AN EXPONENTIAL INTERPOLATION SERIES

By

W. E. Howell

Thesis submitted to the Graduate Faculty of the

Virginia Polytechnic Institute

in candidacy for the degree of

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in

Mathematics

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III. ACKNOWLEDGMENTS

I would like to acknowledge the contribution of Dr. J. W. Layman to this paper. The original conjecture as to the validity of the interpolating series developed here and the method of attack are his. His assistance and guidance throughout the preparation of this paper has been invaluable. I would also like to acknowledge a long-standing debt to Mr. K. A. Thompson, who first introduced me to mathematics.

IV. INTRODUCTION

The problem of interpolating a function between given points is a very fundamental one with some of the earliest work dating back to that of James Gregory in 1670 [6]. The most general form of the Gregory-Newton series is given by Norlund [5] as

$$f(Z) = \sum_{S=0}^{\infty} \binom{Z - a}{S} \Delta^S f(a), \tag{1}$$

Normally the value of a is taken as an integer. For example, if a = 0, equation (1) yields

$$f(Z) = f(0) + Z \,\Delta f(0) + \frac{Z(Z - 1)}{2!} \Delta^2 f(0) + \dots \tag{2}$$

Layman [4] and [4a] with Pitts [4b] has investigated an interpolating series somewhat similar to equation (2) as follows:

$$f(Z) = 1^Z f(0) + (2^Z - 1^Z) \Delta f(0) + \frac{3^Z - 2.2^Z + 1^Z}{2!} \Delta(\Delta - 1) f(0) + \dots \tag{3}$$

Equation (3) can be written more compactly as

$$f(Z) = \sum_{n=0}^{\infty} \frac{U_n(Z)}{n!} \Delta^{(n)} f(0), \tag{4}$$

where

$$U_n(Z) = \sum_{m=0}^{n} (-1)^m \binom{n}{m} (n - m + 1)^Z, \tag{5}$$

$$\Delta^{(n)} = \Delta(\Delta - 1) \ . \ . \ . \ (\Delta - n + 1).$$

A fundamental question concerning equation (4) is how large a class of functions does the series represent. That this class of functions is not empty is evident from the following example.

$$f(Z) = 2^Z$$

$$f(0) = 1$$

$$\Delta f(0) = f(1) - f(0) = 2 - 1 = 1$$

$$\Delta(\Delta - 1)f(0) = \Delta^2 f(0) - \Delta f(0) = f(2) - 3f(1) + 2f(0) = 4 - 6 + 2 = 0$$

All higher $\Delta^{(n)} f(0)$ are also zero, and substituting into equation (3) we get

$$2^Z = 2^Z .$$

The purpose of this paper is to determine a set of conditions on $f(Z)$ which will guarantee that equation (4) does indeed represent a large class of functions. To establish this fact, the general function will be represented by a kernel expansion; its validity will rest on the uniform convergence of a particular Gregory-Newton series.

V. THE KERNEL EXPANSION

The purpose of the following discussion is to review the tools necessary to use the kernel expansion. These tools include the basic technique of representing a given function by the method of kernel expansion and particular examples which lead to results that will be needed later in the paper. These examples include the development of the Cauchy integral from the kernel expansion and the development of the McClaurin and Gregory-Newton series from the Cauchy integral.

The kernel expansion method is a very general technique by which analytic functions may be represented in an integral form. Several sources have treated the kernel expansion technique in considerable detail with the individual and collective works of Boas and Buck [1], [1a], [2], and [2a] being among the most complete. The following discussion generally follows the development in references [1a] and [2] which gives the following general form:

$$f(Z) = \frac{1}{2\pi i} \int_{\Gamma} K(Z,\omega)F(\omega)d\omega. \tag{6}$$

A few comments about each of these terms is in order. The function $f(Z)$ is regular in a prescribed domain and can be written as

$$f(Z) = \sum_{n=0}^{\infty} f_n Z^n.$$

The function $K(Z,\omega)$ is the kernel of the expansion and is analytic for (Z,ω) in an open set Λ containing the plane $Z = 0$. The function $K(Z,\omega)$ is normally restricted to the form

$$K(Z,\omega) = \Psi(Z\omega),$$

with

$$\Psi(Z\omega) = \sum_{n=0}^{\infty} \psi_n \left[Z\omega\right]^n, \qquad \psi_n \neq 0.$$

With these two definitions we may now define

$$F(\omega) = \sum_{n=0}^{\infty} \frac{f_n}{\psi_n \omega^{n+1}}.$$

The contour Γ is taken as a contour outside of which and on which $F(\omega)$ is regular.

