

**On a Moderate Rotation Theory for Anisotropic Shells**

by

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**(ABSTRACT)**

The present work discusses a new moderate rotation theory for anisotropic shells, proposed by Schmidt and Reddy. All aspects of the derivations are explicitly covered and a finite element formulation of the theory is developed for the solution of test cases. Specific forms of the equations for rectangular plates, cylindrical and spherical shells are derived and the respective finite elements are implemented in a computer code.

In order to compare the results, two other theories are implemented: a refined von Kármán type shell theory and a shell theory proposed by Librescu. A finite element computer code based on a degenerate 2-D shell theory is also used.

A set of cases involving anisotropic shells in bending, buckling and postbuckling permit an evaluation of all these models and form a basis for future developments.

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# DEDICATION

To my family and  
to my Family

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# 1. INTRODUCTION

## *1.1 Motivation*

The development of shell theories is still a field of intense research due to some basic factors:

- the need of structural elements to work to the limit of their load carrying capacity, where the material or the structure, or both, may be far from the linear regime;
- subsequent use of new materials;
- economic constraints preventing full use of a three-dimensional analysis, which are not efficient even with increasingly faster computers.

These factors motivated the development of more general and rigorous shell theories to offer a better representation of the kinematics of shells. An important aspect of these developments is the modelling of composite structures, frequently used in aerospace applications. To fulfil this need, more general theories are necessary to model the more complex behavior of the new material-structure integration process.

From this broad scenario, we will focus on geometrically non-linear shell theories, with the fundamental work done by von Kármán [1]. Nowadays, a variety of shell theories can be found in the literature, based on an equally large number of specific kinematic assumptions. Some of these are outlined as follows:

- the inclusion of transverse shear and transverse normal effects, in order to analyze thick structures. In the case of composite materials, even for thin walls, the above mentioned effects may not be negligible, due to their lower transverse rigidity compared to the in-plane rigidity;

- the inclusion of high order nonlinear effects;
- the inclusion of anisotropic constitutive equations;
- the observation of continuity requirements for displacements and tractions for the laminae interfaces;
- the inclusion of moderate, large or unrestricted rotations of the tangents and normal to the shell.

Some of these theories remain in a very complex form, making their use difficult for practical applications. A great deal of effort has been expended to keep the balance between an accurate representation and simplicity of formulation. This is, in fact, the motivation upon which the present work is based.

## ***1.2 Literature review***

This section is composed of two distinct portions, tracing the development of the shell theories up to the present work. The first portion covers the evolution from the Kirchhoff-Love theory to refined theories and from von Kármán non-linearity to full non-linearity. Due to the vast amount of information in this field, we will mention only pioneering contributions, presented in schematic form in the Figure 1 on page 3. For a more detailed presentation of these developments, we refer the reader to information in references [33,34].

The second portion deals with the review of shell theories with assumed magnitudes of strains and rotations, in the full non-linear equations. Such a procedure has resulted in shell theories which are valid for certain classes of problems. Although this idea was used in the past, based on intuition, only recently have Librescu [30] and Pietrzakiewicz [17] introduced it in a formal manner, so that a set of governing equations and boundary conditions can be obtained from a variational principle.

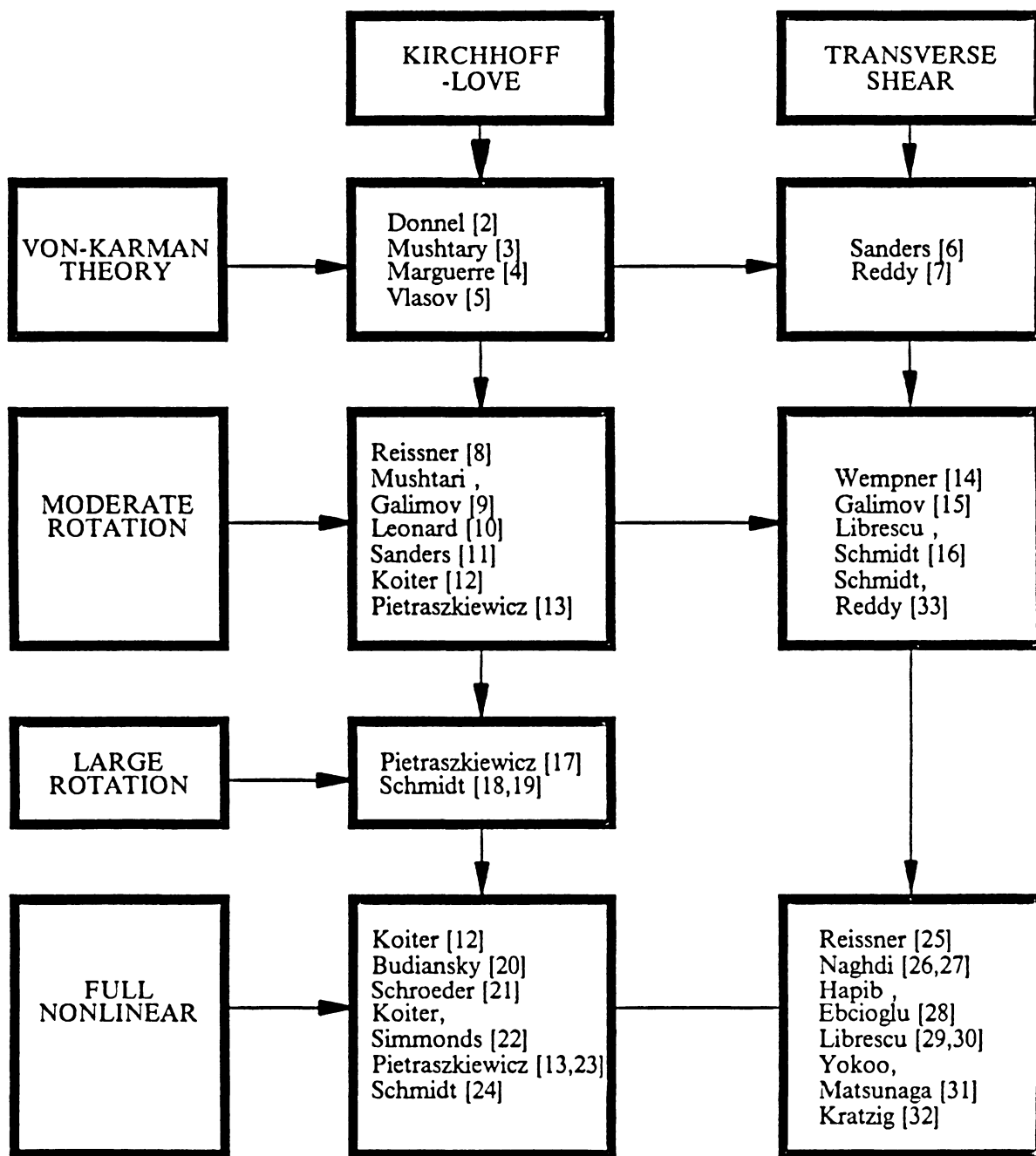


Figure 1. Literature review

Pietrazkiewicz developed both Total Lagrangian and Updated Lagrangian formulations of geometrically non-linear shells, based on the Kirchhoff-Love assumptions. The strains and rotations about the normal to the surface are assumed to be of the order  $\theta^2$ , where  $\theta$  is small compared to unity. The rotations about the tangents to the surface,  $\omega$ , are classified as follows:

- small for  $\omega \leq O(\theta^2)$
- moderate for  $\omega = O(\theta)$
- large for  $\omega = O(\sqrt{\theta})$
- finite for  $\omega \geq O(1)$

For each range of magnitude of the rotations, specific shell equations are obtained.

Librescu [30] developed a refined geometrically non-linear theory for anisotropic laminated shells, based on the expansion of the displacement field with respect to the thickness coordinate. In his work, the substantiation of a general theory for laminated composite shells is sought, such that transverse effects and high order dynamic effects are present. The set of equations of motions and boundary conditions are obtained from the Hu-Washizu variational principle. Moreover, Librescu [30] discusses the specialization of the derived general equations to the following multilayered shells theories:

- linearized high order theory;
- refined theory of von Kármán type;
- Kirchhoff-Love theory;
- Koiter and Sanders theories, for small strains and moderate rotations.

From the general work of Librescu [30], there followed a number of applications to small strains and moderate rotation theories of plates [35,36,37,38] and moderate rotation theories of shells [16,33]. Reddy and Schmidt [33] developed a moderate rotation theory in the context of the first-order shear deformation theories. This assumption results in simpler strain-displacement relations

if compared to those found in reference [30]. The kinematic relations are further simplified by invoking the order-of-magnitude assumptions. The identification of terms of the lower order theories is more apparent in this new relations. Classical theories like the Kirchhoff-Love, Donnell-Mushtary-Vlasov and refined von Kármán theory can be obtained from the new moderate rotation theory (MRT). In contrast to Librescu [30], where the independent parameters are the stress resultants, in MRT the governing equations are expressed in terms of displacements, making the formulation suitable for the use of a displacement finite element model.

In the course of this work, we repeatedly refer to reference [39] for comparison of results. It is useful to discuss the theoretical background of that reference to permit a better understanding of such comparisons. Reference [39] uses a total Lagrangian formulation of the geometric nonlinear problem, in incremental form. The stress and strain measures are the Second Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensor. The shell finite element results from the degeneration of a 3-D solid element upon which two constraints are imposed, namely: (a) normals to the mid-surface of the shell remain straight but not necessarily normal to the deformed mid-surface, (b) the transverse normal strain and stress are ignored. The formulation permits only small strains but displacements and rotations can be large. At each Gauss point, all displacement increments and derivatives are calculated with respect to the reference Cartesian system of coordinates. The remaining five components of the Green-Lagrange strain tensor (three in-plane components and two in transverse shear) are then obtained.

### *1.3 Present study*

The present work is the first application of the development by Reddy and Schmidt [33] to specific shell problems. Our motivation is to assess the performance of MRT in the prediction of the geometrically nonlinear behavior of anisotropic shells. We want to find limits where MRT starts to diverge from results obtained by the use of theories with higher degree of nonlinearity. To accomplish this study we define the following objectives for this work:

- to apply MRT to rectangular plates, circular cylindrical shells and spherical shells;
- to study the non-linear bending, buckling and post-buckling of the above shells, by the finite element method;
- to analyze structures made of laminated composite materials, with a general lamination scheme;
- to compare the results with those obtained from other theories;
- to implement refined von Kármán type shell equations (RVK) to help perform such comparisons;
- to offer conclusions on the applicability of the new moderate rotation theory, based on the results.

We should mention that some problems will not be included in the scope of this work, like local effects, due to concentrated loads, material non-linearity or failure and snap-back type behavior.

## 2. FORMULATION OF THE MODERATE ROTATION THEORY

### 2.1 Introduction

The primary objective of studying the moderate rotation theory is to gain insight into the behavior geometrically nonlinear shell problems, where small strains, but moderate rotations, may occur. This class of problems may be formulated in the framework of either a Lagrangian or a Eulerian description. The former one is more appropriate for most studies in solid mechanics.

The solution process in the Lagrangian description can be of two different types:

- Total Lagrangian formulation, where all variables refer to the initial undeformed configuration;
- Updated Lagrangian formulation, where a solution step refers to the previous step.

In this work, the Total Lagrangian description will be used.

Still in the continuum mechanics level, a choice must be made with regard to stress and strain components and constitutive relations. We shall use the Second Piola-Kirchhoff stress tensor and the Green strain tensor, for which the following observations apply:

- the Second Piola-Kirchhoff stress tensor and Green strain tensor are energy conjugates of the Cauchy stress tensor and the infinitesimal strain tensor, in the following sense:

$$\int_{V_\sigma} \sigma^{ij} e_{ij} dV_\sigma = \int_{V_0} S^{ij} E_{ij} dV_0$$

where :

$\sigma^{ij}$  = contravariant components of the Cauchy stress tensor;

$e_{ij}$  = infinitesimal strain tensor, given by  $e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i})$ ;

$S^{ij}$  = contravariant components of the Second Piola-Kirchhoff stress tensor;

$E_{ij}$  = covariant components of the Green strain tensor;

$V_0$  = volume in the reference configuration;

$V_t$  = volume in the deformed configuration;

- for large rotations, but small strains, the Second Piola-Kirchhoff stress tensor and the Green strain tensor are the engineering stress and strain measures, if we use the generalized Hooke's Law, as constitutive equations (see reference 41, p. 181).

With these facts in mind, the derivation of this moderate rotation theory is developed in detail in the next sections.

## 2.2 Preliminary Concepts

The geometry of a portion of a shell, in its undeformed configuration, is shown in Figure 2 on page 9 and the following entities are defined:

$V$  = volume

$S$  = surface of the shell, comprising the top surface  $S^+$ , where  $\theta^3 = h/2$ , and the bottom surface  $S^-$ , where  $\theta^3 = -h/2$

$A$  = lateral surface

$\Omega$  = midsurface, defined by  $\theta^3 = 0$



$a$  = determinant of the 2D metric tensor

$g_{ij}$  = covariant components of the 3D metric tensor

$g$  = determinant of the 3D metric tensor

$c_i^j \equiv \delta_i^j - \theta^3 b_i^j \equiv$  mixed components of the shifter tensor.

$c$  = determinant of the shifter tensor  $\equiv \sqrt{g/a}$

$( )_{,i}$  = differentiation with respect to  $\theta^i$

$( )_{;i}$  = covariant derivative in  $V$  with respect to  $\theta^i$

$|_{\bullet}$  = covariant derivative in  $\Omega$

$\delta_{\beta}^{\alpha}$  = mixed Kronecker delta

$g_i$  = covariant components of the base vectors in  $V$ , given by  $g_i = R_{,i}$

$a_{\bullet}$  = covariant components of the base vectors, in  $\Omega$ , given by  $a_{\bullet} = r_{,\bullet}$

$r = r(\theta^1, \theta^2)$  = position vector of a point  $(\theta^1, \theta^2)$ , in  $\Omega$

$R = R(\theta^1, \theta^2, \theta^3)$  = position vector of a point  $(\theta^1, \theta^2, \theta^3)$  in  $V$

$n$  = unit normal to the midsurface at a point  $(\theta^1, \theta^2)$

The following results, from tensor analysis, will be useful:

$$n = R_{,3} |_{\theta^3=0} = g_3 = g^3$$

$$R = r + \theta^3 n$$

$$g_{\alpha 3} = g^{\alpha 3} = 0$$

$$g_{33} = g^{33} = 1$$

$b_{\bullet\beta} \equiv r_{,\bullet\beta} \cdot n \equiv$  covariant components of the curvature tensor

$$g_{\alpha} = c_{\alpha}^{\beta} a_{\beta}$$

$$g_{\alpha\beta} = c_{\alpha}^{\lambda} c_{\beta}^{\mu} a_{\lambda\mu}$$

$$g^{\alpha\beta} = a^{\alpha\lambda} \sum_{n=0}^{\infty} (B)_{\lambda}^{\beta} (n+1) (\theta^3)^n$$

$$(B)_{\beta}^{\alpha} \equiv b_{\lambda}^{\alpha} (B)_{\beta}^{\lambda} = b_{\beta}^{\lambda} (B)_{\lambda}^{\alpha}$$

$$(B)_{\beta}^{\alpha} \equiv \delta_{\beta}^{\alpha}$$

$$(B)_{\beta}^{\alpha} \equiv 0 \quad \text{for } n < 0$$

$$dV = c \, d\theta^3 \, d\Omega$$

$$dS = c \, d\Omega$$

$$n_{\alpha} dA = v_{\alpha} \, c \, d\theta^3 \, d\Gamma$$

A vector can be referred to any of the defined bases; in particular, the displacement vector  $\mathbf{u}$  can be written in  $V$  as:

$$\mathbf{u} = U^l \mathbf{g}_l = U_l \mathbf{g}^l \quad , \quad U^l = U^l(\theta^1, \theta^2, \theta^3)$$

or referred to  $\Omega$  as:

$$\mathbf{u} = u^{\alpha} \mathbf{a}_{\alpha} + u^3 \mathbf{n} = u_{\alpha} \mathbf{a}^{\alpha} + u_3 \mathbf{n} \quad , \quad u_{\alpha} \equiv u_{\alpha}(\theta^1, \theta^2)$$

The relationship between covariant derivatives in both systems can be written as:

$$U_{\alpha i; \beta} = c_{\alpha}^{\lambda} (u_{\lambda} |_{\beta} - b_{\lambda\beta} u_3)$$

$$U_{\alpha i; 3} = c_{\alpha}^{\lambda} u_{\lambda; 3}$$

$$U_{3i;\alpha} = u_{3i,\alpha} + b_{\alpha}^{\lambda} u_{\lambda}$$

$$U_{3i3} = u_{3i3}$$

### 2.3 Order of Magnitude Assumptions

The first fundamental step, leading to the proposed moderate rotation theory, is centered on the limitation of the magnitude of the strains to a small value of order  $\varepsilon^2$ , where  $\varepsilon \ll 1$  (see also [30]).

The Green strain tensor components are:

$$E_{ij} = \frac{1}{2} (U_{ij} + U_{ji} + U_{mi} U^m_{.j})$$

and the components of the linear strain and rotation tensors are:

$$e_{ij} = \frac{1}{2} (U_{ij} + U_{ji}) \quad (1)$$

$$\omega_{ij} = \frac{1}{2} (U_{ij} - U_{ji})$$

We can rewrite the Green strain tensor components in terms of  $e_{ij}$  and  $\omega_{ij}$  as:

$$E_{ij} = e_{ij} + \frac{1}{2} \omega_{kl} \omega^k_{.j} + \frac{1}{2} (e_{kj} \omega^k_{.i} + e_{ki} \omega^k_{.j}) + \frac{1}{2} e_{kl} e^k_{.j} \quad (2)$$

It is assumed that  $E_{ij} = O(\varepsilon^2)$ , from which it follows immediately that:

$$e_{ij} = O(\varepsilon^2) \quad (3)$$

$$\omega_{kl} \omega^k_{.j} = \omega_{\gamma\alpha} \omega^{\gamma}_{.\beta} + \omega_{3\alpha} \omega^3_{.\beta} + \omega_{3\alpha} \omega^{\gamma}_{3.} = O(\varepsilon^2)$$

From equation (3) we conclude that:

$$e_{kl} e^k_{.j} = O(\varepsilon^4)$$

Also we conclude that  $\omega_{3\alpha}$  can be of order  $\varepsilon$ , and these are the moderate rotations the shell will undergo. Notice that  $\omega_{\alpha\beta}$  is small and will be assumed to be of the order  $\varepsilon^2$ .

The second step in building the theory is to keep only terms of the order  $\varepsilon^2$  and  $\varepsilon^3$  in  $E_{ij}$ . The reason for keeping the terms of the order  $\varepsilon^3$  resides in the fact that they contribute to the equations of motion, in terms of the same order of magnitude as those coming from terms of order  $\varepsilon^2$ . A detailed discussion is presented in [35].

Let's consider  $E_{\alpha\beta}$ :

$$E_{\alpha\beta} = e_{\alpha\beta} + \frac{1}{2} \omega_{3\alpha} \omega_{3\beta} + \frac{1}{2} (e_{k\beta} \omega^k_{.\alpha} + e_{k\alpha} \omega^k_{.\beta})$$

which was obtained from (2) by eliminating terms of order  $O(\varepsilon^4)$ , i.e.,  $\omega_{\gamma\alpha} \omega^{\gamma}_{.\beta}$  and  $e_{\gamma\alpha} e^{\gamma}_{.\beta}$ .

We can further simplify the equations if we notice that when  $k = \gamma = 1, 2$  then:

$$e_{\gamma\beta} \omega^{\gamma}_{.\alpha} = O(\varepsilon^4)$$

Finally:

$$E_{\alpha\beta} = e_{\alpha\beta} + \frac{1}{2} \omega_{3\alpha} \omega_{3\beta} + \frac{1}{2} (e_{3\beta} \omega_{3\alpha} + e_{3\alpha} \omega_{3\beta}) \quad (4)$$

where  $\omega^3_{.\alpha} = g^{3k} \omega_{k\alpha} = g^{33} \omega_{3\alpha} = \omega_{3\alpha}$

Applying the same reasoning to  $E_{\alpha 3}$  and  $E_{33}$ , we find:

$$E_{\alpha 3} = e_{\alpha 3} + \frac{1}{2} (\omega_{\beta 3} \omega^{\beta}_{.\alpha} + e_{\alpha\beta} \omega^{\beta}_{.3} + e_{33} \omega^3_{.\alpha}) \quad (5)$$

$$E_{33} = e_{33} + \frac{1}{2} \omega_{\alpha 3} \omega_{.3}^{\alpha} + \underline{e_{\alpha 3} \omega_{.3}^{\alpha}} \quad (6)$$

where the underlined terms have order  $\epsilon^3$ .

## 2.4 Kinematics of Deformation

In this section, a displacement field is assumed and the associated strain-displacement equations are obtained.

### 2.4.1 Power expansion of the displacements

A common procedure, in derivation of high order shell theories, is to expand the displacements as a function of powers of the normal coordinate  $\theta^3$ , as:

$$u_i(\theta^\alpha, \theta^3, t) = \sum_{n=0}^{\infty} (\theta^3)^n u_i^{(n)}(\theta^\alpha, t) \quad (7)$$

If we substitute (7) into (1) and the result in (2), we find:

$$E_{ij} = \sum_{n=0}^{\infty} \theta^n E_{ij}^{(n)} \quad (8)$$

### 2.4.2 Power expansion of the strains

In order to obtain the expansion for the strains, we first notice that if  $f(x)$  and  $g(x)$  are functions of  $x$ , and are expanded in the same form as in equation (7):

$$f(x) = \sum_{n=0}^{\infty} x^n f_n \quad , \quad g(x) = \sum_{n=0}^{\infty} x^n g_n$$

then :

$$f(x)g(x) = \sum_{n=0}^{\infty} x^n \sum_{m=0}^n f_m g_{n-m}$$

From (1) and (7), we can write:

$$e_{ij} = \sum_{n=0}^{\infty} (\theta^3)^n e_{ij}^{(n)}$$

$$\omega_{ij} = \sum_{n=0}^{\infty} (\theta^3)^n \omega_{ij}^{(n)}$$

After substitution in (2) and considering (4), (5), (6), it follows (see also [16]):

$$E_{\alpha\beta}^{(n)} = e_{\alpha\beta}^{(n)} + \frac{1}{2} \sum_{m=0}^n (\omega_{3\alpha}^{(m)} \omega_{3\beta}^{(p)} + \omega_{3\beta}^{(p)} e_{3\alpha}^{(m)} + \omega_{3\alpha}^{(p)} e_{3\beta}^{(m)}) ; p = n - m$$

$$E_{\alpha 3}^{(n)} = e_{\alpha 3}^{(n)} + \frac{1}{2} \sum_{m=0}^n (\omega_{\lambda 3}^{(m)} \omega_{\cdot \alpha}^{(p)\lambda} + e_{\lambda \alpha}^{(p)\lambda} \omega_{\cdot 3}^{(m)} + e_{33}^{(p)\lambda} \omega_{\cdot \alpha}^{(m)}) \quad (9)$$

$$E_{33}^{(n)} = e_{33}^{(n)} + \frac{1}{2} \sum_{m=0}^n (\omega_{\lambda 3}^{(m)} \omega_{\cdot 3}^{(p)\lambda} + 2e_{\lambda 3}^{(p)\lambda} \omega_{\cdot 3}^{(m)})$$

We now need to find explicit relations between  $e_{ij}^{(n)}$  and  $\omega_{ij}^{(n)}$  and the displacements. Detailed calculations are shown in Appendix A, and only final results are compiled here:

$$e_{\alpha\beta}^{(n)} = \frac{1}{2} (\Phi_{\alpha\beta}^{(n)} + \Phi_{\beta\alpha}^{(n)} - b_{\alpha}^{\lambda} \Phi_{\lambda\beta}^{(n-1)} - b_{\beta}^{\lambda} \Phi_{\lambda\alpha}^{(n-1)})$$

$$e_{\beta}^{\lambda} = \frac{1}{2} \sum_{m=0}^n (B)^{m\alpha\lambda} \left[ \Phi_{\lambda\beta}^{(p)} + \Phi_{\beta\lambda}^{(p)} + m \Phi_{\beta\lambda}^{(p)} - (m+1) b_{\beta}^{\gamma} \Phi_{\gamma\lambda}^{(n-m-1)} \right]$$

$$e_{\alpha 3}^{(n)} = \frac{1}{2} \left[ \Phi_{\alpha 3}^{(n)} + (n+1) u_{\alpha}^{(n+1)} - n b_{\alpha}^{\lambda} u_{\lambda}^{(n)} \right]$$

$$e_{3}^{\lambda} = \frac{1}{2} \sum_{m=0}^n (B)^{m\alpha\lambda} \left[ (n-m+1) u_{\lambda}^{(n-m+1)} + (m+1) \Phi_{\lambda 3}^{(p)} \right]$$

$$e_{33}^{(n)} = (n+1) u_3^{(n+1)}$$

$$\omega_{\alpha\beta}^{(n)} = \frac{1}{2} (\Phi_{\alpha\beta}^{(n)} - \Phi_{\beta\alpha}^{(n)} - b_{\alpha}^{\lambda} \Phi_{\lambda\beta}^{(n-1)} + b_{\beta}^{\lambda} \Phi_{\lambda\alpha}^{(n-1)}) \quad (10)$$

$$\omega_{\cdot\beta}^{\alpha} = \frac{1}{2} \sum_{m=0}^n (B)^{m\alpha\lambda} \left[ \Phi_{\lambda\beta}^{(p)} - \Phi_{\beta\lambda}^{(p)} - m \Phi_{\beta\lambda}^{(p)} + (m+1) b_{\beta}^{\gamma} \Phi_{\gamma\lambda}^{(n-m-1)} \right]$$

$$\omega_{3\alpha}^{(n)} = \frac{1}{2} \left[ \Phi_{\alpha 3}^{(n)} - (n+1) u_{\alpha}^{(n+1)} + n b_{\alpha}^{\lambda} u_{\lambda}^{(n)} \right]$$

$$\omega_{\alpha 3}^{(n)} = \frac{1}{2} \sum_{m=0}^n (B)^{\alpha\lambda} \left[ (n-m+1) u_{\lambda}^{(n-m+1)} - (m+1) \Phi_{\lambda 3}^{(p)} \right]$$

where:

$$\Phi_{\alpha 3}^{(n)} = u_{3, \alpha}^{(n)} + b_{\alpha}^{\lambda} u_{\lambda}^{(n)}$$

$$\Phi_{\alpha\beta}^{(n)} = u_{\alpha, \beta}^{(n)} - b_{\alpha\beta} u_3^{(n)}$$

$$\Phi_{\alpha\beta}^{(n)} = 0, \text{ for } n < 0$$

### 2.4.3 Strain-displacement relations

The final step, in building this theory, is given by keeping only the first two terms, in the power expansion of the displacement field, leading to a first order shear deformation theory. We write:

$$u_i = u_i^{(0)} + \theta^3 u_i^{(1)} \quad (11)$$

Based on this representation we obtain:

$$e_{ij} = e_{ij}^{(0)} + \theta^3 e_{ij}^{(1)}$$

$$\omega_{ij} = \omega_{ij}^{(0)} + \theta^3 \omega_{ij}^{(1)} \quad (12)$$

$$\Phi_{ij} = \Phi_{ij}^{(0)} + \theta^3 \Phi_{ij}^{(1)}$$

Notice that:

$$e_{ij}^{(0)}, \omega_{\alpha\beta}^{(0)}, \Phi_{\alpha\beta}^{(0)} = O(\varepsilon^2)$$

$$e_{ij}^{(1)}, \omega_{\alpha\beta}^{(1)}, \Phi_{\alpha\beta}^{(1)} = O(\varepsilon^2/\hbar)$$

$$\omega_{\alpha 3}^{(0)}, \Phi_{\alpha 3}^{(0)} = O(\varepsilon)$$

$$\omega_{\alpha 3}^{(1)}, \Phi_{\alpha 3}^{(1)} = O(\varepsilon/\hbar)$$

Taking equation (12) into account, equations (9) can be reduced to:

$$\begin{aligned} E_{\alpha\beta}^{(0)} &= e_{\alpha\beta}^{(0)} + \frac{1}{2} (\omega_{3\alpha}^{(0)} \omega_{3\beta}^{(0)} + \omega_{3\beta}^{(0)} e_{3\alpha}^{(0)} + \omega_{3\alpha}^{(0)} e_{3\beta}^{(0)}) \\ E_{\alpha\beta}^{(1)} &= e_{\alpha\beta}^{(1)} + \frac{1}{2} (\omega_{3\alpha}^{(0)} \omega_{3\beta}^{(1)} + \omega_{3\alpha}^{(1)} \omega_{3\beta}^{(0)} + \omega_{3\beta}^{(0)} e_{3\alpha}^{(1)} + \omega_{3\beta}^{(1)} e_{3\alpha}^{(0)} + \omega_{3\alpha}^{(0)} e_{3\beta}^{(1)} + \omega_{3\alpha}^{(1)} e_{3\beta}^{(0)}) \\ E_{\alpha\beta}^{(2)} &= \frac{1}{2} (\omega_{3\alpha}^{(1)} \omega_{3\beta}^{(1)} + \omega_{3\beta}^{(1)} e_{3\alpha}^{(1)} + \omega_{3\alpha}^{(1)} e_{3\beta}^{(1)}) \\ E_{\alpha 3}^{(0)} &= e_{\alpha 3}^{(0)} + \frac{1}{2} (\omega_{\lambda 3}^{(0)} \omega_{\cdot \alpha}^{(0)\lambda} + e_{\lambda \alpha}^{(0)} \omega_{\cdot 3}^{(0)\lambda} + e_{33}^{(0)} \omega_{\cdot \alpha}^{(0)3}) \\ E_{\alpha 3}^{(1)} &= e_{\alpha 3}^{(1)} + \frac{1}{2} (\omega_{\lambda 3}^{(0)} \omega_{\cdot \alpha}^{(1)\lambda} + \omega_{\lambda 3}^{(1)} \omega_{\cdot \alpha}^{(0)\lambda} + e_{\lambda \alpha}^{(0)} \omega_{\cdot 3}^{(1)\beta} + e_{\lambda \alpha}^{(1)} \omega_{\cdot 3}^{(0)\beta} + e_{33}^{(0)} \omega_{\cdot \alpha}^{(1)3} + e_{33}^{(1)} \omega_{\cdot \alpha}^{(0)3}) \\ E_{\alpha 3}^{(2)} &= \frac{1}{2} (\omega_{\lambda 3}^{(1)} \omega_{\cdot \alpha}^{(1)\lambda} + e_{\lambda \alpha}^{(1)} \omega_{\cdot 3}^{(1)\lambda} + e_{33}^{(1)} \omega_{\cdot \alpha}^{(1)3}) \\ E_{33}^{(0)} &= e_{33}^{(0)} + \frac{1}{2} (\omega_{\lambda 3}^{(0)} \omega_{\cdot \alpha}^{(0)\lambda} + e_{\lambda 3}^{(0)} \omega_{\cdot 3}^{(0)\lambda}) \\ E_{33}^{(1)} &= e_{33}^{(1)} + \frac{1}{2} (\omega_{\lambda 3}^{(0)} \omega_{\cdot 3}^{(1)\lambda} + \omega_{\lambda 3}^{(1)} \omega_{\cdot 3}^{(0)\lambda} + e_{\lambda 3}^{(0)} \omega_{\cdot 3}^{(1)\lambda} + e_{\lambda 3}^{(1)} \omega_{\cdot 3}^{(0)\lambda}) \\ E_{33}^{(2)} &= \frac{1}{2} (\omega_{\lambda 3}^{(1)} \omega_{\cdot 3}^{(1)\lambda} + e_{\lambda 3}^{(1)} \omega_{\cdot 3}^{(1)\lambda}) \end{aligned} \tag{13}$$

Equations (13) still contain terms of higher order than  $\varepsilon^3$ . This fact can be demonstrated, by a simple example. From equations (10) we can write:

$$\omega_{3\alpha}^{(0)} = \frac{1}{2} (\Phi_{\alpha 3}^{(0)} - u_{\alpha}^{(1)}) \quad , \quad e_{3\alpha}^{(0)} = \frac{1}{2} (\Phi_{\alpha 3}^{(0)} + u_{\alpha}^{(1)})$$

Then:  $\omega_{3\alpha}^{(0)} = \Phi_{\alpha 3}^{(0)} - e_{3\alpha}^{(0)}$

and:  $\omega_{3\alpha}^{(0)} \omega_{3\beta}^{(0)} = e_{3\alpha}^{(0)} e_{3\beta}^{(0)} + \Phi_{\alpha 3}^{(0)} \Phi_{\beta 3}^{(0)} - e_{3\alpha}^{(0)} \Phi_{\beta 3}^{(0)} - e_{3\beta}^{(0)} \Phi_{\alpha 3}^{(0)}$

The term  $e_{3\alpha}^{(0)} e_{3\beta}^{(0)} = O(\varepsilon^4)$ , must be eliminated, in order to maintain consistency with the assumptions of the theory.

The result of application of this idea to all strain equations, will give the final form of the strains.

Appendix B discusses in more detail how to obtain the final equations:

$$E_{\alpha\beta}^{(0)} = \theta_{\alpha\beta} + \frac{1}{2} \Phi_{\alpha 3}^{(0)} \Phi_{\beta 3}^{(0)}$$

$$E_{\alpha\beta}^{(1)} = \theta_{\alpha\beta} - \frac{1}{2} (b_{\alpha}^{\lambda} \Phi_{\lambda\beta}^{(0)} + b_{\beta}^{\lambda} \Phi_{\lambda\alpha}^{(0)}) + \frac{1}{2} (\Phi_{\alpha 3}^{(0)} \Phi_{\beta 3}^{(1)} + \Phi_{\beta 3}^{(0)} \Phi_{\alpha 3}^{(1)})$$

$$E_{\alpha\beta}^{(2)} = -\frac{1}{2} (b_{\alpha}^{\lambda} \Phi_{\lambda\beta}^{(1)} + b_{\beta}^{\lambda} \Phi_{\lambda\alpha}^{(1)} - \Phi_{\alpha 3}^{(1)} \Phi_{\beta 3}^{(1)})$$

$$2E_{\alpha 3}^{(0)} = \Phi_{\alpha 3}^{(0)} + u_{\alpha}^{(1)} + u^{\lambda} \Phi_{\lambda\alpha}^{(0)} + \frac{1}{2} u_3^{(1)} (\Phi_{\alpha 3}^{(0)} - u_{\alpha}^{(1)})$$

$$2E_{\alpha 3}^{(1)} = u_{3,\alpha}^{(1)} + u^{\lambda} u_{\lambda|\alpha}^{(1)} - \frac{1}{2} u_{3,\lambda}^{(1)} \Phi_{\alpha}^{\lambda} \quad (14)$$

$$2E_{\alpha 3}^{(2)} = 0$$

$$E_{33}^{(0)} = u_3 + \frac{1}{2} u_{\lambda}^{(1)} u^{\lambda}$$

$${}^{(1)}2E_{33} = 0$$

$${}^{(2)}2E_{33} = 0$$

$${}^{(n)}\theta_{\alpha\beta} = \frac{1}{2} ({}^{(n)}u_{\alpha} |_{\beta} + {}^{(n)}u_{\beta} |_{\alpha} - 2b_{\alpha\beta} {}^{(n)}u_3) = \frac{1}{2} ({}^{(n)}\Phi_{\alpha\beta} + {}^{(n)}\Phi_{\beta\alpha})$$

Equations (14) fully express the moderate rotation theory under investigation. Before we proceed, a number of observations are in order:

- the in-plane strains contain three terms in its power expansion.  $\theta_{\alpha\beta}^{(0)}$  and  $\theta_{\alpha\beta}^{(1)}$  are the linear terms. All terms in the expansion contain non-linearities;
- the presence of  $E_{\alpha 3}^{(0)}$ ,  $E_{\alpha 3}^{(1)}$  accounts for shear deformation;
- $E_{33}^{(0)}$  is the transverse normal strain component, resulting from the linear variation of the normal displacement with respect to the normal coordinate  $\theta^3$  and nonlinear contributions from the first order approximation of the in-plane displacements;
- a von Kármán type shell theory, with shear deformation, can be obtained from the above equations, if we eliminate all nonlinear terms, with the exception of products  $u_{3,\alpha} u_{3,\beta}$  in  $E_{\alpha\beta}^{(0)}$ ;
- if we neglect shear deformation and through the thickness expansion, by making  $\Phi_{\alpha 3}^{(0)} = -u_{3,\alpha}^{(0)}$  and  $u_{\alpha}^{(1)} = -u_{3,\alpha}^{(0)}$ , we obtain von Kármán nonlinear classical shell theory with Kirchhoff-Love assumptions.

