Moment Sequences and Their Applications

by

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(Abstract)

In this dissertation, we first present a unified treatment of compact moment problems, both the truncated and full moment cases. Second, we define the lower and upper functions $V_{\pm}(x_1, \cdots, x_n)$ on the convex hull of the curve $\Gamma_n = \{(t, \cdots, t^n) : \quad t \in [0,1] \}$ for each positive integer n. Explicit formulas of these functions are derived and applied to the study of the subnormal completion problem in operator theory. Last, we show that certain power functions are the building blocks of completely positive functions; by our definition, these functions are the continuous functions on the interval [0, 1] that map each Hausdorff moment sequence of a probability measure into another one.

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INTRODUCTION

Moment theory originated from probability investigation in the work of P. L. Chebyshev in 1874. Since then, it has found significant applications to other areas. Various connections with many branches of mathematics were established, such as with continued fraction theory in P. Chebyshev [36] [37] [35], A. Markov [115] [114], K. Posse [132] and T. Stieltjes [171] [172], with convex analysis in F. Riesz [141], with function theory in C. Caratheodory [26], Carleman [27], H. Hamburger [70], F. Hausdorff [72], E. Hellinger [75], G. Herglotz [76], M. Krein and P. Rekhtman [102], R. Nevanlinna [124], G. Pick [129] [128], F. Riesz [141] and I. Schur [156] [157], with functional analysis in M. Riesz [144] [143] and E. Haviland [73], and with operator theory in M. Stone [173]. Modern authors have contributed a great amount of work to addressing the interplay between these branches of mathematics, in particular, several mathematicians have demonstrated the significant interaction between functional analysis and moment theory. For specifics, consult Akhiezer [1] and [2], Akhiezer and Krein [3], Ando [5], Atzmon [7], Bellman and Blackwell [9], Berberian [10], Berezanskii [11], Berg [12], Berg and Christensen [13], Berg, Christensen and Jensen [14], Berg and Maserick [15], Berg and Thill [16], Bergman and Martin [17], Borwein [21], Buchwalter and Cassier [23] [24], Burgess [25], Carleson [28], Carling [29], Cassier [33] [30] [32] [31], Challifour and Slinker [34], Contantinescu [38], Cotlar [40], Cotlar and Sadosky [41], Coxeter [42], Cumakin [43], Curto [44], Curto and Fialkow [46] [48] [45] [47], Curto and Putinar [50] [49], Devinatz [53], deLaudenfels [52], Ersov [59], Fialkow [61], Fuchs [62], Gaer [63], Galstjan [64], Gavruta and Paunescu [65], Gellar and Wallen [66], Geronimus [67], Hamburger [71], Hedge [74], Hildebrandt and Schoenberg [79], Hunter [80], Ilmushkin [81], Ilmushkin and Turitsyn [82], Inin [83], Jorgensen [84], Kantorovitch [86], Karlin and Shapley [87] [88], Karlin and Studden [89], Katsnelson [90], Kemperman [92], Kocubei [93], Korkina [94], Kovalishina and Potapov [95], Krein [98] [96] [97], Krein and Kranoselskii [99], Krein and Nudelman [101], Krein and Rekhtman [102], Lambert [103], Landau [104], Leininger [105], Leviatan [106], Leviatan and Ramanujan [107], Livshits [109], Loskarev [110], Luks [111] [112], Mardia and Thompson [113], Mccullough and Paulsen [116], McGregor [117], Naimark [120], Nakonecnii [121], Natanson [122], Neustadt [123], Nudelman [125], Ovcharenko [127] [126], Putinar [135] [134], Ramanujan [138] [137], Rukshin [147], Sakai [148], Saks [149], Sarason [151] [150], Schaefer [152], Schmudgen [153], Schoenberg [155], Sebestyen [160] [159] [158], Sherman [162], Shields [163], Shohat and Tamarkin [164] Shonkwiler [165], Skibinsky [168] [167], Smilansky [169], Stampfli [170], Sz.-Nagy [174], Toeplitz [175], Upreti [176], Us [177], Wall [178], Weyl [179] and Whitford [180]. (No implication of completeness of literature should be assumed.)

In this thesis, we will explore moment problems by using the method of extending positive linear functionals, that was initiated in special cases by M. Riesz in [144] and by Haviland in [73] and was generalized by Akhiezer in [1, p69]. In Chapter 1, our exploration will unify the treatment of compact moment problems. In Chapter 2, we will introduce the convex analysis approach of F. Riesz and establish the geometry for Hausdorff moment problems, this approach is of particular interest to operator theory and the study of subnormal operators. We then apply the results to the subnormal completion problem, that was first touched upon by J. Stampfli in [170, p372] and was later solved by R. Curto and L. Fialkow in [45, Theorem 4.1 p622, Theorem 4.3 p623]; we note that we give a solution from an entirely different perspective. In the last chapter, we consider a class of continuous functions on the interval [0, 1] that preserve Hausdorff moment sequences. A general form of such functions is given by using a technique of Bernstein [18].

The following notations, definitions and theorems will be used.

We shall use $\mathbf{R}$ and $\mathbf{C}$ to denote the sets of all real and complex numbers, respectively. The real and imaginary parts of a complex number are referred to by $\Re$ and $\Im$. The symbol $\mathbf{D}$ is used for the open unit disk in the complex plane.

A binary relation $\prec$ on a set S is called a *partial order* if for $a, b, c \in S$, the following three statements hold.

(i) $a \prec b$ and $b \prec c$ implies $a \prec c$.

(ii) $a \prec a$.

(iii) $a \prec b$ and $b \prec a$ implies $a = b$.

Let $(S, \prec)$ be a partially ordered set. A subset E of S is said to be *totally ordered* if for all $a, b \in E$, we have either $a \prec b$ or $b \prec a$. An element B of S is an *upper bound* of a subset E if $a \prec B$ for all $a \in E$. An element M of S is *maximal* if $M \prec a$ for some $a \in S$ implies $M = a$.

Theorem 0.1 (Zorn's Lemma) *If any totally ordered subset of a nonempty partially ordered set has an upper bound in it, then it has a maximal element.*

This fact can be found in standard abstract algebra or general topology text books. For reader's convenience, consult N. Dunford and J. T. Schwartz [55, p6] or J. Kelly and T. Srinivasan [91, p7].

Let K be a topological space, and we denote the real (complex) vector space of all real(complex)-valued continuous functions on K by $\mathbf{C}_r(K)$ (by $\mathbf{C}(K)$). The notation $M(K)$ is used for the Banach space of all complex regular Borel measures on a compact Hausdorff space K. The set of all non-negative continuous functions on K will be denote by $\mathbf{C}_+(K)$. A linear map from $\mathbf{C}_r(K)$ or $\mathbf{C}(K)$ into $\mathbf{R}$ or $\mathbf{C}$ is called a *positive linear functional* if it maps $\mathbf{C}_+(K)$ into the set of all non-negative numbers. Positive linear functionals on $\mathbf{C}(K)$ may be represented or recovered by positive regular Borel measures on K. F. Riesz [140] [141] gave the first such representation theorem and its proof for K being a compact interval in $\mathbf{R}$. J. Radon [136, p1333] extended the theorem onto a compact set in $\mathbf{R}^n$ for $n \geq 2$. S. Kakutani [85, p1009] generalized the theorem further for any compact Hausdorff space K. For related representation theory, consult E. Hewitt [77, pp161-168], N. Dunford and J. Schwarz [55, pp132-155, pp380-382], S. Saks [149, pp43-52, Theorem 2 p326], I. Segal and

R. Kunze [161, p129], J. Kelly and T. Srinivasan [91, p77] and W. Rudin [146, Theorem 2.14, Theorem 2.17].

Theorem 0.2 (Kakutani – F. Riesz Representation Theorem) *Let ϕ be a positive linear functional on $\mathbf{C}(K)$ for some compact Hausdorff topological space K. Then there exists a unique positive* regular Borel measure *μ on K such that*

$$\phi(f) = \int_K f \, d\mu \tag{0.1}$$

for all $f \in \mathbf{C}(K)$.

Remark. If K is only a completely regular topological space, Hewitt showed that there exist a positive *Baire measure* μ so that (0.1) holds in [77, pp161-168]. It is unknown whether or not we can hope for such a *regular Borel measure* to exist.

A non-negative valued function ρ on a vector space X is called a *seminorm* on X if the following hold

(a) $\rho(x + y) \leq \rho(x) + \rho(y)$

(b) $\rho(\alpha x) = |\alpha| \, \rho(x)$

for all $x, y \in X$ and $\alpha \in \mathbf{R}$ (or $\mathbf{C}$).

A topological vector space X is said to be *locally convex* if there is a set $\mathcal{P}$ of seminorms such that:

(a) If $\rho(x) = 0$ for some $x \in X$ and all $\rho \in \mathcal{P}$, then $x = 0$.

(b) Any neighborhood of the zero vector in X contains a open subset of the form

$$\{ \, x \in X : \rho_1(x) < 1/m, \cdots, \rho_n(x) < 1/m \, \}$$

for some positive integer m and seminorms $\rho_1, \cdots, \rho_n \in \mathcal{P}$.

A subset V of a vector space X is called *convex* if $tv_1 + (1-t)v_2 \in V$ for all $v_1, v_2 \in V$ and $0 \leq t \leq 1$. Let S be a subset of a vector space X. The smallest convex set containing

S (the intersection of all convex sets that contain S) is called the *convex hull* of S, denoted by co(S). It is easy to verify that

$$\text{co}(S) = \{\sum_{j=1}^{m} \alpha_j s_j : \ s_j \in S, \ \alpha_j > 0 \text{ for } j = 1, \cdots, m, \ \sum_{1}^{m} \alpha_j = 1, \ m = 1, 2, \cdots\}.$$

Theorem 0.3 (Strict Separation Theorem) *Let X be a locally convex topological vector space, and A, B disjoint, nonempty, convex subsets of X. If A is compact and B is closed, then there exists a continuous linear functional ϕ on X such that*

$$\sup \{\Re\, \phi(a) : \ a \in A\} < \inf \{\Re\, \phi(b) : \ b \in B\}.$$

This can be found in W. Rudin's book [145, p58] or standard text books on topological vector spaces.

Let X be a topological vector space. It is called *separable* if is has a countable dense subset. The *dual space* of X, denoted by X^*, is the vector space of all continuous linear functionals on X.

Theorem 0.4 (Banach – Alaoglu Theorem) *Let V be a neighborhood of X, and*

$$K = \{\Lambda \in X^* : \ |\Lambda(x)| \leq 1 \textit{ for } x \in V\}.$$

Then K is weak star compact.

The theorem is due to Banach [8] and Alaoglu [4]. For a proof, consult Rudin [145, p66]. The special version when $X = \mathbf{C}(K)$ and $X^* = M(K)$ will be used to prove the generalized F. Riesz theorem on moment theory and convex analysis in Chapter 2.

Theorem 0.5 (Helly's Selection Theorem) *Let V be a neighborhood of the origin in a* separable *topological vector space X, and $\{\Lambda_n\}_1^\infty$ a sequence in the dual space X^* such that*

$$|\Lambda(x)| \leq 1$$

for $x \in V$ and all positive integers n. There exists a subsequence $\{\Lambda_{n_j}\}_1^\infty$ and a continuous linear functional $\Lambda \in X^*$ such that

$$\Lambda(x) = \lim_{j \to \infty} \Lambda_{n_j}(x)$$

for $x \in X$.

When $X = \mathbf{C}[a,b]$ and $X^* = BV[a,b]$, the theorem is called the First Theorem of Helly. Here $BV[a,b]$ is the Banach space of all Lebesgue measurable functions of bounded variation on the finite interval $[a,b]$ that can be identified with $M([a,b])$. We will only use this special case of the theorem in Chapter 3. For a proof, consult Rudin [145, p68].

Theorem 0.6 (Sierpinski Theorem) *Let ϕ be a convex function on an open G set in $\mathbf{R}^n$ for some positive integer n; that is,*

$$\phi(\frac{\mathbf{x}+\mathbf{y}}{2}) \leq \frac{\phi(\mathbf{x})+\phi(\mathbf{y})}{2}$$

for $\mathbf{x}$, $\mathbf{y}$ and $\frac{\mathbf{x}+\mathbf{y}}{2}$ in G. Then ϕ is continuous.

