\cos 2a\xi > 0$ in $\mathcal{S}_+$. The results for the other cases will only be stated and briefly explained later. As a result of the above estimates, we have that, for small $\gamma$,

$$F_1(\psi;\gamma,\mu) = \psi_1 + O(\gamma), \quad F_2(\psi;\gamma,\mu) = -\sqrt{1-\psi_1^2} - \psi_2 + O(\gamma).$$

Hence, if $\gamma=0$, the solution of (4.85) is

$$\psi^{(0)} = (0,-1).$$

For large $\mu$ (small $\gamma$), we expect to find a solution of (4.85) close to $\psi^{(0)}$. To prove the existence of such a solution, we use a fixed point argument. We let, for any $\rho>0$,

$$B(\psi^{(0)};\rho) = \{\psi \in \mathbb{R}^2 : \|\psi - \psi^{(0)}\| \leq \rho\}$$

($\|\,\|$ denotes the Euclidean vector norm and later also the operator norm) and define $F^{(0)}: B(\psi^{(0)};1/2) \mapsto \mathbb{R}^2$ by

$$F^{(0)}(\psi) = (\psi_1, -\sqrt{1-\psi_1^2} - \psi_2).$$

Then for the Jacobian of $F^{(0)}$, we have (in the usual matrix form)

$$DF^{(0)}(\psi) = \begin{pmatrix} 1 & 0 \\ \psi_1 / \sqrt{1-\psi_1^2} & -1 \end{pmatrix}. \tag{4.88}$$

Note that

$$DF^{(0)}(\psi^{(0)}) = J, \tag{4.89}$$

where $J$ is defined in (4.13). We also define $F(\psi;\gamma,\mu) = (F_1(\psi;\gamma,\mu), F_2(\psi;\gamma,\mu))$ and we will denote the solution of $F(\psi;\gamma,\mu)=0$ by $\psi^{(0)}(\gamma,\mu)$. Let

$$\mathcal{T}: B(\psi^{(0)};1/2) \mapsto \mathbb{R}^2$$

be defined by

$$\mathcal{T}(\psi) = \psi - JF(\psi;\gamma,\mu).$$

Note that the mapping $\mathcal{T}$ depends on $\gamma$ and $\mu$.

*Lemma 4.10: Assume p satisfies the assumptions of Lemma 4.6 and suppose that $\xi$ lies in a compact subinterval of* $(0,\infty)$. *Then for sufficiently large* $\mu$, $\mathcal{T}$ *is a contraction mapping on* $B(\psi^{(0)};\rho_0)$, *where* $\rho_0 = \sqrt{3}/4$.

*Proof:* Write

$$D\mathcal{T}(\psi) = I - JDF(\psi;\gamma,\mu) = I - JDF^{(0)}(\psi) + J[DF^{(0)}(\psi) - DF(\psi;\gamma,\mu)], \tag{4.90}$$

where $I$ is the $2\times 2$ identity mapping. There is a constant $K_1>0$ such that

$$\|DF^{(0)}(\psi) - DF(\psi;\gamma,\mu)\| \leq K_1\gamma \tag{4.91}$$

for $\psi \in B(\psi^{(0)};1/2)$, all $\gamma \geq 0$, and $\mu \geq \mu_1$, where $\mu_1$ is defined in (2.3). This follows from Lemma 4.9 and (4.88). Furthermore, by (4.88) and (4.89),

$$I - JDF^{(0)}(\psi) = -J[DF^{(0)}(\psi) - DF^{(0)}(\psi^{(0)})] = -J\begin{pmatrix} 0 & 0 \\ \psi_1/\sqrt{1-\psi_1^2} & 0 \end{pmatrix}.$$

Then

$$\|I - JDF^{(0)}(\psi)\| \leq \frac{1}{\sqrt{3}}\|\psi - \psi^{(0)}\|, \quad \psi \in B(\psi^{(0)};1/2),$$

since on $B(\psi^{(0)};1/2)$, $\sup(|\psi_1|/\sqrt{1-\psi_1^2}) = 1/\sqrt{3}$. Thus, by (4.90),

$$\|D\mathcal{T}(\psi)\| \leq \frac{1}{\sqrt{3}}\|\psi - \psi^{(0)}\| + K_1\gamma, \tag{4.92}$$

and hence

$$\|\mathcal{T}(\psi) - \mathcal{T}(\psi^{(0)})\| \leq \left(\frac{1}{\sqrt{3}}\|\psi - \psi^{(0)}\| + K_1\gamma\right)\|\psi - \psi^{(0)}\|. \tag{4.93}$$

Also, there is a constant $K_2$ such that

$$\|\mathcal{T}(\psi^{(0)}) - \psi^{(0)}\| = \|F(\psi^{(0)};\gamma,\mu)\| \leq K_2\gamma. \tag{4.94}$$

This follows on inserting $\psi^{(0)}$ in (4.86) and (4.87) and using Lemma 4.9. Combining (4.93) and (4.94) yields

$$\|\mathcal{T}(\psi) - \psi^{(0)}\| \leq \left(\frac{1}{\sqrt{3}}\|\psi - \psi^{(0)}\| + K_1\gamma\right)\|\psi - \psi^{(0)}\| + K_2\gamma, \tag{4.95}$$

and so, if we choose

$$\rho_0 = \frac{\sqrt{3}}{4}, \quad \gamma_0 = \min\left\{\frac{1}{4K_1}, \frac{\sqrt{3}}{8K_2}\right\},$$

then, for $\gamma \leq \gamma_0$ and $\psi \in B(\psi^{(0)};\rho_0)$, we have

$$\|\mathcal{T}(\psi) - \psi^{(0)}\| \leq \rho_0.$$

Hence $\mathcal{T}$ maps $B(\psi^{(0)};\rho_0)$ into itself. Also, from (4.92) we have that $\|(D\mathcal{T})(\psi)\| \leq 1/2$ on $B(\psi^{(0)};\rho_0)$, and so $\mathcal{T}$ is a contraction. □

