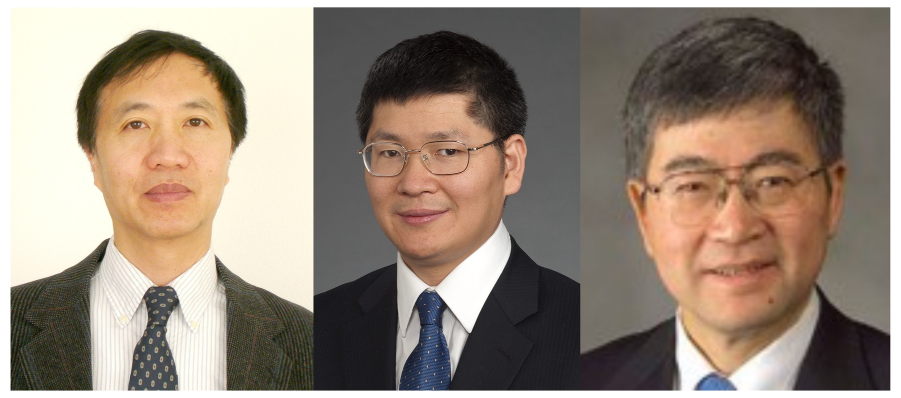
The University of Iowa Department of Mathematics

Gel’fand-Graev’s Reconstruction Formula in the 3D Real Space

— A Framework towards a General Interior Tomography Theory



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Abstract

In [1-4], I. M. Gel’fand and M. I. Graev proposed inversion formulas for x-ray transforms in different spaces. In particular, Gel’fand-Graev’s inversion formula [1] is a fundamental relationship linking projection data to the Hilbert transform of an image to be reconstructed. This finding was re-discovered in the CT field; see [5-9]. It has wide applications, including local reconstruction [10-11], backprojection filtration (BPF) [12], interior tomography [13-17], and limited-angle tomography [18]. For a survey, see [19, 20]. Despite its high information density, Gel’fand-Graev’s inversion formula [1] was cast in high dimensions and specialized terms, and difficult to follow for a well-trained engineer. In this poster, we represent this formula and its proof for the 1D x-ray transform in a 3D real space for easy access and further extension.

Notations

With the notations in [1], set $n = 3$, $k = 1$, $\alpha = \alpha_1 \in \mathbb{R}^3$ non zero, and $\beta \in \mathbb{R}^3$. Then, $h: x = \alpha t + \beta$, $t \in \mathbb{R}$, is a straight line. Let f be a function on $\mathbb{R}^3$. The x-ray projection is an integral transform $f \mapsto \mathcal{R}f$ given by

$$(1) \qquad \mathcal{R}f(\alpha,\beta) = \int_{\mathbb{R}} f(\alpha t + \beta)\, dt$$

Denote by $H_{1,3}$ the **manifold of lines** h in $\mathbb{R}^3$. Then, $\dim H_{1,3} = 4$. Denote by $G_{1,3}$ the **Grassman manifold of 1D subspaces** in $\mathbb{R}^3$, which defines a natural mapping $\pi: H_{1,3} \to G_{1,3}$ by translating a line $h \in H_{1,3}$ to the line $a \in G_{1,3}$ through the system origin. Clearly, $\dim G_{1,3} = 2$.

Let $E_{1,3} \triangleq \mathbb{R}^3 - \{O\}$, which is the manifold of 1-frames in the sense that any $\alpha \in E_{1,3}$ defines a 1D coordinate system in the line $a: x = \alpha t$, $t \in \mathbb{R}$, in $G_{1,3}$. Then any $(\alpha,\beta) \in E_{1,3} \times \mathbb{R}^3$ defines a line $h \in H_{1,3}$ by $x = \alpha t + \beta$, $t \in \mathbb{R}$ as before, and we denote this by $\sigma: (\alpha,\beta) \mapsto h$. Two $(\alpha,\beta), (\alpha',\beta') \in E_{1,3} \times \mathbb{R}^3$ define the same h if and only if

$$(2) \qquad \alpha' = A\alpha,\ \beta' = \beta + \alpha t_0 \text{ for some } A \in \mathbb{R} - \{O\},\ t_0 \in \mathbb{R}.$$

Consider a function $\varphi = \mathcal{R}f$ on $H_{1,3}$. Denote by σ^* the mapping $\sigma^*: \varphi(h) \mapsto \varphi(\alpha,\beta) = \varphi(\sigma(\alpha,\beta))$. The image of σ^* is the set of functions on $E_{1,3} \times \mathbb{R}^3$ that are invariant under (2).