A particularly significant result occurs for the case

$$\Psi(Z\omega) = e^{Z\omega},$$

for which we have

$$f(Z) = \frac{1}{2\pi i} \int_\Gamma e^{Z\omega} F(\omega) d\omega .$$

This gives the relationship between the function $f(Z)$ and its Borel (or in this case Laplace) transform. This expression for $f(Z)$ is also known as the Polya representation and any entire function of exponential type has a representation of this form.

Sufficient information has now been developed to justify this approach and to outline the remaining necessary steps. Consider now a set of functions, $Q_n(Z)$, from which we wish to construct a given function $f(Z)$ as

$$f(Z) = \sum_n C_n Q_n(Z) \,. \tag{7}$$

The method of kernel expansion provides a means of selecting the proper C_n's in a systematic manner to ensure convergence of the summation to the given function. This is accomplished by selecting a second sequence of functions $P_n(\omega)$ such that

$$\Psi(Z\omega) = \sum_n Q_n(Z)P_n(\omega) \tag{8}$$

converges uniformly on the contour Γ. We may now use $\Psi(Z\omega)$ as the kernel of an expansion as in equation (6). This yields

$$\begin{aligned} f(Z) &= \frac{1}{2\pi i} \int_\Gamma \Psi(Z\omega)F(\omega)d\omega \\ &= \frac{1}{2\pi i} \int_\Gamma \sum Q_n(Z)P_n(\omega)F(\omega)d\omega \\ &= \sum_n Q_n(Z) \frac{1}{2\pi i} \int_\Gamma P_n(\omega)F(\omega)d\omega \,, \end{aligned}$$

where the interchange of the summation and integration is justified by the previously specified uniform convergence of the left-hand side of equation (8). This can be further reduced to

$$f(Z) = \sum Q_n(Z)L_n(f),$$

where

$$L_n(f) = \frac{1}{2\pi i} \int_\Gamma P_n(\omega)F(\omega)d\omega$$

is a linear functional which assigns a number to the function f. The value of $L_n(f)$ is the desired C_n in equation (7).

As an example of this, consider the sequence of functions $\{Z^k\}$, which takes the role of the Q_n's, and an associated set $\{\omega^k\}$, which takes the role of the P_n's; these functions result in the kernel

$$\Psi(Z\omega) = \sum_0^\infty Z^k\omega^k = \frac{1}{1 - Z\omega}, \qquad |Z\omega| < 1 .$$

This notation can be directly compared with equation (8). Continuing in the same manner, we obtain

$$f(Z) = \frac{1}{2\pi i} \int_\Gamma \frac{F(\omega)}{1 - Z\omega} d\omega .$$

By the definitions following equation (6), we have

$$\Psi(Z\omega) = \sum_{k=0}^{\infty} (Z\omega)^k, \text{ with } \psi_n = 1,$$

$$f(Z) = \sum_{n=0}^{\infty} f_n Z^n,$$

$$F(\omega) = \sum_{n=0}^{\infty} \frac{f_n}{\psi_n \omega^{n+1}} = \frac{1}{\omega} \sum_{n=0}^{\infty} f_n\left(\frac{1}{\omega}\right)^n = \frac{1}{\omega} f\left(\frac{1}{\omega}\right).$$

Substituting $F(\omega)$ into the above equation for $F(Z)$ yields

$$f(Z) = \frac{1}{2\pi i} \int_\Gamma \frac{f\left(\frac{1}{\omega}\right)}{\omega(1 - Z\omega)} d\omega.$$

Substituting $\tau = \frac{1}{\omega}$ into the last integrand, noting that $d\omega = -\frac{1}{\tau^2} d\tau$ and that this substitution reverses the path of integration, we get

$$f(Z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\tau)}{\tau - Z} d\tau. \qquad (9)$$

This is the Cauchy representation of an analytic function. Rearranging the terms of equation (9) and returning to the ω notation, we have

$$f(Z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\omega)}{\omega} \sum_{k=0}^{\infty} \left(\frac{Z}{\omega}\right)^k d\omega.$$

Since we have uniform convergence of the series, we may write

$$f(Z) = \sum_{k=0}^{\infty} Z^k \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\omega)}{\omega^{k+1}} d\omega$$

$$= \sum_{k=0}^{\infty} Z^k L_k(f),$$

where

$$L_k(f) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\omega)}{\omega^{k+1}} d\omega = \frac{f^{(K)}(0)}{K!}$$

by the basic property of the Cauchy integral. Thus we see that by proper selection of a kernel we have arrived at the MacLaurin expansion. In the process of this development we have obtained the Cauchy representation through a kernel expansion. We have also seen that the linear functional, $L_n(f)$, can be interpreted as the Nth derivative evaluated at zero and divided by $N!$; these are the numerical coefficients of the McClaurin expansion.