The theorem belongs to Sierpinski [166] when G is an interval in $\mathbf{R}$. For a proof, consult Rudin [146, p63] (that proof applies to the multi-dimensional case).

Let $\mathcal{H}$ be a separable Hilbert space with inner product $<,>$. (We shall only consider such Hilbert spaces in the thesis.) A continuous linear map T from $\mathcal{H}$ into itself is called an *operator.* Its *conjugate* operator, denoted by T^*, is defined by $< x, Ty > = < T^*x, y >$ for all $x, y \in \mathcal{H}$. It is called a *contraction* or is said to be *contractive* if its norm is less than or equal to 1. A closed subspace $\mathcal{M}$ of $\mathcal{H}$ is called *invariant* for T if $Tm \in \mathcal{M}$ for all vectors $m \in \mathcal{M}$. In this case, we will also say that T *leaves $\mathcal{M}$ invariant.*

Let T be an operator on a Hilbert space $\mathcal{H}$. Then T is called *normal* if for any $x \in \mathcal{H}$, $T^*(Tx) = T(T^*x)$. If there exists a Hilbert space $\tilde{\mathcal{H}}$ containing $\mathcal{H}$ as a subspace and a normal operator $\tilde{T}$ on $\tilde{\mathcal{H}}$ such that $\tilde{T}$ leaves $\mathcal{H}$ invariant and equals T on $\mathcal{H}$, then we say T is a *subnormal* operator. Fix an orthonormal basis $\{\mathbf{e}_n\}_0^\infty$ of $\mathcal{H}$. If $T\mathbf{e}_n = \alpha_n \mathbf{e}_{n+1}$ for each $n = 0, 1, \cdots$, we shall say T is a *weighted shift (operator)* with the *weight sequence* $\{\alpha_n\}_0^\infty$.

A finite sequence is said to have a *(contractive) subnormal completion* if it can be extended into a weight sequence of some (contractive) subnormal weighted shift. The concept of subnormal completion was introduced by J. Stampfli in [170, p372]. The subnormal completion problem was solved by R. Curto and Fialkow in [45, Theorem 4.1 p622, Theorem 4.3 p623]. We will give another solution, that is to give necessary and sufficient conditions for a finite sequence to have a contractive subnormal completion.

For general references on functional analysis and operator theory, consult N. Dunford and J. Schwartz [55] [56], P. Halmos [68] and F. Riesz and B. Sz.-Nagy [142]. For normal operator theory, consult P. Halmos [68, pp 926-936]. For subnormal operator theory, consult J. Conway [39]. For weighted shift operator theory, consult A. Shields [163].

Let $\{f_k(x)\}_0^\infty$ be a sequence of functions in the vector space $\mathbf{C}(K)$ for some locally compact Hausdorff topological space K. A sequence $\{\mu_n\}_0^\infty$ of complex numbers is called a *positive moment sequence* with respect to the given sequence $\{f_k(x)\}_0^\infty$ of functions, if there exists a positive regular Borel measure μ (may not be finite) on K so that

$$\mu_n = \int_K f_n(x)\, d\mu(x)$$

for all integers $n = 0, 1, 2, \cdots$. We shall refer a positive moment sequence simply as a *moment sequence*. A *truncated moment sequence* is a finite sequence of complex numbers that can be extended into a *full* moment sequence, that is, a moment sequence. In this thesis, the *(truncated) moment problem* is to give necessary and sufficient conditions that determine whether a given (finite) sequence is a (truncated) moment sequence. Some authors study in addition certain extremal functions. For example, given a finite sequence $\{\mu_k\}_0^n$ of positive numbers for some positive integer n and a continuous function $f(x)$ on the interval $[-1, 1]$, A. Markov considered the extremal functions $\sup\{\int_{-1}^x f(t)\, d\mu(t) : \mu \in \mathcal{M}_n\}$ and $\inf\{\int_{-1}^x f(t)\, d\mu(t) : \mu \in \mathcal{M}_n\}$, where $\mathcal{M}_n$ consists of measures ν such that $\mu_k = \int_{-1}^1 x^k\, d\nu(x)$ for $k = 0, \cdots, n$. We will not investigate such extremal problems at all.

If the set $\{f_j(x)\}$ of functions is closed under complex conjugation, i.e., there exists one-to-one map π either from the finite set $\{0, 1, \cdots, n\}$ of integers into itself or from the set of all non-negative integers into itself so that $\overline{f_k(x)} = f_{\pi(k)}(x)$ for each index k, then the corresponding moment sequence (problem) is called a *symmetric moment sequence (problem).*

Fix the sequence of functions $f_n(x) = x^n$ on the complex plane $\mathbf{C}$ for $n = 0, 1, \cdots$. For K being $\mathbf{R}$, $[0, \infty)$, $[0, 1]$ and the unit circle, the corresponding (truncated) moment sequences are called *(truncated) Hamburger moment sequences, (truncated) Stieltjes moment sequences, (truncated) Hausdorff moment sequences,(truncated) trigonometric moment sequences*, respectively.

The book by N. I. Akhiezer [1] is an excellent general reference for moment theory. Consult also N. Akhiezer and M. Krein [3], M. Krein and A. Nudelman [101], S. Karlin and W. Studden [89] and J. Shohat and J. Tamarkin [164].

The following theorem of A. Lambert [103, Theorem 3.1 p477] established the direct connection between subnormal theory and the classical moment theory. For related results, consult J. Bram [22] and M. Embry [58] or J. Conway [39, p31].

Theorem 0.7 (Lambert Theorem) *Let S be a contraction on a Hilbert space $\mathcal{H}$ such that its kernel $\{x \in \mathcal{H} : Sx = 0\}$ contains only the zero vector. Then S is subnormal if and only if for each unit vector $x \in \mathcal{H}$, the sequence $\{\|S^n x\|^2\}_0^\infty$ is a Hausdorff moment sequence.*

This theorem is a paraphrase of Theorem 3.1 of A. Lambert [103, p477] by using the following Berger's Theorem (consult P. Halmos [69, p117] and A. Shields [163, pp895-896] for the origin and a proof). Berger's Theorem reveals an intimate relation between the contractive subnormal completion problem and the truncated Hausdorff moment problem.

Lemma 0.8 (Berger's Theorem) *A necessary and sufficient condition for a given sequence $\{\alpha_n\}_0^\infty$ of* positive *numbers to be a weight sequence of some contractive subnormal weighted shift is that its Berger sequence*

$$1,\ \alpha_0^2,\ \alpha_0^2\alpha_1^2,\ \alpha_0^2\alpha_1^2\alpha_2^2,\ \alpha_0^2\alpha_1^2\alpha_2^2\alpha_3^2,\ \cdots$$

is a Hausdorff moment sequence.

If the sequence of functions is given by the doubly indexed sequence

$$f_{m,n}(z) = z^m \bar{z}^n, \quad z \in \mathbf{C}$$

for all non-negative integers m and n and K is a set in the complex plane, then the corresponding (truncated) moment (sequences) problems are called the *(truncated) K-moment (sequences) problems.*

The $\overline{\mathbf{D}}$-moment problems was solved by A. Atzmon [7, Theorem 2.1, p319]. Also consult J. McGregor [117, Theorem 1, p318]. A. Atzmon pointed out the connection between the invariant subspace problem and the $\overline{\mathbf{D}}$-moment problem, and gave the solution of the latter problem by using subnormal operator theory. G. Cassier [33] had a very important idea of considering a certain special cone and its positive linear functionals. That went one step farther than M. Riesz's method. Following Cassier's idea, K. Schmudgen [153] solved K-moment problem for a class of compact sets K's. For closely related work, consult C. Berg, J. Christensen and C. Jensen [14], C. Berg and P. Maserick [15], L. Hedge [74] and M. Putinar [135].

R. Curto and L. Fialkow [47] and L. Fialkow [61] took a different approach in their work on the truncated $\overline{\mathbf{D}}$-moment problem.

We shall focus mainly on the compact moment problem.

Compact moment problem: Fix a (finite) sequence $\{f_n(x)\}$ of complex-valued functions on a *compact* Hausdorff topological space K. When is a (finite) sequence $\{\mu_n\}$ of complex numbers a (truncated) moment sequence?

Answers to the compact moment problems and the symmetric moment problems will be given and be applied to (truncated) Hausdorff and trigonometric moment problems. The (truncated) $\overline{\mathbf{D}}$-moment problems will be briefly discussed. The solutions indicate that the structure of finite complex linear combinations of the functions $\{f_n(x)\}_0^\infty$ that have non-negative real parts on K plays a vital role in the study of the compact moment problem.

When the compact set K is contained in the real line $\mathbf{R}$ and $f_n(x) = x^n$ for $x \in K$ and $n = 0, 1, \cdots$, the corresponding moment problems can be solved through representation theorems of non-negative polynomials. In the one dimensional case, a non-negative polynomial on $\mathbf{R}$ can be written as the sum of two squares of polynomials. This fact can be proved by using induction on the degree of the polynomial and the *fundamental theorem of algebra.* There is no corresponding fundamental theorem of algebra in the several variable case. That is one of the major difficulties in dealing with the multi-dimensional moment problems.

In fact, non-negative polynomials in two variables can no longer be written as sums of squares of polynomials in general. The first non-constructive proof for the existence of such a polynomial was given by D. Hilbert in [78]. Explicit examples of such polynomials were discovered long after that. For examples, the polynomial $1 + x^2y^4 + x^4y^2 - 3x^2y^2$ was given by T. Motzkin in [119], the polynomial $1 + x_1^2x_2^2(x_1^2 + x_2^2 - 1)$ by C. Berg, J. Christensen and C. Jensen in [14] and the polynomial

$$200(x_1^3 - 4x_1)^2 + 200(x_2^3 - 4x_2)^2 + (x_2^2 - x_1^2)(x_1^2 + 2x_1)[x_1(x_1 - 2) + 2(x_2^2 - 4)]$$

by K. Schmudgen in [154].

Hilbert's 17th problem is concerned, in particular, with whether or not non-negative polynomials in several variables can be expressed as sums of squares of rational functions. E. Artin answered this affirmatively in [6]. Along this line of study, consult J. Bochnak and G. Efroymson [19], J. Dieudonne [54] and C. Procesi [133] beside C. Berg [12], G. Cassier [33] and M. Putinar [135].

It seems that very little is done on multi-dimensional non-compact moment problems. From the view point of applying M. Riesz's method, the difficulties come from two major factors.

1. *In the several variable case, it is much more complicated to represent non-negative polynomials in terms of squares of* polynomials *rather than rational functions, that is one of the subject matters of algebraic geometry, a rich branch of mathematics.*

2. *In the case where K is not compact in $\mathbf{R}^n$, we may not be able to control the growth of positive linear functionals on the set of functions $\{f_n(x)\}_0^\infty$. But in comparison with the previous obstacle, this one may be overcome by extending positive linear functionals onto a vector space different from $\mathbf{C}_r(K)$.*

In Chapter 2, we introduce the convex analysis approach of F. Riesz [141, p56] to the compact moment problems. Then combining the two methods together, we give the geometric characterizations of both the truncated Hausdorff moment problem and the full Hausdorff moment problem in terms of the lower and upper functions, by our definition, they represent the bottom and top surfaces of the convex solid in $\mathbf{R}^{n+1}$ generated by the *moment curve* $\{(f_0(x), \cdots, f_n(x)) : x \in K\}$.

In the author's paper [108, Theorem 2.8, Theorem 2.10, Corollary 2.11], several explicit formulas of the lower and upper functions were derived. The computation was lengthy and could not be extended to general cases easily. Professor Curto pointed out the connection between those formulas and certain Hankel matrices to me through the author's advisor — Professor Robert Olin. This subtle link leads the author to the discovery of all formulas of lower and upper functions by combining the functional analysis method of M. Riesz with the convex analysis method of F. Riesz.

The author would like to thank Professor Curto for the valuable suggestion.

In the last chapter, we will consider a natural class of functions in $\mathbf{C}_r([0,1])$, called completely positive, that map the set of certain Hausdorff moment sequences into itself.

