The Combinatorial Curve Neighborhoods of Affine Flag Manifold in Type $A_{n-1}^{(1)}$

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(ABSTRACT)

Let $\mathcal{X}$ be the affine flag manifold of Lie type $A_{n-1}^{(1)}$ where $n \geq 3$ and let W_{aff} be the associated affine Weyl group. The moment graph for $\mathcal{X}$ encodes the torus fixed points (which are elements of the affine Weyl group W_{aff}) and the torus stable curves in $\mathcal{X}$. Given a fixed point $u \in W_{\text{aff}}$ and a degree $\mathbf{d} = (d_0, d_1, ..., d_{n-1}) \in \mathbb{Z}_{\geq 0}^n$, the combinatorial curve neighborhood is the set of maximal elements in the moment graph of $\mathcal{X}$ which can be reached from $u' \leq u$ by a chain of curves of total degree $\leq \mathbf{d}$. In this thesis we give combinatorial formulas and algorithms for calculating these elements.

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(GENERAL AUDIENCE ABSTRACT)

The study of curves on flag manifolds is motivated by questions in enumerative geometry and physics. To a space of curves and incidence conditions one can associate a combinatorial object called the 'combinatorial curve neighborhood'. For a fixed degree d and a (Schubert) cycle, the curve neighborhood consists of the set of all elements in the Weyl group which can be reached from the given cycle by a path of fixed degree in the moment graph of the flag manifold, and are also Bruhat maximal with respect to this property. For finite dimensional flag manifolds, a description of the curve neighborhoods helped answer questions related to the quantum cohomology and quantum K theory rings, and ultimately about enumerative geometry of the flag manifolds.

In this thesis we study the situation of the affine flag manifolds, which are infinite dimensional. We obtain explicit combinatorial formulas and recursions which characterize the curve neighborhoods for flag manifolds of affine Lie type A. Among the conclusions, we mention that, unlike in the finite dimensional case, the curve neighborhoods have more than one component, although all components have the same length. In general, calculations reflect a close connection between the curve neighborhoods and the Lie combinatorics of the affine root system, especially the imaginary roots.

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Chapter 1

Introduction

Let $X = G/P$ be a flag manifold defined by a semisimple complex Lie group G and a parabolic subgroup P and let $\Omega \subset X$ be a Schubert variety. Fix an effective degree $\mathbf{d} \in H_2(X)$. The (geometric) *curve neighborhood* $\Gamma_{\mathbf{d}}(\Omega)$ is the closure of the union of all rational curves of degree $\mathbf{d}$ in X which intersect Ω. The curve neighborhood $\Gamma_{\mathbf{d}}(\Omega)$ was originally studied by Buch, Chaput, Mihalcea and Perrin; see [4]. It plays a key role in the study of the quantum cohomology and quantum K-theory ring of X. It was proved in [4] that $\Gamma_{\mathbf{d}}(\Omega)$ is irreducible whenever Ω is irreducible. In particular, if Ω is a Schubert variety in X then $\Gamma_{\mathbf{d}}(\Omega)$ is also a Schubert variety. In [5], Buch and Mihalcea provided an explicit combinatorial formula for the Weyl group element corresponding to $\Gamma_{\mathbf{d}}(\Omega)$ when Ω is a Schubert variety in X. It has been also shown that the calculation of the curve neighborhoods is encoded in the *moment graph* of X which is a graph encoding the T-fixed points and the T-stable curves in X where T is a maximal torus of G. In [12] Withrow has studied curve neighborhoods to compute a presentation for the small quantum cohomology ring of a particular Bott-Samelson variety in Type A. Moreover, Mihalcea and Shifler use the technique of curve neighborhoods to prove a Chevalley formula in the equivariant quantum cohomology of the odd symplectic Grassmannian, see [11]. The curve neighborhood $\Gamma_{\mathbf{d}}(\Omega)$ has also been studied in the case when X is an affine flag manifold by Mare and Mihalcea, see [9]. Mare and Mihalcea have defined an affine version of the quantum cohomology ring and gave a combinatorial description of the curve neighborhood for "small" degrees. Furthermore, a combinatorial formula for the curve neighborhood of affine flag manifold of Lie type A_1^1 has been recently given by Norton and Mihalcea; see [10]. In this dissertation we give a combinatorial formula for the curve neighborhoods of Schubert varieties in the affine flag manifold of Lie type $A_{n-1}^{(1)}$ for any degree where $n \geq 3$.

1.1 Statement of results

Here, we will introduce some notation and recall some definitions before we give an overview of our results. Let G be the special linear group $SL_n(\mathbb{C})$. Let $B \subset G$ be the Borel subgroup, the set of upper triangular matrices, and let $T \subset B$ be the maximal torus, the set of diagonal matrices. We denote by $\mathfrak{g}$ the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ of G. Let Π be the standard root system associated to the triple (T, B, G). Also, let $\Delta = \{\alpha_1, \alpha_2, ..., \alpha_{n-1}\} \subset \Pi$ be the set of simple roots. This determines a partition of $\Pi = \Pi^+ \sqcup \Pi^-$ such that $\Pi^+ = \Pi \cap \bigoplus_{i=1}^{n-1} \mathbb{Z}_{\geq 0}\, \alpha_i$ is the set of positive roots and $\Pi^- = \Pi \cap \bigoplus_{i=1}^{n-1} \mathbb{Z}_{\leq 0}\, \alpha_i$ is the set of negative roots (these roots are described in chapter 2). Denote by W the Weyl group. The Weyl group W is generated by the simple reflections, $s_1, s_2, ..., s_{n-1}$, where $s_i := s_{\alpha_i}$, $i = 1, 2, ..., n-1$. For $w \in W$, the *length* of w is denoted by $\ell(w)$. We set $\ell(id) = 0$. Now, let $\mathfrak{g}_{\text{aff}}$ be the affine Kac-Moody Lie algebra associated to the Lie algebra $\mathfrak{g}$. Denote by Π_{aff} the affine root system associated to $\mathfrak{g}_{\text{aff}}$. Let $\Delta_{\text{aff}} = \{\alpha_0, \alpha_1, ..., \alpha_{n-1}\} \subset \Pi_{\text{aff}}$ be the affine simple roots. This determines a partition of Π_{aff} into positive and negative affine roots: $\Pi_{\text{aff}} = \Pi_{\text{aff}}^{+} \sqcup \Pi_{\text{aff}}^{-}$ where $\Pi_{\text{aff}}^{-} = -\Pi_{\text{aff}}^{+}$. We denote by $\Pi_{\text{aff}}^{\text{re},+}$ the set of affine positive real roots. Let W_{aff} be the affine Weyl group of type $A_{n-1}^{(1)}$ which is generated by the simple reflections $s_0, s_1, .., s_{n-1}$ where $s_i := s_{\alpha_i}$, $i = 0, 1, ..., n-1$. For $w \in W_{\text{aff}}$, the *length* of w is denoted by $\ell(w)$. We set $\ell(id) = 0$. Now, let $\mathcal{X}$ be the affine flag manifold in type $A_{n-1}^{(1)}$. The (undirected) *moment graph* for $\mathcal{X}$ is given by the following two conditions:

- The set of vertices is W_{aff}.
- There is an edge between $u, v \in W_{\text{aff}}$ in the moment graph if and only if there exists an affine positive real root α such that $v = us_\alpha$. This situation is denoted by

$$u \xrightarrow{\alpha} v$$

  We say that the *degree* of this edge is α.

A *chain* from u to v in the moment graph is a succession of adjacent edges, starting with u and ending with v;

$$\pi : u = u_0 \xrightarrow{\beta_0} u_1 \xrightarrow{\beta_1} ... \xrightarrow{\beta_{k-2}} u_{k-1} \xrightarrow{\beta_{k-1}} u_k = v.$$

The *degree* of the chain π is $\deg(\pi) = \beta_0 + \beta_1 + \ldots + \beta_{k-1}$. A chain is called *increasing* if the length is increasing at each step i.e $\ell(u_i) > \ell(u_{i-1})$ for all i. There is a partial order on the elements of W_{aff} which is known as Bruhat partial order. It is defined by $u < v$ if and only if there exists an increasing chain starting with u and ending with v.

Example 1.1.1. Assume that $\mathcal{X}$ is the affine flag manifold associated to $A_2^{(1)}$. Then the moment graph for $\mathcal{X}$ up to elements that can be obtained by a chain of degree at most $\alpha_0 + \alpha_1 + \alpha_2$ is given in the following figure with each edge labelled by its degree;

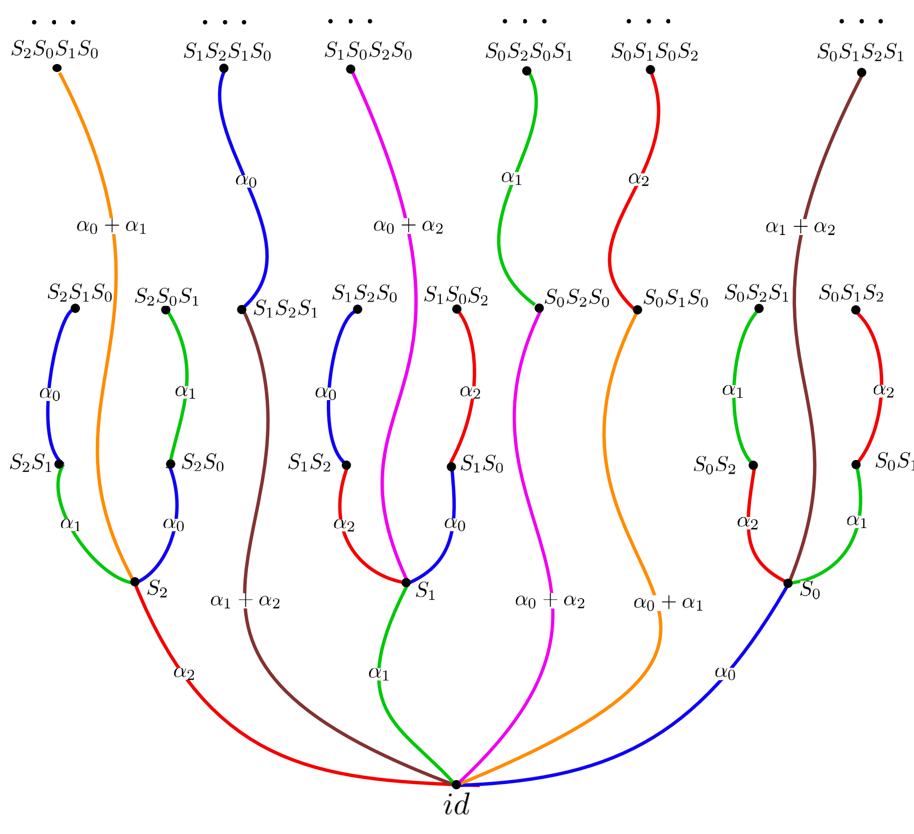

Figure 1.1: The moment graph for the affine flag manifold of type $A_2^{(1)}$.

Remark 1.1.2. The vertices and edges of this graph correspond to the T-fixed points and T-stable curves in the affine flag manifold, see [7, §12] for details.

A *degree* $\mathbf{d}$ is an n-tuple of nonnegative integers $(d_0, d_1, ..., d_{n-1})$. There is a natural partial order on degrees: If $\mathbf{d} = (d_0, ..., d_{n-1})$ and $\mathbf{d}' = (d'_0, ..., d'_{n-1})$ then $\mathbf{d} \geq \mathbf{d}'$ if and only if $d_i \geq d'_i$ for all $i \in \{0, 1, ..., n-1\}$.

Now, let a degree $\mathbf{d}$ and an element u of W_{aff} be given. Then inspired by the geometric definition of curve neighborhoods in [5] and [9], the (combinatorial) *curve neighborhood* $\Gamma_{\mathbf{d}}(u)$ is the set of elements v in W_{aff} such that:

1) v can be joined to $u' \leq u$ (in the moment graph) by a chain of degree $\leq \mathbf{d}$;

2) The elements v are maximal among all elements satisfying (1).

In other words, to compute $\Gamma_{\mathbf{d}}(u)$ we first consider all the paths in the moment graph that start with u' such that $u' \leq u$ and have a total degree at most $\mathbf{d}$. Then the curve neighborhood is given by the set of the elements with the maximal length that can be reached by the paths.

Example 1.1.3. Assume that $\mathcal{X}$ is the affine flag manifold associated to $A_2^{(1)}$. The moment graph for $\mathcal{X}$ is shown in Figure 1.1. Now, let's consider all the paths in the moment graph that start with the identity element and have a degree at most $\alpha_0+\alpha_1+\alpha_2$. Then the elements which can be reached by the paths and have the maximal length are the top elements in the moment graph so

$$\Gamma_{(\alpha_0+\alpha_1+\alpha_2)}(id) = \{s_2s_0s_1s_0, s_1s_2s_1s_0, s_1s_0s_2s_0, s_0s_2s_0s_1, s_0s_1s_0s_2, s_0s_1s_2s_1\}.$$

Now, we will state our first result which allows us to reduce the calculation of $\Gamma_{\mathbf{d}}(w)$ for any given degree $\mathbf{d}$ and $w \in W_{\text{aff}}$ to the calculation of $\Gamma_{\mathbf{d}}(id)$.

Theorem 1.1.4. *Let $w \in W_{aff}$ and $\mathbf{d}$ be any degree. Then*

$$\Gamma_{\mathbf{d}}(w) = max\{w \cdot u : u \in \Gamma_{\mathbf{d}}(id)\}\,.$$

Next, we will consider the curve neighborhood of the identity element at degree $\mathbf{d} = (d_1, d_2, ..., d_{n-1})$ where $d_i = 0$ for some $i = 1, 2, ..., n-1$. Here, using the automorphism of the Dynkin diagram φ, $\mathbf{d}$ can be identified with a finite degree, see Sections 2.3 and 5.1 for details. In this case, by [5], the curve neighborhood $\Gamma_{\mathbf{d}}(id)$ consists of a unique element;

$$\Gamma_{\mathbf{d}}(id) = \{z_{\mathbf{d}}\} \tag{1.1}$$

where $z_{\mathbf{d}} := s_\alpha \cdot z_{\mathbf{d}-\alpha}$ such that α is a maximal root which is smaller than and equal to $\mathbf{d}$, we refer to definition 3.1.1 for details. One might also see Theorem 3.1.2 where we show how z_d can be simplified.

Our next result is a corollary of Equation 1.1. Let $\alpha \in \Pi^{\text{re},+}_{\text{aff}}$ such that $\alpha < c$. Then

$$\Gamma_\alpha(id) = \{s_\alpha\}. \tag{1.2}$$

Example 1.1.5. Assume that the affine Weyl group is associated to $A_4^{(1)}$ and $\alpha = \alpha_0 + \alpha_4$. Then $\Gamma_\alpha(id) = \{s_{\alpha_0+\alpha_4}\}$ by Equation 1.2.

The rest of our results are about the instances where the degrees are not finite and proven by using some techniques which are different than those one can find in [5] and [9].

Now, we will state the result for the curve neighborhood of the identity element at c which is the degree corresponding to the imaginary coroot. In this case the neighborhood is described in terms of translations. See Section 2.2 in chapter 2 for details.

Theorem 1.1.6. *We have*

1) $\Gamma_c(id) = \{t_\gamma : \gamma \in \Pi\}$.

2) $|\Gamma_c(id)| = n(n-1)$.

3) For all $w \in \Gamma_c(id)$, $\ell(w) = 2(n-1)$.

Example 1.1.7. Suppose that the affine Weyl group W_{aff} is associated to $A_2^{(1)}$. Then by Theorem 1.1.6

$$\Gamma_c(id) = \{t_{\alpha_1}, t_{\alpha_2}, t_{\alpha_1+\alpha_2}, t_{-\alpha_1}, t_{-\alpha_2}, t_{-(\alpha_1+\alpha_2)}\}.$$

Moreover, for any $w \in \Gamma_c(id)$, $\ell(w) = 2(n-1) = 2 \cdot 2 = 4$.

In the next theorem, we will generalize the case in Theorem 1.1.6 by considering a constant m multiple of c where $m \geq 2$ is a positive integer.

Theorem 1.1.8. *Let* $m \geq 2$ *be a positive integer. Then we have*

1) $\Gamma_{mc}(id) = \{t_{m\gamma} : \gamma \in \Pi\}$.

2) $|\Gamma_{mc}(id)| = n(n-1)$.

3) For all $w \in \Gamma_{mc}(id)$, $\ell(w) = 2m(n-1)$.

Example 1.1.9. Suppose that the affine Weyl group W_{aff} is associated to $A_2^{(1)}$. Then by Theorem 1.1.8

$$\Gamma_{10c}(id) = \{t_{10\alpha_1}, t_{10\alpha_2}, t_{10(\alpha_1+\alpha_2)}, t_{-10\alpha_1}, t_{-10\alpha_2}, t_{-10(\alpha_1+\alpha_2)}\}.$$

Moreover, for any $w \in \Gamma_{10c}(id)$, $\ell(w) = 2m(n-1) = 2 \cdot 10 \cdot 2 = 40$.

In the following theorem we will provide the result for the curve neighborhood of the identity element at degree $c + \alpha$ where $\alpha \in \Pi_{\text{aff}}^{\text{re},+}$ and $\alpha < c$ which plays a key role in calculating the most general case in this thesis.

Theorem 1.1.10. *Let* $\alpha \in \Pi_{aff}^{re,+}$ *be such that* $\alpha < c$. *Then*

1) $\Gamma_{c+\alpha}(id) = \{t_{\beta'} s_\alpha : \beta' \in \Pi_{aff}^{re,+}(\alpha)\} \cup \{t_{\beta'-c} s_\alpha : \beta' \in \Pi_{aff}^{re,+}(\alpha)\}$

2) $|\Gamma_{c+\alpha}(id)| = |\{\beta' : \beta' \in \Pi_{aff}^{re,+}(\alpha)\}|$

3) For all $w \in \Gamma_{c+\alpha}(id)$, $\ell(w) = 2(n-1) + \ell(s_\alpha)$

where $\Pi_{aff}^{re,+}(\alpha) := \{\beta' \in \Pi_{aff}^{re,+} : \beta' < c$ *and either* $\beta' \leq c - \alpha$ *or both* $\beta' > \alpha$ *and* $\beta' \perp \alpha\}$.

Example 1.1.11. Let W_{aff} be the Weyl group of type $A_4^{(1)}$. We compute $\Gamma_{c+\alpha}(id)$ where $\alpha = \alpha_0 + \alpha_4$. Note that, we have six positive real roots which are smaller than $c - \alpha = \alpha_1 + \alpha_2 + \alpha_3$;

$\beta'_1 = \alpha_1, \beta'_2 = \alpha_2, \beta'_3 = \alpha_3, \beta'_4 = \alpha_1 + \alpha_2, \beta'_5 = \alpha_2 + \alpha_3, \beta'_6 = \alpha_1 + \alpha_2 + \alpha_3$. Also, we have only one positive root which is smaller than c, strictly bigger than α and perpendicular to α; $\beta'_7 = \alpha_0 + \alpha_1 + \alpha_3 + \alpha_4$. So by Theorem 1.1.10, we get

$$\Gamma_{c+\alpha}(id) = \{t_{\beta'_1}s_\alpha, t_{\beta'_2}s_\alpha, t_{\beta'_3}s_\alpha, t_{\beta'_4}s_\alpha, t_{\beta'_5}s_\alpha, t_{\beta'_6}s_\alpha, t_{\beta'_7-c}s_\alpha\}$$

Moreover, for any $w \in \Gamma_{c+\alpha}(id)$, we have $\ell(w) = 2(n-1) + \ell(s_\alpha) = 2(n-1) + 2\,\mathrm{supp}(\alpha) - 1 = 2 \cdot 4 + 2 \cdot 2 - 1 = 11$.

Next, we will consider the generalization of the case in the previous theorem to the instance where the degree is given by $mc + \alpha$ such that $m \in \mathbb{Z}^{>0}$, $\alpha \in \Pi^{\mathrm{re},+}_{\mathrm{aff}}$ and $\alpha < c$.

Theorem 1.1.12. *Let $\alpha < c$ be an affine positive real root and m be a positive integer. Then*

1) $\Gamma_{mc+\alpha}(id) = \{t_{m\beta'}s_\alpha : \beta' \in \Pi^{re,+}_{aff}(\alpha)\} \cup \{t_{m(\beta'-c)}s_\alpha : \beta' \in \Pi^{re,+}_{aff}(\alpha)\}$

2) $|\Gamma_{mc+\alpha}(id)| = |\{\beta' : \beta' \in \Pi^{re,+}_{aff}(\alpha)\}|$

3) For all $w \in \Gamma_{mc+\alpha}(id)$, $\ell(w) = 2m(n-1) + \ell(s_\alpha)$

where $\Pi^{re,+}_{aff}(\alpha)$ is defined in Theorem 1.1.10.

Example 1.1.13. Suppose that the affine Weyl group W_{aff} is of type $A_4^{(1)}$. We compute $\Gamma_{12c+\alpha}(id)$ where $\alpha = \alpha_0 + \alpha_4$. Then, by Theorem 1.1.12, we get

$$\Gamma_{12c+\alpha}(id) = \{t_{12\beta'_1}s_\alpha, t_{12\beta'_2}s_\alpha, t_{12\beta'_3}s_\alpha, t_{12\beta'_4}s_\alpha, t_{12\beta'_5}s_\alpha, t_{12\beta'_6}s_\alpha, t_{12(\beta'_7-c)}s_\alpha\}$$

where $\beta'_1 = \alpha_1, \beta'_2 = \alpha_2, \beta'_3 = \alpha_3, \beta'_4 = \alpha_1 + \alpha_2, \beta'_5 = \alpha_2 + \alpha_3, \beta'_6 = \alpha_1 + \alpha_2 + \alpha_3$ and $\beta'_7 = \alpha_0 + \alpha_1 + \alpha_3 + \alpha_4$, see Example 5.3.6. Moreover, for any $w \in \Gamma_{12c+\alpha}(id)$, we have $\ell(w) = 2m(n-1) + \ell(s_\alpha) = 2m(n-1) + 2\operatorname{supp}(\alpha) - 1 = 2 \cdot 12 \cdot 4 + 2 \cdot 2 - 1 = 99$.

Last, we will state the result for the most general case: the curve neighborhood of the identity element at any degree $\mathbf{d}$ which is strictly bigger than c.

Theorem 1.1.14. *Let $\mathbf{d} = (d_0, d_1, ..., d_{n-1}) > c$ be a degree and $m = min\{d_0, d_1, ..., d_{n-1}\}$. Also, assume that $z_{\mathbf{d}-mc} = s_{\gamma_1} s_{\gamma_2} ... s_{\gamma_k}$ for some k, where this expression is obtained in Theorem 3.1.2. Then*

1) $\Gamma_{\mathbf{d}}(id) = \{t_{m\beta'} z_{\mathbf{d}-mc} : \beta' \in \Pi^{re,+}_{aff}(\mathbf{d} - mc)\} \cup \{t_{m(\beta'-c)} z_{\mathbf{d}-mc} : \beta' \in \Pi^{re,+}_{aff}(d - mc)\}$

2) $|\Gamma_{\mathbf{d}}(id)| = |\{\beta' : \beta' \in \Pi^{re,+}_{aff}(\mathbf{d} - mc)\}|$

3) For all $w \in \Gamma_{\mathbf{d}}(id)$, $\ell(w) = 2m(n-1) + \ell(z_{\mathbf{d}-mc})$

where $\Pi^{re,+}_{aff}(\mathbf{d} - mc)$ is the set of $\beta' \in \Pi^{re,+}_{aff}$ such that $\beta' < c$ and either $\beta' \cap \gamma_i = \emptyset$ or both $\beta' > \gamma_i$ and $\beta' \perp \gamma_i$ for any $i = 1, ..., k$.

Example 1.1.15. Let W_{aff} be the affine Weyl group associated to $A_3^{(1)}$ and $\mathbf{d} = (6, 5, 8, 5)$. Then $\mathbf{d} = 5c + (1, 0, 3, 0)$ so $m = 5$ and $\mathbf{d} - 5c = (1, 0, 3, 0)$. Also, $z_{\mathbf{d}-5c} = s_{\alpha_0} \cdot s_{\alpha_2} \cdot s_{\alpha_2} \cdot s_{\alpha_2} = s_{\alpha_0} s_{\alpha_2}$. Hence by Theorem 1.1.14 we get

$$\Gamma_{\mathbf{d}}(id) = \{t_{5\beta'_1} z_{\mathbf{d}-5c}, t_{5\beta'_2} z_{\mathbf{d}-5c}, t_{5\beta'_4} z_{\mathbf{d}-5c}, t_{5(\beta'_3-c)} z_{\mathbf{d}-5c}\}$$

where $\beta'_1 = \alpha_1$, $\beta'_2 = \alpha_3$, $\beta'_3 = \alpha_0 + \alpha_1 + \alpha_3$, $\beta'_4 = \alpha_1 + \alpha_2 + \alpha_3$. Moreover, for all $w \in \Gamma_{\mathbf{d}}(id)$, $\ell(w) = 2m(n-1) + \ell(z_{\mathbf{d}-mc}) = 2 \cdot 5 \cdot 3 + 2 = 32$.

1.2 Sketch of Proofs

In this section, we will summarize some of the proofs of the results. In all cases below, we first consider a path $id \xrightarrow{\beta_1} s_{\beta_1} \xrightarrow{\beta_2} s_{\beta_1}s_{\beta_2}... \xrightarrow{\beta_r} w = s_{\beta_1}s_{\beta_2}...s_{\beta_r}$ in the moment graph such that $\sum_{i=1}^{r} \beta_i \leq \mathbf{d}$. Note that, by Remark 4.0.2 we can assume that $w = s_{\beta_1} \cdot s_{\beta_2} \cdot ... \cdot s_{\beta_r}$ where $\sum_{i=1}^{r} \beta_i = \mathbf{d}$ and $\beta_i < c$ for all i.

Proof of $\Gamma_c(id)$: Let $w = s_{\beta_1} \cdot s_{\beta_2} \cdot ... \cdot s_{\beta_r}$ such that $\sum_{i=1}^{r} \beta_i = c$ and $\beta_i < c$ for all i. Now, note that $\sum_{i=2}^{r} \beta_i = c - \beta_1$ is an affine positive real root. Moreover, $s_{\beta_2} \cdot s_{\beta_3} \cdot ... \cdot s_{\beta_r} \leq s_{c-\beta_1}$ by Corollary 3.2.2. Thus $w = s_{\beta_1} \cdot (s_{\beta_2} \cdot ... \cdot s_{\beta_r}) \leq s_{\beta_1} \cdot s_{c-\beta_1}$. But by Lemma 3.4.3, $s_{\beta_1}s_{c-\beta_1}$ is reduced so $s_{\beta_1} \cdot s_{c-\beta_1} = s_{\beta_1}s_{c-\beta_1}$. We also know that $s_{\beta_1}s_{c-\beta_1} = t_\gamma$ for some $\gamma \in \Pi$, see the proof of Lemma 3.4.1.

Proof of $\Gamma_{c+\alpha}(id)$: Let $w = s_{\beta_1} \cdot s_{\beta_2} \cdot ... \cdot s_{\beta_r}$ be such that $\sum_{i=1}^{r} \beta_i = c + \alpha$ and $\beta_i < c$ for all i. By Lemma 5.3.2, we can also suppose that $\sum_{i=1}^{t} \beta_i = c$ and $\sum_{i=t+1}^{r} \beta_i = \alpha$ for some t. Then

$$w = (s_{\beta_1} \cdot ... \cdot s_{\beta_t}) \cdot (s_{\beta_{t+1}} \cdot s_{\beta_{t+2}} \cdot ... \cdot s_{\beta_r}) \leq (s_\mu \cdot s_{\mu'}) \cdot s_\alpha$$

for some $\mu, \mu' \in \Pi_{\text{aff}}^{\text{re},+}$ such that $\mu + \mu' = c$ by by Equation 5.1 and Theorem 5.1.3. Furthermore, by using Lemma 5.3.3 we are able to show that $s_\mu \cdot s_{\mu'} \cdot s_\alpha \leq s_\beta \cdot s_{\beta'} \cdot s_\alpha$ where $\beta + \beta' = c$ and either $\beta' \leq c - \alpha$ or both $\beta' > \alpha$ and $\beta' \perp \alpha$. In addition, $s_\beta s_{\beta'} s_\alpha$ is reduced so $s_\beta \cdot s_{\beta'} \cdot s_\alpha = s_\beta s_{\beta'} s_\alpha$ by Lemma 3.4.6, and 3.4.8. Last, $s_\beta s_{\beta'} = t_{\beta'}$ if $\alpha_0 \notin \text{supp}(\beta')$ and $s_\beta s_{\beta'} = t_{\beta'-c}$ if $\alpha_0 \in \text{supp}(\beta')$, see Section 5.3 for details.

Proof of $\Gamma_{mc}(id)$: Let $w = s_{\beta_1} \cdot s_{\beta_2} \cdot ... \cdot s_{\beta_r}$ be such that $\sum_{i=1}^{r} \beta_i = mc$ and $\beta_i < c$ for all i. Now, note that $\sum_{i=1}^{r-1} \beta_i = mc - \beta_r = (m-1)c + c - \beta_r$. Let $\alpha := c - \beta_r$ and $\alpha' := \beta_r$. Then by Equation 5.7 we have $s_{\beta_1} s_{\beta_2} ... s_{\beta_{r-1}} \leq (s_\beta s_{\beta'})^{m-1} s_\alpha$ for some positive real roots, β and β' such that $\beta + \beta' = c$ and either $\beta' \leq c - \alpha$ or both $\beta' > \alpha$ and $\beta' \perp \alpha$. Then, we get

$$w = s_{\beta_1} s_{\beta_2} ... s_{\beta_r} \leq (s_{\beta_1} s_{\beta_2} ... s_{\beta_{r-1}}) \cdot s_{\beta_r} \leq (s_\beta s_{\beta'})^{m-1} s_\alpha \cdot s_{\alpha'}.$$

Furthermore, by Lemmas 5.4.1 and 5.4.2 we get $(s_\beta s_{\beta'})^{m-1} s_\alpha \cdot s_{\alpha'} \leq (s_\nu \cdot s_{\nu'})^m$ for some $\nu, \nu' \in \Pi_{\text{aff}}^{\text{re},+}$ such that $\nu + \nu' = c$. Last, $(s_\nu s_{\nu'})^m = t_{m\gamma}$ for some $\gamma \in \Pi$, see Section 5.5 for details.