Therefore, $\mathcal{T}$ has a unique fixed point $(\tilde{\psi}_1(\gamma,\mu), \tilde{\psi}_2(\gamma,\mu)) \in B(\psi^{(0)};\rho_0)$ and we can solve the equation $\mathcal{T}(\psi) = \psi$ by iteration starting with $\psi = \psi^{(0)}$ as the initial guess. By induction, we see that

$$\|\mathcal{T}^n(\psi^{(0)}) - \mathcal{T}^{n-1}(\psi^{(0)})\| \leq C_n\gamma^n, \quad n = 1,2, \ldots,$$

where $C_n = (2K_2/\sqrt{3} + K_1)^{n-1}K_2$, and therefore, $\|\tilde{\psi}(\gamma,\mu) - \psi^{(0)}\| \leq 2K_2\gamma$. A calculation (we recommend using MATHEMATICA) gives

$$\mathcal{T}^2(\psi^{(0)}) = \begin{pmatrix} (\gamma/p(d)) - (\beta\gamma/2\mu) + o(\gamma^2) \\ -1 + (\gamma^2/p(a)p(d)) + o(\gamma^2) \end{pmatrix},$$

where $\beta$ is given in (4.72), and thus,

$$\tilde{\psi}_1(\gamma,\mu) = \frac{\gamma}{p(d)} - \frac{\beta\gamma}{2\mu} + o(\gamma^2), \tag{4.96}$$

$$\tilde{\psi}_2(\gamma,\mu) = -1 + \frac{\gamma^2}{p(a)p(d)} + o(\gamma^2). \tag{4.97}$$

We mention that the term $-\beta\gamma/(2\mu)$ in (4.96) arises from expanding

$$2i\hat{B} = -\gamma p'(a)/(2\mu p(a)^3) + O(\gamma^3) = -\gamma\beta/(2\mu) + O(\gamma^3),$$

where $\hat{B}$ is defined in (4.80). Further computations show that the orders of the error terms in (4.96) and (4.97) are consequences of the estimates (4.67).

For the proof of Theorem 4.8, we will also need bounds on the partial derivatives of $\tilde{\psi}_1(\gamma,\mu)$ and $\tilde{\psi}_2(\gamma,\mu)$. To obtain these, we use the relation

$$\begin{pmatrix} \tilde{\psi}_{1;\mu}(\gamma,\mu) \\ \tilde{\psi}_{2;\mu}(\gamma,\mu) \end{pmatrix} = -[DF(\tilde{\psi};\gamma,\mu)]^{-1} \begin{pmatrix} F_{1;\mu}(\tilde{\psi};\gamma,\mu) \\ F_{2;\mu}(\tilde{\psi};\gamma,\mu) \end{pmatrix},$$

and a similar relation for the partial derivative with respect to $\gamma$. Notice that $[DF(\tilde{\psi};\gamma,\mu)]^{-1}$ exists if $\mu$ is large enough on account of (4.89), (4.91), (4.96), and (4.97). Thus, bounds on the partial derivatives of $F_1$ and $F_2$ with respect to $\gamma$ and $\mu$ give rise to bounds of the same form (except for unimportant constants) on the partial derivatives of $\tilde{\psi}_1$ and $\tilde{\psi}_2$. Using (4.86) and (4.87), we obtain

$$\tilde{\psi}_{1;\mu}(\gamma,\mu) = o(1/\mu^2), \tag{4.98}$$

$$\tilde{\psi}_{1;\gamma}(\gamma,\mu) = \frac{1}{p(d)} + O(1/\mu), \tag{4.99}$$

$$\tilde{\psi}_{2;\mu}(\gamma,\mu) = o(1/\mu^2), \tag{4.100}$$

$$\tilde{\psi}_{2;\gamma}(\gamma,\mu) = O(1/\mu). \tag{4.101}$$

We remark that (4.98) comes from an estimate of the form

$$|\tilde{\psi}_{1;\mu}(\gamma,\mu)| \leqslant \frac{C\gamma}{\mu}(\gamma + o(1)) + \frac{C\gamma}{\mu^2}.$$

Here the first summand comes from (4.68) and the second comes from differentiating the quantities $\hat{A}$ and $\hat{B}$ in (4.80) with respect to $\mu$. Since $\gamma = O(1/\mu)$, the result follows. The justification for (4.99)–(4.101) is similar to that for (4.98).

*Proof of Theorem 4.8:* Substituting (4.96) and (4.97) in (4.79), we obtain two equations for $\gamma$ and $\mu$. We give the detailed proof only for $\mathcal{S}_+$ if $\cos(2a\xi) > 0$. First we prove (i). For small $\gamma$, setting the right-hand side of (4.81) equal to zero gives

$$\psi_1 - i(\sqrt{1-\psi_1^2} + \psi_2) + O(\gamma) = 0. \tag{4.102}$$

For the real part in (4.102), we thus obtain

$$\psi_1 = \sin(2a\xi) = O(\gamma). \tag{4.103}$$

Since $\gamma = \xi/\mu$, the right-hand side of (4.103) is bounded by $C\xi/\mu$. Thus, given $\xi_0 \in (0, \pi/(2a))$, choose $\mu_0 = C\xi_0(\sin(2a\xi_0))^{-1}$. Then, if $\xi \in (0, \xi_0]$ and $\mu > \mu_0$, $\sin(2a\xi) \geqslant (\sin(2a\xi_0)/\xi_0)\xi = (C/\mu_0)\xi > C(\xi/\mu)$, a contradiction; so assertion (i) follows. Now we turn to the proof of part (ii). We look at solutions $\xi(\gamma)$ of [see (4.79)]