The following natural question (inspired from reading S. Richter's paper [139, Theorem 1, pp152-154]) is the source of interest for this inquiry:

Let W be a subnormal operator on a Hilbert space having an orthonormal basis $\{e_n\}_0^\infty$ such that

$$We_n = a_n e_{n+1}, \quad n = 0, 1, \cdots,$$

where a_n is positive for each $n = 0, 1, \cdots$.

Fix a positive number p. Define the operator W_p on the Hilbert space by

$$W_p e_n = a_n^p e_{n+1}, \quad n = 0, 1, \cdots.$$

Is this operator W_p still subnormal?

Loosely the question is: What changes on the weight sequences preserve subnormality? It turns out that answering the question for W_p is the key to the resolution of the question in general. By design, completely positive functions correspond to those changes that preserve subnormality. We will establish the formula for completely positive functions.

Chapter 1

A UNIFIED APPROACH TO MOMENT PROBLEMS

In this chapter, we will first introduce some basic terminology and then give a complete algorithm for the solution of the compact moment problems, both the truncated and full moment problems. This algorithm is described using the method of extending positive linear functionals on a given cone, that was first devised specifically for the cone of all non-negative polynomials on the real axis by M. Riesz in [144] and was later explored for the study of multivariate moment problems by E. K. Haviland in [73]. N. I. Akhiezer [1, p69] extracted the key ingredients from this process and established an extension theorem for positive linear functionals on general vector spaces, that is similar to the Hahn-Banach theorem in Functional Analysis. In Section 1, we will prove the extension theorem and apply it to unify the treatment of compact moment problems. In Section 3, we will demonstrate the beauty of this method by solving the classical moment problems, such as Hausdorff moment problems and trigonometric moment problems, through the corresponding non-negative polynomial representation theorems obtained earlier in Section 2.

1.1 The Extension Theorem

Let X be a vector space (either over the real numbers $\mathbf{R}$ or the complex numbers $\mathbf{C}$). A subset $\mathcal{K}$ of X is called a *cone* if $ak_1 + bk_2 \in \mathcal{K}$ for all non-negative real numbers a and b and all $k_1, k_2 \in \mathcal{K}$; in addition, we require that if $k_1 + k_2 = 0$ and $k_1, k_2 \in \mathcal{K}$, then $k_1 = k_2 = 0$. We shall write $x_1 \leq x_2$ or $x_2 \geq x_1$ for $x_1, x_2 \in X$ if $x_2 - x_1 \in \mathcal{K}$. Now the binary relations $\leq$ and $\geq$ on X induced by a cone $\mathcal{K}$ are partial orders on X. We are mainly interested in the cone $\mathbf{C}_+(K)$ of all non-negative continuous functions on a compact Hausdorff space K.

A $\mathcal{K}$*-positive real (complex) linear functional* ϕ on a subspace $\mathcal{M}$ of X is a linear map from $\mathcal{M}$ into the real numbers $\mathbf{R}$ (the complex numbers $\mathbf{C}$) such that $\phi(m) \geq 0$ whenever $m \in \mathcal{K} \bigcap \mathcal{M}$. Such a functional will be simply called a *positive real (complex) linear functional* on K if $\mathcal{K}$ is the cone $\mathbf{C}_+(K)$.

For simplicity, we shall say the pair $(\Phi, \mathcal{N})$ is an *extension* of the given pair $(\phi, \mathcal{M})$ or Φ *extends* ϕ if the maps Φ, ϕ are linear functionals on the (either real or complex) vector spaces $\mathcal{N}$, $\mathcal{M}$, respectively, $\mathcal{M}$ is a subspace of $\mathcal{N}$ and the two linear functionals coincide on $\mathcal{M}$.

Our goal in this section is to establish the general form of extension theorems for $\mathcal{K}$-positive real linear functionals.

Theorem 1.1 (Extension Theorem for Positive Real Linear Functionals) *Let X be a real vector space, $\mathcal{M}$ a subspace of X and $\mathcal{K}$ a cone of X. Suppose for any vector $x_0 \in X$, we have*

$$(x_0 + \mathcal{K}) \bigcap \mathcal{M} \neq \emptyset \text{ and } (x_0 - \mathcal{K}) \bigcap \mathcal{M} \neq \emptyset. \tag{1.1}$$

Then any $\mathcal{K}$-positive real *linear functional ϕ on $\mathcal{M}$ can be extended into a $\mathcal{K}$-positive* real *linear functional Φ on the whole space X.*

Proof. Let $\mathcal{S}$ be the set of all pairs $(\psi, \mathcal{N})$ that ψ is a $\mathcal{K}$-positive real linear functional on $\mathcal{N}$ and extends ϕ. The partial order on $\mathcal{S}$ is defined as the binary relation of extension. We shall use Zorn's lemma 0.1 on the partially ordered set $\mathcal{S}$ and show that a maximal element of $\mathcal{S}$ corresponds to a desired extension.

By hypothesis, $(\phi, \mathcal{M}) \in \mathcal{S}$, so $\mathcal{S}$ is not empty. Let $(\psi_n, \mathcal{N}_n)$ be a totally ordered subset of $\mathcal{S}$. Then the extension ψ on the union $\bigcup \mathcal{N}_n$ of all linear functionals ψ_n can be defined naturally. So Zorn's lemma applies. Let $(\Phi, \mathcal{N})$ be a maximal element of $\mathcal{S}$. It suffices to show that $\mathcal{N} = X$.

Suppose not, choose $x_0 \in X \setminus \mathcal{N}$, we want to construct a $\mathcal{K}$-positive real linear functional ψ on the subspace $\tilde{\mathcal{N}}$ generated by $\mathcal{N}$ and the vector x_0 that extends Φ. This contradicts the maximality of the pair $(\Phi, \mathcal{N})$ in $\mathcal{S}$. Thus the proof is complete.

Construction of ψ:

Let p be the supremum of $\Phi(y)$ over the set of vectors $y \in (x_0 - \mathcal{K}) \bigcap \mathcal{N} \supset (x_0 - \mathcal{K}) \bigcap \mathcal{M} \neq \emptyset$, and q the infimum of $\Phi(y)$ over the other set of vectors $y \in (x_0 + \mathcal{K}) \bigcap \mathcal{N} \supset (x_0 + \mathcal{K}) \bigcap \mathcal{M} \neq \emptyset$. Let $x_0 - k_1 = n_1, x_0 + k_2 = n_2$ for $k_1, k_2 \in \mathcal{K}$ and $n_1, , n_2 \in \mathcal{N}$. We have $\Phi(n_2 - n_1) = \Phi(k_2 + k_1) \geq 0$ since Φ is $\mathcal{K}$-positive. Thus $-\infty < p \leq q < +\infty$. Let r_0 be a number in the interval $[p, q]$. Define $\psi(n + ax_0) = \Phi(n) + ar_0$ for $n \in \mathcal{N}$ and $a \in \mathbf{R}$, it remains to check that ψ is $\mathcal{K}$-positive since it is clearly a real linear functional on the subspace $\tilde{\mathcal{N}}$.

Let $k = n + ax_0$ belong to the cone $\mathcal{K}$ for some real number a. If $a = 0$, then $\psi(k) = \Phi(k) \geq 0$ since Φ is $\mathcal{K}$-positive. If $a < 0$, then

$$\begin{aligned} \psi(k) &= \Phi(k - ax_0) + ar_0 \\ &= (-a)\Phi(x_0 + \frac{k}{-a}) + ar_0 \\ &\geq (-a)r_0 + ar_0 \\ &= 0. \end{aligned}$$

Similarly if $a > 0$, we also have $\psi(k) \geq 0$. This establishes the $\mathcal{K}$-positivity of ψ on $\tilde{\mathcal{N}}$. $\square$

Remark. This theorem and its proof is from Akhiezer [1, Theorem 2.6.2 pp69-70].

Notice that if X is the n-dimensional Euclidean space $\mathbf{R}^n$ and $\mathcal{K}$ is the cone generated by the vector $(1, 0, \cdots, 0)$, then the only subspace $\mathcal{M}$ of X that satisfies the condition of the extension theorem is the whole space X. Yet the projection into the first coordinate on any subspace of X has a $\mathcal{K}$-positive extension onto X. It is desirable to improve this extension theorem.

Remark. I have observed the following short proof of the extension theorem. It sheds some light on the nature of extensions of $\mathcal{K}$-positive linear functionals.

Define $\phi_+(x) = \inf\{\phi(m) : m \geq x, m \in \mathcal{M}\}$ and $\phi_-(x) = \sup\{\phi(m) : m \leq x, m \in \mathcal{M}\}$ for any vector $x \in X$. By hypothesis, both are well-defined *finite* numbers. It is not difficult to prove that the following assertions.

(i) $\phi_+(x+y) \le \phi_+(x) + \phi_+(y)$ for $x, y \in X$.

(ii) $\phi_+(\alpha x) = \alpha\phi_+(x)$ for $x \in X$ and $\alpha \ge 0$.

(iii) $\phi_+(-x) = -\phi_-(x)$ for $x \in X$.

(iv) $\phi_-(x) \ge 0$ for $x \ge 0$.

Now applying Theorem 3.2 in Rudin's book [145, p56], we get an extension Φ onto X of ϕ such that

$$\phi_-(x) \le \Phi(x) \le \phi_+(x)$$

for $x \in X$, which is clearly $\mathcal{K}$-positive.

Remark. Unconditional extension of $\mathcal{K}$-positive real linear functional in general is not possible.

Consider the following example: Let X be the vector space $\mathbf{C}_r([0,1])$, let $\mathcal{M}$ be the subspace $\{f \in X : |f(x)| \le C(f)\ |x|\}$ where $C(f)$ is a constant that depends only on the function f, and let $\mathcal{K} = \mathbf{C}_+([0,1])$. Define the linear functional ϕ on $\mathcal{M}$ by

$$\phi(f) = \int_0^1 \frac{f(x)}{x}\, dx$$

for $f \in \mathcal{M}$; clearly ϕ is positive on $\mathcal{M}$. There is no positive extension of ϕ onto the whole space X. To see this, assume such an extension Φ onto X exists. Thus, sup $\{\Phi(f) : f \le 1,\ f \in X\} = \Phi(1) < +\infty$. Notice that $f_n(x) = 1-(1-x)^n \in \mathcal{M}$ and $f_n \le 1$ for all positive integer n, but $\Phi(f_n) = \phi(f_n) = \int_0^1 \sum_0^{n-1}(1-x)^j\, dx = \sum_1^n \frac{1}{j} \to \infty$ as $n \to \infty$. This contradiction completes the assertion.

Remark. Known theorems on complex-valued moment sequence suggest the possibility of extending $\mathcal{K}$-positive *complex* linear functionals. We shall see from the following example that the extension theorem is **not** true for positive complex linear functionals.

Let X be the Banach space of all complex-valued continuous functions on the unit square $[0,1]\times[0,1]$ in the complex plane, $\mathcal{M}$ the one-dimensional subspace generated by the identity function z and $\mathcal{K}$ the cone of all non-negative functions in X. Since $\mathcal{K} \bigcap \mathcal{M} = \{0\}$, we know that any complex linear functional ϕ on $\mathcal{M}$ is $\mathcal{K}$-positive. If the extension theorem were true for such functionals, then for *any* given complex number z_0 there corresponds a $\mathcal{K}$-positive *complex* linear functional Φ on X such that $\Phi(\alpha z) = \alpha z_0$, where $\alpha \in \mathbf{C}$. On the other hand, we also know there is a positive regular Borel measure μ on the unit square so that

$$\Phi(f) = \int f \, d\mu$$

for any function $f \in X$ by using Theorem 0.2. In particular, we have

$$\Re z_0 = \Re\Phi(z) = \int_{x\in[0,1],y\in[0,1]} x \, d\mu \geq 0,$$

that contradicts to the fact that z_0 is an arbitrarily chosen complex number.