If a slightly different approach had been taken after equation (9), the Gregory-Newton series would have followed instead of the McClaurin series. To arrive at this latter result, the following substitution is required in equation (9) [5]

$$\frac{1}{\omega - Z} = \frac{1}{\omega} + \frac{Z}{\omega(\omega - 1)} + \frac{Z(Z - 1)}{\omega(\omega - 1)(\omega - 2)} + \ldots \frac{Z(Z - 1)\ldots(Z - n)}{\omega(\omega - 1)\ldots(\omega - n)(\omega - Z)}.$$

Since there are only a finite number of terms, we may substitute this into equation (9) to obtain

$$f(Z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\omega)}{\omega - Z}\, d\omega \tag{9}$$

$$= \frac{1}{2\pi i} \int_\Gamma \frac{f(\omega)}{\omega} + \frac{Zf(\omega)}{\omega(\omega - 1)} + \ldots \frac{Z(Z - 1)\ldots(Z - n)f(\omega)}{\omega(\omega - 1)\ldots(\omega - n)(\omega - Z)}\, d\omega, \tag{10}$$

and integrate term by term. The first term yields $f(0)$ by the basic property of the Cauchy integral. The second term yields

$$\frac{1}{2\pi i} \int_\Gamma \frac{Zf(\omega)}{\omega(\omega - 1)}\, d\omega = \frac{Z}{2\pi i} \int_\Gamma f(\omega)\left[-\frac{1}{\omega} + \frac{1}{\omega - 1}\right] d\omega$$

$$= \frac{Z}{2\pi i} \int_\Gamma -\frac{f(\omega)}{\omega}\, d\omega + \frac{Z}{2\pi i} \int_\Gamma \frac{f(\omega)}{\omega - 1}\, d\omega$$

$$= -Zf(0) + Zf(1)$$

$$= Z\, \Delta f(0).$$

Considering each term individually in equation (10), we obtain the following result

$$f(Z) = f(0) + Z\, \Delta f(0) + \ldots \frac{Z(Z - 1)\ldots(Z - n + 1)}{n!} \Delta^n f(0)$$

$$+ \frac{1}{2\pi i} \int_\Gamma \frac{Z(Z - 1)\ldots(Z - n)f(\omega)}{\omega(\omega - 1)\ldots(\omega - Z)}\, d\omega. \tag{11}$$

Equation (11) is the Gregory-Newton expansion with remainder.

VI. THE GREGORY-NEWTON SERIES AND ITS CHARACTERISTICS

To make later development more straightforward and to develop some required characteristics of the Gregory-Newton series, it is appropriate to look at an alternate development. This approach utilizes the kernel expansion directly and closely follows the work of Buck [2]. This development leads to several theorems on the convergence properties of the Gregory-Newton series, but the principal advantage is to acquaint the reader with certain steps that will be used in developing the exponential interpolating series.

Let K be the class of entire functions of exponential type, $\{T_n\}$ be a sequence of linear functionals defined on all or part of K, and $C \subset K$ be a uniqueness class for $\{T_n\}$. ($f \in C$ and $T_n(f) = 0$ for all n implies $f \equiv 0$.) Now further suppose it is possible to find a sequence of functions $\{Q_n(Z)\}$ which are orthogonal to $\{T_n\}$, that is, $T_n(Q_m) = 0$ for all $n \neq m$, while $T_n\{Q_n\} = 1$. Then we may attempt to represent any function $f(Z)$ by the series

$$\sum_{0}^{\infty} Q_n(Z)T_n(f)\,.$$

This is referred to as an interpolating series for $\{T_n\}$, [2], provided it converges.

From the similarity of forms between this and equation (8), it is appropriate to define T_n analogous to L_n as

$$T_n(f) = \frac{1}{2\pi i}\int_{\Gamma} P_n(\omega)F(\omega)d\omega$$

where $F(\omega)$ is the Borel transform of $f(Z)$, $P_n(\omega)$ is known as the generating function for $T_n(f)$, and Γ is a circle outside of which and on which $F(\omega)$ is regular.

To obtain the Gregory-Newton series from this approach, we take as the generating function $P_n(\omega) = (e^{\omega} - 1)^n$ and the functions $Q_n(Z)$ as the polynomials $Q_n(Z) = Z(Z-1)\ldots(Z-n+1)/n!$. This will yield a functional $\{T_n\}$, $T_n = \Delta^n f(0)$ as can be seen by the following development.