Define a **differential 1-form** $\varkappa\varphi$ on $E_{1,3} \times \mathbb{R}^3$ by

$$(3) \qquad \varkappa\varphi = \sum_{j=1}^{3} \frac{\partial\varphi(\alpha,\beta)}{\partial\beta^j} d\alpha^j,$$

where $\alpha = (\alpha^1,\alpha^2,\alpha^3)$ and $\beta = (\beta^1,\beta^2,\beta^3)$.

We have $\mathcal{R}f(\alpha',\beta') = \int_{\mathbb{R}} f(\alpha' t + \beta')\, dt = \int_{\mathbb{R}} f(A\alpha t + \beta + \alpha t_0)\, dt = \int_{\mathbb{R}} f(\alpha(At + t_0) + \beta)\, dt = \frac{1}{A}\int_{\mathbb{R}} f(\alpha t + \beta)\, dt = \frac{1}{A}\mathcal{R}f(\alpha,\beta)$. Thus,

$\sum_{j=1}^{3} \frac{\partial\varphi(\alpha',\beta')}{\partial\beta'^j} d\alpha'^j = \frac{1}{A}\sum_{j=1}^{3} \frac{\partial\varphi(\alpha,\beta)}{\partial\beta'^j} d\alpha'^j = \sum_{j=1}^{3} \frac{\partial\varphi(\alpha,\beta)}{\partial\beta^j} d\alpha^j$. That is, the differential form $\varkappa\varphi$ depends on $h = \sigma(\alpha,\beta)$ only and is independent of the choice of (α,β). In other words, $\varkappa\varphi$ is a differential form on $H_{1,3}$.

Let H_0 be the **manifold of oriented 1D lines** in $\mathbb{R}^3$ through the origin. Since the oriented lines have directions, H_0 is diffeomorphic to the unit sphere S^2 in $\mathbb{R}^3$. Fix a 1D oriented submanifold $\gamma \subset H_0$. Then, γ is an oriented curve on S^2. For any vector $x \in \mathbb{R}^3$, denote by $x + \gamma$ the manifold of oriented lines shifted from those in γ to x. That is, $x + \gamma$ is diffeomorphic to an oriented curve on the unit shpere centered at x. Define an **operator** $\boldsymbol{\mathcal{J}_\gamma}$ on $\varphi = \mathcal{R}f$ by

$$(4) \qquad (\mathcal{J}_\gamma\varphi)(x) = \frac{1}{2\pi i}\int_{x+\gamma} \varkappa\varphi.$$

Note that $\mathcal{J}_\gamma = 0$ for any closed curve γ. Hence, $\mathcal{J}_\gamma$ depends only on the end points of γ.

Take $\xi \in E_{1,3}$. Denote by G_ξ the **manifold of all subspaces** $h \in H_0$ of $\mathbb{R}^3$ belonging to the plane $\langle\xi,x\rangle = 0$. Then, G_ξ is diffeomorphic to the large circle on the sphere S^2 intersecting the plane $\langle\xi,x\rangle = 0$, oriented by the right-hand rule with ξ pointing to the direction of the thumb. Note that $\dim G_\xi = 1$.

Now let us define the **intersection index** $\gamma \cdot G_\xi$ of γ and G_ξ. An intersection of γ and G_ξ contributes 1 to the intersection index if γ and ξ point to the same side of G_ξ looking from the origin. An intersection contributes -1 if if γ and ξ point to opposite sides of G_ξ looking from the origin. Define the **Crofton function** by $\mathrm{Crf}_\gamma(\xi) = \gamma \cdot G_\xi$. Note that $\mathrm{Crf}_\gamma(\xi) = 0, \pm 1$, and $\mathrm{Crf}_\gamma(-\xi) = -\mathrm{Crf}_\gamma(\xi)$.

