

Research Article

A Positive Answer on Nirenberg's Problem on Expansive Mappings in Hilbert Spaces

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Nirenberg proposed a problem as to whether or not a continuous and expansive operator $T : X \rightarrow X$ (where X is a Hilbert space) is surjective if $R(T)^\circ \neq \emptyset$. I shall give a positive answer for the problem provided that $R(T)^\circ$ is unbounded. For contents related to this paper, the reader is referred to the remarks and the study of Asfaw (2021). The present paper gives a complete answer for the problem that has been open for about 47 years.

1. Introduction and Preliminaries

In what follows, X denotes a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$, and the symbol $\|\cdot\|$ denotes the norm in X . An operator $T : X \rightarrow X$ is called " α -expansive" if there exists $\alpha > 0$ such that $\|Tx - Ty\| \geq \alpha\|x - y\|$ for all $x \in X$ and $y \in X$. The operator is called expansive if $\alpha = 1$. In 1974, Nirenberg [1] proposed a problem as to whether or not a continuous and expansive operator is surjective provided that $R(T)^\circ \neq \emptyset$.

The main goal of the present paper is to give a positive answer for the problem provided that $R(T)^\circ$ is unbounded. Clearly, T cannot be surjective if $R(T)^\circ$ is bounded.

Remark 1. Recently, I published a paper in [2] planning to give a proof of the problem. A minor error was found after publication. The editors decided a retraction based on their belief that the error is profound and not fixable. However, the error was minor, not profound and fixable. I submitted a corrigendum about the published paper but the editors decided to retract it based on the error in the original paper. Thus, the present paper reuses contents from [2]. The differences between the current and the retracted papers are given below.

- (i) In Theorem 7, it is assumed that $R(T)$ has unbounded interior which was not assumed in [2].

It is obvious to see that T cannot be surjective if $R(T)^\circ$ is bounded

- (ii) The sequences $p(n, m)$ and $q(n, m)$ used in Step 1 in [2] are not required (in the current paper) to show the (S_+) property of the operator $S + \lambda I$. Instead, the (S_+) property of $S + \lambda I$ is achieved based on Kirszbraun-Valentine [3, 4] extension theorem (c.f., $L^{10} - L^{27}$ on page 5 of the current paper). It is essential to mention that Valentine's [4] result is a generalization of Kirszbraun [3] extension theorem

Morel and Steinlein [5] constructed an example of such an operator T defined from $\ell^1(N)$ into $\ell^1(N)$ satisfying the stated conditions but fails to be surjective. In a Hilbert space setting (i.e., if $H = \ell^2(N)$), Szczepanski [6] constructed a family of nonsurjective continuous maps $\{F_\varepsilon : H \rightarrow H\}_{\varepsilon > 0}$ such that $R(F_\varepsilon)$ contains $B_1(0)$ and F_ε is expansive on each sphere and $\|F_\varepsilon x - F_\varepsilon y\| \geq \|x - y\|$ for all $x \in P_\varepsilon$ or $y \in P_\varepsilon$, where $P_\varepsilon = \{x \in H : 1 < \|x\| < 1 + \varepsilon\}$. In addition, Szczepanski [7] gave a negative answer for the problem in $X = \ell^2(N)$ by constructing a continuous and nonsurjective operator T satisfying weak expansive condition. Chang and Shujie [8], Theorem 7, gave a positive answer for the problem provided that $T : X \rightarrow Y$ is Fréchet differentiable on X (where X is a real Banach space and Y is reflexive), and its derivative is locally bounded.

Recently, Kartsatos [9], Theorem 6 (with $\alpha = 1$ and $c \in (0, 1)$), and Xiang [10], Theorem 2.12 (with $c \in (-\infty, \alpha)$), gave surjectivity results if the condition $R(T)^\circ \neq \emptyset$ is replaced by strong monotonicity of $T + \alpha I$.

The following remarks on the recent result due to Ives and Preiss [11], are essential.

Remark 2. Ives and Preiss [11], Theorem 4, constructed an operator $\mathcal{P} : X \rightarrow X$ (where $X = L^2(0, \infty)$) satisfying the following properties.