The Polya representation gives

$$f(Z) = \frac{1}{2\pi i} \int_{\Gamma} e^{Z\omega} F(\omega)d\omega .$$

Likewise,

$$f(Z+1) = \frac{1}{2\pi i} \int_{\Gamma} e^{(Z+1)\omega} F(\omega)d\omega .$$

By definition

$$\Delta f(0) = f(Z+1) - f(Z)\Big|_{Z=0} ,$$

or

$$\Delta f(0) = \frac{1}{2\pi i} \int_{\Gamma} e^{(Z+1)\omega} F(\omega)d\omega\Big|_{Z=0} - \frac{1}{2\pi i} \int_{\Gamma} e^{Z\omega} F(\omega)d\omega\Big|_{Z=0}$$

$$= \frac{1}{2\pi i} \int_{\Gamma} (e^{\omega} - 1)F(\omega)d\omega$$

$$= T_1(f) . \qquad (12)$$

The desired result of

$$\Delta^n f(0) = T_n(f) = \frac{1}{2\pi i} \int_\Gamma (e^\omega - 1)^n F(\omega) d\omega \tag{13}$$

therefore follows directly.

Based on this type of development, Buck [2] has proved several theorems relating the growth properties of entire functions and the convergence properties of the Gregory-Newton series. Since they only apply to entire functions, they will not be applicable to the problem that must be considered later. A theorem that will apply is that of Nörlund [5]. The series (1) referred to below is essentially equation (1) of this paper, and N_o is a bounded sector in the complex plane lying to the right of an arbitrary, noninteger point $Z_o = \sigma_o + i\tau_o$.

<u>Theorem</u> (Nörlund [5])

Si la série (1) converge pour $Z = Z_o = \sigma_o + i\tau_o$:
I Elle converge en tout point situé á droite de $R(Z) = \sigma_o$
II Elle converge uniformément dans le secteur N_o

Si une fonction $F(Z)$, holomorphe dans le demiplane $\sigma \geq \alpha$, y satisfait á l'inégalité

$$|F(\alpha + re^{i\nu})| < e^{r\psi(\nu)} (1 + r)^{\beta+\epsilon(r)},$$

$$\psi(\nu) = \cos \nu \, ln(2 \cos \nu) + \nu \sin \nu, \qquad \left(-\frac{\pi}{2} \leq \nu \leq \frac{\pi}{2}\right),$$

oú la fonction $\epsilon(r)$ tend uniformément vers zéro quand r augmente indéfiniment, elle admet un developpement de la formi (1), dont l'abscisse de convergence est inferience ou égale au plus grand des nombres $\alpha, \beta + 1/2$.

A translation of these theorems is as follows:

If the series (1) converges for $Z = Z_0 = \sigma_0 + i\tau_0$:

I It converges at all points lying on the right of $R(Z) = \sigma_0$

II It converges uniformly within the sector N_0

If $F(Z)$ is an analytic function, holomorphic in the semiplane $\sigma \geq \alpha$ and there satisfies the inequality

$$\left|F(\alpha + re^{i\nu})\right| < e^{r\psi(\nu)} (1 + r)^{\beta+\epsilon(r)},$$

$$-\frac{\pi}{2} \leq \nu \leq \frac{\pi}{2},$$

when the function $\epsilon(r)$ tends uniformly to zero as $r \to \infty$, then the function can be represented by the Newton expansion, and its abscissa of convergence is less than or equal to the greater of the two numbers α, $\beta + 1/2$.

As Nörlund points out several places in his book, the provision $Z \neq$ integer must be made in each of these theorems. Also the usual definition of a holomorphic function [7] as one which is single valued, continuous, and differentiable in the domain under consideration is intended. Another definition is required for "the abscissa of convergence."

Translating directly from Nörlund [5], we have

> It results from Theorem I that a real number λ exists such that series I converges at $\sigma > \lambda$, and diverges at $\sigma < \lambda$. The domain of convergence is therefore a semiplane limited at the left by the line $\sigma = \lambda$. This line is called the line of convergence, and λ the abscissa of convergence.

Since uniform convergence is of major importance as required by the interchange of summation and integration in the kernel expansion, this region has been illustrated in figure 1 for the Gregory-Newton series.

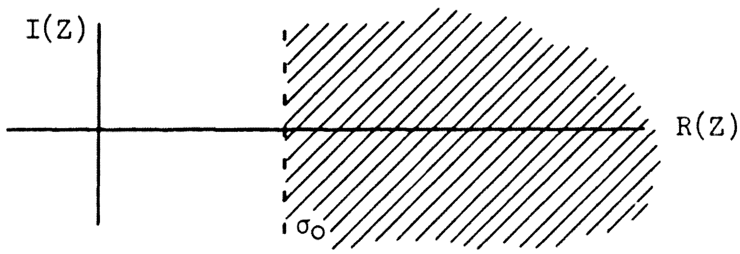

Figure 1.- Region of uniform convergence of the Gregory-Newton series.