1.2 Representations of Non-negative Polynomials

In this section, we will present several classical theorems on the representations of non-negative polynomials on various sets in the plane, such as the unit interval [0, 1], the unit circle and the unit disk. As we have commented in the introduction that a non-negative polynomial on the real line can be written as

$$p^2(x) + q^2(x)$$

for some polynomials $p(x)$ and $q(x)$. It is also well-known that a non-negative polynomial on the half line $[0, \infty)$ can be represented as

$$p^2(x) + xq^2(x)$$

for some polynomials $p(x)$ and $q(x)$. Consult Polya and Szego [130, p78] for a weaker version of this fact, and Akhiezer [1, p77] for a proof. The reader can also give his own proof by imitating the proof of Markov's Theorem that follows.

The following theorem of Markov will be used to solve both the truncated and full Hausdorff moment problems.

Theorem 1.2 (Markov's Theorem) *Let $f(x)$ be a real polynomial of degree n, and non-negative on the unit interval [0, 1]. There are real polynomials $p(x)$ and $q(x)$ of degree $\leq [\frac{n}{2}]$ and $\leq [\frac{n-1}{2}]$ ($[x]$ refers to the biggest integer $\leq x$) such that*

$$f(x) = xp^2(x) + (1-x)q^2(x) \tag{1.2}$$

if n is odd and

$$f(x) = p^2(x) + x(1-x)q^2(x) \tag{1.3}$$

if n is even.

Proof. We shall use induction on the degree n of the polynomial $f(x)$.

Let $n = 1$, say $f(x) = ax + b$ for some real numbers a and b, we have $0 \leq b = f(0)$ and $0 \leq a + b = f(1)$. Define $p(x) = \sqrt{a+b}$ and $q(x) = \sqrt{b}$; we see that claim (1.2) is true.

Let $n = 2$, say $f(x) = ax^2 + bx + c$ for some real numbers a, b and c, we have $0 \leq c = f(0)$ and $0 \leq a + b + c = f(1)$. Define $p(x) = (\sqrt{a+b+c} + \sqrt{c})x - \sqrt{c}$ and $q(x) = \sqrt{A}$, where $A = (\sqrt{a+b+c} + \sqrt{c})^2 - a$; we see (1.3) holds if we show that $A \geq 0$. To see this we may assume $a > c$ and $b < 0$ without loss of generality. Now $A \geq 0$ is equivalent to $b^2 - 4ac \leq 0$. Suppose $b^2 > 4ac$, then the minimum value of $f(x)$ attained at $x = \frac{-b}{2a}$ is negative, that implies $\frac{-b}{2a} \geq 1$. Thus, $0 \leq a + b + c = (2a + b) + (c - a) \leq c - a$. This is a contradiction with the assumption $a > c$, hence $A \geq 0$.

Now let $n \geq 3$ be odd. Using the fundamental theorem of algebra, we can write $f(x) = f_1(x)f_2(x)$ for two non-negative polynomials on the interval [0, 1]. Where $f_1(x)$ has odd degree n_1 and $f_2(x)$ has even degree n_2. By the inductive hypothesis, we have $f_1(x) = xp_1^2(x) + (1-x)q_1^2(x)$ and $f_2(x) = p_2^2(x) + x(1-x)q_2^2(x)$ for some polynomials $p_1(x)$ of degree $(n_1 - 1)/2$, $p_2(x)$ of degree $n_2/2$, $q_1(x)$ of degree $(n_1 - 1)/2$ and $q_2(x)$ of degree $(n_2 - 2)/2$. Thus we obtain that

$$f(x) = x(p_1(x)p_2(x) + q_1(x)q_2(x))^2 + (1-x)(p_2(x)q_1(x) - xp_1(x)q_2(x))^2,$$

which validate the claim for odd degree polynomials.

Now let $n \geq 4$ be even. Again using the fundamental theorem of algebra, we can write $f(x) = f_1(x)f_2(x)$ for two non-negative polynomials on the interval [0, 1]. Where $f_1(x)$ and $f_2(x)$ *both* have even degrees $n_1, n_2 < n$. By the inductive hypothesis, we get $f_1(x) = p_1^2(x) + x(1-x)q_1^2(x)$ and $f_2(x) = p_2^2(x) + x(1-x)q_2^2(x)$ for some polynomials $p_1(x)$ of degree $n_1/2$, $p_2(x)$ of degree $n_2/2$, $q_1(x)$ of degree $(n_1 - 2)/2$ and $q_2(x)$ of degree $(n_2 - 2)/2$. Similarly we have that

$$f(x) = (p_1(x)p_2(x) + x(1-x)q_1(x)q_2(x))^2 + x(1-x)(p_2(x)q_1(x) - p_1(x)q_2(x))^2,$$

which validate the claim for even degree polynomials. The proof is complete. □

Remark. Markov showed this theorem in [115] only for the case n being even. Akhiezer [1, p74] mentioned this fact without giving a direct proof. Similar representations for non-negative polynomials on several intervals was considered by Krein, Levin and Nudelman in [100, Theorem 5.1 p45].

The following representation theorem for the so-called quasi-polynomials that are non-negative on the unit circle will be applied to solve the (truncated) trigonometric moment problems in the next section.

Theorem 1.3 (Fejer's Theorem) *Let $f(z) = \sum_{-n}^{n} a_j z^j$ be non-negative on the unit circle $\partial\mathbf{D}$. Then there exists a polynomial $p(z)$ of degree $\leq n$ such that*

$$f(z) = |p(z)|^2$$

for each z on the unit circle.

Proof. Let $q(z) = z^n f(z)$. By hypothesis, for each $|z| = 1$, we have

$$f(z) = |f(z)| = |q(z)|. \tag{1.4}$$

Now $q(z)$ is a polynomial on the complex plane of degree $N \leq 2n$, we want to consider its roots. We note first that for z belonging to the unit circle $\partial\mathbf{D}$, we have $\overline{f(z)} = f(z)$; that is,

$$\sum_{-n}^{n} (a_j - \overline{a_{-j}})z^j = 0.$$

So we get $\overline{a_j} = a_{-j}$ for each j. Hence $f(\frac{1}{\overline{z}}) = \overline{f(z)}$ for any $z \neq 0$, which implies that if z_0 is a root of $q(z)$, then so is $\frac{1}{\overline{z_0}}$.

Suppose that $z_0 \in \partial\mathbf{D}$ is a root of $q(z)$. Considering the function $f(z_0 e^{ix})$ of the variable x on the interval $(-1, 1)$, we obtain that $f(z_0 e^{ix}) = x^{2k} f_0(x)$ for a positive integer k and some non-negative real analytic function $f_0(x)$ (i.e., has a Taylor series expansion on the interval (-1, 1)) such that $f_0(0) > 0$. This gives $f^{(j)}(z_0) = 0$ for each $j < 2k$ and $f^{(2k)}(z_0) \neq 0$, and consequently, $q^{(j)}(z_0) = 0$ for each $j < 2k$ and $q^{(2k)}(z_0) \neq 0$. Thus each root of $q(z)$ on the unit circle has even multiplicity.

Using the fundamental theorem of algebra, we conclude that

$$q(z) = c\, z^{N-2l-2m} \prod_1^l (z - w_j)(z - \frac{1}{\overline{w_j}}) \prod_1^m (z - u_j)^2,$$

where c is constant, $u_j \in \partial\mathbf{D}$ for $j = 1, \cdots, m$ and $0 < |w_j| < 1$ for $j = 1, \cdots, l$.

Let $z \in \partial\mathbf{D}$ and note that $|z - \frac{1}{\overline{w_j}}| = \frac{|z - w_j|}{|w_j|}$ for $j = 1, \cdots, l$. We derive from (1.4) that

$$f(z) = |c| \prod_1^l \frac{|z - w_j|^2}{|w_j|} \prod_1^m |z - u_j|^2$$

and $l + m \leq \frac{N}{2} \leq n$, this complete the proof by setting

$$p(z) = \sqrt{\frac{|c|}{\prod_1^l |w_j|}}\ \prod_1^l (z - w_j) \prod_1^m (z - u_j).$$

$\square$

Remark. The theorem is from Fejer [60]. Also consult Polya and Szego [130, p259].

It is crucial that we can control the degrees of polynomials $p(z)$ and $q(z)$ as in Markov's Theorem and Fejer's Theorem. We shall see in the next section that this enables us to solve the *truncated* moment problems along with the full moment cases. Various representations of non-negative polynomials in z and $\overline{z}$ or in x and y on the unit disk are considered by Cassier [33], Berg [12], Berg , Christensen and Jensen [14], Berg and Maserick [15], Schmudgen [153] and Putinar [135], but the control on degrees of polynomials is not handled in these works.

1.3 Compact Moment Problems

We will give solutions to the compact moment problem, and apply the Symmetric Moment Theorem to solve the Hausdorff and trigonometric moment problems. We will also briefly discuss the $\overline{\mathbf{D}}$-moment problem.

Now fix the (finite) sequence $(\{f_k(x)\}_0^n)$ $\{f_k(x)\}_0^\infty$ in $\mathbf{C}(K)$ for some *compact* Hausdorff space K throughout this section except where it is stated otherwise.

Theorem 1.4 (Compact Moment Theorem) *Let $f_0(x) = 1$ for every $x \in K$. Then a necessary and sufficient condition for the (finite) sequence $(\{\mu_k\}_0^n)$ $\{\mu_k\}_0^\infty$ to be a (truncated) moment sequence is the following: whenever we have complex numbers α_j and a non-negative integer $m (\leq n)$ such that $\Re \sum_0^m \alpha_j f_j(x) \geq 0$ for every $x \in K$, then $\Re \sum_0^m \alpha_j \mu_j \geq 0$.*

Proof. Necessity of the condition is obvious. We will prove the sufficiency of the condition.

Let $X = \mathbf{C}_r(K)$, and let $\mathcal{M}$ be the subspace of X consisting of all *finite* real linear combinations of the real and imaginary parts of all functions f_j's. Define the real linear functional ϕ on $\mathcal{M}$ by $\phi(\Re f_j) = \Re \mu_j$ and $\phi(\Im f_j) = \Im \mu_j$ for all j's and extend ϕ in the obvious way, we want to show that ϕ is well-defined and positive on $\mathcal{M}$.

Suppose that $\sum (a_j \Re f_j(x) + b_j \Im f_j(x)) \geq 0$ on K for some real numbers a_j's and b_j's, then

$$\Re \sum (a_j - ib_j) f_j(x)) \geq 0$$

on K. By hypothesis, we obtain $\Re \sum (a_j - ib_j)\mu_j \geq 0$, that is, $\sum (a_j \Re \mu_j + b_j \Im \mu_j) \geq 0$. Thus ϕ is positive on $\mathcal{M}$. Consequently it is also well-defined since $\phi(f) \geq 0$ and $\phi(-f) \geq 0$ if $f \in \mathcal{M}$ is the zero vector.

First note that $1 \in \mathcal{M}$ and both sets $\{m \in \mathcal{M} : m \geq f\}$ and $\{m \in \mathcal{M} : m \leq f\}$ are non-empty for any $f \in X$. Now applying the Extension Theorem 1.1, we find a positive real linear functional Φ on X that extends ϕ on $\mathcal{M}$. By using Theorem 0.2, we get a positive regular Borel measure μ on K such that

$$\Phi(f) = \int_K f(x)d\mu(x)$$

for any $f \in \mathbf{C}_r(K)$. In particular, we have for each j that

$$\begin{aligned}
\mu_j &= \Re\mu_j + i\Im\mu_j \\
&= \phi(\Re f_j) + i\phi(\Im f_j) \\
&= \Phi(\Re f_j) + i\Phi(\Im f_j) \\
&= \int_K \Re f_j(x)d\mu(x) + i\int_K \Im f_j(x)d\mu(x) \\
&= \int_K f_j(x)d\mu(x).
\end{aligned}$$

This concludes the proof. □

Remark. The assumption $f_0 = 1$ on K was used only to assure that the condition of Theorem 1.1 is satisfied. So the Compact Moment Theorem is still valid if we assume that either f_0 or $-f_0$ is bounded below (i.e., the minimum value on K is positive). We will not consider in detail how the assumption may be optimized.