## 2.5 Equations of Motion and Boundary Conditions

In the derivation of the equations of motion and boundary conditions, use will be made of the Hamilton's principle:

$$0 = \int_0^T \left( - \int_V \rho \dot{u}_i \delta \dot{u}_i^t dV + \int_V S^{ij} \delta E_{ij} dV - \int_V f^i \delta u_i dV - \int_{A_t} \hat{S}^i \delta u_i dA - \int_S P^i \delta u_i dS \right) dt \quad (15)$$

where:

$t$  = time;

$\dot{u}_i$  = components of the velocity;

$\rho$  = density of the undeformed body;

$\delta$  = variational operator;

$S^{ij}$  = components of the second Piola-Kirchhoff stress tensor;

$f_i$  = body forces measured per unit volume of the undeformed body;

$\hat{S}^i$  = prescribed components of the stress vector, per unit area of the undeformed surface A;

$P^i$  = prescribed components of the stress vector per unit area of the surface S;

We recall that:

$$u_i = {}^{(0)}u_i + \theta^3 {}^{(1)}u_i$$

$$E_{\alpha\beta} = {}^{(0)}E_{\alpha\beta} + \theta^3 {}^{(1)}E_{\alpha\beta} + (\theta^3)^2 {}^{(2)}E_{\alpha\beta} \quad (16)$$

$$E_{\alpha 3} = {}^{(0)}E_{\alpha 3} + \theta^3 {}^{(1)}E_{\alpha 3}$$

$$E_{33} = E_{33}^{(0)}$$

Substituting equations (16) into (15) and using results from section 2.2, it is possible to perform integration through the thickness of the shell and reduce the problem to an equivalent two dimensional one. The form of the functional becomes:

$$0 = \int_0^T \int_{\Omega} \left( Y^{\alpha} \delta u_{\alpha}^{(0)} + Y^3 \delta u_3^{(0)} + Y^{\alpha} \delta u_{\alpha}^{(1)} + Y^3 \delta u_3^{(1)} \right) d\Omega dt$$

$$+ \int_0^T \int_{\Omega} \left[ R_{(0)}^{\alpha\beta} \delta E_{\alpha\beta}^{(0)} + R_{(1)}^{\alpha\beta} \delta E_{\alpha\beta}^{(1)} + R_{(2)}^{\alpha\beta} \delta E_{\alpha\beta}^{(2)} + 2R_{(0)}^{\alpha 3} \delta E_{\alpha 3}^{(0)} + 2R_{(1)}^{\alpha 3} \delta E_{\alpha 3}^{(1)} + 2R_{(0)}^{33} \delta E_{33}^{(0)} \right] d\Omega dt$$

$$- \int_0^T \int_{\Gamma} \left( \hat{S}_{(0)}^{\alpha} \delta u_{\alpha}^{(0)} + \hat{S}_{(0)}^3 \delta u_3^{(0)} + \hat{S}_{(1)}^{\alpha} \delta u_{\alpha}^{(1)} + \hat{S}_{(1)}^3 \delta u_3^{(1)} \right) d\Gamma dt$$

where:

$$Y^i = I_1 \dot{u}^i + I_2 \ddot{u}^i - P_0^i - F_0^i$$

$$Y^i = I_1 \dot{u}^i + I_2 \ddot{u}^i - P_1^i - F_1^i$$

$$R_{(n)}^{ij} = \int_{-\frac{h}{2}}^{\frac{h}{2}} c S^{ij}(\theta^3)^n d\theta^3$$

$$I_l = \int_{-\frac{h}{2}}^{\frac{h}{2}} c \rho (\theta^3)^{l-1} d\theta^3$$

$$F_{(n)}^\alpha = \int_{-\frac{h}{2}}^{\frac{h}{2}} c f^\alpha (\theta^3)^n d\theta^3$$

$$F_{(n)}^3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} c f^3 (\theta^3)^n d\theta^3$$

$$\hat{S}_{(n)}^\alpha = \int_{-\frac{h}{2}}^{\frac{h}{2}} c c_\lambda^\alpha \hat{i}^{\beta\lambda} \nu_\beta (\theta^3)^n d\theta^3$$

$$\hat{S}_{(n)}^3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} c \hat{i}^{\beta 3} \nu_\beta (\theta^3)^n d\theta^3$$

$$P_{(n)}^l = [(\theta^3)^n c P^l]_{-h/2}^{h/2}$$

and  $\hat{i}^{\beta l}$  = are the components of the First Piola-Kirchhoff stress tensor.

If we substitute the variations of the strains in terms of displacements and perform the standard operations, the following equilibrium equations are obtained:

$$\delta u_\alpha^{(0)} : T_{(0)}^{\alpha\beta} |_\beta - b_\beta^\alpha T_{(0)}^{3\beta} = I_1 \dot{u}_\alpha^{(0)} + I_2 \dot{u}_\alpha^{(1)} - P_{(0)}^\alpha - F^\alpha$$

$$\delta u_3^{(0)} : T_{(0)}^{3\beta} |_\beta + b_{\alpha\beta} T_{(0)}^{\alpha\beta} = I_1 \dot{u}_3^{(0)} + I_2 \dot{u}_3^{(1)} - P_{(0)}^3 - F^3$$

$$\delta u_\alpha^{(1)} : T_{(1)}^{\alpha\beta} |_\beta - b_\beta^\alpha \tilde{T}_{(1)}^{3\beta} + Q_{(1)}^{\alpha 3} = I_2 \dot{u}_\alpha^{(0)} + I_3 \dot{u}_\alpha^{(1)} - P_{(1)}^\alpha - F^\alpha$$

$$\delta u_3^{(1)} : T_{(1)}^{3\beta} |_{\beta} + b_{\alpha\beta} \tilde{T}_{(1)}^{\alpha\beta} + Q_{(1)}^{33} = I_1 u^{(0)3} + I_2 u^{(1)3} - P_{(0)}^3 - F_{(0)}^3$$

The natural boundary conditions are:

$$\delta u_{\alpha}^{(0)} : T_{(0)}^{\alpha\beta} v_{\beta} - \hat{S}^{\alpha} = 0$$

$$\delta u_3^{(0)} : T_{(0)}^{\alpha 3} v_{\alpha} - \hat{S}^3 = 0$$

$$\delta u_{\alpha}^{(1)} : T_{(1)}^{\alpha\beta} v_{\beta} - \hat{S}^{\alpha} = 0$$

$$\delta u_3^{(1)} : T_{(1)}^{\alpha 3} v_{\alpha} - \hat{S}^3 = 0$$

where:

$$T_{(0)}^{\alpha\beta} = R_{(0)}^{\alpha\beta} - b_{\lambda}^{\alpha} R_{(1)}^{\lambda\beta} + u^{\lambda} R_{(0)}^{\beta 3} - \frac{1}{2} u_3 |^{\alpha} R_{(1)}^{\beta 3}$$

$$T_{(0)}^{3\beta} = (1 + \frac{1}{2} u_3) R_{(0)}^{\beta 3} + \Phi_{\lambda 3}^{(0)} R_{(0)}^{\lambda\beta} + \Phi_{\lambda 3}^{(1)} R_{(1)}^{\lambda\beta}$$

$$T_{(1)}^{\alpha\beta} = R_{(1)}^{\alpha\beta} - b_{\lambda}^{\alpha} R_{(2)}^{\lambda\beta} + u^{\lambda} R_{(1)}^{\beta 3}$$

$$\tilde{T}_{(1)}^{\alpha\beta} = R_{(1)}^{\alpha\beta} - b_{\lambda}^{\alpha} R_{(2)}^{\lambda\beta}$$

$$T_{(1)}^{3\beta} = R_{(1)}^{\beta 3} - \frac{1}{2} \Phi_{\lambda}^{(0)} R_{(1)}^{\lambda 3} + \Phi_{\lambda 3}^{(0)} R_{(1)}^{\lambda\beta} + \Phi_{\lambda 3}^{(1)} R_{(2)}^{\lambda\beta}$$

$$\tilde{T}_{(1)}^{3\beta} = \Phi_{\lambda 3}^{(0)} R_{(1)}^{\lambda\beta} + \Phi_{\lambda 3}^{(1)} R_{(1)}^{\lambda\beta}$$

$$Q_{(1)}^{\alpha 3} = - \left[ (1 - \frac{1}{2} u_3) R_{(0)}^{\alpha 3} + \Phi_{\lambda}^{(0)} R_{(0)}^{\lambda 3} + u^{\lambda} |_{\lambda} R_{(1)}^{\lambda 3} + u^{\lambda} R_{(0)}^{\lambda 3} \right]$$

$$Q_{(1)}^{33} = - \left[ R_{(0)}^{33} + \frac{1}{2} (\Phi_{\alpha 3}^{(0)} - u_{\alpha}^{(1)}) R_{(0)}^{\alpha 3} \right]$$

In Appendix C, the form of the equations for strains, equilibrium equations and boundary conditions, obtained for rectangular plates, circular cylindrical shells and spherical shells are displayed.

## 2.6 Constitutive Equations

### 2.6.1 Material Stiffness Matrix

As mentioned before, this study will focus on linear materials subjected to small strains. The natural choice for constitutive equations is the generalized Hooke's Law, i.e.:

$$S^{ij} = C^{ijkl} E_{kl}$$

We will limit this study to laminated materials, composed of orthotropic laminae, which require nine elastic constants. The equations for the terms in the material stiffness matrix, and their transformation due to a rotation about the normal, will not be given here, but they can be found in the literature (see reference 41 , pp. 51 and 55).

### 2.6.2 Laminate Stiffness Matrix

In the course of the analytical integration through the thickness of the shell, we have defined the stress resultants, for example:

$$R_{(n)}^{11} = \int_{-h/2}^{h/2} c S^{11} (\theta^3)^n d\theta^3$$

Using equation ( 17 ) and expanding the strains, we find:

$$R_{(n)}^{11} = \int_{-h/2}^{h/2} c \left[ (\theta^3)^n \left( \bar{C}_{11}^{(0)} E_{11} + \bar{C}_{12}^{(0)} E_{22} + \bar{C}_{13}^{(0)} (2E_{12}) \right) + (\theta^3)^{n+1} \left( \bar{C}_{11}^{(1)} E_{11} + \bar{C}_{12}^{(1)} E_{22} + \bar{C}_{13}^{(1)} (2E_{12}) \right) + (\theta^3)^{n+2} \left( \bar{C}_{11}^{(2)} E_{11} + \bar{C}_{12}^{(2)} E_{22} + \bar{C}_{13}^{(2)} (2E_{12}) \right) \right] d\theta^3$$

where  $\bar{C}_{11}$  is the rotated  $C^{1111}$ , etc. The relationship between stress resultants and strains can be written, after rearrangement, as follows:

$$\begin{bmatrix} \mathbf{N} \\ \mathbf{M} \\ \mathbf{P} \\ \mathbf{Q} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{D} & \mathbf{0} \\ \mathbf{B}^T & \bar{\mathbf{D}} & \mathbf{E} & \mathbf{0} \\ \mathbf{D}^T & \mathbf{E}^T & \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{S} \end{bmatrix} \begin{bmatrix} \mathbf{E}^{(0)} \\ \mathbf{E}^{(1)} \\ \mathbf{E}^{(2)} \\ \mathbf{G} \end{bmatrix}$$

where:

$$\mathbf{N}^T = (R_{(0)}^{11}, R_{(0)}^{22}, R_{(0)}^{12}, R_{(0)}^{33})^T$$

$$\mathbf{M}^T = (R_{(1)}^{11}, R_{(1)}^{22}, R_{(1)}^{12})^T$$

$$\mathbf{P}^T = (R_{(2)}^{11}, R_{(2)}^{22}, R_{(2)}^{12})^T$$

$$\mathbf{Q}^T = (R_{(0)}^{23}, R_{(0)}^{13}, R_{(1)}^{23}, R_{(1)}^{13})^T$$

$$\mathbf{E}^{(0)T} = (E_{11}^{(0)}, E_{22}^{(0)}, 2E_{12}^{(0)}, E_{33}^{(0)})^T$$

$$\mathbf{E}^{(1)T} = (E_{11}^{(1)}, E_{22}^{(1)}, 2E_{12}^{(1)})^T$$

$$\mathbf{E}^{(2)T} = (E_{11}^{(2)}, E_{22}^{(2)}, 2E_{12}^{(2)})^T$$

$$\mathbf{G}^T = (2E_{23}^{(0)}, 2E_{13}^{(0)}, 2E_{23}^{(1)}, 2E_{13}^{(1)})^T$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{16} & a_{13} \\ a_{12} & a_{22} & a_{26} & a_{23} \\ a_{16} & a_{26} & a_{66} & a_{63} \\ a_{13} & a_{23} & a_{63} & a_{33} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{16} \\ b_{12} & b_{22} & b_{26} \\ b_{16} & b_{26} & b_{66} \\ b_{13} & b_{23} & b_{63} \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} a_{44} & a_{45} & b_{44} & b_{45} \\ a_{45} & a_{55} & b_{45} & b_{55} \\ b_{44} & b_{45} & d_{44} & d_{45} \\ b_{45} & b_{55} & d_{45} & d_{55} \end{bmatrix}$$

$$(a_{ij}, b_{ij}, d_{ij}, e_{ij}, f_{ij}) = \int_{-h/2}^{h/2} c \bar{C}_{ij} (\theta^3)^n d\theta^3 \quad n = 0,1,2,3,4 \quad ij = 1,6$$

$\mathbf{D}$  has the same form as  $\mathbf{B}$ .  $\bar{\mathbf{D}}$  is a  $3 \times 3$  matrix obtained from  $\mathbf{D}$ , by eliminating its last row.  $\mathbf{E}$  and  $\mathbf{F}$  have the same form as  $\bar{\mathbf{D}}$ .

At this point we notice that:

- transverse shear terms are isolated, making the equations suitable for reduced integration;
- the shifter tensor determinant,  $c$ , is dependent on  $\theta^3$ . For the geometries of interest of this work, we have:

$c = 1$  for rectangular plates;

$c = 1 + \frac{\theta^3}{R}$  for circular cylinders, where  $R$  is the radius;

$c = (1 + \frac{\theta^3}{R})^2$  for spheres.

We will assume that the shell is thin enough, to make  $\theta^3/R$  negligible, compared to 1. Appendix D shows the form of the stress resultants-strain relations, when this assumption is not made.

- the wall of the shell being a laminate, we can write:

$$(a_{ij}, b_{ij}, d_{ij}, e_{ij}, f_{ij}) = \sum_{k=1}^n \frac{1}{m+1} [(\theta_{k+1}^3)^{m+1} - (\theta_k^3)^{m+1}] C_{ij}$$

where:

$n$  is the number of layers;

$m = 1, 2, 3, 4, 5$  give  $a_{ij}$ ,  $b_{ij}$ ,  $d_{ij}$ ,  $e_{ij}$ ,  $f_{ij}$ , respectively;

$\theta_k^3$  are the coordinates of each interface and the angle between the  $n$ -th lamina axes (1-2)

and the structural axes ( $x$ - $y$ ) are shown in Figure 3 on page 30.

### 2.6.3 Reduced Laminate Stiffness.

In the next chapters, finite element models for MRT and also for a refined von Kármán type theory (RVK) will be developed, having elements with 5 or 6 degrees of freedom per node. For the case of a 5 dof model, we will assume that  $S^{33}$  is negligible. In this case, the laminate stiffness matrix simplifies to its most used form ( see reference 41 page 83, equation 1.7.46a). The previously defined matrices  $A$ ,  $B$ ,  $D$ ,  $\bar{D}$ , in the stress resultants-strains relations, will now be the usual (3x3) matrices.

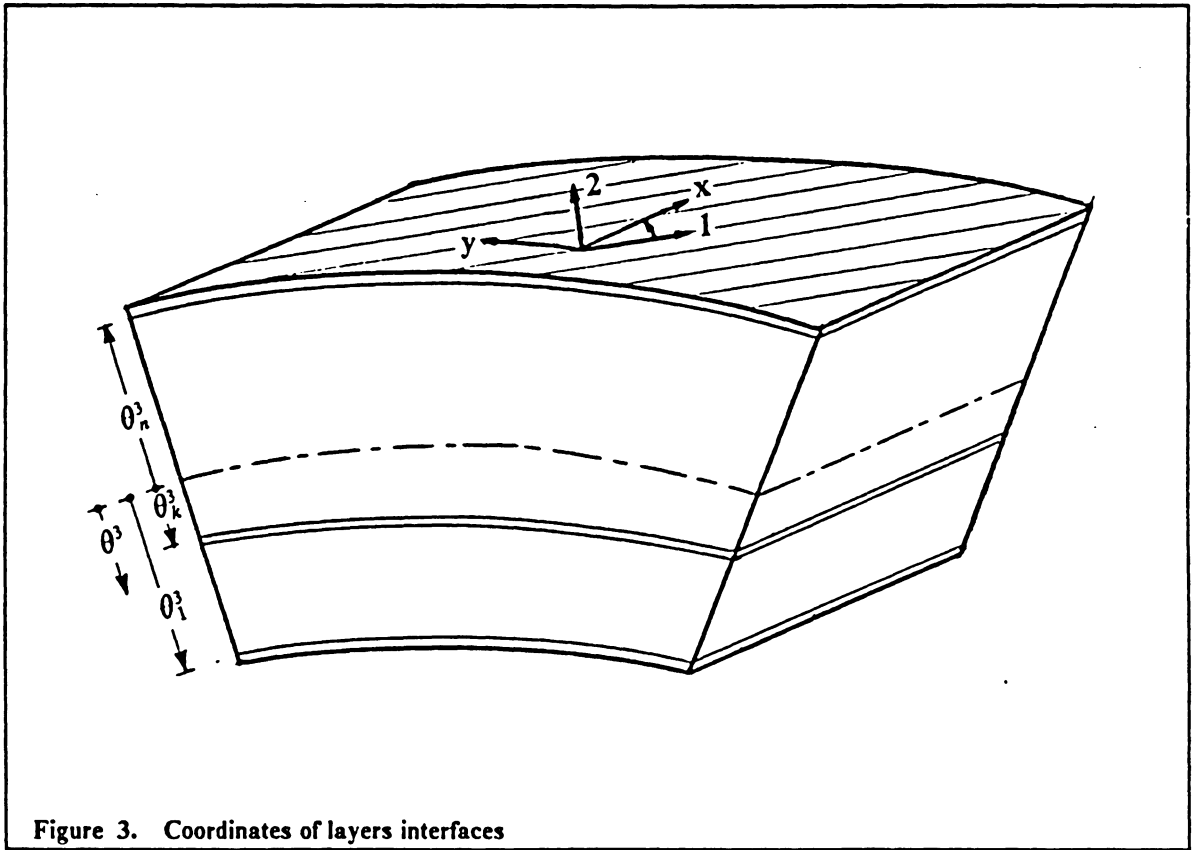


Figure 3. Coordinates of layers interfaces

### 2.7 Comments

In this paragraph, we want to emphasize the assumptions of the moderate rotation proposed by Reddy and Schmidt, as well as other assumptions related to the application of the theory to engineering problems.

Basic assumptions:

- small strains ( of order  $\epsilon^2$  ,  $\epsilon \ll 1$  . );
- in the strain-displacement equations, only terms of order  $\epsilon^2$  and  $\epsilon^3$  will be kept;
- displacements are expanded as :  $u_i = u_i^{(0)} + \theta^3 u_i^{(1)}$ .

Further assumptions:

- the material is transversely isotropic;
- shells are thin, i.e.,  $h/R < 20$ .

Notice that the last two assumptions are not imposed by the moderate rotation theory, but have been adopted to make the formulation less complex, but still useful for the study of a large class of problems.

The form of the derived equations leads to the following observations:

- von Kármán type nonlinearity is present in the equations, as shown in Appendix C, for the geometries in study;
- the equations are very general with respect to the geometry of the shell, and can be particularized to most cases of practical interest;
- the degree of nonlinearity should produce less expensive calculations than a full nonlinear model and yet provide an accurate description within the limits of its assumptions;
- the inclusion of the transverse shear effect makes the theory suitable for studies on anisotropic shells.

## 3. FINITE ELEMENT FORMULATION

### *3.1 Introduction*

After determining the general equations of MRT, we present solutions to specific problems. We make no attempt in this work to look for analytic solutions; therefore, numerical methods need to be selected in order to solve the governing equations of the test problems.

The selection of the finite element method is due to its generality and ease of use, when many different boundary conditions and boundary geometries should be analyzed. After this first definition, we must make other choices and justify them in the sequel.

A first consideration relates to the shape functions. Each problem has a set of functions that provides the best solution. In this work, we want to study very diverse structures and, for convenience, we select a set of function general enough to represent well a wide range of problems, namely the Lagrangian family of interpolation functions in their linear and quadratic form.

Another good reason for using the above mentioned functions has to do with the numerical integration of the approximated equations. Gaussian integration, in conjunction with Lagrangian shape functions, results in a very efficient scheme for element generation. All the details of the implementation of these techniques can be found in the literature and will not be repeated here. Throughout this work, all the results are relative to the Gauss points, unless otherwise specified.

The governing equations, being nonlinear, will generate a nonlinear finite element problem that will be solved iteratively by Newton's method. For some shell cases, this method may break down if the displacement path contains any limit points. Special techniques to handle this situation are

discussed in this chapter. In the next sections, we present the formulation of element matrices in detail.

### 3.2 Direct and tangent stiffness matrices

In the first phase of this work, the direct and tangent stiffness matrices were obtained for plates in explicit form, before integration. Both derivation and element implementation were fairly involved and prone to errors. With even more complex shell equations, it was decided to formulate the problem in matrix form, where no explicit determination of the element matrices is required. This procedure is efficient from the point of view of derivation and implementation of the element matrices, but leads to an inefficient program, if some work is not done to eliminate sparse matrix computations.

The derivations of the next subsections are based on reference 42.

#### 3.2.1 Direct stiffness matrix

Let us write the strain energy variation in matrix form:

$$\delta U = \int_{\Omega} \delta \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} d\Omega$$

where:

$$\boldsymbol{\varepsilon}^T = (\overset{(0)}{E}_{11}, \overset{(0)}{E}_{22}, \overset{(0)}{2E}_{12}, \overset{(0)}{E}_{33}, \overset{(1)}{E}_{11}, \overset{(1)}{E}_{22}, \overset{(1)}{2E}_{12}, \overset{(2)}{E}_{11}, \overset{(2)}{E}_{22}, \overset{(2)}{2E}_{12}, \overset{(0)}{2E}_{23}, \overset{(0)}{2E}_{13}, \overset{(1)}{2E}_{23}, \overset{(1)}{2E}_{13})^T$$

$$\boldsymbol{\sigma} = \mathbf{H} \boldsymbol{\varepsilon}$$

$\mathbf{H}$  = material stiffness matrix

The strains can be written as the sum of the linear plus nonlinear part as:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_L + \boldsymbol{\varepsilon}_{NL} = \mathbf{B}_0 \cdot \mathbf{u} + \frac{1}{2} \mathbf{A}(\mathbf{u}) \boldsymbol{\theta}(\mathbf{u})$$

where:

$\mathbf{u}$  = displacements

$\mathbf{u}$  = displacement vector

$$\boldsymbol{\theta}(\mathbf{u}) = \mathbf{G} \mathbf{u}$$

$\mathbf{A}$  ,  $\mathbf{B}_0$ ,  $\mathbf{G}$  = differential operator matrices.

Then:

$$\boldsymbol{\varepsilon} = (\mathbf{B}_0 + \frac{1}{2} \mathbf{A} \mathbf{G}) \mathbf{u}$$

and:

$$\delta \boldsymbol{\varepsilon} = (\mathbf{B}_0 + \frac{1}{2} \mathbf{A} \mathbf{G}) \delta \mathbf{u} + \frac{1}{2} (\delta \mathbf{A}) \mathbf{G} \mathbf{u}$$

As will be seen, in Appendix E,  $\mathbf{A}$  and  $\boldsymbol{\theta}$  can be built such that  $\delta \mathbf{A} \boldsymbol{\theta} = \mathbf{A} \delta \boldsymbol{\theta}$ . Then:

$$\delta \boldsymbol{\varepsilon} = (\mathbf{B}_0 + \mathbf{A} \mathbf{G}) \delta \mathbf{u}$$

Next, we want to introduce the approximation:

$$\mathbf{u} \approx \mathbf{N} \mathbf{u}$$

where:

$\mathbf{u}$  = displacements at any point in the element;

$\mathbf{u}$  = displacements at the nodes;

$\mathbf{N}$  = matrix of shape functions.

If we call:

$$\tilde{\mathbf{B}} = (\mathbf{B}_0 + \frac{1}{2} \mathbf{A} \mathbf{G})\mathbf{N} \quad , \quad \bar{\mathbf{B}} = (\mathbf{B}_0 + \mathbf{A} \mathbf{G})\mathbf{N}$$

the variation of the strain energy becomes:

$$\delta U \simeq \delta \mathbf{u}^T \left[ \int_{\Omega} \bar{\mathbf{B}} \mathbf{H} \tilde{\mathbf{B}} d\Omega \right] \mathbf{u} = \delta \mathbf{u}^T \mathbf{K}_D \mathbf{u}$$

where  $\mathbf{K}_D$  is the direct stiffness matrix.

### 3.2.2 Tangent stiffness matrix

Consider the variation of the strain energy in the form:

$$\delta U \simeq \delta \mathbf{u}^T \int_{\Omega} \bar{\mathbf{B}} \sigma d\Omega$$

Taking the second variation of  $U$  and neglecting second variations of the displacements, we find:

$$\delta^2 U = \delta \mathbf{u}^T \int_{\Omega} (\delta \bar{\mathbf{B}}' \sigma + \bar{\mathbf{B}} \delta \sigma) d\Omega = \delta \mathbf{u}^T \mathbf{K}_T \delta \mathbf{u}$$

where  $\mathbf{K}_T$  is the tangent stiffness matrix, composed of the sum of two matrices, that will be now determined.

Notice that:

$$\delta \sigma = \mathbf{H} \delta \varepsilon = \mathbf{H} \bar{\mathbf{B}} \delta \mathbf{u}$$

Then one of the matrices is simply:

$$\bar{\mathbf{K}} = \bar{\mathbf{B}}^T \mathbf{H} \bar{\mathbf{B}}$$

Let us consider now the first integral, in the second variation of U, where:

$$\delta \bar{\mathbf{B}}^T = \delta [\mathbf{N}^T (\mathbf{B}_0^T + \mathbf{G}^T \mathbf{A}^T)] = (\mathbf{G} \mathbf{N})^T \delta \mathbf{A}^T$$

As will be seen in Appendix E, the above equation can be rearranged, such that:

$$\delta \mathbf{A}^T \sigma = \mathbf{S} \delta \theta \simeq \mathbf{S} \mathbf{G} \mathbf{N} \mathbf{u}$$

where the elements of matrix  $\mathbf{S}$  are the stress resultants. Then the so called stress matrix is:

$$\mathbf{K}_\sigma = (\mathbf{G} \mathbf{N})^T \mathbf{S} (\mathbf{G} \mathbf{N})$$

and finally:

$$\mathbf{K}_T = \bar{\mathbf{K}} + \mathbf{K}_\sigma$$

Notice that  $\mathbf{H}$  and  $\mathbf{S}$ , being symmetric, will lead to a symmetric tangent stiffness matrix, as expected.

### 3.3 Reduced integration

In many circumstances, it is necessary to resort to reduced integration as a remedy for locking. To do so, we need to separate the terms involving transverse shear and transverse normal effects from the in-plane and bending terms. It is necessary to consider the respective material stiffness matrices, with the  $\mathbf{A}$ ,  $\mathbf{B}_0$ ,  $\mathbf{G}$  and  $\mathbf{S}$  for each case given in Appendix E.

**In-plane and bending terms:**

$$\mathbf{H} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{D} \\ \mathbf{B} & \mathbf{D} & \mathbf{E} \\ \mathbf{D} & \mathbf{E} & \mathbf{F} \end{bmatrix}$$

**Transverse shear terms:**

$$\mathbf{H} = \begin{bmatrix} a_{44} & a_{45} & b_{44} & b_{45} \\ a_{45} & a_{55} & b_{45} & b_{55} \\ b_{44} & b_{45} & d_{44} & d_{45} \\ b_{45} & b_{55} & d_{45} & d_{55} \end{bmatrix}$$

**Transverse normal terms:**

$$\mathbf{H} = \begin{bmatrix} 0 & 0 & 0 & a_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{36} & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{13} & a_{23} & a_{36} & a_{33} & b_{13} & b_{23} & b_{36} & d_{13} & d_{23} & d_{36} \\ 0 & 0 & 0 & b_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{36} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_{36} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

### 3.4 Displacement control methods

When the buckling of shells is considered, as a general rule, a limit point separates the pre-buckling and post-buckling regions. At that point, the tangent stiffness matrix is singular and Newton's method does not converge to the softening path, jumping to the next hardening path, from A to B in Figure 4 on page 39. As illustrated, the shell may even undergo a snap-back, between points C and D of the postbuckling path.

To overcome this difficulty, we have investigated a number of methods found in the literature, all based on some form of displacement control, i.e. , displacements are incremented and the corresponding load is obtained at the end of each step. An outstanding one is the modified Ricks-Wempner method (see reference 43), which can follow a path like the one shown, with the additional feature of performing a uniform number of iterations throughout the calculation.

We made use of a less general technique (from reference 44) for the following reasons:

- its implementation is relatively easy;
- the objective of the work is to test MRT, rather than to provide a general framework for shell calculations;
- there are a sufficient number of cases of shell post-buckling, without snap-back, to study how MRT performs.

The adopted displacement control method consists of specifying a certain component of the displacement vector and obtaining the load corresponding to that state of displacements. In a typical Newton iteration, we now have:

$$\mathbf{K}_T \Delta \mathbf{u} = \mathbf{r} + \Delta \lambda \mathbf{f}_0$$

where:

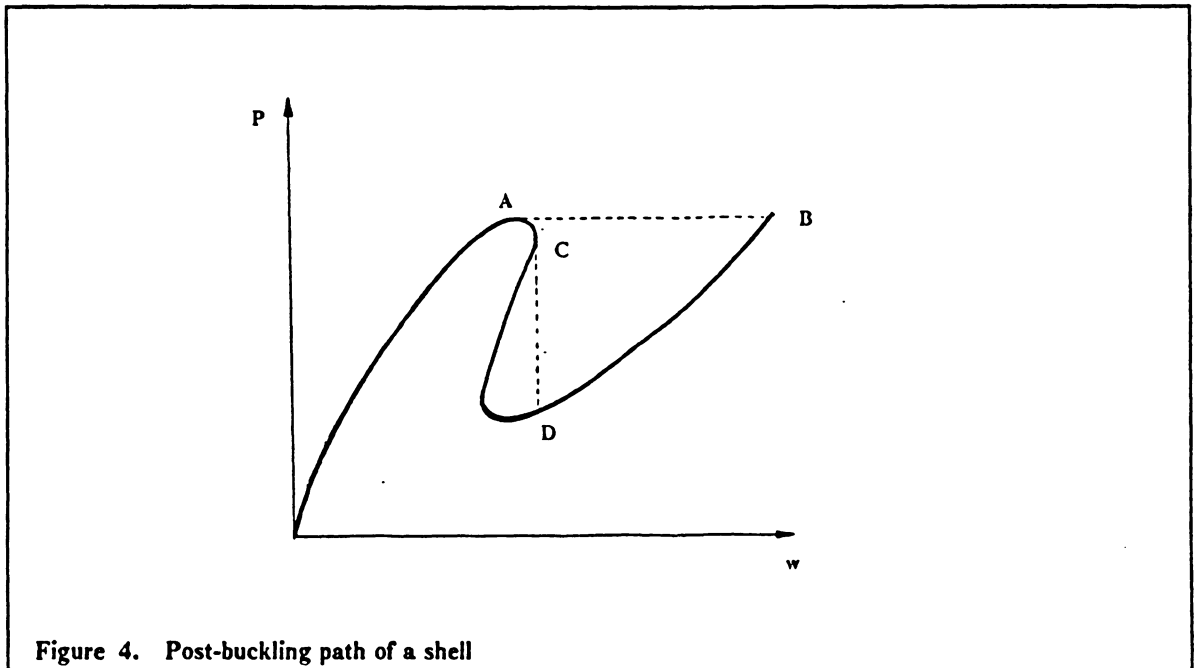


Figure 4. Post-buckling path of a shell

$K_T$  = tangent stiffness matrix;

$r$  = residue due to previous iteration;

$f_0$  = unit vector, parallel to the load increment;

$\Delta\lambda$  = increment in load;

$\Delta u$  = increment in displacements .

The problem to be solved is:

$$K_T \Delta u_r = r$$

$$K_T \Delta u_f = f_0$$

and  $\Delta u = \Delta u_r + \Delta\lambda \Delta u_f$

Suppose the n-th component of  $\mathbf{u}$  is specified. Then  $(\Delta \mathbf{u})_n = (\Delta \mathbf{u}_r)_n + \Delta \lambda (\Delta \mathbf{u}_f)_n = 0$ , resulting in the following:

$$\Delta \lambda = - \frac{(\Delta \mathbf{u}_r)_n}{(\Delta \mathbf{u}_f)_n}$$

Finally:

$$\mathbf{u}_i = \mathbf{u}_{i-1} + \Delta \mathbf{u} ; \mathbf{f}_i = \mathbf{f}_{i-1} + \Delta \lambda \mathbf{f}_0$$

The normal procedure for checking for convergence follows.

### ***3.5 Criterion for MRT calculations***

In the course of the derivations of MRT equations, some assumptions were made with regard to the magnitude of the strains and rotations. An interesting point of investigation is to verify when these assumptions will begin to be trespassed in the various tests. This information can be a helpful tool, in comparing results from other theories, and in justifying possible divergences. We recall here the assumptions on the order of magnitude of the linear strains and rotations:

$$e_{ij} = \omega_{\alpha\beta} = O(\varepsilon^2) \quad , \quad \omega_{3\alpha} = O(\varepsilon)$$

where  $\varepsilon^2 \ll 1$ . A value of  $\varepsilon^2 = 0.01$  will be tentatively taken. Note that, in this case,  $\omega_{3\alpha}$  can be of order 0.1 . Appendix F lists the equations for the linear strains and rotations derived for this purpose.

# 4. NUMERICAL EVALUATION OF THE MODERATE ROTATION THEORY

## 4.1 Introduction

The purpose of this chapter is to test MRT equations, by applying them to a variety of cases of practical interest. Some of the examples to be discussed are taken from the literature and the others are variations of the former ones. As a general rule, we want to compare results from MRT against RVK and a fully nonlinear formulation [39], expecting that these two models will provide upper and lower bounds for MRT.

### 4.1.1. Number of degrees of freedom per node

In almost all cases, we will use five degrees of freedom per node, i.e.,  $u$ ,  $v$ ,  $w$ ,  $\Psi_x$ ,  $\Psi_y$ , due to the following reasons:

- economy in the calculations;
- both refined von Kármán and the fully nonlinear formulations use 5 dof;
- $\Psi_z$  is the linear term in the transverse normal component of the strain,  $E_{33}^{(0)}$ , as we can see in the strain-displacement equations, in Appendix C. We can constrain this dof in a clamped edge, but for other boundary conditions it is not clear if that strain component should or not be zero. It was observed that leaving  $\Psi_z$  free in the entire model may generate a uniform value of this dof, with high order of magnitude. This is a numerical problem equivalent to a rigid body motion, where a dof is unconstrained.

#### 4.1.2. Reduced integration, number of nodes per element and mesh size

Reduced integration was used in all examples, to prevent locking. All calculations were performed using 9-node elements.

#### 4.1.3. Boundary conditions

This is an important subject, especially since we want to make comparisons with results from the literature. There are two sources of misunderstanding, namely:

- the simply supported boundary conditions are defined in different ways. In some cases, no explicit statement of the constrained degrees of freedom is given;
- the use of appropriate symmetry boundary conditions to model only part of a structure.

As reference 45 shows, depending on the lay-up and the kind of support at the edges, the model of a quarter of plate requires appropriate boundary conditions along with the geometric symmetry lines, different from those used for isotropic materials. In the present study, the above mentioned reference will be followed. We therefore define two sets of boundary conditions, as follows:

##### Boundary Condition 1 (BC1):

$$v = w = \Psi_y = 0 \text{ at } x = a$$

$$u = w = \Psi_x = 0 \text{ at } y = b$$

$$v = \Psi_y = 0 \text{ at } y = 0$$

$$u = \Psi_x = 0 \text{ at } x = 0$$

### Boundary Condition 2 (BC2):

$$u = w = \Psi_y = 0 \text{ at } x = a$$

$$v = w = \Psi_x = 0 \text{ at } y = b$$

$$u = \Psi_y = 0 \text{ at } y = 0$$

$$v = \Psi_x = 0 \text{ at } x = 0$$

where the origin of the coordinate system is taken at the center of the plate of dimensions  $2a$  and  $2b$ .

We note the correctness of the nonlinear solution cannot be judged by the correctness of the linear solution with respect to boundary conditions on the geometric symmetry lines. In section 4.2.3, we discuss this matter in more detail.

#### 4.1.4. Material considerations for the 6 dof model

When using the 6 dof MRT, the full 3D Hooke's Law must be used to obtain the material stiffness matrix. To have the same material coefficients for both 5 dof and 6 dof models we must set  $\nu_{13} = \nu_{23} = 0$ . Notice that if we do not set  $\nu_{13} = \nu_{23} = 0$ , the material stiffness will induce stiffer results for the 6 dof model.

The equations of MRT do not impose either plane stress or plane strain states, and, in fact, we could retain  $E_x^{(0)}$  and  $N_x$ , even with only 5 dof. However, we obtain stiffer results in this case. We decided, therefore, to assume a plane stress state to obtain a better correlation with other formulations which, in general, have this assumption.

## ***4.2 Numerical Results for plates***

### **4.2.1. A simply supported (BC1) orthotropic plate under uniform load.**

Figure 5 on page 47 contains the geometry and material used for this problem, and the results. For a displacement of 0.238 in, the corresponding loads resulting from each formulation are:

Liao [39]: 2.048 psi

MRT: 2.040 psi

RVK: 2.033 psi

Notice that, although all the values are very close together, Moderate Rotation falls between the other two results, as expected.