Pseudo-differential Operators

Let $P(D) = \sum_n a_n D^n$ be a linear differential operator with constant coefficients a_n, acting on smooth functions on $\mathbb{R}^3$. Here $P(\xi) = \sum_n a_n \xi^n$ is a polynomial of three variables with $n = (n_1,n_2,n_3)$, $\xi = (\xi_1,\xi_2,\xi_3)$, $\xi^n = \xi_1^{n_1}\xi_2^{n_2}\xi_3^{n_3}$, $D = (-i\partial_1,-i\partial_2,-i\partial_3)$, $D^n = (-i\partial)_1^{n_1}(-i\partial)_2^{n_2}(-i\partial)_3^{n_3}$.

Then, by the Fourier inversion formula $P(D)u(x) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} d\xi \int_{\mathbb{R}^3} e^{i(x-y)\cdot\xi} P(\xi)u(y)\, dy$, **pseudo-differential operator** $C(D)$ on $\mathbb{R}^3$ is defined by

$$(5) \qquad C(D)u(x) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} d\xi \int_{\mathbb{R}^3} e^{i(x-y)\cdot\xi} C(\xi)u(y)\, dy = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} e^{ix\cdot\xi} C(\xi)\hat{u}(\xi)d\xi,$$

where $C(\xi)$ is the **symbol of the pseudo-differential operator** $C(D)$.

Theorem 1. (Gel’fand and Graev [1]) ***The composition operator*** $\mathcal{J}_\gamma\mathcal{R}$ ***defined by (1) and (4) is a pseudo-differential operator on*** $\mathbb{R}^3$ ***with symbol*** $\mathrm{Crf}_\gamma(\xi)$**.**

Proof: By Eqs. (1), (3) and (4),

$$(6) \qquad (\mathcal{J}_\gamma\mathcal{R}f)(x) = \frac{1}{2\pi i}\int_\gamma \sum_{j=1}^{3} d\alpha^j \int_{\mathbb{R}} \frac{\partial f(\alpha t + x)}{\partial x^j} dt.$$

Since $\hat{f}(\xi) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x)dx$, $f(x) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} e^{ix\cdot\xi}\hat{f}(\xi)d\xi$, we have $\sum_{j=1}^{3} \frac{\partial f(\alpha t + x)}{\partial x^j} d\alpha^j = \frac{i}{(2\pi)^{3/2}}\sum_{j=1}^{3}\left(\int_{\mathbb{R}^3} e^{i(\alpha t + x)\cdot\xi}\hat{f}(\xi)\xi^j d\xi\right) d\alpha^j$. By Eq. (6), we have

$$(7) \qquad (\mathcal{J}_\gamma\mathcal{R}f)(x) = \frac{1}{(2\pi)^{\frac{5}{2}}}\int_{\mathbb{R}} dt \int_\gamma \langle\xi, d\alpha\rangle \int_{\mathbb{R}^3} e^{i(\alpha t + x)\cdot\xi}\hat{f}(\xi)d\xi = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_\gamma \int_{\mathbb{R}^3} e^{i(\alpha t + x)\cdot\xi}\hat{f}(\xi)\delta(\langle\xi,\alpha\rangle)d\xi \wedge \langle\xi, d\alpha\rangle.$$

Here we used the fact that $\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\alpha\cdot\xi t}dt = \delta(\langle\xi,\alpha\rangle)$. Therefore, by Eq. (7) $\mathcal{J}_\gamma\mathcal{R}$ is a pseudo-differential operator with symbol $C(\xi) = \int_\gamma \delta(\langle\xi,\alpha\rangle)\langle\xi,d\alpha\rangle = \frac{1}{2\pi}\int_{\mathbb{R}} dt \int_\gamma e^{i\alpha\cdot\xi t}\langle\xi,d\alpha\rangle$.

We want to prove that $C(\xi) = \mathrm{Crf}_\gamma(\xi)$ for almost all $\xi \in E_{1,3}$. To that effect, we only need to prove $C(\xi) = \mathrm{Crf}_\gamma(\xi)$ for a small local piece of γ. Now, we can use a local coordinate system for γ. Let s, $s_1 \le s \le s_2$, be a variable. For each oriented line $h(s) \in \gamma$, with $\alpha(s)$ being its basis depending on s smoothly. Set $u(s) = \langle\xi,\alpha(s)\rangle$. Then $C(\xi) = \int_{s_1}^{s_2}\delta(u(s))\frac{du}{ds}ds = \frac{1}{2\pi}\int_{\mathbb{R}} dt \int_{s_1}^{s_2} e^{itu(s)}\frac{du}{ds}ds$. If γ does not intersect G_ξ, $u(s)$ is never zero, and hence $\int_{s_1}^{s_2}\delta(u(s))\frac{du}{ds}ds = 0$. If γ intersects G_ξ once, from $\int_{s_1}^{s_2}\delta(u(s))\frac{du}{ds}ds = \int_a^b \delta(u)du$, where $a = u(s_1) = \langle\xi,\alpha(s_1)\rangle$ and $b = u(s_2) = \langle\xi,\alpha(s_2)\rangle$, we get $C(\xi) = 1$ if $a < 0 < b$, and $C(\xi) = -1$ if $a > 0 > b$, which is exactly $\mathrm{Crf}_\gamma(\xi)$. ■