Theorem 1.5 *A necessary and sufficient condition for the (finite) sequence ($\{\mu_k\}_0^n$) $\{\mu_k\}_0^\infty$ to be a (truncated) moment sequence of a* finite *positive measure is the following:*

$$\sup_{0\le m(\le n)} \{\Re \sum_0^m \alpha_j\mu_j : \Re \sum_0^m \alpha_j f_j(x) \le 1 \ \textit{for } x \in K,\ \alpha_j \in \mathbf{C}\} < \infty \tag{1.5}$$

and whenever we have complex numbers α_j and a non-negative integer m ($\le n$) such that $\Re\sum_0^m \alpha_j f_j(x) \ge 0$ for every $x \in K$, then $\Re\sum_0^m \alpha_j\mu_j \ge 0$.

Proof. Necessity follows easily from the fact that the supremum in the equation (1.5) $\le \mu(K)$, where μ is the finite positive measure. We will show the conditions are sufficient.

Using the same notation introduced in the proof of the previous theorem, we know that ϕ is a positive real linear functional on $\mathcal{M}$. Without loss of generality, we assume that $1 \notin \mathcal{M}$ (since K is compact and any function in $\mathbf{C}_r(K)$ has its maximum and minimum values). Let r_0 be the supremum in the equation (1.5), we define $\tilde{\phi}$ on $\tilde{\mathcal{M}}$ generated by $\mathcal{M}$ and the constant functions on K by

$$\tilde{\phi}(m+a) = \phi(m) + ar_0,$$

where $m \in \mathcal{M}$ and $a \in \mathbf{R}$. Similar to the proof of Theorem 1.1, we get $\tilde{\phi}$ is positive on $\tilde{\mathcal{M}}$. Now applying the Extension Theorem 1.1 to the positive real linear functional $\tilde{\phi}$ on $\tilde{\mathcal{M}}$, then we follow the proof of the previous theorem to complete the proof. □

Remark. This proof reveals the intimate relation between extending positive linear functionals and solving moment problems, even for K being locally compact. Sometimes a proof of a theorem is more valuable than the theorem itself, this is such an example.

Theorem 1.6 (Symmetric Moment Theorem) *Let $f_0(x) = 1$ for every $x \in K$ and $\overline{f_j(x)} = f_{\pi(j)}(x)$ for some one-to-one map π on ($\{0, \cdots, n\}$) the set of all non-negative integers. Then a necessary and sufficient condition for the (finite) sequence ($\{\mu_k\}_0^n$) $\{\mu_k\}_0^\infty$ to be a (truncated) moment sequence is the following: whenever we have complex numbers α_j and a non-negative integer m ($\leq n$) such that $\sum_0^m \alpha_j f_j(x) \geq 0$ for every $x \in K$, then $\sum_0^m \alpha_j \mu_j \geq 0$,*

Proof. The condition is clearly necessary. We have to show its sufficiency.

First we show that

$$\overline{\mu_j} = \mu_{\pi(j)} \tag{1.6}$$

for all j's. Since $f_0(x) \geq 0$, we have $\mu_0 \geq 0$ by the hypothesis. Fix an index j; note that $1 + zf_j(x) + \overline{z}f_{\pi(j)}(x)$ is real-valued and non-negative for $|z|$ being sufficiently small. We obtain by hypothesis that

$$\mu_0 + z\mu_j + \overline{z}\mu_{\pi(j)} \geq 0$$

for z in some disk centered at the origin. Thus $\overline{z}(\overline{\mu_j} - \mu_{\pi(j)}) \in \mathbf{R}$ for z in that disk, so (1.6) holds.

If $\Re \sum_0^m a_j f_j(x) \geq 0$ for each $x \in K$, some $a_j \in \mathbf{C}$ and some m, then we get

$$\sum_0^m (a_j f_j(x) + \overline{a_j} f_{\pi(j)}(x)) \geq 0.$$

By hypothesis again, we conclude

$$\sum_0^m (a_j\mu_j + \overline{a_j}\mu_{\pi(j)}) \geq 0,$$

that is, $\Re \sum_0^m a_j\mu_j \geq 0$ by using the equation (1.6). The proof is completed by using the Compact Moment Theorem. □

Remark. Notice that if each $f_j(x)$ is real-valued, then the necessary and sufficient condition may be replaced by " whenever we have *real* numbers a_j and a non-negative integer m $(\leq n)$ such that $\sum_0^m a_j f_j(x) \geq 0$ for every $x \in K$, then $\sum_0^m a_j\mu_j \geq 0$".

Next we will apply the Symmetric Moment Theorem to solve several moment problems. We will use the notation $(\alpha(j,k))_0^m$ to denote the matrix having (j,k)th entry $\alpha(j,k)$, where both indices vary from 0 to m. Recall the well-known fact that a matrix $(\alpha(j,k))_0^m$ is non-negative definite if and only if the determinants, det $(\alpha(j,k))_0^l$, of the major matrices of it are non-negative for $l = 0, \cdots, m$. Also recall for the next corollary that the truncated Hausdorff moment problem corresponds to $K = [0,1]$ and $f_j(x) = x^j$ for appropriate j's.

Corollary 1.7 (Hausdorff Moment Problem) *A finite sequence $\{\mu_j\}_0^n$ is a truncated Hausdorff moment sequence if and only if the matrices $(\mu_{j+k})_0^m$ and $(\mu_{j+k+1} - \mu_{j+k+2})_0^{m-1}$ are non-negative definite when $n = 2m$ is even, or the matrices $(\mu_{j+k+1})_0^m$ and $(\mu_{j+k} - \mu_{j+k+1})_0^m$ are non-negative definite when $n = 2m+1$ is odd.*

Proof. Note that a real matrix $(\alpha_{j,k})_0^m$ is non-negative definite means that the quadratic form $\sum_{j,k=0}^m x_j x_k \alpha_{j,k}$ is non-negative for all real numbers $x_0, \cdots, x_m$.

Let $n = 2m$ be even. To see the necessity of the conditions, we note that for arbitrarily chosen real numbers $a_0, \cdots, a_m$, both polynomials $(\sum_0^m a_j x^j)^2$ and $x(1-x)(\sum_0^m a_j x^j)^2$ are non-negative on the interval [0, 1]. So applying the Symmetric Moment Theorem, we get both forms $\sum_0^m a_j a_k \mu_{j+k}$ and $\sum_0^m a_j a_k(\mu_{j+k+1} - \mu_{j+k+2})$ are non-negative. Thus the conditions are necessary.

To show the sufficiency of the conditions, we use again the Symmetric Moment Theorem. Let $p(x) = \sum_0^n c_j x^j$ be an arbitrary non-negative real polynomial on the interval [0, 1], we want to show that $\sum_0^n c_j\mu_j \geq 0$. If the degree of $p(x)$ equals $n = 2m$, then by Markov's Theorem 1.2, we have

$$p(x) = (\sum_0^m a_j x^j)^2 + x(1-x)(\sum_0^{m-1} b_j x^j)^2$$

for some real numbers a_j's and b_j's. Therefore

$$\begin{aligned} \sum_0^n c_j\mu_j &= \sum_{j,k=0}^m a_j a_k \mu_{j+k} + \sum_{j,k=0}^{m-1} b_j b_k (\mu_{j+k+1} - \mu_{j+k+2}) \\ &\geq 0, \end{aligned}$$

by the hypothesis. If the degree of $p(x)$ is less than n, (that is, $c_n = 0$), then we consider the polynomial $p(x) + \epsilon x^n$ of degree n for any positive number ϵ. We obtain from the even case that

$$\sum_0^n c_j\mu_j + \epsilon\mu_n \geq 0,$$

thus $\sum_0^n c_j\mu_j \geq 0$ by taking the limit as $\epsilon \to 0$. This implies that the conditions are sufficient.

When $n = 2m+1$ is odd, we prove the theorem using the same argument just given. □

Remark. It is interesting to note that as a corollary of this theorem, we get the following fact:

If both matrices $(\mu_{j+k})_0^m$ and $(\mu_{j+k+1} - \mu_{j+k+2})_0^{m-1}$ are non-negative definite, then so is $(\mu_{j+k} - \mu_{j+k+1})_0^{m-1}$.

Remark. The author is unable to determine the source of this theorem. cf Akhiezer [1], Shohat and Tamarkin [164] and Karlin and Shapley [87].

Remark. In essence, we have proved the following theorem on full Hausdorff moment problem:

A sequence $\{\mu_j\}_0^\infty$ is a full Hausdorff moment sequence if and only if the matrices $(\mu_{j+k})_0^m$ and $(\mu_{j+k+1} - \mu_{j+k+2})_0^m$ are non-negative definite for any non-negative integer m if and only if the matrices $(\mu_{j+k+1})_0^m$ and $(\mu_{j+k} - \mu_{j+k+1})_0^m$ are non-negative definite for any non-negative integer m.

For other characterizations of Hausdorff moment sequences, consult Hausdorff [72] and Hildebrandt and Schoenberg [79].

Recall that the (truncated) trigonometric moment problem corresponds to $K = \partial\mathbf{D}$ and $f_j(z) = z^j$ for all integers j's ($|j| \leq n$). Here n is a positive integer.

Corollary 1.8 (Trigonometric Moment Problem) *A (finite) sequence ($\{\mu_j\}_0^n$) $\{\mu_j\}_0^\infty$ is a (truncated) trigonometric moment sequence if and only if the matrices $(\mu_{j-k})_0^m$ are non-negative definite for all non-negative integers m ($\leq n$).*

Proof. Necessity follows from the Symmetric Moment Theorem and the observation that

$$\begin{aligned} \sum_{j,k=0}^{m} \alpha_j \overline{\alpha_k} z^{j-k} &= |\sum_{0}^{m} \alpha_j z^j|^2 \\ &\geq 0 \end{aligned}$$

for z on the unit circle $\partial\mathbf{D}$ and arbitrary complex numbers α_j's. So if the (finite) sequence ($\{\mu_j\}_0^n$) $\{\mu_j\}_0^\infty$ is a (truncated) trigonometric moment sequence, then

$$\sum_{j,k=0}^{m} \alpha_j \overline{\alpha_k} \mu_{j-k} \geq 0$$

for any non-negative integer ($m \leq n$) m and arbitrary complex numbers α_j's; that is, the matrix $(\mu_{j-k})_0^m$ is non-negative definite. Hence, the condition is necessary.

To see this condition is also sufficient, it suffices by using the Symmetric Moment Theorem to show that whenever we have $\sum_{-m}^{m} \alpha_j z^j \geq 0$ on $\partial\mathbf{D}$ for some non-negative integer m ($\leq n$) and complex numbers α_j's, then $\sum_{-m}^{m} \alpha_j \mu_j \geq 0$. For this, applying Fejer's Theorem 1.3, we get $\sum_{-m}^{m} \alpha_j z^j = |\sum_0^m \beta_j z^j|^2 = \sum_{j,k=0} \beta_j \overline{\beta_k} z^{j-k}$ on $\partial\mathbf{D}$ for some complex numbers β_j's. Thus,

$$\begin{aligned} \sum_{-m}^{m} \alpha_j \mu_j &= \sum_{j,k=0} \beta_j \overline{\beta_k} \mu_{j-k} \\ &\geq 0 \end{aligned}$$

by hypothesis. The proof is complete. □

Remark. This theorem belongs to F. Riesz [141] and G. Herglotz [76]. Consult also C. Caratheodory [26] and O. Toeplitz [175].

Corollary 1.9 ($\overline{\mathbf{D}}$-Moment Problem) *A (finite) doubly indexed sequence ($\{\mu_{j,k}\}_0^n$) $\{\mu_{j,k}\}_0^\infty$ is a (truncated) $\overline{\mathbf{D}}$-moment sequence if and only if for any complex polynomial $\sum_{j+k(\leq n)} \alpha_{j,k} z^j \overline{z}^k$ in z and $\overline{z}$ that is non-negative on the open unit disk $\mathbf{D}$, we have*

$$\sum_{j+k(\leq n)} \alpha_{j,k}\mu_{j,k} \geq 0.$$

Proof. This follows immediately from the Symmetric Moment Theorem since the map $\pi : (j,k) \longrightarrow (k,j)$ satisfies its condition. □

Remark. Atzmon gave an operator theoretic proof to his solution of the $\overline{\mathbf{D}}$-moment problems in [7]. That solution follows directly from this corollary and the following theorem of Cassier [33] and Putinar [135]:

If $p(z,\overline{z})$ is a *positive* polynomial of degree n on the closed unit disk $\overline{\mathbf{D}}$, then there are finitely many polynomials $A_j(z,\overline{z})$ and $B_k(z)$ such that

$$p(z,\overline{z}) = \sum_j |A_j(z,\overline{z})|^2 + (1-|z|^2)\sum_k |B_k(z)|^2.$$

Consult also Berg [12].