Proof of $\Gamma_{mc+\alpha}(id)$: In this case we prove the result by induction on m. Note that, if $m = 1$ then we have nothing but the case $\Gamma_{c+\alpha}(id)$. Now, assume that the statement is true for $m = k$. We prove that it is also true for $m = k + 1$. Let $w = s_{\beta_1} \cdot s_{\beta_2} \cdot ... \cdot s_{\beta_r}$ be such that $\sum_{i=1}^{r} \beta_i = (k+1)c + \alpha$ and $\beta_i < c$ for all i. Moreover, by Lemma 5.3.2, we can suppose that there is an integer p such that $\sum_{i=1}^{p} \beta_i = c$ and $\sum_{i=p+1}^{r} \beta_i = kc + \alpha$. Then by Equation 5.1 we have $s_{\beta_1} \cdot s_{\beta_2} \cdot ... \cdot s_{\beta_p} \leq s_\gamma s_{\gamma'}$ for some $\gamma, \gamma' \in \Pi_{\text{aff}}^{\text{re},+}$ such that $\gamma + \gamma' = c$ and $s_{\beta_{p+1}} \cdot ... \cdot s_{\beta_r} \leq (s_\nu s_{\nu'})^k s_\alpha$ for some $\nu, \nu' \in \Pi_{\text{aff}}^{\text{re},+}$ such that $\nu + \nu' = c$ where either $\nu' \leq c - \alpha$ or both $\nu' > \alpha$ and $\nu' \perp \alpha$ by the induction assumption. Thus,

$$w = (s_{\beta_1} \cdot s_{\beta_2} \cdot ... \cdot s_{\beta_p}) \cdot (s_{\beta_{p+1}} \cdot ... \cdot s_{\beta_r}) \leq (s_\gamma s_{\gamma'}) \cdot ((s_\nu s_{\nu'})^k s_\alpha) \leq (s_\gamma \cdot s_{\gamma'}) \cdot ((s_\nu \cdot s_{\nu'})^k \cdot s_\alpha).$$

Then by using Lemmas 5.3.3, 5.4.1, 5.4.3 and 5.4.2 we are able to prove that

$$(s_\gamma \cdot s_{\gamma'}) \cdot ((s_\nu \cdot s_{\nu'})^k \cdot s_\alpha) \leq (s_\beta \cdot s_{\beta'})^{k+1} \cdot s_\alpha$$

for some $\beta, \beta' \in \Pi_{\text{aff}}^{\text{re},+}$ such that $\beta + \beta' = c$ and either $\beta' \leq c - \alpha$ or both $\beta' > \alpha$ and $\beta' \perp \alpha$. Moreover, by Lemmas 3.4.6 and 3.4.8, $(s_\beta s_{\beta'})^{k+1} s_\alpha$ is reduced so $(s_\beta \cdot s_{\beta'})^{k+1} \cdot s_\alpha = (s_\beta s_{\beta'})^{k+1} s_\alpha$. Last, the result follows by the fact that $s_\beta s_{\beta'} = t_{\beta'}$ if $\alpha_0 \notin \text{supp}(\beta')$ and $s_\beta s_{\beta'} = t_{\beta'-c}$ if $\alpha_0 \in \text{supp}(\beta')$, see Section 5.4 for details.

Proof of $\Gamma_{\mathbf{d}}(id)$ for $\mathbf{d} > c$: Let $w = s_{\beta_1} \cdot s_{\beta_2} \cdot ... \cdot s_{\beta_r}$ such that $\sum_{i=1}^r \beta_i = \mathbf{d}$ and $\beta_i < c$ for all i. In addition, we can suppose that there is an integer p such that $\sum_{i=1}^p \beta_i = c$ and $\sum_{i=p+1}^r \beta_i = \mathbf{d} - c$, by Lemma 5.3.2. We will prove the statement by induction on m.

First, suppose that $m = 1$. Then by Equation 5.1, $s_{\beta_1} \cdot s_{\beta_2} \cdot ... \cdot s_{\beta_p} \leq s_\alpha \cdot s_{\alpha'}$ for some $\alpha, \alpha' \in \Pi_{\text{aff}}^{\text{re},+}$ such that $\alpha + \alpha' = c$. Moreover, $s_{\beta_{p+1}} \cdot ... \cdot s_{\beta_r} \leq z_{d-c}$, by Theorem 5.1.1. Thus

$$w = (s_{\beta_1} \cdot s_{\beta_2} \cdot ... \cdot s_{\beta_p}) \cdot (s_{\beta_{p+1}} \cdot ... \cdot s_{\beta_r}) \leq (s_\alpha \cdot s_{\alpha'}) \cdot z_{d-c}.$$

Now, by Lemma 5.6.1 we have $s_\alpha \cdot s_{\alpha'} \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot ... \cdot s_{\gamma_k} \leq s_\beta \cdot s_{\beta'} \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot ... \cdot s_{\gamma_k}$ for some affine positive real roots, $\beta, \beta' \in \Pi_{\text{aff}}^{\text{re},+}$ such that we have either $\beta' \cap \gamma_i = \emptyset$ or both $\beta' > \gamma_i$ and $\beta' \perp \gamma_i$ for any $i = 1, 2, ..., k$.

Now, we will suppose that the statement is true for m and prove that it is also true for $m+1$. Again, by Equation 5.1 we have $s_{\beta_1} \cdot s_{\beta_2} \cdot ... \cdot s_{\beta_p} \leq s_\alpha \cdot s_{\alpha'}$ for some $\alpha, \alpha' \in \Pi_{\text{aff}}^{\text{re},+}$ such that $\alpha + \alpha' = c$ and $s_{\beta_{p+1}} \cdot ... \cdot s_{\beta_r} \leq (s_\mu \cdot s_{\mu'})^m \cdot z_{d-(m+1)c}$ for some $\mu, \mu' \in \Pi_{\text{aff}}^{\text{re},+}$ such that $\mu + \mu' = c$ where either $\mu' \cap \gamma_i = \emptyset$ or $\mu' > \gamma_j$ and $\mu' \perp \gamma_j$ by induction assumption. Thus

$$w = (s_{\beta_1} \cdot s_{\beta_2} \cdot ... \cdot s_{\beta_p}) \cdot (s_{\beta_{p+1}} \cdot ... \cdot s_{\beta_r}) \leq (s_\alpha \cdot s_{\alpha'}) \cdot (s_\mu \cdot s_{\mu'})^m \cdot z_{d-(m+1)c}.$$

Now, observe that by Equation 5.2 and Lemma 5.6.2 we have $(s_\alpha \cdot s_{\alpha'}) \cdot (s_\mu \cdot s_{\mu'})^m \cdot z_{d-(m+1)c} \leq (s_\beta \cdot s_{\beta'})^{m+1} \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot ... \cdot s_{\gamma_k}$ for some affine positive real roots, β, β' such that $\beta + \beta' = c$ where either $\beta' \cap \gamma_i = \emptyset$ or both $\beta' > \gamma_i$ and $\beta' \perp \gamma_i$ for any $i = 1, 2, ..., k$. Furthermore, by Lemma 3.4.9 , $z_{d-(m+1)c} = s_{\gamma_1} s_{\gamma_2} ... s_{\gamma_k}$ is reduced so $s_{\gamma_1} s_{\gamma_2} ... s_{\gamma_k} = s_{\gamma_1} \cdot s_{\gamma_2} \cdot ... \cdot s_{\gamma_k}$ which follows by $(s_\beta \cdot s_{\beta'})^{m+1} \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot ... \cdot s_{\gamma_k} = (s_\beta \cdot s_{\beta'})^{m+1} \cdot (s_{\gamma_1} s_{\gamma_2} ... s_{\gamma_k}) = (s_\beta \cdot s_{\beta'})^{m+1} \cdot z_{d-(m+1)c}$. Now by Lemma 3.4.10, the element $(s_\beta s_{\beta'})^{m+1} z_{d-(m+1)c}$ is reduced so $(s_\beta \cdot s_{\beta'})^{m+1} \cdot z_{d-(m+1)c} = (s_\beta s_{\beta'})^{m+1} z_{d-(m+1)c}$.

Here, we will provide an outline of the rest of the thesis. Chapter 2 covers the background on the Weyl and the affine Weyl group with the associated root systems corresponding to the affine flag manifold as well as the Hecke product that we will need throughout the thesis. In addition, we prove several preliminary lemmas which will be used for proving the results.

In chapter 3 we recall some basic facts about the affine flag manifold of type $A_{n-1}^{(1)}$ and give the definition of the moment graph for the affine flag manifold and the (combinatorial) curve neighborhoods.

Chapter 4 is dedicated to the results. We discuss $\Gamma_{\mathbf{d}}(id)$ in Sections 5.1 through 5.6. In particular, in Section 5.1 we prove Equations 1.1 and 1.2 which are the results for the finite degrees. We provide the proof of Theorem 1.1.6 in Section 5.2 which states the result for the degree c. The result has a central role in proving the rest of the results in this thesis. In Section 5.3 we prove Theorem 1.1.10 which covers the result for the degree $c + \alpha$ where $\alpha \in \Pi_{\text{aff}}^{\text{re},+}$ and $\alpha < c$. The result is generalized in Theorem 1.1.12 which is proved in Section 5.4. We then present the proof of Theorem 1.1.8 which generalizes the case in Theorem 1.1.6 in Section 5.5. Furthermore, we prove Theorem 1.1.14 that provides the result for the most general case in Section 5.6. In Section 5.7 we prove Theorem 1.1.4 which implies that for any given $w \in W_{\text{aff}}$ and degree $\mathbf{d}$, calculation of $\Gamma_{\mathbf{d}}(w)$ can be reduced to calculation of $\Gamma_{\mathbf{d}}(id)$. Each result is followed by an example of the corresponding case.

Chapter 2

Preliminaries

In this chapter we will set up notations and recall some basic facts about the affine root system in type $A_{n-1}^{(1)}$. We refer to [2], [7, §1], [8, §3], and chapters 1 through 8 in [6] for further details.

2.1 Weyl Group

Let G be the Lie group $SL_n(\mathbb{C})$. Let $B \subset G$ be the Borel subgroup, the set of upper triangular matrices, and $T \subset B$ be the maximal torus, the set of diagonal matrices. We denote by $\mathfrak{g}$ the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ of G. Let E be the subspace of $\mathbb{R}^n$ which consists of n-tuples for which the coordinates sum to 0. Let $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n$ be the standard basis for $\mathbb{R}^n$ and let $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq n-1$. Let $\Delta = \{\alpha_1, \alpha_2, ..., \alpha_{n-1}\} \subset \mathfrak{h}^*$ be the simple roots and $\Delta^\vee = \{\alpha_1^\vee, \alpha_2^\vee, ..., \alpha_{n-1}^\vee\} \subset \mathfrak{h}$ be the corresponding coroot simple roots, where $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$. Let $Q = \bigoplus_{i=1}^{n-1} \mathbb{Z}\alpha_i \subset \mathfrak{h}^*$ and $Q^\vee = \bigoplus_{i=1}^{n-1} \mathbb{Z}\alpha_i^\vee \subset \mathfrak{h}$ be the root and coroot lattice. Let $\Pi = \{\varepsilon_i - \varepsilon_j : i \neq j\}$ be the standard root system associated to the triple (T, B, G) corresponding to Δ, where $\Pi = \Pi^+ \sqcup \Pi^-$ such that $\Pi^+ = \Pi \cap \bigoplus_{i=1}^{n-1} \mathbb{Z}_{\geq 0}\, \alpha_i = \{\varepsilon_i - \varepsilon_j : i < j\}$ is the set of positive roots and $\Pi^- = \Pi \cap \bigoplus_{i=1}^{n-1} \mathbb{Z}_{\leq 0}\, \alpha_i = \{\varepsilon_i - \varepsilon_j : i > j\}$ is the set of negative roots. Due to the fact that $\alpha_i = \alpha_i^\vee$ for all i in type A, we may identify Q with $Q^\vee$, and $\mathfrak{h}^*$ with $\mathfrak{h}$. Let $\langle , \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$ be the natural pairing. Denote by W the Weyl group. The Weyl group W is generated by the simple reflections, $s_1, s_2, ..., s_{n-1}$, where $s_i := s_{\alpha_i}$, $i = 1, 2, ..., n-1$. For $w \in W$, the *length* of w, denoted $\ell(w)$, is the smallest positive integer k such that $w = s_{i_1} s_{i_2} ... s_{i_k}$, $1 \leq i_j \leq n-1$. Such an expression is called a *reduced expression* of w. We set $\ell(id) = 0$. W acts on $\mathfrak{h}^*$ by

$$s_i(\mu) = \mu - \langle \mu, \alpha_i^\vee \rangle \alpha_i \quad \text{for } \mu \in \mathfrak{h}^*$$

For all $w \in W, \mu, \lambda \in \mathfrak{h}^*$, we have $\langle w \cdot \mu, w \cdot \lambda \rangle = \langle \mu, \lambda \rangle$. For each $\alpha \in \Pi$ there is a $w \in W$ and $1 \leq i \leq n-1$ such that $\alpha = w \cdot \alpha_i$. The reflection of α is given by $s_\alpha = w s_i w^{-1}$ which is independent of the choice of w and i. Moreover, it is well-known that $\ell(w) = |\{\alpha \in \Pi^+ : w \cdot \alpha < 0\}|$ for all $w \in W$. We can identify W with the symmetric group S_n via the map $s_i \mapsto (i, i+1)$.

2.2 Affine Weyl Group

Let $\mathfrak{g}_{\text{aff}}$ be the affine Kac-Moody Lie algebra associated to the Lie algebra $\mathfrak{g}$. We have $\mathfrak{g}_{\text{aff}} = \mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}c \oplus \mathbb{C}d$, where $\mathcal{L}(\mathfrak{g}) := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]\, (t \in \mathbb{C}^*)$ is the space of all Laurent polynomials in $\mathfrak{g}$ and c is a central element with respect to the Lie bracket in $\mathfrak{g}_{\text{aff}}$. We denote by $\mathfrak{h}_{\text{aff}}$ the Cartan subalgebra of $\mathfrak{g}_{\text{aff}}$ which is given by $\mathfrak{h}_{\text{aff}} := \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$. Let $\langle , \rangle : \mathfrak{h}^*_{\text{aff}} \times \mathfrak{h}_{\text{aff}} \to \mathbb{C}$ be the natural pairing. The affine root system Π_{aff} associated to $\mathfrak{g}_{\text{aff}}$ consists of

- $m\delta + \alpha$, where $\alpha \in \Pi$ and $m \in \mathbb{Z}$, which are called the *affine real roots.*
- $m\delta$, $m \in \mathbb{Z} \setminus \{0\}$, which are called the *imaginary roots.*

For any root $\alpha \in \Pi \subset \Pi_{\text{aff}}$ we identify α as a linear function on $\mathfrak{h}^*$ whose restriction to $\mathfrak{h}$ is α and $\langle \alpha, c\rangle = \langle \alpha, d\rangle = 0$ and the imaginary root $\delta \in \mathfrak{h}^*$ is defined by $\delta|\mathfrak{h} \oplus \mathbb{C}c = 0$ and $\langle \delta, d\rangle = 1$. Let $\Delta_{\text{aff}} = \{\alpha_0 := \delta - \theta, \alpha_1, ..., \alpha_{n-1}\} \subset \Pi_{\text{aff}}$ be the affine simple roots where $\theta = \alpha_1 + ... + \alpha_{n-1} \in \Pi$ is the highest root. This determines a partition of Π_{aff} into positive and negative affine roots: $\Pi_{\text{aff}} = \Pi^+_{\text{aff}} \sqcup \Pi^-_{\text{aff}}$ where $\Pi^-_{\text{aff}} = -\Pi^+_{\text{aff}}$ and Π^+_{aff} consists of the elements $m\delta + \alpha$ such that either $m > 0$ or both $\alpha \in \Pi^+$ and $m = 0$. Denote by $\Pi^{\text{re},+}_{\text{aff}}$ the set of affine positive real roots. We denote by $Q_{\text{aff}} = \bigoplus_{i=0}^{n-1} \mathbb{Z}\alpha_i \subset \mathfrak{h}^*_{\text{aff}}$ and $Q^\vee_{\text{aff}} = \bigoplus_{i=0}^{n-1} \mathbb{Z}\alpha_i^\vee \subset \mathfrak{h}_{\text{aff}}$ the affine root and coroot lattices. For $a, b \in \bigoplus_{i=0}^{n-1} \mathbb{Z}\alpha_i$ we say that $a \leq b$ if $b - a$ is a linear combination of non-negative coefficients of $\alpha_0, \alpha_1, ..., \alpha_{n-1}$. Denote by $a < b$ if $a \leq b$ and $a \neq b$. Furthermore, in type A we identify δ with c so we set $c = \alpha_0 + \theta = \sum_{i=0}^{n-1} \alpha_i$.

Let $W_{\text{aff}} = W \ltimes Q^\vee$ be the affine Weyl group corresponding to W. We denote by t_μ the image of $\mu \in Q^\vee$ in W_{aff}. For all $w \in W$ and $\mu \in Q^\vee$, we have $t_{w \cdot \mu} = w t_\mu w^{-1}$. W_{aff} is generated by the simple reflections, $s_0, s_1, ..., s_{n-1}$, where $s_i := s_{\alpha_i}$, $i = 0, 1, ..., n-1$ and

$$s_i(\mu) = \mu - \langle \mu, \alpha_i^\vee \rangle \, \alpha_i \quad \text{for } \mu \in \mathfrak{h}^*_{\text{aff}} \text{ and } 0 \leq i \leq n-1.$$

The affine root system Π_{aff} is W_{aff}-invariant. For an affine real root α, we have $\alpha = w \cdot \alpha_i$ for some $w \in W_{\text{aff}}$ and $0 \leq i \leq n-1$. Then the associated reflection s_α is independent of the choice of w and i. Also, $s_\alpha = w s_i w^{-1}$ and $s_\alpha(\mu) = \mu - \langle \mu, \alpha^\vee \rangle \alpha$ for $\mu \in \mathfrak{h}^*_{\text{aff}}$. For an affine real root $\beta = \alpha + m\delta$ we have $s_\beta = s_\alpha t_{m\alpha^\vee} = s_\alpha t_{m\alpha}$ and, in particular $s_0 = s_\theta t_{-\theta^\vee} = s_\theta t_{-\theta}$, see [8, §3] for further details.

Let $S = \{s_0, s_1, s_2, ..., s_{n-1}\}$. Then (W_{aff}, S) is a Coxeter system with the following relations;

$$s_i^2 = 1 \quad \text{for all } i$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad \text{for all } i$$

$$s_i s_j = s_j s_i \quad \text{for } i, j \text{ not adjacent, } i \neq j$$

where the indices are taken modulo n; see [3, p.263]. Each $w \neq e$ in W_{aff} can be written in the form $w = s_{i_1} s_{i_2} ... s_{i_k}$ for some s_{i_j} (not necessarily distinct) in S. If k is as small as possible, call it the *length* of w, written $l(w)$, and call any expression of w as a product of k elements of S a *reduced expression.* We set $\ell(e) = 0$. For an $\alpha \in \Pi_{\text{aff}}$ if $\alpha = \sum_{i=0}^{n-1} a_i \alpha_i$ then the support of α is $\text{supp}(\alpha) = \{\alpha_i : a_i \neq 0\}$. We have the following equations:

- If $\alpha \in \Pi_{\text{aff}}^{\text{re},+}$ such that $\alpha < c$ then $\ell(s_\alpha) = 2|\text{supp}(\alpha)| - 1$.
- If $w \in W_{\text{aff}}$ then $\ell(w) = |\{\alpha \in \Pi^+_{\text{aff}} : w \cdot \alpha < 0\}|$.
- If $x = w t_\lambda \in W_{\text{aff}}$ then by Lemma 3.1 in [8], we have

$$\ell(x) = \sum_{\gamma \in \Pi^+} |\chi(w \cdot \gamma < 0) + \langle \lambda, \gamma \rangle| \tag{2.1}$$

where $\chi(P) = 1$ if P is true and $\chi(P) = 0$ otherwise.

Given $u, v \in W_{\text{aff}}$, we say that the product uv is *reduced* if $\ell(uv) = \ell(u) + \ell(v)$.

Next, we will set up some notations for affine positive real roots which are smaller than c; let $p_{i,i} := \alpha_i$ for $0 \leq i \leq n-1$, and $p_{i,j} := \alpha_i + \alpha_{i+1} + ... + \alpha_{j-1} + \alpha_j$ for $0 \leq i < j \leq n-1$ such that $j - i < n-1$ and $p_{i,j} := \alpha_0 + \alpha_1 + ... + \alpha_j + \alpha_i + \alpha_{i+1} + ... + \alpha_{n-1}$ where $0 \leq j < i-1 \leq n-2$. Note that, $s_{p_{i,i}} = s_i$ and $s_{p_{i,j}} = s_i s_{i+1} ... s_{j-1} s_j s_{j-1} ... s_{i-1} s_i = s_j s_{j-1} ... s_{i+1} s_i s_{i+1} ... s_{j-1} s_j$ if $i < j$, and

$$\begin{aligned} s_{p_{i,j}} &= s_j s_{j-1} ... s_1 s_0 s_i s_{i+1} ... s_{n-2} s_{n-1} s_{n-2} ... s_{i+1} s_i s_0 s_1 ... s_{j-1} s_j \\ &= s_i s_j s_{j-1} ... s_1 s_0 s_{i+1} ... s_{n-2} s_{n-1} s_{n-2} ... s_{i+1} s_0 s_1 ... s_{j-1} s_j s_i \end{aligned}$$

if $i > j$, are some of the reduced expressions for the reflections which will be used throughout this paper.

2.3 Dynkin Diagram

In this section, we will consider the Dynkin diagram for Π_{aff} which is given in the figure below.

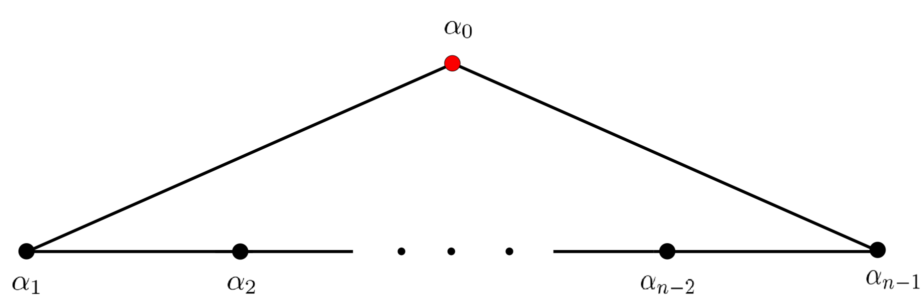

Figure 2.1: Dynkin diagram of affine root system in type $A_{n-1}^{(1)}$.

Let $\varphi : \Delta_{\text{aff}} \to \Delta_{\text{aff}}$ be a map given by $\varphi(\alpha_i) = \alpha_{i+1}$ if $0 \leq i \leq n-2$ and $\varphi(\alpha_{n-1}) = \alpha_0$. Note that, $\langle \alpha_i, \alpha_j^\vee \rangle = \langle \varphi(\alpha_i), \varphi(\alpha_j)^\vee \rangle$ for all i, j, since the Dynkin diagram for Π_{aff} is simply laced and circular. Thus φ is an automorphism of the diagram. It is clear that the order of φ is n and φ preserves the partial order "$\leq$" on the roots.

2.4 Hecke Product

We describe some of the curve neighborhoods in terms of the Hecke product. For that reason, we recall the definition of the Hecke product and some of its properties in this section. We refer to [5, §3] for further details. For $u \in W_{\text{aff}}$ and $i \in \{0, 1, ..., n-1\}$, define

$$u \cdot s_i = \begin{cases} us_i & \text{If } \ell(us_i) > \ell(u) \\ u & \text{otherwise.} \end{cases}$$

Let $u, v \in W_{\text{aff}}$ and let $v = s_{i_1} s_{i_2} ... s_{i_k}$ be any reduced expression for v. Define the *Hecke product* of u and v by

$$u \cdot v = u \cdot s_{i_1} \cdot s_{i_2} \cdot ... \cdot s_{i_k},$$

where the simple reflections are multiplied to u in left to right order. This product is independent of the chosen reduced expressions for v. It provides a monoid structure on the affine Weyl group W_{aff}. Furthermore, we have the following properties of the Hecke product: Let $u, v, v', w \in W_{\text{aff}}$.

a) The Hecke product is associative, i.e. $(u \cdot v) \cdot w = u \cdot (v \cdot w)$.

b) If $v \leq v'$ then $u \cdot v \cdot w \leq u \cdot v' \cdot w$.

c) We have $u \leq u \cdot v$, $v \leq u \cdot v$, $uv \leq u \cdot v$, and $\ell(u \cdot v) \leq \ell(u) + \ell(v)$.

d) If uv is reduced then $uv = u \cdot v$ and $\ell(u \cdot v) = \ell(u) + \ell(v)$.

e) The element $v = (w \cdot u)u^{-1}$ satisfies $v \leq w$ and $vu = v \cdot u = w \cdot u$.

Chapter 3

Preliminary Lemmas and $z_{\mathbf{d}}$

In this chapter we will prove some lemmas which are used throughout the thesis. We will also discuss the element $z_{\mathbf{d}}$ and prove a theorem about it that is crucial for some of the results.

3.1 The element $z_{\mathbf{d}}$

In this section we give the definition of $z_{\mathbf{d}}$ which is the unique element in $\Gamma_{\mathbf{d}}(id)$ where $\mathbf{d} = (d_0, d_1, ..., d_{n-1}) \in Q_{\text{aff}}$ such that $d_i = 0$ for some i. Furthermore, the most general case of the curve neighborhood $\Gamma_{\mathbf{d}}(id)$ where $\mathbf{d} > c$ is described in terms of $z_{\mathbf{d}}$. This section also includes a theorem where we show how one can simplify $z_{\mathbf{d}}$. We refer the reader to [5, §4] and [1] for more details.

Definition 3.1.1. *Let* $\mathbf{d} = (d_0, d_1, ..., d_{n-1}) \in Q_{aff}$ *such that* $d_i = 0$ *for some* i*. If* $\mathbf{d} = 0$ *then set* $z_{\mathbf{d}} = id$*. Otherwise we set* $z_{\mathbf{d}} = s_\alpha \cdot z_{\mathbf{d}-\alpha}$ *where* α *is any maximal root which is smaller than and equal to* $\mathbf{d}$*.*

$z_{\mathbf{d}}$ is well-defined by induction on $\mathbf{d}$.

Theorem 3.1.2. *Let* $\mathbf{d} = (d_0, d_1, ..., d_{n-1})$ *be a degree such that* $d_i = 0$ *for some* i*. Then* $z_{\mathbf{d}} = s_{\gamma_1} s_{\gamma_2} ... s_{\gamma_r}$ *for some positive real roots,* $\gamma_1, \gamma_2, ..., \gamma_r$ *such that either supp*(γ_i) *and supp*(γ_j) *are disconnected or both* $\gamma_i \perp \gamma_j$ *and* γ_i, γ_j *are comparable, for any* $1 \leq i, j \leq r$ *such that* $i \neq j$*. Moreover,* $z_{\mathbf{d}} = s_{\gamma_1} s_{\gamma_2} ... s_{\gamma_r}$ *is reduced and* $\ell(z_{\mathbf{d}}) = \sum_{i=1}^{r} (2\,|supp(\gamma_i)| - 1)$*.*

Proof. Let $z_{\mathbf{d}} = s_{\beta_1} \cdot s_{\beta_2} \cdot ... \cdot s_{\beta_r}$ for some positive real roots, $\beta_1, \beta_2, ..., \beta_r$. By definition, we have either supp(β_i) and supp(β_j) are disconnected or β_i, β_j are comparable, for any $1 \leq i, j \leq r$ such that $i \neq j$, see [5, §4]. So $s_{\beta_i} \cdot s_{\beta_j} = s_{\beta_j} \cdot s_{\beta_i}$ for any $1 \leq i, j \leq r$, by Lemma 3.2.1. Now, assume that β_i and β_j are comparable but not perpendicular for some $1 \leq i, j \leq r$ such that $i \neq j$. Here, we can suppose that $\beta_i \geq \beta_j$, without loss of generality. Now, if $\beta_i = p_{k,l}$ for some $0 \leq k, l \leq n-1$ then we have three cases; either $\beta_j = p_{k,q}$ where $q \neq l$ or $\beta_j = p_{q,l}$ where $q \neq k$ or $\beta_j = p_{k,l}$. First, suppose that $\beta_j = p_{k,q}$ where $q \neq l$. Note that, both s_{β_i} and s_{β_j} have a reduced expression which starts and ends with the simple reflection, s_k this implies that $s_{\beta_i} \cdot s_k = s_{\beta_i}$. We will show that $s_{\beta_i} \cdot s_{\beta_j} = s_{\beta_i} \cdot s_{\beta_j - \alpha_k}$ where α_k is the simple root. Here, we will use the convention, $s_a := id$ if $a = 0$. Now, observe that $s_{\beta_j} = s_k \cdot s_{\beta_j - \alpha_k} \cdot s_k$. Furthermore, $s_{\beta_i} \cdot s_{\beta_j - \alpha_k} = s_{\beta_j - \alpha_k} \cdot s_{\beta_i}$ since $\beta_j - \alpha_k < \beta_i$ by Lemma

3.2.1. Thus

$$\begin{aligned} s_{\beta_i} \cdot s_{\beta_j} &= s_{\beta_i} \cdot s_k \cdot s_{\beta_j - \alpha_k} \cdot s_k = s_{\beta_i} \cdot s_{\beta_j - \alpha_k} \cdot s_k = s_{\beta_j - \alpha_k} \cdot s_{\beta_i} \cdot s_k \\ &= s_{\beta_j - \alpha_k} \cdot s_{\beta_i} = s_{\beta_i} \cdot s_{\beta_j - \alpha_k}. \end{aligned}$$

Here, notice that β_i and $\beta_j - \alpha_k$ are perpendicular. Second, suppose that $\beta_j = p_{q,l}$ where $q \neq k$. Then, one can show that $s_{\beta_i} \cdot s_{\beta_j} = s_{\beta_i} \cdot s_{\beta_j - \alpha_l}$ where α_l is the simple root, by using similar arguments in the previous case. Also, note that β_i and $\beta_j - \alpha_l$ are perpendicular. Last, assume that $\beta_j = p_{k,l}$. Then again by the previous two cases one can show that $s_{\beta_i} \cdot s_{\beta_j} = s_{\beta_i} \cdot s_{\beta_j - \alpha_k - \alpha_l}$ where α_k and α_l are the simple reflections. Moreover, β_i and $\beta_j - \alpha_k - \alpha_l$ are perpendicular. Hence, we can assume that either $\mathrm{supp}(\beta_i)$ and $\mathrm{supp}(\beta_j)$ are disconnected or both β_i, β_j are comparable and $\beta_i \perp \beta_j$, for any $1 \leq i, j \leq r$ such that $i \neq j$.

Next, we will show that $z_{\mathbf{d}} = s_{\beta_1} s_{\beta_2} ... s_{\beta_r}$ is reduced. Note that $\beta_i < c$ for all i, by definition. So $\ell(s_{\beta_i}) = 2|\mathrm{supp}(\beta_i)| - 1$ and we have $\ell(z_{\mathbf{d}}) = \ell(s_{\beta_1} s_{\beta_2} ... s_{\beta_r}) \leq \sum_{i=1}^{r} \ell(s_{\beta_i}) = \sum_{i=1}^{r} 2|\mathrm{supp}(\beta_i)| - 1$. Furthermore, for a root ν, we have $s_{\beta_1} s_{\beta_2} ... s_{\beta_r}(\nu) = \nu - \sum_{i=1}^{r} < \nu, \beta_i^{\vee} > \beta_i$ since $\beta_i \perp \beta_j$, for any $1 \leq i, j \leq r$ such that $i \neq j$. Now, assume that ν is an affine positive real root such that $\nu \leq \beta_i$ for some i. Also, assume that ν and β_i are not perpendicular. This implies that $< \nu, \beta_i^{\vee} >$ is either equal to 2 or 1. if $< \nu, \beta_i^{\vee} >= 2$ then $\beta_i = \nu$. But then $\nu \perp \beta_j$ for all $j \in \{1, 2, ..., r\} \setminus \{i\}$, which follows by $s_{\beta_1} s_{\beta_2} ... s_{\beta_r}(\nu) = -\nu < 0$. Now, suppose that $< \nu, \beta_i^{\vee} >= 1$ then we have $\nu \perp \beta_j$ for all j such that $\mathrm{supp}(\beta_i)$ and $\mathrm{supp}(\beta_j)$ are disconnected since $\nu \leq \beta_i$. So if ν and β_j are not perpendicular for some β_j where $\beta_j \neq \beta_i$ we have to have $\beta_j < \beta_i$ which implies that $< \nu, \beta_j^{\vee} >$ is either equal to 1 or -1 and there is at most one such root as β_j. Now, if $< \nu, \beta_j^{\vee} >= 1$ then either $\beta_j < \nu$

or $\nu < \beta_j$ but since $\beta_j < \beta_i$ and $\beta_j \perp \beta_i$ we have to have $\beta_j < \nu$. If $<\nu, \beta_j^\vee>=-1$ then $\beta_j \cap \nu = \emptyset$. Thus we have either $s_{\beta_1}s_{\beta_2}...s_{\beta_r}(\nu) = \nu - \beta_i - \beta_j$ or $s_{\beta_1}s_{\beta_2}...s_{\beta_r}(\nu) = \nu - \beta_i + \beta_j$. Here, observe that in either case $s_{\beta_1}s_{\beta_2}...s_{\beta_r}(\nu) < 0$. Now, let $\beta_i = p_{k,l}$ and $\nu = p_{t,q}$. Then $t = k$ or $q = l$. If $k \leq l$ then there are $2l - 2k + 1 = 2|\mathrm{supp}(\beta_i)| - 1$ such ν and if $k > l$ then there are $2n + 2l - 2k + 1 = 2|\mathrm{supp}(\beta_i)| - 1$ such ν. Hence, if A is the set of all positive real roots, ν such that $s_{\beta_1}s_{\beta_2}...s_{\beta_r}(\nu) < 0$ where $\nu \leq \beta_i$ and, ν and β_i are not perpendicular for some $i = 1, 2, .., r$ then $|A| = \sum_{i=1}^{r} 2|\mathrm{supp}(\beta_i)| - 1$. Also, note that $A \subseteq \{\gamma \in \Pi_{\mathrm{aff}}^{\mathrm{re},+} : s_{\beta_1}s_{\beta_2}...s_{\beta_r}(\gamma) < 0\}$ and we have

$$\begin{aligned} |A| = \textstyle\sum_{i=1}^{r} 2|\mathrm{supp}(\beta_i)| - 1 &\leq |\{\gamma \in \Pi_{\mathrm{aff}}^{\mathrm{re},+} : s_{\beta_1}s_{\beta_2}...s_{\beta_r}(\nu) < 0\}| \\ &= \ell(s_{\beta_1}s_{\beta_2}...s_{\beta_l}) \leq \textstyle\sum_{i=1}^{r} 2|\mathrm{supp}(\beta_i)| - 1 \end{aligned}$$

which follows by $\ell(s_{\beta_1}s_{\beta_2}...s_{\beta_r}) = \sum_{i=1}^{r} 2|\mathrm{supp}(\beta_i)| - 1$. Thus $s_{\beta_1}s_{\beta_2}...s_{\beta_r}$ is reduced.