### **4.2.2. Simply supported (BC1) (0/90) plate under uniform load.**

Figure 6 on page 48 contains the geometric and material parameters for the problem. When the center deflection is 1.317 in, the respective loads are:

Liao [39]: 96.2 Pa

MRT: 93.9 Pa

RVK: 93.9 Pa

In this case, the MRT solution coincides exactly with that of the RVK.

### **4.2.3. A simply supported (BC2) (45/-45) plate under uniform load.**

The results are presented in Figure 7 on page 49. For the center displacement of 0.864 cm, the corresponding load is 194 Pa in all formulations.

Up to this point, we can see that the RVK is a very good model and that the MRT did not contribute significantly to the deflection.

A further comment on symmetry boundary conditions is useful at this point. If we take a (45/-45) plate clamped on all edges and subjected to uniform pressure, the linear solutions of the full model and the quarter plate model with the following symmetry conditions:

$$v = \Psi_x = 0 \text{ at } x = 0$$

$$u = \Psi_y = 0 \text{ at } y = 0$$

are the same. But, in the nonlinear range, as we can see from Figure 8 on page 50 the solution obtained using the quarter plate model drifts away from that of the full plate model. For a load of 25 ksi, the central displacements obtained by the two models are:

full plate: 0.7918 in

quarter plate: 0.8840 in (+ 11.4 %)

It is clear from this example that, for some lamination schemes, the quarter plate models do not give correct results.

Figure 9 on page 51 shows the result of the same type of analysis, where the lamination scheme is now (45/ - 45)<sub>s</sub>. Apart from the increased stiffness of the plate, we can observe that the quarter plate model and the full plate models give essentially the same results. Under a load of 25 ksi, the central deflections are:

full plate: 0.694 in

quarter plate: 0.753 in (+ 8.5 %)

We observe in this last case that the two responses are closer to each other. This is due to the reduction of  $B_{16}$  and  $B_{26}$ , in the material stiffness matrix, by a factor of 4 with respect to the (45/-45)

case. We conclude that care must be exercised when the lamination is not symmetric and leads to nonzero  $B_{16}$  and  $B_{26}$ .

#### 4.2.4. Supported orthotropic plate under combined load.

In this example, we want to excite the nonlinear contributions of the in-plane displacements and the rotations  $\Psi_x$ ,  $\Psi_y$ . We recall that the RVK contains nonlinear terms related to the transverse displacements only. We expect, then, to observe some divergence between RVK and models which contain nonlinear terms in the above mentioned degrees of freedom.

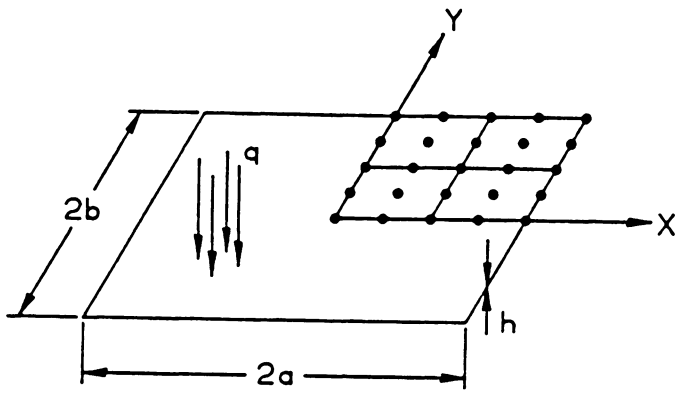
A few words are in order, about the modelling of this problem. Only half of the structure was modelled using symmetry boundary conditions and a 10x1 mesh. The procedure of reference 39 updates all load components for each new step. In order to compare results, we adopted this method and defined a load parameter that multiplies both the transverse load and the inplane load. The first solution step was obtained with a transverse load of -4.6 N and a horizontal load of -9.2 N. To reduce the size of the problem, all displacements  $u$  and rotations  $\Psi_x$  were set to zero.

Figure 10 on page 52 shows a good performance of the MRT element. When the load parameter is 3.002, the resulting displacements at the center of the plate are:

Liao [39]: 1.423 cm

MRT: 1.501 cm

RVK: 1.844 cm



**Boundary conditions :**

$$u = \Psi_x = 0 \text{ at } x = 0$$

$$v = \Psi_y = 0 \text{ at } y = 0$$

$$v = w = \Psi_x = 0 \text{ at } x = a, -a$$

$$u = w = \Psi_y = 0 \text{ at } y = b, -b$$

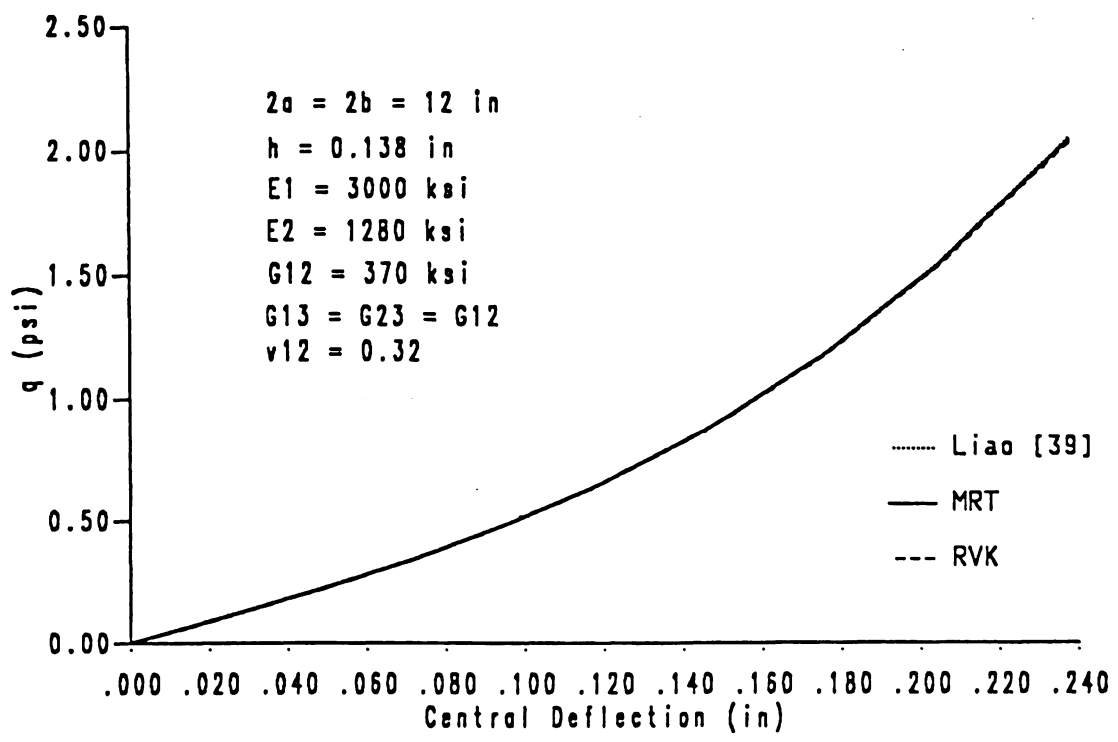
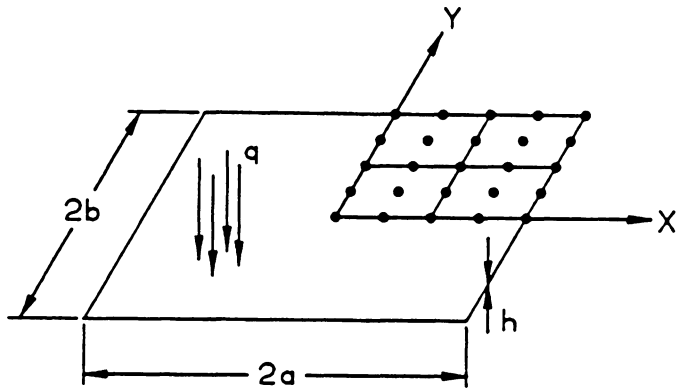


Figure 5. Simply supported (BC1) orthotropic plate under uniform load



**Boundary conditions :**

$$u = \Psi_x = 0 \text{ at } x = 0$$

$$v = \Psi_y = 0 \text{ at } y = 0$$

$$v = w = \Psi_y = 0 \text{ at } x = a, -a$$

$$u = w = \Psi_x = 0 \text{ at } y = b, -b$$

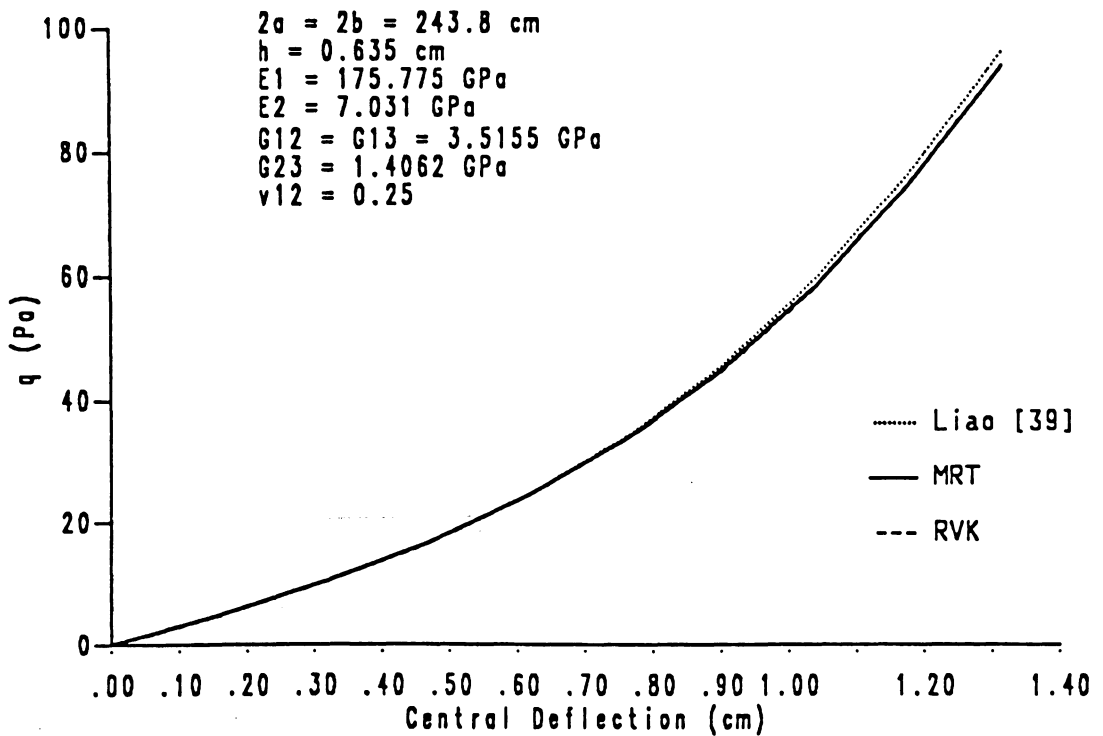
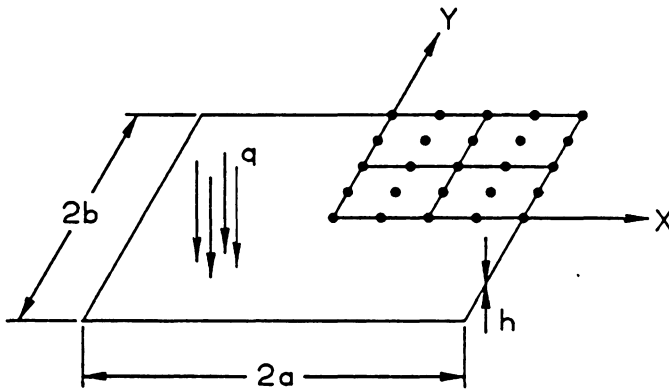


Figure 6. Simply supported (BC1) (0/90) plate under uniform load



**Boundary conditions :**

$$v = \Psi_x = 0 \text{ at } x = 0$$

$$u = \Psi_y = 0 \text{ at } y = 0$$

$$u = w = \Psi_y = 0 \text{ at } x = a, -a$$

$$v = w = \Psi_x = 0 \text{ at } y = b, -b$$

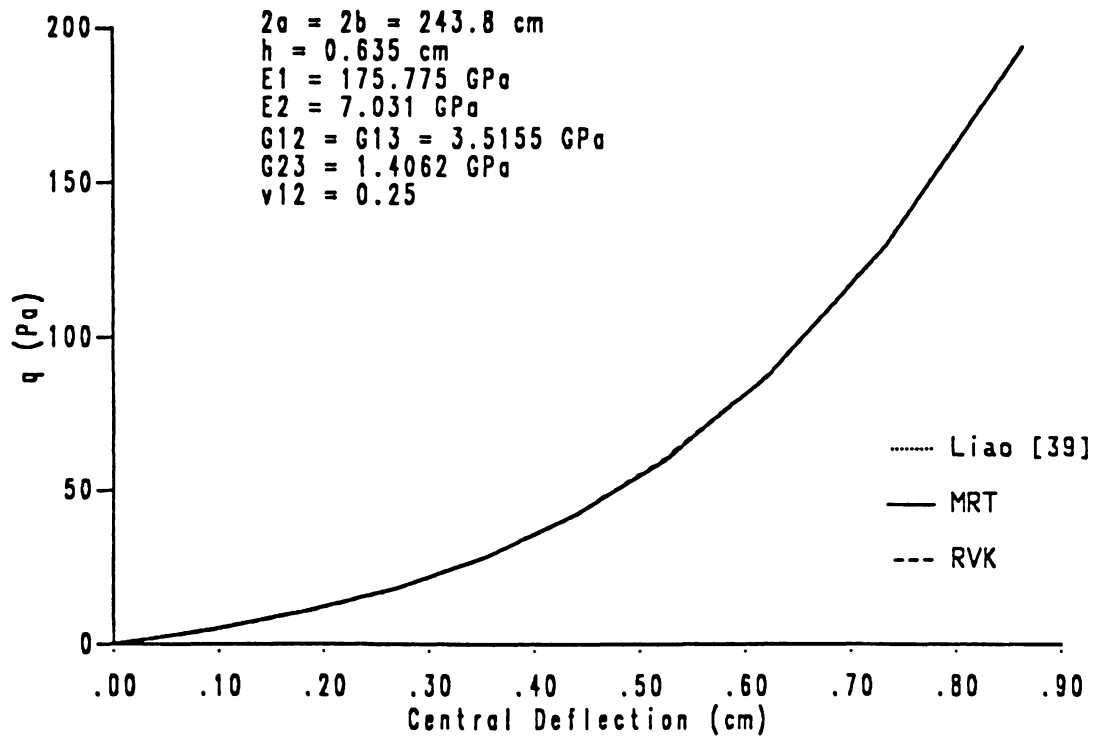


Figure 7. Simply supported (BC2) (45/-45) plate under uniform load

**Boundary conditions :**

$$v = \Psi_x = 0 \text{ at } x = 0$$

$$u = \Psi_y = 0 \text{ at } y = 0$$

$$u = v = w = \Psi_x = \Psi_y = 0 \text{ at } x = a, -a \text{ and } y = a, -a$$

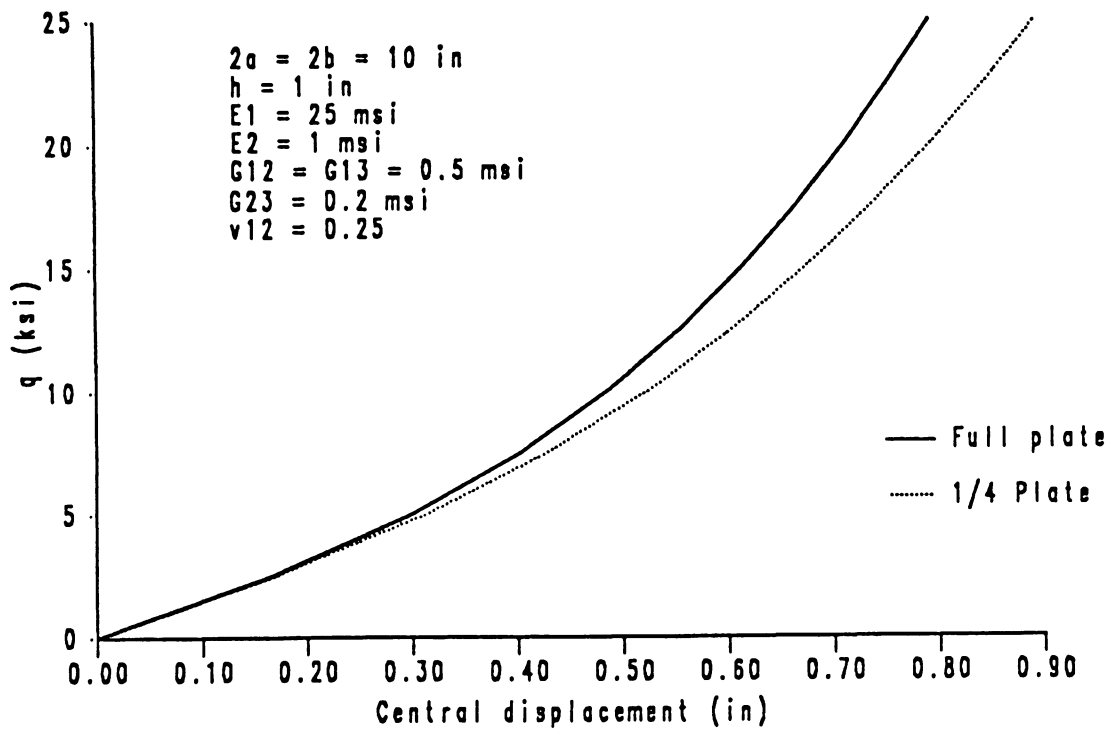
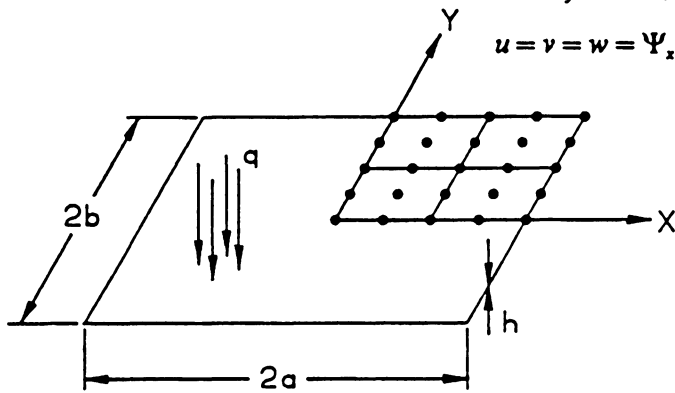


Figure 8. Clamped (45/-45) plate under uniform load

**Boundary conditions :**

$$v = \Psi_x = 0 \text{ at } x = 0$$

$$u = \Psi_y = 0 \text{ at } y = 0$$

$$u = v = w = \Psi_x = \Psi_y = 0 \text{ at } x = a, -a \text{ and } y = a, -a$$

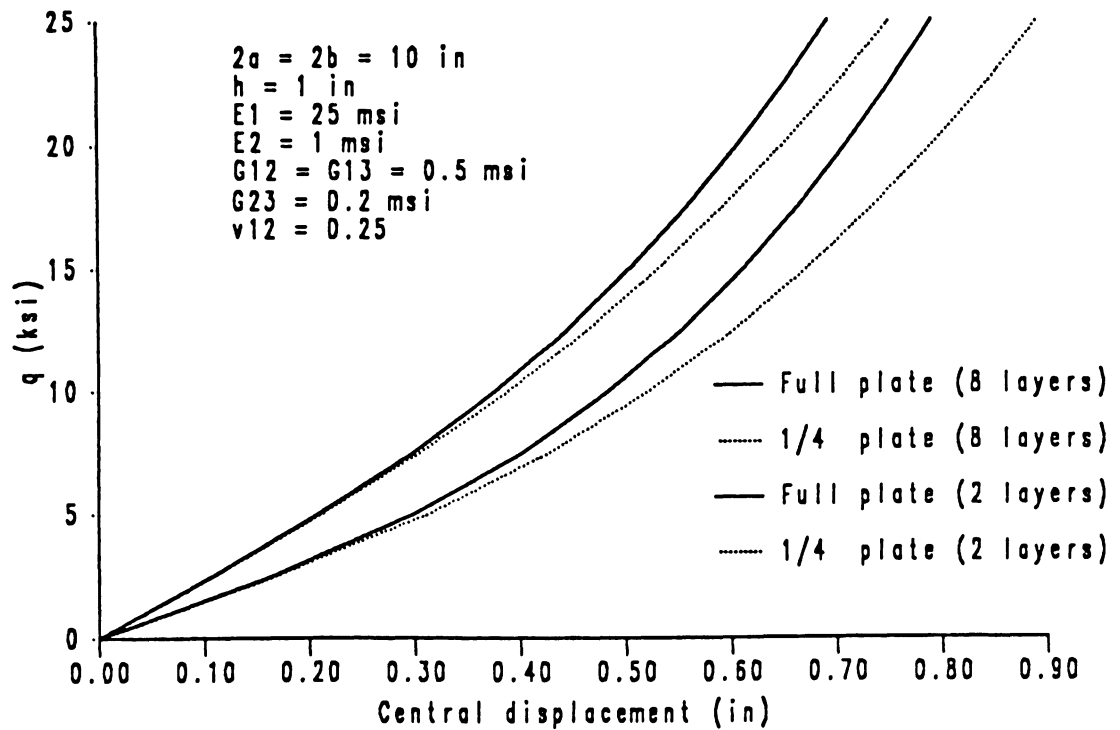
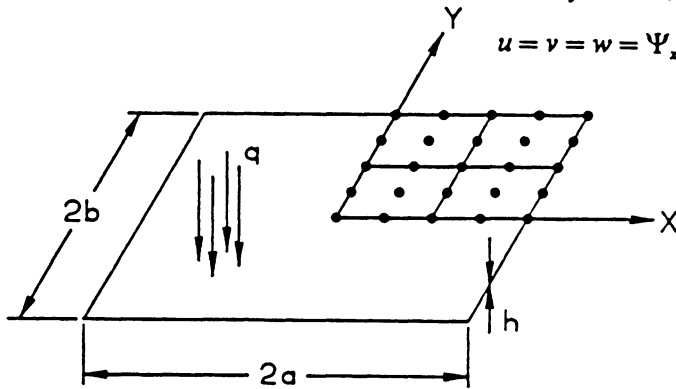
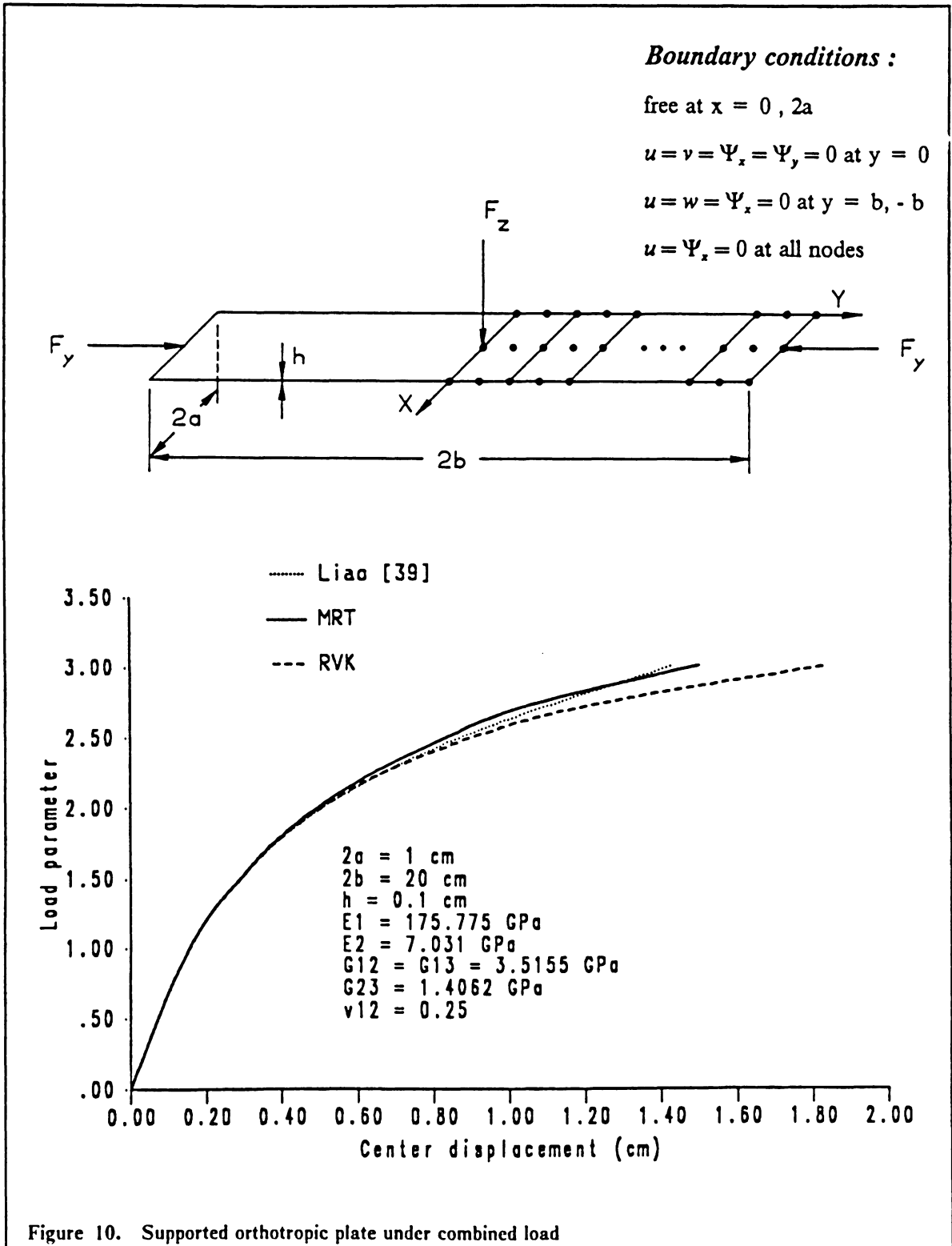


Figure 9. Clamped  $(45/-45)_4$  plate under uniform load



### ***4.3 Numerical Results for cylindrical shells***

#### **4.3.1. Introduction**

All examples of this section consist of shallow portions of cylindrical shells, called cylindrical roofs in the literature. The specific geometry, material and boundary conditions are defined in the same figure, where results are shown for that specific case.

#### **4.3.2. Isotropic hinged cylindrical roof ( thickness = 1.0 in )**

Sabir and Lock [46] reported results for a family of isotropic cylindrical roofs, where the thickness is varied, to show the increasing complexity of the postbuckling region. Sabir and Lock adopted von Kármán type nonlinearity, conjugated with shape functions derived from generalized strain functions and a 5x5 mesh, in a quarter of the structure.

Figure 11 on page 57 indicates the geometry and material for this first example. The boundary conditions used, in this case, are:

$$u = v = w = \Psi_x = 0 \text{ at } y = \pm S \text{ ( hinged )}$$

$$\text{free at } x = \pm L$$

The results show a plate-like behavior and fairly good comparison, between the solutions obtained by all the formulations. The loads for a center displacement of 1.1 in are:

MRT: 1159 lb

Sabir and Lock [46]: 1197 lb

RVK: 1145 lb

#### 4.3.3. Isotropic hinged cylindrical roof ( thickness = 0.5 in )

Figure 12 on page 58 shows the results for this case, where the softening of the postbuckling region is very pronounced. In this case a 2x2 mesh was not sufficient to provide a converged solution, so we used a 3x3 mesh. The general agreement is good, but again, the MRT solution is very close to that of the RVK, even in the final stiffening of the shell.

This example is also a good test for the displacement control technique used here.

#### 4.3.4. Simply supported (BC1) quasi-isotropic cylindrical roof

A quasi-isotropic laminate (0/45/ - 45/90), is the first example of a composite shell. Comparison of the load-deflection behavior with that of reference 39 shows a good agreement in Figure 13 on page 59; however, the RVK shows a slightly better trend than the MRT at the end of the curve.

#### 4.3.5. Simply supported (BC1) orthotropic cylindrical roof

For this problem the RVK gives a closer result to the full nonlinear model, as we can see in Figure 14 on page 60. For the center displacement of 2.026 in the loads are:

Liao [39]: 78420 lb  $\times 4$   
MRT: 82310 lb (+ 4.9 %)  $\times 6$   
RVK: 80640 lb (+ 2.8 %)  $\times 4$

As a prelude to further discussion, we performed the MRT calculations, where all nonlinear contributions from the transverse shear terms are neglected. The load for the above-mentioned displacement is now 80030 lb (+ 2.1 %).

#### 4.3.6. Simply supported (BC1) (0/90) cylindrical roof.

This example differs from the previous one only in the lamination scheme. Figure 15 on page 61 shows the same type of results as in the orthotropic case. Now the solution of the modified MRT (i.e. the nonlinear transverse shear strains are neglected) coincides exactly with that of the von Kármán theory. For the center displacement of 2.222 in, the loads are:

Liao [39]: 62370 lb

MRT: 66420 lb (+ 6.5%)

RVK: 63690 lb (+ 2.1%)

#### 4.3.7. Hinged orthotropic cylindrical roof.

We employ the same geometry and boundary conditions of Example 4.3.2, but use an orthotropic material, as shown in Figure 16 on page 62. Here the MRT model and the RVK model yield the same equilibrium path, diverging from the full nonlinear formulation, mainly in the final stiffening branch of the curve.

#### 4.3.8. Hinged (0/90) cylindrical roof

This example shows the most divergence in results, and, in a certain sense, helps us to understand some of the MRT features. As in the preceding examples, we first compare the full nonlinear, von Kármán, MRT with 5 dof (MRT5) and modified MRT5 formulations, using a 2x2 mesh of finite elements. From Figure 17 on page 63 we conclude that:

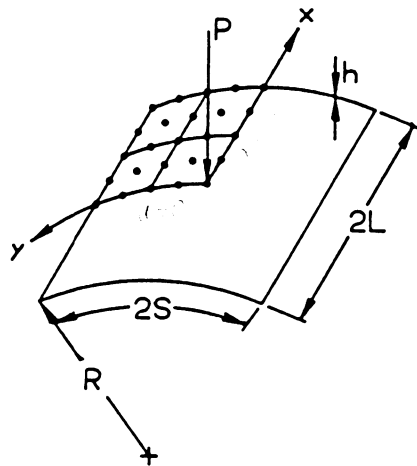
- neither RVK nor MRT5 can follow correctly the post buckling of the shell;
- the RVK solution is closer to the full nonlinear one than the MRT5 solution;
- the modified MRT5 solution is essentially the RVK solution.

Now we compare the MRT model with 6 dof (MRT6) with RVK and the full nonlinear model. We notice from Figure 18 on page 64 that:

- there is no difference between MRT6 and its modified version;
- MRT6 gives slightly better results before the limit point and stiffens faster in the final part of the curve, but has a poor softening behavior.

Next we use a  $3 \times 3$  mesh. We can conclude from the results shown in Figure 19 on page 65 the following:

- the mesh refinement leads to improvement only in first bending portion of the curve, for the RVK , MRT5 and the full nonlinear models;
- this is a converged solution for the above-mentioned results, in the postbuckling region;
- both forms of MRT5 converge to the same solution, with mesh refinement;
- the RVK and MRT5 models do not give good results for this problem;
- MRT6 shows a good trend for the  $3 \times 3$  mesh.



**Boundary conditions :**

$$u = \Psi_x = 0 \text{ at } x = 0$$

$$v = \Psi_y = 0 \text{ at } y = 0$$

free at  $x = a, -a$

$$u = v = w = \Psi_x = 0 \text{ at } y = b, -b$$

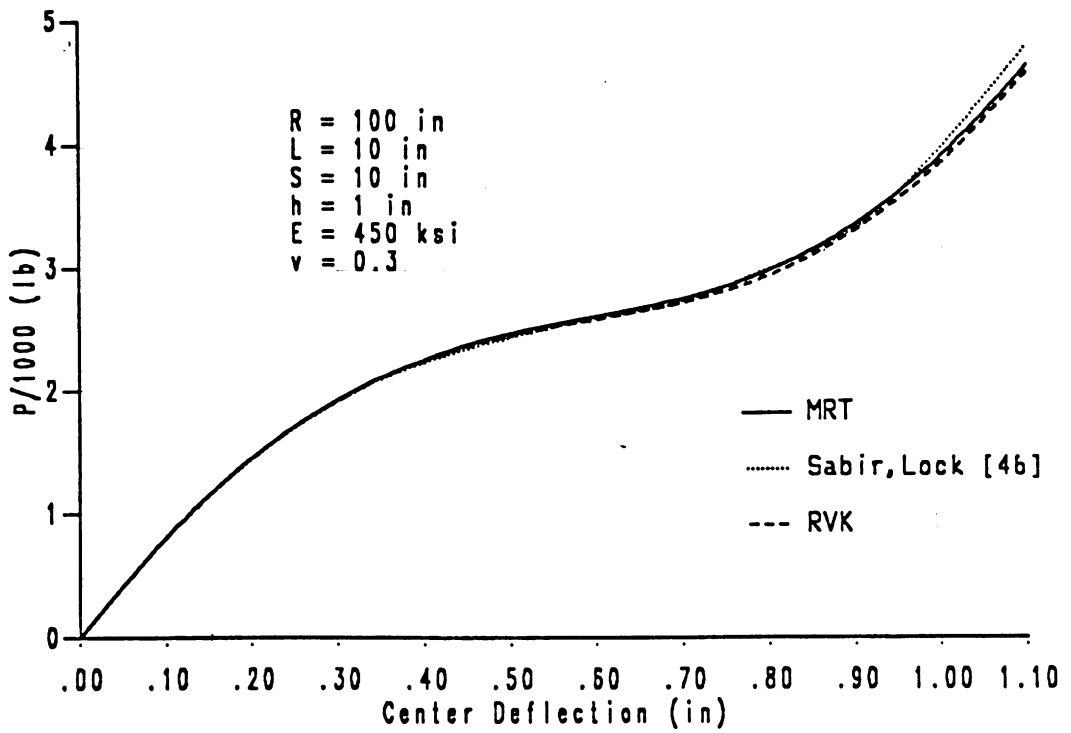
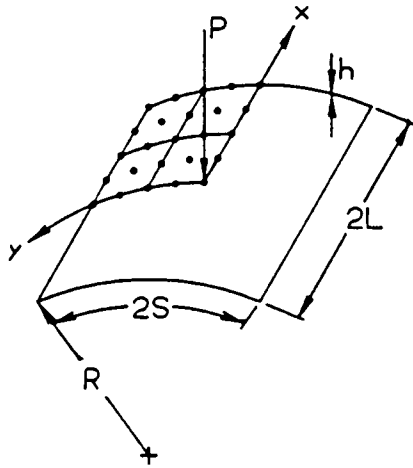


Figure 11. Hinged cylindrical roof (thickness = 1.0 in)



**Boundary conditions :**

$$u = \Psi_x = 0 \text{ at } x = 0$$

$$v = \Psi_y = 0 \text{ at } y = 0$$

free at  $x = a, -a$

$$u = v = w = \Psi_x = 0 \text{ at } y = b, -b$$

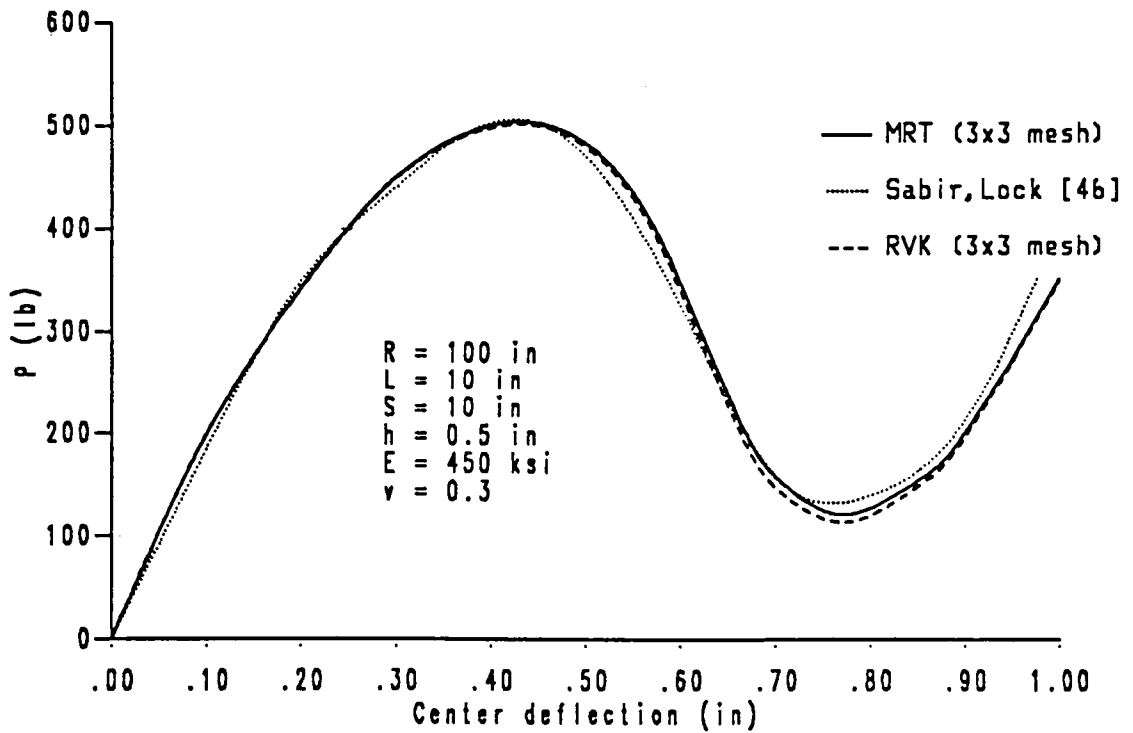
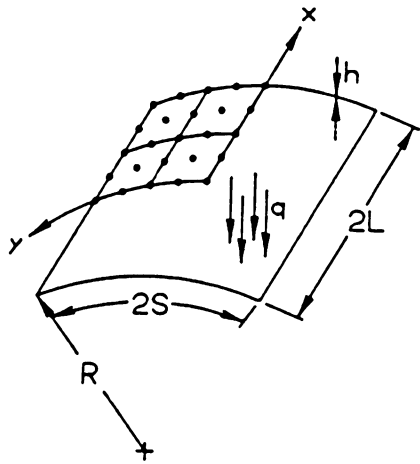