We will not discuss this decomposition (representation) theorem in great detail. Notice that we do not know whether or not the degrees of A_j's are no more than $[n/2]$; in particular, we do not know the minimal degree of these polynomials. This difficulty prevents us from solving the truncated moment problem easily. (Consult the remark after Fejer's Theorem 1.3.)

The following example (due to McGregor [117]) shows that other means of representing polynomials may be necessary:

$$1 + x = \frac{(1+x)^2}{2} + \frac{y^2}{2} + (1 - x^2 - y^2) \cdot \frac{1}{2},$$

where $x^2 + y^2 \leq 1$.

A vector $\alpha = (\alpha_1, \cdots, \alpha_n)$ in $\mathbf{R}^n$ is called a multi-index if each coordinate α_j is a non-negative integer for $j = 1, \cdots, n$. We will use $|\alpha|$ to denote the sum $\alpha_1 + \cdots + \alpha_n$, and x^α to denote the product $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, where $x = (x_1, \cdots, x_n)$ is a vector in $\mathbf{R}^n$.

Corollary 1.10 (Haviland's Theorem) *Given a multi-indexed (finite) sequence ($\{\mu_\alpha\}_{|\alpha|=0}^N$) $\{\mu_\alpha\}_0^\infty$. It is necessary and sufficient for a positive regular Borel measure on a compact subset K of $\mathbf{R}^n$ to exist such that*

$$\mu_\alpha = \int x^\alpha d\mu(x)$$

for each multi-index α that

$$\sum_\alpha a_\alpha \mu_\alpha \geq 0$$

whenever $\sum_{0 \leq |\alpha| (\leq N)} a_\alpha x^\alpha \geq 0$ for each $x \in K$ and some real numbers a_α's.

Proof. It is an immediate corollary of the Symmetric Moment Theorem. □

Remark. Haviland's original theorem in [73] applies also to unbounded set K. (Consult Devinatz [53], Hedge [74] and Nakonecnii [121].)

Chapter 2

LOWER AND UPPER FUNCTIONS

We will first introduce F. Riesz's theorem that relate a moment sequence with a sequence of convex sets; this geometrical property leads us to our investigation of the convex hull of the curve $\{(t, \cdots, t^n) : 0 \leq t \leq 1\}$ for positive integers. These convex sets are characterized by the so-called *lower* and *upper* functions; they represent the top and bottom surfaces of the convex sets. Our contribution is to give explicit formulas for these functions; these results are used to solve the contractive subnormal completion problem.

2.1 F. Riesz's Approach

We need the fundamental theorem of Caratheodory in finite dimensional convex analysis; it can be found in H. Eggleston [57, p35] and L. Danzer, B. Grunbaum and V. Klee [51, p103]. Serious reader should also consult T. Bonnesen and W. Fenchel [20], C. Caratheodory [26], H. Minkowski [118], T. Popoviciu [131]. Further references may be found in the beautiful expository paper on convexity by L. Danzer, B. Grunbaum and V. Klee [51].

Theorem 2.1 (Caratheodory Theorem) *Let E be a subset of $\mathbf{R}^n$ for some positive integer n. Then any vector in the convex hull of E can be written as a convex combination of at most $(n+1)$ vectors in E.*

Proof. Let $\mathbf{x}$ be a vector in $\text{co}(E)$; we know that

$$\mathbf{x} = \sum_1^m a_j \mathbf{x}_j \quad \text{where} \quad \sum_1^m a_j = 1$$

for some positive numbers a_j's and vectors $\mathbf{x}_j$'s in E. We assume that the positive integer m is the smallest among these expressions. It suffices to show $m \leq n + 1$.

Suppose the contrary $m > n+1$, then the $(m-1)$ vectors $\mathbf{x}_2 - \mathbf{x}_1,\ \cdots,\ \mathbf{x}_m - \mathbf{x}_1$ are linearly dependent. Thus, there are real numbers α_j's such that

$$\sum_1^m \alpha_j \mathbf{x}_j = \mathbf{0}, \quad \sum_1^m \alpha_j = 0$$

and at least one of the numbers α_j's is non-zero. Now note that the set $\Lambda = \{\lambda : -a_j \leq \lambda\alpha_j,\ j = 1, \cdots, m\}$ is a non-empty and proper subset of $\mathbf{R}$; in fact, it is either a finite closed interval or a half line. Now note for all $\lambda \in \Lambda$ that

$$\mathbf{x} = \sum_1^m (a_j + \lambda\alpha_j)\mathbf{x}_j, \quad \sum_1^m (a_j + \lambda\alpha_j) = 1. \tag{2.1}$$

Let λ_0 be a boundary point of Λ; at least one of the numbers $a_j + \lambda_0\alpha_j$'s is zero. Thus the corresponding equation (2.1) contradicts with the minimality of m. The proof is complete. $\square$

Remark. The proof is from H. Eggleston [57, pp35-36].

Corollary 2.2 *If K is a compact subset of $\mathbf{R}^n$ for some positive integer n, then so is its convex hull* co(K).

Proof. Consider the following map ϕ from $\mathbf{R}^{(n+1)^2} = \mathbf{R}^{n+1} \times \mathbf{R}^n \times \cdots \times \mathbf{R}^n$ to $\mathbf{R}^n$

$$(\boldsymbol{\lambda}, \mathbf{x}_1, \cdots, \mathbf{x}_{n+1}) \longrightarrow \sum_1^{n+1} \lambda_j \mathbf{x}_j,$$

where $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_{n+1})$. Note the set $B = \{(\lambda_1, \cdots, \lambda_{n+1}) : \sum \lambda_j = 1,\ \lambda_j \geq 0\}$ is a compact subset of $\mathbf{R}^{n+1}$, so $E_0 = B \times E \times \cdots \times E \subset \mathbf{R}^{(n+1)^2}$ is compact.

Using Theorem 2.1, we see that the map ϕ maps E_0 *onto* co(E). Therefore the image co(E) is also compact. $\square$

Remark. The proof is also from H. Eggleston [57, p38].

This corollary is *not* valid for closed subsets of Euclidean spaces, for example, the set $K = \{(n, \frac{1}{n}) : n = 1, 2, \cdots\} \subset \mathbf{R}^2$ is closed, but its convex hull co(K) is not closed.

The next theorem is the inspiration for our investigation of convex hulls of certain curves. Its proof is due, in essence, to F. Riesz [141, pp55-57]. Also consult [178, pp265-267].

We fix a (finite) sequence ($\{f_j(x)\}_0^n$) $\{f_j(x)\}_0^\infty$ of functions in $\mathbf{C}_r(K)$ for some compact Hausdorff space K.

Theorem 2.3 (F. Riesz) *Let $f_0(x) = 1$ for each $x \in K$. For a (finite) sequence ($\{\mu_j\}_0^n$) $\{\mu_j\}_0^\infty$ to be a moment sequence of a* probability measure, *it is necessary and sufficient that*

$$(\mu_0, \cdots, \mu_m) \in \mathrm{co}(\{(f_0(t), \cdots, f_m(t)) : t \in K\})$$

for all non-negative integers m ($\leq n$).

Proof. First we establish the sufficiency of the condition.

For each positive integer ($m \leq n$) m, choose positive numbers a_{jm} and points t_{jm} in K such that

$$\begin{aligned} \mu_k &= \textstyle\sum_j a_{jm} f_k(t_{jm}), \\ \textstyle\sum_j a_{jm} &= 1, \end{aligned}$$

for $k = 0, \cdots, m$.

Define the measure $\nu_m = \sum_j a_{jm}\delta_{t_{jm}}$, where the notation δ_t represents the point mass measure at t on K. The last set of equalities implies that ν_m is a probability measure and

$$\mu_k = \int_K f_k(t) d\nu_m(t),$$

for $k = 0, 1, \cdots, m$. Let the measure ν be a weak-star limit of a subnet of $\{\nu_m : m = 1, 2, \cdots\}$; such a limit exists by Banach-Alaoglu Theorem 0.4. It is not difficult to check that ν is a positive measure, and

$$\mu_k = \int_K f_k(t) d\nu(t)$$

for $0 \leq k$ ($\leq n$). When $k = 0$, the equation gives $\nu(K) = 1$. Hence, ν is a probability measure.

For the other direction, let ν be a probability measure such that

$$\mu_k = \int_K f_k(t) d\nu(t),$$

for $0 \le k$ ($\le n$). Now suppose that $(\mu_0, \cdots, \mu_m) \notin \mathrm{co}(\{(f_0(t), \cdots, f_m(t)) : t \in K\})$ for some non-negative integer ($m \le n$) m; we want to derive a contradiction from this supposition.

According to Corollary 2.2, the set $\mathrm{co}(\{(f_0(t), \cdots, f_m(t)) : t \in K\})$ is compact; using the *strict* separation theorem 0.3 for $\mathbf{R}^{m+1}$, we find a hyperplane with a normal vector $(p_0, \cdots, p_m)$ passing through the point $(\mu_0, \cdots, \mu_m)$ such that for all $t \in K$

$$p_0(f_0(t) - \mu_0) + \cdots + p_m(f_m(t) - \mu_m) \ge \epsilon > 0.$$

Integrating with respect to the measure ν on both sides of the last inequality, we arrive at the contradiction $0 \ge \epsilon > 0$. □

Remark. F. Riesz showed this theorem for $K = [0, 1]$. We will not pursue further generalizations of the theorem.

We denote the curve $\{(t, \cdots, t^n) : 0 \le t \le 1\}$ in $\mathbf{R}^n$ by Γ_n. The next corollary follows immediately from the previous theorem. It will be used along with Corollary 1.7 to give the formulas for the lower and upper functions.

Corollary 2.4 *For a (finite) sequence ($\{x_j\}_0^n$) $\{x_j\}_0^\infty$ to be a (truncated) Hausdorff moment sequence of a* probability measure, *it is necessary and sufficient that*

$$(x_1, \cdots, x_m) \in \mathrm{co}(\Gamma_m)$$

for each positive integer m ($\le n$).

Remark. Note that if $(x_1, \cdots, x_m) \in \mathrm{co}(\Gamma_m)$, then $(x_1, \cdots, x_k) \in \mathrm{co}(\Gamma_k)$ for each positive integer $k < m$.

The following well-known fact, which is usually derived via the theorem of Hausdorff in [72] (cf also [1, p74]), will be needed in the next chapter.