This contrasts rather sharply with the power series which exhibits a circle of convergence. The requirement that Z not be an integer is necessary because the series truncates for positive integer values of Z. Nörlund [5] gives the following example:

$$(1 + a)^Z = \sum_{S=0}^{\infty} a^S \frac{Z(Z - 1)\ldots(Z - S + 1)}{S!}.$$

If $|a| > 1$, the series will converge for any positive integer value Z since the summation will eventually truncate; however, it will diverge for all other values of Z.

VII. DEVELOPMENT OF THE EXPONENTIAL INTERPOLATING SERIES

Let us now consider the following example of the Gregory-Newton expansion (i.e., the expansion of the function)

$$f(U) = U^Z.$$

Note that this is to be looked upon as a function of U, not of Z. This will be expanded according to equation (1) with a = 1. The first few coefficients are

$$f(1) = 1^Z,$$

$$\Delta f(1) = f(2) - f(1) = 2^Z - 1^Z,$$

$$\Delta^2 f(1) = f(3) - 2f(2) + f(1) = 3^Z - 2.2^Z + 1^Z.$$

The series takes the form

$$f(U) = U^Z = 1^Z + (2^Z - 1^Z)(U - 1) + \frac{3^Z - 2.2^Z + 1^Z}{2!}(U - 1)(U - 2) + \ldots, \tag{14}$$

or more compactly

$$f(U) = \sum_{n=0}^{\infty} \frac{U_n(Z)(U - 1)^{(n)}}{n!}, \tag{15}$$

where $U_n(Z)$ is given by equation (5) and

$$U^{(n)} = U(U - 1)(U - 2). \ . \ .(U - n + 1). \tag{16}$$

Instead of considering the variable U, we may write

$$U = e^{\omega}$$

and consider ω as the variable. Then we may further write

$$U^Z = (e^{\omega})^Z = e^{Z\omega}$$

and formally substitute into equation (15), obtaining

$$e^{Z\omega} = \sum_{n=0}^{\infty} \frac{U_n(Z)}{n!}(e^{\omega} - 1)^{(n)}. \tag{17}$$

If this is uniformly convergent, it may be used as a kernel in the sense of equation (18) and we obtain

$$f(Z) = \frac{1}{2\pi i} \int_{\Gamma} e^{Z\omega} F(\omega)d\omega .$$

Substituting equation (17) into the above gives

$$f(Z) = \sum_{n=0}^{\infty} \frac{U_n(Z)}{n!} \frac{1}{2\pi i} \int_{\Gamma} (e^{\omega} - 1)^{(n)} F(\omega)d\omega$$

$$= \sum_{n=0}^{\infty} \frac{U_n(Z)}{n!} M_n(f), \tag{18}$$

where

$$M_n(f) = \frac{1}{2\pi i} \int_\Gamma (e^\omega - 1)^{(n)} F(\omega)d\omega . \tag{19}$$

As in the previous examples (MacLaurin expansion and the Gregory-Newton expansion using $L_n(f)$ and $T_n(f)$), a more useful interpretation can be obtained for $M_n(f)$. The difference between equations (19) and (13) should be noted. In equation (13), "n" indicates exponentiation, while in equation (19), "(n)" is defined by equation (16). Consider $M_2(f)$ (note that $M_0(f)$ and $M_1(f)$ are equal to $T_1(f)$ and $T_2(f)$, respectively)

$$M_2(f) = \frac{1}{2\pi i} \int_\Gamma (e^\omega - 1)^{(2)} F(\omega)d\omega$$

$$= \frac{1}{2\pi i} \int_\Gamma (e^\omega - 1)(e^\omega - 2)F(\omega)d\omega$$

$$= \frac{1}{2\pi i} \int_\Gamma (e^\omega - 1)\left[(e^\omega - 1) - 1\right]F(\omega)d\omega$$

$$= \frac{1}{2\pi i} \int_\Gamma \left[(e^\omega - 1)^2 - (e^\omega - 1)\right]F(\omega)d\omega .$$

By virtue of the relationship established in equation (13), we now obtain

$$M_2(f) = \Delta^2 f(0) - \Delta f(0) = \Delta(\Delta - 1)f(0).$$

Proceeding similarly, $M_n(f)$ is then

$$M_n(f) = \Delta(\Delta - 1). \ . \ .(\Delta - n + 1)f(0)$$

$$= \Delta^{(n)} f(0). \tag{20}$$

Combining the results of equation (20) with equation (18), we may write

$$f(Z) = \sum_{n=0}^{\infty} \frac{U_n(Z)}{n!} \Delta^{(n)} f(0)$$

$$= 1^Z f(0) + (2^Z - 1^Z)\Delta f(0) + \frac{(3^Z - 2.2^Z + 1^Z)}{2!}\Delta(\Delta - 1)f(0) + . \ . \ . \ .$$

These are equations (4) and (3), respectively, which we set out to investigate.