$$\sin(2a\xi) = \tilde{\psi}_1(\gamma, \xi/\gamma) \tag{4.104}$$

for $\gamma$ near zero (note that we have replaced $\mu$ by $\xi/\gamma$ on the right-hand side). We write (4.104) in the form

$$\xi = \frac{n\pi}{2a} + \frac{1}{2a}\arcsin(\tilde{\psi}_1(\gamma, \xi/\gamma)), \tag{4.105}$$

where $n \geqslant 2$ is even. Since, by (4.98),

$$\left| \frac{\partial \arcsin(\tilde{\psi}_1(\gamma,\xi/\gamma))}{\partial \xi} \right| = \frac{|\tilde{\psi}_{1;\mu}(\gamma,\xi/\gamma)|}{\gamma\sqrt{1-\tilde{\psi}_1(\gamma,\xi/\gamma)^2}} = o\left(\frac{1}{\mu}\right),$$

we can solve (4.105) by iteration and obtain [using (4.96)]

$$\xi(\gamma) = \frac{n\pi}{2a} + \frac{\gamma}{2ap(d)} - \frac{\beta\gamma^2}{2n\pi} + o(\gamma^2). \tag{4.106}$$

Now we turn to the second equation in (4.79), that is,

$$\cot\left(\frac{\xi(\gamma)}{\gamma}\int_a^d \sqrt{p(\tau)^2+\gamma^2}d\tau\right) = \tilde{\psi}_2\left(\gamma, \frac{\xi(\gamma)}{\gamma}\right), \tag{4.107}$$

which is to be solved for $\gamma$. We claim that the argument of the cotangent increases monotonically to $+\infty$ as $\gamma\to 0$. Differentiating the argument we see that it suffices to show that $\xi'(\gamma)<\xi(\gamma)/\gamma$ for $\gamma$ small. This would be obvious from (4.106) if we knew that we could differentiate the $o(\gamma^2)$ remainder term. To overcome this difficulty, we show that $\lim_{\gamma\to 0}\xi'(\gamma)$ exists and is equal to $1/(2ap(d))$, which is what we obtain by formally differentiating (4.106). By (4.105) and the implicit function theorem, we have

$$\xi'(\gamma) = \frac{\tilde{\psi}_{1;\gamma}(\gamma,\xi(\gamma)/\gamma) + (\gamma^{-2}\tilde{\psi}_{1;\mu}(\gamma,\xi(\gamma)/\gamma)[\gamma\xi'(\gamma)-\xi(\gamma)]}{2a\sqrt{1-\tilde{\psi}_1(\gamma,\xi(\gamma)/\gamma)^2}}. \tag{4.108}$$

Now notice that according to (4.98),

$$\gamma^{-2}\tilde{\psi}_{1;\mu}(\gamma,\xi(\gamma)/\gamma) = o(1),$$

and, by (4.99),

$$\tilde{\psi}_{1;\gamma}(\gamma,\xi(\gamma)/\gamma) \to \frac{1}{p(d)}.$$

Moreover, by (4.96), $\tilde{\psi}_1(\gamma,\xi(\gamma)/\gamma)=o(1)$. Upon solving (4.108) for $\xi'(\gamma)$, we conclude that $\xi'(\gamma)=1/(2ap(d))+o(1)$ as $\mu\to\infty$; thus, $\xi'(\gamma)<\xi(\gamma)/\gamma$ for small $\gamma$. Furthermore, due to (4.100) and (4.101), the total derivative of $\tilde{\psi}_2(\gamma,\xi(\gamma)/\gamma)$ with respect to $\gamma$ is $o(1)$. Hence, for $\gamma$ near zero the branches of the left-hand side of (4.107) intersect the graph of $\tilde{\psi}_2(\gamma,\xi(\gamma)/\gamma)$ at points $\check{\gamma}_j$ which form a decreasing sequence converging to zero. From (4.107), we see that

$$\frac{\xi(\check{\gamma}_j)}{\check{\gamma}_j}\int_a^d \sqrt{p(\tau)^2+\check{\gamma}_j^2}d\tau = \frac{3\pi}{4} + j\pi - \frac{\check{\gamma}_j^2}{2p(a)p(d)} + o(\check{\gamma}_j^3). \tag{4.109}$$

Expanding the square root on the left-hand side to order $O(\check{\gamma}_j^2)$, we find that

$$\check{\gamma}_j = \frac{\alpha_1}{j+3/4} + \frac{\alpha_2}{(j+3/4)^2} + \frac{\alpha_3}{(j+3/4)^3} + o(1/j^3),$$

where

$$\alpha_1 = \frac{nA}{2a}, \quad \alpha_2 = \frac{nA^2}{4\pi a^2 p(d)},$$

$$\alpha_3 = \frac{nA^2}{16\pi^2 a^3 p(d)^2}\left[2A - 2A\beta a p(d)^2 + n^2\pi^2 p(d)^2\left(\int_a^d \frac{d\tau}{p(\tau)}\right)\right].$$