Figure 12. Hinged cylindrical roof (thickness = 0.5 in)



**Boundary conditions :**

$$u = \Psi_x = 0 \text{ at } x = 0$$

$$v = \Psi_y = 0 \text{ at } y = 0$$

$$v = w = \Psi_y = 0 \text{ at } x = a, -a$$

$$u = w = \Psi_x = 0 \text{ at } y = b, -b$$

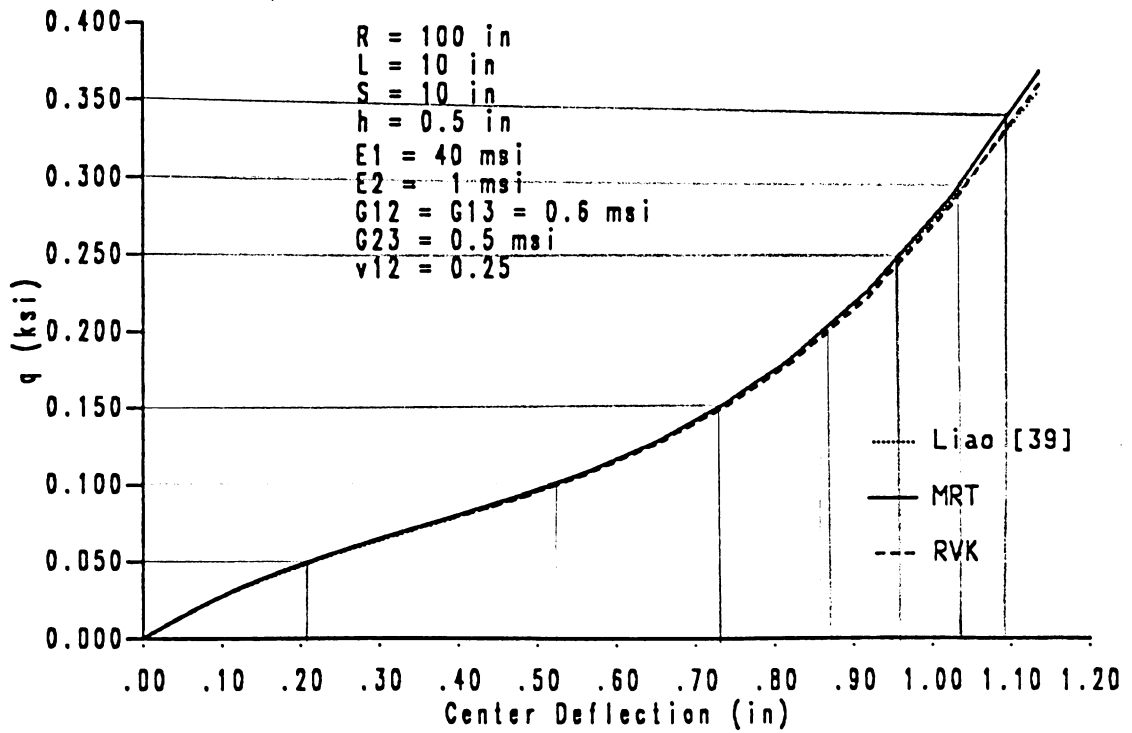
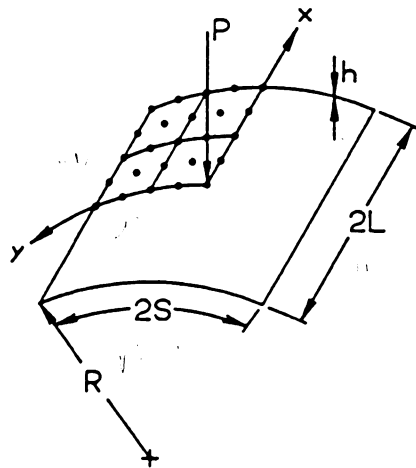


Figure 13. Simply supported (BC1) quasi-isotropic cylindrical roof



**Boundary conditions :**

$$u = \Psi_x = 0 \text{ at } x = 0$$

$$v = \Psi_y = 0 \text{ at } y = 0$$

$$v = w = \Psi_y = 0 \text{ at } x = a, -a$$

$$u = w = \Psi_x = 0 \text{ at } y = b, -b$$

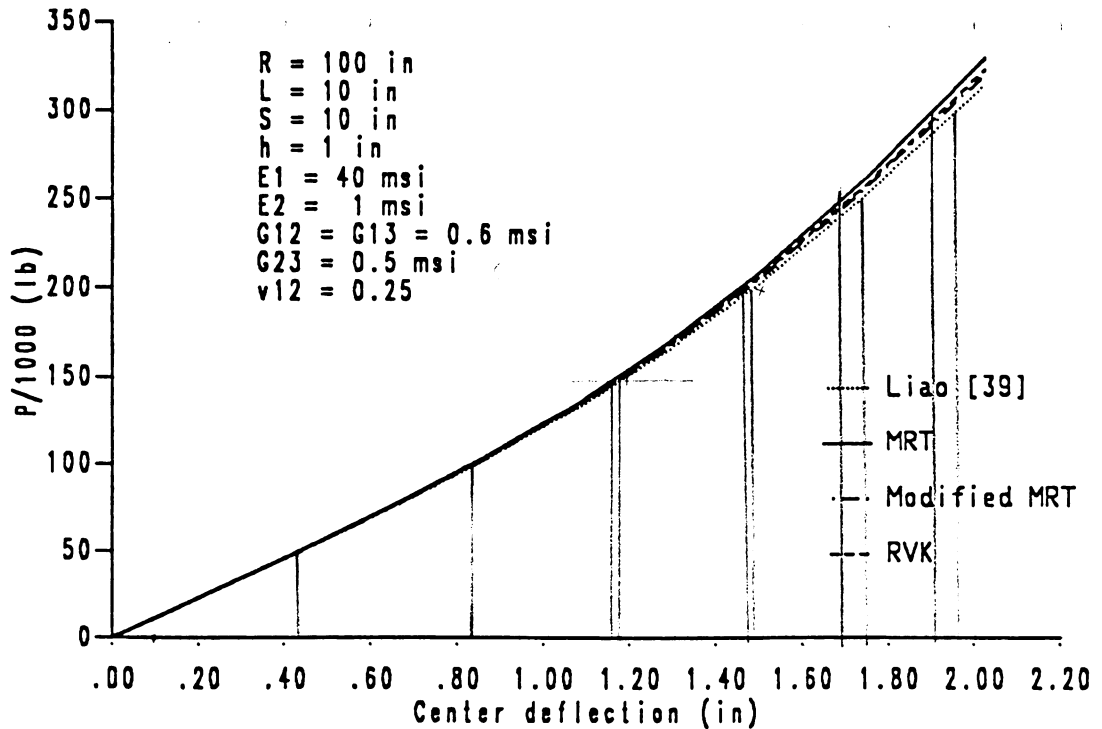
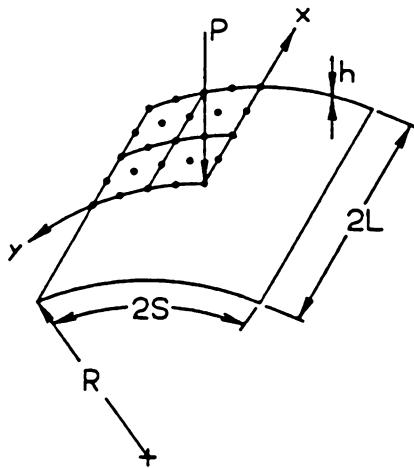


Figure 14. Simply supported (BC1) orthotropic cylindrical roof



**Boundary conditions :**

$$u = \Psi_x = 0 \text{ at } x = 0$$

$$v = \Psi_y = 0 \text{ at } y = 0$$

$$v = w = \Psi_y = 0 \text{ at } x = a, -a$$

$$u = w = \Psi_x = 0 \text{ at } y = b, -b$$

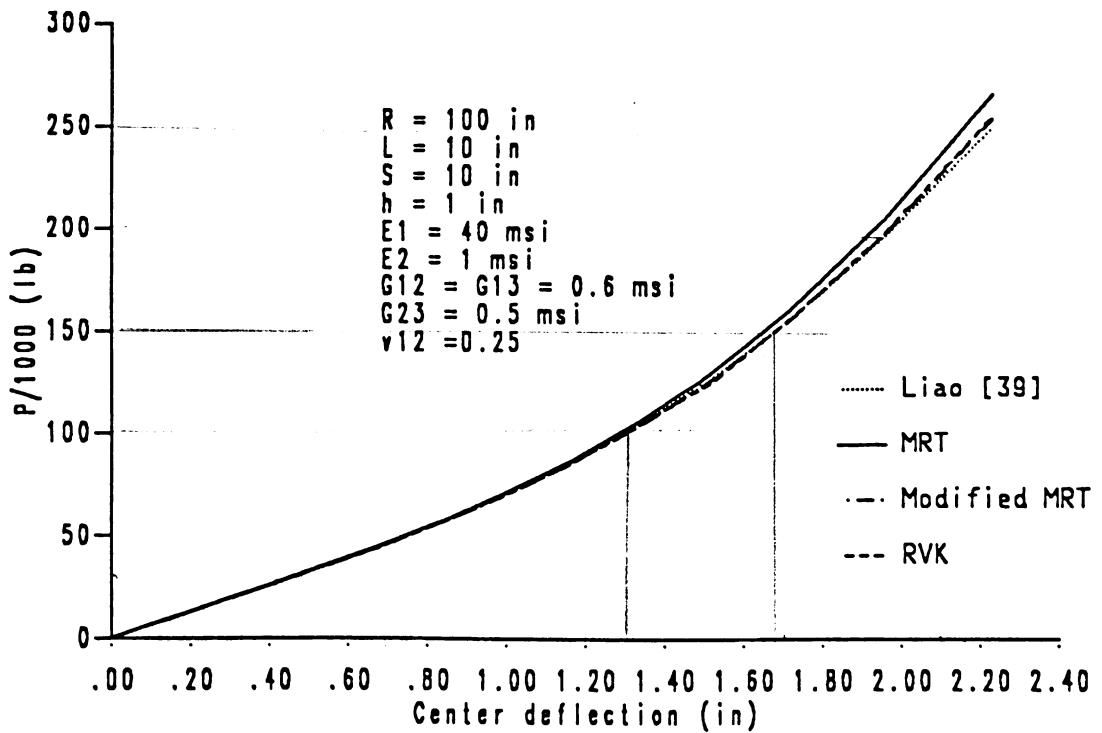
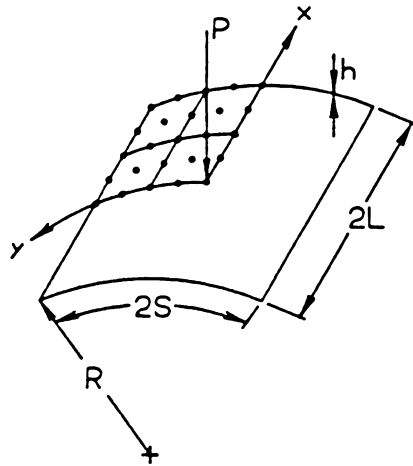


Figure 15. Simply supported (BC1) (0/90) cylindrical roof



**Boundary conditions :**

$$u = \Psi_x = 0 \text{ at } x = 0$$

$$v = \Psi_y = 0 \text{ at } y = 0$$

free at  $x = a, -a$

$$u = v = w = \Psi_x = 0 \text{ at } y = b, -b$$

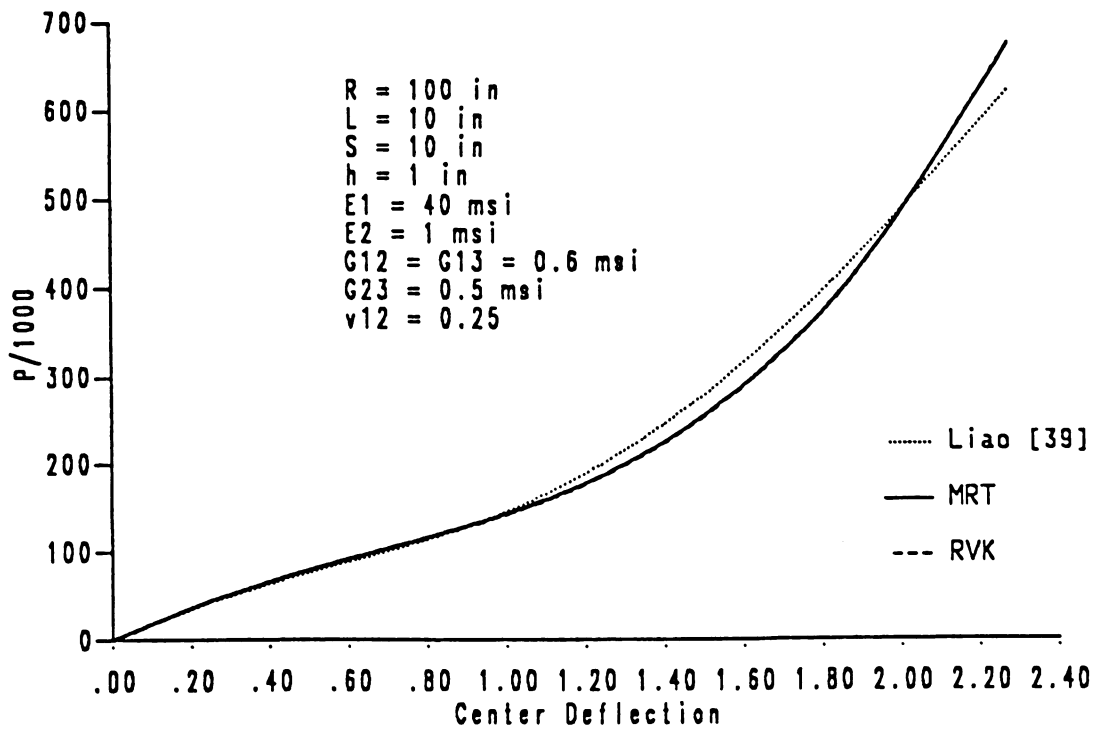
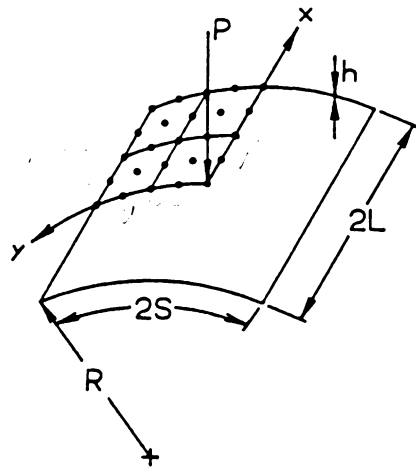


Figure 16. Hinged orthotropic cylindrical roof



**Boundary conditions :**

$$u = \Psi_x = 0 \text{ at } x = 0$$

$$v = \Psi_y = 0 \text{ at } y = 0$$

free at  $x = a, -a$

$$u = v = w = \Psi_x = 0 \text{ at } y = b, -b$$

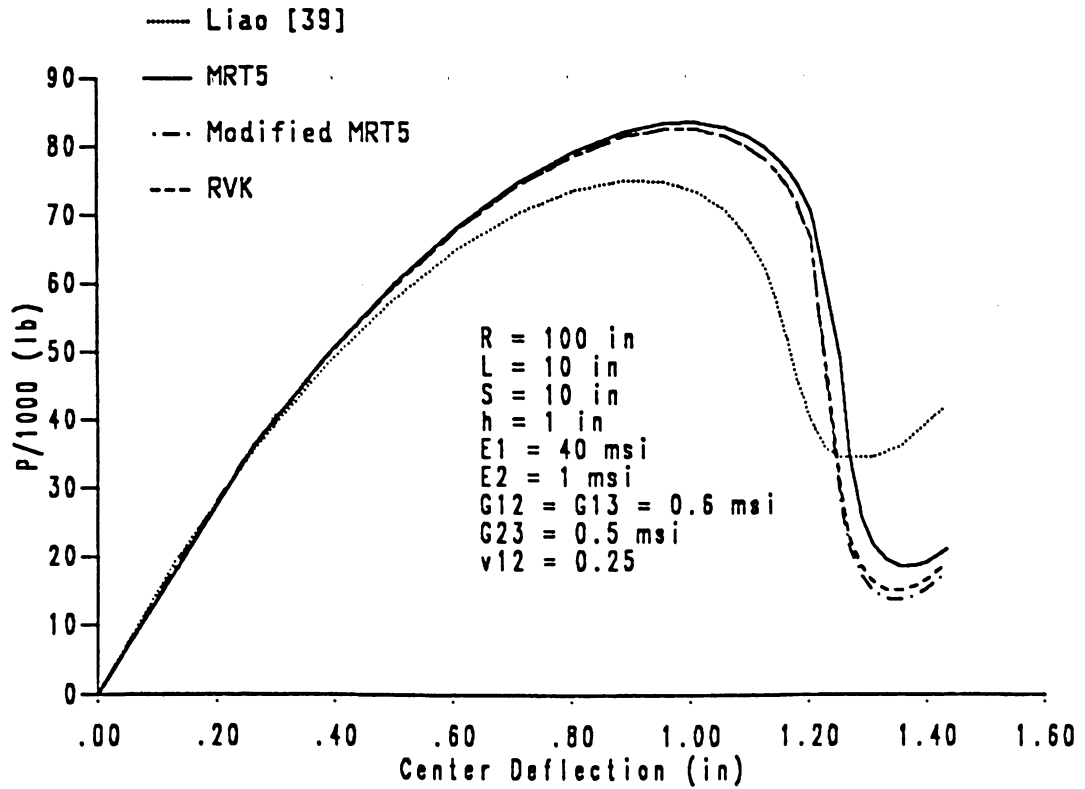
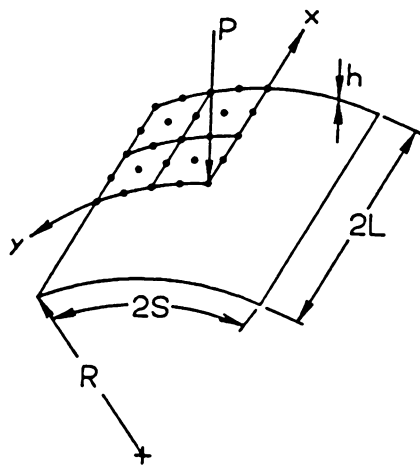


Figure 17. Hinged (0/90) cylindrical roof



**Boundary conditions :**

$$u = \Psi_x = 0 \text{ at } x = 0$$

$$v = \Psi_y = 0 \text{ at } y = 0$$

free at  $x = a, -a$

$$u = v = w = \Psi_x = 0 \text{ at } y = b, -b$$

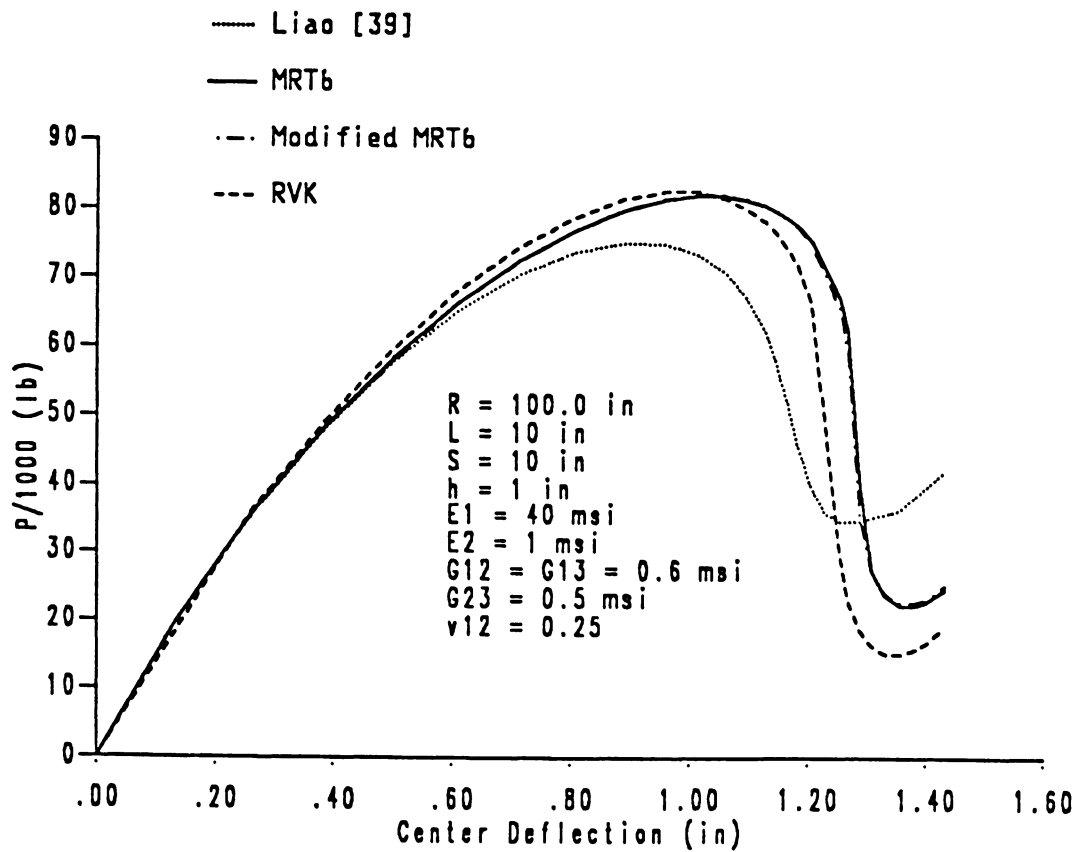
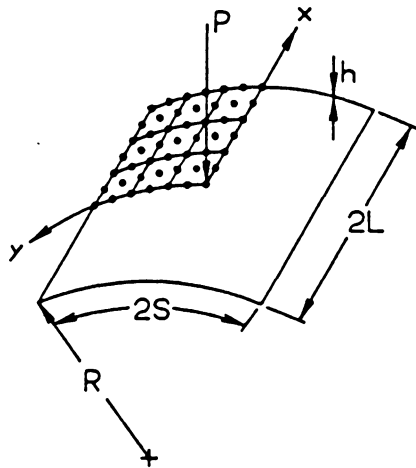


Figure 18. Hinged (0/90) cylindrical roof



**Boundary conditions :**

$$u = \Psi_x = 0 \text{ at } x = 0$$

$$v = \Psi_y = 0 \text{ at } y = 0$$

free at  $x = a, -a$

$$u = v = w = \Psi_x = 0 \text{ at } y = b, -b$$

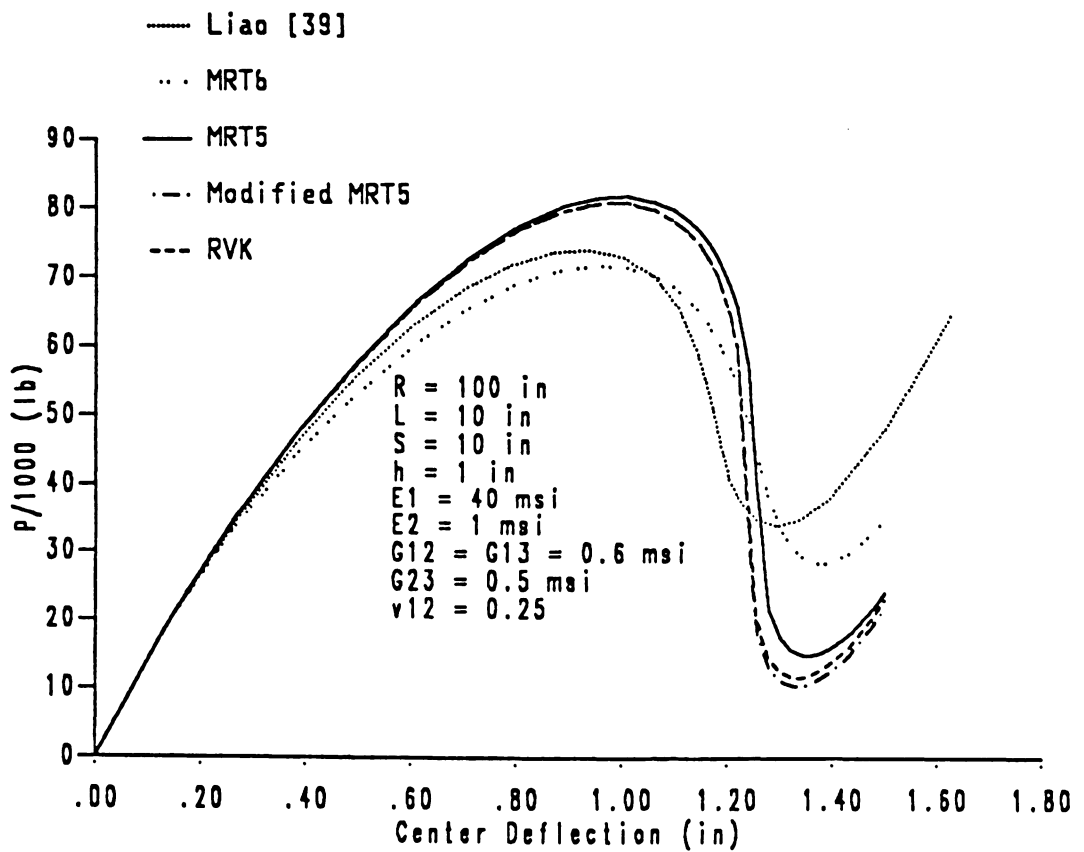


Figure 19. Hinged (0/90) cylindrical roof (3x3) mesh

## ***4.4 Numerical Results for spherical shells***

### **4.4.1. Introduction**

The spherical shell calculations require a few comments. First, as in the cylindrical roof cases, we take only shallow caps from the spherical shells. A second observation involves terms in  $\cot \varphi$ , in the spherical shell equations ( $\varphi$  is indicated in Figure 20 on page 67). In the neighborhood of cap's apex,  $\varphi$  is small and the above-mentioned terms may attain high values, even if the Gauss points are not exactly at the singular point. Guided by the fact that  $\cot \varphi$  multiplies terms that tend to zero, at the apex, we will simply eliminate them from the calculations. We should remark, however, that this is not a general procedure, especially if the problem to be analyzed is not symmetric near the singular point.

Two types of spherical geometries will be studied. The first is the circular cap, modelled by the three meshes shown in Figure 20 on page 67. The other type of cap is shown in the same figure, where the geometry and mesh for a quarter of the structure are indicated. This cap has a rectangular projection on the XY plane.

### **4.4.2. Clamped spherical cap under concentrated load**

This example was taken from reference 42, p. 520, where the complete Green-Lagrange strain tensor was used. As this is a circular cap, we used appropriate finite element meshes, as seen in Figure 20 on page 67, for a quarter of the structure. The results shown in Figure 21 on page 70 were obtained by progressively refining the mesh. For the coarse mesh, the results obtained agree only for the initial points of the curve. The intermediate mesh provides agreement up to a load of 34 lb. Only the finer mesh could give the converged result. At the last point of the calculations, the central deflection is 0.16 in, and the corresponding loads are as follows:

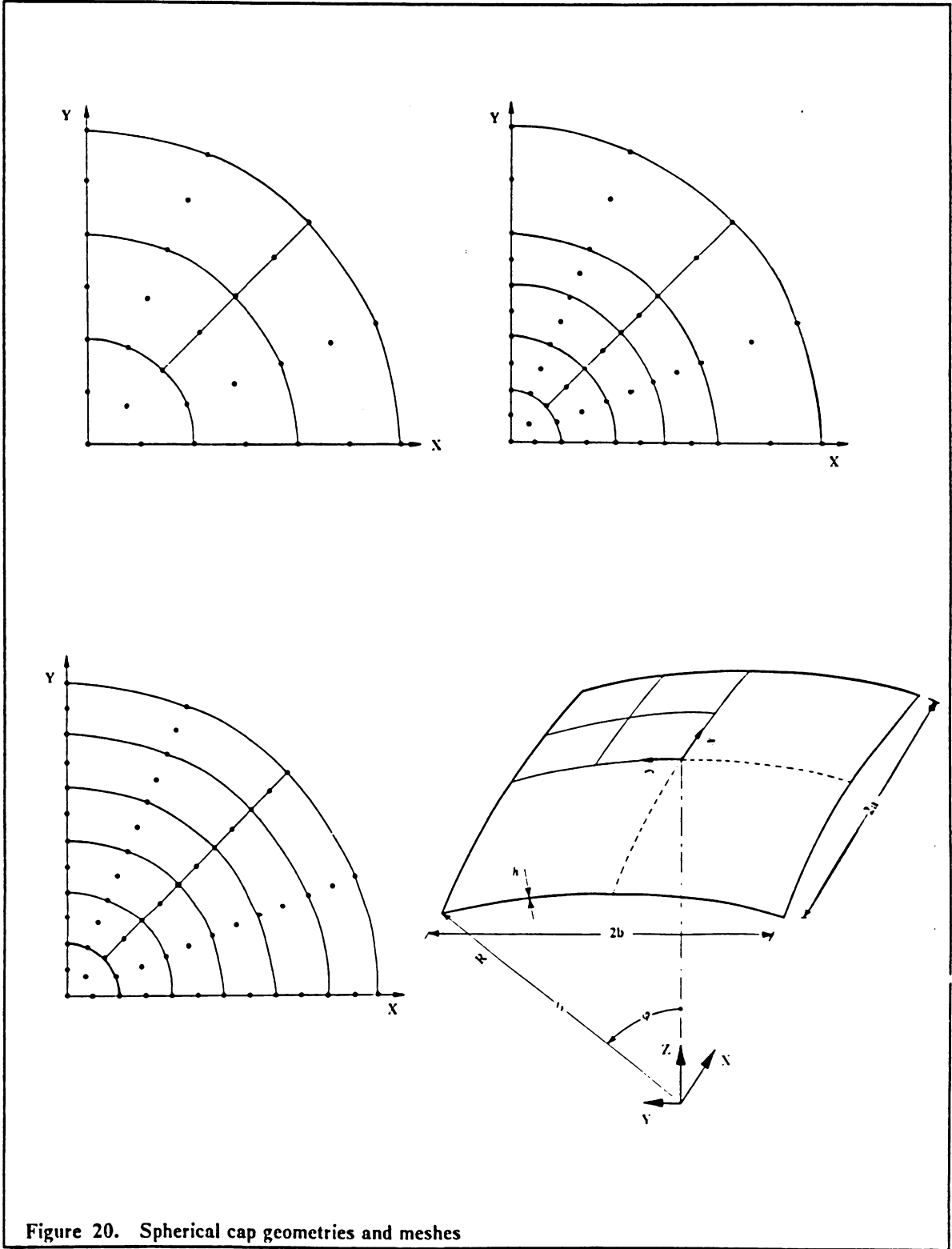


Figure 20. Spherical cap geometries and meshes

Zienkiewicz [42]: 69.2 lb

MRT: 69.2 lb

RVK: 72 lb

This constitutes a good result for the MRT model, as well as for RVK.

#### **4.4.3. Hinged spherical cap under concentrated load**

This example is taken from reference 39, where the cap has a rectangular projection on the X-Y plane. From Figure 22 on page 71 we can see that the MRT does not model this case well. Its modified version closely follows the RVK results, except in the final stiffening part of the equilibrium path.