To establish the validity of this representation (eqs. (3) or (4)), it is necessary to establish the uniform convergence of the series in (17). The convergence of this series is dependent upon the region of convergence of equation (14); hence, the validity of equation (14) must be determined.

The function

$$f(U) = U^Z,$$

considered as a function of U, is not an entire function since it is not analytic at $U = 0$ for all values of Z. The theorem of Nörlund requires that the function be holomorphic in a semiplane. The function U^Z has this property. If U is restricted to the right half plane, Nörlund's theorem may be applied.

It is, therefore, necessary to establish the following inequality:

$$\left|F(\alpha + re^{i\theta})\right| < e^{r\psi(\theta)} (1 + r)^{\beta+\epsilon(r)}, \tag{21}$$

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.$$

The quantity α is an arbitrarily small number since the holomorphic region extends arbitrarily close to the imaginary axis. The abscissa of convergence is less than or equal to the greater of the two numbers α and $\beta + 1/2$; therefore, we take $\beta \leq -1/2$ so that this term will not restrict the abscissa of convergence, provided we can make equation (21) hold. Furthermore, the term $\psi(\theta)$ has a minimum value [5]

$$\psi(\theta)_{min} = \ln 2.$$

Substituting these values into relationship (21) gives

$$\left|F(\alpha + re^{i\theta})\right| < e^{r \ln 2} (1 + r)^{-1/2 + \epsilon(r)}, \tag{22}$$

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},$$

as the desired inequality. For the function under consideration, we have

$$\left|F(\alpha + re^{i\theta})\right| = \left|(\alpha + re^{i\theta})^Z\right| . \tag{23}$$

With the usual substitutions

$$U \equiv re^{i\theta} = u + iv ,$$

$$r^2 = u^2 + v^2 ,$$

$$\theta = \tan^{-1} \frac{v}{u}, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},$$

$$Z = a + ib ,$$

and

$$r' = \sqrt{(\alpha + u)^2 + v^2} ,$$

$$\theta' = \tan^{-1} \frac{v}{\alpha + u} ,$$

and

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \Rightarrow -\frac{\pi}{2} < \theta' < \frac{\pi}{2}.$$

We may now write

$$\alpha + re^{i\theta} = r'e^{i\theta'} .$$

Then,

$$\left|(\alpha + re^{i\theta})^Z\right| = \left|(r'e^{i\theta'})^Z\right|, \quad -\frac{\pi}{2} < \theta' < \frac{\pi}{2}$$

$$= \left|e^{Z\,[ln\ r' + i(\theta' + 2n\pi)]}\right|.$$

Choosing the branch $n = 0$, we have

$$\left|(\alpha + re^{i\theta})^Z\right| = \left|e^{Z[\ln r' + i\theta']}\right|$$

$$< \left|e^{Z[\ln(r+1)+i(\theta+1)]}\right|$$

since α was arbitrarily small.

Making the substitution for Z and collecting real and imaginary terms, we obtain

$$\left|(\alpha + re^{i\theta})^Z\right| < \left|e^{a\ \ln(r+1)-b(\theta+1)}\right| \left|e^{i\left[a(\theta+1)+b\ \ln(r+1)\right]}\right|, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.$$

Since

$$\left|e^{ix}\right| = 1 \quad \text{for all } x$$

and

$$e^y \geq 0 \quad \text{for all real } y$$

we have

$$\left|(\alpha + re^{i\theta})^Z\right| < e^{a\ \ln(r+1)-b(\theta+1)}.$$

Up to this point it has been established that

$$\left|F(\alpha + re^{i\theta})\right| < e^{a\ \ln(r+1)-b(\theta+1)}, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.$$

We now let θ range over the values

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},$$

and the maximum value of the preceding function is

$$e^{a\ \ln(r+1)-b(\theta+1)} \leq e^{a\ \ln(r+1)+\frac{|b|(\pi+2)}{2}}.$$

Note that

$$e^{|b|(\pi+2)/2} = e^{\left\{\frac{|b|(\pi+2)}{2}\left[\frac{1}{\ln(r+1)}\right]\ln(r+1)\right\}}$$

$$= e^{\ln(r+1)\left\{\frac{|b|(\pi+2)/2}{\ln(r+1)}\right\}}$$

$$= (r+1)^{\left\{\frac{|b|(\pi+2)/2}{\ln(r+1)}\right\}}$$

So that

$$e^{\left\{a\ \ln(r+1)+\frac{|b|(\pi+2)}{2}\right\}} = e^{a\ \ln(r+1)}\ e^{|b|(\pi+2)/2}$$