From this and (4.106), we obtain (4.73) and then, using $\check{\mu}_j=\check{\xi}_j/\check{\gamma}_j$, we get (4.74) with (4.75). By the way, in deriving (4.74) some surprising cancellations occur; in particular, the coefficient $\beta$ disappears. To prove the last assertion of (ii), we use (4.81) which links $\check{\xi}_j$ to $\check{\mu}_j$, since the right-hand side must be zero when $\mu=\check{\mu}_j$ and $\xi=\check{\xi}_j$. Evaluating the partial derivatives with respect to $\xi$ and $\mu$ of the right-hand side of (4.81) and using (2.4), we obtain, through a MATHEMATICA supported calculation, that $\lim_{j\to\infty}\check{\xi}_j'(\mu_j)=-iA/a$. In doing this calculation, we also used the fact that owing to the estimates in Lemma 4.7 the dominant contribution to $\check{\xi}_j'(\check{\mu}_j)$ for large $j$ comes from $\mathcal{U}_1$ alone. Thus (ii) is proved. Parts (iii)–(v) are proved along the same lines. We only mention some of the changes. In (iii) we are in $\mathcal{S}_+$ but have $\cos(2a\xi)<0$. Then

$$F^{(0)}(\psi)=(\psi_1,\sqrt{1-\psi_1^2}-\psi_2),$$

and this is zero when $\psi=\psi^{(0)}=(0,1)$. The mapping $\mathcal{T}$ remains the same since it is still true that $DF^{(0)}(\psi^{(0)})=J$. As a result, iteration yields

$$\tilde{\psi}_1(\gamma,\mu)=\frac{\gamma}{p(d)}+\frac{\beta\gamma}{2\mu}+o(1/\mu^2),\quad \tilde{\psi}_2(\gamma,\mu)=1-\frac{\gamma^2}{p(a)p(d)}+o(1/\mu^2). \tag{4.110}$$

In part (iv) we are in $\mathcal{S}_-$ with $\cos(2a\xi)<0$. This case can be accomodated by replacing $w_k^{(+)}$ by $-w_k^{(+)}$ for $k=1,2$ in (4.82)–(4.84), and $\sqrt{1-\psi_1^2}$ by $-\sqrt{1-\psi_1^2}$ [since $\cos(2a\xi)<0$]. Then we get

$$F^{(0)}(\psi)=(-\psi_1,-\sqrt{1-\psi_1^2}-\psi_2),$$

which is zero when $\psi=\psi^{(0)}=(0,-1)$. In contrast to (ii), and (iii), we now have that $DF^{(0)}(\psi^{(0)})=-J$. This means that we have to change the definition of $\mathcal{T}$ to $\mathcal{T}(\psi)=\psi+JF(\psi;\gamma,\mu)$. Then we obtain

$$\tilde{\psi}_1(\gamma,\mu)=-\frac{\gamma}{p(d)}+\frac{\beta\gamma}{2\mu}+o(1/\mu^2),\quad \tilde{\psi}_2(\gamma,\mu)=-1+\frac{\gamma^2}{p(a)p(d)}+o(1/\mu^2). \tag{4.111}$$

In (v) we are in $\mathcal{S}_-$ with $\cos(2a\xi)>0$. Then

$$F^{(0)}(\psi)=(-\psi_1,\sqrt{1-\psi_1^2}-\psi_2),$$

which is zero when $\psi=\psi^{(0)}=(0,1)$. Again $DF^{(0)}(\psi^{(0)})=-J$ and

$$\tilde{\psi}_1(\gamma,\mu)=-\frac{\gamma}{p(d)}-\frac{\beta\gamma}{2\mu}+o(1/\mu^2),\quad \tilde{\psi}_2(\gamma,\mu)=1-\frac{\gamma^2}{p(a)p(d)}+o(1/\mu^2). \tag{4.112}$$

Using (4.110), (4.11), and (4.112) in (4.104) and (4.107), we establish the assertions of (iii)–(v). For $\lim_{j\to\infty}\xi_j'(\mu_j)$, we get $iA/a$ in cases (iii) and (v), and $-iA/a$ in case (iv). □

We conclude this section with a comparison of analytical and numerical results pertaining to the graph in Fig. 5 as an example; similar results hold for the graphs in Figs. 2–4. As we move from right to left, the five spectral singularities (intersections of the curve with the real axis) correspond to $n=5$, 4, 3, 2, and 1, respectively, in (4.3). These $n$-values are listed—in this order—in the first column of Table I. In the adjacent column are the $j$-values that are to be used in (4.73) and (4.74) when $n$ is even, and in (4.76) and (4.77) when $n$ is odd. For even $n$, $j$ was determined from (4.109) and for odd $n$ by using the corresponding formula which has the terms $\pi/4+j\pi$ on the right-hand side. On the left-hand side, we used the numerical values for $\check{\gamma}_j$ and $\xi(\check{\gamma}_j)=\xi_j(\check{\gamma}_j)$. At each spectral singularity, the value of $j$ was unmistakably close to an integer. The remaining columns show the spectral singularities and the corresponding coupling constants; the approximate values are obtained from (4.73)–(4.77) and the exact values are calculated numerically. Note that as the coupling constant increases, the eigenvalue moves from right to left and that the slopes at the spectral singularities are in agreement with the results of Theorem 4.8 about the sign of Im $\xi_j'(\mu_j)$.

## APPENDIX: PROOF OF THEOREM 3.3

We set $\xi=is$ $(s>0)$, $z=s/\mu$ (so that $\gamma=iz$), and

$$\phi(t;z)=\sqrt{p(t)^2-z^2},$$

$$\Lambda_1(z,\mu) = e^{-2z\mu a}, \quad \Lambda_2(z,\mu) = \cot(\Sigma_a(d;z,\mu)), \tag{A1}$$

where $\Sigma_a(d;z,\mu)$ is defined by (4.47). Also, remember that we are in $\mathcal{S}_+$ and may assume that $v_1(0;z,\mu)=v_2(0;z,\mu)=1$. As before, to shorten the notation we will display the variables $z$ and $\mu$ only when needed for clarity. Following the method of Sec. IV, we use the transformation (4.10) and write

$$w_1(t) = e^{-i\Sigma_a(t)+\alpha_a(t)} w_1(a)(1+\delta_1(t)), \quad a \le t \le d, \tag{A2}$$

where (since $v(a)=(e^{az\mu},e^{-az\mu})^T$)

$$w_1(a) = e^{az\mu}\left(\frac{-ip(a)}{2\phi(a)} + \frac{\phi(a)-iz}{2\phi(a)}\Lambda_1\right).$$