#### **4.4.4. (0/90/0/90/0)s simply supported (BC1) cap under uniform load**

Figure 23 on page 72 defines the geometry and the material for this problem. In this case we used a 3x3 mesh to get results that agree quite well with those in reference 39 which, in their turn, compare well with those in reference 47. For the central deflection of 3.982 in, we have:

Liao [39]: 10.47 psi

MRT: 10.44 psi

RVK: 10.31 psi

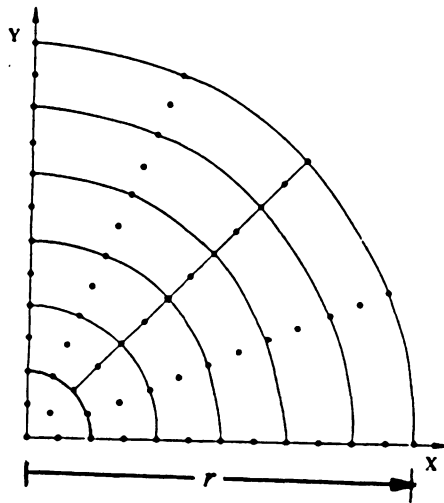
#### **4.4.5. (0/90) simply supported (BC1) cap under uniform load**

In this case we have the same geometry and boundary conditions as those in the preceding example. The material is defined in Figure 24 on page 73 for this nonsymmetric lay-up. Here again a 3x3 mesh was used to find better agreement with reference 39. The overall comparison is very good, where the RVK is a good model for the problem. At a central deflection of 3.918 in, the loads are:

Liao [39]: 3.53 psi

MRT: 3.47 psi

RVK: 3.42 psi



**Boundary conditions :**

$$u = \Psi_x = 0 \text{ at } x = 0$$

$$v = \Psi_y = 0 \text{ at } y = 0$$

$$u = v = w = \Psi_x = \Psi_y = 0 \text{ at } \sqrt{x^2 + y^2} = r$$

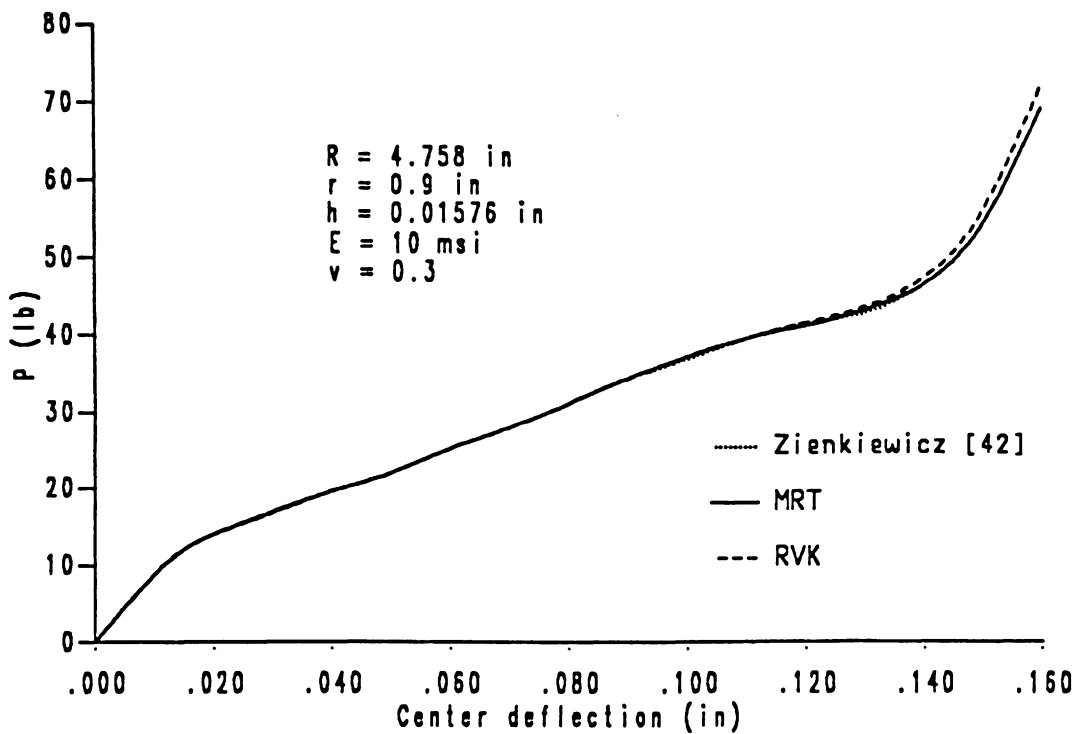
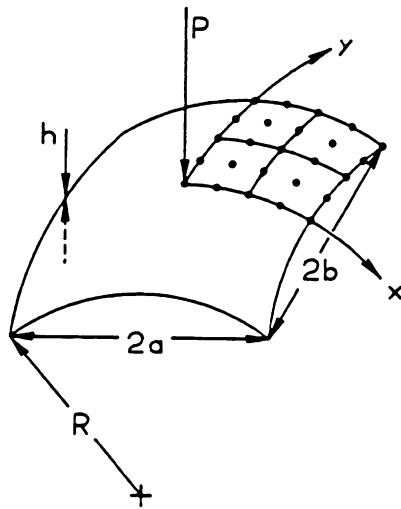


Figure 21. Clamped spherical cap under concentrated load



**Boundary conditions :**

$$u = \Psi_x = 0 \text{ at } x = 0$$

$$v = \Psi_y = 0 \text{ at } y = 0$$

$$u = v = w = \Psi_y = 0 \text{ at } x = a, -a$$

$$u = v = w = \Psi_x = 0 \text{ at } y = b, -b$$

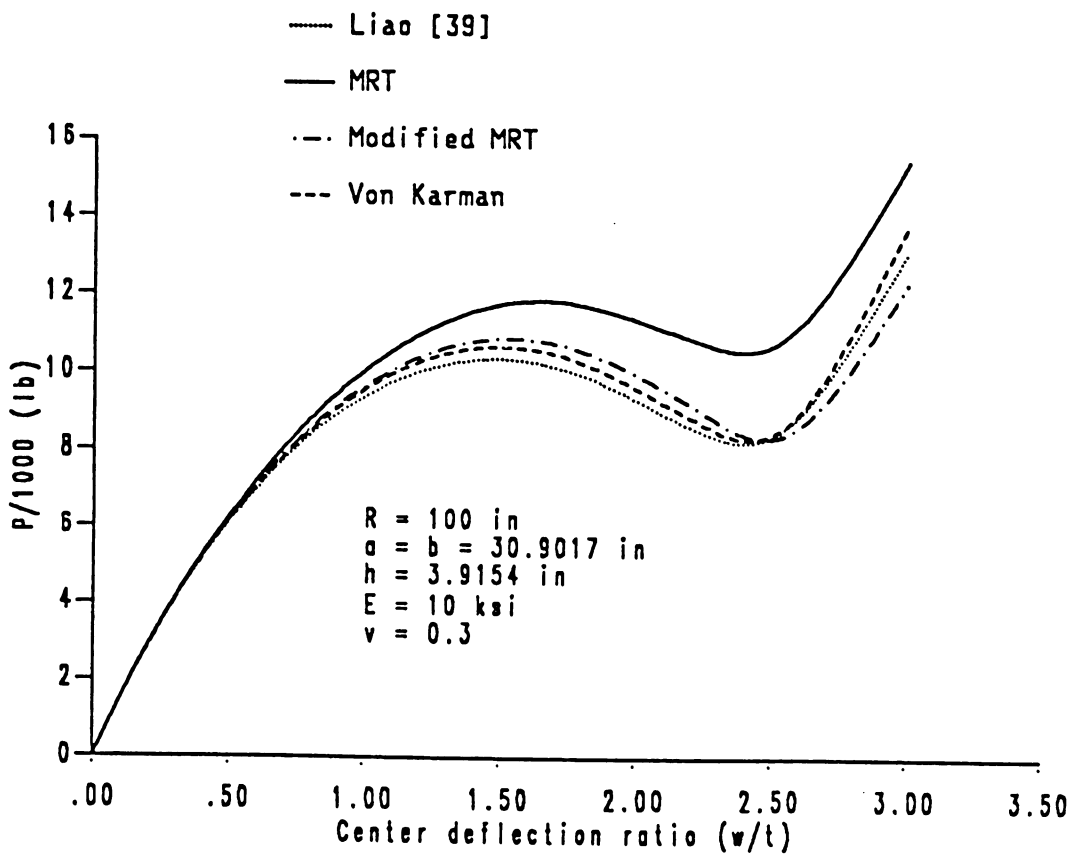
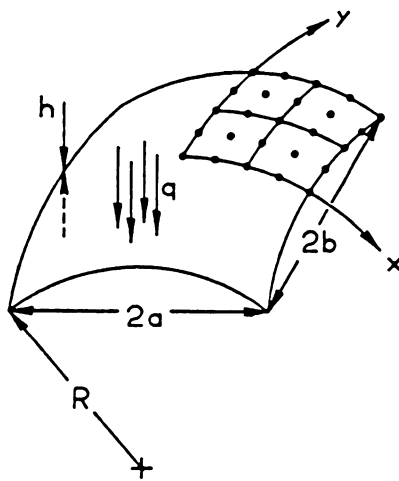


Figure 22. Hinged spherical cap under concentrated load



**Boundary conditions :**

$$u = \Psi_x = 0 \text{ at } x = 0$$

$$v = \Psi_y = 0 \text{ at } y = 0$$

$$v = w = \Psi_y = 0 \text{ at } x = a, -a$$

$$u = w = \Psi_x = 0 \text{ at } y = b, -b$$

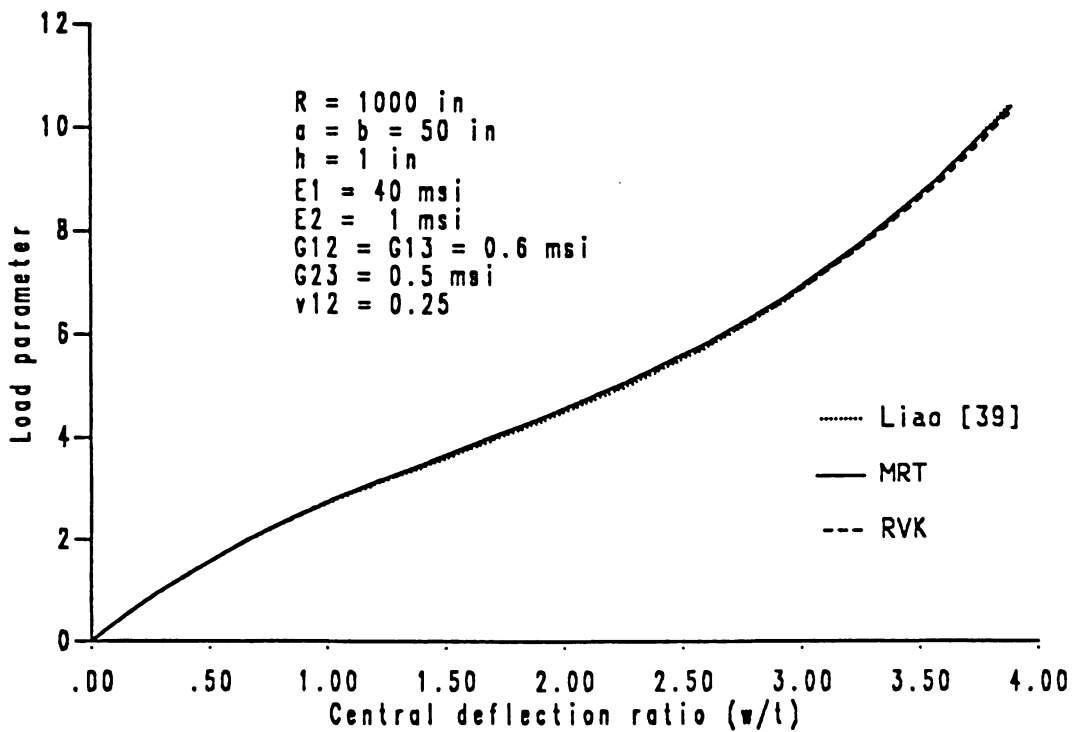
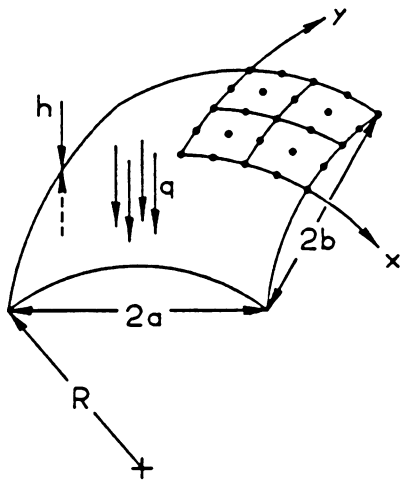


Figure 23. 9-layer simply supported (BC1) cap under uniform load



**Boundary conditions :**

$$u = \Psi_x = 0 \text{ at } x = 0$$

$$v = \Psi_y = 0 \text{ at } y = 0$$

$$v = w = \Psi_y = 0 \text{ at } x = a, -a$$

$$u = w = \Psi_x = 0 \text{ at } y = b, -b$$

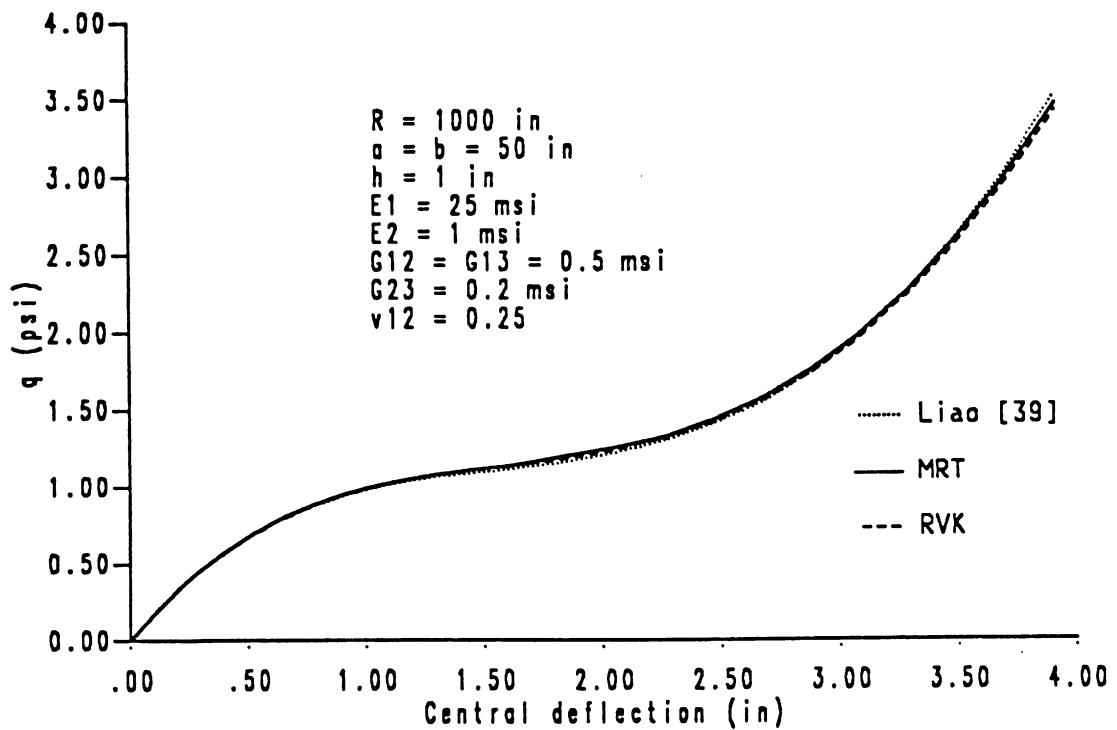


Figure 24. Simply supported (BC1) (0/90) cap under uniform load

## *4.5 Discussion of the results*

### **4.5.1 Introduction**

In this section, we discuss the performance of the MRT, based on the examples presented in the previous sections. We first explain why some of the results diverge from what we expected and, second suggest ways of improving them.

The following conclusions can be drawn from the examples:

- the refined von Kármán theory gives very good results, even in postbuckling, for plates and shallow shells;
- the non-linear transverse shear terms of the MRT can lead to negative effects. Elimination of this terms may improve results substantially;
- since the refined von Kármán theory has non-linear terms related to transverse deflection only, when it fails we conclude that lack of the nonlinear terms in the surface displacements contributed to the discrepancy;
- the same can be said about the MRT, when non-linear transverse shear terms are absent. In this situation, if the MRT and the RVK model yield the same results, then the extra terms in the MRT do not contribute.

At this point, it seems that the MRT, although derived in a logical and consistent manner, does not contain certain terms of physical importance. Following this line of thought, we want to consider the effect of extra terms in the non-linear strain-displacements equations. To do so, we start with the Green-Lagrange strain tensor as a function of displacements, and, inspired by reference 30, we substitute the power expansion of the displacements to obtain the power expansion of the strain

components. Identities used in the derivations are given in appendix G, together with the shell element matrices. The resulting strain components are:

$$E_{\alpha\beta}^{(n)} = e_{\alpha\beta}^{(n)} + \frac{1}{2} \sum_{m=0}^n \left[ \binom{m}{\Phi_{\gamma\alpha}} \binom{n-m}{\Phi_{\beta}^{\gamma}} + \binom{m}{\Phi_{\alpha 3}} \binom{n-m}{\Phi_{\beta}^3} - (b^{\lambda\gamma}) \binom{m-1}{\Phi_{\gamma\alpha}} \binom{n-m}{\Phi_{\lambda\beta}} + b_{\gamma}^{\lambda} \binom{m-1}{\Phi_{\lambda\alpha}} \binom{n-m}{\Phi_{\beta}^{\gamma}} + b_{\gamma}^{\lambda} b^{\rho\gamma} \binom{m-2}{\Phi_{\lambda\alpha}} \binom{n-m}{\Phi_{\rho\beta}} \right]$$

$$E_{33}^{(n)} = e_{33}^{(n)} + \frac{1}{2} \sum_{m=0}^n (m+1) \left[ (n-m+1) \left( u_{\alpha}^{\binom{m+1}{n-m+1}} u_{\alpha}^{\binom{m+1}{n-m+1}} + u_3^{\binom{m+1}{n-m+1}} u_3^{\binom{m+1}{n-m+1}} \right) - \right. \\ \left. (n-m) (b^{\lambda\alpha} u_{\lambda}^{\binom{m+1}{n-m}} u_{\alpha}^{\binom{m+1}{n-m}} + b_{\alpha}^{\lambda} u_{\lambda}^{\binom{m+1}{n-m}} u^{\alpha}) + \right. \\ \left. (n-m+1) (b_{\alpha}^{\lambda} b^{\gamma\alpha} u_{\lambda}^{\binom{m+1}{n-m-1}} u_{\gamma}^{\binom{m+1}{n-m-1}}) \right]$$

$$E_{\alpha 3}^{(n)} = e_{\alpha 3}^{(n)} + \frac{1}{2} \sum_{m=0}^n (m+1) \left[ u^{\binom{m+1}{n-m}} \Phi_{\gamma\alpha}^{\binom{m+1}{n-m}} + u^{\binom{m+1}{n-m}} \Phi_{\alpha 3}^{\binom{m+1}{n-m}} - (b^{\lambda\gamma} u_{\gamma}^{\binom{m+1}{n-m-1}} \Phi_{\lambda\alpha}^{\binom{m+1}{n-m-1}} + b^{\lambda\gamma} u_{\lambda}^{\binom{m+1}{n-m-1}} \Phi_{\gamma\alpha}^{\binom{m+1}{n-m-1}}) \right. \\ \left. + b_{\gamma}^{\lambda} b^{\rho\gamma} \Phi_{\lambda\alpha}^{\binom{m-2}{m+1}} u_{\rho}^{\binom{m-2}{m+1}} \right]$$

where  $e_{ij}^{(n)}$  are the linear strain components. Notice that reference [30] keeps only the non-linear terms not affected by the curvature tensor and we will do the same in this discussion. If the above equations are specialized to the case of the first order spherical shell theory, one obtains:

$$E_{xx}^{(0)} = e_{xx}^{(0)} + \frac{1}{2} \left[ (u_{,x} + \eta w)^2 + (v_{,x} - \xi v)^2 + (w_{,x} - \eta u)^2 \right]$$

$$E_{yy}^{(0)} = e_{yy}^{(0)} + \frac{1}{2} \left[ (u_{,y} - \xi v)^2 + (v_{,y} + \xi u + Kw)^2 + (w_{,y} - Kv)^2 \right]$$

$$2E_{xy}^{(0)} = 2e_{xy}^{(0)} + [(u_{,x} + \eta w)(u_{,y} - \xi v) + (v_{,x} - \xi v)(v_{,y} + \xi u + Kw) + (w_{,x} - \eta u)(w_{,y} - Kv)]$$

$$E_{xx}^{(1)} = e_{xx}^{(1)} + (u_{,x} + \eta w)(\Psi_{x,x} + \eta\Psi_2) + (v_{,x} - \xi v)(\Psi_{y,x} - \xi\Psi_y) + (w_{,x} - \eta u)(\Psi_{z,x} - \eta\Psi_x)$$

$$E_{yy}^{(1)} = e_{yy}^{(1)} + (u_{,y} - \xi v)(\Psi_{x,y} - \xi \Psi_y) + (v_{,y} + \xi u + Kw)(\Psi_{y,y} + \xi \Psi_x + K\Psi_z) + (w_{,y} - Kv)(\Psi_{z,y} - K\Psi_y)$$

$$2E_{xy}^{(1)} = 2e_{xy}^{(1)} + (u_{,x} + \eta w)(\Psi_{x,y} - \xi \Psi_y) + (v_{,x} - \xi v)(\Psi_{y,y} + \xi \Psi_x + K\Psi_z) + (u_{,y} - \xi v)(\Psi_{x,x} + \eta \Psi_z) \\ + (v_{,y} + \xi u + Kw)(\Psi_{y,x} - \xi \Psi_y) + (w_{,x} - \eta u)(\Psi_{z,y} - K\Psi_y) + (\Psi_{z,x} - \eta \Psi_x)(w_{,y} - Kv)$$

$$E_{xx}^{(2)} = e_{xx}^{(2)} + \frac{1}{2} [(\Psi_{x,x} + \eta \Psi_z)^2 + (\Psi_{y,x} - \xi \Psi_y)^2 + (\Psi_{z,x} - \eta \Psi_x)^2]$$

$$E_{yy}^{(2)} = e_{yy}^{(2)} + \frac{1}{2} [(\Psi_{x,y} - \xi \Psi_y)^2 + (\Psi_{y,y} + \xi \Psi_x + K\Psi_z)^2 + (\Psi_{z,y} - K\Psi_y)^2]$$

$$2E_{xy}^{(2)} = 2e_{xy}^{(2)} + (\Psi_{x,x} + \eta \Psi_z)(\Psi_{x,y} - \xi \Psi_y) + (\Psi_{y,x} - \xi \Psi_y)(\Psi_{y,y} + \xi \Psi_x + K\Psi_z) \\ + (\Psi_{z,x} - \eta \Psi_x)(\Psi_{z,y} - K\Psi_y)$$

$$2E_{yz}^{(0)} = 2e_{yz}^{(0)} + \Psi_x(u_{,y} - \xi v) + \Psi_y(\xi u + v_{,y} + Kw) + \Psi_z(w_{,y} - Kv)$$

$$2E_{xz}^{(0)} = 2e_{xz}^{(0)} + \Psi_x(u_{,x} + \eta w) + \Psi_y(v_{,x} - \xi v) + \Psi_z(w_{,x} - \eta u)$$

$$2E_{yz}^{(1)} = 2e_{yz}^{(1)} + \Psi_x(\Psi_{x,y} - \xi \Psi_y) + \Psi_y(\Psi_{y,y} + \xi \Psi_x + K\Psi_z) + \Psi_z(\Psi_{z,y} - K\Psi_y)$$

$$2E_{xz}^{(1)} = 2e_{xz}^{(1)} + \Psi_x(\Psi_{x,x} + \eta \Psi_z) + \Psi_y(\Psi_{y,x} - \xi \Psi_y) + \Psi_z(\Psi_{z,x} - \eta \Psi_x)$$

It is interesting to note that the above non-linear transverse shear terms are exactly the same as in the MRT, if we neglect  $\Psi_.$  In contrast, the bending terms contain substantially more elements, eliminated in the MRT when the order of magnitude assumptions were imposed. We proceed to reexamine some of the examples and discuss new results. In some of the examples, we also include results where the nonlinear transverse shear terms were eliminated from the new formulation.

#### 4.5.2 Hinged (0/90) cylindrical roof

We repeat example 4.3.8 here. The results in Figure 25 on page 79 show that the theory of reference 30 gives a better agreement with the refined von Kármán theory. We conclude that the trend is right, but the result is not yet ideal.

#### 4.5.3 Hinged spherical cap under concentrated load

This is the same as Example 4.4.3, where a more significant contribution from the theory of reference 30 is now noticed in Figure 26 on page 80. In this case, the refined von Kármán theory still yields to a better solution.

#### 4.5.4 Clamped shallow arch

Reference 39 modeled this arch using a mesh of five three-node beam elements. We used a mesh of five nine-node cylindrical shell elements. Figure 27 on page 81 shows the geometry and the discretization, where the half angle of the arch is 0.245 rad. The load factor was defined by:

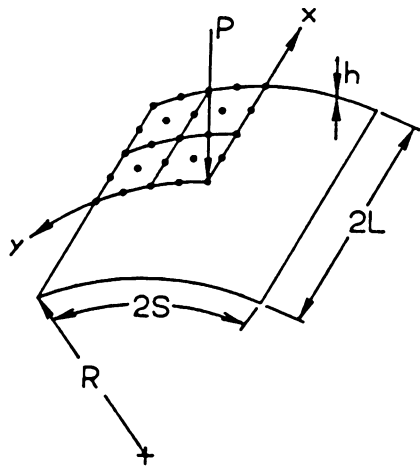
$$\bar{P} = \frac{\pi^2 E h^3}{12 \theta R^2} P$$

where  $P$  is the concentrated load at the center of the arch. Notice that we could have used the same mesh to model half of the arch. The reason we modeled one quarter of the structure was to compare with solutions from [ 39 ], using the shell element, if necessary. Figure 28 on page 82 shows the results for this case. We can see that the trend is the same as in the previous cases.

#### 4.5.5 Symmetric buckling of a clamped shallow arch

In this example, the geometry is slightly different from the one just examined, where now the half angle is 0.707 rad, such that the postbuckling region is more pronounced. As we can see from Figure 29 on page 83, the MRT diverges, reference 30 provides a better trend and the refined von Kármán theory gives the closest solution to the fully non-linear model. Results were shown only up to a deflection of 0.3 in, approximately, due to the cost of the calculations that required 280 load steps and 40 minutes of CPU time, for each model. This illustrates how expensive it is to analyze arches, especially if we are using extra degrees of freedom of shell elements.

For all these cases, we notice that the theory of reference 30 matches the results from the refined von Kármán theory and the full nonlinear model better than the MRT. When the nonlinear transverse shear terms are eliminated, a good comparison is obtained from [30], except for the last test, where this feature was not investigated.



**Boundary conditions :**

$u = \Psi_x = 0$  at  $x = 0$

$v = \Psi_y = 0$  at  $y = 0$

free at  $x = a, -a$

$u = v = w = \Psi_x = 0$  at  $y = b, -b$

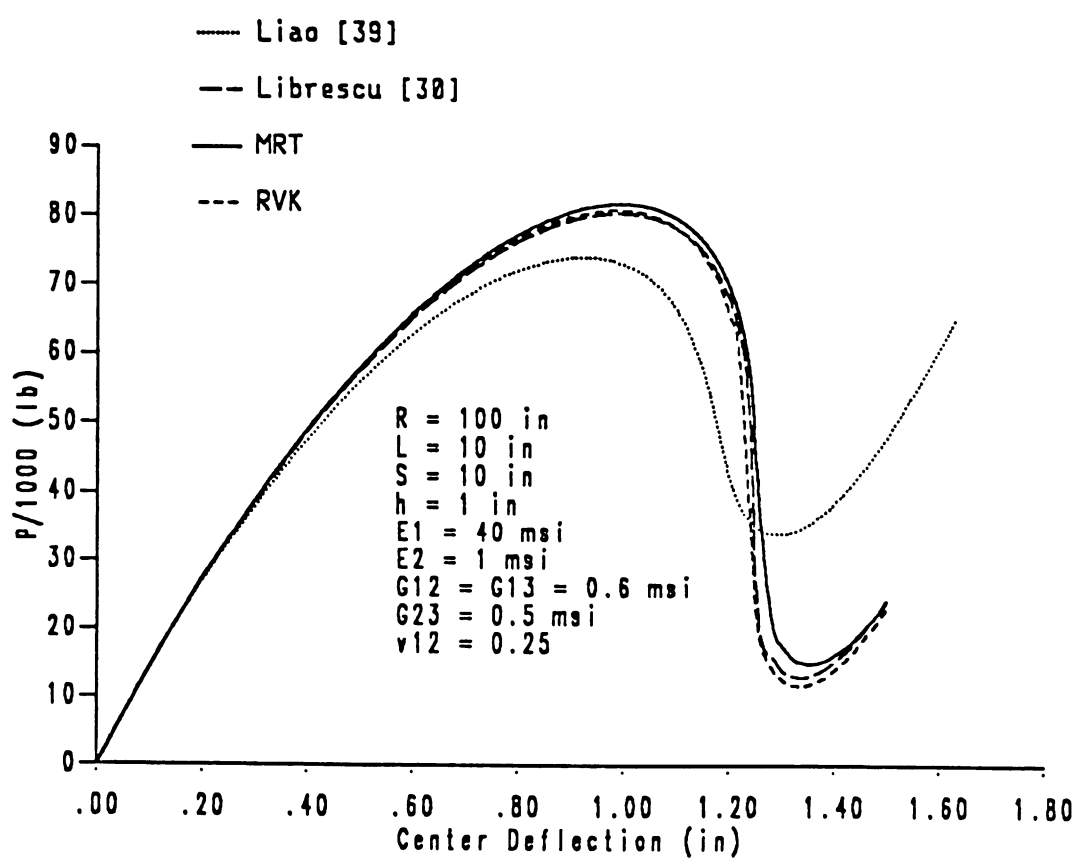
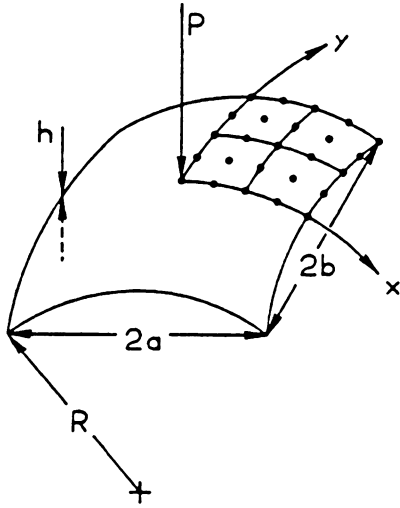


Figure 25. (0/90) hinged cylindrical roof



**Boundary conditions :**

$u = \Psi_x = 0$  at  $x = 0$

$v = \Psi_y = 0$  at  $y = 0$

$u = v = w = \Psi_x = 0$  at  $x = a, -a$

$u = v = w = \Psi_x = 0$  at  $y = b, -b$

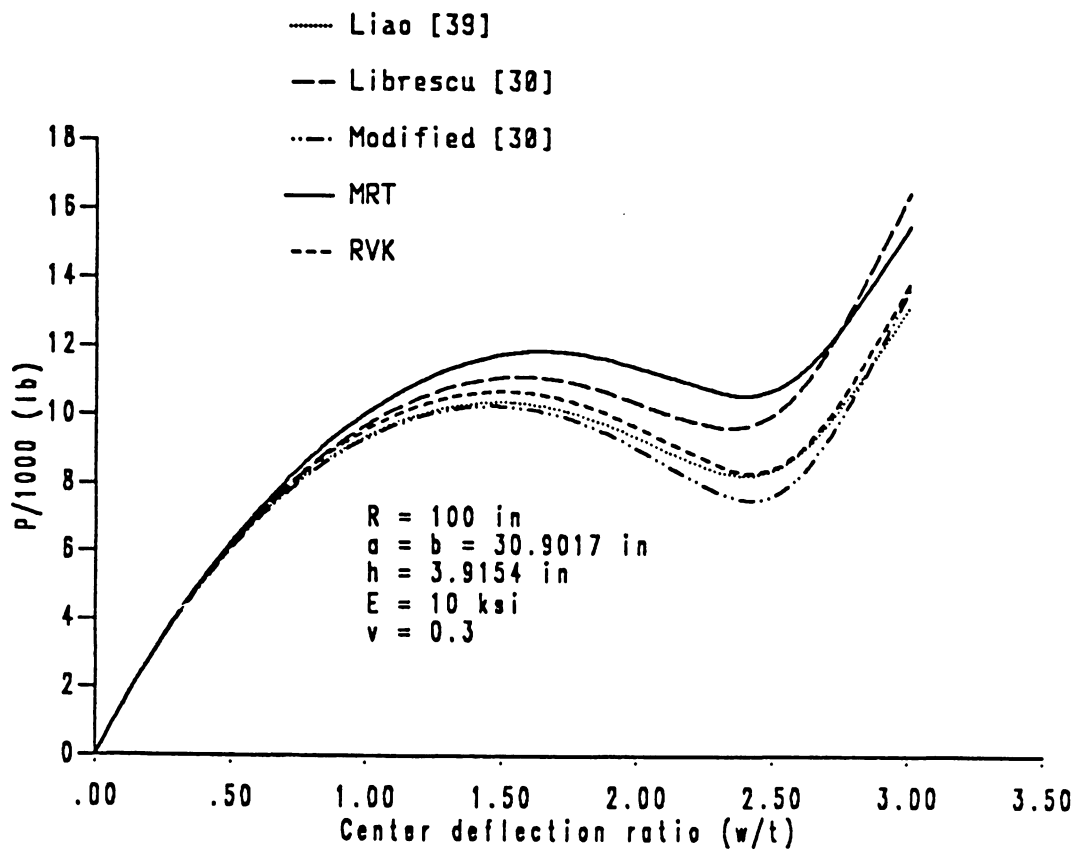


Figure 26. Hinged spherical cap under concentrated load

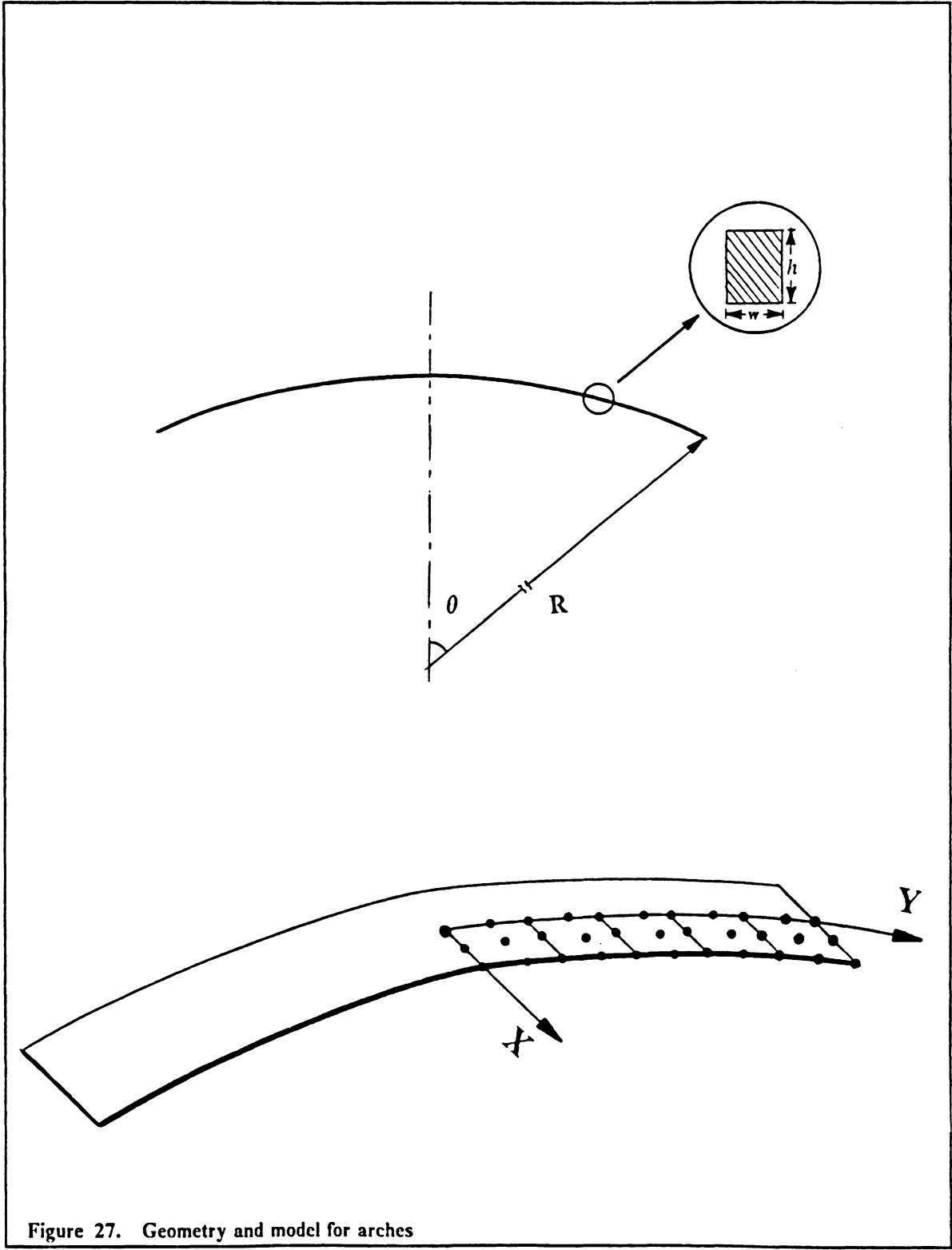


Figure 27. Geometry and model for arches

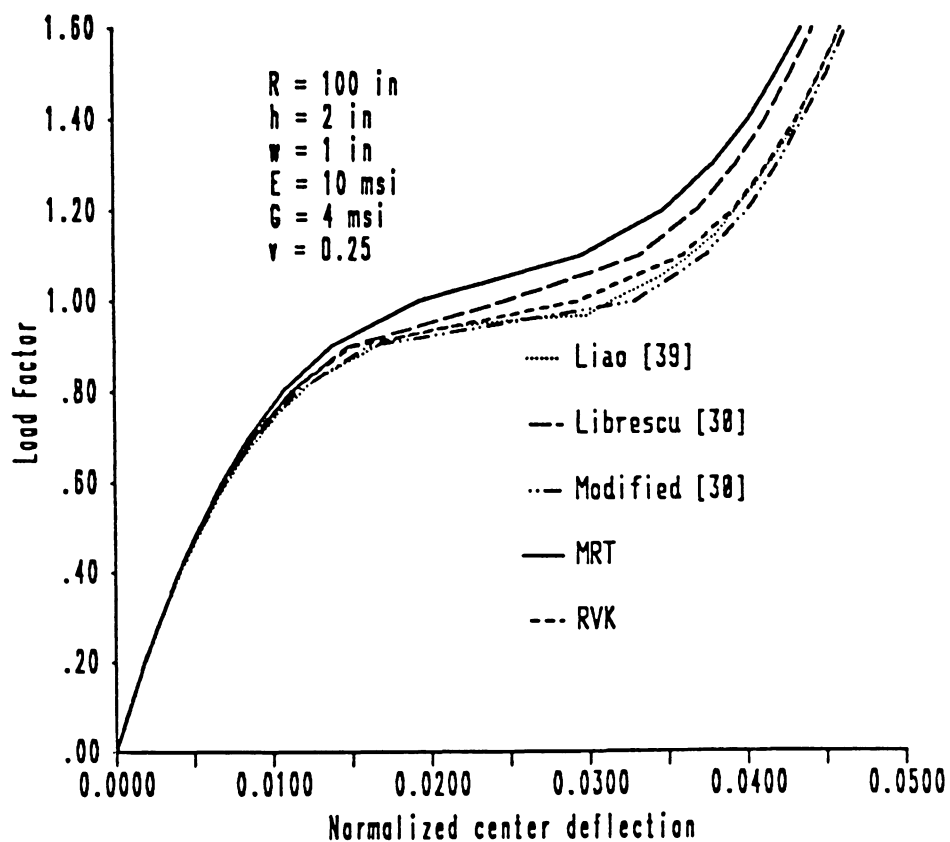
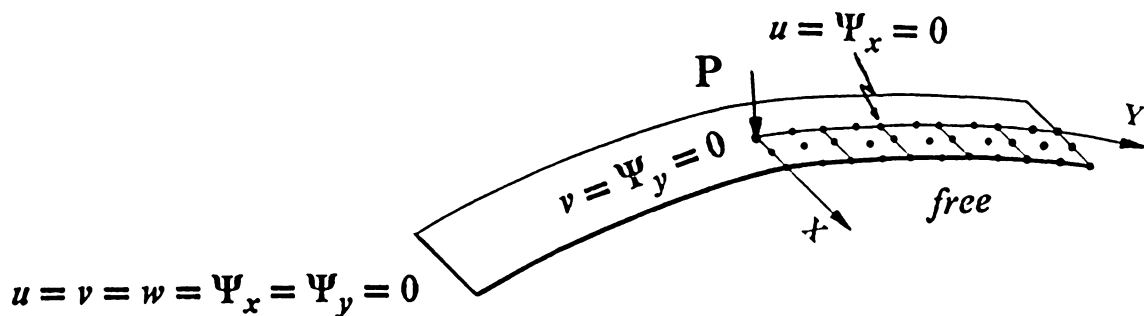
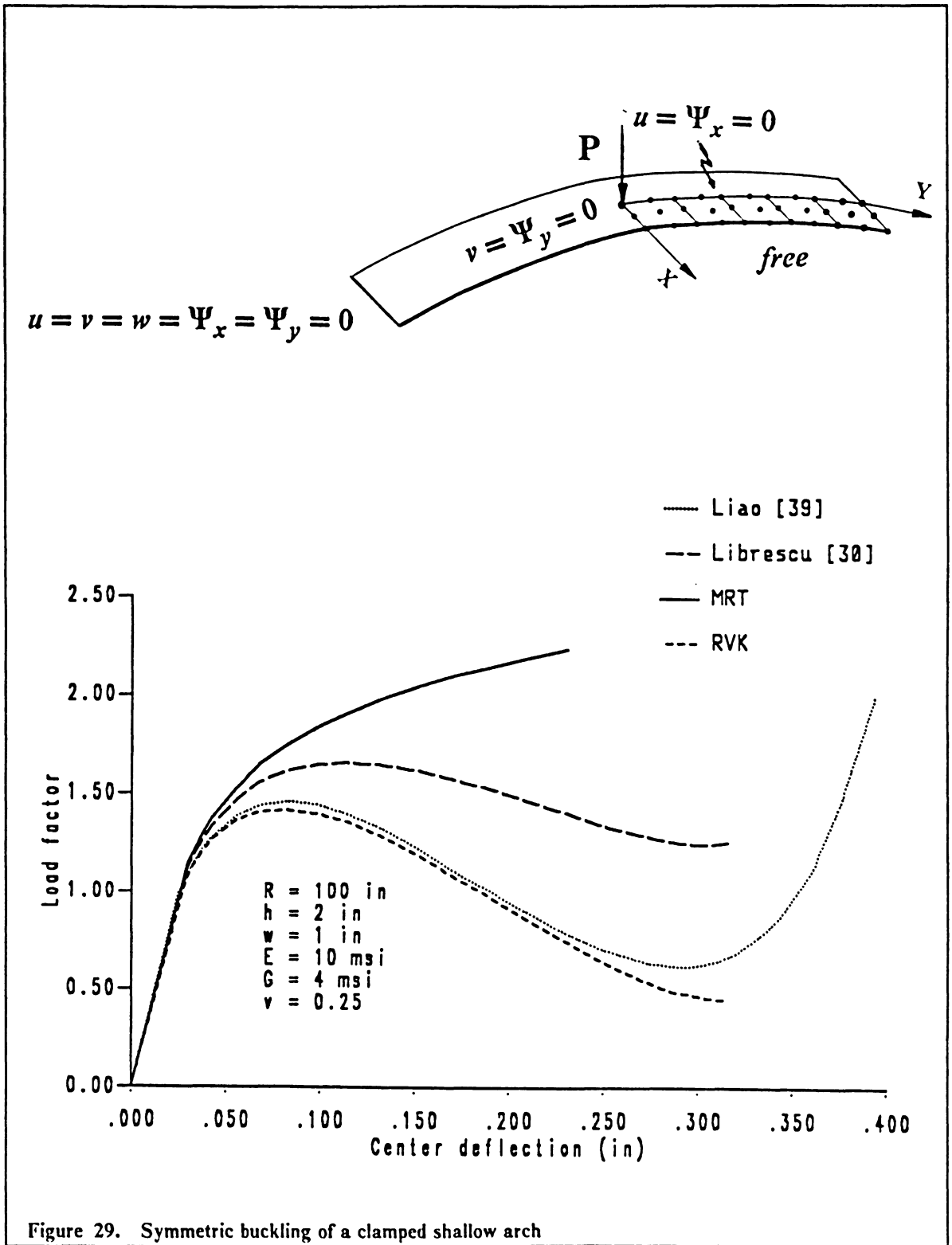


Figure 28. Clamped shallow arch



## 5. CONCLUSIONS AND RECOMMENDATIONS

### 5.1 Conclusions

In this work, the first complete theoretical development and finite element analysis of the moderate rotation theory, proposed in reference 33, has been presented. In order to form a basis of comparison of results, two other models have been considered, namely, a refined von Kármán type shell theory and a formulation proposed by Librescu [30]. In addition, the examples and computer code by Liao [39] were used.