$$= (r+1)^a(r+1)^{\left\{\frac{|b|(\pi+2)/2}{\ln(r+1)}\right\}}.$$

To satisfy the requirements of Nörlund's theorem, we must establish the inequality

$$(r+1)^a(r+1)^{\frac{|b|(\pi+2)/2}{\ln(r+1)}} < e^{r\ \ln 2}\ (r+1)^{-1/2+\epsilon(r)},$$

where $\epsilon(r)$ tends uniformly to zero as r approaches infinity. Since the exponent

$$\frac{|b|(\pi+2)/2}{\ln(r+1)}$$

satisfies this restriction on $\epsilon(r)$, we may substitute for $\epsilon(r)$ the quantity

$$\epsilon(r) = \frac{|b|(\pi + 2)/2}{\ln(r + 1)} + \epsilon'(r),$$

where $\epsilon'(r)$ is yet to be determined. When this is done, and like terms canceled, the inequality reduces to

$$(r + 1)^a < (r + 1)^{\epsilon'(r)-1/2} e^{r \ln 2} .$$

The existence of such an $\epsilon'(r)$ satisfying the previous restriction is not obvious; however, solving the relationship for $\epsilon'(r)$ we obtain

$$a \ln(r + 1) < \left(\epsilon'(r) - \frac{1}{2}\right)\ln(r + 1) + r \ln 2$$

or

$$\epsilon'(r) > \xi(r) = a + \frac{1}{2} - \frac{r \ln 2}{\ln(r + 1)} = \xi - \frac{r \ln 2}{\ln(1 + r)} . \qquad (24)$$

If equation (24) is satisfied, the inequality is established. Furthermore, we see that regardless of the choice of ξ, $\xi(r)$ will eventually be negative. Hence, the existence of an $\epsilon'(r) > 0$ satisfying the restriction that $\epsilon'(r)$ tends uniformly to zero is guaranteed. A plot of $\xi(r)$ versus r is shown in figure 2.

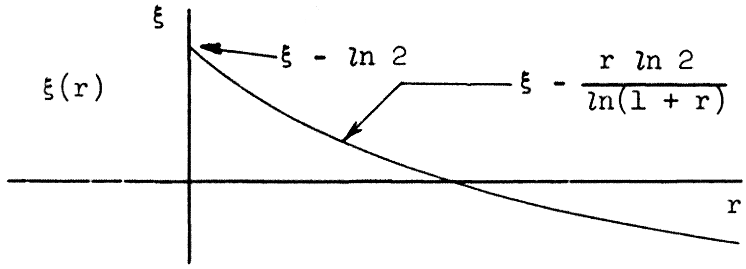

Figure 2.- Plot of $\xi(r)$ versus r; initial value of $\xi(r) = \xi - \ln 2$; final value of $\xi(r) = -\infty$.

Since the inequality required by the theorem is satisfied, equation (14) is uniformly convergent in any bounded sector of the right half plane for all values of Z. In particular, the series is convergent for e^{ω} lying in the right half plane. This places a restriction on ω, however, for consider

$$e^{\omega} = e^{u+iv} = e^{u}(\cos v + i \sin v),$$

which will have a negative or zero real part unless $|v| < \frac{\pi}{2}$. Hence, we must place the restriction $|I(\omega)| < \frac{\pi}{2}$ on ω.

VIII. A CONVERGENCE THEOREM FOR THE EXPONENTIAL INTERPOLATING SERIES

The preceding sections of this paper have reviewed or developed all the necessary tools to state and prove the following convergence theorem for the exponential interpolating series; however, before doing this it will simplify the statement of the theorem to define the convex hull, D(f), of a set as consisting of the smallest, closed, convex polygon containing every element of the set and its interior.

Theorem

Any entire function of exponential type such that the convex hull of the set of singularities of its Borel transform lies in the strip $|I(\omega)| < \frac{\pi}{2}$ admits the convergent exponential interpolation series expansion

$$f(Z) = \sum_{n=0}^{\infty} \frac{U_n(Z)}{n!} \Delta^{(n)} f(0), \text{ for all } Z.$$