Since $\xi$ is imaginary, we have that $w_2(t)=w_1(t)^*$; this follows easily from (4.14) and the fact that $\alpha_a(t)=\beta_a(t)^*$. The term $\delta_1(t)=\delta_1(t;z,\mu)$ and its partial derivatives can be controlled by estimates like those in Lemma 4.7. Since in the present context $z$ is small (as a consequence of Theorem 3.2) and $\mu$ is large, the inverse powers of $1+z$ that correspond to the inverse powers of $1+\gamma$ appearing in (4.66)–(4.69) can be omitted. For example, we have the simpler estimate $|\delta_1(d;z,\mu)| \le Cz/\mu$. Using (A2), its complex conjugate, and (4.10) yields

$$-ie^{-\alpha_a(d)}e^{-z\mu a}\frac{v_1(d;is,\mu)}{\sin(\Sigma_a(d))} = G_1(\Lambda_1,\Lambda_2;z,\mu),$$

where

$$G_1(\Lambda_1,\Lambda_2;z,\mu) = \frac{iz+\phi(d)}{p(d)}(\Lambda_2-i)\left[\frac{-ip(a)}{2\phi(a)} + \frac{\phi(a)-iz}{2\phi(a)}\Lambda_1\right](1+\delta_1(d)) + \frac{iz-\phi(d)}{p(d)}(\Lambda_2+i)\eta_a(d)$$
$$\times\left[\frac{ip[a]}{2\phi(a)} + \frac{\phi(a)+iz}{2\phi(a)}\Lambda_1\right](1+\delta_1(d)^*).$$

The notation with four arguments is intended to convey the idea that $\Lambda_1$ and $\Lambda_2$ should be viewed as "primary unknowns" which are to be determined first as functions of $z$ and $\mu$; then (A1) constitutes two equations for $z$ and $\mu$, the solution being a pair $(z_j^c,\mu_j^c)$ representing an eigenvalue collision. The equations for $\Lambda_1$ and $\Lambda_2$ come from the two conditions $v_1(d;is,\mu)=0$ and $v_{1;s}(d;is,\mu)=0$ which we can recast as follows: Let

$$G_2(\Lambda_1,\Lambda_2;z,\mu) = \mu^{-1}G_{1;z}(\Lambda_1,\Lambda_2;z,\mu), \tag{A3}$$

where the purpose of the factor $\mu^{-1}$ is to divide out factors of $\mu$ that appear because of the relations

$$\Lambda_{1;z} = -2a\mu\Lambda_1, \quad \Lambda_{2;z} = \mu z\left(\int_a^d \frac{dt}{\sqrt{p(t)^2-z^2}}\right)(1+\Lambda_2^2). \tag{A4}$$

The explicit expression for $G_2$ is too lengthy to be written out here. Then an eigenvalue collision occurs when $z=z_j^c$ and $\mu=\mu_j^c$ are such that

$$G_1(\Lambda_1,\Lambda_2;z_j^c,\mu_j^c) = G_2(\Lambda_1,\Lambda_2;z_j^c,\mu_j^c) = 0. \tag{A5}$$

The roots $(\Lambda_1,\Lambda_2)$ of these equations can be constructed by a fixed point argument similar to that used in Sec. IV (proof of Lemma 4.10) which also yields the asserted uniqueness. However, since the computations leading to (3.10) and (3.11) are already cumbersome enough, we will not give the details of the fixed point argument here but instead prove uniqueness by a separate argument based on the Brouwer degree.[14] To do this, we consider in the $(z,\mu)$-plane the rectangular regions

$$D_j = \{(z,\mu):0 \leqslant z \leqslant K \ln \hat{\mu}_j/\hat{\mu}_j, |\mu - \hat{\mu}_j| < \delta,\}, \quad j = 1,2,3, \ldots ,$$

where $\hat{\mu}_j$ is defined in (3.8), $0<\delta<\pi/(2\omega_1)$, and $K>c_2$, the constant in Theorem 3.2; clearly, the $D_j$ are disjoint. It follows from Theorem 3.2 that if $(z^c,\mu^c)$ marks a collision and $\mu^c$ is sufficiently large, then $\Lambda_1(z^c,\mu^c)\leqslant(\mu^c)^{-2ac_1}$. So, if $(z_n^c,\mu_n^c)$ is a sequence of such points with $\mu_n^c$ converging to infinity, then $\Lambda_1(z^c,\mu^c)\to 0$. Inspection of the equation $G_1(\Lambda_1,\Lambda_2;z_n^c,\mu_n^c)=0$ shows that therefore $\Lambda_2(z_n^c,\mu_n^c)\to 0$. Consequently, there is a sequence of integers $j_n$ converging to infinity such that $|\mu_n^c-\hat{\mu}_{j_n}|\to 0$ Hence for $n$ large enough the points $(z_n^c,\mu_n^c)$ are contained in $\cup D_j$. Our goal now is to show that for sufficiently large $j$, each region $D_j$ contains exactly one point $(z_j^c,\mu_j^c)$. First, in order to bring the equations (A5) into conformity with the standard literature on degree theory, we define $V_1=ie^{\alpha_a(d)}G_1$ and $V_2=i\mu^{-1}(e^{\alpha_a(d)}G_1)_z$ which makes $V_1$ and $V_2$ real-valued; then (A5) is equivalent to $V_1=V_2=0$. By somewhat tedious calculations, one shows that

$$V_1(z,\mu) = \Lambda_1 + \Lambda_2 + O(\ln \hat{\mu}_j/\hat{\mu}_j), \tag{A6}$$

$$V_2(z,\mu) = 2\omega_2 z(1+\Lambda_2^2) - 2a\Lambda_1(1+O(\ln \hat{\mu}_j/\hat{\mu}_j)) + O(1/\hat{\mu}_j). \tag{A7}$$