□

Next, we will give an example for computing $z_{\mathbf{d}}$.

Example 3.1.3. Let $\mathbf{d} = (5, 0, 2, 2, 3, 0, 4)$. Observe that $\alpha_0 + \alpha_6$ and $\alpha_2 + \alpha_3 + \alpha_4$ are maximal roots of $\mathbf{d}$ which have disconnected supports and $\mathbf{d} = 4(\alpha_0 + \alpha_6) + 2(\alpha_2 + \alpha_3 + \alpha_4) + \alpha_0 + \alpha_4$ so by definition we get

$$z_{\mathbf{d}} = s_{\alpha_0+\alpha_6} \cdot s_{\alpha_0+\alpha_6} \cdot s_{\alpha_0+\alpha_6} \cdot s_{\alpha_0+\alpha_6} \cdot s_{\alpha_2+\alpha_3+\alpha_4} \cdot s_{\alpha_2+\alpha_3+\alpha_4} \cdot s_{\alpha_0} \cdot s_{\alpha_4}.$$

But notice that $s_{\alpha_0+\alpha_6} \cdot s_{\alpha_0+\alpha_6} = s_{\alpha_0+\alpha_6}$ and $s_{\alpha_2+\alpha_3+\alpha_4} \cdot s_{\alpha_2+\alpha_3+\alpha_4} = s_{\alpha_2+\alpha_3+\alpha_4} \cdot s_{\alpha_3}$, see the proof of Lemma 3.2.1, so we have $z_{\mathbf{d}} = s_{\alpha_0+\alpha_6} \cdot s_{\alpha_2+\alpha_3+\alpha_4} \cdot s_{\alpha_3} \cdot s_{\alpha_0} \cdot s_{\alpha_4}$. Now, also note that any two reflections that appear in $z_{\mathbf{d}}$ Hecke commute by Lemma 3.2.1. Moreover,

$s_{\alpha_0+\alpha_6} \cdot s_{\alpha_0} = s_{\alpha_0+\alpha_6}$ and $s_{\alpha_2+\alpha_3+\alpha_4} \cdot s_{\alpha_4} = s_{\alpha_2+\alpha_3+\alpha_4}$, again see the proof of Lemma 3.2.1. Thus $z_{\mathbf{d}} = s_{\alpha_0+\alpha_6} \cdot s_{\alpha_2+\alpha_3+\alpha_4} \cdot s_{\alpha_3}$. But this multiplication for $z_{\mathbf{d}}$ is reduced; see the proof of Theorem 3.1.2. So $z_{\mathbf{d}} = s_{\alpha_0+\alpha_6} s_{\alpha_2+\alpha_3+\alpha_4} s_{\alpha_3}$. Furthermore,

$$\ell(z_{\mathbf{d}}) = 2|\mathrm{supp}(\alpha_0 + \alpha_6)| - 1 + 2|\mathrm{supp}(\alpha_2 + \alpha_3 + \alpha_4)| - 1 + 2|\mathrm{supp}(\alpha_3)| - 1 = 9.$$

3.2 Lemma about the Hecke Product

Here, we will consider a lemma which allows us to manipulate a given Hecke multiplication of two reflections without changing the sum of the corresponding roots.

Lemma 3.2.1. *Let $\alpha, \beta \in \Pi_{\mathit{aff}}^{re,+}$ such that $\alpha, \beta < c$. Then*

1) Assume that $\alpha \leq \beta$. Then

$$s_\alpha \cdot s_\beta = s_\beta \cdot s_\alpha$$

2) Suppose that $\gamma = \alpha \cap \beta \neq \emptyset$ and $\gamma \neq \alpha, \beta$. Then

a) If γ is a root then $\alpha - \gamma$ and $\beta - \gamma$ are also roots. Moreover,

$$s_\alpha \cdot s_\beta = s_\alpha \cdot s_{\beta-\gamma} \cdot s_\gamma = s_\gamma \cdot s_{\alpha-\gamma} \cdot s_\beta$$

b) If γ is not a root then $\gamma = \gamma_1 + \gamma_2$ for some positive real roots γ_1, γ_2. Also, $\alpha - \gamma$,$\alpha - \gamma_1$, $\alpha - \gamma_2$, $\beta - \gamma$, $\beta - \gamma_1$ and $\beta - \gamma_2$ are all roots. Moreover,

$$s_\alpha \cdot s_\beta = s_\alpha \cdot s_{\beta-\gamma} \cdot s_{\gamma_1} \cdot s_{\gamma_2} = s_{\gamma_1} \cdot s_{\gamma_2} \cdot s_{\alpha-\gamma} \cdot s_\beta$$

3) Assume that $\alpha \cap \beta = \emptyset$. Then

a) If $\alpha + \beta$ is a root then $s_\alpha \cdot s_\beta \leq s_{\alpha+\beta}$.

b) If $\alpha + \beta$ is not a root then $s_\alpha \cdot s_\beta = s_\beta \cdot s_\alpha$.

Proof. Let a simple reflection s_q and a positive real root $p_{i,j} < c$ be given. We will show that $s_q \cdot s_{p_{i,j}} = s_{p_{i,j}} \cdot s_q$ if $\alpha_q \in \text{supp}(p_{i,j})$. First, assume that $q = i$. Then we have three cases; either $i = j$ or $i < j$ or $i > j$. If $i = j$ then the equality is clear since $p_{i,j} = p_{i,i} = \alpha_i$ and $s_q = s_{p_{i,j}} = s_i$ in this case. Now, suppose that $i < j$. Then since $s_i \cdot s_i = s_i$ we have

$$\begin{aligned} s_q \cdot s_{p_{i,j}} &= s_i \cdot s_i \cdot s_{i+1} \cdot \ldots \cdot s_{j-1} \cdot s_j \cdot s_{j-1} \cdot \ldots \cdot s_{i-1} \cdot s_i \\ &= s_i \cdot s_{i+1} \cdot \ldots \cdot s_{j-1} \cdot s_j \cdot s_{j-1} \cdot \ldots \cdot s_{i-1} s_i \cdot s_i = s_{p_{i,j}} \cdot s_q. \end{aligned}$$

The case, $i > j$ is similar. If $q = j$ then we have the equalities by the same argument since $s_{p_{i,j}}$ has a reduced expression which starts and ends with s_j in all sub cases. Now assume that q is neither i nor j. Here, we have two sub cases; $i < j$ or $i > j$. First, assume that $i < j$. Then $i < q < j$ and we have

$$\begin{aligned} s_q \cdot s_{p_{i,j}} &= s_q \cdot s_i \cdot s_{i+1} \cdot \ldots \cdot (s_{q-1} \cdot s_q) \cdot \ldots \cdot s_{j-1} \cdot s_j \cdot s_{j-1} \cdot \ldots \cdot (s_q \cdot s_{q-1}) \cdot \ldots \cdot s_{i-1} \cdot s_i \\ &= s_i \cdot s_{i+1} \cdot \ldots \cdot (s_q \cdot s_{q-1} \cdot s_q) \cdot \ldots \cdot s_{j-1} \cdot s_j \cdot s_{j-1} \cdot \ldots \cdot (s_q \cdot s_{q-1}) \cdot \ldots \cdot s_{i-1} \cdot s_i \\ &= s_i \cdot s_{i+1} \cdot \ldots \cdot (s_{q-1} \cdot s_q \cdot s_{q-1}) \cdot \ldots \cdot s_{j-1} \cdot s_j \cdot s_{j-1} \cdot \ldots \cdot (s_q \cdot s_{q-1}) \cdot \ldots \cdot s_{i-1} \cdot s_i \\ &= s_i \cdot s_{i+1} \cdot \ldots \cdot (s_{q-1} \cdot s_q) \cdot \ldots \cdot s_{j-1} \cdot s_j \cdot s_{j-1} \cdot \ldots \cdot (s_{q-1} \cdot s_q \cdot s_{q-1}) \cdot \ldots \cdot s_{i-1} \cdot s_i \\ &= s_i \cdot s_{i+1} \cdot \ldots \cdot (s_{q-1} \cdot s_q) \cdot \ldots \cdot s_{j-1} \cdot s_j \cdot s_{j-1} \cdot \ldots \cdot (s_q \cdot s_{q-1} \cdot s_q) \cdot \ldots \cdot s_{i-1} \cdot s_i \\ &= s_i \cdot s_{i+1} \cdot \ldots \cdot (s_{q-1} \cdot s_q) \cdot \ldots \cdot s_{j-1} \cdot s_j \cdot s_{j-1} \cdot \ldots \cdot (s_q \cdot s_{q-1}) \cdot \ldots \cdot s_{i-1} \cdot s_i \cdot s_q \\ &= s_{p_{i,j}} \cdot s_q \end{aligned}$$

The case, $i > j$ is similar.

1) Assume that $\alpha \leq \beta$. Suppose that $\alpha = s_{i_1} \cdot s_{i_2} \cdot ... \cdot s_{i_r}$ is a reduced expression for α. Now, since $\alpha \leq \beta$ we have $\text{supp}(\alpha) = \{i_1, i_2, ..., i_r\} \subset \text{supp}(\beta)$. Thus the reflection s_{i_k} will Hecke commute with β for all k by the argument above. So we will have

$$\alpha \cdot \beta = s_{i_1} \cdot s_{i_2} \cdot ... \cdot s_{i_r} \cdot \beta = \beta \cdot s_{i_1} \cdot s_{i_2} \cdot ... \cdot s_{i_r} = \beta \cdot \alpha.$$

2) *Case a:* Let $\gamma = \alpha \cap \beta$ be a root and $\gamma \neq \alpha, \beta$. First, we will show that $s_\alpha = u \cdot s_{\alpha-\gamma} \cdot u^{-1}$ and $s_\beta = u^{-1} \cdot s_{\beta-\gamma} \cdot u$ for some permutation u such that $\gamma = u \cdot u^{-1}$. We have several cases here;

Case a1: Assume that $\alpha = p_{i,j}$ and $\beta = p_{k,l}$ where $0 \leq i < k < j < l \leq n-1$. Then $\gamma = p_{k,j}$. Furthermore, $\alpha - \gamma = p_{i,k-1}$ and $\beta - \gamma = p_{j+1,l}$, so both are roots. Now, note that

$$s_\alpha = s_j \cdot s_{j-1} \cdot ... \cdot s_{k+1} \cdot s_k \cdot s_{k-1} \cdot ... \cdot s_{i+1} \cdot s_i \cdot s_{i+1} \cdot ... \cdot s_{k-1} \cdot s_k \cdot s_{k+1} \cdot ... \cdot s_{j-1} \cdot s_j,$$

$$s_\beta = s_k \cdot s_{k+1} \cdot ... \cdot s_{j-1} \cdot s_j \cdot s_{j+1} \cdot ... \cdot s_{l-1} \cdot s_l \cdot s_{l-1} \cdot ... s_{j+1} \cdot s_j \cdot s_{j-1} \cdot ... \cdot s_{k+1} \cdot s_k,$$

$s_\gamma = s_j \cdot s_{j-1} \cdot ... \cdot s_{k+1} \cdot s_k \cdot s_{k+1} \cdot ... \cdot s_{j-1} \cdot s_j$ and $s_{\alpha-\gamma} = s_{k-1} \cdot ... \cdot s_{i+1} \cdot s_i \cdot s_{i+1} \cdot ... \cdot s_{k-1}$ and $s_{\beta-\gamma} = s_{j+1} \cdot ... \cdot s_{l-1} \cdot s_l \cdot s_{l-1} \cdot ... s_{j+1}$ are some reduced expressions for the reflections. So we can take $u = s_j \cdot s_{j-1} \cdot ... \cdot s_{k+1} \cdot s_k$ which will imply that $s_\gamma = u \cdot u^{-1}$ since $s_k \cdot s_k = s_k$.

Case a2: Suppose that $\alpha = p_{i,j}$ and $\beta = p_{k,l}$ where $0 \leq k < j < l \leq i \leq n-1$. Then $\gamma = p_{k,j}$. Moreover, $\alpha - \gamma = p_{i,n-1}$ and $\beta - \gamma = p_{j+1,l}$, hence both are roots. Again, if we consider the reduced expressions;

$$s_\alpha = s_j \cdot s_{j-1} \cdot ... \cdot s_1 \cdot s_0 \cdot s_i \cdot s_{i+1} \cdot ... \cdot s_{n-1} \cdot s_n \cdot s_{n-1} \cdot ... \cdot s_{i+1} \cdot s_i \cdot s_0 \cdot s_1 ... \cdot s_{j-1} \cdot s_j,$$

$$s_\beta = s_0 \cdot s_1 ... \cdot s_{j-1} \cdot s_j \cdot s_{j+1} \cdot ... s_{l-1} \cdot s_l \cdot s_{l-1} ... \cdot s_{j+1} \cdot s_j \cdot s_{j-1} ... \cdot s_1 \cdot s_0,$$

$s_\gamma = s_j \cdot s_{j-1} \cdot ... \cdot s_1 \cdot s_0 \cdot s_1 ... \cdot s_{j-1} \cdot s_j$ and $s_{\alpha-\gamma} = s_i \cdot s_{i+1} \cdot ... \cdot s_{n-1} \cdot s_n \cdot s_{n-1} \cdot ... \cdot s_{i+1} \cdot s_i$ and $s_{\beta-\gamma} = s_{j+1} \cdot ... s_{l-1} \cdot s_l \cdot s_{l-1} ... \cdot s_{j+1}$ we can take $u = s_j \cdot s_{j-1} \cdot ... \cdot s_1 \cdot s_0$.

Case a3: Suppose that $\alpha = p_{i,j}$ and $\beta = p_{k,l}$ where $0 \leq j \leq k < i < l \leq n-1$. Then $\gamma = p_{i,l}$. Also, $\alpha - \gamma = p_{0,j}$ and $\beta - \gamma = p_{k,i-1}$, thus both are roots. Similar to $a2$).

Case a4: Suppose that $\alpha = p_{i,j}$ and $\beta = p_{l,k}$ where $0 \leq k < j \leq l < i \leq n-1$. Then $\gamma = p_{i,k}$. Moreover, $\alpha - \gamma = p_{k+1,j}$ and $\beta - \gamma = p_{l,i-1}$, so both are roots. Note that,

$$\begin{aligned}
s_\alpha &= s_j \cdot ... \cdot s_k \cdot ... \cdot s_1 \cdot s_0 \cdot s_i \cdot s_{i+1} \cdot ... \cdot s_{n-1} \cdot ... \cdot s_{i-1} \cdot s_i \cdot s_0 \cdot s_1 \cdot ... \cdot s_k \cdot ... \cdot s_j \\
&= s_j \cdot ... \cdot s_k \cdot ... \cdot s_1 \cdot s_i \cdot s_{i+1} \cdot ... \cdot s_0 \cdot s_{n-1} \cdot s_0 \cdot ... \cdot s_{i-1} \cdot s_i \cdot s_1 \cdot ... \cdot s_k \cdot ... \cdot s_j \\
&= s_j \cdot ... \cdot s_k \cdot ... \cdot s_1 \cdot s_i \cdot s_{i+1} \cdot ... \cdot s_{n-1} \cdot s_0 \cdot s_{n-1} \cdot ... \cdot s_{i-1} \cdot s_i \cdot s_1 \cdot ... \cdot s_k \cdot ... \cdot s_j \\
&= s_i \cdot s_{i+1} \cdot ... \cdot s_{n-1} \cdot s_j \cdot ... \cdot s_k \cdot ... \cdot s_1 \cdot s_0 \cdot s_1 \cdot ... \cdot s_k \cdot ... \cdot s_j \cdot s_{n-1} \cdot ... \cdot s_{i-1} \cdot s_i \\
&= s_i \cdot s_{i+1} \cdot ... \cdot s_{n-1} \cdot s_0 \cdot s_1 \cdot ... \cdot s_k \cdot ... \cdot s_j \cdot ... \cdot s_k \cdot ... \cdot s_1 \cdot s_0 \cdot s_{n-1} \cdot ... \cdot s_{i-1} \cdot s_i,
\end{aligned}$$

$$\begin{aligned}
s_\beta &= s_k \cdot ... \cdot s_1 \cdot s_0 \cdot s_l \cdot s_{l+1} \cdot ... \cdot s_i \cdot ... s_{n-1} \cdot ... \cdot s_i \cdot ... \cdot s_{l+1} \cdot s_l \cdot s_0 \cdot s_1 \cdot ... \cdot s_k \\
&= s_k \cdot ... \cdot s_0 \cdot s_{n-1} \cdot ... s_i \cdot s_{i-1} \cdot ... \cdot s_{l+1} \cdot s_l \cdot s_{l+1} \cdot ... \cdot s_{i-1} \cdot s_i \cdot ... \cdot s_{n-1} \cdot s_0 \cdot ... \cdot s_k,
\end{aligned}$$

$$\begin{aligned}
s_\gamma &= s_k \cdot .. \cdot s_1 \cdot s_0 \cdot s_i \cdot s_{i+1} \cdot ... \cdot s_{n-1} \cdot ... \cdot s_{i+1} \cdot s_i \cdot s_0 \cdot s_1 \cdot ... \cdot s_k \\
&= s_k \cdot .. \cdot s_1 \cdot s_i \cdot s_{i+1} \cdot ... \cdot s_0 \cdot s_{n-1} \cdot s_0 \cdot ... \cdot s_{i+1} \cdot s_i \cdot s_1 \cdot ... \cdot s_k \\
&= s_k \cdot .. \cdot s_1 \cdot s_i \cdot s_{i+1} \cdot ... \cdot s_{n-1} \cdot s_0 \cdot s_{n-1} \cdot ... \cdot s_{i+1} \cdot s_i \cdot s_1 \cdot ... \cdot s_k \\
&= s_i \cdot s_{i+1} \cdot ... \cdot s_{n-1} \cdot s_k \cdot ... \cdot s_1 \cdot s_0 \cdot s_1 \cdot ... \cdot s_k \cdot s_{n-1} \cdot ... \cdot s_{i+1} \cdot s_i \\
&= s_i \cdot s_{i+1} \cdot ... \cdot s_{n-1} \cdot s_0 \cdot s_1 \cdot ... \cdot s_k \cdot ... \cdot s_1 \cdot s_0 \cdot s_{n-1} \cdot ... \cdot s_{i+1} \cdot s_i,
\end{aligned}$$

and $s_{\alpha-\gamma} = s_{p_{k+1,j}} = s_{k+1} \cdot s_{k+2} \cdot \ldots \cdot s_j \cdot .. \cdot s_{k+2} \cdot s_{k+1}$, and

$$\begin{aligned} s_{\beta-\gamma} &= s_{p_{l,i-1}} = s_l \cdot s_{l+1} \cdot \ldots \cdot s_{i-1} \cdot \ldots \cdot s_{l+1} \cdot s_l \\ &= s_{i-1} \cdot s_{i-2} \cdot \ldots \cdot s_{l+1} \cdot s_l \cdot s_{l+1} \cdot \ldots \cdot s_{i-2} \cdot s_{i-1}. \end{aligned}$$

Thus, we can take $u = s_i \cdot s_{i+1} \cdot \ldots \cdot s_{n-1} \cdot s_0 \cdot s_1 \cdot \ldots \cdot s_k$ since $s_k \cdot s_k = s_k$.

Case a5: Assume that $\alpha = p_{i,j}$ and $\beta = p_{l,k}$ where $0 \leq j < k \leq i < l \leq n-1$. Then $\gamma = p_{l,j}$. Furthermore, $\alpha - \gamma = p_{i,l-1}$ and $\beta - \gamma = p_{j+1,k}$, hence both are roots. This case is similar to $a4$).

Now, note that, in all cases above we have $s_\alpha = u \cdot s_{\alpha-\gamma} \cdot u^{-1}$, $s_\beta = u^{-1} \cdot s_{\beta-\gamma} \cdot u$ where $s_\gamma = u \cdot u^{-1}$. Also, note that $u^{-1} \cdot s_\beta = s_\beta \cdot u^{-1}$ since each simple root that appears in u^{-1} is in the support of β since $\gamma < \beta$. Hence,

$$\begin{aligned} s_\alpha \cdot s_\beta &= u \cdot s_{\alpha-\gamma} \cdot u^{-1} \cdot s_\beta = u \cdot s_{\alpha-\gamma} \cdot s_\beta \cdot u^{-1} \\ &= u \cdot s_{\alpha-\gamma} \cdot u^{-1} \cdot s_{\beta-\gamma} \cdot u \cdot u^{-1} = s_\alpha \cdot s_{\beta-\gamma} \cdot s_\gamma \end{aligned}$$

Case b: Now, assume that $\gamma = \alpha \cap \beta$ is not a root. First, we will show that $\gamma = \gamma_1 + \gamma_2$ for some $\gamma_1, \gamma_2 \in \Pi_{\text{aff}}^{\text{re},+}$ and $\alpha - \gamma_1, \alpha - \gamma_2, \alpha - \gamma, \beta - \gamma_1, \beta - \gamma_2$ and $\beta - \gamma$ are all roots. Here, we have two cases;

Case b1: Suppose that $\alpha = p_{i,j}$ and $\beta = p_{k,l}$ where $0 \leq k \leq j < i-1 \leq l-1 < n-2$. Note that, $\gamma = \alpha \cap \beta = p_{k,j} + p_{i,l}$ so we can take $\gamma_1 = p_{k,j}$ and $\gamma_2 = p_{i,l}$. Also, note that $\alpha - \gamma_1 = p_{i,k-1}$, $\alpha - \gamma_2 = p_{l+1,j}$, $\alpha - \gamma = p_{l+1,k-1}$, $\beta - \gamma_1 = p_{j+1,l}$, $\beta - \gamma_2 = p_{k,i-1}$, and $\beta - \gamma = p_{j+1,i-1}$, thus all are roots.

Case b2: Assume that $\alpha = p_{i,j}$ and $\beta = p_{k,l}$ where $0 < k \leq j < i-1 \leq l-1 \leq n-2$. Again,

$\gamma = \alpha \cap \beta = p_{k,j} + p_{i,l}$ so we can take $\gamma_1 = p_{k,j}$ and $\gamma_2 = p_{i,l}$. Furthermore, $\alpha - \gamma_1 = p_{i,k-1}$, $\alpha - \gamma_2 = p_{l+1,j}$, $\alpha - \gamma = p_{l+1,k-1}$, $\beta - \gamma_1 = p_{j+1,l}$, $\beta - \gamma_2 = p_{k,i-1}$, and $\beta - \gamma = p_{j+1,i-1}$, so all are roots.

Now, note that, by the same arguments in part a) above we can get some reduced expressions for s_α and s_β such that $s_\alpha = u_1 \cdot s_{\alpha-\gamma_1} \cdot u_1^{-1}$ and $s_{\beta-\gamma_1} = u_1^{-1} \cdot s_{\beta-\gamma_1} \cdot u_1$ for a permutation u_1 such that $s_{\gamma_1} = u_1 \cdot u_1^{-1}$ since both $\alpha - \gamma_1$ and $\beta - \gamma_1$ are roots. Similarly, we can write $s_{\alpha-\gamma_1} = u_2 \cdot s_{\alpha-\gamma_1-\gamma_2} \cdot u_2^{-1}$ and $s_{\beta-\gamma_1} = u_2^{-1} \cdot s_{\beta-\gamma_1-\gamma_2} \cdot u_2$ for a permutation u_2 such that $s_{\gamma_2} = u_2 \cdot u_2^{-1}$ since both $\alpha - \gamma_1 - \gamma_2$ and $\beta - \gamma_1 - \gamma_2$ are roots. Also, note that $u_1^{-1} \cdot s_\beta = s_\beta \cdot u_1^{-1}$ and $u_2^{-1} \cdot s_\beta = s_\beta \cdot u_2^{-1}$ since each simple root that appears in u_1^{-1} and u_2^{-1} is in the support of β by the fact that $\gamma_1, \gamma_2 < \beta$. Furthermore, $\mathrm{supp}(\gamma_1)$ and $\mathrm{supp}(\gamma_2)$ are disconnected so $s_{\gamma_1} \cdot s_{\gamma_2} = s_{\gamma_2} \cdot s_{\gamma_1}$ and more specifically, $u_1 \cdot s_{\gamma_2} = s_{\gamma_2} \cdot u_1$ since any simple root which appears in u_1 is in the support of γ_1 by the equation $s_{\gamma_1} = u_1 \cdot u_1^{-1}$. Thus,

$$
\begin{aligned}
s_\alpha \cdot s_\beta &= u_1 \cdot s_{\alpha-\gamma_1} \cdot u_1^{-1} \cdot s_\beta = u_1 \cdot u_2 \cdot s_{\alpha-\gamma_1-\gamma_2} \cdot u_2^{-1} \cdot u_1^{-1} \cdot s_\beta \\
&= u_1 \cdot u_2 \cdot s_{\alpha-\gamma_1-\gamma_2} \cdot s_\beta \cdot u_2^{-1} \cdot u_1^{-1} \\
&= u_1 \cdot u_2 \cdot s_{\alpha-\gamma_1-\gamma_2} \cdot u_2^{-1} \cdot s_{\beta-\gamma_2} \cdot u_2 \cdot u_2^{-1} \cdot u_1^{-1} \\
&= u_1 \cdot u_2 \cdot s_{\alpha-\gamma_1-\gamma_2} \cdot u_2^{-1} \cdot u_1^{-1} \cdot s_{\beta-\gamma_2-\gamma_1} \cdot u_1 \cdot u_2 \cdot u_2^{-1} \cdot u_1^{-1} \\
&= u_1 \cdot s_{\alpha-\gamma_1} \cdot u_1^{-1} \cdot s_{\beta-\gamma_2-\gamma_1} \cdot u_1 \cdot s_{\gamma_2} \cdot u_1^{-1} = s_\alpha \cdot s_{\beta-\gamma} \cdot s_{\gamma_2} \cdot u_1 \cdot u_1^{-1} \\
&= s_\alpha \cdot s_{\beta-\gamma} \cdot s_{\gamma_2} \cdot s_{\gamma_1} = s_\alpha \cdot s_{\beta-\gamma} \cdot s_{\gamma_1} \cdot s_{\gamma_2}
\end{aligned}
$$

3) *Case a:* Note that, $\mathrm{supp}(\alpha) \cap \mathrm{supp}(\beta) = \emptyset$ in this case and since both α and β are smaller than c, the multiplication $s_\alpha s_\beta$ is reduced so $s_\alpha s_\beta = s_\alpha \cdot s_\beta$. Now, by Corollary 5.1.5 we get $s_\alpha \cdot s_\beta = s_\alpha s_\beta \leq s_{\alpha+\beta}$.

Case b: Here, supp(α) and supp(β) are disconnected. Hence, the multiplication, $s_\alpha s_\beta$ is reduced and by the braid relations any simple reflection that appears in s_α will commute with all simple reflections which appear in s_β so $s_\alpha s_\beta = s_\beta s_\alpha$. Thus, $s_\alpha \cdot s_\beta = s_\alpha s_\beta = s_\beta s_\alpha = s_\beta \cdot s_\alpha$. □

Corollary 3.2.2. *Let $w = s_{\beta_1} \cdot s_{\beta_2} \cdot \ldots \cdot s_{\beta_r}$ be such that $\beta_i \in \Pi_{aff}^{re,+}$ for all i and $\sum_{i=1}^{r} \beta_i = \alpha$ where $\alpha < c$. Then $w \leq s_\alpha$.*

Proof. The proof follows by using Lemma 3.2.1 part 3) repeatedly.

□

3.3 Lemma about Decomposition

Lemma 3.3.1. *Let $\alpha, \beta \in \Pi_{aff}^{re,+}$ such that $\beta < c$ and $\alpha = mc + \beta$ for an integer $m \geq 1$. Then $s_\alpha = (s_\beta s_{\beta'})^m s_\beta$ where $\beta + \beta' = c$.*

Proof. First, we will show that $s_\alpha = s_\beta s_\gamma s_\beta$ where $\gamma = mc - \beta$. Note that, $\langle \gamma, \beta^\vee \rangle = \langle mc - \beta, \beta^\vee \rangle = -2$, hence $s_\beta(\gamma) = \gamma - \langle \gamma, \beta^\vee \rangle \beta = \gamma + 2\beta = \alpha$. So $s_\alpha = s_{s_\beta(\gamma)} = s_\beta s_\gamma s_\beta^{-1} = s_\beta s_\gamma s_\beta$. Next, we will prove the statement by induction on m. First, suppose that $m = 1$. Then $s_\alpha = s_\beta s_\gamma s_\beta$ where $\gamma = mc - \beta = c - \beta$. Now, assume that the statement is true for $m = k$ for a positive integer k. We will prove that it is also true for $m = k + 1$. Let $\alpha = (k+1)c + \beta$ where $\beta \in \Pi_{\text{aff}}^{\text{re},+}$ and $\beta < c$. Again, we can write $s_\alpha = s_\beta s_\gamma s_\beta$ where $\gamma = (k+1)c - \beta$. Note that, $\gamma = (k+1)c - \beta = kc + c - \beta$ and $\beta' := c - \beta$ is a positive root which is smaller than c. So by the induction assumption we can write $s_\gamma = (s_{\beta'}s_\beta)^k s_{\beta'}$. Thus $s_\alpha = s_\beta s_\gamma s_\beta = s_\beta(s_{\beta'}s_\beta)^k s_{\beta'}s_\beta = (s_\beta s_{\beta'})^{k+1}s_\beta$. □

Corollary 3.3.2. *Let $\alpha \in \Pi_{aff}^{re,+}$ such that $\alpha > c$. Then $s_\alpha = s_{\beta_1}s_{\beta_2}...s_{\beta_k}$ for some k where $\beta_i \in \Pi_{aff}^{re,+}$ such that $\beta_i < c$ for all i and $\sum_{i=1}^k \beta_i = \alpha$.*

Proof. Now, suppose that $\alpha = mc + \beta$ for some integer m such that $m \geq 1$ and an affine positive real root β which is smaller than c. Then by Lemma 3.3.1 we have $s_\alpha = (s_\beta s_{\beta'})^m s_\beta$ where $\beta + \beta' = c$. □

3.4 Lemmas about Lengths

The main goal of this section is to compute the lengths of some elements of the affine Weyl group W_{aff} which appear in the curve neighborhoods.