Corollary 2.5 *If both sequences $\{x_j\}_0^\infty$ and $\{y_j\}_0^\infty$ are Hausdorff moment sequences, then so is the product sequence $\{x_j\, y_j\}_0^\infty$.*

Proof. We may assume $x_0 = y_0 = 1$ without loss of generality. Fix an arbitrary positive integer n; using Corollary 2.4, we find sequences $a_{jn} > 0$, $b_{jn} > 0$ and $t_{jn} \in [0,1]$, $\tilde{t}_{jn} \in [0,1]$ so that

$$x_k = \sum_j a_{jn} t_{jn}^k,$$

$$\sum_j a_{jn} = 1,$$

$$y_k = \sum_j b_{jn} \tilde{t}_{jn}^k,$$

$$\sum_j b_{jn} = 1,$$

for $k = 1, \cdots, n$. Hence, $x_k y_k = \sum_{j,l} a_{jn} b_{ln} (t_{jn}\tilde{t}_{ln})^k$ for $k = 1, \cdots, n$, and $\sum_{j,l} a_{jn} b_{ln} = 1$. Using Corollary 2.4 again, we see that the product sequence $\{x_k y_k\}_0^\infty$ is a Hausdorff moment sequence. □

2.2 Lower and Upper Functions

In view of Corollary 2.4, the geometric understanding of Hausdorff moment sequences lies in depicting the convex sets $\mathrm{co}(\Gamma_n)$. This motivates the following:

Definition 2.6 *Given any positive integer n, we call the functions defined on the convex set* $\mathrm{co}(\Gamma_n)$ *by*

$$V_-(x_1, \cdots, x_n) = \sup_{(a_0, \cdots, a_n) \in E_n^+} (a_0 + a_1 x_1 + \cdots + a_n x_n)$$

and

$$V_+(x_1, \cdots, x_n) = \inf_{(a_0, \cdots, a_n) \in E_n^-} (a_0 + a_1 x_1 + \cdots + a_n x_n)$$

the lower and upper function *(of n variables); respectively, where*

$$E_n^+ = \{(a_0, \cdots, a_n) : t^{n+1} - a_n t^n - \cdots - a_1 t - a_0 \geq 0 \ \text{for} \ 0 \leq t \leq 1\}$$

and

$$E_n^- = \{(a_0, \cdots, a_n) : \ t^{n+1} - a_n t^n - \cdots - a_1 t - a_0 \leq 0 \ \text{for} \ 0 \leq t \leq 1\}.$$

Remarks. These two functions have important geometrical interpretations. They are used to describe the set $\mathrm{co}(\Gamma_{n+1}) \in \mathbf{R}^{n+1}$ in terms of $\mathrm{co}(\Gamma_n) \in \mathbf{R}^n$. Sierpinski's Theorem 0.6 says that there are two *continuous* functions $V_\pm$ defined on the interior of $\mathrm{co}(\Gamma_n)$ so that $\mathrm{co}(\Gamma_{n+1})$ is the convex body lying below the surface

$$\{(x_1, \cdots, x_n;\ V_+(x_1, \cdots, x_n)) :\ (x_1, \cdots, x_n) \in \mathrm{co}(\Gamma_n)\}$$

and above the surface

$$\{(x_1, \cdots, x_n;\ V_-(x_1, \cdots, x_n)) :\ (x_1, \cdots, x_n) \in \mathrm{co}(\Gamma_n)\}.$$

That is,

$$\begin{aligned} \mathrm{co}(\Gamma_{n+1}) \quad = \quad & \{(x_1, \cdots, x_n;\ y) :\ (x_1, \cdots, x_n) \in \mathrm{co}(\Gamma_n), \\ & V_-(x_1, \cdots, x_n) \le y \le V_+(x_1, \cdots, x_n)\}. \end{aligned} \tag{2.2}$$

(It is worth noting that we could have chosen any other direction to designate two other continuous functions defined on the projection of the convex set $\mathrm{co}(\Gamma_{n+1})$ along that direction. The surfaces represented by the two functions still enclose the convex body $\mathrm{co}(\Gamma_{n+1})$. I suspect that these functions take simple forms if we choose our directions cleverly.)

To see that these two functions are equal to the functions as we have defined is merely a matter of using the separation theorem 0.3. First of all we will identify (throughout the paper) the hyperplane in $\mathbf{R}^{n+1}$

$$x_{n+1} = a_0 + a_1 x_1 + \cdots + a_n x_n$$

with the point $(a_0, a_1, \cdots, a_n)$. In particular, note that E_n^- consists of all hyperplanes

$$x_{n+1} = a_0 + a_1 x_1 + \cdots + a_n x_n$$

that are above the curve Γ_{n+1}; that is,

$$a_0 + a_1 t + \cdots + a_n t^n \ge t^{n+1}$$

for $0 \leq t \leq 1$. Now to compute the upper function $V_+(x_1, \cdots, x_n)$ one takes the infimum of such hyperplanes evaluated at a fixed point $(x_1, \cdots, x_n) \in \mathrm{co}(\Gamma_n)$. The lower function $V_-(x_1, \cdots, x_n)$ has a similar explanation in terms of the hyperplanes E_n^+ lying below the curve Γ_{n+1}.

We extend the definition of V_- and V_+ to those points $(x_1, \cdots, x_n) \notin \mathrm{co}(\Gamma_n)$ by defining

$$V_-(x_1, \cdots, x_n) = +\infty$$

and

$$V_+(x_1, \cdots, x_n) = -\infty.$$

The following theorem establishes the relation between the Hausdorff moment sequences and the lower and upper functions just defined

Theorem 2.7 *Let* $(\{x_j\}_0^n)$ $\{x_j\}_0^\infty$ *be a (finite) sequence of positive numbers and* $x_0 = 1$. *Necessary and sufficient conditions for the (finite) sequence* $(\{x_j\}_0^n)$ $\{x_j\}_0^\infty$ *to be a (truncated) Hausdorff moment sequence are that* $0 \leq x_1 \leq 1$ *and*

$$V_-(x_1, \cdots, x_m) \leq x_{m+1} \leq V_+(x_1, \cdots, x_m)$$

for all positive integer m $(< n)$.

Proof. The crucial ingredient for the proof comes from the previous remark, especially the equation 2.2, besides corollary 2.4.

Claim. $(x_1, \cdots, x_m) \in \mathrm{co}(\Gamma_m)$ *if and only if* $x_1 \in [0, 1]$ *and*

$$V_-(x_1, \cdots, x_k) \leq x_{k+1} \leq V_+(x_1, \cdots, x_k),$$

for $k = 1, \cdots, m-1$.

Note that $\mathrm{co}(\Gamma_1) = [0, 1]$; thus the claim is true when $m = 1$. Using induction on m and the equation 2.2, we can easily prove the claim.

Now the theorem follows directly from Corollary 2.4. □

The following theorem gives the explicit formulas for the lower and upper functions, and, hence, positively answers Question 1 in [108]. In addition, we will also determine the interior and boundary points of the convex set $\mathrm{co}(\Gamma_n)$ for positive integer n.

Theorem 2.8 *Let $(x_1, \cdots, x_n)$ be an interior point of* $\mathrm{co}(\Gamma_n)$.

1. *The upper function $V_+(x_1, \cdots, x_n)$ solves the linear equation in x_{n+1}:*

$$\det\,(x_{j+k+1} - x_{j+k+2})_0^{m-1} = 0$$

when $n = 2m - 1$ is odd or

$$\det\,(x_{j+k} - x_{j+k+1})_0^{m} = 0$$

when $n = 2m$ is even.

2. *The lower function $V_-(x_1, \cdots, x_n)$ solves the linear equation in x_{n+1}:*

$$\det\,(x_{j+k})_0^{m} = 0$$

when $n = 2m - 1$ is odd or

$$\det\,(x_{j+k+1})_0^{m} = 0$$

when $n = 2m$ is even.

Proof. We will establish the theorem in several steps.

Claim 1. Let $(z_1, \cdots, z_p) \in \mathrm{co}(\Gamma_p)$ for some positive integer p. If $z_p = V_+(z_1, \cdots, z_{p-1})$ or $z_p = V_-(z_1, \cdots, z_{p-1})$, then the point $(z_1, \cdots, z_p)$ belongs to the boundary $\partial(\mathrm{co}(\Gamma_p))$ of $\mathrm{co}(\Gamma_p)$

Suppose not, then that point must be in the interior of $\mathrm{co}(\Gamma_p)$. Thus there is a positive number ϵ so that both points $(z_1, \cdots, z_{p-1}, z_p + \epsilon)$ and $(z_1, \cdots, z_{p-1}, z_p - \epsilon)$ belong to the set $\mathrm{co}(\Gamma_p)$. By the equation 2.2, we get

$$V_-(z_1, \cdots, z_{p-1}) \leq z_p - \epsilon < z_p + \epsilon \leq V_+(z_1, \cdots, z_{p-1}),$$

which contradicts our hypothesis. Claim 1 is established.

Claim 2. Let μ be a measure supported on a set $\{t_j\}_1^{p/2}$ when p is even and on a set $\{t_j\}_1^{(p-1)/2} \bigcup \{0\}$ when p is odd, where p is a positive integer and $t_j \in [0,1]$ for $j = 1, \cdots, [p/2]$, and

$$z_k = \int_0^1 t^k d\mu(t)$$

for $k = 0, 1, \cdots, p$. Then

$$z_p = V_-(z_1, \cdots, z_{p-1}) = V_+(z_1, \cdots, z_{p-1}).$$

To prove this, we have by hypothesis that

$$\mu = \sum_0^{[p/2]} a_j \delta_{t_j}, \ \sum_0^{[p/2]} a_j = 1,$$

where $t_0 = 0$, $a_j \geq 0$ for all j's, and $a_0 = 0$ if p is even. Therefore

$$z_k = \sum_{j=0}^{[p/2]} a_j t_j^k$$

for $k = 0, 1, \cdots, p$. Obviously we have

$$V_-(z_1, \cdots, z_{p-1}) \leq z_p \leq V_+(z_1, \cdots, z_{p-1}).$$

We define constants $c_0, \cdots, c_{p-1}$ by the following equality:

$$t^p - \sum_0^{p-1} c_j t^j = \pm\, t^{p-2[p/2]} \prod_0^{[p/2]} (t - t_j)^2.$$

We obtain that

$$\begin{aligned} \sum_0^{p-1} c_k z_k &= \sum_{j=0}^{[p/2]} a_j \sum_{k=0}^{p-1} c_k t_j^k \\ &= \sum_{j=0}^{[p/2]} a_j t_j^p \\ &= z_p. \end{aligned}$$

Now using the definition of the lower and upper functions, we get $V_+(z_1, \cdots, z_{p-1}) \leq z_p$ and $V_-(z_1, \cdots, z_{p-1}) \geq z_p$. This completes the proof of Claim 2.

Claim 3. If $n = 2m-1$ *is odd, then both determinants* $\det\, (x_{j+k})_0^{m-1}$ *and* $\det\, (x_{j+k+1} - x_{j+k+2})_0^{m-2}$ *are* positive. *If* $n = 2m$ *is even, then both determinants* $\det\, (x_{j+k+1})_0^{m-1}$ *and* $\det\, (x_{j+k} - x_{j+k+1})_0^{m-1}$ *are* positive.

Let us consider the case where $n = 2m$ is even; the other case can be dealt with in a similar manner.

Assume $\det\, (x_{j+k+1})_0^{m-1} = 0$. So for some non-zero vector $\mathbf{c} = (c_0, \cdots, c_{m-1})$, we have

$$\mathbf{c}\ \ (x_{j+k+1})_0^{m-1}\ \mathbf{c}^T = 0,$$

where $\mathbf{c}^T$ refers to the transpose column vector of $\mathbf{c}$. That is, $\sum c_j c_k x_{j+k+1} = 0$. Let μ be a probability measure on $[0,1]$ such that $x_k = \int t^k d\mu(t)$ for $k = 1, \cdots, 2m$; such a measure exists by using Corollary 2.4. We conclude that

$$\int t\, (\sum_0^{m-1} c_j t^j)^2\, d\mu(t) = 0.$$

This fact implies the measure μ is supported on the set of zeros of the polynomial $\sum_0^{m-1} c_j t^{j+1}$ of degree $\leq m = n/2$. Applying Claim 2 and then Claim 1, we arrive at the conclusion that the point $(x_1, \cdots, x_n)$ is on the boundary $\partial(\mathrm{co}(\Gamma_n))$. This contradiction says that $\det\, (x_{j+k+1})_0^{m-1} > 0$ because the matrix $(x_{j+k+1})_0^{m-1}$ is non-negative definite according to Corollary 1.7.

Using the same reasoning, we can show that $\det\, (x_{j+k} - x_{j+k+1})_0^{m-1} > 0$.

Finally we are ready to prove the theorem. Assume $n = 2m-1$ for some integer m; again the other case follows from a similar argument. Let x_{n+1}^- and x_{n+1}^+ be the rational functions in $x_1, \cdots, x_n$ that solve the linear equations in x_{n+1}: $\det\, (x_{j+k})_0^m = 0$ and $\det\, (x_{j+k+1} - x_{j+k+2})_0^{m-1} = 0$, respectively. The *existence* of $x_{n+1}^\pm$ follows from Claim 3.