For the rectangular plates and shallow cylindrical and spherical shells analyzed in the present study, we conclude the following:

- the refined von Kármán and fully non-linear model correlate very well in almost all cases;
- for plates, the MRT correlates very well in bending. In case of inplane loads, the MRT and [39] agree and the refined von Kármán theory diverges;
- for cylindrical shells in bending, the MRT provides good agreement;
- for postbuckling of cylindrical shells with a (0/90) lay-up, the refined von Kármán and MRT models agree, but diverge from [39]. Introduction of more non-linear bending terms, from [30], improves the results, but not significantly;
- from the two arch cases, we conclude that the MRT lacks nonlinear terms and the application of [30] shows a great improvement in the response;

- for spheres, in all three hardening cases, the MRT shows a good correlation. In the example involving postbuckling, again application of the theory of reference 30 shows that MRT is not non-linear enough;
- in the cases where the MRT shows stiffer results, the elimination of the non-linear transverse shear terms produces the flexibility necessary to match the results from the refined von Kármán theory and the theory of reference 39. This was observed with respect to the theory of reference 30, as well.

## ***5.2 Recommendations***

There are two lines of research that might be followed in order to extend and improve the results presented here: on one side, a review of the MRT with regard to the magnitude of the terms to be neglected; on the other, a study of the shell element formulation, where alternative concepts, like the degenerated solid, may be able to capture the shell behavior better.

This work has dealt primarily with bending due to the limitations found in the MRT. Given the performance of the refined von Kármán theory, we conclude that the true use of more complex non-linear theories is found in cases of inplane and nonsymmetric loads, as discussed in reference 48.

We have also found that, for certain lamination schemes, even when the structure can be analyzed by the one quarter model in the linear range, it may require the use of a full model in the nonlinear range. This feature deserves a more complete investigation in order to find the class of lamination schemes that can use the reduced (i.e. half or quarter) model.

## *Appendix A. Linear strains-displacements relations*

In this appendix, details of algebraic manipulations, necessary to obtain equations 10, are given.

1)  $e_{\alpha\beta}$

From equation 1 and the definitions of  $U_{i;j}$  and  $c_{\alpha}^{\lambda}$ , we write :

$$e_{\alpha\beta} = \frac{1}{2} (U_{\alpha;\beta} + U_{\beta;\alpha}) \quad (A1)$$

$$U_{\alpha;\beta} = c_{\alpha}^{\lambda} (u_{\lambda|\beta} - b_{\lambda\beta} u_3) \quad (A2)$$

$$c_{\alpha}^{\lambda} = \delta_{\alpha}^{\lambda} - \theta^3 b_{\alpha}^{\lambda} \quad (A3)$$

Substituting A2 into A1 and A3 into the resulting equation, we find :

$$e_{\alpha\beta} = \frac{1}{2} (\Phi_{\alpha\beta} + \Phi_{\beta\alpha} - \theta^3 b_{\alpha}^{\lambda} \Phi_{\lambda\beta} - \theta^3 b_{\beta}^{\lambda} \Phi_{\lambda\alpha})$$

Now :

$$\theta^3 \Phi_{\alpha\beta} = \theta^3 \sum_{n=0}^{\infty} (\theta^3)^n \overset{(n)}{\Phi}_{\alpha\beta} = \sum_{n=0}^{\infty} (\theta^3)^n \overset{(n-1)}{\Phi}_{\alpha\beta}$$

where  $\overset{(n)}{\Phi}_{\alpha\beta} = 0$  for  $n < 0$

Substituting this last result into the strains' expansion :

$$e_{\alpha\beta} = \sum_{n=0}^{\infty} (\theta^3)^n e_{\alpha\beta}^{(n)}$$

one obtains  $e_{\alpha\beta}^{(n)}$  of section 2.4.2.

2)  $e_{\alpha 3}$

Proceeding as before, we can write :

$$e_{\alpha 3} = \frac{1}{2} (\Phi_{\alpha 3} + u_{\alpha,3} - \theta^3 b_{\alpha}^{\lambda} u_{\lambda,3})$$

Now :

$$u_{\alpha,3} = \sum_{n=0}^{\infty} [(\theta^3)^n u_{\alpha}]_{,3} = \sum_{n=0}^{\infty} n(\theta^3)^{n-1} u_{\alpha} = \sum_{n=0}^{\infty} (n+1)(\theta^3)^n u_{\alpha}$$

Also :

$$\theta^3 u_{\alpha,3} = \sum_{n=0}^{\infty} n(\theta^3)^n u_{\alpha}$$

With these elements given, it is simple to obtain the expression for  $e_{\alpha 3}^{(n)}$  of 2.4.2.

3)  $\omega_{\cdot\beta}^{\alpha}$

$$\omega_{\cdot\beta}^{\alpha} = g^{\alpha\lambda} \omega_{\lambda\beta} = \frac{1}{2} g^{\alpha\lambda} [\Phi_{\lambda\beta} - \Phi_{\beta\lambda} + \theta^3 (b_{\beta}^{\gamma} \Phi_{\gamma\lambda} - b_{\lambda}^{\gamma} \Phi_{\gamma\beta})]$$

or :

$$\omega_{,\beta}^{\alpha} = \frac{1}{2} g^{\alpha\lambda} \sum_{n=0}^{\infty} (\theta^3)^n (\Phi_{\lambda\beta}^{(n)} - \Phi_{\beta\lambda}^{(n)} + b_{\beta}^{\gamma} \Phi_{\gamma\lambda}^{(n-1)} - b_{\lambda}^{\gamma} \Phi_{\gamma\beta}^{(n-1)})$$

but :

$$g^{\alpha\lambda} = a^{\alpha\rho} \sum_{n=0}^{\infty} \binom{n}{\rho}^{\lambda} (n+1) (\theta^3)^n = \sum_{n=0}^{\infty} \binom{n}{\alpha\lambda} (n+1) (\theta^3)^n$$

then

$$\omega_{,\beta}^{(n)\alpha} = \frac{1}{2} \sum_{m=0}^n \binom{m}{\alpha\lambda} (m+1) \left( \binom{n-m}{\lambda\beta} - \binom{n-m}{\beta\lambda} + b_{\beta}^{\gamma} \binom{n-m-1}{\gamma\lambda} - b_{\lambda}^{\gamma} \binom{n-m-1}{\gamma\beta} \right)$$

or,

$$\begin{aligned} \omega_{,\beta}^{(n)\alpha} &= \frac{1}{2} \sum_{m=0}^n \binom{m}{\alpha\lambda} \left( \binom{n-m}{\lambda\beta} - (1+m) \binom{n-m}{\beta\lambda} + (m+1) b_{\beta}^{\gamma} \binom{n-m-1}{\gamma\lambda} \right) + \\ &\quad \frac{1}{2} \sum_{m=0}^n \binom{m}{\alpha\lambda} \left( m \binom{n-m}{\lambda\beta} - (m+1) b_{\lambda}^{\gamma} \binom{n-m-1}{\gamma\beta} \right) \end{aligned}$$

If the second summation is expanded, it can be verified that it is identically zero, which results in the equation of section 2.4.2. All remaining strain and rotation equations can be found by using similar steps to those presented in this appendix.

## *Appendix B. Simplification of the strain-displacement equations*

The purpose of this appendix is to discuss ways of simplifying the strains' equations 13, of chapter 2. In reference 33, these simplifications were performed by expanding the equations and grouping of like terms. The resulting expressions contained terms that could be neglected, leading to equations 14. Observing that these terms were composed of products of linear strain and rotation components, we approached the simplification task by making the mentioned products explicit from the beginning, so that the final form of the equations could be obtained in a more straight forward fashion.

We first write equations 10 in the following manner :

$$e_{\alpha\beta}^{(n)} + \omega_{\alpha\beta}^{(n)} = \Phi_{\alpha\beta}^{(n)} - b_{\alpha}^{\lambda} \Phi_{\lambda\beta}^{(n-1)} \quad (B.1)$$

$$e_{\alpha 3}^{(n)} + \omega_{\alpha 3}^{(n)} = (n+1) u_{\alpha}^{(n+1)} - n b_{\alpha}^{\lambda} u_{\lambda}^{(n)} \quad (B.2)$$

$$e_{\alpha 3}^{(n)} + \omega_{3\alpha}^{(n)} = \Phi_{\alpha 3}^{(n)} \quad (B.3)$$

$$e_{\beta}^{(n)} + \omega_{\cdot\beta}^{(n)} = \sum_{m=0}^n (B)^{m\alpha\lambda} \Phi_{\lambda\beta}^{(p)} \quad (B.4)$$

$$e_3^{(n)} + \omega_{\cdot 3}^{(n)} = \sum_{m=0}^n (n-m+1) (B)^{m\alpha\lambda} u_{\lambda}^{(n-m+1)} \quad (B.5)$$

The following specializations of the above equations will be useful :

$$e_{\alpha\beta}^{(0)} + \omega_{\alpha\beta}^{(0)} = \Phi_{\alpha\beta}^{(0)} \quad (B.6)$$

$$e_{\alpha\beta}^{(1)} + \omega_{\alpha\beta}^{(1)} = \Phi_{\alpha\beta}^{(1)} - b_{\alpha}^{\lambda} \Phi_{\lambda\beta}^{(0)} \quad (B.7)$$

$$e_{\alpha\beta}^{(2)} + \omega_{\alpha\beta}^{(2)} = -b_{\alpha}^{\lambda} \Phi_{\lambda\beta}^{(1)} \quad (B.8)$$

$$e_{\alpha 3}^{(0)} + \omega_{\alpha 3}^{(0)} = u_{\alpha}^{(1)} \quad ; \quad e_{\alpha 3}^{(0)} + \omega_{3\alpha}^{(0)} = \Phi_{\alpha 3}^{(0)} \quad (B.9)$$

$$e_{\alpha 3}^{(1)} + \omega_{\alpha 3}^{(1)} = -b_{\alpha}^{\lambda} u_{\lambda}^{(1)} \quad ; \quad e_{\alpha 3}^{(1)} + \omega_{3\alpha}^{(1)} = \Phi_{\alpha 3}^{(1)} \quad (B.10)$$

$$e_{\beta}^{\alpha(0)} + \omega_{\beta}^{\alpha(0)} = \Phi_{\beta}^{\alpha(0)} \quad (B.11)$$

$$e_{\beta}^{\alpha(1)} + \omega_{\beta}^{\alpha(1)} = \Phi_{\beta}^{\alpha(1)} + b^{\alpha\lambda} \Phi_{\lambda\beta}^{(0)} \quad (B.12)$$

$$e_{\beta}^{\alpha(2)} + \omega_{\beta}^{\alpha(2)} = b^{\alpha\lambda} \Phi_{\lambda\beta}^{(1)} + b_{\gamma}^{\alpha} b^{\gamma\lambda} \Phi_{\lambda\beta}^{(0)} \quad (B.13)$$

$$e_3^{\alpha(0)} + \omega_{\alpha 3}^{(0)} = u_{\alpha}^{(0)} \quad (B.14)$$

$$e_3^{\alpha(1)} + \omega_{\alpha 3}^{(1)} = b^{\alpha\lambda} u_{\lambda}^{(1)} \quad (B.15)$$

$$e_3^{\alpha(2)} + \omega_{\alpha 3}^{(2)} = b_{\gamma}^{\alpha} b^{\gamma\lambda} u_{\lambda}^{(1)} \quad (B.16)$$

Let's consider first the simplification of  $\overset{(n)}{E}_{\alpha\beta}$ . From B.3, we can say that  $\overset{(n)}{\omega}_{\alpha 3} = \overset{(n)}{\Phi}_{\alpha 3} - \overset{(n)}{e}_{\alpha 3}$ . Substituting in the equation for  $\overset{(n)}{E}_{\alpha\beta}$  and simplifying, we obtain :

$$\overset{(n)}{E}_{\alpha\beta} = \overset{(n)}{e}_{\alpha\beta} + \frac{1}{2} \sum_{m=0}^n \left( \overset{(n)}{\Phi}_{\alpha 3} \overset{(p)}{\Phi}_{\beta 3} - \overset{(m)}{\Phi}_{\alpha 3} \overset{(p)}{e}_{\beta 3} + \overset{(p)}{\Phi}_{\alpha 3} \overset{(m)}{e}_{\beta 3} + \overset{(m)}{e}_{3\beta} \overset{(p)}{e}_{3\alpha} \right) ; p = n - m$$

Recognizing that the second and third terms, inside parenthesis, cancel we find :

$$E_{\alpha\beta}^{(n)} = e_{\alpha\beta}^{(n)} + \frac{1}{2} \sum_{m=0}^n (\Phi_{\alpha 3}^{(n)} \Phi_{\beta 3}^{(p)} - \underline{e_{3\beta}^{(m)} e_{3\alpha}^{(p)}})$$

Notice that this expression is general, in terms of n, and can be specialized to the form of equations 14, by making n equal to 0, 1 or 2, as follows :

$$E_{\alpha\beta}^{(0)} = e_{\alpha\beta}^{(0)} + \frac{1}{2} \Phi_{\alpha 3}^{(0)} \Phi_{\beta 3}^{(0)} - \underline{\frac{1}{2} e_{3\alpha}^{(0)} e_{3\beta}^{(0)}}$$

$$E_{\alpha\beta}^{(1)} = e_{\alpha\beta}^{(1)} + \frac{1}{2} (\Phi_{\alpha 3}^{(0)} \Phi_{\beta 3}^{(1)} + \Phi_{\alpha 3}^{(1)} \Phi_{\beta 3}^{(0)}) - \underline{\frac{1}{2} (e_{3\alpha}^{(1)} e_{3\beta}^{(0)} + e_{3\alpha}^{(0)} e_{3\beta}^{(1)})}$$

$$E_{\alpha\beta}^{(2)} = e_{\alpha\beta}^{(2)} + \frac{1}{2} \Phi_{\alpha 3}^{(1)} \Phi_{\beta 3}^{(1)} - \underline{\frac{1}{2} e_{3\alpha}^{(1)} e_{3\beta}^{(1)}}$$

The underlined terms have order  $\varepsilon^4$  ,  $\varepsilon^4/h$  ,  $\varepsilon^4/h^2$  , respectively, and can be neglected. The remaining terms of the above equations can be easily written in the form of equations 14.

We concentrate now on  $E_{\alpha 3}^{(n)}$ . In contrast to the previous case, these strain components do not have a unique simplified form, as will be shown. The following identities will be used :

$$\sum_{m=0}^n e_{\lambda\alpha}^{(m)} \omega_{\cdot 3}^{(p)\lambda} \equiv \sum_{m=0}^n e_{\alpha}^{(m)\lambda} \omega_{\lambda 3}^{(p)} \equiv \sum_{m=0}^n e_{\alpha}^{(p)\lambda} \omega_{\lambda 3}^{(m)} \quad ; \quad p = n - m$$

$$\omega_{\cdot \alpha}^{(p)3} \equiv \omega_{3\alpha}^{(p)}$$

Then :

$$E_{\alpha 3}^{(n)} = e_{\alpha 3}^{(n)} + \frac{1}{2} \sum_{m=0}^n \left[ \omega_{\lambda 3}^{(m)} (e_{\alpha}^{(m)\lambda} + \omega_{\alpha}^{(p)\lambda}) + e_{33}^{(m)} \omega_{3\alpha}^{(p)} \right]$$

Obtaining  $\omega_{\lambda 3}^{(m)}$  from B.2,  $\omega_{3\alpha}^{(p)}$  from B.3, and substituting  $e_{33}^{(m)}$ , we find :

$$E_{\alpha 3}^{(n)} = e_{\alpha 3}^{(n)} + \frac{1}{2} \sum_{m=0}^n \left[ [(m+1) u_{\lambda}^{(m+1)} - m b_{\lambda}^{\gamma} u_{\gamma}^{(m)}] (e_{\alpha}^{(m)\lambda} + \omega_{\alpha}^{(p)\lambda}) + (m+1) u_3^{(m+1)} \Phi_{\alpha 3}^{(p)} \right] \\ - \frac{1}{2} \sum_{m=0}^n \left[ \underline{e_{\lambda 3}^{(m)} (e_{\alpha}^{(p)\lambda} + \omega_{\alpha}^{(p)\lambda}) - e_{33}^{(m)} e_{3\alpha}^{(p)}} \right]$$

or,

$$E_{\alpha 3}^{(n)} = e_{\alpha 3}^{(n)} + \frac{1}{2} \sum_{m=0}^n \left[ [(m+1) u_{\lambda}^{(m+1)} - m b_{\lambda}^{\gamma} u_{\gamma}^{(m)}] (e_{\lambda\alpha}^{(p)} + \omega_{\lambda\alpha}^{(p)}) + (m+1) u_3^{(m+1)} \Phi_{\alpha 3}^{(p)} \right] \\ - \frac{1}{2} \sum_{m=0}^n \left[ \underline{e_{\lambda 3}^{(m)} (e_{\alpha}^{(p)\lambda} + \omega_{\alpha}^{(p)\lambda}) - e_{33}^{(m)} e_{3\alpha}^{(p)}} \right]$$

For  $n=0$ , we find :

$$E_{\alpha 3}^{(0)} = e_{\alpha 3}^{(0)} + \frac{1}{2} (u_{\lambda}^{(1)} \Phi_{\lambda\alpha}^{(0)} + u_3 \Phi_{\alpha 3}^{(0)}) - \frac{1}{2} \underline{e_{\lambda 3}^{(0)} (e_{\alpha}^{(0)\lambda} + \omega_{\alpha}^{(0)\lambda}) - e_{33}^{(0)} e_{3\alpha}^{(0)}}$$

The difference between this equation and the one equation 14, is due to the product  $e_{33}^{(0)} \omega_{3\alpha}^{(0)}$ , that in reference 33 was not expanded to eliminate  $e_{33}^{(0)}$ .

For  $n=1$ , we find :

$$E_{\alpha 3}^{(1)} = e_{\alpha 3}^{(1)} + \frac{1}{2} (u^{(1)\lambda} \Phi_{\lambda\alpha}^{(1)} - b^{\gamma\lambda} u_{\gamma}^{(1)(0)} \Phi_{\lambda\alpha}^{(0)} + u_3^{(1)(1)} \Phi_{\alpha 3}^{(1)}) - \left[ \frac{1}{2} e_{\lambda 3}^{(0)} (e_{\alpha}^{(1)\lambda} + \omega_{\alpha}^{(1)\lambda}) + e_{\lambda 3}^{(1)} (e_{\alpha}^{(0)\lambda} + \omega_{\alpha}^{(0)\lambda}) - e_{33}^{(0)} e_{3\alpha}^{(1)} \right]$$

Note that  $e_{33}^{(1)} = 2u_3^{(2)} = 0$ . Substituting  $\Phi_{\lambda\alpha}^{(1)}$  and  $\Phi_{\alpha 3}^{(1)}$  and using the identity :

$$b^{\lambda\gamma} u_{\gamma}^{(1)(0)} \Phi_{\lambda\alpha}^{(0)} \equiv b_{\lambda}^{\gamma} u_{\gamma}^{(1)(0)} \Phi_{\alpha}^{(0)} \equiv (\Phi_{\lambda 3}^{(1)} - u_{3,\lambda}^{(1)}) \Phi_{\alpha}^{(0)}$$

we obtain :

$$E_{\alpha 3}^{(1)} = e_{\alpha 3}^{(1)} + \frac{1}{2} (u^{(1)\lambda} u_{\lambda|\alpha}^{(1)} + u_3^{(1)(1)} u_{3,\alpha}^{(1)} - \Phi_{\lambda 3}^{(1)} \Phi_{\alpha}^{(0)} + u_{3,\lambda}^{(1)} \Phi_{\alpha}^{(0)})$$

where underlined terms were neglected.

We believe that the above equation cannot be rendered to the same form of its counterpart in equations 14, concluding that this is an alternative form of  $E_{\alpha 3}^{(1)}$ . The same observation applies to  $E_{\alpha 3}^{(2)}$  and  $E_{33}^{(2)}$ . Although some of the equations obtained in this appendix are different from their equivalents in equations 14, their errors should be of the same order of magnitude.

## *Appendix C. Equations for plates, cylinders and spheres.*

In this appendix, all necessary operations to particularize the general equations, of section 2.2, to the specific geometries of this work, are discussed.

### **1) Rectangular Plates**

Let the geometry of a plate be defined as illustrated in Figure 30 on page 95, where  $x$  and  $y$  are the Cartesian in-plane coordinates, coinciding with curvilinear coordinates  $u^1$ ,  $u^2$  ( and with  $z^1$ ,  $z^2$ , from the 3D Cartesian system of coordinates).

The position vector of a point on the mid-plane of the plate is then :

$$\mathbf{r} = (z^1, z^2, z^3) = (x, y, 0)$$

The following quantities can be obtained :

- tangent vectors:

$$\mathbf{T}_1 = \frac{\partial z^i}{\partial u^1} \mathbf{g}_i = \mathbf{g}_1$$

$$\mathbf{T}_2 = \frac{\partial z^i}{\partial u^2} \mathbf{g}_i = \mathbf{g}_2$$

where  $\mathbf{g}_i$  are the base vectors in the 3D system.

- surface metric

$$a_{\alpha\beta} = \mathbf{T}_\alpha \cdot \mathbf{T}_\beta \rightarrow a_{\alpha\beta} = \delta_{\alpha\beta}, \quad a^{\alpha\beta} = \delta^{\alpha\beta}$$

- curvature tensor :

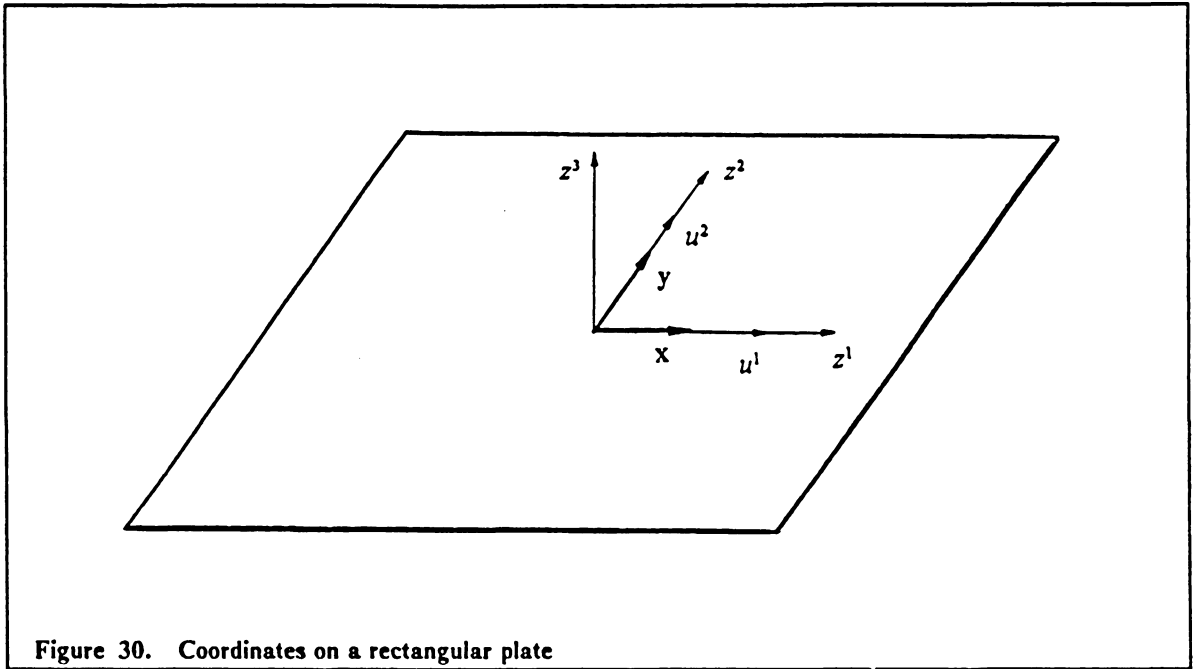


Figure 30. Coordinates on a rectangular plate

$$b_{\alpha\beta} = \mathbf{n} \cdot \frac{\partial \Gamma_{\alpha}}{\partial u^{\beta}} = 0 \rightarrow b_{\beta}^{\alpha} = 0$$

- shifter tensor :  $c_{\beta}^{\alpha} = \delta_{\beta}^{\alpha}$
- Christoffel symbols :

$$\Gamma_{(\alpha)\gamma}^{(\alpha)} = \frac{1}{2a_{(\alpha)(\alpha)}} \frac{\partial a_{(\alpha)(\alpha)}}{\partial u^{\gamma}}, \quad \forall \alpha, \gamma$$

$$\Gamma_{(\beta)(\beta)}^{\alpha} = -\frac{1}{2a_{(\alpha)(\alpha)}} \frac{\partial a_{(\beta)(\beta)}}{\partial u^{\alpha}}, \quad \alpha \neq \beta$$

where  $(\alpha)$  means no summation with respect to  $\alpha$ . Then:  $\Gamma_{\beta\gamma}^{\alpha} = 0 \quad \forall \alpha, \beta, \gamma$

- Derivatives and other results :

$$u_{\alpha} |_{\beta} = u_{\alpha, \beta}$$

$$\Phi_{\alpha\beta} = u_{\alpha, \beta}$$

$$\Phi_{\alpha 3} = u_{3, \alpha}$$

$$u_{\alpha} = u^{\alpha}$$

Physical and curvilinear quantities are identical.

- Change of notation

$$(u, v, w) = ({}^{(0)}u_1, {}^{(0)}u_2, {}^{(0)}u_3)$$

$$(\Psi_x, \Psi_y, \Psi_z) = ({}^{(1)}u_1, {}^{(1)}u_2, {}^{(1)}u_3)$$

$$(R_{(0)}^{11}, R_{(0)}^{22}, R_{(0)}^{12}, R_{(0)}^{33}) = (N_x, N_y, N_{xy}, N_z)$$

$$(R_{(1)}^{11}, R_{(1)}^{22}, R_{(1)}^{12}) = (M_x, M_y, M_{xy})$$

$$(R_{(2)}^{11}, R_{(2)}^{22}, R_{(2)}^{12}) = (P_x, P_y, P_{xy})$$

$$(R_{(0)}^{23}, R_{(0)}^{13}, R_{(1)}^{23}, R_{(1)}^{13}) = (Q_y, Q_x, R_y, R_x)$$

$$(P_{(0)}^1, P_{(0)}^2, P_{(0)}^3) = (P_{0x}, P_{0y}, P_{0z})$$

$$(F_0^1, F_0^2, F_0^3) = (F_{0x}, F_{0y}, F_{0z})$$

$$(\hat{S}_1, \hat{S}_2, \hat{S}_3) = (\hat{X}_0, \hat{Y}_0, \hat{Z}_0)$$

$$(\hat{S}_1, \hat{S}_2, \hat{S}_3) = (\hat{X}_1, \hat{Y}_1, \hat{Z}_1)$$

- Weak form of the functional

$$\begin{aligned}
0 = \int_0^T \int_{\Omega} \left\{ (N_x + \Psi_x Q_x - \frac{1}{2} \Psi_{z,x} R_x) \delta u_{,x} + (N_{xy} + \Psi_x Q_y - \frac{1}{2} \Psi_{z,x} R_y) \delta u_{,y} + \right. \\
(N_{xy} + \Psi_y Q_x - \frac{1}{2} \Psi_{z,y} R_x) \delta v_{,x} + (N_y + \Psi_y Q_y - \frac{1}{2} \Psi_{z,y} R_y) \delta v_{,y} + \\
[(1 + \frac{1}{2} \Psi_z) Q_x + w_{,x} N_x + w_{,y} N_{xy} + \Psi_{z,x} M_x + \Psi_{z,y} M_{xy}] \delta w_{,x} + \\
[(1 + \frac{1}{2} \Psi_z) Q_y + w_{,x} N_{xy} + w_{,y} N_y + \Psi_{z,x} M_{xy} + \Psi_{z,y} M_y] \delta w_{,y} + \\
[(1 + u_{,x} - \frac{1}{2} \Psi_z) Q_x + u_{,y} Q_y + \Psi_{x,x} R_x + \Psi_{x,y} R_y + \Psi_x N_z] \delta \Psi_x + \\
[(1 + v_{,y} - \frac{1}{2} \Psi_z) Q_y + v_{,x} Q_x + \Psi_{y,x} R_x + \Psi_{y,y} R_y + \Psi_y N_z] \delta \Psi_y + \\
[\frac{1}{2} (w_{,x} - \Psi_x) Q_x + \frac{1}{2} (w_{,y} - \Psi_y) Q_y + N_z] \delta \Psi_z + \\
(M_x + \Psi_x R_x) \delta \Psi_{x,x} + (M_{xy} + \Psi_x R_y) \delta \Psi_{x,y} + (M_{xy} + \Psi_y R_x) \delta \Psi_{y,x} + (M_y + \Psi_y R_y) \delta \Psi_{y,y} + \\
[(1 - \frac{1}{2} u_{,x}) R_x - \frac{1}{2} u_{,y} R_y + w_{,x} M_x + w_{,y} M_{xy} + \Psi_{z,x} P_x + \Psi_{z,y} P_{xy}] \delta \Psi_{z,x} + \\
\left. [(1 - \frac{1}{2} v_{,y}) R_y - \frac{1}{2} v_{,x} R_x + w_{,x} M_{xy} + w_{,y} M_y + \Psi_{z,x} P_{xy} + \Psi_{z,y} P_y] \delta \Psi_{z,y} \right\} d\Omega dt
\end{aligned}$$

- Equilibrium equations

$$\delta u : (N_x + \Psi_x Q_x - \frac{1}{2} \Psi_{z,x} R_x)_{,x} + (N_{xy} + \Psi_x Q_y - \frac{1}{2} \Psi_{z,x} R_y)_{,y} = I_1 \ddot{u} + I_2 \ddot{\Psi}_x - P_{0x} - F_{0x}$$

$$\delta v : (N_{xy} + \Psi_y Q_x - \frac{1}{2} \Psi_{z,y} R_x)_{,x} + (N_y + \Psi_y Q_y - \frac{1}{2} \Psi_{z,y} R_y)_{,y} = I_1 \ddot{v} + I_2 \ddot{\Psi}_y - P_{0y} - F_{0y}$$

$$\delta w : [(1 + \frac{1}{2} \Psi_z) Q_x + w_{,x} N_x + w_{,y} N_{xy} + \Psi_{z,x} M_x + \Psi_{z,y} M_{xy}]_{,x} + \\ [(1 + \frac{1}{2} \Psi_z) Q_y + w_{,x} N_{xy} + w_{,y} N_y + \Psi_{z,x} M_{xy} + \Psi_{z,y} M_y]_{,y} = I_1 \ddot{w} + I_2 \ddot{\Psi}_z - P_{0z} - F_{0z}$$

$$\delta \Psi_x : [(1 + u_{,x} - \frac{1}{2} \Psi_z) Q_x + u_{,y} Q_y + \Psi_{x,x} R_x + \Psi_{x,y} R_y + \Psi_x N_z] + \\ (M_x + \Psi_x R_x)_{,x} + (M_{xy} + \Psi_x R_y)_{,y} = I_2 \ddot{u} + I_3 \ddot{\Psi}_x - P_{1x} - F_{1x}$$

$$\delta \Psi_y : [(1 + v_{,y} - \frac{1}{2} \Psi_z) Q_y + v_{,x} Q_x + \Psi_{y,x} R_x + \Psi_{y,y} R_y + \Psi_y N_z] + \\ (M_{xy} + \Psi_y R_x)_{,x} + (M_y + \Psi_y R_y)_{,y} = I_2 \ddot{v} + I_3 \ddot{\Psi}_y - P_{1y} - F_{1y}$$

$$\delta \Psi_z : [\frac{1}{2} (w_{,x} - \Psi_x) Q_x + \frac{1}{2} (w_{,y} - \Psi_y) Q_y + N_z] + \\ [(1 - \frac{1}{2} u_{,x}) R_x - \frac{1}{2} u_{,y} R_y + w_{,x} M_x + w_{,y} M_{xy} + \Psi_{z,x} P_x + \Psi_{z,y} P_{xy}]_{,x} + \\ [(1 - \frac{1}{2} v_{,y}) R_y - \frac{1}{2} v_{,x} R_x + w_{,x} M_{xy} + w_{,y} M_y + \Psi_{z,x} P_{xy} + \Psi_{z,y} P_y]_{,y} = I_2 \ddot{w} + I_3 \ddot{\Psi}_z - P_{1z} - F_{1z}$$

- Natural boundary conditions

$$\delta u : (N_x + \Psi_x Q_x - \frac{1}{2} \Psi_{z,x} R_x) n_x + (N_{xy} + \Psi_x Q_y - \frac{1}{2} \Psi_{z,x} R_y) n_y = \hat{X}_0$$

$$\delta v : (N_{xy} + \Psi_y Q_x - \frac{1}{2} \Psi_{z,y} R_x) n_x + (N_y + \Psi_y Q_y - \frac{1}{2} \Psi_{z,y} R_y) n_y = \hat{Y}_0$$

$$\delta w : [(1 + \frac{1}{2} \Psi_z) Q_x + w_{,x} N_x + w_{,y} N_{xy} + \Psi_{z,x} M_x + \Psi_{z,y} M_{xy}] n_x + \\ [(1 + \frac{1}{2} \Psi_z) Q_y + w_{,x} N_{xy} + w_{,y} N_y + \Psi_{z,x} M_{xy} + \Psi_{z,y} M_y] n_y = \hat{Z}_0$$

$$\delta \Psi_x : (M_x + \Psi_x R_x) n_x + (M_{xy} + \Psi_x R_y) n_y = \hat{X}_1$$

$$\delta\Psi_y : (M_{xy} + \Psi_y R_x)n_x + (M_y + \Psi_y R_y)n_y = \hat{Y}_1$$

$$\delta\Psi_z : \left[ \left(1 - \frac{1}{2} u_{,x}\right)R_x - \frac{1}{2} u_{,y}R_y + w_{,x}M_x + w_{,y}M_{xy} + \Psi_{z,x}P_x + \Psi_{z,y}P_{xy} \right]n_x +$$

$$\left[ \left(1 - \frac{1}{2} v_{,y}\right)R_y - \frac{1}{2} v_{,x}R_x + w_{,x}M_{xy} + w_{,y}M_y + \Psi_{z,x}P_{xy} + \Psi_{z,y}P_y \right]n_y = \hat{Z}_1$$

- Strains

$$E_{xx}^{(0)} = u_{,x} + \frac{1}{2} (w_{,x})^2$$

$$E_{yy}^{(0)} = v_{,y} + \frac{1}{2} (w_{,y})^2$$

$$2E_{xy}^{(0)} = u_{,y} + v_{,x} + w_{,x}w_{,y}$$

$$E_{zz}^{(0)} = \Psi_z + \frac{1}{2} [(\Psi_x)^2 + (\Psi_y)^2]$$

$$E_{xx}^{(1)} = \Psi_{x,x} + w_{,x}\Psi_{z,x}$$

$$E_{yy}^{(1)} = \Psi_{y,y} + w_{,y}\Psi_{z,y}$$

$$2E_{xy}^{(1)} = \Psi_{x,y} + \Psi_{y,x} + w_{,x}\Psi_{z,y} + w_{,y}\Psi_{z,x}$$

$$E_{xx}^{(2)} = \frac{1}{2} (\Psi_{z,x})^2$$

$$E_{yy}^{(2)} = \frac{1}{2} (\Psi_{z,y})^2$$

$$2E_{xy}^{(2)} = \Psi_{z,x}\Psi_{z,y}$$

$$2E_{yz}^{(0)} = w_{,y} + u_{,y}\Psi_x + (1 + v_{,y})\Psi_y + \frac{1}{2}\Psi_z(w_{,y} - \Psi_y)$$

$$2E_{xz}^{(0)} = w_{,x} + v_{,x}\Psi_y + (1 + u_{,x})\Psi_x + \frac{1}{2}\Psi_z(w_{,x} - \Psi_x)$$

$$2E_{yz}^{(1)} = \Psi_x\Psi_{x,y} + \Psi_y\Psi_{y,y} - \frac{1}{2}u_{,y}\Psi_{z,x} + (1 - \frac{1}{2}v_{,y})\Psi_{z,y}$$

$$2E_{xz}^{(1)} = \Psi_x\Psi_{x,x} + \Psi_y\Psi_{y,x} - \frac{1}{2}v_{,x}\Psi_{z,y} + (1 - \frac{1}{2}u_{,x})\Psi_{z,x}$$

## 2) Circular cylindrical shells

Let the geometry of a cylinder be defined as illustrated in Figure 31 on page 101, where  $x$  and  $\theta$  are curvilinear coordinates on the shell, coinciding with curvilinear coordinates  $u^1$ ,  $u^2$  and  $x = -z^3$ .