Lemma 3.4.1. *Let $\beta, \beta', \alpha, \alpha' \in \Pi_{aff}^{re,+}$ such that $\alpha + \alpha' = \beta + \beta' = c$. Also, suppose that $\beta \neq \alpha$. Then $(s_\beta s_{\beta'})^m \neq (s_\alpha s_{\alpha'})^m$ for any positive integer m.*

Proof. First, assume that $\alpha_0 \notin \text{supp}(\beta)$. Now, $s_{\beta'} = s_{c-\beta} = s_{-\beta}t_{-\beta} = s_\beta t_{-\beta}$. Thus $s_\beta s_{\beta'} = s_\beta s_{c-\beta} = s_\beta s_\beta t_{-\beta} = t_{-\beta}$. So $(s_\beta s_{\beta'})^m = (t_{-\beta})^m = t_{-m\beta}$. Second, assume that $\alpha_0 \in \text{supp}(\beta)$. Then $s_\beta = s_{c-(c-\beta)} = s_{-(c-\beta)}t_{-(c-\beta)} = s_{(c-\beta)}t_{-(c-\beta)}$. Hence

$$\begin{aligned} s_\beta s_{\beta'} &= s_{(c-\beta)}t_{-(c-\beta)}s_{c-\beta} = t_{s_{c-\beta}(-(c-\beta))} \\ &= t_{(c-\beta)} \end{aligned}$$

So $(s_\beta s_{\beta'})^m = (t_{c-\beta})^m = t_{m(c-\beta)}$. Similarly, we have either $(s_\alpha s_{\alpha'})^m = t_{-m\alpha}$ or $(s_\alpha s_{\alpha'})^m = t_{m(c-\alpha)}$. So if $(s_\beta s_{\beta'})^m = (s_\alpha s_{\alpha'})^m$ then either $t_{-m\beta} = t_{-m\alpha}$, $t_{-m\beta} = t_{m(c-\alpha)}$, $t_{m(c-\beta)} = t_{-m\alpha}$, or $t_{m(c-\beta)} = t_{m(c-\alpha)}$ so we have either $\beta = \alpha$, $\beta - \alpha = c$, or $\alpha - \beta = c$, but this is a contradiction. □

Lemma 3.4.2. *Let $\beta \in \Pi^{re,+}_{aff}$ such that $\beta < c$. Then $\sum_{\gamma\in\Pi^+} |\langle\beta,\gamma\rangle| = 2(n-1)$.*

Proof. First, assume that $\alpha_0 \notin \mathrm{supp}(\beta)$. Let $\beta = p_{i,j} = \varepsilon_i - \varepsilon_{j+1}$ such that $1 \le i < j \le n-1$ and $\gamma \in \Pi^+$. We have several cases here;

- If $\gamma = \beta$ then $\langle\beta,\gamma\rangle = 2$. If $\gamma < \beta$ then $\gamma = p_{k,l}$ such that either $i < k < l < j$ which implies $\langle\beta,\gamma\rangle = 0$, or $k = i < l < j$ or $i < k < l = j$ which implies $\langle\beta,\gamma\rangle = 1$ and there are $2(j-i)$ such γ.

- If $\mathrm{supp}(\gamma)$ and $\mathrm{supp}(\beta)$ are disconnected then $\langle\beta,\gamma\rangle = 0$.

- If $\beta+\gamma$ is a root then $\gamma = p_{k,l}$ such that either $1 \le k < l = i-1$ or $k = j+1 < l \le n-1$ and so $\langle\beta,\gamma\rangle = -1$. We have $n-j+i-2$ such γ.

- If $\beta\cap\gamma \ne \emptyset$ and $\beta\cap\gamma \ne \beta,\gamma$ then $\gamma = p_{k,l}$ such that either $1 \le k < i < l < j$ or $i < k < j < l \le n-1$ which implies $\langle\beta,\gamma\rangle = 0$.

- If $\gamma > \beta$ then $\gamma = p_{k,l}$ such that either $1 \le k < i < j < l \le n-1$ which implies $\langle\beta,\gamma\rangle = 0$, or $k = i < j < l \le n-1$ or $1 \le k < i < j = l$ which follows by $\langle\beta,\gamma\rangle = 1$ and there are $n-j+i-2$ such γ.

Thus $\sum_{\gamma\in\Pi^+}|\langle\beta,\gamma\rangle| = 2+(n-j+i-2)+2(j-i)+(n-j+i-2) = 2(n-1)$. Next, if $\alpha_0 \in \mathrm{supp}(\beta)$ then the proof is similar.

□

Lemma 3.4.3. *Let m be a positive integer and $\beta, \beta' \in \Pi_{aff}^{re,+}$ such that $\beta+\beta' = c$. Then $(s_\beta s_{\beta'})^m$ is reduced for any positive integer m. In particular, $\ell((s_\beta s_{\beta'})^m) = 2m(n-1)$.*

Proof. First, note that, since both β and β' are smaller than c we have $\ell(\beta) = 2|\mathrm{supp}(\beta)|-1$ and $\ell(\beta') = 2|\mathrm{supp}(\beta')|-1$. Also, $|\mathrm{supp}(\beta)|+|\mathrm{supp}(\beta')| = n$. So

$$\ell((s_\beta s_{\beta'})^m) \le m(\ell(\beta)+\ell(\beta')) = m(2|\mathrm{supp}(\beta)|-1+2|\mathrm{supp}(\beta')|-1) = 2m(n-1).$$

Now, assume that $\alpha_0 \notin \mathrm{supp}(\beta)$. Then we have $(s_\beta s_{\beta'})^m = t_{-m\beta}$; see the proof of Lemma 3.4.1. Moreover,

$$\ell(t_{-m\beta}) = \sum_{\gamma\in\Pi^+} |\chi(\gamma<0)+\langle -m\beta,\gamma\rangle|$$

by Equation (2.1). Here, note that, $\chi(\gamma<0) = 0$ for all $\gamma \in \Pi^+$ and $\langle -m\beta,\gamma\rangle = -m\langle\beta,\gamma\rangle$ so

$$\ell(t_{-m\beta}) = \sum_{\gamma\in\Pi^+} |-m\langle\beta,\gamma\rangle| = m\sum_{\gamma\in\Pi^+}|\langle\beta,\gamma\rangle|.$$

By Lemma 3.4.2, $\sum_{\gamma\in\Pi^+}|\langle\beta,\gamma\rangle| = 2(n-1)$. So $\ell(t_{-m\beta}) = 2m(n-1)$. Next, assume that $\alpha_0 \in \mathrm{supp}(\beta)$. Then we have $(s_\beta s_{\beta'})^m = t_{m(c-\beta)}$, see the proof of Lemma 3.4.1. By the similar arguments above, one can show that $\ell(t_{m(c-\beta)}) = 2m(n-1)$. Hence $(s_\beta s_{\beta'})^m$ is reduced and $\ell((s_\beta s_{\beta'})^m) = 2m(n-1)$. □

Remark 3.4.4. Alternatively, one can prove Lemma 3.4.3 by using the following arguments: For any $\beta^\vee \in Q^\vee$ and $w \in W$ we have $\ell(t_{\beta^\vee}) = \ell(t_{w(\beta)})$. So one may restrict to the case where $\beta^\vee$ is a dominant root. Then we have $\ell(t_{\beta^\vee}) = \langle \beta^\vee, 2\rho \rangle$ where $\rho = \sum_{i=1}^{n-1} \omega_i$ is the sum of the fundamental weights, see [8]. But note that $\langle \alpha_i^\vee, \omega_j \rangle = \delta_{ij}$ for $1 \leq i, j \leq n-1$ and if we take $\beta = \theta = \sum_{i=1}^{n-1} \alpha_i$ we get $\langle \beta^\vee, 2\rho \rangle = 2(n-1)$.

Lemma 3.4.5. *Let $\beta, \alpha \in \Pi_{aff}^{re,+}$ and $\gamma \in \Pi^+$ such that $\alpha \leq \beta < c$ and $s_\alpha(\gamma) < 0$. Then $\langle s_\alpha(\beta), \gamma \rangle \leq 0$.*

Proof. First, assume that $\alpha_0 \notin \text{supp}(\beta)$. Also, assume that $\beta = p_{i,j}$ such that $1 \leq i < j \leq n-1$. Now, let $\alpha = p_{k,l}$. Then $\beta \geq \alpha$ implies that either $1 \leq i < k < l < j \leq n-1$, $1 \leq i = k < l < j \leq n-1$ or $1 \leq i < k < l = j \leq n-1$ and $\langle \beta, \alpha \rangle \geq 0$, in either case. Now, $s_\alpha(\gamma) < 0$ implies that $\langle \alpha^\vee, \gamma \rangle = \langle \alpha, \gamma \rangle > 0$ since both α and γ are positive roots. So either $\langle \alpha, \gamma \rangle = 2$ or $\langle \alpha, \gamma \rangle = 1$. If $\langle \alpha, \gamma \rangle = 2$ then $\alpha = \gamma$ which follows by $\langle s_\alpha(\beta), \gamma \rangle = \langle \beta, s_\alpha(\gamma) \rangle = \langle \beta, s_\alpha(\alpha) \rangle = \langle \beta, -\alpha \rangle = -\langle \beta, \alpha \rangle \leq 0$. Now, if $\langle \alpha, \gamma \rangle = 1$ then $s_\alpha(\gamma) = \gamma - \alpha < 0$ and so $\gamma < \alpha$. Thus, if $\gamma = p_{s,q}$ then we either have $k < s < q < l$ or $k = s < q < l$ or $k < s < q = l$. But in either case we get $\langle \beta, \gamma \rangle \leq \langle \beta, \alpha \rangle$; see the cases in the proof of Lemma 3.4.3. So $\langle s_\alpha(\beta), \gamma \rangle = \langle \beta, s_\alpha(\gamma) \rangle = \langle \beta, \gamma - \alpha \rangle = \langle \beta, \gamma \rangle - \langle \beta, \alpha \rangle \leq 0$. Assume that β is a simple root. Then $\beta \geq \alpha \geq \gamma$ implies that $\beta = \alpha = \gamma$ which implies $\langle s_\alpha(\beta), \gamma \rangle = \langle s_\alpha(\alpha), \alpha \rangle = \langle -\alpha, \alpha \rangle = -2$.

Second, if $\alpha_0 \in \text{supp}(\beta)$ then the statement follows by similar arguments to the previous case. ∎

Lemma 3.4.6. *Let $\beta, \beta', \alpha \in \Pi_{aff}^{re,+}$ such that $\beta + \beta' = c$ and $\beta' \leq c - \alpha$. Then $(s_\beta s_{\beta'})^m s_\alpha$ is reduced for any positive integer m. In particular, $\ell((s_\beta s_{\beta'})^m s_\alpha) = 2m(n-1) + \ell(s_\alpha)$.*

Proof. First, observe that we can assume that $\alpha_0 \notin \mathrm{supp}(\beta)$ by the automorphism of the Dynkin diagram in Section 2.3. Then $(s_\beta s_{\beta'})^m = t_{-m\beta}$; see the proof of Lemma 3.4.1. So $(s_\beta s_{\beta'})^m s_\alpha = t_{-m\beta} s_\alpha$. We also have $t_{-m\beta} s_\alpha = s_\alpha t_{s_\alpha(-m\beta)}$ since $s_\alpha t_{-m\beta} s_\alpha = t_{s_\alpha(-m\beta)}$. Now, by Equation (2.1), we have

$$\begin{aligned} \ell(s_\alpha t_{s_\alpha(-m\beta)}) &= \textstyle\sum_{\gamma \in \Pi^+} |\chi(s_\alpha(\gamma) < 0) + \langle s_\alpha(-m\beta), \gamma \rangle| \\ &= \textstyle\sum_{\gamma \in \Pi^+} |\chi(s_\alpha(\gamma) < 0) - m\langle s_\alpha(\beta), \gamma \rangle|. \end{aligned}$$

Now, note that, $\chi(s_\alpha(\gamma) < 0) = 0$ if $s_\alpha(\gamma) > 0$ and $\chi(s_\alpha(\gamma) < 0) = 1$ otherwise. Thus

$$\ell(s_\alpha t_{s_\alpha(-m\beta)}) = m \sum_{s_\alpha(\gamma) > 0} |\langle s_\alpha(\beta), \gamma \rangle| + \sum_{s_\alpha(\gamma) < 0} |1 - m\langle s_\alpha(\beta), \gamma \rangle|$$

where $\gamma \in \Pi^+$. Moreover, $\beta \geq \alpha$ since $\beta' \leq c - \alpha$. Here, notice that, for $\gamma \in \Pi^+$ if $s_\alpha(\gamma) < 0$ then $\langle s_\alpha(\beta), \gamma \rangle \leq 0$ by Lemma 3.4.5, which implies that $|1 - m\langle s_\alpha(\beta), \gamma \rangle| = 1 + m|\langle s_\alpha(\beta), \gamma \rangle|$. Thus

$$\begin{aligned} \ell(s_\alpha t_{s_\alpha(-m\beta)}) &= m \textstyle\sum_{s_\alpha(\gamma) > 0} |\langle s_\alpha(\beta), \gamma \rangle| + \sum_{s_\alpha(\gamma) < 0} |1 - m\langle s_\alpha(\beta), \gamma \rangle| \\ &= m \textstyle\sum_{s_\alpha(\gamma) > 0} |\langle s_\alpha(\beta), \gamma \rangle| + \sum_{s_\alpha(\gamma) < 0} (1 + m|\langle s_\alpha(\beta), \gamma \rangle|) \\ &= m \textstyle\sum_{\gamma \in \Pi^+} |\langle s_\alpha(\beta), \gamma \rangle| + |\{\gamma \in \Pi^+ : s_\alpha(\gamma) < 0\}| \end{aligned}$$

Now, observe that, s_α can be considered as an element of W since $\alpha_0 \notin \mathrm{supp}(\alpha)$ due to the fact that $\beta \geq \alpha$, so $|\{\gamma \in \Pi^+ : s_\alpha(\gamma) < 0\}| = \ell(s_\alpha)$. Also, $\beta \geq \alpha$ implies that $\langle \beta, \alpha^\vee \rangle$ is either 0, 1, or 2. Then $s_\alpha(\beta)$ is either $\pm\beta$ or $\beta-\alpha$ and so $|s_\alpha(\beta)|$ is a positive real root which is smaller than c. Thus, by Lemma 3.4.2, we get $\sum_{\gamma\in\Pi^+}|\langle s_\alpha(\beta),\gamma\rangle| = 2(n-1)$ which follows by $\ell(s_\alpha t_{s_\alpha(-m\beta)}) = 2m(n-1)+\ell(s_\alpha)$ so $\ell((s_\beta s_{\beta'})^m s_\alpha) = 2m(n-1)+\ell(s_\alpha)$.

On the other hand, $\ell((s_\beta s_{\beta'})^m s_\alpha) \le \ell((s_\beta s_{\beta'})^m) + \ell(s_\alpha) = 2m(n-1)+\ell(s_\alpha)$ by Lemma 3.4.3. □

Lemma 3.4.7. *Let $\beta,\alpha \in \Pi_{\mathit{aff}}^{re,+}$ and $\gamma\in\Pi^+$ such that $\beta < c-\alpha < c$ and $\beta\perp\alpha$. Also, suppose that $s_\alpha(\gamma)<0$. Then $\langle s_\alpha(\beta),\gamma\rangle = 0$.*

Proof. First, note that $s_\alpha(\beta)=\beta$ since $\beta\perp\alpha$. Moreover, $\langle\gamma,\alpha^\vee\rangle>0$ since α and γ are both positive roots and $s_\alpha(\gamma)<0$. This implies that either $\gamma=\alpha$ or $\gamma<\alpha$. But in either case we have $\beta\perp\gamma$ since $\beta<c-\alpha$ and $\beta\perp\alpha$. So $\langle s_\alpha(\beta),\gamma\rangle = \langle\beta,\gamma\rangle = 0$.

□

Lemma 3.4.8. *Let $\beta,\beta',\alpha\in\Pi_{\mathit{aff}}^{re,+}$ such that $\beta+\beta'=c$. Also, suppose that $\beta'>\alpha$ and $\beta'\perp\alpha$. Then $(s_\beta s_{\beta'})^m s_\alpha$ is reduced for any positive integer m. In particular, $\ell((s_\beta s_{\beta'})^m s_\alpha) = 2m(n-1)+\ell(s_\alpha)$.*

Proof. First, note that we can assume that $\alpha_0\notin\mathrm{supp}(\alpha)$ by the automorphism of the Dynkin diagram in Section 2.3. Observe that, $\beta'>\alpha$ and $\beta'\perp\alpha$ imply that $\beta<c-\alpha$ and $\beta\perp\alpha$ since $\beta+\beta'=c$. Now, assume that $\alpha_0\notin\mathrm{supp}(\beta)$ then $(s_\beta s_{\beta'})^m = t_{-m\beta}$; see the proof of Lemma 3.4.1. So $(s_\beta s_{\beta'})^m s_\alpha = t_{-m\beta}s_\alpha$. We also have $t_{-m\beta}s_\alpha = s_\alpha t_{s_\alpha(-m\beta)}$ since

$s_\alpha t_{-m\beta} s_\alpha = t_{s_\alpha(-m\beta)}$. Now, by Equation (2.1), we have

$$\begin{aligned}\ell(s_\alpha t_{s_\alpha(-m\beta)}) &= \textstyle\sum_{\gamma\in\Pi^+} |\chi(s_\alpha(\gamma)<0) + \langle s_\alpha(-m\beta),\gamma\rangle| \\ &= \textstyle\sum_{\gamma\in\Pi^+} |\chi(s_\alpha(\gamma)<0) - m\langle s_\alpha(\beta),\gamma\rangle|.\end{aligned}$$

Now, note that, $\chi(s_\alpha(\gamma)<0)=0$ if $s_\alpha(\gamma)>0$ and $\chi(s_\alpha(\gamma)<0)=1$ otherwise. Thus

$$\ell(s_\alpha t_{s_\alpha(-m\beta)}) = m\sum_{s_\alpha(\gamma)>0} |\langle s_\alpha(\beta),\gamma\rangle| + \sum_{s_\alpha(\gamma)<0} |1-m\langle s_\alpha(\beta),\gamma\rangle|$$

where $\gamma\in\Pi^+$. Furthermore, since $\beta < c-\alpha$ and $\beta\perp\alpha$ we have $\langle s_\alpha(\beta),\gamma\rangle = 0$ if $\gamma\in\Pi^+$ and $s_\alpha(\gamma)<0$ by Lemma 3.4.7, which implies that $|1-m\langle s_\alpha(\beta),\gamma\rangle| = 1$. Thus

$$\begin{aligned}\ell(s_\alpha t_{s_\alpha(-m\beta)}) &= m\textstyle\sum_{s_\alpha(\gamma)>0} |\langle s_\alpha(\beta),\gamma\rangle| + \sum_{s_\alpha(\gamma)<0} |1-m\langle s_\alpha(\beta),\gamma\rangle| \\ &= m\textstyle\sum_{s_\alpha(\gamma)>0} |\langle s_\alpha(\beta),\gamma\rangle| + \sum_{s_\alpha(\gamma)<0}(1) \\ &= m\textstyle\sum_{\gamma\in\Pi^+} |\langle s_\alpha(\beta),\gamma\rangle| + |\{\gamma\in\Pi^+ : s_\alpha(\gamma)<0\}|\end{aligned}$$

Now, observe that, since $\alpha_0\notin \mathrm{supp}(\alpha)$ we can write $s_\alpha\in W$ so $|\{\gamma\in\Pi^+ : s_\alpha(\gamma)<0\}| = \ell(s_\alpha)$. Also, note that $\langle\beta,\alpha^\vee\rangle = 0$ since $\beta\perp\alpha$, which follows by $s_\alpha(\beta)=\beta$. Thus, we get $\sum_{\gamma\in\Pi^+} |\langle s_\alpha(\beta),\gamma\rangle| = \sum_{\gamma\in\Pi^+} |\langle\beta,\gamma\rangle|$. Now, by Lemma 3.4.2, we also have $\sum_{\gamma\in\Pi^+} |\langle\beta,\gamma\rangle| = 2(n-1)$ which follows by $\ell(s_\alpha t_{s_\alpha(-m\beta)}) = 2m(n-1)+\ell(s_\alpha)$ so $\ell((s_\beta s_{\beta'})^m s_{s_\alpha}) = 2m(n-1)+\ell(s_\alpha)$. Next, assume that $\alpha_0\in\mathrm{supp}(\beta)$ then one can prove the statement by the similar arguments above.

On the other hand, $\ell((s_\beta s_{\beta'})^m s_\alpha) \le \ell((s_\beta s_{\beta'})^m) + \ell(s_\alpha) = 2m(n-1)+\ell(s_\alpha)$ by Lemma 3.4.3. $\square$

Lemma 3.4.9. *Let* $\beta, \gamma_1, \gamma_2, ..., \gamma_k \in \Pi_{aff}^{re,+}$ *such that* $\beta < c$ *and for any* $1 \leq i, j \leq k$ *such that* $i \neq j$*, either supp*(γ_i) *and supp*(γ_j) *are disconnected or both* $\gamma_i \perp \gamma_j$ *and* γ_i, γ_j *are comparable. Also, suppose that either* $\gamma_i \leq \beta$ *or both* $\beta \cap \gamma_i = \emptyset$ *and* $\beta \perp \gamma_i$ *for any* $1 \leq i \leq k$*. Moreover, let* $\gamma \in \Pi^+$ *such that* $s_{\gamma_1} s_{\gamma_2} ... s_{\gamma_k}(\gamma) < 0$*. Then* $< s_{\gamma_k} ... s_{\gamma_2} s_{\gamma_1}(\beta), \gamma > \leq 0$*.*

Proof. First, observe that $< s_{\gamma_k} ... s_{\gamma_2} s_{\gamma_1}(\beta), \gamma > = < \beta, s_{\gamma_1} s_{\gamma_2} ... s_{\gamma_k}(\gamma) >$. Now, note that $< \beta, \gamma_i >$ is either 2, 1 or 0 since we have either $\gamma_i \leq \beta$ or both $\beta \cap \gamma_i = \emptyset$ and $\beta \perp \gamma_i$. Also, note that $s_{\gamma_1} s_{\gamma_2} ... s_{\gamma_k}(\gamma) = \gamma - \sum_{i=1}^{k} < \gamma, \gamma_i^\vee > \gamma_i$ since $\gamma_i \perp \gamma_j$ for any $1 \leq i, j \leq k$ such that $i \neq j$. Now, if $< \gamma, \gamma_i^\vee > \leq 0$ for all $i = 1, 2, ..., k$ then $s_{\gamma_1} s_{\gamma_2} ... s_{\gamma_k}(\gamma) \geq \gamma > 0$. So $< \gamma, \gamma_i^\vee >> 0$ for some i. This implies that either $< \gamma, \gamma_i^\vee > = 2$ or $< \gamma, \gamma_i^\vee > = 1$. Now, if $< \gamma, \gamma_i^\vee > = 2$ then $\gamma = \gamma_i$. But then $\gamma \perp \gamma_j$ for all $j \in \{1, 2, ..., k\} \setminus \{i\}$, which follows by $s_{\gamma_1} s_{\gamma_2} ... s_{\gamma_k}(\gamma) = -\gamma_i$. Then $< \beta, s_{\gamma_1} s_{\gamma_2} ... s_{\gamma_k}(\gamma) > = < \beta, -\gamma_i > = - < \beta, \gamma_i >$. Here, note that if $< \beta, \gamma_i > \neq 0$ then β is not perpendicular to γ_i which implies that $\gamma_i \leq \beta$ and $< \beta, \gamma_i >$ is either 2 or 1 hence we are done with this case. Now, suppose that $< \gamma, \gamma_i^\vee > = 1$. Then either $\gamma < \gamma_i$ or $\gamma_i < \gamma$. Now suppose that $\gamma < \gamma_i$. Then we have three cases;

- $\gamma \perp \gamma_j$ for all $j \in \{1, 2, ..., k\} \setminus \{i\}$ which follows by $s_{\gamma_1} s_{\gamma_2} ... s_{\gamma_k}(\gamma) = \gamma - \gamma_i < 0$. Then $< \beta, s_{\gamma_1} s_{\gamma_2} ... s_{\gamma_k}(\gamma) > = < \beta, \gamma - \gamma_i > = < \beta, \gamma > - < \beta, \gamma_i >$. If $< \beta, \gamma_i > = 2$ then $\beta = \gamma_i$ and $< \beta, \gamma > = 1$ in this case. If $< \beta, \gamma_i > = 1$ then either $< \beta, \gamma > = 1$ or $< \beta, \gamma > = 0$. Last, if $< \beta, \gamma_i > = 0$ then $< \beta, \gamma > = 0$. In all cases, we get $< \beta, \gamma > - < \beta, \gamma_i > \leq 0$.

- For an index q, $< \gamma, \gamma_q^\vee > = -1$ and $\gamma \perp \gamma_j$ for all $j \in \{1, 2, ..., k\} \setminus \{i, q\}$. Thus

$s_{\gamma_1}s_{\gamma_2}...s_{\gamma_k}(\gamma) = \gamma - \gamma_i + \gamma_q < 0$. Now, we have either supp(γ_i) and supp(γ_q) are disconnected or $\gamma_q < \gamma_i$ or $\gamma_i < \gamma_q$. Now, by the facts that $\gamma < \gamma_i$ and $<\gamma, \gamma_q^\vee>= -1$ we have to have $\gamma_q < \gamma_i$. So

$$<\beta, s_{\gamma_1}s_{\gamma_2}...s_{\gamma_k}(\gamma)>=<\beta, \gamma - \gamma_i + \gamma_q>=<\beta, \gamma> - <\beta, \gamma_i> + <\beta, \gamma_q> .$$

If $<\beta, \gamma_i>= 2$ then $\beta = \gamma_i$ which follows by $<\beta, \gamma>= 1$ and $<\beta, \gamma_q>= 0$. If $<\beta, \gamma_i>= 1$ then $<\beta, \gamma_q>= 0$ since γ_q is smaller than γ_i and $\gamma_q \perp \gamma_i$. Also, we have either $<\beta, \gamma>= 1$ or $<\beta, \gamma>= 0$ in this case since $\gamma < \gamma_i \leq \beta$. Last, if $<\beta, \gamma_i>= 0$ then $<\beta, \gamma>=<\beta, \gamma_q>= 0$ since both γ_q and γ are smaller than γ_i. Note that, in all cases, we get $<\beta, \gamma> - <\beta, \gamma_i> + <\beta, \gamma_q> \leq 0$.

- For an index $r \neq i$, $<\gamma, \gamma_q^\vee>= 1$ and $\gamma \perp \gamma_j$ for all $j \in \{1, 2, ..., k\} \setminus \{i, r\}$. Then either $\gamma < \gamma_r$ or $\gamma_r < \gamma$. But since either supp(γ_i) and supp(γ_r) are disconnected or both $\gamma_i \perp \gamma_r$ and γ_i and γ_r are comparable we have to have $\gamma_r < \gamma$ which implies that $\gamma_r < \gamma_i$. Thus $s_{\gamma_1}s_{\gamma_2}...s_{\gamma_k}(\gamma) = \gamma - \gamma_i - \gamma_r$. So

$$<\beta, s_{\gamma_1}s_{\gamma_2}...s_{\gamma_k}(\gamma)>=<\beta, \gamma - \gamma_i + \gamma_r>=<\beta, \gamma> - <\beta, \gamma_i> + <\beta, \gamma_r> .$$

If $<\beta, \gamma_i>= 2$ then $\beta = \gamma_i$ which follows by $<\beta, \gamma>= 1$ and $<\beta, \gamma_r>= 0$. If $<\beta, \gamma_i>= 1$ then $<\beta, \gamma_r>= 0$ since γ_r is smaller than γ_i and $\gamma_r \perp \gamma_i$. Also, we have either $<\beta, \gamma>= 1$ or $<\beta, \gamma>= 0$ in this case since $\gamma < \gamma_i \leq \beta$. Last, if $<\beta, \gamma_i>= 0$ then $<\beta, \gamma>=<\beta, \gamma_r>= 0$ since both γ_r and γ are smaller than γ_i. Note that, in all cases, we get $<\beta, \gamma> - <\beta, \gamma_i> + <\beta, \gamma_r> \leq 0$.

The case where $\gamma_i < \gamma$ is similar.

□

Lemma 3.4.10. *Let* $\beta, \beta' \in \Pi_{aff}^{re,+}$ *such that* $\beta + \beta' = c$ *and let* $\mathbf{d} = (d_0, d_1, ..., d_{n-1})$ *be a degree where* $d_i = 0$ *for some* i. *Also, assume that* $z_{\mathbf{d}} = s_{\gamma_1} s_{\gamma_2} ... s_{\gamma_k}$ *for some* k *where this expression is obtained in Theorem 3.1.2. If either* $\beta' \cap \gamma_i = \emptyset$ *or both* $\beta' > \gamma_i$ *and* $\beta' \perp \gamma_i$ *for any* $1 \leq i \leq k$ *then* $w = (s_\beta s_{\beta'})^m z_{\mathbf{d}}$ *is reduced for any positive integer* m. *In particular,* $\ell((s_\beta s_{\beta'})^m z_{\mathbf{d}}) = 2m(n-1) + \ell(z_{\mathbf{d}})$.

Proof. Note that by Lemma 3.4.3 we have $\ell((s_\beta s_{\beta'})^m) = 2m(n-1)$ so $\ell((s_\beta s_{\beta'})^m z_{\mathbf{d}}) \leq \ell((s_\beta s_{\beta'})^m) + \ell(z_{\mathbf{d}}) = 2m(n-1) + \ell(z_{\mathbf{d}})$. We will show that $\ell((s_\beta s_{\beta'})^m z_{\mathbf{d}}) = 2m(n-1) + \ell(z_{\mathbf{d}})$ to prove that $w = (s_\beta s_{\beta'})^m z_{\mathbf{d}}$ is reduced for any positive integer m.