Here and below, the notation $f(z,\mu)=O(h_j)$ for a function $f:\cup D_j\to\mathbb{R}$ is to be understood in the sense that $\sup_{D_j}|f(z,\mu)|=O(h_j)$ as $j\to\infty$. Now let $\mathcal{V}:\cup D_j\to\mathbb{R}^2$ denote the function $\mathcal{V}(z,\mu)=(V_1(z,\mu),V_2(z,\mu))^T$ and let $\deg_j(\mathcal{V},D_j,0)$ denote its Brouwer degree at zero relative to $D_j$ (which is the same as the index of the boundary of $D_j$ with respect to the vector field $\mathcal{V}$). First we show that $\deg_j(\mathcal{V},D_j,0)\neq 0$, provided $j$ is sufficiently large. To see this we look at how the vector field $\mathcal{V}$ is pointing as we walk along the boundary of $D_j$. On the left vertical side, we have $z=0$, $\Lambda_1=1$, and thus $V_2<0$. On the right vertical side, $\Lambda_1=O(\hat{\mu}_j^{-2aK})=o(\hat{\mu}_j^{-1})$, because $K>c_2$ by assumption and in the proof of Theorem 3.2, it was shown that $c_2>(2a)^{-1}$. Hence the first term on the right-hand side of (A7) dominates and thus $V_2>0$. Considering the bottom and top sides of $D_j$ we assume, for simplicity of the argument, that $\delta$ is so close to $\pi/(2\omega_1)$ that $\Lambda_2>2$ (say) on the bottom and $\Lambda_2<-2$ on the top. This is possible since

$$\Sigma_a(d) = \mu\int_a^d \sqrt{p(t)^2 - z^2}dt = \hat{\mu}_j\omega_1 + (\mu - \hat{\mu}_j)\omega_1 + O\left(\frac{(\ln \hat{\mu}_j)^2}{\hat{\mu}_j^2}\right)$$

on $D_j$ and $\cot(\hat{\mu}_j\omega_1)=0$. Hence, since $0\leqslant\Lambda_1\leqslant 1$, we have that $V_1>0$ on the bottom side and $V_1<0$ on the top side. It follows that the index (degree) of $\mathcal{V}$ is equal to 1. This means that $\mathcal{V}$ has a critical point in $D_j$; so $V_1=V_2=0$ has a solution in $D_j$.

To show uniqueness, we consider the Jacobian determinant of $\mathcal{V}$. The calculations yield

$$V_{1,\mu} = -\omega_1(1+\Lambda_2^2) + O(\ln \hat{\mu}_j/\hat{\mu}_j),$$

$$V_{2,z} = 2\omega_2(1+\Lambda_2^2) + 4a^2\Lambda_1(\mu + O(\ln \hat{\mu}_j)) + O((\ln \hat{\mu}_j)^2/\hat{\mu}_j),$$

$$V_{2,\mu} = O(\ln \hat{\mu}_j/\hat{\mu}_j),$$

and we already know that $V_{1,z}=\mu V_2$. In these calculations, we also used the fact that $|\delta_{1,z}|,|\delta_{1,\mu}|$, $|\delta_{1,zz}|$, and $|\delta_{1,z\mu}|$ are all bounded by $Cz$ for small $z$ (and large $\mu$); these estimates are not optimal but sufficient and they can be proved as those in Lemma 4.7. Consequently, if $D\mathcal{V}$ denotes the Jacobian matrix of $\mathcal{V}$, then

$$\det D\mathcal{V} = (\mu V_2)V_{2,\mu} - V_{2,z}V_{1,\mu} = 4a^2\omega_1\mu\Lambda_1(1+\Lambda_2^2)(1+o(1)) + 2\omega_1\omega_2(1+\Lambda_2^2)^2 + o(1).$$

Since this determinant is strictly positive on $D_j$ if $j$ is large, we conclude (by the definition of the degree, see p. 3 of Ref. 14) that there is exactly one root $(z_j^c,\mu_j^c)$ of the equations $V_1=V_2=0$ in each $D_j$, provided $j$ is large enough.

Now we turn to the derivation of (3.10) and (3.11) but we omit a detailed discussion of the error estimates, since this has been done in the proof of Lemma 4.10. We return to the functions $G_1$ and $G_2$ and emphasize that by the use of (A4), $G_2$ as a function of $\Lambda_1$, $\Lambda_2$, $z$, and $\mu$ is unambiguously defined. Now let us think of these variables as being independent of each other. Then we find that, as $z \to 0$ and $\mu \to \infty$ (independently), $G_1$ and $G_2$ converge to

$$G_1^{(0)}(\Lambda_1,\Lambda_2) = -i(\Lambda_1 + \Lambda_2), \quad G_2^{(0)}(\Lambda_1,\Lambda_2) = 2ia\Lambda_1,$$

which vanish simultaneously if and only if $\Lambda_1 = \Lambda_2 = 0$. Let $G = (G_1, G_2)$, $G^{(0)} = (G_1^{(0)}, G_2^{(0)})$, and think of $G$ and $G^{(0)}$ as mappings from a neighborhood of (0,0) in $(\Lambda_1,\Lambda_2)$-space to $\mathbb{R}^2$. Let $P_0$ denote the Jacobian matrix of $G^{(0)}$ at (0,0), that is,