The position vector of a point on the plate is then :

$$\mathbf{r} = (z^1, z^2, z^3) = (R \cos \theta, R \sin \theta, -x)$$

The following quantities can be obtained :

- tangent vectors and normal :

$$\mathbf{T}_1 = -\mathbf{s}_3$$

$$\mathbf{T}_2 = R(-\sin \theta \mathbf{s}_1 + \cos \theta \mathbf{s}_2)$$

$$\mathbf{n} = (\cos \theta \mathbf{s}_1 + \sin \theta \mathbf{s}_2)$$

- surface metric

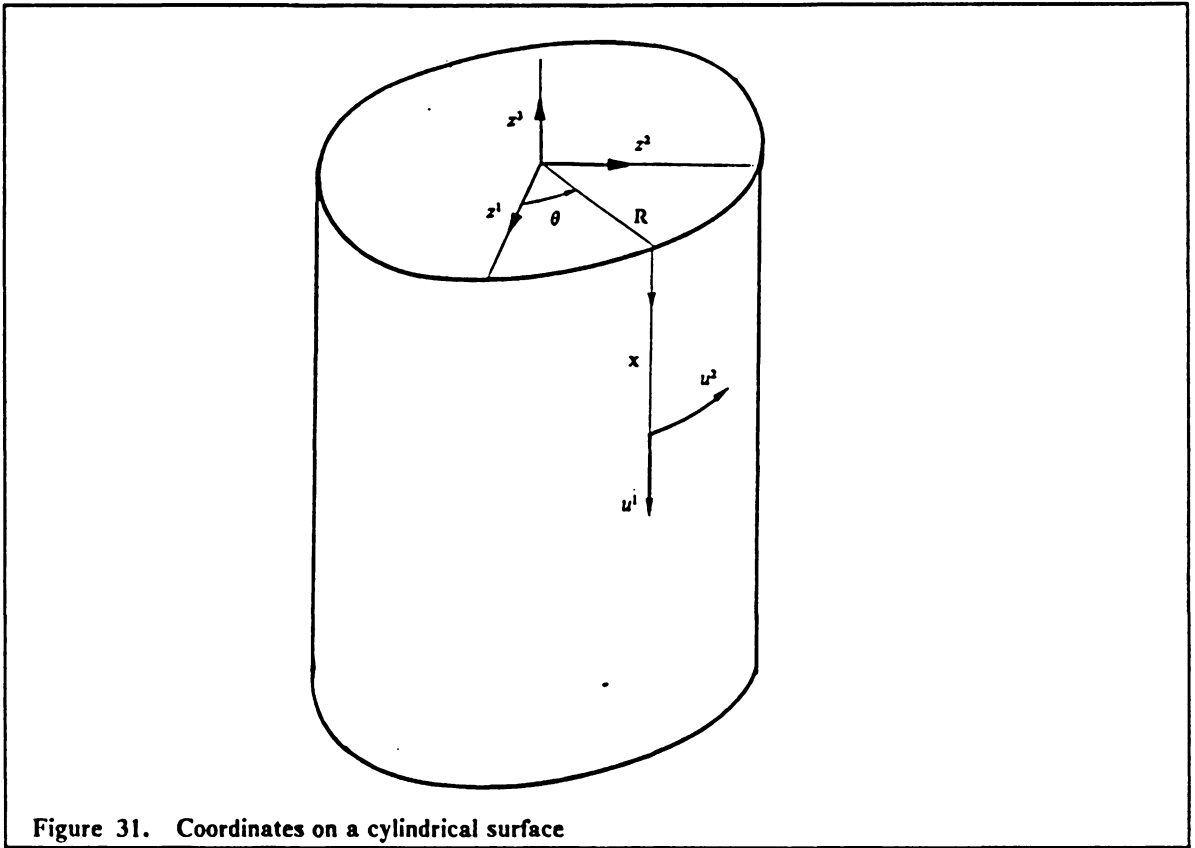


Figure 31. Coordinates on a cylindrical surface

$$[a_{\alpha\beta}] = \begin{bmatrix} 1 & 0 \\ 0 & R^2 \end{bmatrix}, \quad [a^{\alpha\beta}] = \begin{bmatrix} 1 & 0 \\ 0 & 1/R^2 \end{bmatrix}$$

- curvature tensor :

$$[b_{\alpha\beta}] = \begin{bmatrix} 0 & 0 \\ 0 & -R \end{bmatrix}, \quad [b^{\alpha}_{\beta}] = \begin{bmatrix} 0 & 0 \\ 0 & -1/R \end{bmatrix}$$

- shifter tensor :

$$[c^{\alpha}_{\beta}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 + \frac{\theta^2}{R} \end{bmatrix}$$

- Christoffel symbols :

$$\Gamma_{\beta\gamma}^{\alpha} = 0 \quad \forall \alpha, \beta, \gamma$$

- Physical components

$$\hat{u}_1 = u_1, \quad \hat{u}_2 = \frac{1}{R} u_2, \quad \hat{u}_3 = u_3$$

$$\hat{T}^{11} = T^{11}, \quad \hat{T}^{12} = R T^{12}, \quad \hat{T}^{21} = R T^{21}, \quad \hat{T}^{22} = R^2 T^{22},$$

$$\hat{T}^{31} = T^{31}, \quad \hat{T}^{32} = R T^{32}$$

- Derivatives and other results :

These results are given in terms of the physical components, where the hats were dropped.

$$u_1|_1 = u_{1,1}, \quad u_1|_2 = u_{1,2}, \quad u_2|_1 = R u_{2,1}, \quad u_2|_2 = R u_{2,2}$$

$$u^1 = u_1, \quad u^2 = \frac{1}{R} u_2$$

$$u_1|^1 = u_{1,1}, \quad u_1|^2 = \frac{1}{R^2} u_{1,2}, \quad u_2|^1 = R u_{2,1}, \quad u_2|^2 = \frac{1}{R} u_{2,2}$$

$$u^1|_1 = u_{1,1}, \quad u^1|_2 = u_{1,2}, \quad u^2|_1 = \frac{1}{R} u_{2,1}, \quad u^2|_2 = \frac{1}{R} u_{2,2}$$

$$\Phi_{11} = u_{1,1}, \quad \Phi_{12} = u_{1,2}, \quad \Phi_{21} = R u_{2,1}, \quad \Phi_{22} = R (u_{2,2} + u_3)$$

$$\Phi_{.1}^1 = u_{1,1}, \quad \Phi_{.2}^1 = u_{1,2}, \quad \Phi_{.1}^2 = \frac{1}{R} u_{2,1}, \quad \Phi_{.2}^2 = \frac{1}{R} (u_{2,2} + u_3)$$

$$\Phi_{13} = u_{3,1}, \quad \Phi_{23} = u_{3,2} - u_2$$

Only the expressions for the strains will be presented here, since the explicit weak form was not used for shells.

- Physical strains

$$E_{11}^{(0)} = u_{,1} + \frac{1}{2} (w_{,1})^2$$

$$E_{22}^{(0)} = \frac{1}{R} (v_{,2} + w) + \frac{1}{2R^2} (w_{,2} - v)^2$$

$$2E_{12}^{(0)} = \frac{1}{R} u_{,2} + v_{,1} + \frac{1}{R} w_{,1} (w_{,2} - v)^2$$

$$E_{33}^{(0)} = \Psi_3 + \frac{1}{2} [(\Psi_1)^2 + (\Psi_2)^2]$$

$$E_{11}^{(1)} = \Psi_{1,1} + w_{,1} \Psi_{3,1}$$

$$E_{22}^{(1)} = \frac{1}{R^2} (v_{,2} + w) + \frac{1}{R} (\Psi_{2,2} + \Psi_3) + \frac{1}{R^2} (w_{,2} - v)(\Psi_{3,2} - \Psi_2)$$

$$2E_{12}^{(1)} = \frac{1}{R} (v_{,1} + \Psi_{1,2}) + \Psi_{2,1} + \frac{1}{R} [w_{,1}(\Psi_{3,2} - \Psi_2) + \Psi_{3,1}(w_{,2} - v)]$$

$$E_{11}^{(2)} = \frac{1}{2} (\Psi_{3,1})^2$$

$$E_{22}^{(2)} = \frac{1}{R^2} (\Psi_{2,2} + \Psi_3) + \frac{1}{R^2} (\Psi_{3,2} - \Psi_2)^2$$

$$2E_{12}^{(2)} = \frac{1}{R} \Psi_{2,1} + \frac{1}{R} \Psi_{3,1} (\Psi_{3,2} - \Psi_2)$$

$$2E_{23}^{(0)} = \frac{1}{R} (w_{,2} - v) + \Psi_2 + \frac{1}{R} \left[ u_{,2} \Psi_1 + (v_{,2} + w) \Psi_2 + \frac{1}{2} \Psi_3 (w_{,2} - v - R \Psi_2) \right]$$

$$2E_{13}^{(0)} = w_{,1} + \Psi_1 + u_{,1}\Psi_1 + v_{,1}\Psi_2 + \frac{1}{2}\Psi_3(w_{,1} - \Psi_1)$$

$$2E_{23}^{(1)} = \frac{1}{R}\Psi_{3,2} + \frac{1}{R}\left[\Psi_1\Psi_{1,2} + \Psi_2\Psi_{2,2} - \frac{1}{2}(u_{,2}\Psi_{3,1} + \frac{1}{R}\Psi_{3,2}(v_{,2} + w))\right]$$

$$2E_{13}^{(1)} = \Psi_{3,1} + \Psi_1\Psi_{1,1} + \Psi_2\Psi_{2,1} - \frac{1}{2}(u_{,1}\Psi_{3,1} + \frac{1}{R}v_{,1}\Psi_{3,2})$$

If we redefine the curvilinear coordinates as  $(x, y = R\theta, z)$ , the strain-displacement equations become :

$$E_{xx}^{(0)} = u_{,x} + \frac{1}{2}(w_{,x})^2$$

$$E_{yy}^{(0)} = v_{,y} + \frac{w}{R} + \frac{1}{2}(w_{,y} - \frac{1}{R}v)^2$$

$$2E_{xy}^{(0)} = u_{,y} + v_{,x} + w_{,x}(w_{,y} - \frac{1}{R}v)^2$$

$$E_{zz}^{(0)} = \Psi_z + \frac{1}{2}[(\Psi_x)^2 + (\Psi_y)^2]$$

$$E_{xx}^{(1)} = \Psi_{x,x} + w_{,x}\Psi_{z,x}$$

$$E_{yy}^{(1)} = \Psi_{y,y} + \frac{1}{R}(v_{,y} + \frac{1}{R}w + \Psi_z) + (w_{,y} - \frac{1}{R}v)(\Psi_{z,y} - \frac{1}{R}\Psi_y)$$

$$2E_{xy}^{(1)} = \frac{1}{R}v_{,x} + \Psi_{x,y} + \Psi_{y,x} + w_{,x}(\Psi_{z,y} - \frac{1}{R}\Psi_y) + \Psi_{z,x}(w_{,y} - \frac{1}{R}v)$$

$$E_{xx}^{(2)} = \frac{1}{2}(\Psi_{z,x})^2$$

$$E_{yy}^{(2)} = \frac{1}{R}(\Psi_{y,y} + \frac{1}{R}\Psi_z) + \frac{1}{2}(\Psi_{z,y} - \frac{1}{R}\Psi_y)^2$$

$$2E_{xy}^{(2)} = \frac{1}{R} \Psi_{y,x} + \Psi_{z,x}(\Psi_{z,y} - \frac{1}{R} \Psi_z)$$

$$2E_{yz}^{(0)} = w_{,y} - \frac{1}{R} v + \Psi_y + u_{,y} \Psi_x + (v_{,y} + \frac{1}{R} w) \Psi_y + \frac{1}{2} \Psi_z (w_{,y} - \frac{1}{R} v - \Psi_y)$$

$$2E_{xz}^{(0)} = w_{,x} + \Psi_x + u_{,x} \Psi_x + v_{,x} \Psi_y + \frac{1}{2} \Psi_z (w_{,x} - \Psi_x)$$

$$2E_{yz}^{(1)} = \Psi_{z,y} + \Psi_x \Psi_{x,y} + \Psi_y \Psi_{y,y} - \frac{1}{2} \left[ u_{,y} \Psi_{z,x} + \Psi_{z,y} (v_{,y} + \frac{1}{R} w) \right]$$

$$2E_{xz}^{(1)} = \Psi_{z,x} + \Psi_x \Psi_{x,x} + \Psi_y \Psi_{y,x} - \frac{1}{2} (u_{,x} \Psi_{z,x} + v_{,x} \Psi_{z,y})$$

Note that the additional terms, with respect to plates, are those divided by R.

### 3) Spherical shells

Let the geometry of a sphere be defined as illustrated in Figure 32 on page 106, where  $\phi$  and  $\theta$  are curvilinear coordinates on the shell, coinciding with the curvilinear coordinates  $u^1$ ,  $u^2$ , respectively.

The position vector of a point on the plate is then :

$$\mathbf{r} = (z^1, z^2, z^3) = (R \sin \phi \cos \theta, R \sin \phi \sin \theta, R \cos \phi)$$

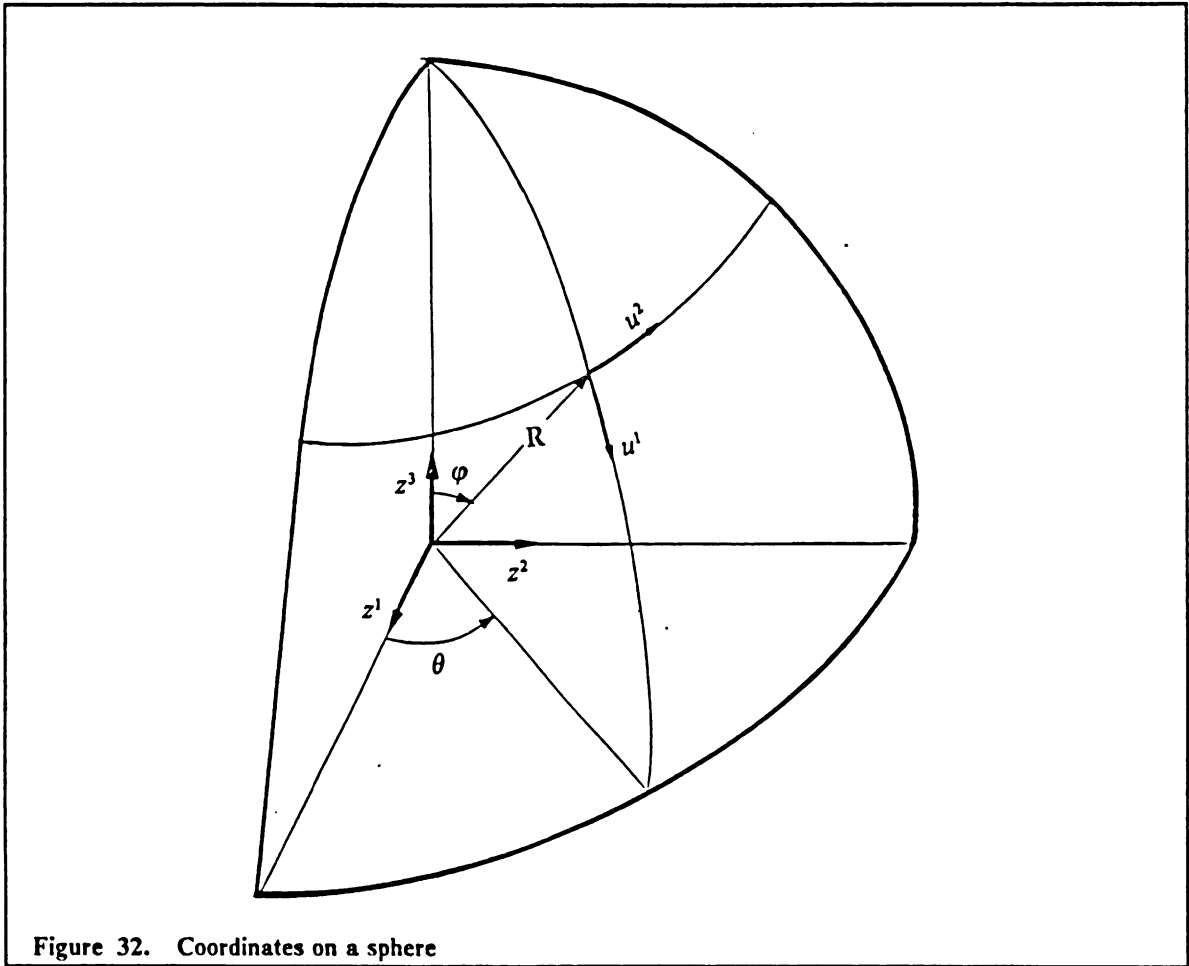
The following quantities can be obtained :

- tangent vectors and normal :

$$\mathbf{T}_1 = \cos \phi \cos \theta \mathbf{e}_1 + \cos \phi \sin \theta \mathbf{e}_2 - \sin \phi \mathbf{e}_3$$

$$\mathbf{T}_2 = R(-\sin \phi \sin \theta \mathbf{e}_1 + \sin \phi \cos \theta \mathbf{e}_2)$$

$$\mathbf{n} = \sin \phi (\cos \theta \mathbf{e}_1 + \sin \theta \mathbf{e}_2) + \cos \phi \mathbf{e}_3$$



- surface metric

$$[a_{\alpha\beta}] = \begin{bmatrix} 1 & 0 \\ 0 & (R \sin \phi)^2 \end{bmatrix}, \quad [a^{\alpha\beta}] = \begin{bmatrix} 1 & 0 \\ 0 & 1/(R \sin \phi)^2 \end{bmatrix}$$

- curvature tensor :

$$[b_{\alpha\beta}] = \begin{bmatrix} -1/R & 0 \\ 0 & -R(\sin \phi)^2 \end{bmatrix}, \quad [b_{\beta}^{\alpha}] = \begin{bmatrix} -1/R & 0 \\ 0 & -1/R \end{bmatrix}$$

- shifter tensor :

$$[c_{\beta}^{\alpha}] = \begin{bmatrix} 1 + \frac{\theta^3}{R} & 0 \\ 0 & 1 + \frac{\theta^3}{R} \end{bmatrix}$$

- Christoffel symbols :

$$\Gamma_{11}^1 = \Gamma_{12}^1 = \Gamma_{11}^2 = \Gamma_{22}^2 = 0$$

$$\Gamma_{21}^2 = \Gamma_{12}^2 = \frac{\cot \phi}{R} , \quad \Gamma_{22}^1 = -R \sin \phi \cos \phi$$

- Physical components

$$\hat{u}_1 = u_1 , \quad \hat{u}_2 = \frac{u_2}{R \sin \phi} , \quad \hat{u}_3 = u_3$$

$$\hat{T}^{11} = T^{11} , \quad \hat{T}^{12} = R \sin \phi T^{12} , \quad \hat{T}^{21} = R \sin \phi T^{21} , \quad \hat{T}^{22} = (R \sin \phi)^2 T^{22} ,$$

$$\hat{T}^{31} = T^{31} , \quad \hat{T}^{32} = R \sin \phi T^{32}$$

- Derivatives and other results :

These results are given in terms of the physical components, where the hats were dropped.

$$u_1|_1 = u_{1,1}$$

$$u_2|_1 = R \sin \phi u_{2,1} - u_2 \cos \phi$$

$$u^1 = u_1$$

$$u_1|_1 = u_{1,1}$$

$$u_2|_1 = R \sin \phi u_{2,1} - u_2 \cos \phi$$

$$u^1|_1 = u_{1,1}$$

$$u^2|_1 = \frac{1}{(R \sin \phi)^2} (R \sin \phi u_{2,1} - u_2 \cos \phi)$$

$$\Phi_{11} = u_{1,1} + \frac{1}{R} u_3$$

$$u_1|_2 = u_{1,2} - u_2 \cos \phi$$

$$u_2|_2 = R \sin \phi (u_{2,2} + u_1 \cos \phi)$$

$$u^2 = \frac{u_2}{R \sin \phi}$$

$$u_1|_2 = \frac{1}{(R \sin \phi)^2} (u_{1,2} - u_2 \cos \phi)$$

$$u_2|_2 = \frac{1}{R \sin \phi} (u_{2,2} + u_1 \cos \phi)$$

$$u^1|_2 = u_{1,2} - u_2 \cos \phi$$

$$u^2|_2 = \frac{1}{R \sin \phi} (u_{2,2} + u_1 \cos \phi)$$

$$\Phi_{12} = u_{1,2} - u_2 \cos \phi$$

$$\Phi_{21} = R \sin \phi u_{2,1} - u_2 \cos \phi$$

$$\Phi_{22} = R \sin \phi (u_{2,2} + u_1 \cos \phi + u_3 \sin \phi)$$

$$\Phi_{,1}^1 = u_{1,1} + \frac{1}{R} u_3$$

$$\Phi_{,2}^1 = u_{1,2} - u_2 \cos \phi$$

$$\Phi_{,1}^2 = \frac{1}{(R \sin \phi)^2} (R \sin \phi u_{2,1} - u_2 \cos \phi)$$

$$\Phi_{,2}^2 = \frac{1}{(R \sin \phi)} (u_{2,2} + u_1 \cos \phi - u_3 \sin \phi)$$

$$\Phi_{13} = u_{3,1} - \frac{1}{R} u_1$$

$$\Phi_{23} = u_{3,2} - u_2 \sin \phi$$

Only the expressions for the strains will be presented here, since the explicit weak form was not used for shells.

- Physical strains

$$E_{11}^{(0)} = u_{,1} + \frac{1}{R} w + \frac{1}{2} (w_{,1} - \frac{1}{R} u)^2$$

$$E_{22}^{(0)} = \frac{1}{R} (u \cot \phi + \frac{1}{\sin \phi} v_{,2} + w) + \frac{1}{2(R \sin \phi)^2} (w_{,2} - v \sin \phi)^2$$

$$2E_{12}^{(0)} = \frac{1}{R} \left( \frac{1}{\sin \phi} u_{,2} + R v_{,1} - 2v \cot \phi \right) + \frac{1}{R \sin \phi} \left( w_{,1} - \frac{1}{R} u \right) (w_{,2} - v \sin \phi)$$

$$E_{33}^{(0)} = \Psi_3 + \frac{1}{2} [(\Psi_1)^2 + (\Psi_2)^2]$$

$$E_{11}^{(1)} = \Psi_{1,1} + \frac{1}{R} (u_{,1} + \frac{1}{R} w + \Psi_3) + (w_{,1} - \frac{1}{R} u) (\Psi_{3,1} - \frac{1}{R} \Psi_1)$$

$$E_{22}^{(1)} = \frac{1}{R^2} \left( \frac{1}{\sin \phi} v_{,2} + u \cot \phi + w \right) + \frac{1}{R} \left( \frac{1}{\sin \phi} \Psi_{2,2} + \Psi_1 \cot \phi + \Psi_3 \right) + \frac{1}{(R \sin \phi)^2} (w_{,2} - v \sin \phi) (\Psi_{3,2} - \Psi_2 \sin \phi)$$

$$2E_{12}^{(1)} = \frac{1}{R^2} \left( \frac{1}{\sin \phi} u_{,2} + R v_{,1} - 2v \cot \phi \right) + \frac{1}{R} \left( \frac{1}{\sin \phi} \Psi_{1,2} + R \Psi_{2,1} - 2\Psi_2 \cot \phi \right) + \frac{1}{R \sin \phi} \left[ (w_{,1} - \frac{1}{R} u) (\Psi_{3,2} - \Psi_2 \sin \phi) + (w_{,2} - v \sin \phi) (\Psi_{3,1} - \frac{1}{R} \Psi_1) \right]$$

$$E_{11}^{(2)} = \frac{1}{R} (\Psi_{1,1} + \frac{1}{R} \Psi_3) + \frac{1}{2} (\Psi_{3,1} - \frac{1}{R} \Psi_1)^2$$

$$E_{22}^{(2)} = \frac{1}{R^2} \left( \frac{1}{\sin \phi} \Psi_{2,2} + \Psi_1 \cot \phi + \Psi_3 \right) + \frac{1}{2(R \sin \phi)^2} (\Psi_{3,2} - \Psi_2 \sin \phi)^2$$

$$2E_{12}^{(2)} = \frac{1}{R^2} \left( \frac{1}{\sin \phi} \Psi_{1,2} + R \Psi_{2,1} - 2\Psi_2 \cot \phi \right) + \frac{1}{R \sin \phi} (\Psi_{3,1} - \frac{1}{R} \Psi_1) (\Psi_{3,2} - \Psi_2 \sin \phi)$$

$$E_{23}^{(0)} = \frac{1}{R} \left( \frac{1}{\sin \phi} w_{,2} - v \right) + \Psi_2 + \frac{1}{R \sin \phi} \Psi_1 (u_{,2} - v \cos \phi) +$$

$$\frac{1}{R} \Psi_2 \left( \frac{1}{\sin \phi} v_{,2} + u \cot \phi + w \right) + \frac{1}{2} \Psi_3 \left( \frac{1}{R \sin \phi} w_{,2} - \frac{v}{R} - \Psi_2 \right)$$

$$2E_{13}^{(0)} = w_{,1} - \frac{1}{R} u + \Psi_1 (1 + u_{,1} + \frac{1}{R} w) + \Psi_2 (v_{,1} - \frac{1}{R} v \cot \phi) + \frac{1}{2} \Psi_3 (w_{,1} - \frac{1}{R} u - \Psi_1)$$

$$2E_{23}^{(1)} = \frac{1}{R \sin \phi} \Psi_{3,2} + \frac{1}{R \sin \phi} \Psi_1 (\Psi_{1,2} - \Psi_2 \cos \phi) + \frac{1}{R \sin \phi} \Psi_2 (\Psi_{2,2} + \Psi_1 \cos \phi) -$$

$$\frac{1}{2R \sin \phi} \left[ \Psi_{3,1} (u_{,2} - v \cos \phi) + \frac{1}{R} \Psi_{3,2} \left( \frac{1}{\sin \phi} v_{,2} + u \cot \phi + w \right) \right]$$

$$2E_{13}^{(1)} = \Psi_{3,1} + \Psi_1 \Psi_{1,1} + \Psi_2 (\Psi_{2,1} - \frac{1}{R} \Psi_2 \cot \phi)$$

$$- \frac{1}{2} \left[ \Psi_{3,1} (u_{,1} + \frac{1}{R} w) + \frac{1}{R^2 \sin \phi} \Psi_{3,2} (R v_{,1} - v \cot \phi) \right]$$

If we now call :

$$( )_{,1} = ( )_{,x} \quad , \quad \frac{1}{R \sin \phi} ( )_{,2} = ( )_{,y} \quad , \quad \eta = 1/R \quad , \quad \xi = \cot \phi / R$$

the strain-displacement equations become :

$$E_{xx}^{(0)} = u_{,x} + \eta w + \frac{1}{2} (w_{,x} - \eta u)^2$$

$$E_{yy}^{(0)} = v_{,y} + \frac{1}{R} w + \xi u + \frac{1}{2} (w_{,y} - \frac{1}{R} v)^2$$

$${}^{(0)}2E_{xy} = u_{,y} + v_{,x} - 2\xi v + (w_{,x} - \eta u)(w_{,y} - \frac{1}{R} v)$$

$${}^{(0)}E_{zz} = \Psi_z + \frac{1}{2} [(\Psi_x)^2 + (\Psi_y)^2]$$

$${}^{(1)}E_{xx} = \Psi_{x,x} + \eta(u_{,x} + \frac{1}{R} w + \Psi_z) + (w_{,x} - \eta u)(\Psi_{z,x} - \eta \Psi_x)$$

$${}^{(1)}E_{yy} = \Psi_{y,y} + \frac{1}{R}(v_{,y} + \frac{1}{R} w + \Psi_z) + \xi(\frac{1}{R} u + \Psi_x) + (w_{,y} - \frac{1}{R} v)(\Psi_{z,y} - \frac{1}{R} \Psi_y)$$

$${}^{(1)}2E_{xy} = \frac{1}{R} v_{,x} + \Psi_{x,y} + \Psi_{y,x} + \eta u_{,y} - 2\xi(\frac{1}{R} v + \Psi_y) + (w_{,x} - \eta u)(\Psi_{z,y} - \frac{1}{R} \Psi_y) + (\Psi_{z,x} - \eta \Psi_x)(w_{,y} - \frac{1}{R} v)$$

$${}^{(2)}E_{xx} = \eta(\Psi_{x,x} + \frac{1}{R} \Psi_z) + \frac{1}{2} (\Psi_{z,x} - \eta \Psi_x)^2$$

$${}^{(2)}E_{yy} = \frac{1}{R} (\xi \Psi_x + \Psi_{y,y} + \frac{1}{R} \Psi_z) + \frac{1}{2} (\Psi_{z,y} - \frac{1}{R} \Psi_y)^2$$

$${}^{(2)}2E_{xy} = \frac{1}{R} \Psi_{y,x} + \eta(\Psi_{x,y} - 2\xi \Psi_y) + (\Psi_{z,x} - \eta \Psi_x)(\Psi_{z,y} - \frac{1}{R} \Psi_y)$$

$${}^{(0)}2E_{yz} = w_{,y} - \frac{1}{R} v + \Psi_y + \Psi_x(u_{,y} - \xi v) + \Psi_y(\xi u + v_{,y} + \frac{1}{R} w) + \frac{1}{2} \Psi_z(w_{,y} - \frac{1}{R} v - \Psi_y)$$

$${}^{(0)}2E_{xz} = w_{,x} + \Psi_x - \eta u + \Psi_x(u_{,x} + \eta w) + \Psi_y(v_{,x} - \xi v) + \frac{1}{2} \Psi_z(w_{,x} - \eta u - \Psi_x)$$

$${}^{(1)}2E_{yz} = \Psi_{z,y} + \Psi_x(\Psi_{x,y} - \xi \Psi_y) + \Psi_y(\Psi_{y,y} + \xi \Psi_x) - \frac{1}{2} [\Psi_{z,x}(u_{,y} - \xi v) + \Psi_{z,y}(\xi u + v_{,y} + \frac{1}{R} w)]$$

$${}^{(1)}2E_{xz} = \Psi_{z,x} + \Psi_x \Psi_{x,x} + \Psi_y(\Psi_{y,x} - \xi \Psi_y) - \frac{1}{2} [\Psi_{z,x}(u_{,x} + \eta w) + \Psi_{z,y}(v_{,x} - \xi v)]$$

Note that if  $\eta$  and  $\xi$  are equal to zero we recover the expressions for the strains of cylindrical shells.

### Appendix D. Laminate stiffness in full form

In this appendix, no assumption, with respect to the shell thickness, is made, in the derivation of the laminate stiffness coefficients.

#### 1) Cylindrical shell

$$a_{ij}^* = \int_{-h/2}^{h/2} \left(1 + \frac{\theta^3}{R}\right) \bar{C}_{ij} d\theta^3 = \int_{-h/2}^{h/2} \bar{C}_{ij} d\theta^3 + \frac{1}{R} \int_{-h/2}^{h/2} \theta^3 \bar{C}_{ij} d\theta^3 = a_{ij} + \frac{1}{R} b_{ij}$$

$$b_{ij}^* = \int_{-h/2}^{h/2} \theta^3 \left(1 + \frac{\theta^3}{R}\right) \bar{C}_{ij} d\theta^3 = b_{ij} + \frac{1}{R} d_{ij}$$

$$d_{ij}^* = \int_{-h/2}^{h/2} (\theta^3)^2 \left(1 + \frac{\theta^3}{R}\right) \bar{C}_{ij} d\theta^3 = d_{ij} + \frac{1}{R} e_{ij}$$

$$e_{ij}^* = \int_{-h/2}^{h/2} (\theta^3)^3 \left(1 + \frac{\theta^3}{R}\right) \bar{C}_{ij} d\theta^3 = e_{ij} + \frac{1}{R} f_{ij}$$

$$f_{ij}^* = \int_{-h/2}^{h/2} (\theta^3)^4 \left(1 + \frac{\theta^3}{R}\right) \bar{C}_{ij} d\theta^3 = f_{ij} + \frac{1}{R} g_{ij}$$

0 where :

$$g_{ij}^* = \int_{-h/2}^{h/2} (\theta^3)^5 \bar{C}_{ij} d\theta^3$$

## 2) Spherical shell

$$a_{ij}^* = \int_{-h/2}^{h/2} \left(1 + \frac{\theta^3}{R}\right)^2 \bar{C}_{ij} d\theta^3 = a_{ij} + \frac{2}{R} b_{ij} + \frac{1}{R^2} d_{ij}$$

$$b_{ij}^* = b_{ij} + \frac{2}{R} d_{ij} + \frac{1}{R^2} e_{ij}$$

$$d_{ij}^* = d_{ij} + \frac{2}{R} e_{ij} + \frac{1}{R^2} f_{ij}$$

$$e_{ij}^* = e_{ij} + \frac{2}{R} f_{ij} + \frac{1}{R^2} g_{ij}$$

$$f_{ij}^* = f_{ij} + \frac{2}{R} g_{ij} + \frac{1}{R^2} h_{ij}$$

and :

$$h_{ij}^* = \int_{-h/2}^{h/2} (\theta^3)^6 \bar{C}_{ij} d\theta^3$$

## *Appendix E. Element Matrices*

In this appendix, we will determine all matrices necessary to build the direct and tangent stiffness matrices, for RVK and MRT, and the geometries of interest to this work. Notice that all derivations will be based on the spherical shell equations, where appropriate values for  $1/R$  ,  $\eta$  ,  $\xi$  should be taken for plates and cylinders.