First, we can suppose that $d_0 = 0$ by the automorphism of the Dynkin diagram, φ; see Section (2.3) which implies that $\alpha_0 \notin \text{supp}(\gamma_i)$ for any $i = 1, 2, ..., k$ so $\gamma_i \in \Pi^+$ for all i. Now, suppose that $\alpha_0 \notin \text{supp}(\beta)$. Then by Lemma 3.4.1, we have $(s_\beta s_{\beta'})^m = t_{-m\beta}$. So $w = (s_\beta s_{\beta'})^m z_{\mathbf{d}} = (s_\beta s_{\beta'})^m s_{\gamma_1} s_{\gamma_2} ... s_{\gamma_k} = t_{-m\beta} s_{\gamma_1} s_{\gamma_2} ... s_{\gamma_k}$. We also have $t_{-m\beta} s_{\gamma_1} s_{\gamma_2} ... s_{\gamma_k} = s_{\gamma_1} s_{\gamma_2} ... s_{\gamma_k} t_{s_{\gamma_k} ... s_{\gamma_2} s_{\gamma_1}(-m\beta)}$. Now, by Lemma 3.1 in [8], we get

$$\begin{aligned} \ell(w) &= \textstyle\sum_{\gamma \in \Pi^+} |\chi(s_{\gamma_1} ... s_{\gamma_k}(\gamma) < 0) + < s_{\gamma_k} ... s_{\gamma_1}(-m\beta), \gamma > | \\ &= \textstyle\sum_{\gamma \in \Pi^+} |\chi(s_{\gamma_1} ... s_{\gamma_k}(\gamma) < 0) - m < s_{\gamma_k} ... s_{\gamma_1}(\beta), \gamma > |. \end{aligned}$$

Now, note that $\chi(s_{\gamma_1} ... s_{\gamma_k}(\gamma) < 0) = 0$ if $s_{\gamma_1} ... s_{\gamma_k}(\gamma) > 0$ and $\chi(s_{\gamma_1} ... s_{\gamma_k}(\gamma) < 0) = 1$ otherwise. Thus

$$\ell(w) = m \sum_{s_{\gamma_1} ... s_{\gamma_k}(\gamma) > 0} | < s_{\gamma_k} ... s_{\gamma_1}(\beta), \gamma > | + \sum_{s_{\gamma_1} ... s_{\gamma_k}(\gamma) < 0} |1 - m < s_{\gamma_k} ... s_{\gamma_1}(\beta), \gamma > |$$

where $\gamma \in \Pi^+$. Moreover, either $\gamma_i \leq \beta$ or both $\beta \cap \gamma_i = \emptyset$ and $\beta \perp \gamma_i$ for any $1 \leq i \leq k$ since we have either $\beta' \cap \gamma_i = \emptyset$ or both $\beta' > \gamma_i$ and $\beta' \perp \gamma_i$ for any $1 \leq i \leq k$. Here, notice that, for $\gamma \in \Pi^+$ if $s_{\gamma_1} s_{\gamma_2} ... s_{\gamma_k}(\gamma) < 0$ then $< s_{\gamma_k} ... s_{\gamma_2} s_{\gamma_1}(\beta), \gamma > \leq 0$ by Lemma 3.4.9, which implies that $|1 - m < s_{\gamma_k} ... s_{\gamma_1}(\beta), \gamma > | = 1 + m| < s_{\gamma_k} ... s_{\gamma_1}(\beta), \gamma > |$. Hence,

$$\begin{aligned}
\ell(w) &= m \textstyle\sum_{s_{\gamma_1} ... s_{\gamma_k}(\gamma) > 0} | < s_{\gamma_k} ... s_{\gamma_1}(\beta), \gamma > | + \sum_{s_{\gamma_1} ... s_{\gamma_k}(\gamma) < 0} |1 - m < s_{\gamma_k} ... s_{\gamma_1}(\beta), \gamma > | \\
&= m \textstyle\sum_{s_{\gamma_1} ... s_{\gamma_k}(\gamma) > 0} | < s_{\gamma_k} ... s_{\gamma_1}(\beta), \gamma > | + \sum_{s_{\gamma_1} ... s_{\gamma_k}(\gamma) < 0} (1 + m| < s_{\gamma_k} ... s_{\gamma_1}(\beta), \gamma > |) \\
&= m \textstyle\sum_{\gamma \in \Pi^+} | < s_{\gamma_k} ... s_{\gamma_1}(\beta), \gamma > | + |\{\gamma \in \Pi^+ : s_{\gamma_1} ... s_{\gamma_k}(\gamma) < 0\}|.
\end{aligned}$$

Here, the fact that $\gamma_i \leq \beta$ for all i and the assumptions on γ_i's force us to have $s_{\gamma_k} ... s_{\gamma_1}(\beta) = \pm\beta$, $s_{\gamma_k} ... s_{\gamma_1}(\beta) = \beta - \gamma_i$ or $s_{\gamma_k} ... s_{\gamma_1}(\beta) = \beta - \gamma_i - \gamma_q$ for some $1 \leq i, q \leq k$. In all these cases $|s_{\gamma_k} ... s_{\gamma_1}(\beta)|$ is a positive real root which is smaller than c. Thus, we get $\sum_{\gamma \in \Pi^+} | < s_{\gamma_k} ... s_{\gamma_1}(\beta), \gamma > | = 2(n-1)$ by Lemma 3.4.2. Furthermore, $|\{\gamma \in \Pi^+ : s_{\gamma_1} ... s_{\gamma_k}(\gamma) < 0\}| = \ell(s_{\gamma_1} ... s_{\gamma_k}) = \ell(z_{\mathbf{d}})$ since $\gamma_i \in \Pi^+$ for all i.

If $\alpha_0 \in \text{supp}(\beta)$ then the proof is similar.

□

Chapter 4

The Moment Graph and Curve Neighborhoods

In this chapter we recall some basic facts about the affine flag manifold of type $A_{n-1}^{(1)}$ and give the definition of the moment graph and the (combinatorial) curve neighborhoods. We refer to[7] especially §12 and §13 for further details. Let G be the Lie group $SL_n(\mathbb{C})$. Let $B \subset G$ be the Borel subgroup, the set of upper triangular matrices. Now, let $G(\mathbb{C}[t, t^{-1}])$ be the group of Laurent polynomial loops from $\mathbb{C}^*$ to G and $\mathcal{G} := \mathbb{C}^* \ltimes G(\mathbb{C}[t, t^{-1}])$ where $\mathbb{C}^*$ acts by loop rotation. Let $\mathcal{B}$ be the standard Iwahori subgroup of $\mathcal{G}$ determined by B. The affine flag manifold in type $A_{n-1}^{(1)}$ is given by $\mathcal{X} := \mathcal{G}/\mathcal{B}$. The Schubert varieties and the curve neighborhoods of Schubert varieties in this context were defined in [9, §5]. In the following we give a different, but equivalent definition, based on the moment graph, which generalizes the observations from [5, §5.2].

The (undirected) **moment graph** for $\mathcal{X}$ is the graph given by the following data:

- The set of V of **vertices** is the group W_{aff};
- Let $u, v \in V$ be vertices. Then there is an edge from u to v iff there exists an affine root α such that $v = us_{\alpha}$. We denote this situation by

  $$u \xrightarrow{\alpha} v$$

  and we say that the **degree** of this edge is α.

A *chain* between u and v in the moment graph is a succession of adjacent edges starting with u and ending with v

$$\pi : u = u_0 \xrightarrow{\beta_0} u_1 \xrightarrow{\beta_1} \ldots \xrightarrow{\beta_{k-2}} u_{k-1} \xrightarrow{\beta_{k-1}} u_k = v.$$

The *degree* of the chain π is $\deg(\pi) = \beta_0 + \beta_1 + \ldots + \beta_{k-1}$. A chain is called *increasing* if $\ell(u_i) > \ell(u_{i-1})$ for all i. Define the (Bruhat) partial ordering on the elements of W_{aff} by $u < v$ iff there exists an increasing chain starting with u and ending with v.

A **degree** is a tuple of nonnegative integers $\mathbf{d} = (d_0, \ldots, d_{n-1})$. Notice that it has n components, corresponding to the n affine simple roots $\alpha_0, \ldots, \alpha_{n-1}$. There is a natural partial order on degrees: If $\mathbf{d} = (d_0, \ldots, d_{n-1})$ and $\mathbf{d'} = (d'_0, \ldots, d'_{n-1})$ then $\mathbf{d} \geq \mathbf{d'}$ iff $d_i \geq d'_i$ for all $i \in \{0, 1, \ldots, n-1\}$.

Definition 4.0.1. *Fix a degree* $\mathbf{d}$ *and* $u \in W_{\textit{aff}}$. *The* (**combinatorial**) **curve neighborhood** *is the set* $\Gamma_{\mathbf{d}}(u)$ *consisting of elements* $v \in W_{\textit{aff}}$ *such that:*

1) There exists a chain of some degree $\mathbf{d}' \leq \mathbf{d}$ *from* $u' \leq u$ *to* v *in the moment graph of* $\mathcal{X}$;

2) The elements v *are maximal among all of those satisfying the condition in* 1).

Remark 4.0.2. Let $id \xrightarrow{\beta_1} s_{\beta_1} \xrightarrow{\beta_2} s_{\beta_1}s_{\beta_2}... \xrightarrow{\beta_q} w = s_{\beta_1}s_{\beta_2}...s_{\beta_q}$ be a path in the moment graph such that $\sum_{i=1}^{q} \beta_i < \mathbf{d}$. Then there are some $\beta_{q+1}, \beta_{q+2}, ..., \beta_p \in \Pi_{\text{aff}}^{\text{re},+}$ such that $\sum_{i=1}^{p} \beta_i = \mathbf{d}$. Note that, by Corollary 3.3.2 we can assume that $\beta_i < c$ for all i. Also, observe that

$$w = s_{\beta_1}s_{\beta_2}...s_{\beta_q} \leq s_{\beta_1} \cdot s_{\beta_2} \cdot ... \cdot s_{\beta_q} \leq s_{\beta_1} \cdot s_{\beta_2} \cdot ... \cdot s_{\beta_q} \cdot s_{\beta_{q+1}} \cdot s_{\beta_{q+2}} \cdot ... \cdot s_{\beta_p}.$$

Now, if one can show that $s_{\beta_1} \cdot s_{\beta_2} \cdot ... \cdot s_{\beta_q} \cdot s_{\beta_{q+1}} \cdot s_{\beta_{q+2}} \cdot ... \cdot s_{\beta_p} \leq u$ for some $u \in W_{\text{aff}}$ such that u can be reached by a path of degree at most $\mathbf{d}$ that starts with id then it will follow that $u \in \Gamma_{\mathbf{d}}(id)$.

Chapter 5

Calculation of the Curve Neighborhoods

In this chapter we will prove our results. In Section 5.7 we will state Theorem 5.7.1 which implies that it is sufficient to calculate $\Gamma_{\mathbf{d}}(id)$ to compute $\Gamma_{\mathbf{d}}(w)$ for any given degree $\mathbf{d}$ and $w \in W_{\text{aff}}$. For that reason we will mainly focus on the (combinatorial) curve neighborhood of the identity element at some degree $\mathbf{d}$.

5.1 Curve Neighborhood $\Gamma_{\mathbf{d}}(id)$ for Finite Degrees

Let $\mathbf{d} = (d_0, d_1, ..., d_{n-1}) \in Q_{\text{aff}}$. If $d_0 = 0$ then $\mathbf{d} \in Q$. If $d_0 \neq 0$ but $d_i = 0$ for some $i \neq 0$ then for an integer $1 \leq k \leq n-1$ we will have $\varphi^k(\mathbf{d}) = \mathbf{d}'$ where $\mathbf{d}' = (d'_0, d'_1, ..., d'_{n-1})$ such that $d'_0 = 0$ and φ is the automorphism of the Dynkin diagram; see Section (2.3). So we will consider $\mathbf{d}$ as an element of the finite root lattice, Q, whenever $d_i = 0$ for some i.

Theorem 5.1.1. *Let $\mathbf{d} = (d_0, d_1, ..., d_{n-1})$ be a degree such that $d_i = 0$ for some i. Then $\Gamma_{\mathbf{d}}(id) = \{z_{\mathbf{d}}\}$.*

Proof. Here, we are in the finite case since $d_i = 0$ for some i. So we get the conclusion by the finite case, see [5]. □

Example 5.1.2. Let W_{aff} be the affine Weyl group associated to $A_6^{(1)}$ and $\mathbf{d} = (5, 0, 2, 2, 3, 0, 4)$. Then by Theorem 5.1.1, we get $\Gamma_{\mathbf{d}}(id) = \{z_{\mathbf{d}}\}$ where $z_{\mathbf{d}} = s_{\alpha_0+\alpha_6} s_{\alpha_2+\alpha_3+\alpha_4} s_{\alpha_3}$, see Example 3.1.3.

Theorem 5.1.3. *Let $\alpha \in \Pi_{aff}^{re,+}$ such that $\alpha < c$. Then $\Gamma_\alpha(id) = \{s_\alpha\}$.*

Proof. Let $\alpha = \sum_{i=0}^{n-1} a_i \alpha_i$. Then $a_i = 0$ for some i since $\alpha < c$ and $\alpha \in \Pi_{\text{aff}}^{\text{re},+}$. So $\Gamma_\alpha(id) = \{z_\alpha\}$. Now, observe that $z_\alpha = s_\alpha$ by definition. □

Example 5.1.4. Assume that the affine Weyl group is associated to $A_4^{(1)}$ and $\alpha = \alpha_0 + \alpha_4$. Then $\Gamma_\alpha(id) = \{s_{\alpha_0+\alpha_4}\}$ by Theorem 5.1.3.

Corollary 5.1.5. *Let $\beta \in \Pi_{aff}^{re,+}$ such that $\beta < c$. If $\sum_{i=1}^{r} \beta_i \leq \beta$ for some $\beta_i \in \Pi_{aff}^{re,+}$ then $s_{\beta_1} s_{\beta_2} ... s_{\beta_r} \leq s_\beta$.*

Proof. Let $w = s_{\beta_1} s_{\beta_2} ... s_{\beta_r}$. Then w can be reached by a chain of degree $\sum_{i=1}^{r} \beta_i = \beta$ that starts with the identity element in the moment graph. Thus $w \leq s_\beta$ since $\Gamma_\beta(id) = \{s_\beta\}$, by Theorem 5.1.3.

□

5.2 Curve Neighborhood $\Gamma_c(id)$

Here we will consider one of our basic yet crucial results. The result will play a key role in proving the rest of the results in this thesis.

Theorem 5.2.1. *We have*

1) $\Gamma_c(id) = \{t_\gamma : \gamma \in \Pi\}$.

2) $|\Gamma_c(id)| = n(n-1)$.

3) For all $w \in \Gamma_c(id)$, $\ell(w) = 2(n-1)$.

Proof. 1) Let $id \xrightarrow{\beta_1} s_{\beta_1} \xrightarrow{\beta_2} s_{\beta_1}s_{\beta_2}... \xrightarrow{\beta_r} w = s_{\beta_1}s_{\beta_2}...s_{\beta_r}$ be a path in the moment graph such that $\sum_{i=1}^r \beta_i \leq c$. Note that, by Remark 4.0.2 we can assume that $w = s_{\beta_1} \cdot s_{\beta_2} \cdot ... \cdot s_{\beta_r}$. where $\sum_{i=1}^r \beta_i = c$ and $\beta_i < c$ for all i. Now, note that $\sum_{i=2}^r \beta_i = c - \beta_1$ is an affine positive real root. Moreover, $s_{\beta_2} \cdot s_{\beta_3} \cdot ... \cdot s_{\beta_r} \leq s_{c-\beta_1}$ by Corollary 3.2.2. Thus $w = s_{\beta_1} \cdot (s_{\beta_2} \cdot ... \cdot s_{\beta_r}) \leq s_{\beta_1} \cdot s_{c-\beta_1}$. But by Lemma 3.4.3, $s_{\beta_1}s_{c-\beta_1}$ is reduced so $s_{\beta_1} \cdot s_{c-\beta_1} = s_{\beta_1}s_{c-\beta_1}$. We also know that $s_{\beta_1}s_{c-\beta_1} = t_\gamma$ for some $\gamma \in \Pi$, see the proof of Lemma 3.4.1.

2) Let $\beta, \beta', \mu, \mu' \in \Pi_{\text{aff}}^{\text{re},+}$ such that $\mu + \mu' = \beta + \beta' = c$. Then $s_\beta s_{\beta'} \neq s_\mu s_{\mu'}$ if $\beta \neq \mu$ by Lemma 3.4.1. Also, note that $|\{\beta \in \Pi_{\text{aff}}^{\text{re},+} : \beta < c\}| = n(n-1)$.

3) Let $w = s_\beta s_{\beta'} \in \Gamma_c(id)$ for some positive real roots, β, β' such that $\beta + \beta' = c$. Then $\ell(w) = 2(n-1)$, by Lemma 3.4.3. ☐

Remark 5.2.2. Note that, $t_{\beta'} = s_\beta s_{\beta'}$ if $\alpha_0 \in \text{supp}(\beta)$ and $t_{\beta'-c} = s_\beta s_{\beta'}$ if $\alpha_0 \notin \text{supp}(\beta)$, where $\beta, \beta' \in \Pi_{\text{aff}}^{\text{re},+}$ such that $\beta + \beta' = c$; see the proof of Lemma 3.4.1. Also,

$$\{\beta' : \beta' \in \Pi_{\text{aff}}^{\text{re},+},\, \beta' < c,\, \alpha_0 \notin \text{supp}(\beta')\} \sqcup \{\beta' - c : \beta' \in \Pi_{\text{aff}}^{\text{re},+},\, \beta' < c,\, \alpha_0 \in \text{supp}(\beta')\} = \Pi.$$

Thus we have

$$\Gamma_c(id) = \{s_\beta s_{\beta'} : \beta, \beta' \in \Pi_{\text{aff}}^{\text{re},+} \text{ such that } \beta + \beta' = c\}. \tag{5.1}$$

Example 5.2.3. Suppose that the affine Weyl group W_{aff} is associated to $A_2^{(1)}$. Then

$$\Gamma_c(id) = \{t_{\alpha_1}, t_{\alpha_2}, t_{\alpha_1+\alpha_2}, t_{-\alpha_1}, t_{-\alpha_2}, t_{-(\alpha_1+\alpha_2)}\}.$$

Moreover, for any $w \in \Gamma_c(id)$, $\ell(w) = 2(n-1) = 2 \cdot 2 = 4$.

5.3 Curve Neighborhood $\Gamma_{c+\alpha}(id)$

Here, we will state a result which inspires us to calculate the most general case. First, we will have some preliminary lemmas.

Lemma 5.3.1. *let $\alpha < c$ be a positive real root. Then $s_\alpha \cdot s_\alpha \cdot s_{c-\alpha} \leq s_\alpha \cdot s_{c-\alpha} \cdot s_\alpha$ and $s_{c-\alpha} \cdot s_\alpha \cdot s_\alpha \leq s_\alpha \cdot s_{c-\alpha} \cdot s_\alpha$.*

Proof. First, assume that $\alpha = p_{i,i} = \alpha_i$ for some i. Then $s_\alpha = s_i$ and since $s_i \cdot s_i = s_i$ we get

$$s_\alpha \cdot s_\alpha \cdot s_{c-\alpha} = s_i \cdot s_i \cdot s_{c-\alpha} = s_i \cdot s_{c-\alpha} \leq s_i \cdot s_{c-\alpha} \cdot s_i = s_\alpha \cdot s_{c-\alpha} \cdot s_\alpha.$$

Now, assume that $\alpha = p_{i,j}$ where $i \neq j$. Then $s_\alpha = s_i \cdot s_{p_{i,j}-\alpha_i} \cdot s_i$ and $s_{p_{i,j}-\alpha_i} = s_j \cdot s_{p_{i,j}-\alpha_i-\alpha_j} \cdot s_j$. Note that $s_\alpha \cdot s_i = s_\alpha$ and similarly $s_\alpha \cdot s_j = s_\alpha$. Also, by Lemma 3.2.1, s_α Hecke commutes with $s_{p_{i,j}-\alpha_i}$ and $s_{p_{i,j}-\alpha_i-\alpha_j}$ since $p_{i,j}-\alpha_i<\alpha$ and $p_{i,j}-\alpha_i-\alpha_j<\alpha$. We also have $s_{p_{i,j}-\alpha_i-\alpha_j}<s_\alpha$ by Corollary 5.1.5. Furthermore, $\mathrm{supp}(p_{i,j}-\alpha_i-\alpha_j)$ and $\mathrm{supp}(c-p_{i,j})$ are disconnected so $s_{p_{i,j}-\alpha_i-\alpha_j}$ Hecke commutes with $s_{c-p_{i,j}}$. Thus,

$$\begin{aligned}
s_\alpha\cdot s_\alpha\cdot s_{c-\alpha} &= s_\alpha\cdot s_i\cdot s_{p_{i,j}-\alpha_i}\cdot s_i\cdot s_{c-\alpha}=s_\alpha\cdot s_{p_{i,j}-\alpha_i}\cdot s_i\cdot s_{c-\alpha}\\
&= s_{p_{i,j}-\alpha_i}\cdot s_\alpha\cdot s_i\cdot s_{c-\alpha}=s_{p_{i,j}-\alpha_i}\cdot s_\alpha\cdot s_{c-\alpha}\\
&= s_j\cdot s_{p_{i,j}-\alpha_i-\alpha_j}\cdot s_j\cdot s_\alpha\cdot s_{c-\alpha}=s_j\cdot s_{p_{i,j}-\alpha_i-\alpha_j}\cdot s_\alpha\cdot s_{c-\alpha}\\
&= s_j\cdot s_\alpha\cdot s_{p_{i,j}-\alpha_i-\alpha_j}\cdot s_{c-\alpha}=s_\alpha\cdot s_{p_{i,j}-\alpha_i-\alpha_j}\cdot s_{c-\alpha}\\
&= s_\alpha\cdot s_{c-\alpha}\cdot s_{p_{i,j}-\alpha_i-\alpha_j}\leq s_\alpha\cdot s_{c-\alpha}\cdot s_\alpha.
\end{aligned}$$

The proof of the inequality $s_{c-\alpha}\cdot s_\alpha\cdot s_\alpha\leq s_\alpha\cdot s_{c-\alpha}\cdot s_\alpha$ is similar. □

Lemma 5.3.2. *Let $w=s_{\beta_1}\cdot s_{\beta_2}\cdot\ldots\cdot s_{\beta_r}$ where $\beta_i\in\Pi_{aff}^{re,+}$ and $\beta_i<c$ for all i. Also, assume that $\mathbf{d}=\sum_{i=1}^r\beta_i=\mathbf{d}'+\mathbf{d}''$ where $\mathbf{d}'$ is the biggest degree that $0<\mathbf{d}'\leq c$ and $\mathbf{d}''=\mathbf{d}-\mathbf{d}'>0$ may or may not be the degree of a root. Then there are some positive real roots, $\beta_1',\beta_2',\ldots,\beta_{r'}'$ such that $w\leq s_{\beta_1'}\cdot s_{\beta_2'}\cdot\ldots\cdot s_{\beta_{r'}'}$ where $\sum_{i=1}^t\beta_i'=\mathbf{d}'$ and $\sum_{i=t+1}^{r'}\beta_i'=\mathbf{d}''$ for some t and $\beta_i'<c$ for all i.*

Proof. First, assume that there is an integer b such that $\sum_{i=1}^b\beta_i=\mathbf{d}'$. Then $\sum_{i=b+1}^{r'}\beta_i'=\mathbf{d}''$ so we can take $t=b$ and $\beta_i=\beta_i'$ for all i. Now, suppose that for any integer b such that $1\leq b\leq r$ we have $\sum_{i=1}^b\beta_i\neq\mathbf{d}'$. We will prove the statement by induction on $\mathbf{d}$. Here, for the base case we have $\mathbf{d}=2\beta_i$, for some simple root β_i. So $\mathbf{d}'=\mathbf{d}''=\beta_i$. Note that, if $w=s_{\beta_1}\cdot s_{\beta_2}\cdot\ldots\cdot s_{\beta_r}$ where $\beta_i\in\Pi_{\mathrm{aff}}^{\mathrm{re},+}$ and $\beta_i<c$ for all i such that $\sum_{i=1}^r\beta_i=d=2\beta_i$ then we have to have $w=s_{\beta_i}\cdot s_{\beta_i}$. But, notice that, w satisfies the statement, so we are done

with the base case. Now assume that the statement is true for all degrees that are smaller than $\mathbf{d}$. We will prove that it is also true for $\mathbf{d}$.

1) Suppose that $\beta_r < \mathbf{d}''$. Then since $\mathbf{d}-\beta_r < \mathbf{d}$ and $\mathbf{d}-\beta_r = \mathbf{d}'+\mathbf{d}''-\beta_r$ where $\mathbf{d}''-\beta_r > 0$ by the induction assumption we have some real positive roots, $\beta'_1, \beta'_2, ..., \beta'_{r'}$, such that

$$w = (s_{\beta_1} \cdot s_{\beta_2} \cdot ... \cdot s_{\beta_{r-1}}) \cdot s_{\beta_r} \leq (s_{\beta'_1} \cdot s_{\beta'_2} \cdot ... \cdot s_{\beta'_{r'}}) \cdot s_{\beta_r}$$

where $\sum_{i=1}^{t} \beta'_i = \mathbf{d}'$ and $\sum_{i=t+1}^{r'} \beta'_i = \mathbf{d}'' - \beta_r$ for some t and $\beta'_i < c$ for all i. This implies that we are done with this case since $\sum_{i=1}^{t} \beta'_i = \mathbf{d}'$ and $(\sum_{i=t+1}^{r'} \beta'_i) + \beta_r = \mathbf{d}''$.

2) Assume that $\beta_1 < \mathbf{d}''$. Then again since $\mathbf{d} - \beta_1 < \mathbf{d}$ and $\mathbf{d} - \beta_1 = \mathbf{d}' + \mathbf{d}'' - \beta_1$ where $\mathbf{d}'' - \beta_1 > 0$ by induction assumption we have some real positive roots, $\beta'_1, \beta'_2, ..., \beta'_{r'}$ such that

$$w = s_{\beta_1} \cdot (s_{\beta_2} \cdot ... \cdot s_{\beta_{r-1}} \cdot s_{\beta_r}) \leq s_{\beta_1} \cdot (s_{\beta'_1} \cdot s_{\beta'_2} \cdot ... \cdot s_{\beta'_{r'}})$$

where $\sum_{i=1}^{t} \beta'_i = \mathbf{d}'$ and $\sum_{i=t+1}^{r'} \beta'_i = \mathbf{d}'' - \beta_1$ for some t and $\beta'_i < c$ for all i. Now, observe that $\beta'_{r'} < \mathbf{d}'' - \beta_1 < \mathbf{d}''$. Thus, this case follows by the previous case.

3) Suppose that $\beta_1 \cap \mathbf{d}'' = \emptyset$. Then the statement will follow by the same arguments in the previous cases.

4) Assume that $\beta_i \cap \beta_j \neq \emptyset$ for some i, j such that $2 \leq i < j \leq r$. Then $\sum_{i=2}^{r} \beta_i = \mathbf{d} - \beta_1 < \mathbf{d}$ can be written as $\mathbf{D} + \mathbf{D}'$ where $\mathbf{D}'$ is the biggest degree such that $0 < \mathbf{D}' \leq \mathbf{d}' \leq c$ and $0 < \mathbf{D}'' < \mathbf{d}''$. Thus by the induction assumption there are some real positive

roots, $\beta'_1, \beta'_2, ..., \beta'_{r'}$, such that

$$s_{\beta_2} \cdot s_{\beta_3} \cdot ... \cdot s_{\beta_r} \leq s_{\beta'_1} \cdot s_{\beta'_2} \cdot ... \cdot s_{\beta'_{r'}}$$

where $\sum_{i=1}^{t} \beta'_i = \mathbf{D}'$ and $\sum_{i=t+1}^{r'} \beta'_i = \mathbf{D}''$ for some t and $\beta'_i < c$ for all i. This follows by

$$w = s_{\beta_1} \cdot (s_{\beta_2} \cdot ... \cdot s_{\beta_r}) \leq s_{\beta_1} \cdot (s_{\beta'_1} \cdot s_{\beta'_2} \cdot ... \cdot s_{\beta'_{r'}})$$

Note that $\beta'_{r'} \leq \mathbf{D}'' \leq \mathbf{d}''$ thus we are done with this case.

5) Suppose that $\beta_i \cap \beta_j = \emptyset$ for any i, j such that $2 \leq i < j \leq r$. This implies that $\sum_{i=2}^{r} \beta_i \leq c$ so $\mathbf{d}'' \leq \beta_1$ and $\mathbf{d} < 2c$. We have two main cases here:

5a) Assume that $\sum_{i=2}^{r} \beta_i < c$. Now, for any i such that $2 \leq i \leq r-1$ if $\beta_i + \beta_{i+1}$ is a root then by Lemma 3.2.1 we have $s_{\beta_i} \cdot s_{\beta_{i+1}} \leq s_{\beta_i + \beta_{i+1}}$. So if we repeatedly combine consecutive roots in $s_{\beta_2} \cdot ... \cdot s_{\beta_r}$ which add up to a root we can assume that no two consecutive roots in the multiplication add up to a root. By the same lemma, this implies that any consecutive roots of the multiplication Hecke commute. So, we can assume that for any i, j such that $2 \leq i, j \leq r$, $s_{\beta_i} \cdot s_{\beta_j} = s_{\beta_j} \cdot s_{\beta_i}$. Thus, we can also suppose that $\beta_1 \cap \beta_2 \neq \emptyset$. Now, if $\beta_1 \cap \beta_2 \leq \mathbf{d}''$ is not a root and $\beta_1 \cap \beta_2 = \gamma_1 + \gamma_2$ for some positive real roots, γ_1, γ_2 we will have $s_{\beta_1} \cdot s_{\beta_2} = s_{\gamma_2} \cdot s_{\gamma_1} \cdot s_{\beta_1 - \gamma_1 - \gamma_2} \cdot s_{\beta_2}$ by Lemma 3.2.1. So we get

$$w = (s_{\beta_1} \cdot s_{\beta_2}) \cdot s_{\beta_3} \cdot ... \cdot s_{\beta_r} \leq (s_{\gamma_2} \cdot s_{\gamma_1} \cdot s_{\beta_1 - \gamma_1 - \gamma_2} \cdot s_{\beta_2}) \cdot s_{\beta_3} \cdot ... \cdot s_{\beta_r}.$$

Note that $\gamma_2 < \mathbf{d}''$ so we are done with this case by case 2) above. Now assume that $\beta_1 \cap \beta_2 \leq \mathbf{d}''$ is a root. Let $\gamma := \beta_1 \cap \beta_2$. Then by Lemma 3.2.1 we have $s_{\beta_1} \cdot s_{\beta_2} = s_\gamma \cdot s_{\beta_1 - \gamma} \cdot s_{\beta_2}$. So,

$$w = (s_{\beta_1} \cdot s_{\beta_2}) \cdot s_{\beta_3} \cdot ... \cdot s_{\beta_r} \leq (s_\gamma \cdot s_{\beta_1-\gamma} \cdot s_{\beta_2}) \cdot s_{\beta_3} \cdot ... \cdot s_{\beta_r}.$$

Now, if $\gamma < \mathbf{d}''$ then again we are done by case 2) above. Assume that $\gamma = \mathbf{d}''$ which implies that for any $i \geq 3$ we have $\beta_1 \cap \beta_i = \emptyset$ and $\mathbf{d}'' \cap \beta_i = \emptyset$. Here, we will consider the multiplication, $s_{\beta_1} \cdot s_{\beta_2} \cdot s_{\beta_3} \cdot ... \cdot s_{\beta_r}$. By Hecke commutation, we can assume that s_{β_2} is the last reflection of the multiplication. Now, if the sum of β_1 and β_3 is not a root then they will Hecke commute by Lemma 3.2.1 and we will have $w = s_{\beta_1} \cdot s_{\beta_3} \cdot s_{\beta_4} \cdot ... \cdot s_{\beta_r} \cdot s_{\beta_2} = s_{\beta_3} \cdot s_{\beta_1} \cdot s_{\beta_4} \cdot ... \cdot s_{\beta_r} \cdot s_{\beta_2}$. But this implies that we are done with this case by case 3) above since $\beta_3 \cap \mathbf{d}'' = \emptyset$. Now, if the sum is a root then we will replace $s_{\beta_1} \cdot s_{\beta_3}$ with $s_{\beta_1+\beta_3}$ by the same lemma to get $w = s_{\beta_1} \cdot s_{\beta_3} \cdot s_{\beta_4} \cdot ... \cdot s_{\beta_r} \cdot s_{\beta_2} \leq s_{\beta_1+\beta_3} \cdot s_{\beta_4} \cdot ... \cdot s_{\beta_r} \cdot s_{\beta_2}$. Next, we will apply the same argument to $\beta_1 + \beta_3$ and β_4 etc. So either we will be done by case 3) above that is because w can be written in a way that the first root which appears in it is s_{β_i} such that $\beta_i \cap \mathbf{d}'' = \emptyset$ or we will get $w = s_{\beta_1} \cdot s_{\beta_3} \cdot s_{\beta_4} \cdot ... \cdot s_{\beta_r} \cdot s_{\beta_2} \leq s_{\beta_1+\beta_3+\beta_4+...+\beta_r} \cdot s_{\beta_2}$. Last, note that, $(\beta_1 + \beta_3 + \beta_4 + ... + \beta_r) \cap \beta_2 = \gamma = \mathbf{d}''$. By Lemma 3.2.1 we can write $s_{\beta_1+\beta_3+\beta_4+...+\beta_r} \cdot s_{\beta_2} = s_{\beta_1+\beta_3+\beta_4+...+\beta_r} \cdot s_{\beta_2-\gamma} \cdot s_\gamma$. Hence, we have done with this case as well.