- (i) $\mathcal{P}x = x$ for all $x \in \bar{B}_1(0)$
- (ii) The \mathcal{P} image of $\{x \in X : \|x\| \geq 1\}$ is nowhere dense in X
- (iii) \mathcal{P} is not surjective

We shall show that (i) and (ii) imply the boundedness of $\mathcal{R}(\mathcal{P})^\circ$. Indeed, let $F = \mathcal{P}(X \setminus B_1(0))$. The continuity and expansiveness of \mathcal{P} and the closedness of $X \setminus B_1(0)$ show that F is closed. Since F is nowhere dense, it follows that $F^\circ = \emptyset$, i.e., $F = \partial F$ and $X \setminus F = X$. Let $V = R(\mathcal{P})^\circ$. Then, for each $x \in V$, there exists a sequence $\{x_n\}$ in $X \setminus F$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$. Since V is nonempty and open containing x (V is a neighbourhood of x), there exists a positive integer N such that $x_n \in V$ for all $n \geq N$. In addition, we see that

$$R(\mathcal{P}) = \mathcal{P}(B_1(0)) \cup \mathcal{P}(X \setminus B_1(0)) = B_1(0) \cup F, \quad (1)$$

i.e., we have $V = V \cap (B_1(0) \cup F) = B_1(0) \cup (F \cap V)$. Since $x_n \in F$ for all n (i.e., $x_n \in F \cap V$ for all n) and $B_1(0)$ and $F \cap V$ are disjoint, we conclude that $x_n \in B_1(0)$ for all $n \geq N$. This shows that $x \in \bar{B}_1(0)$, i.e., $V \subseteq \bar{B}_1(0)$. Thus, the set V is bounded, i.e., \mathcal{P} is not surjective.

Remark 3. Remark 2 shows that conditions (i) through (v) in Theorem 4 due to Ives and Preiss [11] imply that $R(T)$ has a bounded interior, i.e., T is nonsurjective. It is essential to mention that the result in [11] did not give a negative answer for Nirenberg’s problem if $R(T)^\circ$ is unbounded. In addition, we point out that the case of boundedness of $R(T)^\circ$ can be ignored from the statement of Nirenberg’s problem.

The following definition is needed in the sequel.

Definition 4. Let Y be a real reflexive Banach space and Y^* be its dual space. Let G be a nonempty, bounded, and open subset of Y and $S : \bar{G} \rightarrow Y^*$ be bounded on finite dimensional subsets of Y . Then, S is said to be “of type (S_+) ” if a sequence $\{x_n\}$ in \bar{G} satisfies $x_n \rightarrow x_0$ and $\limsup_{n \rightarrow \infty} \langle Sx_n, x_n - x_0 \rangle \leq 0$; then, $x_n \rightarrow x_0$ and $Sx_n \rightarrow Sx_0$ as $n \rightarrow \infty$.

Browder [12], Definition 5, introduced the following definition.

Definition 5. Let G be a nonempty, bounded and open subset of a reflexive Banach space Y . A family of operators $\{S^t\}_{t \in [0,1]}$, defined from \bar{G} into Y^* , is called a “homotopy of

class (S_+) ” if for sequences $\{x_n\}$ in \bar{G} and $\{t_n\}$ in $[0, 1]$ such that $x_n \rightarrow x_0$, $t_n \rightarrow t_0$ and $\limsup_{n \rightarrow \infty} \langle S^{t_n}x_n, x_n - x_0 \rangle \leq 0$, we have $x_n \rightarrow x_0$ and $S^{t_n}x_n \rightarrow S^{t_0}x_0$ as $n \rightarrow \infty$.

The operator $J : Y \rightarrow 2^{Y^*}$ defined by

$$J(y) = \{y^* \in Y^* : \langle y^*, y \rangle = \|y\|^2, \|y^*\| = \|y\|\}, \quad (2)$$

is called the normalized duality mapping. It is well-known that J is single-valued, bounded, monotone, bicontinuous, surjective, and of type (S_+) if Y and Y^* are locally uniformly convex and real reflexive Banach spaces. The operator $J = I$ (where I is the identity operator on Y) if Y is a real Hilbert space. Browder [12], Theorems 3 and 4, constructed a degree mapping d for the class of operators of type (S_+) that satisfies the basic properties listed in Lemma 6.

Lemma 6. Let G be a nonempty, bounded, and open subset of a real reflexive Banach space Y . Let $f^* \in Y^*$. Then, the following basic properties hold.