Inversion Formula

Corollary. (Gel’fand and Graev [1]) *For two oriented 1D submanifolds* γ_1 *and* γ_2*, the symbol of the pseudo-differential operator* $(\mathcal{J}_{\gamma_1}\mathcal{R})(\mathcal{J}_{\gamma_2}\mathcal{R})$ *is* $C(\xi) = \mathrm{Crf}_{\gamma_1}(\xi)\mathrm{Crf}_{\gamma_2}(\xi)$.

Proof: This is the composition law of pseudo-differential operators. In fact, by (5) we have

$$C_1(D)C_2(D)u(x) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} e^{ix\cdot\xi}C_1(\xi)(C_2(D)u)^\wedge(\xi)d\xi = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} e^{ix\cdot\xi}C_1(\xi)C_2(\xi)\hat{u}(\xi)d\xi. \blacksquare$$

A 1D submanifold $\gamma \subset H_0$ is called **a quasicycle** if $|\mathrm{Crf}_\gamma(\xi)|$ is a constant function for almost every ξ. In particular, γ is a quasicycle if $|\gamma \cap G_\xi| = 1$ for almost all $\xi \in \mathbb{R}^3 - \{O\}$. This is the case when γ is a smooth curve on S^2 whose end points are diametrically opposite: $\gamma(s_2) = -\gamma(s_1)$. For a quasicycle γ, denote the constant $|\mathrm{Crf}_\gamma(\xi)|$ by $c(\gamma)$. Note that in this case $\mathrm{Crf}_\gamma(\xi)$ is not a constant in general.

By the Corollary, if γ is a quasicycle, then the symbol of $(\mathcal{J}_\gamma\mathcal{R})^2$ is $(\mathrm{Crf}_\gamma(\xi))^2 = |\mathrm{Crf}_\gamma(\xi)|^2 = c(\gamma)^2$, which is a constant. A pseudo-differential operator with a constant symbol $c(\gamma)^2$ is indeed $c(\gamma)^2E$, where E is the identity operator, by the Fourier inversion formula. Therefore, we have proved an inversion formula for the x-ray transform $\mathcal{R}$.

Theorem 2. (Gel’fand and Graev[1]) If γ ***is a quisicycle in*** H_0***, then*** $(\mathcal{J}_\gamma\mathcal{R})^2 = c(\gamma)^2E$***. Thus, for the integral transform*** $f \mapsto \varphi = \mathcal{R}f$***, one has the inversion formula***

$$(9) \qquad \mathcal{J}_\gamma\mathcal{R}\mathcal{J}_\gamma\varphi = c(\gamma)^2 f.$$

Proof: By Eqs. (7) and (8), we have

$$(\mathcal{J}_\gamma\mathcal{R}f)(x) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} e^{ix\cdot\xi}\hat{f}(\xi)d\xi \frac{1}{2\pi i}\int_{\mathbb{R}}\left(e^{it\langle\xi,\alpha(s_2)\rangle} - e^{it\langle\xi,\alpha(s_1)\rangle}\right)\frac{dt}{t}.$$

By the Fourier inversion formula,

$$(\mathcal{J}_\gamma\mathcal{R}f)(x) = \frac{1}{2\pi i}\mathrm{PV}\int_{\mathbb{R}}\left(f(\alpha(s_2)t + x) - f(\alpha(s_1)t + x)\right)\frac{dt}{t} \qquad (10)$$

which shows a Hilbert transform. ■

Note that Eq. (9) is essentially the backprojection filtration (BPF) formula for any scanning curve that is a quasicycle. After [1], Eq. (10) was extended by Rullgård for SPECT [5], and rediscovered in the CT field [6-9, 21-32].

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