5b) Suppose that $\sum_{i=2}^{r} \beta_i = c$ then $\mathbf{d}' = c$ and $\beta_1 = \mathbf{d}''$. Now, note that, by Equation 5.1 there is a positive real root, $\beta < c$ such that $s_{\beta_2} \cdot s_{\beta_3} \cdot ... \cdot s_{\beta_r} \leq s_\beta \cdot s_{c-\beta}$. So we get

$$w = s_{\beta_1} \cdot (s_{\beta_2} \cdot \ldots \cdot s_{\beta_r}) \leq s_{\beta_1} \cdot s_\beta \cdot s_{c-\beta}.$$

Here, we can assume that $w = s_{\beta_1} \cdot s_\beta \cdot s_{c-\beta}$;

i) If $\beta_1 \cap \beta = \emptyset$ then $\beta_1 \leq c - \beta$. Now, first assume that $\beta_1 + \beta = c$ then $c - \beta = \beta_1 = \mathbf{d}''$ and this implies that w is already in the desired form. Second, if $\beta_1 + \beta < c$ is not a root then by Lemma 3.2.1 we have $s_{\beta_1} \cdot s_\beta = s_\beta \cdot s_{\beta_1}$. Also, by the same lemma we have $s_{\beta_1} \cdot s_{c-\beta} = s_{c-\beta} \cdot s_{\beta_1}$ since $\beta_1 \leq c - \beta$. Hence,

$$w = s_{\beta_1} \cdot s_\beta \cdot s_{c-\beta} = s_\beta \cdot s_{c-\beta} \cdot s_{\beta_1}.$$

So we are done with this case as well. Last, suppose that $\beta_1 + \beta < c$ is a root then by Lemma 3.2.1 we have $s_{\beta_1} \cdot s_\beta \leq s_{\beta_1+\beta}$. So we get

$$w = s_{\beta_1} \cdot s_\beta \cdot s_{c-\beta} \leq s_{\beta_1+\beta} \cdot s_{c-\beta}.$$

Now, note that $(\beta_1 + \beta) \cap (c - \beta) = \beta_1$ and by the same lemma we have $s_{\beta_1+\beta} \cdot s_{c-\beta} = s_{\beta_1+\beta} \cdot s_{c-\beta-\beta_1} \cdot s_{\beta_1}$. Thus, the statement follows.

ii) If $\beta_1 \cap \beta \neq \emptyset$ then we have several cases here; first, suppose if $\beta_1 = \beta$ then by Lemma 5.3.1 we have

$$w = s_{\beta_1} \cdot s_\beta \cdot s_{c-\beta} = s_{\beta_1} \cdot s_{\beta_1} \cdot s_{c-\beta_1} \leq s_{\beta_1} \cdot s_{c-\beta_1} \cdot s_{\beta_1}.$$

Thus, we are done with case. Next, if $\beta_1 \neq \beta$ then either $\beta_1 < \beta$, $\beta_1 > \beta$ or β_1 is incomparable with β. Now, if $\beta_1 < \beta$ or $\beta_1 > \beta$ then s_{β_1} and s_β will

Hecke commute by Lemma 3.2.1 so have $w = s_{\beta_1} \cdot s_\beta \cdot s_{c-\beta} = s_\beta \cdot s_{\beta_1} \cdot s_{c-\beta}$. Now, if $\beta_1 < \beta$ then $\beta_1 + c - \beta < c$ so we are done with this case by case 5a) above. Now, if $\beta_1 > \beta$ then $\beta_1 \cap (c-\beta) \neq \emptyset$ so the statement follows by case 4) above. Last, assume that β_1 is incomparable with β. Then β_1 intersects with both β and $c-\beta$ which implies that $\beta_1 \cap \beta < \mathbf{d}''$. Then by Lemma 3.2.1 we can replace the multiplication, $s_{\beta_1} \cdot s_\beta$ in w with $s_{\gamma_2} \cdot s_{\gamma_1} \cdot s_{\beta_1-\gamma_1-\gamma_2} \cdot s_\beta$ if $\beta_1 \cap \beta$ is not a root and $\beta_1 \cap \beta = \gamma_1 + \gamma_2$ for some positive real roots, γ_1, γ_2 and with $s_\gamma \cdot s_{\beta_1-\gamma} \cdot s_\beta$ if $\beta_1 \cap \beta$ is a root and $\beta_1 \cap \beta = \gamma$. So, the first root of the permutation, w is either γ_2 or γ, and both roots are strictly smaller than $\mathbf{d}''$. But then the statement follows by case 2) above.

□

Lemma 5.3.3. *Let μ, μ' and $\alpha \in \Pi_{aff}^{re,+}$ such that $\mu + \mu' = c$ and $\alpha < c$. Then there are some positive real roots β and β' such that $s_\mu \cdot s_{\mu'} \cdot s_\alpha \leq s_\beta \cdot s_{\beta'} \cdot s_\alpha$ where $\beta + \beta' = c$ and either $\beta' \leq c - \alpha$ or both $\beta' > \alpha$ and $\beta' \perp \alpha$.*

Proof. We have three cases;

1) If $\mu' \leq c - \alpha$ then take $\beta' = \mu'$.

2) If $\mu' > c - \alpha$ then $\mu' \cap \alpha \neq \emptyset$. Note that, $\mu' \cap \alpha \neq \mu', \alpha$. We have two cases here;

   a) If $\mu' \cap \alpha$ is a root then let $\gamma := \mu' \cap \alpha$. Now, by Lemma 3.2.1 we get $s_\mu \cdot s_{\mu'} \cdot s_\alpha = s_\mu \cdot s_\gamma \cdot s_{\mu'-\gamma} \cdot s_\alpha$. Here, note that $\mu + \gamma$ is a root since $\mu + \gamma = c - \mu' + \gamma = c - (\mu' - \gamma)$

and $\mu' - \gamma$ is a root. Thus

$$s_\mu \cdot s_{\mu'} \cdot s_\alpha = s_\mu \cdot s_\gamma \cdot s_{\mu'-\gamma} \cdot s_\alpha \leq s_{\mu+\gamma} \cdot s_{\mu'-\gamma} \cdot s_\alpha.$$

Note that, $\mu' - \gamma = c - \alpha$ and $(\mu + \gamma) + (\mu' - \gamma) = c$ so we will take $\beta' = \mu' - \gamma$ in this case.

b) If $\mu' \cap \alpha$ is not a root then by Lemma 3.2.1 we have $\mu' \cap \alpha = \gamma_1 + \gamma_2$ for some positive real roots γ_1 and γ_2 and $s_\mu \cdot s_{\mu'} \cdot s_\alpha = s_\mu \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot s_{\mu'-\gamma} \cdot s_\alpha$. Again, here $\mu + \gamma$ is a root since both $\mu + \alpha = c - \mu' + \gamma = c - (\mu' - \gamma)$ and $\mu' - \gamma$ are roots. Thus

$$s_\mu \cdot s_{\mu'} \cdot s_\alpha = s_\mu \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot s_{\mu'-\gamma} \cdot s_\alpha \leq s_{\mu+\gamma_1+\gamma_2} \cdot s_{\mu'-\gamma} \cdot s_\alpha.$$

Now, observe that, $\mu' - \gamma = c - \alpha$ and $(\mu + \gamma_1 + \gamma_2) + (\mu' - \gamma) = c$ so again we will take $\beta' = \mu' - \gamma$ in this case.

3) If μ' is incomparable with $c - \alpha$ then again $\mu' \cap \alpha \neq \emptyset$. But then we will either have the same two cases in the second case which then we will be done by the arguments in the case or $\mu' \leq \alpha$ or $\mu' > \alpha$.

a) Assume that $\mu' \leq \alpha$. If $\mu' = \alpha$ then $s_\mu \cdot s_{\mu'} \cdot s_\alpha = s_{c-\alpha} \cdot s_\alpha \cdot s_\alpha$. Now, by Lemma 5.3.1, we have $s_{c-\alpha} \cdot s_\alpha \cdot s_\alpha \leq s_\alpha \cdot s_{c-\alpha} \cdot s_\alpha$. So we can take $\beta' = c - \alpha$ in this case. Now, suppose that $\mu' < \alpha$. Then $s_{\mu'}$ and s_α Hecke commute by Lemma 3.2.1 part a). Also, $\mu \cap \alpha \neq \emptyset$. Note that the intersection of two positive roots is either a root or the sum of two positive roots. Now, assume that $\mu \cap \alpha$ is a root. Let $\gamma := \mu \cap \alpha$. Then again by Lemma 3.2.1 we have $s_\mu \cdot s_\alpha = s_\gamma \cdot s_{\mu-\gamma} \cdot s_\alpha$. Thus

$s_\mu \cdot s_{\mu'} \cdot s_\alpha = s_\mu \cdot s_\alpha \cdot s_{\mu'} = s_\gamma \cdot s_{\mu-\gamma} \cdot s_\alpha \cdot s_{\mu'} = s_\gamma \cdot s_{\mu-\gamma} \cdot s_{\mu'} \cdot s_\alpha$. Here, note that $\mu + \mu' - \gamma = c - \gamma$ so $s_{\mu-\gamma} \cdot s_{\mu'} \leq s_{c-\gamma}$ by Corollary 5.1.5. So $s_\gamma \cdot s_{\mu-\gamma} \cdot s_{\mu'} \cdot s_\alpha \leq s_\gamma \cdot s_{c-\gamma} \cdot s_\alpha$. Note that, $c - \gamma > c - \alpha$, hence this case falls into the case 2) above. So we are done with this case. If $\mu \cap \alpha$ is not a root then the proof is similar.

b) Assume that $\mu' > \alpha$. Now, if $\mu' \perp \alpha$ then we can take $\beta' = \mu'$. Now, assume that μ' is not perpendicular to α. Then $< \mu', \alpha >= 1$ which implies $s_\alpha(\mu') = \mu' - < \mu', \alpha^\vee > \alpha = \mu' - \alpha$. So $\mu' - \alpha$ is a root. Also, $s_{\mu'}$ and s_α Hecke commute by Lemma 3.2.1. Furthermore, $\mu + \alpha = c - \mu' + \alpha = c - (\mu' - \alpha)$ is also a root and $(\mu + \alpha) \cap \mu' = \alpha$ since μ and μ' are disjoint and $\mu' > \alpha$. We also have $s_\mu \cdot s_\alpha \leq s_{\mu+\alpha}$ by Corollary 5.1.5. Thus $s_\mu \cdot s_{\mu'} \cdot s_\alpha = s_\mu \cdot s_\alpha \cdot s_{\mu'} \leq s_{\mu+\alpha} \cdot s_{\mu'}$. Here, note that $s_{\mu+\alpha} \cdot s_{\mu'} = s_{\mu+\alpha} \cdot s_{\mu'-\alpha} \cdot s_\alpha$ by case 2) in Lemma 3.2.1. Now, observe that $(\mu + \alpha) + (\mu' - \alpha) = c$ and $\mu' - \alpha \leq c - \alpha$. Thus, we can take $\beta' = \mu' - \alpha$ in this case.

□

We will state the result in the following theorem.

Theorem 5.3.4. *Let $\alpha \in \Pi_{aff}^{re,+}$ be such that $\alpha < c$. Then*

1) $\Gamma_{c+\alpha}(id) = \{t_{\beta'} s_\alpha : \beta' \in \Pi_{aff}^{re,+}(\alpha)\} \cup \{t_{\beta'-c} s_\alpha : \beta' \in \Pi_{aff}^{re,+}(\alpha)\}$

2) $|\Gamma_{c+\alpha}(id)| = |\{\beta' : \beta' \in \Pi_{aff}^{re,+}(\alpha)\}|$

3) For all $w \in \Gamma_{c+\alpha}(id)$, $\ell(w) = 2(n-1) + \ell(s_\alpha)$

where $\Pi_{aff}^{re,+}(\alpha) := \{\beta' \in \Pi_{aff}^{re,+} : \beta' < c$ *and either* $\beta' \leq c - \alpha$ *or both* $\beta' > \alpha$ *and* $\beta' \perp \alpha\}$.

Proof. 1) Let $id \xrightarrow{\beta_1} s_{\beta_1} \xrightarrow{\beta_2} s_{\beta_1}s_{\beta_2}... \xrightarrow{\beta_r} w = s_{\beta_1}s_{\beta_2}...s_{\beta_r}$ be a path in the moment graph such that $\sum_{i=1}^{r} \beta_i \leq c+\alpha$. Note that, by Remark 4.0.2 we can assume that $w = s_{\beta_1} \cdot s_{\beta_2} \cdot ... \cdot s_{\beta_r}$. where $\sum_{i=1}^{r} \beta_i = c + \alpha$ and $\beta_i < c$ for all i. Now, by Lemma 5.3.2, we can also suppose that $\sum_{i=1}^{t} \beta_i = c$ and $\sum_{i=t+1}^{r} \beta_i = \alpha$ for some t. Then

$$w = (s_{\beta_1} \cdot ... \cdot s_{\beta_t}) \cdot (s_{\beta_{t+1}} \cdot s_{\beta_{t+2}} \cdot ... \cdot s_{\beta_r}) \leq (s_\mu \, s_{\mu'}) \cdot s_\alpha$$

for some $\mu, \mu' \in \Pi_{\text{aff}}^{\text{re},+}$ such that $\mu + \mu' = c$ by Equation 5.1 and Corollary 3.2.2. Here, we have $(s_\mu s_{\mu'}) \cdot s_\alpha \leq (s_\mu \cdot s_{\mu'}) \cdot s_\alpha$. Furthermore, $s_\mu \cdot s_{\mu'} \cdot s_\alpha \leq s_\beta \cdot s_{\beta'} \cdot s_\alpha$ where $\beta + \beta' = c$ and either $\beta' \leq c - \alpha$ or both $\beta' > \alpha$ and $\beta' \perp \alpha$, by Lemma 5.3.3. Now, by Lemmas 3.4.6 and 3.4.8 $s_\beta s_{\beta'} s_\alpha$ is reduced so $s_\beta \cdot s_{\beta'} \cdot s_\alpha = s_\beta s_{\beta'} s_\alpha$. Hence $\Gamma_{c+\alpha}(id) = \{(s_\beta s_{\beta'}) s_\alpha : \beta, \beta' \in \Pi_{\text{aff}}^{\text{re},+}\}$ where $\beta + \beta' = c$ such that either $\beta' \leq c - \alpha$ or both $\beta' > \alpha$ and $\beta' \perp \alpha$. Now, also note that $s_\beta s_{\beta'} = t_{\beta'}$ if $\alpha_0 \notin \text{supp}(\beta')$ and $s_\beta s_{\beta'} = t_{\beta' - c}$ if $\alpha_0 \in \text{supp}(\beta')$, see the proof of Lemma 3.4.1.

2) Let $s_\beta s_{\beta'} s_\alpha$ and $s_\nu s_{\nu'} s_\alpha \in \Gamma_{c+\beta}(id)$ be given. We need to show that $s_\beta s_{\beta'} s_\alpha \neq s_\nu s_{\nu'} s_\alpha$ if $\beta' \neq \nu'$. Now assume that $\beta' \neq \nu'$ but $s_\beta s_{\beta'} s_\alpha = s_\nu s_{\nu'} s_\alpha$. By Lemmas 3.4.6 and 3.4.8, both $s_\beta s_{\beta'} s_\alpha$ and $s_\nu s_{\nu'} s_\alpha$ are reduced so the equality, $s_\beta s_{\beta'} s_\alpha = s_\nu s_{\nu'} s_\alpha$ implies that $s_\beta s_{\beta'} = s_\nu s_{\nu'}$ but this is a contradiction by Lemma 3.4.1.

3) Let $w = s_\beta s_{\beta'} s_\alpha \in \Gamma_{c+\alpha}(id)$. Then by Lemmas 3.4.6 and 3.4.8, $\ell(w) = 2(n-1) + \ell(s_\alpha)$. □

Remark 5.3.5. Let $\alpha \in \Pi_{\text{aff}}^{\text{re},+}$ be such that $\alpha < c$. Now, note that, $t_{\beta'} = s_\beta s_{\beta'}$ if $\alpha_0 \in \text{supp}(\beta)$ and $t_{\beta' - c} = s_\beta s_{\beta'}$ if $\alpha_0 \notin \text{supp}(\beta)$ where $\beta, \beta' \in \Pi_{\text{aff}}^{\text{re},+}$ such that $\beta + \beta' = c$, see the proof of

Lemma 3.4.1. So by Theorem 5.3.4 we get

$$\Gamma_{c+\alpha}(id) = \{s_\beta s_{\beta'} s_\alpha : \beta' \in \Pi_{\text{aff}}^{\text{re},+}(\alpha) \text{ such that } \beta + \beta' = c\}. \tag{5.2}$$

Example 5.3.6. Let W_{aff} be the Weyl group of type $A_4^{(1)}$. We compute $\Gamma_{c+\alpha}(id)$ where $\alpha = \alpha_0 + \alpha_4$. Note that, we have six positive real roots which are smaller than $c - \alpha = \alpha_1 + \alpha_2 + \alpha_3$; $\beta_1' = \alpha_1, \beta_2' = \alpha_2, \beta_3' = \alpha_3, \beta_4' = \alpha_1 + \alpha_2, \beta_5' = \alpha_2 + \alpha_3, \beta_6' = \alpha_1 + \alpha_2 + \alpha_3$. Also, we have only one positive root which is smaller than c, strictly bigger than α and perpendicular to α; $\beta_7' = \alpha_0 + \alpha_1 + \alpha_3 + \alpha_4$. So by Theorem 5.3.4, we get

$$\Gamma_{c+\alpha}(id) = \{t_{\beta_1'} s_\alpha, t_{\beta_2'} s_\alpha, t_{\beta_3'} s_\alpha, t_{\beta_4'} s_\alpha, t_{\beta_5'} s_\alpha, t_{\beta_6'} s_\alpha, t_{\beta_7' - c} s_\alpha\}$$

Moreover, for any $w \in \Gamma_{c+\alpha}(id)$, we have $\ell(w) = 2(n-1) + \ell(s_\alpha) = 2(n-1) + 2\operatorname{supp}(\alpha) - 1 = 2 \cdot 4 + 2 \cdot 2 - 1 = 11$.

5.4 Curve Neighborhood $\Gamma_{mc+\alpha}(id)$

This section is devoted to a generalization of the result in the previous section. We will begin with some lemmas.

Lemma 5.4.1. *Let β, β', ν and $\nu' \in \Pi_{aff}^{re,+}$ such that $\nu + \nu' = \beta + \beta' = c$ and $\nu' \leq \beta'$. Then $s_\nu \cdot s_{\nu'} \cdot s_\beta \cdot s_{\beta'} \leq (s_\nu \cdot s_{\nu'})^2$.*

Proof. Here we have two main cases;

1) $\nu' + \beta$ is a root. Then $c - (\nu' + \beta) = c - \nu' - \beta = \nu - \beta = \beta' - \nu'$ is also a root. We have $s_{\nu'} \cdot s_\beta \leq s_{\nu'+\beta}$ and $(\nu' + \beta) \cap \beta' = \nu'$ so by Lemma 3.2.1 part $2a$) we get

$$s_\nu \cdot s_{\nu'} \cdot s_\beta \cdot s_{\beta'} \leq s_\nu \cdot s_{\nu'+\beta} \cdot s_{\beta'} \leq s_\nu \cdot s_{\nu'+\beta} \cdot s_{\beta'-\nu'} \cdot s_{\nu'}.$$

Also, we have $\nu \cap (\nu' + \beta) = \beta$ and $\beta + \beta' - \nu' = c - \nu' = \nu$ which implies

$$s_\nu \cdot s_{\nu'+\beta} \cdot s_{\beta'-\nu'} \cdot s_{\nu'} \leq s_\nu \cdot s_{\nu'} \cdot s_\beta \cdot s_{\beta'-\nu'} \cdot s_{\nu'} \leq s_\nu \cdot s_{\nu'} \cdot s_\nu \cdot s_{\nu'}.$$

by Lemma 3.2.1 part $2a$).

2) $\nu' + \beta$ is not a root. Note that, since $\nu' + \beta < c$ and $\beta < \nu$ we have $s_{\nu'} \cdot s_\beta = s_\beta \cdot s_{\nu'}$ and $s_\nu \cdot s_\beta = s_\beta \cdot s_\nu$ which implies that $s_\nu \cdot s_{\nu'} \cdot s_\beta \cdot s_{\beta'} = s_\beta \cdot s_\nu \cdot s_{\nu'} \cdot s_{\beta'}$. Also note that, $\nu' + \beta = c - \nu + c - \beta' = 2c - (\nu + \beta')$ and since $\nu' + \beta$ is not a root $\nu + \beta'$ is also not a root. Then $s_\nu \cdot s_{\beta'} = s_{\beta'} \cdot s_\nu$. Furthermore, $s_{\nu'} \cdot s_{\beta'} = s_{\beta'} \cdot s_{\nu'}$ since $\nu' < \beta'$. We get $s_\beta \cdot s_\nu \cdot s_{\nu'} \cdot s_{\beta'} = s_\beta \cdot s_\nu \cdot s_{\beta'} \cdot s_{\nu'}$. Note that, $\nu \cap \beta' \neq \emptyset$. Here we have two cases;

   a) If $\gamma := \nu \cap \beta'$ is a root then $\nu - \gamma$ and $\beta' - \gamma$ are also roots. By Lemma 3.2.1 we get $s_\beta \cdot s_\nu \cdot s_{\beta'} \cdot s_{\nu'} = s_\beta \cdot s_\nu \cdot s_{\beta'-\gamma} \cdot s_\gamma \cdot s_{\nu'}$. Also note that, $\nu' = \beta' - \gamma$ and $\beta + \gamma = c - \beta' + \gamma = c - (\beta' - \gamma) = c - \nu' = \nu$. Again, if we also use the facts that $s_\beta \cdot s_{\nu'} = s_{\nu'} \cdot s_\beta$ and $s_\beta \cdot s_\nu = s_\nu \cdot s_\beta$ we will obtain

$$\begin{aligned} s_\beta \cdot s_\nu \cdot s_{\beta'-\gamma} \cdot s_\gamma \cdot s_{\nu'} &= s_\beta \cdot s_\nu \cdot s_{\nu'} \cdot s_\gamma \cdot s_{\nu'} = s_\nu \cdot s_\beta \cdot s_{\nu'} \cdot s_\gamma \cdot s_{\nu'} \\ &= s_\nu \cdot s_{\nu'} \cdot s_\beta \cdot s_\gamma \cdot s_{\nu'} \leq s_\nu \cdot s_{\nu'} \cdot s_\nu \cdot s_{\nu'} \end{aligned}$$

   since $s_\beta \cdot s_\gamma \leq s_{\beta+\gamma} = s_\nu$.

   b) If $\nu \cap \beta'$ is not a root then there are some roots γ_1 and γ_2 such that $\nu \cap \beta' = \gamma_1 + \gamma_2$. Now, by Lemma 3.2.1 part $2b$) we have $s_\nu \cdot s_{\beta'} = s_\nu \cdot s_{\beta'-\gamma_1-\gamma_2} \cdot s_{\gamma_1} \cdot s_{\gamma_2}$. We also have $\beta' - \gamma_1 - \gamma_2 = \nu'$ and $\beta + \gamma_1 + \gamma_2 = \nu$ which implies that

$$\begin{aligned} s_\nu \cdot s_{\nu'} \cdot s_\beta \cdot s_{\beta'} &= s_\beta \cdot s_\nu \cdot s_{\beta'} \cdot s_{\nu'} = s_\beta \cdot s_\nu \cdot s_{\beta'-\gamma_1-\gamma_2} \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot s_{\nu'} \\ &= s_\beta \cdot s_\nu \cdot s_{\nu'} \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot s_{\nu'} \end{aligned}$$

Again, s_β commutes with s_ν and $s_{\nu'}$. Thus,

$$\begin{aligned} s_\beta \cdot s_\nu \cdot s_{\nu'} \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot s_{\nu'} &= s_\nu \cdot s_{\nu'} \cdot s_\beta \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot s_{\nu'} \leq s_\nu \cdot s_{\nu'} \cdot s_{\beta+\gamma_1+\gamma_2} \cdot s_{\nu'} \\ &= s_\nu \cdot s_{\nu'} \cdot s_\nu \cdot s_{\nu'} \end{aligned}$$

due to the fact that $s_\beta \cdot s_{\gamma_1} \cdot s_{\gamma_2} \leq s_{\beta+\gamma_1+\gamma_2} = s_\nu$.

□

Lemma 5.4.2. *Let β, β', ν and $\nu' \in \Pi_{aff}^{re,+}$ such that $\nu + \nu' = \beta + \beta' = c$. Also, assume that $\nu' > \beta$ and $\nu' \perp \beta$. Then $s_\nu \cdot s_{\nu'} \cdot s_\beta \cdot s_{\beta'} \leq (s_\beta \cdot s_{\beta'})^2$.*

Proof. First, note that $s_{\nu'}$ and s_β Hecke commute by part 1) in Lemma 3.2.1. Also, by the assumptions on ν' and β, the support of ν and β are disconnected which implies that s_ν and s_β Hecke commute. So we get $s_\nu \cdot s_{\nu'} \cdot s_\beta \cdot s_{\beta'} = s_\beta \cdot s_\nu \cdot s_{\nu'} \cdot s_{\beta'}$. Here, the fact that the support of ν and β are disconnected forces us to have $\nu' \cap \beta' \neq \emptyset$ and $\nu' \cap \beta'$ not to be a root. But then $\nu' \cap \beta' = \gamma_1 + \gamma_2$ for some $\gamma_1, \gamma_2 \in \Pi_{\text{aff}}^{\text{re},+}$. By part 2) in Lemma 3.2.1, we get $s_{\nu'} \cdot s_{\beta'} = s_{\gamma_1} \cdot s_{\gamma_2} \cdot s_{\nu'-\gamma_1-\gamma_2} \cdot s_{\beta'}$. Now, observe that $\nu + \gamma_1 + \gamma_2 = \beta'$ and $\nu' - \gamma_1 - \gamma_2 = \beta$. Thus

$$s_\beta \cdot s_\nu \cdot s_{\nu'} \cdot s_{\beta'} = s_\beta \cdot s_\nu \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot s_{\nu'-\gamma_1-\gamma_2} \cdot s_{\beta'} \leq s_\beta \cdot s_{\nu+\gamma_1+\gamma_2} \cdot s_\beta \cdot s_{\beta'} = s_\beta \cdot s_{\beta'} \cdot s_\beta \cdot s_{\beta'}.$$

□

Lemma 5.4.3. *Let β, β', ν and $\nu' \in \Pi_{aff}^{re,+}$ such that $\nu + \nu' = \beta + \beta' = c$. Also, assume that $\nu' \leq \beta'$, $\beta' > \alpha$, and $\beta' \perp \alpha$. Then, for any positive integer m, $s_\nu \cdot s_{\nu'} \cdot (s_\beta \cdot s_{\beta'})^m \cdot s_\alpha \leq$*

$(s_\mu \cdot s_{\mu'})^{m+1} \cdot s_\alpha$ *for some positive real roots* μ, μ' *such that* $\mu + \mu' = c$ *and either* $\mu' \leq c - \alpha$ *or both* $\mu' > \alpha$ *and* $\mu' \perp \alpha$.

Proof. First, observe that $\beta' > \alpha$ and $\beta' \perp \alpha$ imply that $c - \alpha > \beta$ and $(c - \alpha) \perp \beta$. Also, supp(β) and supp(α) are disconnected. So α Hecke commute with both β' and β. We have several cases here;

1) If $\nu' = \beta'$ then take $\mu' = \beta'$.

2) If $\nu' \cap \alpha = \emptyset$ then $\nu' \leq c - \alpha$. Note that, by Lemma 5.4.1, we have $s_\nu \cdot s_{\nu'} \cdot s_\beta \cdot s_{\beta'} \leq (s_\nu \cdot s_{\nu'})^2$ since $\nu' \leq \beta'$ which follows by $s_\nu \cdot s_{\nu'} \cdot (s_\beta \cdot s_{\beta'})^m \cdot s_\alpha \leq (s_\nu \cdot s_{\nu'})^{m+1} \cdot s_\alpha$. So we can take $\mu' = \nu'$.

3) If $\nu' \cap \alpha \neq \emptyset$ then we have several cases;

- If we have both $\nu' > \alpha$ and $\nu' \perp \alpha$ then by the same arguments in case 2) we can take $\mu' = \nu'$.

- If $v' = \alpha$ then $s_\nu \cdot s_{\nu'} \cdot s_\alpha = s_{c-\alpha} \cdot s_\alpha \cdot s_\alpha$. By Lemma 5.3.1, $s_{c-\alpha} \cdot s_\alpha \cdot s_\alpha \leq s_\alpha \cdot s_{c-\alpha} \cdot s_\alpha$. Thus $s_\nu \cdot s_{\nu'} \cdot (s_\beta \cdot s_{\beta'})^m \cdot s_\alpha = s_\nu \cdot s_{\nu'} \cdot s_\alpha \cdot (s_\beta \cdot s_{\beta'})^m \leq s_\alpha \cdot s_{c-\alpha} \cdot s_\alpha \cdot (s_\beta \cdot s_{\beta'})^m = s_\alpha \cdot s_{c-\alpha} \cdot (s_\beta \cdot s_{\beta'})^m \cdot s_\alpha$. Here, observe that by Lemma 5.4.2, $s_\alpha \cdot s_{c-\alpha} \cdot s_\beta \cdot s_{\beta'} \leq (s_\beta \cdot s_{\beta'})^2$ since $c - \alpha > \beta$ and $(c - \alpha) \perp \beta$ which follows by $s_\alpha \cdot s_{c-\alpha} \cdot (s_\beta \cdot s_{\beta'})^m \cdot s_\alpha \leq (s_\beta \cdot s_{\beta'})^{m+1} \cdot s_\alpha$. Hence we can take $\mu' = \beta'$ in this case.

- If $\nu' < \alpha$ then ν' and α Hecke commute by part 1) in Lemma 3.2.1. Also, note that $\nu \cap \alpha \neq \emptyset$. Now, assume that $\nu \cap \alpha$ is a root, let $\gamma := \nu \cap \alpha$. Then, by part 2) in

Lemma 3.2.1 we have $s_\nu \cdot s_\alpha = s_\gamma \cdot s_{\nu-\gamma} \cdot s_\alpha$. Here, note that $\nu - \gamma + \nu' = c - \gamma$ is a root and by the same lemma $s_{\nu-\gamma} \cdot s_{\nu'} \leq s_{c-\gamma}$. Hence $s_\nu \cdot s_{\nu'} \cdot s_\alpha = s_\nu \cdot s_\alpha \cdot s_{\nu'} = s_\gamma \cdot s_{\nu-\gamma} \cdot s_\alpha \cdot s_{\nu'} = s_\gamma \cdot s_{\nu-\gamma} \cdot s_{\nu'} \cdot s_\alpha \leq s_\gamma \cdot s_{c-\gamma} \cdot s_\alpha$. Moreover, $c - \gamma > c - \alpha > \beta$ and $(c-\gamma) \perp \beta$ since $(c-\alpha) \perp \beta$. Then by Lemma 5.4.2, $s_\gamma \cdot s_{c-\gamma} \cdot s_\beta \cdot s_{\beta'} \leq (s_\beta \cdot s_{\beta'})^2$ which follows by $s_\nu \cdot s_{\nu'} \cdot (s_\beta \cdot s_{\beta'})^m \cdot s_\alpha \leq s_\nu \cdot s_{\nu'} \cdot s_\alpha \cdot (s_\beta \cdot s_{\beta'})^m \leq s_\gamma \cdot s_{c-\gamma} \cdot s_\alpha \cdot (s_\beta \cdot s_{\beta'})^m = s_\gamma \cdot s_{c-\gamma} \cdot (s_\beta \cdot s_{\beta'})^m \cdot s_\alpha \leq (s_\beta \cdot s_{\beta'})^{m+1} \cdot s_\alpha$. Hence we can take $\mu' = \beta'$ in this case. Now, if $\nu \cap \alpha$ is not a root then one can show that we can take $\mu' = \beta'$, by the same arguments in this case.