- (i) *Normalization.* $d(J, G, 0) = 1$ if $0 \in G$ and $d(J, G, 0) = 0$ if $0 \notin \bar{G}$.
- (ii) *Existence.* Let $S : \bar{G} \rightarrow Y^*$ be a bounded operator of type (S_+) such that $f^* \in S(\partial G)$ and $d(S, G, f^*) \neq 0$. Then, $f^* \in S(G)$.
- (iii) *Homotopy Invariance.* Let $\{S^t\}_{t \in [0,1]}$ be a homotopy of class (S_+) defined from \bar{G} into Y^* such that $f^* \in S^t(\partial G)$ for all $t \in [0, 1]$. Then, $d(S^t, G, f^*)$ is independent of $t \in [0, 1]$.
- (iv) *Invariance under Affine Homotopy.* Let $S_i : \bar{G} \rightarrow Y^*$ ($i = 1, 2$) be a bounded operator of type (S_+) and $K^t = tS_1 + (1 - t)S_2$ for $t \in [0, 1]$. Then, $\{K^t\}_{t \in [0,1]}$ is a homotopy of class (S_+) , and $d(K^t, G, f^*)$ is independent of $t \in [0, 1]$ provided that $f^* \in K^t(\partial G)$ for all $t \in [0, 1]$.

2. Main Result

Theorem 7 gives a proof of the Nirenberg problem.

Theorem 7. Let X be a real Hilbert space and $T : X \rightarrow X$ be a continuous and α -expansive operator such that $R(T)^\circ$ is unbounded. Then, T is surjective.

Proof. Choose $x_0 \in R(T)^\circ$. Then, there exists $r > 0$ such that $B_r(x_0) \subseteq R(T)$. Let $\tilde{T}x = Tx - x_0$. Then, the operator $\tilde{T} : X \rightarrow X$ is α -expansive and continuous such that $R(T) - x_0 = R(\tilde{T})$. It follows that $R(T)^\circ$ is unbounded if and only if $R(\tilde{T})^\circ$ is unbounded. Clearly, the set $V = B_r(x_0) - x_0$ is open and $0 \in V \subseteq R(\tilde{T})$, i.e., $0 \in R(\tilde{T})^\circ$. In addition, we see that T is surjective if and only if \tilde{T} is surjective. Thus, we assume without loss of generality that $0 \in R(T)^\circ$. Let $U = R(T)^\circ$ and $f \in X$. The continuity and α -expansive conditions imply

that $R(T)$ is closed and

$$\|T^{-1}x - T^{-1}y\| \leq \alpha^{-1}\|x - y\|, \quad (3)$$

for all $x \in R(T)$ and $y \in R(T)$. Then, it follows that

$$\begin{aligned} \langle (T^{-1}x + \alpha^{-1}x) - (T^{-1}y + \alpha^{-1}y), x - y \rangle &= \langle T^{-1}x - T^{-1}y, x - y \rangle \\ &\quad + \alpha^{-1}\|x - y\|^2 \\ &\geq -\|T^{-1}x - T^{-1}y\|\|x - y\| \\ &\quad + \alpha^{-1}\|x - y\|^2 \\ &\geq -\alpha^{-1}\|x - y\|^2 + \alpha^{-1}\|x - y\|^2 \\ &= 0, \end{aligned} \quad (4)$$

for all $x \in R(T)$ and $y \in R(T)$, i.e., $T^{-1} + \alpha^{-1}I : R(T) \rightarrow X$ is a continuous and monotone operator. It holds that $\bar{U} \subseteq R(T)$ because $R(T)$ is closed and $U \subseteq R(T)$. Let $S : \bar{U} \rightarrow X$ be the restriction of $T^{-1} + \alpha^{-1}I$ on \bar{U} . Let $\lambda > 0$, and for each $f \in X$ and $\mu \in (0, 1)$, let $f_\mu = \mu^{-1}f$. Let

$$S_\lambda^f x = Sx + \lambda(x - f_\mu), \quad x \in \bar{U}. \quad (5)$$

The monotonicity of S implies

$$\langle S_\lambda^f x - S_\lambda^f y, x - y \rangle \geq \lambda\|x - y\|^2, \quad (6)$$

for all $x \in \bar{U}$ and $y \in \bar{U}$. The monotonicity of S and the condition $0 \in \bar{U}$ yield

$$\begin{aligned} \langle S_\lambda^f x, x \rangle &= \langle Sx - S0 + S0 + \lambda(x - f_\mu), x \rangle \\ &\geq \lambda\|x\|^2 \\ &\quad - (\lambda\|f_\mu\| + \|S0\|)\|x\|, \end{aligned} \quad (7)$$