$$P_0 = DG^{(0)}(0,0) = \begin{pmatrix} -i & -i \\ 2ia & 0 \end{pmatrix},$$

and note that $P_0$ is nonsingular. Define a mapping $\mathcal{M}: \mathbb{R}^2 \mapsto \mathbb{R}^2$ by

$$\mathcal{M}(q) = q - P_0^{-1} G(q).$$

Then $\mathcal{M}$ can be seen to be a contraction mapping on a small fixed disk about (0,0) in $(\Lambda_1,\Lambda_2)$ space, provided $z$ is sufficiently small and $\mu$ is sufficiently large. As mentioned earlier, since the proof of this fact is similar to that for the mapping $\mathcal{T}$ in Sec. IV, we omit the details. As in Sec. IV we can obtain the solution to (A4) by iteration of $\mathcal{M}$, choosing the zero vector as initial guess. For our purposes, it suffices to iterate twice, which yields

$$\Lambda_1(z,\mu) = \frac{\omega_2 z}{a} + \frac{\omega_3}{2a\mu} + o(1/\mu), \tag{A8}$$

$$\Lambda_2(z,\mu) = -\left(\omega_3 + \frac{\omega_2}{a}\right)z - \frac{\omega_3}{2a\mu} + o(1/\mu), \tag{A9}$$

where

$$\omega_3 = \frac{1}{2}\left(\frac{1}{p(a)} + \frac{1}{p(d)}\right),$$

and $\omega_2$ is defined in (3.9). In order to arrive at a simple form for the remainders in (A8) and (A9), we have assumed that $z = o(1)$ as $\mu \to \infty$ which is true for the points where collisions occur. Writing the first equation of (A1) in the form $e^{-2as} = \Lambda_1(z, s/z)$ and using (A8), we obtain

$$s(z) = -\frac{1}{2a}\ln\left(\frac{\omega_2 z}{a}\right) + O\left(\frac{1}{\ln z}\right). \tag{A10}$$

From the second equation of (A1), we have

$$\Lambda_2(z,\mu) = \Lambda_2(z,s(z)/z) = \cot\left(\frac{s(z)}{z}\int_a^d \sqrt{p(t)^2 - z^2}dt\right), \tag{A11}$$

which, if we substitute (A9) on the left side, determines the sequence $\{z_j^c\}$. Since $\Lambda_2(z,\mu)$ is near zero for small $z$ and large $\mu$, we see that the argument of the cotangent in (A11) must be close to an odd multiple of $\pi/2$. Using this fact, we infer from (A9) and (A11) that

$$\frac{s(z)}{z}\int_a^d \sqrt{p(t)^2 - z^2}dt - \hat{\mu}_j\omega_1 = \left(\omega_3 + \frac{\omega_2}{a}\right)z + \frac{\omega_3}{2a\mu} + o(1/\mu), \tag{A12}$$

where $\hat{\mu}_j = (2j-1)\pi/(2\omega_1)$. From

$$\int_a^d \sqrt{p(t)^2 - z^2}dt = \omega_1 - \omega_2 z^2 + O(z^4)$$

and (A10), we obtain for the roots of (A12), the asymptotic expansions,

$$z_j^c = \frac{\ln(2a\hat{\mu}_j)}{2a\hat{\mu}_j} - \frac{1}{2a\hat{\mu}_j}\ln\left(\frac{\omega_2}{a}\ln(2a\hat{\mu}_j)\right) + O(\ln\ln\hat{\mu}_j/(\hat{\mu}_j\ln\hat{\mu}_j)). \tag{A13}$$

To get this, solve (A12) for $\hat{\mu}_j\omega_1$, multiply the equation by $2az$, replace $1/\mu$ by $z/s(z)$, use (A10), substitute $z=(2a\hat{\mu}_j)^{-1}\ln(2a\hat{\mu}_j)(1+\chi)$, cancel the term $\omega_1\ln(2a\hat{\mu}_j)$, and iteratively solve for $\chi$. Now (3.11) follows on substituting (A13) in (A10). Finally, $\mu_j^c=s(z_j^c)/z_j^c$ is most easily computed directly from (A12) which yields

$$\mu_j^c = \hat{\mu}_j + \frac{1}{\omega_1}\left(\omega_3 + \frac{\omega_2}{a}\right)z_j^c + \frac{\omega_2\hat{\mu}_j}{\omega_1}(z_j^c)^2 + \frac{\omega_3}{2a\omega_1\hat{\mu}_j} + o(1/\hat{\mu}_j). \tag{A14}$$

Inserting $z_j^c$ from (A13) in (A14) leads to (3.10). For the computation of the entries in Table II, we used the two (four) explicit terms on the right-hand sides of (3.11) and (A14), and the two explicit terms of (A13) for $z_j^c$. The downward moving eigenvalue spawned by the collision corresponding to $(z_j^c,\mu_j^c)$ must vanish into the continuous spectrum at $\mu=\mu_{2j}$ because the point $(0,\mu_{2j})$ is in $D_j$ [assuming $\delta>\pi/(4\omega_1)$] and there is no other collision taking place as $\mu$ increases from $\mu_j^c$ to $\mu_{2j}$.