- If $\nu' > \alpha$ but ν' is not perpendicular to α then $< \nu', \alpha^\vee >= 1$ so $s_\alpha(\nu') = \nu' - \alpha$. So $\nu' - \alpha$ is a root which implies that $\nu + \alpha = c - \nu' + \alpha = c - (\nu' - \alpha)$ is also a root. Thus $s_\nu \cdot s_{\nu'} \cdot s_\alpha = s_\nu \cdot s_\alpha \cdot s_{\nu'} = s_{\nu+\alpha} \cdot s_{\nu'}$ by Lemma 3.2.1. Here, note that $(\nu + \alpha) \cap \nu' = \alpha$. So by the same lemma we get $s_{\nu+\alpha} \cdot s_{\nu'} = s_{\nu+\alpha} \cdot s_{\nu'-\alpha} \cdot s_\alpha$. Now, observe that $\nu' - \alpha \leq \beta'$ since both ν' and α are smaller than β'. Then by Lemma 5.4.1, $s_{\nu+\alpha} \cdot s_{\nu'-\alpha} \cdot s_\beta \cdot s_{\beta'} \leq (s_{\nu+\alpha} \cdot s_{\nu'-\alpha})^2$ which follows by $s_\nu \cdot s_{\nu'} \cdot (s_\beta \cdot s_{\beta'})^m \cdot s_\alpha = s_\nu \cdot s_{\nu'} \cdot s_\alpha \cdot (s_\beta \cdot s_{\beta'})^m \leq s_{\nu+\alpha} \cdot s_{\nu'-\alpha} \cdot s_\alpha \cdot (s_\beta \cdot s_{\beta'})^m \leq s_{\nu+\alpha} \cdot s_{\nu'-\alpha} \cdot (s_\beta \cdot s_{\beta'})^m \cdot s_\alpha \leq (s_{\nu+\alpha} \cdot s_{\nu'-\alpha})^{m+1} \cdot s_\alpha$. Last, note that $\nu' - \alpha \leq c - \alpha$ so we can take $\mu' = \nu' - \alpha$ in this case.

- Assume that $\nu' \cap \alpha \neq \emptyset$ and also $\nu' \cap \alpha \neq \nu', \alpha$. Then $\nu' \cap \alpha$ is a root since $\nu' + \alpha$ would be bigger than c otherwise, and that would contradict the facts that $\beta' > \alpha$ and $\beta' > \nu'$. Now, let $\gamma := \nu' \cap \alpha$. Then $\nu' - \gamma$ is a root and $s_{\nu'} \cdot s_\alpha = s_\gamma \cdot s_{\nu'-\gamma} \cdot s_\alpha$ by Lemma 3.2.1. Also, note that $\nu + \gamma = c - \nu' + \gamma = c - (\nu' - \gamma)$ is a root so by the same lemma we have $s_\nu \cdot s_\gamma \leq s_{\nu+\gamma}$. Thus $s_\nu \cdot s_{\nu'} \cdot s_\alpha = s_\nu \cdot s_\gamma \cdot s_{\nu'-\gamma} \cdot s_\alpha \leq s_{\nu+\gamma} \cdot s_{\nu'-\gamma} \cdot s_\alpha$. Here, note that

$\nu' - \gamma \leq \beta'$ since $\nu' \leq \beta'$. Then by Lemma 5.4.1, $s_{\nu+\gamma} \cdot s_{\nu'-\gamma} \cdot s_\beta \cdot s_{\beta'} \leq (s_{\nu+\gamma} \cdot s_{\nu'-\gamma})^2$ which follows by $s_\nu \cdot s_{\nu'} \cdot (s_\beta \cdot s_{\beta'})^m \cdot s_\alpha = s_\nu \cdot s_{\nu'} \cdot s_\alpha \cdot (s_\beta \cdot s_{\beta'})^m \leq s_{\nu+\gamma} \cdot s_{\nu'-\gamma} \cdot s_\alpha \cdot (s_\beta \cdot s_{\beta'})^m \leq s_{\nu+\gamma} \cdot s_{\nu'-\gamma} \cdot (s_\beta \cdot s_{\beta'})^m \cdot s_\alpha \leq (s_{\nu+\gamma} \cdot s_{\nu'-\gamma})^{m+1} \cdot s_\alpha$. Last, note that $\nu' - \gamma \leq c - \alpha$ so we can take $\mu' = \nu' - \gamma$ in this case.

□

The result of this section is:

Theorem 5.4.4. *Let $\alpha < c$ be an affine positive real root and let m be a positive integer. Then*

1) $\Gamma_{mc+\alpha}(id) = \{t_{m\beta'} s_\alpha : \beta' \in \Pi_{aff}^{re,+}(\alpha)\} \cup \{t_{m(\beta'-c)} s_\alpha : \beta' \in \Pi_{aff}^{re,+}(\alpha)\}$

2) $|\Gamma_{mc+\alpha}(id)| = |\{\beta' : \beta' \in \Pi_{aff}^{re,+}(\alpha)\}|$

3) For all $w \in \Gamma_{mc+\alpha}(id)$, $\ell(w) = 2m(n-1) + \ell(s_\alpha)$

where $\Pi_{aff}^{re,+}(\alpha) := \{\beta' \in \Pi_{aff}^{re,+} : \beta' < c$ *and either* $\beta' \leq c - \alpha$ *or both* $\beta' > \alpha$ *and* $\beta' \perp \alpha\}$.

Proof. 1) We will prove the statement by induction on m. Assume that $m = 1$. Then the statement is true by Theorem 5.3.4. Now, assume that the statement is true for $m = k$. We will prove that it is also true for $m = k+1$. Let $id \xrightarrow{\beta_1} s_{\beta_1} \xrightarrow{\beta_2} s_{\beta_1} s_{\beta_2} \ldots \xrightarrow{\beta_r} w = s_{\beta_1} s_{\beta_2} \ldots s_{\beta_r}$ be a path in the moment graph such that $\sum_{i=1}^r \beta_i \leq (k+1)c + \alpha$. Note that, by Remark 4.0.2 we can assume that $w = s_{\beta_1} \cdot s_{\beta_2} \cdot \ldots \cdot s_{\beta_r}$ where $\sum_{i=1}^r \beta_i = (k+1)c + \alpha$ and $\beta_i < c$ for all i. It is enough to show that $w \leq (s_\beta s_{\beta'})^{k+1} s_\alpha$ for some $\beta, \beta' \in \Pi_{\mathrm{aff}}^{\mathrm{re},+}$ such that $\beta + \beta' = c$ where either $\beta' \leq c-\alpha$ or both $\beta' > \alpha$ and $\beta' \perp \alpha$ since $t_{\beta'} = s_\beta s_{\beta'}$ if $\alpha_0 \in \text{supp}(\beta)$ and $t_{\beta'-c} = s_\beta s_{\beta'}$ if $\alpha_0 \notin \text{supp}(\beta)$, see the proof of Lemma 3.4.1. Moreover, by Lemma 5.3.2 we can make the assumption that there is an integer p such that $\sum_{i=1}^{p} \beta_i = c$ and $\sum_{i=p+1}^{r} \beta_i = kc + \alpha$. Then by Equation 5.1 we have $s_{\beta_1} \cdot s_{\beta_2} \cdot \ldots \cdot s_{\beta_p} \leq s_\gamma s_{\gamma'}$ for some $\gamma, \gamma' \in \Pi_{\text{aff}}^{\text{re},+}$ such that $\gamma + \gamma' = c$ and $s_{\beta_{p+1}} \cdot \ldots \cdot s_{\beta_r} \leq (s_\beta s_{\beta'})^k s_\alpha$ for some $\beta, \beta' \in \Pi_{\text{aff}}^{\text{re},+}$ such that $\beta + \beta' = c$ where either $\beta' \leq c - \alpha$ or both $\beta' > \alpha$ and $\beta' \perp \alpha$ by the induction assumption. Here, note that, $s_\gamma s_{\gamma'}$ is reduced by Lemma 3.4.3. Moreover, $(s_\beta s_{\beta'})^k s_\alpha$ is also reduced by Lemmas 3.4.6 and 3.4.8. Thus,

$$w = (s_{\beta_1} \cdot s_{\beta_2} \cdot \ldots \cdot s_{\beta_p}) \cdot (s_{\beta_{p+1}} \cdot \ldots \cdot s_{\beta_r}) \leq (s_\gamma s_{\gamma'}) \cdot ((s_\beta s_{\beta'})^k s_\alpha) = (s_\gamma \cdot s_{\gamma'}) \cdot ((s_\beta \cdot s_{\beta'})^k \cdot s_\alpha).$$

Now note that by Lemma 5.3.3 there are $\nu, \nu' \in \Pi_{\text{aff}}^{\text{re},+}$ such that $s_\gamma \cdot s_{\gamma'} \cdot s_\beta \leq s_\nu \cdot s_{\nu'} \cdot s_\beta$ where $\nu + \nu' = c$ and either $\nu' \leq c - \beta = \beta'$ or both $\nu' > \beta$ and $\nu' \perp \beta$.

- If $\nu' \leq c - \beta = \beta'$ then by Lemma 5.4.1, we get $s_\nu \cdot s_{\nu'} \cdot s_\beta \cdot s_{\beta'} \leq (s_\nu \cdot s_{\nu'})^2$ which follows by

$$w \leq (s_\gamma \cdot s_{\gamma'}) \cdot (s_\beta \cdot s_{\beta'})^k \cdot s_\alpha \leq s_\nu \cdot s_{\nu'} \cdot (s_\beta \cdot s_{\beta'})^k \cdot s_\alpha \leq (s_\nu \cdot s_{\nu'})^{k+1} \cdot s_\alpha.$$

  Now, if we also have $\beta' \leq c - \alpha$ then $c - \alpha \geq \beta' \geq \nu'$ so we are done with this case. Now if $\beta' > \alpha$ and $\beta' \perp \alpha$ then by Lemma 5.4.3, $s_\nu \cdot s_{\nu'} \cdot (s_\beta \cdot s_{\beta'})^m \cdot s_\alpha \leq (s_\mu \cdot s_{\mu'})^{m+1} \cdot s_\alpha$ for some positive real roots, μ, μ' such that $\mu + \mu' = c$ and either $\mu' \leq c - \alpha$ or both $\mu' > \alpha$ and $\mu' \perp \alpha$. Hence we are done with this case too.

- If $\nu' > \beta$ and $\nu' \perp \beta$ then by Lemma 5.4.2 we get $s_\nu \cdot s_{\nu'} \cdot s_\beta \cdot s_{\beta'} \leq (s_\beta \cdot s_{\beta'})^2$ which follows by

$$w \leq (s_\gamma \cdot s_{\gamma'}) \cdot (s_\beta \cdot s_{\beta'})^k \cdot s_\alpha \leq s_\nu \cdot s_{\nu'} \cdot (s_\beta \cdot s_{\beta'})^k \cdot s_\alpha \leq (s_\beta \cdot s_{\beta'})^{k+1} \cdot s_\alpha.$$

Thus we are done with all cases.

Moreover, by Lemmas 3.4.6 and 3.4.8, $(s_\beta s_{\beta'})^{k+1} s_\alpha$ is reduced so $(s_\beta \cdot s_{\beta'})^{k+1} \cdot s_\alpha = (s_\beta s_{\beta'})^{k+1} s_\alpha$.

2) Let $(s_\beta s_{\beta'})^m s_\alpha$ and $(s_\nu s_{\nu'})^m s_\alpha \in \Gamma_{mc+\alpha}(id)$. We need to show that $(s_\beta s_{\beta'})^m s_\alpha \neq (s_\nu s_{\nu'})^m s_\alpha$ if $\beta' \neq \nu'$. Now assume that $\beta' \neq \nu'$ but

$$(s_\beta s_{\beta'})^m s_\alpha = (s_\nu s_{\nu'})^m s_\alpha \tag{5.3}$$

. By Lemmas 3.4.6 and 3.4.8 both $(s_\beta s_{\beta'})^m s_\alpha$ and $(s_\nu s_{\nu'})^m s_\alpha$ are reduced, so Equation 5.3 implies that $(s_\beta s_{\beta'})^m = (s_\nu s_{\nu'})^m$ but this is a contradiction by Lemma 3.4.1.

3) Let $w = (s_\beta s_{\beta'})^m s_\alpha \in \Gamma_{mc+\alpha}(id)$. Then $\ell(w) = 2m(n-1) + \ell(s_\alpha)$, by Lemmas 3.4.6 and 3.4.8.

□

Remark 5.4.5. *a*) Let $\alpha \in \Pi_{\text{aff}}^{\text{re},+}$ be such that $\alpha < c$. Now, note that, $t_{m\beta'} = (s_\beta s_{\beta'})^m$ if $\alpha_0 \in \text{supp}(\beta)$ and $t_{m(\beta'-c)} = (s_\beta s_{\beta'})^m$ if $\alpha_0 \notin \text{supp}(\beta)$ where $\beta, \beta' \in \Pi_{\text{aff}}^{\text{re},+}$ such that $\beta + \beta' = c$ and m is a positive integer, see the proof of Lemma 3.4.1. So by Theorem 5.4.4 we get

$$\Gamma_{mc+\alpha}(id) = \{(s_\beta s_{\beta'})^m s_\alpha : \beta' \in \Pi_{\text{aff}}^{\text{re},+}(\alpha) \text{ such that } \beta + \beta' = c\}. \tag{5.4}$$

b) For any $w = s_{\beta_1} \cdot s_{\beta_2} \cdot ... \cdot s_{\beta_r}$. where $\sum_{i=1}^r \beta_i = mc + \alpha$ we have $w \leq u$ such that $u \in \Gamma_{mc+\alpha}(id)$, see the arguments in the proof of Theorem 5.4.4.

Example 5.4.6. Suppose that the affine Weyl group W_{aff} is of type $A_4^{(1)}$. We compute $\Gamma_{12c+\alpha}(id)$ where $\alpha = \alpha_0 + \alpha_4$. Then, by Theorem 5.4.4, we get

$$\Gamma_{12c+\alpha}(id) = \{t_{12\beta'_1}s_\alpha, t_{12\beta'_2}s_\alpha, t_{12\beta'_3}s_\alpha, t_{12\beta'_4}s_\alpha, t_{12\beta'_5}s_\alpha, t_{12\beta'_6}s_\alpha, t_{12(\beta'_7-c)}s_\alpha\}$$

where $\beta'_1 = \alpha_1, \beta'_2 = \alpha_2, \beta'_3 = \alpha_3, \beta'_4 = \alpha_1 + \alpha_2, \beta'_5 = \alpha_2 + \alpha_3, \beta'_6 = \alpha_1 + \alpha_2 + \alpha_3$ and $\beta'_7 = \alpha_0 + \alpha_1 + \alpha_3 + \alpha_4$, see Example 5.3.6. Moreover, for any $w \in \Gamma_{12c+\alpha}(id)$, we have $\ell(w) = 2m(n-1) + \ell(s_\alpha) = 2m(n-1) + 2\,\text{supp}(\alpha) - 1 = 2 \cdot 12 \cdot 4 + 2 \cdot 2 - 1 = 99$.

5.5 Curve Neighborhood $\Gamma_{mc}(id)$

Theorem 5.5.1. *Let $m \geq 2$ be a positive integer. Then we have*

1) $\Gamma_{mc}(id) = \{t_{m\gamma} : \gamma \in \Pi\}$.

2) $|\Gamma_{mc}(id)| = n(n-1)$.

3) For all $w \in \Gamma_{mc}(id)$, $\ell(w) = 2m(n-1)$.

Proof. 1) Let $id \xrightarrow{\beta_1} s_{\beta_1} \xrightarrow{\beta_2} s_{\beta_1}s_{\beta_2}... \xrightarrow{\beta_r} w = s_{\beta_1}s_{\beta_2}...s_{\beta_r}$ be a path in the moment graph such that $\sum_{i=1}^r \beta_i \leq mc$. Note that, by Remark 4.0.2 we can assume that $w = s_{\beta_1} \cdot s_{\beta_2} \cdot ... \cdot s_{\beta_r}$. where $\sum_{i=1}^r \beta_i = mc$ and $\beta_i < c$ for all i. Now, note that $\sum_{i=1}^{r-1} \beta_i = mc - \beta_r = (m-1)c + c - \beta_r$. Let $\alpha := c - \beta_r$ and $\alpha' := \beta_r$. Then by Remark 5.4.5 parts a) and b) we have $s_{\beta_1} \cdot s_{\beta_2} \cdot ... \cdot s_{\beta_{r-1}} \leq (s_\beta s_{\beta'})^{m-1} s_\alpha$ for some positive real roots β and β' such that $\beta + \beta' = c$ and either $\beta' \leq c - \alpha$

or both $\beta' > \alpha$ and $\beta' \perp \alpha$. Then, we get

$$w = (s_{\beta_1} \cdot s_{\beta_2} \cdot \ldots \cdot s_{\beta_{r-1}}) \cdot s_{\beta_r} \leq (s_\beta s_{\beta'})^{m-1} s_\alpha \cdot s_{\alpha'}.$$

Now, note that $(s_\beta s_{\beta'})^{m-1} s_\alpha = (s_\beta \cdot s_{\beta'})^{m-1} \cdot s_\alpha$, since $= (s_\beta s_{\beta'})^{m-1} s_\alpha$ is reduced by Lemma 3.4.6 and Lemma 3.4.8. Furthermore, if $\beta' \leq c - \alpha$ then $s_\beta \cdot s_{\beta'} \cdot s_\alpha \cdot s_{\alpha'} \leq (s_\beta \cdot s_{\beta'})^2$ by Lemma 5.4.1 which follows by $(s_\beta s_{\beta'})^{m-1} s_\alpha \cdot s_{\alpha'} = (s_\beta \cdot s_{\beta'})^{m-1} \cdot s_\alpha \cdot s_{\alpha'} \leq (s_\beta \cdot s_{\beta'})^m$. If $\beta' > \alpha$ and $\beta' \perp \alpha$ then $s_\beta \cdot s_{\beta'} \cdot s_\alpha \cdot s_{\alpha'} \leq (s_\alpha \cdot s_{\alpha'})^2$ by Lemma 5.4.2 which follows by $(s_\beta s_{\beta'})^{m-1} s_\alpha \cdot s_{\alpha'} = (s_\beta \cdot s_{\beta'})^{m-1} \cdot s_\alpha \cdot s_{\alpha'} \leq (s_\alpha \cdot s_{\alpha'})^m$. We also know that $(s_\alpha s_{\alpha'})^m = t_{m\gamma}$ for some $\gamma \in \Pi$, see the proof of Lemma 3.4.1.

2) Let $\beta, \beta', \nu, \nu' \in \Pi_{\text{aff}}^{\text{re},+}$ such that $\nu + \nu' = \beta + \beta' = c$. Then $(s_\beta s_{\beta'})^m \neq (s_\nu s_{\nu'})^m$ for any positive integer m if $\beta \neq \nu$ by Lemma 3.4.1. Also, note that $|\{\beta \in \Pi_{\text{aff}}^{\text{re},+} : \beta < c\}| = n(n-1)$. □

3) Let $w = (s_\beta s_{\beta'})^m \in \Gamma_{mc}(id)$ for some positive real roots β, β' such that $\beta + \beta' = c$. Then $\ell(w) = 2m(n-1)$ by Lemma 3.4.3.

Remark 5.5.2. Note that, $t_{m\beta'} = s_\beta s_{\beta'}$ if $\alpha_0 \in \text{supp}(\beta)$ and $t_{m(\beta'-c)} = s_\beta s_{\beta'}$ if $\alpha_0 \notin \text{supp}(\beta)$, where $\beta, \beta' \in \Pi_{\text{aff}}^{\text{re},+}$ such that $\beta + \beta' = c$ and m is a positive integer, see the proof of Lemma 3.4.1. Furthermore,

$$\{\beta' : \beta' \in \Pi_{\text{aff}}^{\text{re},+},\, \beta' < c,\, \alpha_0 \notin \text{supp}(\beta')\} \sqcup \{\beta' - c : \beta' \in \Pi_{\text{aff}}^{\text{re},+},\, \beta' < c,\, \alpha_0 \in \text{supp}(\beta')\} = \Pi.$$

Thus we have

$$\Gamma_{mc}(id) = \{(s_\beta s_{\beta'})^m : \beta, \beta' \in \Pi_{\text{aff}}^{\text{re},+} \text{ such that } \beta + \beta' = c\}. \tag{5.5}$$

Example 5.5.3. Suppose that the affine Weyl group W_{aff} is associated to $A_2^{(1)}$. Then

$$\Gamma_{10c}(id) = \{t_{10\alpha_1}, t_{10\alpha_2}, t_{10(\alpha_1+\alpha_2)}, t_{-10\alpha_1}, t_{-10\alpha_2}, t_{-10(\alpha_1+\alpha_2)}\}.$$

Moreover, for any $w \in \Gamma_{10c}(id)$, $\ell(w) = 2m(n-1) = 2 \cdot 10 \cdot 2 = 40$.

5.6 Curve Neighborhood $\Gamma_{\mathbf{d}}(id)$ for any $\mathbf{d} > c$

Finally, we will discuss the most general case in this section. In order to prove the result we will first consider two lemmas.

Lemma 5.6.1. *Let $\alpha, \alpha' \in \Pi_{aff}^{re,+}$ such that $\alpha + \alpha' = c$ Also, let $\gamma_1, \gamma_2, ..., \gamma_k \in \Pi_{aff}^{re,+}$ such that supp(γ_i) and supp(γ_j) are disconnected for any $1 \leq i, j \leq k$ such that $i \neq j$. Then $s_\alpha \cdot s_{\alpha'} \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot ... \cdot s_{\gamma_k} \leq s_\beta \cdot s_{\beta'} \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot ... \cdot s_{\gamma_k}$ for some affine positive real roots, $\beta, \beta' \in \Pi_{aff}^{re,+}$ such that either $\beta' \cap \gamma_i = \emptyset$ or both $\beta' > \gamma_i$ and $\beta' \perp \gamma_i$ for any $i = 1, 2, ..., k$.*

Proof. We will prove the statement by the induction on k. Let $k = 1$. Then by Equation 5.2, $(s_\alpha \cdot s_{\alpha'}) \cdot s_{\gamma_1} \leq (s_\beta \cdot s_{\beta'}) \cdot s_{\gamma_1}$ for some $\beta, \beta' \in \Pi_{\text{aff}}^{\text{re},+}$ such that $\beta + \beta' = c$ where either $\beta' \leq c - \gamma_1$ i.e $\beta' \cap \gamma_1 = \emptyset$ or both $\beta' > \gamma_1$ and $\beta' \perp \gamma_1$. So we are done with this case. Now we will assume that the statement is true for k and prove that it is also true for $k+1$ i.e we will show that $(s_\alpha \cdot s_{\alpha'}) \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot ... \cdot s_{\gamma_{k+1}} \leq (s_\beta \cdot s_{\beta'}) \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot ... \cdot s_{\gamma_{k+1}}$ for some affine positive real roots, $\beta, \beta' \in \Pi_{\text{aff}}^{\text{re},+}$ such that either $\beta' \cap \gamma_i = \emptyset$ or both $\beta' > \gamma_i$ and $\beta' \perp \gamma_i$ for any $i = 1, 2, ..., k+1$. Now note that $(s_\alpha \cdot s_{\alpha'}) \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot ... \cdot s_{\gamma_k} \leq (s_\nu \cdot s_{\nu'}) \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot ... \cdot s_{\gamma_k}$ for some $\nu, \nu' \in \Pi_{\text{aff}}^{\text{re},+}$ such that $\nu + \nu' = c$ where $\nu' \cap \gamma_i = \emptyset$ or both $\nu' > \gamma_i$ and $\nu' \perp \gamma_i$ for any $i = 1, 2, ..., k$ by the induction assumption. So there is a subset I of $\{\gamma_1, \gamma_2, ..., \gamma_k\}$ such that $\nu' \cap \gamma_i = \emptyset$ for all $\gamma_i \in I$ and $\nu' > \gamma_j$ and $\nu' \perp \gamma_j$ for all $\gamma_j \in \{\gamma_1, \gamma_2, ..., \gamma_k\} \setminus I$. Thus $(s_\alpha \cdot s_{\alpha'}) \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot ... \cdot s_{\gamma_k} \cdot s_{\gamma_{k+1}} \leq (s_\nu \cdot s_{\nu'} \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot ... \cdot s_{\gamma_k}) \cdot s_{\gamma_{k+1}}$. Now, if we also have $\nu' \cap \gamma_{k+1} = \emptyset$ or both $\nu' > \gamma_{k+1}$ and $\nu' \perp \gamma_{k+1}$ then we are done. Assume not i.e assume that $\nu' \cap \gamma_{k+1} \neq \emptyset$ and ν' is not strictly bigger than or not perpendicular to γ_{k+1}. Note that s_{γ_i} and s_{γ_j} Hecke commute for any $1 \leq i, j \leq k+1$ so $s_\nu \cdot s_{\nu'} \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot ... \cdot s_{\gamma_k} \cdot s_{\gamma_{k+1}} = s_\nu \cdot s_{\nu'} \cdot s_{\gamma_{k+1}} \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot ... \cdot s_{\gamma_k}$. Here we have several cases;

a) Assume that $\nu' \cap \gamma_{k+1} \neq \nu', \gamma_{k+1}$ and $\nu' \cap \gamma_{k+1}$ is a root. Let $\gamma := \nu' \cap \gamma_{k+1}$. Then $\nu' - \gamma$ is a root and $s_{\nu'} \cdot s_{\gamma_{k+1}} = s_\gamma \cdot s_{\nu'-\gamma} \cdot s_{\gamma_{k+1}}$ by Lemma 3.2.1. Also, note that $\nu + \gamma = c - \nu' + \gamma = c - (\nu' - \gamma)$ is a root so by the same lemma we have $s_\nu \cdot s_\gamma \leq s_{\nu+\gamma}$. Thus $s_\nu \cdot s_{\nu'} \cdot s_{\gamma_{k+1}} = s_\nu \cdot s_\gamma \cdot s_{\nu'-\gamma} \cdot s_{\gamma_{k+1}} \leq s_{\nu+\gamma} \cdot s_{\nu'-\gamma} \cdot s_{\gamma_{k+1}}$ which follows by $s_\nu \cdot s_{\nu'} \cdot s_{\gamma_{k+1}} \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot ... \cdot s_{\gamma_k} \leq s_{\nu+\gamma} \cdot s_{\nu'-\gamma} \cdot s_{\gamma_{k+1}} \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot ... \cdot s_{\gamma_k}$. Here, note that $(\nu' - \gamma) \cap \gamma_{k+1} = \emptyset$ and $(\nu' - \gamma) \cap \gamma_i = \emptyset$ for all $\gamma_i \in I$ and $(\nu' - \gamma) > \gamma_j$ and $(\nu' - \gamma) \perp \gamma_j$ for all $\gamma_j \in \{\gamma_1, \gamma_2, ..., \gamma_k\} \setminus I$ since $\nu' - \gamma < \nu'$ and $\mathrm{supp}(\gamma)$ and $\mathrm{supp}(\gamma_i)$ are disconnected for any $i = 1, 2, ..., k$. So we can take $\beta' = \nu' - \gamma$ in this case. The case where $\nu' \cap \gamma_{k+1}$ is not a root is similar.

b) If $\nu' \cap \gamma_{k+1} = \gamma_{k+1}$ then $\gamma_{k+1} \leq \nu'$.

- Assume that $v' = \gamma_{k+1}$. Then $s_\nu \cdot s_{\nu'} \cdot s_{\gamma_{k+1}} = s_{c-\gamma_{k+1}} \cdot s_{\gamma_{k+1}} \cdot s_{\gamma_{k+1}}$. By Lemma 5.3.1, $s_{c-\gamma_{k+1}} \cdot s_{\gamma_{k+1}} \cdot s_{\gamma_{k+1}} \leq s_{\gamma_{k+1}} \cdot s_{c-\gamma_{k+1}} \cdot s_{\gamma_{k+1}}$ which follows by $s_\nu \cdot s_{\nu'} \cdot s_{\gamma_{k+1}} \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot ... \cdot s_{\gamma_k} \leq s_{\gamma_{k+1}} \cdot s_{c-\gamma_{k+1}} \cdot s_{\gamma_{k+1}} \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot ... \cdot s_{\gamma_k}$. Here, observe that $(c - \gamma_{k+1}) \cap \gamma_{k+1} = \emptyset$. Also, $(c - \gamma_{k+1}) > \gamma_i$ and $(c - \gamma_{k+1}) \perp \gamma_i$ for all $i = 1, 2, ..., k$ since $\mathrm{supp}(\gamma_{k+1})$ and $\mathrm{supp}(\gamma_i)$

are disconnected for any $i = 1, 2, ..., k$. Hence we can take $\beta' = c - \gamma_{k+1}$ in this case.

- Assume that $\gamma_{k+1} < \nu'$. Now since ν' is not perpendicular to γ_{k+1} we have $< \nu', \gamma_{k+1}^{\vee} >= 1$ so $s_{\gamma_{k+1}}(\nu') = \nu' - \gamma_{k+1}$. So $\nu' - \gamma_{k+1}$ is a root which implies that $\nu + \gamma_{k+1} = c - \nu' + \gamma_{k+1} = c - (\nu' - \gamma_{k+1})$ is also a root. Thus $s_\nu \cdot s_{\nu'} \cdot s_{\gamma_{k+1}} = s_\nu \cdot s_{\gamma_{k+1}} \cdot s_{\nu'} = s_{\nu+\gamma_{k+1}} \cdot s_{\nu'}$ by Lemma 3.2.1. Here, note that $(\nu + \gamma_{k+1}) \cap \nu' = \gamma_{k+1}$. So by the same lemma we get $s_{\nu+\gamma_{k+1}} \cdot s_{\nu'} = s_{\nu+\gamma_{k+1}} \cdot s_{\nu'-\gamma_{k+1}} \cdot s_{\gamma_{k+1}}$ which follows by $s_\nu \cdot s_{\nu'} \cdot s_{\gamma_{k+1}} \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot ... \cdot s_{\gamma_k} \leq s_{\nu+\gamma_{k+1}} \cdot s_{\nu'-\gamma_{k+1}} \cdot s_{\gamma_{k+1}} \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot ... \cdot s_{\gamma_k}$. Here, observe that $(\nu' - \gamma_{k+1}) \cap \gamma_{k+1} = \emptyset$. Also, $(\nu' - \gamma_{k+1}) \cap \gamma_i = \emptyset$ for all $\gamma_i \in I$ and $(\nu' - \gamma_{k+1}) > \gamma_j$ and $(\nu' - \gamma_{k+1}) \perp \gamma_j$ for all $\gamma_j \in \{\gamma_1, \gamma_2, ..., \gamma_k\} \setminus I$ since $\nu' - \gamma_{k+1} < \nu'$ and $\text{supp}(\gamma_{k+1})$ and $\text{supp}(\gamma_i)$ are disconnected for any $i = 1, 2, ..., k$. So we can take $\beta' = \nu' - \gamma_{k+1}$ in this case.

c) If $\nu' \cap \gamma_{k+1} = \nu'$ then $\nu' \leq \gamma_{k+1}$. We already consider the case where $\nu' = \gamma_{k+1}$ in case b) above so assume that $\nu' < \gamma_{k+1}$. Then $s_{\nu'}$ and $s_{\gamma_{k+1}}$ Hecke commute by part 1) in Lemma 3.2.1. Also, note that $\nu \cap \gamma_{k+1} \neq \emptyset$. Now, assume that $\nu \cap \gamma_{k+1}$ is a root; let $\gamma := \nu \cap \gamma_{k+1}$. Then, by part 2) in Lemma 3.2.1 we have $s_\nu \cdot s_{\gamma_{k+1}} = s_\gamma \cdot s_{\nu-\gamma} \cdot s_{\gamma_{k+1}}$. Here, note that $\nu - \gamma + \nu' = c - \gamma$ is a root and by the same lemma $s_{\nu-\gamma} \cdot s_{\nu'} \leq s_{c-\gamma}$. Hence $s_\nu \cdot s_{\nu'} \cdot s_{\gamma_{k+1}} = s_\nu \cdot s_{\gamma_{k+1}} \cdot s_{\nu'} = s_\gamma \cdot s_{\nu-\gamma} \cdot s_{\gamma_{k+1}} \cdot s_{\nu'} = s_\gamma \cdot s_{\nu-\gamma} \cdot s_{\nu'} \cdot s_{\gamma_{k+1}} \leq s_\gamma \cdot s_{c-\gamma} \cdot s_{\gamma_{k+1}}$ which follows by $s_\nu \cdot s_{\nu'} \cdot s_{\gamma_{k+1}} \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot ... \cdot s_{\gamma_k} \leq s_\gamma \cdot s_{c-\gamma} \cdot s_{\gamma_{k+1}} \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot ... \cdot s_{\gamma_k}$. Moreover, $(c-\gamma) \cap \gamma_{k+1} \neq \emptyset$ and $(c-\gamma) \cap \gamma_{k+1} \neq c-\gamma, \gamma_{k+1}$ since $\gamma < \gamma_{k+1}$. Also, observe that $(c-\gamma) > \gamma_i$ and $(c-\gamma) \perp \gamma_i$ for all $i = 1, 2, ..., k$ since $\text{supp}(\gamma_{k+1})$ and $\text{supp}(\gamma_i)$ are disconnected for any $i = 1, 2, ..., k$. Thus we can consider this case under case a) above. Now, if $\nu \cap \gamma_{k+1}$ is not a root then the proof is similar.