for all $x \in \bar{U}$. Since \bar{U} is unbounded, the right side of (7) approaches infinity as $\|x\| \rightarrow \infty$. Then, there exists $R = R(f_\mu) > 0$ such that

$$\langle Sx + \lambda(x - f_\mu), x \rangle > 0, \quad (8)$$

for all $x \in \bar{U} \cap \partial B_R(0)$. Let $\Omega = U \cap B_R(0)$. We shall show that the operator $S + \lambda(I - f_\mu) : \bar{\Omega} \rightarrow X$ is of type (S_+) . We shall apply Kirszbraun-Valentine [3, 4] extension theorem. Indeed, the Lipschitz operator T^{-1} (with Lipschitz constant α^{-1}) can be extended to an operator $P : X \rightarrow X$ with the same Lipschitz constant. Thus, the operator $\tilde{S} = P + \alpha^{-1}I + \lambda(I - f_\mu) : X \rightarrow X$ is continuous and uniformly monotone, i.e., \tilde{S} is of type (S_+) . Let \tilde{S}_Ω be the restriction of \tilde{S} on $\bar{\Omega}$. We shall show that \tilde{S}_Ω is of type (S_+) . Indeed, let $x_n \in \bar{\Omega}$

for all n such that $x_n \rightarrow x_0 \in X$ as $n \rightarrow \infty$ and

$$\limsup_{n \rightarrow \infty} \langle \tilde{S}_\Omega x_n, x_n - x_0 \rangle \leq 0. \quad (9)$$

It is enough to show that $x_n \rightarrow x_0 \in \bar{\Omega}$. Since $\tilde{S} = \tilde{S}_\Omega$ on $\bar{\Omega}$, it follows that

$$\limsup_{n \rightarrow \infty} \langle \tilde{S}x_n, x_n - x_0 \rangle \leq 0. \quad (10)$$

The uniform monotonicity of \tilde{S} with domain X (i.e., using (5), (6) and $\tilde{S} = S_\lambda^f$ on $\bar{\Omega}$) shows that

$$\begin{aligned} \langle \tilde{S}x_n - \tilde{S}x_0 + \tilde{S}x_0, x_n - x_0 \rangle &= \langle \tilde{S}x_n - \tilde{S}x_0, x_n - x_0 \rangle \\ &\quad + \langle \tilde{S}x_0, x_n - x_0 \rangle \\ &\geq \lambda\|x_n - x_0\|^2 \\ &\quad + \langle \tilde{S}x_0, x_n - x_0 \rangle, \end{aligned} \quad (11)$$

for all n . Thus, (9)–(11) yield

$$\limsup_{n \rightarrow \infty} (\lambda\|x_n - x_0\|^2 + \langle \tilde{S}x_0, x_n - x_0 \rangle) \leq \limsup_{n \rightarrow \infty} \langle \tilde{S}x_n, x_n - x_0 \rangle \leq 0. \quad (12)$$

Since $\langle \tilde{S}x_0, x_n - x_0 \rangle \rightarrow 0$ as $n \rightarrow \infty$ (because $x_n \rightarrow x_0$), (12) shows that

$$\limsup_{n \rightarrow \infty} \lambda\|x_n - x_0\|^2 \leq 0, \quad (13)$$

i.e., $x_n \rightarrow x_0$ as $n \rightarrow \infty$. Clearly $x_0 \in \bar{\Omega}$ because $x_n \in \bar{\Omega}$ for all n . The continuity of \tilde{S}_Ω implies that $\tilde{S}_\Omega x_n \rightarrow \tilde{S}_\Omega x_0$ as $n \rightarrow \infty$. Thus, the operator $\tilde{S}_\Omega = S + \lambda(I - f_\mu) : \bar{\Omega} \rightarrow X$ is bounded, continuous, and of type (S_+) .

Clearly, the identity operator $I : \bar{\Omega} \rightarrow X$ is bounded and of type (S_+) . For each $\lambda > 0$, let $H_\lambda : [0, 1] \times \bar{\Omega} \rightarrow X$ be given by

$$H_\lambda(t, x) = t(Sx + \lambda x - \lambda f_\mu) + (1 - t)x, \quad (t, x) \in [0, 1] \times \bar{\Omega}. \quad (14)$$

Then, (iv) of Lemma 6 shows that the family $\{H_\lambda(t, \cdot)\}_{t \in [0, 1]}$ is an affine homotopy of class (S_+) . Clearly, $0 \neq H_\lambda(0, x)$ for all $x \in \partial\Omega$. Thus, exactly one of the following holds.