### E.1 Matrices for von Kármán theory

#### E.1.1 Bending and in-plane linear matrix

$$\mathbf{B}_0 \cdot \mathbf{N} = \begin{bmatrix} \phi_{l,x} & 0 & \eta\phi_l & 0 & 0 \\ \xi\phi_l & \phi_{l,y} & K\phi_l & 0 & 0 \\ \phi_{l,y} & \phi_{l,x} - 2\xi\phi_l & 0 & 0 & 0 \\ \eta\phi_{l,x} & 0 & \eta K\phi_l & \phi_{l,x} & 0 \\ K\xi\phi_l & K\phi_{l,y} & K^2\phi_l & \xi\phi_l & \phi_{l,y} \\ \eta\phi_{l,y} & K(\phi_{l,x} - 2\xi\phi_l) & 0 & \phi_{l,y} & \phi_{l,x} - 2\xi\phi_l \\ 0 & 0 & 0 & \eta\phi_{l,x} & 0 \\ 0 & 0 & 0 & K\xi\phi_l & K\phi_{l,y} \\ 0 & 0 & 0 & \eta\phi_{l,y} & K(\phi_{l,x} - 2\xi\phi_l) \end{bmatrix}$$

#### E.1.2 Transverse shear linear matrix

$$\mathbf{B}_0 \cdot \mathbf{N} = \begin{bmatrix} 0 & -K\phi_l & \phi_{l,y} & 0 & \phi_l \\ -\eta\phi_l & 0 & \phi_{l,x} & \phi_l & 0 \end{bmatrix}$$

#### E.1.3 Nonlinear matrix

Only strains related to bending have nonlinear terms, and we can write :

$$\varepsilon_{NL} = \frac{1}{2} \mathbf{A} \boldsymbol{\theta} = \frac{1}{2} \begin{bmatrix} w_{,x} & 0 \\ 0 & w_{,y} \\ w_{,y} & w_{,x} \end{bmatrix} \begin{bmatrix} w_{,x} \\ w_{,y} \end{bmatrix}$$

$$\boldsymbol{\theta} \simeq \mathbf{G} \cdot \mathbf{N} \cdot \underline{u} = \begin{bmatrix} 0 & 0 & \phi_{l,x} & 0 & 0 \\ 0 & 0 & \phi_{l,y} & 0 & 0 \end{bmatrix}$$

Then :

$$\mathbf{A} \cdot \mathbf{G} \cdot \mathbf{N} = \begin{bmatrix} 0 & 0 & w_{,x} \phi_{l,x} & 0 & 0 \\ 0 & 0 & w_{,y} \phi_{l,y} & 0 & 0 \\ 0 & 0 & w_{,y} \phi_{l,x} + w_{,x} \phi_{l,y} & 0 & 0 \end{bmatrix}$$

#### E.1.4 Stress matrix

We need to find matrix S first, by writing :  $(\delta \mathbf{A})^T \boldsymbol{\sigma} = \mathbf{S} \delta \boldsymbol{\theta}$ .

$$(\delta \mathbf{A})^T \boldsymbol{\sigma} = \begin{bmatrix} \delta w_{,x} N_{11} + \delta w_{,y} N_{12} \\ \delta w_{,y} N_{12} + \delta w_{,x} N_{22} \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ N_{12} & N_{22} \end{bmatrix} \begin{bmatrix} \delta w_{,x} \\ \delta w_{,y} \end{bmatrix}$$

The only non zero element of  $\mathbf{K}_e$  will be :

$$k_{33} = N_{11} \phi_{l,x} \phi_{j,x} + N_{12} (\phi_{l,x} \phi_{j,y} + \phi_{l,y} \phi_{j,x}) + N_{22} \phi_{l,y} \phi_{j,y}$$

## E.2 Matrices for Moderate Rotation theory

### E.2.1 Bending and in-plane linear matrix

$$\mathbf{B}_0 \mathbf{N} = [\mathbf{B}_1 \quad \mathbf{B}_2]$$

where :

$$\mathbf{B}_1 = \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \quad , \quad \mathbf{B}_2 = \begin{bmatrix} \mathbf{0} \\ \mathbf{B} \end{bmatrix}$$

and :

$$\mathbf{B} = \begin{bmatrix} \phi_{l,x} & 0 & \eta\phi_l \\ \xi\phi_l & \phi_{l,y} & K\phi_l \\ \phi_{l,y} & \phi_{l,x} - 2\xi\phi_l & 0 \\ \eta\phi_{l,x} & 0 & \eta K\phi_l \\ K\xi\phi_l & K\phi_{l,y} & K^2\phi_l \\ \eta\phi_{l,y} & K(\phi_{l,x} - 2\xi\phi_l) & 0 \end{bmatrix}$$

and  $\mathbf{0}$  is a (3x3) matrix of zeros

### E.2.2 Transverse shear linear matrix

$$\mathbf{B.N} = \begin{bmatrix} 0 & -K\phi_l & \phi_{l,y} & 0 & \phi_l & 0 \\ -\eta\phi_l & 0 & \phi_{l,x} & \phi_l & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \phi_{l,y} \\ 0 & 0 & 0 & 0 & 0 & \phi_{l,x} \end{bmatrix}$$

### E.2.3 Nonlinear matrix in bending

We first define :

$$X1 = w_{,x} - \eta u \quad , \quad X2 = w_{,y} - K v \quad , \quad X3 = \Psi_{z,x} - \eta \Psi_x \quad , \quad X4 = \Psi_{z,y} - K \Psi_y \quad , \quad K = 1/R$$

Then :

$$A = [A_1 \ A_2]$$

where :

$$A_1 = \begin{bmatrix} A_3 \\ 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 \\ A_3 \end{bmatrix}$$

and :

$$A_3 = \begin{bmatrix} X1 & 0 \\ 0 & X2 \\ X2 & X1 \\ X3 & 0 \\ 0 & X4 \\ X4 & X3 \end{bmatrix}$$

and  $0$  is a (3x2) matrix of zeros, and  $\theta^T = (X1, X2, X3, X4)^T$ . Next we obtain :

$$G.N = \begin{bmatrix} -\eta\phi_l & 0 & \phi_{l,x} & 0 & 0 & 0 \\ 0 & -K\phi_l & \phi_{l,y} & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_l & 0 & 0 \\ 0 & 0 & 0 & 0 & \phi_l & 0 \\ 0 & 0 & 0 & -\eta\phi_l & 0 & \phi_{l,x} \\ 0 & 0 & 0 & 0 & -K\phi_l & \phi_{l,y} \end{bmatrix}$$

Then :

$$A . G.N = [A_4 \ A_5]$$

where :

$$A_4 = \begin{bmatrix} A_6 \\ 0 \end{bmatrix} \quad , \quad A_5 = \begin{bmatrix} 0 \\ A_6 \end{bmatrix}$$

and :

$$A_6 = \begin{bmatrix} -\eta\phi_r X1 & 0 & \phi_{l,x} X1 \\ 0 & -K\phi_r X2 & \phi_{l,y} X2 \\ -\eta\phi_r X2 & -K\phi_r X1 & \phi_{l,x} X2 + \phi_{l,y} X1 \\ -\eta\phi_r X3 & 0 & \phi_{l,x} X3 \\ 0 & -K\phi_r X4 & \phi_{l,y} X4 \\ -\eta\phi_r X4 & -K\phi_r X3 & \phi_{l,x} X4 + \phi_{l,y} X3 \end{bmatrix}$$

and  $\mathbf{0}$  is a (3x3) matrix. Notice that it is not necessary to perform the multiplication of  $A$  and  $GN$ , since each term in  $AGN$  is the expansion of a certain degree of freedom from the corresponding nonlinear term in the strain equations. For example, consider :

$$(w_{,x} - \eta u)^2 = X1^2 = -\eta u X1 + w_{,x} X1$$

Then :

$$(A_6)_{11} = -\eta\phi_r X1 \quad , \quad (A_6)_{13} = \phi_{l,x} X1$$

#### E.2.4 Nonlinear matrix in transverse shear

We first define :

$$\begin{aligned} X5 &= u_{,y} - \xi v \quad , \quad X6 = v_{,y} + K w + \xi u \quad , \quad X7 = w_{,y} - \Psi_y - K v \\ X8 &= u_{,x} + \eta w \quad , \quad X9 = v_{,x} - \xi v \quad , \quad X10 = w_{,x} - \Psi_x - \eta u \end{aligned}$$

then :

$$A = [A_7, A_8, A_9]$$

where :

$$A_7 = \begin{bmatrix} X5 & X6 & \frac{1}{2}X7 & \Psi_x & \Psi_y \\ X8 & X9 & \frac{1}{2}X10 & 0 & 0 \\ \Psi_{x,y} & \Psi_{y,y} & 0 & -\frac{1}{2}\Psi_{z,x} & -\frac{1}{2}\Psi_{z,y} \\ \Psi_{x,x} & \Psi_{y,x} - 2\xi\Psi_y & 0 & 0 & 0 \end{bmatrix}$$

$$A_8 = \begin{bmatrix} \frac{1}{2}\Psi_z & 0 & 0 & 0 & 0 \\ 0 & \Psi_x & \Psi_y & \frac{1}{2}\Psi_z & 0 \\ 0 & 0 & 0 & 0 & \Psi_x \\ 0 & -\frac{1}{2}\Psi_{z,x} & -\frac{1}{2}\Psi_{z,y} & 0 & 0 \end{bmatrix}$$

$$A_9 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \Psi_y & -\frac{1}{2}X1 & -\frac{1}{2}X2 & 0 & 0 \\ 0 & -\frac{1}{2}X8 & -\frac{1}{2}X9 & \Psi_x & \Psi_y \end{bmatrix}$$

and :

$$\theta^T = (\Psi_x, \Psi_y, \Psi_z, X5, X6, X7, X8, X9, X10, \Psi_{x,y}, \Psi_{y,y}, \Psi_{z,x}, \Psi_{z,y}, \Psi_{x,x}, \Psi_{x,y})^T$$

Notice that the above organization is arbitrary. There is a chance that a better one can be found, from the point of view of generating more compact matrices, the final A.G.N being the same, though.

Deriving A.G.N directly we find :

$$A.G.N = [A_9, A_{10}]$$

where :

$$A_9 = \begin{bmatrix} \Psi_x \phi_{l,y} + \xi \Psi_y \phi_l & -\xi \Psi_x \phi_l + \Psi_y \phi_{l,y} - \frac{1}{2} K \Psi_z \phi_l & K \Psi_y \phi_l + \frac{1}{2} \Psi_z \phi_{l,y} \\ \Psi_x \phi_{l,x} - \frac{1}{2} \eta \Psi_z \phi_l & \Psi_y (\phi_{l,x} - \xi \phi_l) & \eta \Psi_x \phi_l + \frac{1}{2} \Psi_z \phi_{l,x} \\ -\frac{1}{2} (\Psi_{z,x} \phi_{l,y} + \xi \Psi_{z,y} \phi_l) & \frac{1}{2} (\xi \Psi_{z,x} \phi_l - \Psi_{z,y} \phi_{l,y}) & -\frac{1}{2} K \Psi_{z,y} \phi_l \\ -\frac{1}{2} \Psi_{z,x} \phi_{l,x} & -\frac{1}{2} \Psi_{z,y} (\phi_{l,x} - \xi \phi_l) & -\frac{1}{2} \eta \Psi_{z,x} \phi_l \end{bmatrix}$$

$$A_{10} = \begin{bmatrix} X5 \phi_l & X6 \phi_l - \frac{1}{2} \Psi_z \phi_l & \frac{1}{2} X7 \phi_l \\ X8 \phi_l - \frac{1}{2} \Psi_z \phi_l & X9 \phi_l & \frac{1}{2} X10 \phi_l \\ \Psi_{x,y} \phi_l + \Psi_x \phi_{l,y} & \Psi_y \phi_{l,y} + \Psi_{y,y} \phi_l & -\frac{1}{2} (X5 \phi_{l,x} + X6 \phi_{l,y}) \\ \Psi_{x,x} \phi_l + \Psi_x \phi_{l,x} & \Psi_{y,x} \phi_l + \Psi_y (\phi_{l,x} - 2\xi \Psi_y \phi_l) & -\frac{1}{2} (X8 \phi_{l,x} + X9 \phi_{l,y}) \end{bmatrix}$$

### E.2.5 Stress matrix in bending

Proceeding as before we find :

$$S = \begin{bmatrix} N_{11} & N_{12} & M_{11} & M_{12} \\ N_{12} & N_{22} & M_{12} & M_{22} \\ M_{11} & M_{12} & P_{11} & P_{12} \\ M_{12} & M_{22} & P_{12} & P_{22} \end{bmatrix}$$

then :

$$K_\sigma = \begin{bmatrix} N_\sigma & M_\sigma \\ M_\sigma & P_\sigma \end{bmatrix}$$

where :

$$(N_\sigma)_{11} = \eta^2 N_{11} \phi_i \phi_j$$

$$(N_\sigma)_{12} = \eta K N_{12} \phi_i \phi_j$$

$$(N_\sigma)_{13} = -\eta(N_{11} \phi_i \phi_{j,x} + N_{12} \phi_i \phi_{j,y})$$

$$(N_\sigma)_{22} = K^2 N_{22} \phi_i \phi_j$$

$$(N_\sigma)_{23} = -K(N_{12} \phi_i \phi_{j,x} + N_{22} \phi_i \phi_{j,y})$$

$$(N_\sigma)_{33} = N_{11} \phi_{i,x} \phi_{j,x} + N_{12}(\phi_{i,x} \phi_{j,y} + \phi_{i,y} \phi_{j,x}) + N_{22} \phi_{i,y} \phi_{j,y}$$

$M_\sigma$  and  $P_\sigma$  have the same form as  $N_\sigma$ , where  $N_{\alpha\beta}$  are substituted by  $M_{\alpha\beta}$  and  $P_{\alpha\beta}$  respectively.

### E.2.6 Stress matrix in transverse shear

This matrix is very sparse, so that we will give only the non zero elements in the upper triangle :

$$s_{17} = s_{28} = Q_1, \quad s_{39} = Q_1/2, \quad s_{14} = s_{25} = Q_2, \quad s_{36} = Q_2/2$$

$$s_{114} = s_{215} = R_1, \quad s_{712} = s_{813} = R_1/2, \quad s_{110} = s_{211} = R_2, \quad s_{412} = s_{513} = R_2/2$$

$$s_{22} = -2\xi R_1$$

Then :

$$K_\sigma = \begin{bmatrix} 0 & Q_\sigma \\ (Q_\sigma)^T & R_\sigma \end{bmatrix}$$

where :

$$(Q_\sigma)_{11} = Q_1 \phi_{i,x} \phi_j + Q_2 \phi_{i,y} \phi_j$$

$$(Q_\sigma)_{12} = \xi Q_2 \phi_i \phi_j$$

$$(Q_\sigma)_{13} = -\frac{1}{2} [\eta Q_1 \phi_i \phi_j + R_1 \phi_i \phi_j + R_2(\phi_{i,y} \phi_{j0} - \xi \phi_i \phi_j)]$$

$$(\mathbf{Q}_\sigma)_{21} = -\xi Q_2 \phi_i \phi_j$$

$$(\mathbf{Q}_\sigma)_{22} = Q_1(\phi_{i,x} \phi_j - \xi \phi_i \phi_j) + Q_2 \phi_{i,y} \phi_j$$

$$(\mathbf{Q}_\sigma)_{23} = \frac{1}{2} [KQ_2 \phi_i \phi_j + R_1(\phi_{i,x} \phi_{j,y} - \xi \phi_i \phi_{j,y}) + R_2(\xi \phi_i \phi_{j,x} - \phi_{i,y} \phi_{j,y})]$$

$$(\mathbf{Q}_\sigma)_{31} = \eta Q_1 \phi_i \phi_j$$

$$(\mathbf{Q}_\sigma)_{32} = KQ_2 \phi_i \phi_j$$

$$(\mathbf{Q}_\sigma)_{33} = \frac{1}{2} (Q_1 \phi_{i,x} \phi_j + Q_2 \phi_{i,y} \phi_j - \eta R_1 \phi_i \phi_{j,x} - KR_2 \phi_i \phi_{j,y})$$

$$(\mathbf{R}_\sigma)_{11} = R_1(\phi_{i,x} \phi_j + \phi_i \phi_{j,x}) + R_2(\phi_{i,y} \phi_j + \phi_i \phi_{j,y})$$

$$(\mathbf{R}_\sigma)_{12} = 0$$

$$(\mathbf{R}_\sigma)_{13} = -\frac{1}{2} Q_1 \phi_i \phi_j$$

$$(\mathbf{R}_\sigma)_{22} = R_1(\phi_{i,x} \phi_j + \phi_i \phi_{j,x} - 2\xi \phi_i \phi_j) + R_2(\phi_{i,y} \phi_j + \phi_i \phi_{j,y})$$

$$(\mathbf{R}_\sigma)_{23} = -\frac{1}{2} Q_2 \phi_i \phi_j$$

$$(\mathbf{R}_\sigma)_{33} = 0$$

$\mathbf{0}$  is a (3x3) matrix of zeros.

### E.2.7 Transverse normal matrices

These terms will usually be reduced integrated and, for this purpose, the corresponding material stiffness matrix is very sparse, as seen in paragraph 3.3. As a consequence, it was decided to derive the contributions to the direct and tangent stiffness matrices in explicit form.

### Linear direct stiffness matrix

$$\mathbf{B}_0^T \mathbf{D} \mathbf{B}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & a_{1l} \\ 0 & 0 & 0 & 0 & 0 & a_{2l} \\ 0 & 0 & 0 & 0 & 0 & a_3 \\ 0 & 0 & 0 & 0 & 0 & a_{4l} \\ 0 & 0 & 0 & 0 & 0 & a_{5l} \\ a_{1j} & a_{2j} & a_3 & a_{4j} & a_{5j} & a_6 \end{bmatrix}$$

### Direct stiffness matrix ( including linear terms )

$$\bar{\mathbf{B}} \mathbf{H} \tilde{\mathbf{B}} = \begin{bmatrix} \mathbf{0} & \mathbf{D}_1 \\ \mathbf{D}_2 & \mathbf{D}_3 \end{bmatrix}$$

where  $\mathbf{0}$  is a (3x3) matrix of zeros and :

$$\mathbf{D}_1 = \begin{bmatrix} \frac{1}{2} (a_{1l} + b_1) \Psi_x & \frac{1}{2} (a_{1l} + b_1) \Psi_y & (a_{1l} + b_1) \\ \frac{1}{2} (a_{2l} + b_2) \Psi_x & \frac{1}{2} (a_{2l} + b_2) \Psi_y & (a_{2l} + b_2) \\ \frac{1}{2} (a_3 + b_{3l}) \Psi_x & \frac{1}{2} (a_3 + b_{3l}) \Psi_y & (a_3 + b_{3l}) \end{bmatrix}$$

$$\mathbf{D}_2 = \begin{bmatrix} (a_{1j} + \frac{1}{2} b_1) \Psi_x & (a_{2j} + \frac{1}{2} b_2) \Psi_x & (a_3 + \frac{1}{2} b_{3j}) \Psi_x \\ (a_{1j} + \frac{1}{2} b_1) \Psi_y & (a_{2j} + \frac{1}{2} b_2) \Psi_y & (a_3 + \frac{1}{2} b_{3j}) \Psi_y \\ (a_{1j} + \frac{1}{2} b_1) & (a_{2j} + \frac{1}{2} b_2) & (a_3 + \frac{1}{2} b_{3j}) \end{bmatrix}$$

$$\mathbf{D}_3 = \begin{bmatrix} c_{4j} + \frac{1}{2} (c_{4l} + e_4) & c_{5j} + \frac{1}{2} (d_{4l} + f_4) & a_{4l} + c_6 + \frac{1}{2} \Psi_x b_{6l} \\ d_{4j} + \frac{1}{2} (c_{5l} + e_5) & d_{5j} + \frac{1}{2} (d_{5l} + f_5) & a_{5l} + d_6 + \frac{1}{2} \Psi_y b_{6l} \\ a_{4j} + \frac{1}{2} (c_6 + \Psi_x b_{6j}) & a_{5j} + \frac{1}{2} (d_6 + \Psi_y b_{6j}) & a_6 + b_{6j} + \frac{1}{2} b_{6l} \end{bmatrix}$$

### Tangent stiffness matrix

$$\bar{\mathbf{B}} \cdot \mathbf{H} \cdot \bar{\mathbf{B}} = \begin{bmatrix} \mathbf{0} & \mathbf{T}_1 \\ \mathbf{T}_1^T & \mathbf{T}_2 \end{bmatrix}$$

where  $\mathbf{0}$  is a (3x3) matrix of zeros and :

$$\mathbf{T}_1 = \begin{bmatrix} (a_{1l} + b_1)\Psi_x & (a_{1l} + b_1)\Psi_y & (a_{1l} + b_1) \\ (a_{2l} + b_2)\Psi_x & (a_{2l} + b_2)\Psi_y & (a_{2l} + b_2) \\ (a_{3l} + b_3)\Psi_x & (a_{3l} + b_3)\Psi_y & (a_{3l} + b_3) \end{bmatrix}$$

$$\mathbf{T}_2 = \begin{bmatrix} c_{4l} + c_{4j} + e_4 + g & c_{5l} + d_{4l} + f_4 & a_{4l} + c_6 + \Psi_x b_{6l} \\ d_{4j} + c_{5j} + e_5 & d_{5l} + d_{5j} + f_5 + g & a_{5l} + d_6 + \Psi_y b_{6l} \\ a_{4j} + c_6 + \Psi_x b_{6j} & a_{5j} + d_6 + \Psi_y b_{6j} & a_6 + b_{6l} + b_{6j} \end{bmatrix}$$

where :

$$a_{1l} = \xi(a_{23} + K b_{23})\phi_l\phi_j + (a_{13} + \eta b_{13})\phi_{l,x}\phi_j + (a_{36} + \eta b_{36})\phi_{l,y}\phi_j$$

$$a_{2l} = -2\xi(a_{36} + K b_{36})\phi_l\phi_j + (a_{36} + K b_{36})\phi_{l,x}\phi_j + (a_{23} + K b_{23})\phi_{l,y}\phi_j$$

$$a_3 = [\eta(a_{13} + K b_{13}) + K(a_{23} + K b_{23})]\phi_l\phi_j$$

$$a_{4l} = \xi(b_{23} + K d_{23})\phi_l\phi_j + (b_{13} + \eta d_{13})\phi_{l,x}\phi_j + (b_{36} + \eta d_{36})\phi_{l,y}\phi_j$$

$$a_{5l} = -2\xi(b_{36} + K d_{36})\phi_l\phi_j + (b_{36} + K d_{36})\phi_{l,x}\phi_j + (b_{23} + K d_{23})\phi_{l,y}\phi_j$$

$$a_6 = [a_{33} + 2\eta(b_{13} + K d_{13}) + 2K(b_{23} + K d_{23})]\phi_l\phi_j$$

$$n_1 = a_{13}X1 + a_{36}X2 + b_{13}X3 + b_{36}X4$$

$$n_2 = a_{23}X2 + a_{36}X1 + b_{23}X4 + b_{36}X3$$

$$n_3 = b_{13}X1 + b_{36}X2 + d_{13}X3 + d_{36}X4$$

$$n_4 = b_{23}X2 + b_{36}X1 + d_{23}X4 + d_{36}X3$$

$$b_1 = -\eta n_1 \phi_i \phi_j \quad , \quad b_2 = -K n_2 \phi_i \phi_j$$

$$b_{3j} = n_1 \phi_i \phi_{j,x} + n_2 \phi_i \phi_{j,y} \quad , \quad b_{6i} = n_3 \phi_i \phi_{j,x} + n_4 \phi_i \phi_{j,y}$$

$$c_{1i} = \Psi_x a_{1i} \quad , \quad c_{2i} = \Psi_x a_{2i} \quad , \quad c_3 = \Psi_x a_3 \quad , \quad c_{4i} = \Psi_x a_{4i} \quad , \quad c_{5i} = \Psi_x a_{5i}$$

$$d_{1i} = \Psi_y a_{1i} \quad , \quad d_{2i} = \Psi_y a_{2i} \quad , \quad d_3 = \Psi_y a_3 \quad , \quad d_{4i} = \Psi_y a_{4i} \quad , \quad d_{5i} = \Psi_y a_{5i}$$

$$c_6 = \{\Psi_x [a_{33} + \eta(b_{13} + K d_{13}) + K(b_{23} + K d_{23})] - \eta n_3\} \phi_i \phi_j$$

$$d_6 = \{\Psi_y [a_{33} + \eta(b_{13} + K d_{13}) + K(b_{23} + K d_{23})] - K n_3\} \phi_i \phi_j$$

$$e_4 = \Psi_x (a_{33} \Psi_x - 2\eta n_3) \phi_i \phi_j \quad , \quad e_5 = \Psi_x (a_{33} \Psi_x \Psi_y - \eta \Psi_y n_3 - K \Psi_x n_4) \phi_i \phi_j$$

$$f_4 = e_5 \quad , \quad f_5 = \Psi_y (a_{33} \Psi_y - 2K n_4) \phi_i \phi_j$$

$$g = N_{33} \phi_i \phi_j$$

## *Appendix F. Linear strains and rotations*

In this appendix, we will determine the equations for the linear strains and rotations to be used in criterion for checking the validity of MRT calculations with respect to the order of magnitude assumptions. From Appendix C, we have already the linear strains and we can write :

$$e_{xx} = u_{,x} + \eta w + \theta^3 [\Psi_{x',x} + \eta(u_{,x} + \frac{1}{R} w + \Psi_z)] + (\theta^3)^2 \eta (\Psi_{x',x} + \frac{1}{R} \Psi_z)$$

$$e_{yy} = v_{,y} + \frac{1}{R} w + \xi u + \theta^3 [\Psi_{y',y} + \frac{1}{R} (v_{,y} + \frac{1}{R} w + \Psi_z) + \xi (\frac{1}{R} u + \Psi_x)] + \frac{1}{R} (\theta^3)^2 (\xi \Psi_x + \Psi_{y',y} + \frac{1}{R} \Psi_z)$$

$$2e_{xy} = u_{,y} + v_{,x} - 2\xi v + \theta^3 [\eta u_{,y} - 2\xi (\frac{1}{R} v + \Psi_y) + \frac{1}{R} v_{,x} + \Psi_{x',y} + \Psi_{y',x}] + (\theta^3)^2 [\frac{1}{R} \Psi_{y',x} + \eta (\Psi_{x',y} - 2\xi \Psi_y)]$$

$$e_{zz} = \Psi_z$$

$$2e_{yz} = w_{,y} - \frac{1}{R} v + \Psi_y + \theta^3 \Psi_{z',y}$$

$$2e_{xz} = w_{,x} + \Psi_x - \eta u + \theta^3 \Psi_{z',x}$$

The linear rotations can be obtained in similar way and the final equations are :

$$2\omega_{xy} = u_{,y} - v_{,x} + \theta^3 [\eta u_{,y} - \frac{1}{R} v_{,x} + \Psi_{x',y} + \Psi_{y',x}] + (\theta^3)^2 (\eta \Psi_{x',y} - \frac{1}{R} \Psi_{y',x})$$

$$2\omega_{yz} = w_{,y} - \frac{1}{R} v - \Psi_y + \theta^3 (\Psi_{z',y} - \frac{2}{R} \Psi_y)$$

$$2\omega_{xz} = w_{,x} - \Psi_x - \eta u + \theta^3 (\Psi_{z',x} - 2\eta \Psi_x)$$

## Appendix G. Identities and matrices for reference 30

### 1) Identities

$$\Phi_{\gamma\alpha} \Phi_{\beta}^{\gamma} = \sum_{n=0}^{\infty} \sum_{m=0}^n (\theta^3)^n \binom{(m)}{\gamma\alpha} \binom{(n-m)}{\beta}$$

$$\theta^3 \Phi_{\gamma\alpha} \Phi_{\beta}^{\gamma} = \sum_{n=0}^{\infty} \sum_{m=0}^n (\theta^3)^{n+1} \binom{(m)}{\gamma\alpha} \binom{(n-m)}{\beta} = \sum_{n=0}^{\infty} \sum_{m=0}^n (\theta^3)^n \binom{(m-1)}{\gamma\alpha} \binom{(n-m)}{\beta}$$

$$(\theta^3)^2 \Phi_{\gamma\alpha} \Phi_{\beta}^{\gamma} = \sum_{n=0}^{\infty} \sum_{m=0}^n (\theta^3)^{n+2} \binom{(m)}{\gamma\alpha} \binom{(n-m)}{\beta} = \sum_{n=0}^{\infty} \sum_{m=0}^n (\theta^3)^n \binom{(m-2)}{\gamma\alpha} \binom{(n-m)}{\beta}$$

$$\Phi_{\gamma\alpha}(u^{\gamma})_{,3} = \sum_{n=0}^{\infty} \sum_{m=0}^n (m+1) (\theta^3)^n \binom{(m)}{\gamma\alpha} u^{\gamma(m+1)}$$

$$\theta^3 \Phi_{\gamma\alpha}(u^{\gamma})_{,3} = \sum_{n=0}^{\infty} \sum_{m=0}^n (m+1) (\theta^3)^n \binom{(n-m-1)}{\gamma\alpha} u^{\gamma(m+1)}$$

$$(\theta^3)^2 \Phi_{\gamma\alpha}(u^{\gamma})_{,3} = \sum_{n=0}^{\infty} \sum_{m=0}^n (m+1) (\theta^3)^n \binom{(n-m-2)}{\gamma\alpha} u^{\gamma(m+1)}$$

$$u_{\lambda,3}(u^{\gamma})_{,3} = \sum_{n=0}^{\infty} \sum_{m=0}^n (m+1)(n-m+1) (\theta^3)^n u_{\lambda}^{(m+1)} u^{\gamma(n-m+1)}$$

$$\theta^3 u_{\lambda,3}(u^\alpha)_{,3} = \sum_{n=0}^{\infty} \sum_{m=0}^n (m+1)(n-m)(\theta^3)^n u_\lambda^{(m+1)} u^\alpha^{(n-m)}$$

$$(\theta^3)^2 u_{\lambda,3}(u^\alpha)_{,3} = \sum_{n=0}^{\infty} \sum_{m=0}^n (m+1)(n-m-1)(\theta^3)^n u_\lambda^{(m+1)} u^\alpha^{(n-m-1)}$$

## 2) Nonlinear direct stiffness matrix in bending

We first define :

$$X1 = u_{,x} + \eta w$$

$$X3 = v_{,x} - \xi v$$

$$X5 = w_{,x} - \eta u$$

$$X2 = u_{,y} - \xi v$$

$$X4 = v_{,y} + \xi u + Kw$$

$$X6 = w_{,y} - Kv$$

$$X7 = \Psi_{,xx} + \eta \Psi_{,z}$$

$$X9 = \Psi_{,yx} - \xi \Psi_{,z}$$

$$X11 = \Psi_{,xz} - \eta \Psi_{,z}$$

$$X8 = \Psi_{,xy} - \xi \Psi_{,y}$$

$$X10 = \Psi_{,yy} + \xi \Psi_{,x} + K\Psi_{,z}$$

$$X12 = \Psi_{,yz} - K\Psi_{,y}$$

Then :

$$\mathbf{A} = [ \mathbf{A}_1 \quad \mathbf{A}_2 ]$$

where :

$$\mathbf{A}_1 = \begin{bmatrix} \mathbf{A}_3 \\ \mathbf{0} \end{bmatrix} \quad , \quad \mathbf{A}_2 = \begin{bmatrix} \mathbf{0} \\ \mathbf{A}_3 \end{bmatrix}$$

and :

$$A_3 = \begin{bmatrix} X1 & 0 & X3 & 0 & X5 & 0 \\ 0 & X2 & 0 & X4 & 0 & X6 \\ X2 & X1 & X4 & X3 & X6 & X5 \\ X7 & 0 & X9 & 0 & X11 & 0 \\ 0 & X8 & 0 & X10 & 0 & X12 \\ X8 & X7 & X10 & X9 & X12 & X11 \end{bmatrix}$$

and  $\mathbf{0}$  is a (3x6) matrix of zeros, and  $\theta^T = (X1, X2, X3, X4, X5, X6, X7, X8, X9, X10, X11, X12)^T$ . Next we obtain :

$$G.N_l = \begin{bmatrix} G_1 & \mathbf{0} \\ \mathbf{0} & G_1 \end{bmatrix}$$

where :

$$G_1 = \begin{bmatrix} \phi_{l,x} & 0 & \eta\phi_l \\ \phi_{l,y} & -\xi\phi_l & 0 \\ 0 & \phi_{l,x} - \xi\phi_l & 0 \\ \xi\phi_l & \phi_{l,y} & K\phi_l \\ -\eta\phi_l & 0 & \phi_{l,x} \\ 0 & -K\phi_l & \phi_{l,y} \end{bmatrix}$$

and  $\mathbf{0}$  is a (6x3) matrix of zeros. Then :

$$A . G.N_l = [ A_4 \quad A_5 ]$$

where :

$$A_4 = \begin{bmatrix} A_6 \\ \mathbf{0} \end{bmatrix} , \quad A_5 = \begin{bmatrix} \mathbf{0} \\ A_6 \end{bmatrix}$$

and :

$$(A_6)_{11} = \phi_{l,x}X1 - \eta\phi_lX5$$

$$(A_6)_{21} = \phi_{l,y}X2 + \xi\phi_lX4$$

$$(A_6)_{31} = \phi_{l,x}X2 + \phi_{l,y}X1 + \xi\phi_lX3 - \eta\phi_lX6$$

$$(A_6)_{41} = \phi_{l,x}X7 - \eta\phi_lX11$$

$$(A_6)_{51} = \phi_{l,y}X8 + \xi\phi_lX10$$

$$(A_6)_{61} = \phi_{l,x}X8 + \phi_{l,y}X7 + \xi\phi_lX9 - \eta\phi_lX12$$

$$(A_6)_{12} = (\phi_{l,x} - \xi\phi_l)X3$$

$$(A_6)_{22} = -\xi\phi_lX2 + \phi_{l,y}X4 - K\phi_lX6$$

$$(A_6)_{32} = -\xi\phi_lX1 + (\phi_{l,x} - \xi\phi_l)X4 + \phi_{l,y}X3 - K\phi_lX5$$

$$(A_6)_{42} = (\phi_{l,x} - \xi\phi_l)X9$$

$$(A_6)_{52} = -\xi\phi_lX8 + \phi_{l,y}X10 - K\phi_lX12$$

$$(A_6)_{62} = -\xi\phi_lX7 + (\phi_{l,x} - \xi\phi_l)X10 + \phi_{l,y}X9 - K\phi_lX11$$

$$(A_6)_{13} = \eta\phi_lX1 + \phi_{l,x}X5$$

$$(A_6)_{23} = K\phi_lX4 + \phi_{l,y}X6$$

$$(A_6)_{33} = \eta\phi_lX2 + K\phi_lX3 + \phi_{l,x}X6 + \phi_{l,y}X5$$

$$(A_6)_{43} = \eta\phi_lX7 + \phi_{l,x}X11$$

$$(A_6)_{53} = K\phi_lX10 + \phi_{l,y}X12$$

$$(A_6)_{63} = \eta\phi_lX7 + K\phi_lX9 + \phi_{l,x}X12 + \phi_{l,y}X11$$

and  $\mathbf{0}$  is a  $(3 \times 3)$  matrix.

### 3) Tangent stiffness matrix in bending

$$\mathbf{S} = \begin{bmatrix} \mathbf{N} & \mathbf{M} \\ \mathbf{M} & \mathbf{P} \end{bmatrix}$$

where :

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{N}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{N}_1 \end{bmatrix}, \quad \mathbf{N}_1 = \begin{bmatrix} N_{11} & N_{12} \\ N_{12} & N_{22} \end{bmatrix}$$

and  $\mathbf{0}$  is  $(2 \times 2)$ .  $\mathbf{M}$  and  $\mathbf{P}$  have the same form as  $\mathbf{N}$  where  $N_{\alpha\beta}$  are substituted by  $M_{\alpha\beta}$  and  $P_{\alpha\beta}$  respectively. Then :

$$\mathbf{K}_\sigma = \begin{bmatrix} \mathbf{N}_\sigma & \mathbf{M}_\sigma \\ \mathbf{M}_\sigma & \mathbf{P}_\sigma \end{bmatrix}$$

where :

$$(\mathbf{N}_\sigma)_{11} = (\phi_{i,x}\phi_{j,x} + \eta^2\phi_i\phi_j)N_{11} + (\phi_{i,x}\phi_{j,y} + \phi_{i,y}\phi_{j,x})N_{12} + (\phi_{i,x}\phi_{j,x} + \xi^2\phi_i\phi_j)N_{22}$$

$$(\mathbf{N}_\sigma)_{12} = [\xi(\phi_i\phi_{j,x} - \phi_{i,x}\phi_j) + (\eta K - \xi^2)\phi_i\phi_j]N_{12} + \xi(\phi_i\phi_{j,y} - \phi_{i,y}\phi_j)N_{22}$$

$$(\mathbf{N}_\sigma)_{13} = \eta(\phi_{i,x}\phi_j - \phi_{i,x}\phi_{j,x})N_{11} + \eta(\phi_{i,y}\phi_j - \phi_{i,y}\phi_{j,y})N_{12} + \xi K\phi_i\phi_j N_{22}$$

$$(\mathbf{N}_\sigma)_{22} = [\phi_{i,x}\phi_{j,x} + \xi(\xi\phi_i\phi_j - \phi_{i,x}\phi_j - \phi_i\phi_{j,x})]N_{11} + [\phi_{i,x}\phi_{j,y} + \phi_{i,y}\phi_{j,x} - \xi(\phi_i\phi_{j,y} + \phi_{i,y}\phi_j)]N_{12} + [\phi_{i,y}\phi_{j,y} + (\xi^2 + K^2)\phi_i\phi_j]N_{22}$$

$$(\mathbf{N}_\sigma)_{23} = [-\xi(\eta + K)\phi_i\phi_j + K(\phi_{i,x}\phi_j - \phi_{i,x}\phi_{j,x})]N_{12} + K(\phi_{i,y}\phi_j - \phi_{i,y}\phi_{j,y})N_{22}$$

$$(\mathbf{N}_\sigma)_{33} = (\xi^2\phi_i\phi_j + \phi_{i,x}\phi_{j,x})N_{11} + (\phi_{i,x}\phi_{j,y} + \phi_{i,y}\phi_{j,x})N_{12} + (\phi_{i,y}\phi_{j,y} + K^2\phi_i\phi_j)N_{22}$$

$\mathbf{M}_\sigma$  and  $\mathbf{P}_\sigma$  have the same form as  $\mathbf{N}_\sigma$  where  $N_{\alpha\beta}$  are substituted by  $M_{\alpha\beta}$  and  $P_{\alpha\beta}$  respectively.

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