Proof

As a result of the development from equations (21) through (24), inclusive, we have

$$\left|f(\alpha + re^{i\theta})\right| < e^{r \ln 2} (1 + r)^{-1/2+\epsilon(r)}, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},$$

for $f(U) = U^Z$. This result and Nörlund's theorem yields

$$U^Z = \sum_{n=0}^{\infty} \frac{U_n(Z)}{n!}(U - 1)^{(n)},$$

uniformly in any bounded sector of the right half plane $R(U) > 0$. Letting $U = e^{\omega}$ and noting that $R(U) > 0$ requires $|I(\omega)| < \frac{\pi}{2}$, we have equation (17)

$$(e^{\omega})^Z = e^{Z\omega} = \sum_{n=0}^{\infty} \frac{U_n(Z)}{n!}(e^{\omega} - 1)^{(n)},$$

uniformly in any bounded sector for which

$$|I(\omega)| < \frac{\pi}{2}.$$

The method of kernel expansion [1], [1a], [2], and [2a] reviewed in the second chapter gives us

$$f(Z) = \frac{1}{2\pi i} \int_{\Gamma} e^{Z\omega}\, F(\omega)d\omega, \quad \text{where } \Gamma \text{ encloses } D(f),$$

for any entire function of exponential type. Substituting equation (17) into the above, we get

$$f(Z) = \frac{1}{2\pi i} \int_{\Gamma} \sum_{n=0}^{\infty} \frac{U_n(Z)}{n!}(e^{\omega} - 1)^{(n)}\, F(\omega)d\omega.$$

Since we have uniform convergence in any bounded sector

$$f(Z) = \sum_{n=0}^{\infty} \frac{U_n(Z)}{n!} \frac{1}{2\pi i} \int_{\Gamma} (e^{\omega} - 1)^{(n)}\, F(\omega)d\omega, \quad |I(\omega)| < \frac{\pi}{2}.$$

From the results of equations (19) and (20), we have

$$M_n(f) = \frac{1}{2\pi i} \int_{\Gamma} (e^{\omega} - 1)F(\omega)d\omega = \Delta^{(n)} f(0)$$

which yields

$$f(Z) = \sum_{n=0}^{\infty} \frac{U_n(Z)}{n!} \Delta^{(n)} f(0)$$

for all Z, as desired.

IX. SUMMARY

An investigation of the conditions under which the exponential interpolating series

$$f(Z) = \sum_{n=0}^{\infty} U_n(Z)\, \Delta^{(n)} f(0)$$

is valid has been made where

$$U_n(Z) = \sum_{k=0}^{\infty} (-1)^k \binom{n}{k} (n - k + 1)^k,$$

$$\Delta^{(n)} f(0) = \Delta(\Delta - 1). . .(\Delta - n + 1)f(0).$$

This investigation has led to the following result:

Theorem

Any entire function of exponential type such that the convex hull of the set of singularities of its Borel transform $(F(\omega))$ lies in the strip $|I(\omega)| < \frac{\pi}{2}$ admits the convergent exponential series expansion

$$f(Z) = \sum_{n=0}^{\infty} U_n(Z)\, \Delta^{(n)} f(0)$$

for all Z, where $U_n(Z)$ and $\Delta^{(n)}$ are defined above.

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AN EXPONENTIAL INTERPOLATION SERIES

By

W. E. Howell

ABSTRACT

The convergence properties of the permanent exponential interpolation series

$$f(Z) = 1^Z f(0) + (2^Z - 1^Z)\Delta f(0) + \left(\frac{3^Z - 2.2^Z + 1^Z}{2!}\right)\Delta(\Delta - 1)f(0) + \ldots$$

have been investigated.

Using the following notation

$$U_n(Z) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n - i + 1)^Z,$$

$$\Delta^{(n)} f(0) = \Delta(\Delta - 1). . .(\Delta - n + 1)f(0),$$

the series can be written more compactly as

$$f(Z) = \sum_{0}^{\infty} \frac{U_n(Z)}{n!} \Delta^{(n)} f(0).$$

It is shown that $\Delta^{(n)} f(0)$ can be represented as

$$\Delta^{(n)} f(0) = M_n(f) = \frac{1}{2\pi i} \int_{\Gamma} (e^{\omega} - 1)^{(n)} F(\omega)d\omega,$$

where $F(\omega)$ is the Borel transform of $f(Z)$ and Γ encloses the convex hull of the singularities of $F(\omega)$. It is further shown that the series

$$\sum_{0}^{\infty} \frac{U_n(Z)}{n!}(e^{\omega} - 1)^{(n)}$$

forms a uniformly convergent Gregory-Newton series, convergent to $e^{Z\omega}$ in any bounded region in the strip $|I(\omega)| < \frac{\pi}{2}$. The Polya representation of an entire function of exponential type is then formed, and the method of kernel expansion (R. P. Boas, and R. C. Buck, <u>Polynomial Expansions of Analytic Functions</u>, Springer-Verlag, Berlin, 1964) yields the desired result. This result is summed up in the following:

<u>Theorem</u>

Any entire function of exponential type such that the convex hull of the set of singularities of its Borel transform lies in the strip $|I(\omega)| < \frac{\pi}{2}$ admits the convergent exponential interpolation series expansion

$$f(Z) = \sum_{n=0}^{\infty} \frac{U_n(Z)}{n!} \Delta^{(n)} f(0) \quad \text{for all } Z.$$