□

Lemma 5.6.2. *Let $\nu, \nu', \mu, \mu' \in \Pi_{aff}^{re,+}$ such that $\nu + \nu' = \mu + \mu' = c$ and either $\nu' \leq \mu'$ or both $\nu' > \mu$ and $\nu' \perp \mu$. Also, let $\gamma_1, \gamma_2, ..., \gamma_k \in \Pi_{aff}^{re,+}$ such that supp(γ_i) and supp(γ_j) are disconnected for any $1 \leq i, j \leq k$ such that $i \neq j$. Furthermore, suppose that either $\mu' \cap \gamma_i = \emptyset$ or both $\mu' > \gamma_i$ and $\mu' \perp \gamma_i$ for any $1 \leq i \leq k$. Then $s_\nu \cdot s_{\nu'} \cdot (s_\mu \cdot s_{\mu'})^m \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot ... \cdot s_{\gamma_k} \leq (s_\beta \cdot s_{\beta'})^{m+1} \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot ... \cdot s_{\gamma_k}$ for some affine positive real roots, $\beta, \beta' \in \Pi_{aff}^{re,+}$ such that either $\beta' \cap \gamma_i = \emptyset$ or both $\beta' > \gamma_i$ and $\beta' \perp \gamma_i$ for any $i = 1, 2, ..., k$ where $m \geq 1$ is an integer.*

Proof. First, if $\nu' = \mu'$ then we can take $\beta' = \mu'$. We will assume that $\nu' \neq \mu'$. If both $\nu' > \mu$ and $\nu' \perp \mu$ are true then by Lemma 5.4.2 we get $s_\nu \cdot s_{\nu'} \cdot (s_\mu \cdot s_{\mu'})^m \leq (s_\mu \cdot s_{\mu'})^{m+1}$ which follows by $s_\nu \cdot s_{\nu'} \cdot (s_\mu \cdot s_{\mu'})^m \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot ... \cdot s_{\gamma_k} \leq (s_\mu \cdot s_{\mu'})^{m+1} \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot ... \cdot s_{\gamma_k}$. Thus we can take $\beta' = \mu'$ in this case. Suppose that $\nu' < \mu'$. Now, observe that there is a subset, I of $\{\gamma_1, \gamma_2, ..., \gamma_k\}$ such that $\nu' \cap \gamma_i = \emptyset$ for all $\gamma_i \in I$ and $\nu' > \gamma_j$ and $\nu' \perp \gamma_j$ for all $\gamma_j \in \{\gamma_1, \gamma_2, ..., \gamma_k\} \setminus I$ since we have either $\mu' \cap \gamma_i = \emptyset$ or both $\mu' > \gamma_i$ and $\mu' \perp \gamma_i$ for any $1 \leq i \leq k$. Then $\nu' \cap \gamma_i = \emptyset$ for all $\gamma_i \in I$ since $\nu' < \mu'$. Here, we have two cases;

a) If we have either $\nu' \cap \gamma_j = \emptyset$ or $\nu' > \gamma_j$ and $\nu' \perp \gamma_j$ for any γ_j such that $\gamma_j \in \{\gamma_1, \gamma_2, ..., \gamma_k\} \setminus I$ then we can take $\beta' = \nu'$ since $s_\nu \cdot s_{\nu'} \cdot (s_\mu \cdot s_{\mu'})^m \leq (s_\nu \cdot s_{\nu'})^{m+1}$ by Lemma 5.4.1 which follows by $s_\nu \cdot s_{\nu'} \cdot (s_\mu \cdot s_{\mu'})^m \cdot z_{d-(m+1)c} \leq (s_\nu \cdot s_{\nu'})^{m+1} \cdot z_{\mathbf{d}-(m+1)c}$.

b) Assume that we have neither $\nu' \cap \gamma_j = \emptyset$ nor both $\nu' > \gamma_j$ and $\nu' \perp \gamma_j$ for all $\gamma_j \in B$ for some $B \subseteq \{\gamma_1, \gamma_2, ..., \gamma_k\} \setminus I$. Note that s_{γ_i} and s_{γ_j} Hecke commutes for any $1 \leq i, j \leq k$ so we can assume that $\gamma_1 \in B$. Now, note that supp(μ) and supp(γ_1) are disconnected since $\mu' > \gamma_1$

and $\mu' \perp \gamma_1$ so γ_1 Hecke commute with s_μ and $s_{\mu'}$. Thus $s_\nu \cdot s_{\nu'} \cdot (s_\mu \cdot s_{\mu'})^m \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot \ldots \cdot s_{\gamma_k} = s_\nu \cdot s_{\nu'} \cdot s_{\gamma_1} \cdot (s_\mu \cdot s_{\mu'})^m s_{\gamma_2} \cdot \ldots \cdot s_{\gamma_k}$. Here, we have several cases;

- If $\nu' = \gamma_1$ then $s_\nu \cdot s_{\nu'} \cdot s_{\gamma_1} = s_{c-\gamma_1} \cdot s_{\gamma_1} \cdot s_{\gamma_1}$. By Lemma 5.3.1, $s_{c-\gamma_1} \cdot s_{\gamma_1} \cdot s_{\gamma_1} \leq s_{\gamma_1} \cdot s_{c-\gamma_1} \cdot s_{\gamma_1}$ which follows by $s_\nu \cdot s_{\nu'} \cdot s_{\gamma_1} \cdot (s_\mu \cdot s_{\mu'})^m \cdot s_{\gamma_2} \cdot \ldots \cdot s_{\gamma_k} \leq s_{\gamma_1} \cdot s_{c-\gamma_1} \cdot s_{\gamma_1} \cdot (s_\mu \cdot s_{\mu'})^m \cdot s_{\gamma_2} \cdot \ldots \cdot s_{\gamma_k} = s_{\gamma_1} \cdot s_{c-\gamma_1} \cdot (s_\mu \cdot s_{\mu'})^m \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot \ldots \cdot s_{\gamma_k}$. Here, observe that $(c-\gamma_1) > \mu$ and $(c-\gamma_1) \perp \mu$ since $\mu' > \gamma_1$ and $\mu' \perp \gamma_1$. So by Lemma 5.4.2 we get $s_{\gamma_1} \cdot s_{c-\gamma_1} \cdot (s_\mu \cdot s_{\mu'})^m \leq (s_\mu \cdot s_{\mu'})^{m+1}$ which follows by $s_{\gamma_1} \cdot s_{c-\gamma_1} \cdot (s_\mu \cdot s_{\mu'})^m \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot \ldots \cdot s_{\gamma_k}. \leq (s_\mu \cdot s_{\mu'})^{m+1} \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot \ldots \cdot s_{\gamma_k}$. Thus we can take $\beta' = \mu'$ in this case.

- If $\nu' < \gamma_1$ then $\nu \cap \gamma_1 \neq \emptyset$ since $\nu + \nu' = c$. Assume that $\nu \cap \gamma_1$ is a root and let $\gamma := \nu \cap \gamma_1$. Then, by part 2) in Lemma 3.2.1 we have $s_\nu \cdot s_{\gamma_1} = s_\gamma \cdot s_{\nu-\gamma} \cdot s_{\gamma_1}$. Here, note that $\nu - \gamma + \nu' = c - \gamma$ is a root and by the same lemma $s_{\nu-\gamma} \cdot s_{\nu'} \leq s_{c-\gamma}$. Hence $s_\nu \cdot s_{\nu'} \cdot s_{\gamma_1} = s_\nu \cdot s_{\gamma_1} \cdot s_{\nu'} = s_\gamma \cdot s_{\nu-\gamma} \cdot s_{\gamma_1} \cdot s_{\nu'} = s_\gamma \cdot s_{\nu-\gamma} \cdot s_{\nu'} \cdot s_{\gamma_1} \leq s_\gamma \cdot s_{c-\gamma} \cdot s_{\gamma_1}$ which follows by $s_\nu \cdot s_{\nu'} \cdot s_{\gamma_1} \cdot (s_\mu \cdot s_{\mu'})^m \cdot s_{\gamma_2} \cdot \ldots \cdot s_{\gamma_k} \leq s_\gamma \cdot s_{c-\gamma} \cdot s_{\gamma_1} \cdot (s_\mu \cdot s_{\mu'})^m \cdot s_{\gamma_2} \cdot \ldots \cdot s_{\gamma_k} = s_\gamma \cdot s_{c-\gamma} \cdot (s_\mu \cdot s_{\mu'})^m \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot \ldots \cdot s_{\gamma_k}$. Here, observe that $(c-\gamma) > \mu$ and $(c-\gamma) \perp \mu$ since $\gamma < \gamma_1$ where $\mu' > \gamma_1$ and $\mu' \perp \gamma_1$. So by Lemma 5.4.2 we get $s_\gamma \cdot s_{c-\gamma} \cdot (s_\mu \cdot s_{\mu'})^m \leq (s_\mu \cdot s_{\mu'})^{m+1}$ which follows by $s_\gamma \cdot s_{c-\gamma} \cdot (s_\mu \cdot s_{\mu'})^m \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot \ldots \cdot s_{\gamma_k}. \leq (s_\mu \cdot s_{\mu'})^{m+1} \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot \ldots \cdot s_{\gamma_k}$. Again we can take $\beta' = \mu'$ in this case. If $\nu \cap \gamma_1$ is not a root then the proof is similar.

- If $\gamma_1 < \nu'$ then since ν' is not perpendicular to γ_1 we have $< \nu', \gamma_1^\vee >= 1$ so $s_{\gamma_1}(\nu') = \nu' - \gamma_1$. So $\nu' - \gamma_1$ is a root which implies that $\nu + \gamma_1 = c - \nu' + \gamma_1 = c - (\nu' - \gamma_1)$ is also a root. Thus $s_\nu \cdot s_{\nu'} \cdot s_{\gamma_1} = s_\nu \cdot s_{\gamma_1} \cdot s_{\nu'} = s_{\nu+\gamma_1} \cdot s_{\nu'}$ by Lemma 3.2.1. Here, note that

$(\nu+\gamma_1)\cap\nu'=\gamma_1$. So by the same lemma we get $s_{\nu+\gamma_1}\cdot s_{\nu'}=s_{\nu+\gamma_1}\cdot s_{\nu'-\gamma_1}\cdot s_{\gamma_1}$ which follows by $s_\nu\cdot s_{\nu'}\cdot s_{\gamma_1}\cdot(s_\mu\cdot s_{\mu'})^m\cdot s_{\gamma_2}\cdot\ldots\cdot s_{\gamma_k}\leq s_{\nu+\gamma_1}\cdot s_{\nu'-\gamma_1}\cdot s_{\gamma_1}\cdot(s_\mu\cdot s_{\mu'})^m\cdot s_{\gamma_2}\cdot\ldots\cdot s_{\gamma_k}=s_{\nu+\gamma_1}\cdot s_{\nu'-\gamma_1}\cdot(s_\mu\cdot s_{\mu'})^m\cdot s_{\gamma_1}\cdot s_{\gamma_2}\cdot\ldots\cdot s_{\gamma_k}$. Here, observe that $(\nu'-\gamma_1)\cap\gamma_1=\emptyset$. Now, if $B=\{\gamma_1\}$ then for any $1\leq i\leq k$ we have either $(\nu'-\gamma_1)\cap\gamma_i=\emptyset$ or both $(\nu'-\gamma_1)>\gamma_i$ and $(\nu'-\gamma_1)\perp\gamma_i$. Also, note that $(\nu'-\gamma_1)<\mu'$ since $\nu'<\mu'$ which implies that $s_{\nu+\gamma_1}\cdot s_{\nu'-\gamma_1}\cdot(s_\mu\cdot s_{\mu'})^m\leq(s_{\nu+\gamma_1}\cdot s_{\nu'-\gamma_1})^{m+1}$ which follows by $s_{\nu+\gamma_1}\cdot s_{\nu'-\gamma_1}\cdot(s_\mu\cdot s_{\mu'})^m\cdot s_{\gamma_1}\cdot s_{\gamma_2}\cdot\ldots\cdot s_{\gamma_k}.\leq(s_{\nu+\gamma_1}\cdot s_{\nu'-\gamma_1})^{m+1}\cdot s_{\gamma_1}\cdot s_{\gamma_2}\cdot\ldots\cdot s_{\gamma_k}$. Thus we can take $\beta'=\nu'-\gamma_1$ in this case. Now, suppose that $|B|>1$. Then B can have at most two elements since we have neither $\nu'\cap\gamma_j=\emptyset$ nor both $\nu'>\gamma_j$ and $\nu'\perp\gamma_j$ for all $\gamma_j\in B$ which implies that ν' intersect with γ_j and it is not strictly bigger than and not perpendicular to γ_j for any $\gamma_j\in B$ and also by the fact that supp(γ_i) and supp(γ_j) are disconnected for any $1\leq i,j\leq k$. Now, let $\gamma_q\in B$ such that $\gamma_q\neq\gamma_1$. Then one can show that $s_{\nu+\gamma_1}\cdot s_{\nu'-\gamma_1}\cdot(s_\mu\cdot s_{\mu'})^m\cdot s_{\gamma_1}\cdot s_{\gamma_2}\cdot\ldots\cdot s_{\gamma_k}.\leq(s_\beta\cdot s_{\beta'})^{m+1}\cdot s_{\gamma_1}\cdot s_{\gamma_2}\cdot\ldots\cdot s_{\gamma_k}$ for some affine positive real roots $\beta,\beta'\in\Pi_{\text{aff}}^{\text{re},+}$ such that either $\beta'\cap\gamma_i=\emptyset$ or both $\beta'>\gamma_i$ and $\beta'\perp\gamma_i$ for any $i=1,2,\ldots,k$ by considering the relationships between $\nu'-\gamma_1$ and γ_q which are identical with those between ν' and γ_1 above under case b).

□

The result for the most general case is:

Theorem 5.6.3. *Let* $\mathbf{d}=(d_0,d_1,\ldots,d_{n-1})>c$ *be a degree and* $m=min\{d_0,d_1,\ldots,d_{n-1}\}$. *Also, assume that* $z_{\mathbf{d}-mc}=s_{\gamma_1}s_{\gamma_2}\ldots s_{\gamma_k}$ *for some* k *where this expression is obtained in Theo-*

rem 3.1.2. Then

1) $\Gamma_{\mathbf{d}}(id) = \{t_{m\beta'} z_{\mathbf{d}-mc} : \beta' \in \Pi^{re,+}_{aff}(\mathbf{d}-mc)\} \cup \{t_{m(\beta'-c)} z_{\mathbf{d}-mc} : \beta' \in \Pi^{re,+}_{aff}(\mathbf{d}-mc)\}$

2) $|\Gamma_{\mathbf{d}}(id)| = |\{\beta' : \beta' \in \Pi^{re,+}_{aff}(\mathbf{d}-mc)\}|$

3) For all $w \in \Gamma_{\mathbf{d}}(id)$, $\ell(w) = 2m(n-1) + \ell(z_{\mathbf{d}-mc})$

where $\Pi^{re,+}_{aff}(\mathbf{d}-mc)$ *is the set of* $\beta' \in \Pi^{re,+}_{aff}$ *such that* $\beta' < c$ *and either* $\beta' \cap \gamma_i = \emptyset$ *or both* $\beta' > \gamma_i$ *and* $\beta' \perp \gamma_i$ *for any* $i = 1, ..., k$.

Proof. 1) First, note that we have either $\mathrm{supp}(\gamma_i)$ and $\mathrm{supp}(\gamma_j)$ are disconnected or both $\gamma_i \perp \gamma_j$ and γ_i, γ_j are comparable, for any $1 \leq i, j \leq k$ such that $i \neq j$; see Theorem 3.1.2. Now, we define $B := \{\gamma_{i_1}, \gamma_{i_2}, ..., \gamma_{i_t}\} \subseteq \{\gamma_1, \gamma_2, ..., \gamma_k\}$ such that for any $1 \leq i \leq k$, we have $\gamma_i \leq \gamma_{i_l}$ for some $\gamma_{i_l} \in B$. Note that, any two elements of B have disconnected supports. Now, observe that for a root $\gamma_{i_j} \in B$ and $\beta' \in \Pi^{\mathrm{re},+}_{\mathrm{aff}}$ such that $\beta' \cap \gamma_{i_j} = \emptyset$ or both $\beta' > \gamma_{i_j}$ and $\beta' \perp \gamma_{i_j}$ implies that we have either $\beta' \cap \gamma_i = \emptyset$ or both $\beta' > \gamma_i$ and $\beta' \perp \gamma_i$ for all $1 \leq i \leq k$ such that $\gamma_i \leq \gamma_{i_j}$, respectively. Also, for any $1 \leq i \leq k$ such that $\beta' \cap \gamma_i = \emptyset$ or both $\beta' > \gamma_i$ and $\beta' \perp \gamma_i$ implies that we have either $\beta' \cap \gamma_{i_j} = \emptyset$ or both $\beta' > \gamma_{i_j}$ and $\beta' \perp \gamma_{i_j}$ where $\gamma_i \leq \gamma_{i_j}$, respectively. So we may assume that $B = \{\gamma_1, \gamma_2, ..., \gamma_k\}$ i.e we will suppose that $\mathrm{supp}(\gamma_i)$ and $\mathrm{supp}(\gamma_j)$ are disconnected for any $1 \leq i, j \leq k$ such that $i \neq j$.

Let $id \xrightarrow{\beta_1} s_{\beta_1} \xrightarrow{\beta_2} s_{\beta_1} s_{\beta_2} ... \xrightarrow{\beta_r} w = s_{\beta_1} s_{\beta_2} ... s_{\beta_r}$ be a path in the moment graph such that $\sum_{i=1}^{r} \beta_i \leq \mathbf{d}$. Note that by Remark 4.0.2 we can assume that $w = s_{\beta_1} \cdot s_{\beta_2} \cdot ... \cdot s_{\beta_r}$ where $\sum_{i=1}^{r} \beta_i = \mathbf{d}$ and $\beta_i < c$ for all i. Now, note that $t_{m\beta'}$ and $t_{m(\beta'-c)}$ are given by $(s_\beta s_{\beta'})^m$

for some $\beta, \beta' \in \Pi_{\text{aff}}^{\text{re},+}$ such that $\beta + \beta' = c$; see the proof of Lemma 3.4.1. So it is enough to show that $w \leq (s_\beta s_{\beta'})^m z_{\mathbf{d}-mc}$ where we have either $\beta' \cap \gamma_i = \emptyset$ or both $\beta' > \gamma_i$ and $\beta' \perp \gamma_i$ for any $i = 1, 2, ..., k$. Furthermore, we can assume that there is an integer p such that $\sum_{i=1}^{p} \beta_i = c$ and $\sum_{i=p+1}^{r} \beta_i = \mathbf{d} - c$, by Lemma 5.3.2. We will prove the statement by the induction on m.

First, suppose that $m = 1$. Then by Equation 5.1, $s_{\beta_1} \cdot s_{\beta_2} \cdot ... \cdot s_{\beta_p} \leq s_\alpha \cdot s_{\alpha'}$ for some $\alpha, \alpha' \in \Pi_{\text{aff}}^{\text{re},+}$ such that $\alpha + \alpha' = c$. Moreover, $s_{\beta_{p+1}} \cdot ... \cdot s_{\beta_r} \leq z_{\mathbf{d}-c}$, by Theorem 5.1.1. Thus

$$w = (s_{\beta_1} \cdot s_{\beta_2} \cdot ... \cdot s_{\beta_p}) \cdot (s_{\beta_{p+1}} \cdot ... \cdot s_{\beta_r}) \leq (s_\alpha \cdot s_{\alpha'}) \cdot z_{\mathbf{d}-c}.$$

Now, by Lemma 5.6.1 we have $s_\alpha \cdot s_{\alpha'} \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot ... \cdot s_{\gamma_k} \leq s_\beta \cdot s_{\beta'} \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot ... \cdot s_{\gamma_k}$ for some affine positive real roots, $\beta, \beta' \in \Pi_{\text{aff}}^{\text{re},+}$ such that we have either $\beta' \cap \gamma_i = \emptyset$ or both $\beta' > \gamma_i$ and $\beta' \perp \gamma_i$ for any $i = 1, 2, ..., k$.

Now, we will suppose that the statement is true for m and prove that it is also true for $m+1$. Again, by Equation 5.1 we have $s_{\beta_1} \cdot s_{\beta_2} \cdot ... \cdot s_{\beta_p} \leq s_\alpha \cdot s_{\alpha'}$ for some $\alpha, \alpha' \in \Pi_{\text{aff}}^{\text{re},+}$ such that $\alpha + \alpha' = c$ and $s_{\beta_{p+1}} \cdot ... \cdot s_{\beta_r} \leq (s_\mu \cdot s_{\mu'})^m \cdot z_{\mathbf{d}-(m+1)c}$ for some $\mu, \mu' \in \Pi_{\text{aff}}^{\text{re},+}$ such that $\mu + \mu' = c$ where either $\mu' \cap \gamma_i = \emptyset$ or $\mu' > \gamma_j$ and $\mu' \perp \gamma_j$ by induction assumption. Thus

$$w = (s_{\beta_1} \cdot s_{\beta_2} \cdot ... \cdot s_{\beta_p}) \cdot (s_{\beta_{p+1}} \cdot ... \cdot s_{\beta_r}) \leq (s_\alpha \cdot s_{\alpha'}) \cdot (s_\mu \cdot s_{\mu'})^m \cdot z_{\mathbf{d}-(m+1)c}.$$

Now, observe that $s_\alpha \cdot s_{\alpha'} \cdot s_\mu \leq s_\nu \cdot s_{\nu'} \cdot s_\mu$ for some $\nu, \nu' \in \Pi_{\text{aff}}^{\text{re},+}$ such that $\nu + \nu' = c$ and either $\nu' \leq \mu'$ or both $\nu' > \mu$ and $\nu' \perp \mu$ by Equation 5.2. Hence $s_\alpha \cdot s_{\alpha'} \cdot (s_\mu \cdot s_{\mu'})^m \cdot z_{\mathbf{d}-(m+1)c} \leq (s_\nu \cdot s_{\nu'}) \cdot (s_\mu \cdot s_{\mu'})^m \cdot z_{\mathbf{d}-(m+1)c}$. By Lemma 5.6.2 we have $s_\nu \cdot s_{\nu'} \cdot (s_\mu \cdot s_{\mu'})^m \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot ... \cdot s_{\gamma_k} \leq (s_\beta \cdot s_{\beta'})^{m+1} \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot ... \cdot s_{\gamma_k}$ for some affine positive real roots, β, β' such that $\beta + \beta' = c$ where either $\beta' \cap \gamma_i = \emptyset$ or both $\beta' > \gamma_i$ and $\beta' \perp \gamma_i$ for any $i = 1, 2, ..., k$. Furthermore, by Lemma 3.4.9 , $z_{\mathbf{d}-(m+1)c} = s_{\gamma_1} s_{\gamma_2} ... s_{\gamma_k}$ is reduced so $s_{\gamma_1} s_{\gamma_2} ... s_{\gamma_k} = s_{\gamma_1} \cdot s_{\gamma_2} \cdot ... \cdot s_{\gamma_k}$ which follows by $(s_\beta \cdot s_{\beta'})^{m+1} \cdot s_{\gamma_1} \cdot s_{\gamma_2} \cdot ... \cdot s_{\gamma_k} = (s_\beta \cdot s_{\beta'})^{m+1} \cdot (s_{\gamma_1} s_{\gamma_2} ... s_{\gamma_k}) = (s_\beta \cdot s_{\beta'})^{m+1} \cdot z_{\mathbf{d}-(m+1)c}$. Now by Lemma 3.4.10, the element $(s_\beta s_{\beta'})^{m+1} z_{\mathbf{d}-(m+1)c}$ is reduced so $(s_\beta \cdot s_{\beta'})^{m+1} \cdot z_{\mathbf{d}-(m+1)c} = (s_\beta s_{\beta'})^{m+1} z_{\mathbf{d}-(m+1)c}$.

2) Let $(s_\beta s_{\beta'})^m z_{\mathbf{d}-mc}$ and $(s_\nu s_{\nu'})^m z_{\mathbf{d}-mc} \in \Gamma_{\mathbf{d}}(id)$. We need to show that $(s_\beta s_{\beta'})^m z_{\mathbf{d}-mc} \neq (s_\nu s_{\nu'})^m z_{\mathbf{d}-mc}$ if $\beta' \neq \nu'$. Now assume that $\beta' \neq \nu'$ but

$$(s_\beta s_{\beta'})^m s_{\gamma_1} s_{\gamma_2} ... s_{\gamma_k} = (s_\nu s_{\nu'})^m s_{\gamma_1} s_{\gamma_2} ... s_{\gamma_k}. \tag{5.6}$$

By Lemma 3.4.10 both $(s_\beta s_{\beta'})^m s_{\gamma_1} s_{\gamma_2} ... s_{\gamma_k}$ and $(s_\nu s_{\nu'})^m s_{\gamma_1} s_{\gamma_2} ... s_{\gamma_k}$ are reduced, so Equation 5.6 implies that $(s_\beta s_{\beta'})^m = (s_\nu s_{\nu'})^m$ but this is a contradiction by Lemma 3.4.1.

3) Let $w \in \Gamma_{\mathbf{d}}(id)$. Then by Lemma 3.4.10 we have $\ell(w) = 2m(n-1) + \ell(z_{\mathbf{d}-mc})$.

□

Remark 5.6.4. Now, note that both $t_{m\beta'}$ and $t_{m(\beta'-c)}$ are given by $(s_\beta s_{\beta'})^m$ where $\beta, \beta' \in \Pi_{\text{aff}}^{\text{re},+}$ such that $\beta + \beta' = c$ and m is a positive integer; see the proof of Lemma 3.4.1. So by Theorem 5.6.3 we get

$$\Gamma_{\mathbf{d}}(id) = \{(s_\beta s_{\beta'})^m z_{\mathbf{d}-mc} : \beta' \in \Pi_{\text{aff}}^{\text{re},+}(\mathbf{d} - mc) \text{ such that } \beta + \beta' = c\}. \tag{5.7}$$

Example 5.6.5. Let W_{aff} be the affine Weyl group associated to $A_3^{(1)}$ and $\mathbf{d} = (6, 5, 8, 5)$. Then $\mathbf{d} = 5c + (1, 0, 3, 0)$ so $m = 5$ and $\mathbf{d} - 5c = (1, 0, 3, 0)$. Also, $z_{\mathbf{d}-5c} = s_{\alpha_0} \cdot s_{\alpha_2} \cdot s_{\alpha_2} \cdot s_{\alpha_2} = s_{\alpha_0} s_{\alpha_2}$.

Hence by Theorem 1.1.14 we get

$$\Gamma_{\mathbf{d}}(id) = \{t_{5\beta_1'} z_{\mathbf{d}-5c}, t_{5\beta_2'} z_{\mathbf{d}-5c}, t_{5\beta_4'} z_{\mathbf{d}-5c}, t_{5(\beta_3'-c)} z_{\mathbf{d}-5c}\}$$

where $\beta_1' = \alpha_1$, $\beta_2' = \alpha_3$, $\beta_3' = \alpha_0 + \alpha_1 + \alpha_3$, $\beta_4' = \alpha_1 + \alpha_2 + \alpha_3$. Moreover, for all $w \in \Gamma_{\mathbf{d}}(id)$, $\ell(w) = 2m(n-1) + \ell(z_{\mathbf{d}-mc}) = 2 \cdot 5 \cdot 3 + 2 = 32$.

5.7 Reduction from $\Gamma_{\mathbf{d}}(w)$ to $\Gamma_{\mathbf{d}}(id)$

In this section we will discuss our last result which indicates that to calculate $\Gamma_{\mathbf{d}}(w)$ for any given degree $\mathbf{d}$ and $w \in W_{\text{aff}}$ one only needs to calculate $\Gamma_{\mathbf{d}}(id)$.

Theorem 5.7.1. *Let $w \in W_{\mathit{aff}}$ and $\mathbf{d}$ be any degree. Then*

$$\Gamma_{\mathbf{d}}(w) = max\{w \cdot u : u \in \Gamma_{\mathbf{d}}(id)\}\,.$$

Proof. Let

$$w \xrightarrow{\beta_1} ws_{\beta_1} \xrightarrow{\beta_2} ws_{\beta_1}s_{\beta_2}... \xrightarrow{\beta_k} ws_{\beta_1}s_{\beta_2}...s_{\beta_k}$$

be a path in the moment graph such that $\sum_{i=1}^{k} \beta_i \leq \mathbf{d}$. Then

$$ws_{\beta_1}s_{\beta_2}...s_{\beta_k} \leq w \cdot (s_{\beta_1}s_{\beta_2}...s_{\beta_k}) \leq w \cdot u$$

for some $u \in \Gamma_{\mathbf{d}}(id)$. To complete the proof we need to show that the element $w \cdot u$ can be reached by a path which starts with an element that is smaller than and equal to w and has a degree at most $\mathbf{d}$. Now, note that $w \cdot u = vu$ for some $v \in W_{\text{aff}}$ such that $v \leq w$ by property

e) of the Hecke product in Section 2.4. In addition, $u = s_{\beta'_1} s_{\beta'_2} ... s_{\beta'_t}$ for some positive affine real roots β'_i, $i = 1, 2, ..., t$ such that $\sum_{i=1}^{t} \beta'_i \leq \mathbf{d}$ since $u \in \Gamma_{\mathbf{d}}(id)$. Thus

$$v \xrightarrow{\beta'_1} vs_{\beta'_1} \xrightarrow{\beta'_2} vs_{\beta'_1} s_{\beta'_2} ... \xrightarrow{\beta'_t} vs_{\beta'_1} s_{\beta'_2} ... s_{\beta'_k} = vu$$

is the path we are looking for.

□

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