(i) There exists $\lambda > 0$ such that for each $\lambda' \geq \lambda$, there exist $t_{\lambda'} \in (0, 1]$ and $x_{\lambda'} \in \partial\Omega$ such that $H_{\lambda'}(t_{\lambda'}, x_{\lambda'}) = 0$ for all $\lambda'' \geq \lambda'$

(ii) For each $\lambda > 0$ there exists $\lambda' \geq \lambda$ such that

$$H_{\lambda'}(t, x) \neq 0 \text{ for all } (t, x) \in [0, 1] \times \partial\Omega \text{ and some } \lambda'' \geq \lambda'. \tag{15}$$

Suppose (i) holds, i.e., there exists $n > 0$ such that for each $n' \geq n$, this corresponds $t_{n'} \in (0, 1]$, $x_{n'} \in \partial\Omega \subseteq \bar{\Omega} \subseteq \bar{U} \subseteq R(T)$, and $y_{n'} \in X$ such that $x_{n'} = Ty_{n'}$ and $H_{n'}(t_{n'}, x_{n'}) = 0$ for all $n'' \geq n'$, i.e., for each $n' \geq n$, we have

$$t_{n'} \left(y_{n'} + \alpha^{-1}Ty_{n'} + n''Ty_{n'} - n''f_{\mu} \right) + (1 - t_{n'})Ty_{n'} = 0, \tag{16}$$

for all $n'' \geq n'$. Replacing n by n_0 , n' by n , and n'' by n' in (16), for each $n \geq n_0$, we get

$$t_n \left(y_n + \alpha^{-1}Ty_n + n'Ty_n - n'f_{\mu} \right) + (1 - t_n)Ty_n = 0, \tag{17}$$

for all $n' \geq n$. Next we choose a sequence of natural numbers $\{N_n\}$ such that $t_n \in (1/N_n, 1]$ for all n . Let $\lambda_n = N_n + (n - 1)$ for all n . Clearly, $\lambda_n \geq n$ and $t_n \in (1/\lambda_n, 1]$ for all n (i.e., $t_n \lambda_n > 1$ for all n). Since $\lambda_n \geq n$ for all n , (17) implies

$$t_n \left(y_n + \alpha^{-1}Ty_n + \lambda_nTy_n - \lambda_n f_{\mu} \right) + (1 - t_n)Ty_n = 0, \tag{18}$$

for all n . Clearly, $t_n \neq 0$ for all n . Dividing (18) by t_n for all n , we arrive at

$$\gamma_n^{-1}y_n + Ty_n = \lambda_n \gamma_n^{-1}f_{\mu}, \tag{19}$$

for all n , where $\gamma_n = \alpha^{-1} + \lambda_n + (1 - t_n)t_n^{-1}$. The boundedness of $\{x_n\}$ and T^{-1} shows that $\{y_n\}$ is bounded. Assume without loss of generality that $t_n \rightarrow t_0 \in [0, 1]$ as $n \rightarrow \infty$. Suppose $t_0 = 0$. The sequence $\{t_n \lambda_n\}$ cannot converge to 0 because $t_n \lambda_n > 1$ for all n . Then, we consider either $\{t_n \lambda_n\}$ is unbounded or bounded. Suppose $\{t_n \lambda_n\}$ is unbounded, i.e., there exists a subsequence of $\{t_n \lambda_n\}$, denoted by $\{t_n \lambda_n\}$, such that $t_n \lambda_n \rightarrow \infty$ as $n \rightarrow \infty$. Since $t_0 = 0$, $\{y_n\}$ and $\{Ty_n\}$ are bounded, letting $n \rightarrow \infty$ in (18) implies

$$t_n \lambda_n Ty_n - t_n \lambda_n f_{\mu} + Ty_n \rightarrow 0, \tag{20}$$

as $n \rightarrow \infty$, i.e., $c_n Ty_n \rightarrow f_{\mu}$ as $n \rightarrow \infty$, where $c_n = 1 + t_n \lambda_n / t_n \lambda_n$ for all n . Since $c_n \rightarrow 1^+$ as $n \rightarrow \infty$ and $\{Ty_n\}$ is bounded, we conclude that $x_n = Ty_n \rightarrow f_{\mu} \in R(\bar{T}) = R(T)$ as $n \rightarrow \infty$. Letting $\mu \rightarrow 1^-$ gives $f \in R(T)$.