Now we turn to the case when $p(t)=p$ is constant on $[a,d]$. Then $\delta_1(d)=0$ and, if we put $b=d-a$ and

$$H_1(\Lambda_1,\Lambda_2;z,\mu) = i\sqrt{p^2-z^2}G_1(\Lambda_1,\Lambda_2;z,\mu),$$

$$H_2(\Lambda_1,\Lambda_2;z,\mu) = \frac{1}{\mu}H_{1;z}(\Lambda_1,\Lambda_2;z,\mu),$$

then

$$H_1(\Lambda_1,\Lambda_2;z,\mu) = p\Lambda_1 + z + \sqrt{p^2-z^2}\Lambda_2, \tag{A15}$$

$$H_2(\Lambda_1,\Lambda_2;z,\mu) = \frac{1}{\mu} - 2ap\Lambda_1 + bz - \frac{z\Lambda_2}{\mu\sqrt{p^2-z^2}} + bz\Lambda_2^2. \tag{A16}$$

Setting the right-hand sides of (A15) and (A16) equal to zero, we could solve for $\Lambda_1$ and $\Lambda_2$. Since the expressions so obtained are unwieldy and not needed for this proof, we do not make use of this explicit solution. Instead we proceed as follows. First we set the right-hand side of (A16) equal to zero, solve for $\Lambda_1=e^{-2\mu za}$, and take the logarithm. This yields

$$z = \frac{\ln(2a\mu)}{2a\mu} - \frac{1}{2a\mu}\ln\left(\frac{b\mu z}{p}\right) - \frac{1}{2a\mu}\ln\left(1 + \frac{1}{b\mu z} - \frac{\Lambda_2}{b\mu\sqrt{p^2-z^2}} + \Lambda_2^2\right). \tag{A17}$$

This expression motivates the introduction of the following quantities:

$$\Delta = \mu - \hat{\mu}_j, \tag{A18}$$

$$\mu = \hat{\mu}_j(1+\eta), \tag{A19}$$

$$z = \frac{\ln(2a\hat{\mu}_j)}{2a\hat{\mu}_j}(1+\chi). \tag{A20}$$

From the definition of $\tau_1$, $\tau_2$, and $\tau_3$ in (3.13), we have

$$z = \tau_1\tau_2(1+\chi), \tag{A21}$$

so that we can write

$$\frac{\ln(2a\mu)}{2a\mu} = \tau_1\tau_2 - \frac{\tau_1\tau_2\eta}{1+\eta} + \frac{\tau_1\tau_2^2\ln(1+\eta)}{1+\eta}, \tag{A22}$$

$$\frac{1}{2a\mu}\ln\left(\frac{b\mu z}{p}\right) = \tau_1\tau_2\tau_3 + \frac{\tau_1\tau_2^2\ln(b/(2ap))}{1+\eta} - \frac{\tau_1\tau_2\tau_3\eta}{1+\eta} + \frac{\tau_1\tau_2^2\ln[(1+\chi)(1+\eta)]}{1+\eta}. \tag{A23}$$

Furthermore, since $\cot(bp\hat{\mu}_j)=0$ (note that $bp=\omega_1$),

$$\Lambda_2 = \cot(\mu b\sqrt{p^2-z^2}) = -\tan\left[b\Delta\sqrt{p^2-(\tau_1\tau_2)^2(1+\chi)^2} - \frac{b\tau_1(1+\chi)^2}{2a(\sqrt{p^2-(\tau_1\tau_2)^2(1+\chi)^2}+p)}\right], \tag{A24}$$

where we have used (A18)–(A21). Also, note that

$$\frac{1}{\mu z} = \frac{2a\tau_2}{(1+\eta)(1+\chi)}, \qquad \eta = 2a\Delta\tau_1\tau_2^2. \tag{A25}$$

Using these together with (A18)–(A21), we can write (A17) as

$$F_1(\Delta,\chi;\tau_1,\tau_2,\tau_3)=0,$$

where

$$F_1(\Delta,\chi;\tau_1,\tau_2,\tau_3) = \chi + \tau_3 - \ln(2ap/b)\tau_2 + r(\Delta,\chi;\tau_1,\tau_2,\tau_3), \tag{A26}$$

with

$$r(\Delta,\chi;\tau_1,\tau_2,\tau_3) = \frac{\eta}{1+\eta} + \frac{\tau_2\ln(2ap/b)\eta}{1+\eta} - \frac{\tau_2\ln(1+\eta)}{1+\eta} - \frac{\tau_3\eta}{1+\eta} + \frac{\tau_2\ln[(1+\chi)(1+\eta)]}{1+\eta} + \frac{\tau_2}{1+\eta}\ln(\cdots). \tag{A27}$$

Here the term abbreviated by $(\cdots)$ is the argument of the logarithm in the third term of (A17) expressed in terms of the variables $\Delta$, $\chi$, $\tau_1$, $\tau_2$, and $\tau_3$ [using also (A25)]. As a second equation, we have

$$F_2(\Delta,\chi;\tau_1,\tau_2,\tau_3)=0, \tag{A28}$$

where

$$F_2(\Delta,\chi;\tau_1,\tau_2,\tau_3) = -\sqrt{p^2-z^2}\tan[\cdots] + z + \frac{bz}{2a}\left(1 + \frac{1}{b\mu z} + \frac{\tan[\cdots]}{b\mu\sqrt{p^2-z^2}} + \tan^2[\cdots]\right). \tag{A29}$$

Here the term denoted by $[\cdots]$ is the bracketed term in (A24). Equation (A29) is obtained by equating the right-hand side of (A15) to zero and eliminating the term $p\Lambda_1$ by using (A16) (also set equal to zero). The Jacobian matrix of the mapping

$$(\Delta,\chi) \mapsto (F_1(\Delta,\chi;0,0,0), F_2(\Delta,\chi;0,0,0))$$

at (0, 0) is nonsingular, since its determinant is equal to $bp^2$; this follows from (A26)–(A28). The assertion of the theorem now is a consequence of the implicit function theorem for the two analytic functions $\Delta(\tau_1,\tau_2,\tau_3)$ and $\chi(\tau_1,\tau_2,\tau_3)$. This yields $\Delta$ and $\chi$ as convergent power series in $\tau_1$, $\tau_2$, and $\tau_3$. Inserting these in (A18) and (A21) gives (3.12) for $\mu_j^c$ and $s_j^c$, respectively.


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