Next we assume that $\{t_n \lambda_n\}$ is bounded. Assume without loss of generality that there exists a subsequence, denoted again by $\{t_n \lambda_n\}$, such that $t_n \lambda_n \rightarrow a \in [1, \infty)$ as $n \rightarrow \infty$. Then, $c_n \rightarrow (a + 1)a^{-1}$ and

$$x_n = Ty_n \rightarrow a(a + 1)^{-1}f_{\mu} \in R(\bar{T}) = R(T), \tag{21}$$

as $n \rightarrow \infty$. Since $d = a(a + 1)^{-1} \in (0, 1)$, letting $\mu \rightarrow d$ gives $f \in R(T)$.

Next we consider the case $t_0 \in (0, 1]$. Then, we have $\mu_n = \alpha^{-1} + (1 - t_n)t_n^{-1} \rightarrow \mu_0 = \alpha^{-1} + (1 - t_0)t_0^{-1} \in \mathbb{R}$ and

$$\begin{aligned} \lambda_n \gamma_n^{-1} &= \frac{\lambda_n}{\alpha^{-1} + \lambda_n + (1 - t_n)t_n^{-1}} \\ &= \frac{N_n + (n - 1)}{\mu_n + N_n + (n - 1)} \\ &= \frac{1}{1 + \xi_n \mu_n}, \end{aligned} \tag{22}$$

for all n , where $\xi_n = (N_n + (n - 1))^{-1}$. Since $\xi_n \rightarrow 0$ as $n \rightarrow \infty$ and $\mu_0 \in \mathbb{R}$, it follows that $\lambda_n \gamma_n^{-1} \rightarrow 1$ as $n \rightarrow \infty$. Letting $n \rightarrow \infty$ in (19) yields $x_n = Ty_n \rightarrow f_{\mu} \in R(\bar{T}) = R(T)$. Letting $\mu \rightarrow 1^-$ shows that $f \in R(T)$. In all cases, each $f \in X$ satisfying (i) lies in $R(T)$.

Suppose (ii) holds. Then, there exists a subsequence of $\{n\}$, denoted again by $\{n\}$, such that $0 \in H_n(t, \partial\Omega)$ for all $t \in [0, 1]$ and n . Then, (iv) of Lemma 6. shows that the family $\{H_n(t, \cdot)\}_{t \in [0, 1]}$ is a homotopy of class (S_+) defined from $\bar{\Omega}$ into X such that $0 \in H_n(t, \partial\Omega)$ for all $t \in [0, 1]$ and n . Let d stand for Browder degree mapping. Applying (iii) of Lemma 6., we conclude that $d(H_n(t, \cdot), \Omega, 0)$ is independent of $t \in [0, 1]$, i.e., (i) and (iii) of Lemma 6. give

$$d(S + nI - nf_{\mu}, \Omega, 0) = d(I, \Omega, 0) = 1 \tag{23}$$

for all n . In addition, (ii) of Lemma 6. confirms that $n f_{\mu} \in (S + nI)(\Omega)$ for all n . Thus, for each positive integer n , there exist $x_n \in \Omega$ and $y_n \in X$ such that $x_n = Ty_n$ for all n and

$$\begin{aligned} nf_{\mu} &= Sx_n + nx_n = T^{-1}x_n + \alpha^{-1}x_n + nx_n \\ &= (n + \alpha^{-1})Ty_n + y_n, \end{aligned} \tag{24}$$

for all n . Since $\{x_n\}$ and T^{-1} are bounded, it follows that $\{y_n\}$ is bounded. In addition, (24) gives

$$x_n = Ty_n = n\tau_n f_{\mu} - \tau_n y_n \rightarrow f_{\mu} \in \bar{\Omega} \subseteq \bar{U} \subseteq R(T), \tag{25}$$

as $n \rightarrow \infty$ because $\tau_n \rightarrow 0$ and $n\tau_n \rightarrow 1$, where $\tau_n = (n + \alpha^{-1})^{-1}$. Thus, letting $\mu \rightarrow 1^-$ shows that $f \in R(T)$. Thus, each $f \in X$ satisfying (i) or (ii) lies in $R(T)$. This shows that T is surjective. The proof is completed. \square

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of the paper.

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