Since $(x_1, \cdots, x_n,\ V_-(x_1, \cdots, x_n)) \in \mathrm{co}(\Gamma_{n+1})$, it follows from Corollary 2.4 and Corollary 1.7 that the following matrices (obtained by replacing x_{n+1} with $V_-(x_1, \cdots, x_n)$ and

$V_+(x_1,\cdots,x_n)$ in the matrices $(x_{j+k})_0^m$ and $(x_{j+k+1}-x_{j+k+2})_0^{m-1})$

$$\begin{pmatrix} x_0 & \cdots & x_{m-1} & x_m \\ \vdots & \ddots & \vdots & \vdots \\ x_{m-1} & \cdots & x_{2m-2} & x_{2m-1} \\ x_m & \cdots & x_{2m-1} & V_-(x_1,\cdots,x_n) \end{pmatrix}$$

and

$$\begin{pmatrix} x_1-x_2 & \cdots & x_{m-1}-x_m & x_m-x_{m+1} \\ \vdots & \ddots & \vdots & \vdots \\ x_{m-1}-x_m & \cdots & x_{2m-3}-x_{2m-2} & x_{2m-2}-x_{2m-1} \\ x_m-x_{m+1} & \cdots & x_{2m-2}-x_{2m-1} & x_{2m-1}-V_+(x_1,\cdots,x_n) \end{pmatrix}$$

are non-negative definite; in particular, their determinants are non-negative. Thus, using Claim 3 and the definition of x_{n+1}^- and x_{n+1}^+, we have $x_{n+1}^- \leq V_-(x_1,\cdots,x_n)$ and $V_+(x_1,\cdots,x_n) \leq x_{n+1}^+$.

Using Corollary 2.4 and Corollary 1.7 and the definition of x_{n+1}^- and x_{n+1}^+, we see that the matrices (obtained by replacing x_{n+1} with x_{n+1}^- and x_{n+1}^+ in both matrices $(x_{j+k})_0^m$ and $(x_{j+k+1}-x_{j+k+2})_0^{m-1})$

$$\begin{pmatrix} x_0 & \cdots & x_{m-1} & x_m \\ \vdots & \ddots & \vdots & \vdots \\ x_{m-1} & \cdots & x_{2m-2} & x_{2m-1} \\ x_m & \cdots & x_{2m-1} & x_{n+1}^{\pm} \end{pmatrix}$$

and

$$\begin{pmatrix} x_1-x_2 & \cdots & x_{m-1}-x_m & x_m-x_{m+1} \\ \vdots & \ddots & \vdots & \vdots \\ x_{m-1}-x_m & \cdots & x_{2m-3}-x_{2m-2} & x_{2m-2}-x_{2m-1} \\ x_m-x_{m+1} & \cdots & x_{2m-2}-x_{2m-1} & x_{2m-1}-x_{n+1}^{\pm} \end{pmatrix}$$

are non-negative definite, since all the major minors are non-negative. Again by Corollary 1.7 and Corollary 2.4, we obtain that $(x_1,\cdots,x_n;x_{n+1}^{\pm}) \in \mathrm{co}(\Gamma_{n+1})$. So $V_-(x_1,\cdots,x_n) \leq x_{n+1}^-$ and $x_{n+1}^+ \leq V_+(x_1,\cdots,x_n)$.

The proof is complete. $\square$

Remarks. In the proof of the theorem, we showed that if the point $(x_1, \cdots, x_n)$ is a convex combination of *at most* [n/2] points on the curve Γ_n, then

$$V_-(x_1, \cdots, x_{n-1}) = V_+(x_1, \cdots, x_{n-1}) = x_n.$$

So such points are on the boundary of the convex set $\mathrm{co}(\Gamma_n)$. This is a very interesting phenomenon in light of Caratheodory Theorem 2.1.

The proof also indicates that the theorem is true under the weaker conditions: det $(x_{j+k})_0^{m-1} > 0$ and det $(x_{j+k+1} - x_{j+k+2})_0^{m-2} > 0$ when $n = 2m - 1$ is odd and det $(x_{j+k+1})_0^{m-1} > 0$ and det $(x_{j+k} - x_{j+k+1})_0^{m-1} > 0$ when $n = 2m$ is even. So we can compute the lower and upper functions on the sets $\mathrm{co}(\Gamma_n)$, not just on the interior.

The proof also suggests that it is possible to describe the interior and boundary points of the convex set $\mathrm{co}(\Gamma_n)$.

Let us determine a few concrete formulas of the lower and upper functions.

Corollary 2.9 *Assume the point (x_1, x_2, x_3, x_4) belongs to* $\mathrm{co}(\Gamma_4)$.

1.

$$V_-(x_1) = x_1^2, \quad V_+(x_1) = x_1.$$

2. Let $0 < x_1 < 1$. Then

$$V_-(x_1, x_2) = \frac{x_2^2}{x_1}, \quad V_+(x_1, x_2) = x_2 - \frac{(x_1 - x_2)^2}{1 - x_1}.$$

3. Let $x_1^2 < x_2 < x_1$. Then

$$V_-(x_1, x_2, x_3) = \frac{x_3^2 - 2x_1x_2x_3 + x_2^3}{x_2 - x_1^2}$$

and

$$V_+(x_1, x_2, x_3) = x_3 - \frac{(x_2 - x_3)^2}{x_1 - x_2}.$$

4. Let $V_-(x_1, x_2) < x_3 < V_+(x_1, x_2)$. *Then*

$$V_-(x_1, x_2, x_3, x_4) = \frac{x_1 x_4^2 - 2x_2 x_3 x_4 + x_3^3}{x_1 x_3 - x_2^2}$$

and

$$V_+(x_1, x_2, x_3, x_4) =$$

$$x_4 - \frac{(1 - x_1)(x_3 - x_4)^2 - 2(x_1 - x_2)(x_2 - x_3)(x_3 - x_4) + (x_2 - x_3)^3}{(1 - x_1)(x_2 - x_3) - (x_1 - x_2)^2}.$$

Proof. These are straightforward computations using Theorem 2.8 and the remarks after its proof. □

Remark. This corollary was established in [108, Theorem 2.8, Theorem 2.10 and Corollary 2.11] by the author. There the results are obtained through direct and lengthy computation from the definition of the lower and upper functions.

In [169], Smilansky studied the convex set

$$\text{co}\ \{(\cos(2\pi pt), \sin(2\pi pt), \cos(2\pi qt), \sin(2\pi qt)) : \ 0 \le t \le 1\}$$

for positive numbers p and q.

Corollary 2.10 *Let* $(x_1, \cdots, x_n) \in \text{co}(\Gamma_n)$ *for some positive integer* n. *Then the point* $(x_1, \cdots, x_n)$ *is in the interior of* $\text{co}(\Gamma_n)$ *if and only if*

$$V_-(x_1, \cdots, x_{n-1}) < x_n < V_+(x_1, \cdots, x_{n-1}).$$

Proof. The necessity of the condition can be easily derived.

To prove the sufficiency, we use induction on the dimension n. We want to prove that

$$V_-(x_1, \cdots, x_{n-2}) < x_{n-1} < V_+(x_1, \cdots, x_{n-2}). \tag{2.3}$$

By induction hypothesis, $(x_1, \cdots, x_{n-1})$ is in the interior of $\text{co}(\Gamma_{n-1})$. Using Sierpinski Theorem 0.6 that both the lower and upper functions are continuous in the interior, we conclude the point $(x_1, \cdots, x_n)$ belongs to the interior of $\text{co}(\Gamma_n)$.

It remains to show the inequalities (2.3). Suppose the contrary that $x_{n-1} = V_-(x_1, \cdots, x_{n-2})$ or $x_{n-1} = V_+(x_1, \cdots, x_{n-2})$. We use the same proof of Claim 3 in the proof of Theorem 2.8 to conclude that $x_n = V_-(x_1, \cdots, x_{n-1}) = V_+(x_1, \cdots, x_{n-1})$; this contradicts our hypothesis. This concludes the proof. $\square$

Final Remark. The author learned during the writing of the thesis that the convex sets $\mathrm{co}(\Gamma_n)$ were also investigated by Karlin and Shapley in [87] and by Skibinsky in [168].

2.3 Applications

We will answer the contractive subnormal completion problem. Consult Curto and Fialkow [46], [48] and [45] for the answer of subnormal completion problem.

Theorem 2.11 *Let α_k be positive for $k = 0, \cdots, m-1$. Define $\beta_0 = 1$ and $\beta_k = \alpha_0^2 \cdots \alpha_{k-1}^2$ when $k = 1, \cdots, m$. For $(\alpha_0, \cdots, \alpha_{m-1})$ to have a contractive subnormal completion it is necessary and sufficient that $\beta_1 \in [0, 1]$ and*

$$V_-(\beta_1, \cdots, \beta_k) \leq \beta_{k+1} \leq V_+(\beta_1, \cdots, \beta_k)$$

for $k = 1, \cdots, m-1$.

Proof. The necessity follows immediately from Berger's Theorem 0.8 and Theorem 2.7.

For the sufficiency, we need only to construct a sequence $\{\alpha_{m+p}\}_{p=0}^{\infty}$ of *positive* numbers so that the following sequence

$$1, \alpha_0^2, \alpha_0^2\alpha_1^2, \alpha_0^2\alpha_1^2\alpha_2^2,\ \alpha_0^2\alpha_1^2\alpha_2^2\alpha_3^2,\ \cdots$$

is a Hausdorff moment sequence. For then $(\alpha_0, \cdots, \alpha_{m-1})$ has a contractive subnormal completion by Berger's Theorem 0.8.

Let us define positive numbers α_{m+p} for $p \geq 0$ inductively in such a way that

$$\begin{cases} V_-(\beta_1, \cdots, \beta_m) \leq \beta_m \alpha_m^2 \leq V_+(\beta_1, \cdots, \beta_m), \\ V_-(\beta_1, \cdots, \beta_{m+k}) \leq \beta_{m+k} \alpha_{m+k}^2 \leq V_+(\beta_1, \cdots, \beta_{m+k}), \end{cases}$$

where $\beta_{m+k} = \beta_m \alpha_m^2 \cdots \alpha_{m+k-1}^2$ for $k = 1, 2, \cdots$. The inductive continues indefinitely because $(\beta_1, \cdots, \beta_k) \in \mathrm{co}(\Gamma_k)$; a fact seen from the claim in the proof of Theorem 2.7. It remains to show that the constructed numbers α_{m+p} for $p \geq 0$ are not zero. In fact we will prove that $V_-(\beta_1, \cdots, \beta_k) > 0$ for all $k \geq m$. Suppose otherwise, then for some $k \geq m$ we have $V_-(\beta_1, \cdots, \beta_k) = 0$. Thus $(\beta_1, \cdots, \beta_k; 0) \in \mathrm{co}(\Gamma_{k+1})$ as seen again from the claim in the proof of Theorem 2.7. Hence, from Theorem 2.7 there exists a probability measure μ such that

$$\beta_j = \int_0^1 t^j d\mu(t), \quad j = 1, \cdots, k$$

and

$$0 = \int_0^1 t^{k+1} d\mu(t).$$

It is easily seen that μ is an atom at the origin; hence, $\beta_1 = \cdots = \beta_k = 0$, a contradiction. □

Corollary 2.12 *Let a_0, a_1, a_2, a_3, a_4 be distinct* positive *numbers.*

1. *A necessary and sufficient condition for a_0, a_1, a_2 to have a subnormal completion is $a_0 < a_1 < a_2$.*

2. *Necessary and sufficient conditions for a_0, a_1, a_2, a_3 to have a subnormal completion are $a_0 < a_1 < a_2$ and*
$$\frac{a_1^2}{a_2^2} \cdot \frac{a_2^4 - 2a_0^2 a_2^2 + a_0^2 a_1^2}{a_1^2 - a_0^2} \leq a_3^2.$$

3. Necessary and sufficient conditions for a_0, a_1, a_2, a_3, a_4 to have a subnormal completion are $a_0 < a_1 < a_2$ and

$$\frac{a_1^2}{a_2^2} \cdot \frac{a_2^4 - 2a_0^2a_2^2 + a_0^2a_1^2}{a_1^2 - a_0^2} \leq a_3^2,$$

$$\frac{a_2^2}{a_3^2} \cdot \frac{a_3^4 - 2a_1^2a_3^2 + a_1^2a_2^2}{a_2^2 - a_1^2} \leq a_4^2.$$

Proof. First note that every operator becomes a contraction by multiplying it by a constant. Hence, a sequence of *positive* numbers $a_0, \cdots, a_{m-1}$ has a subnormal completion if and only if there is a positive number a such that the sequence $aa_0, \cdots, aa_{m-1}$ has a contractive subnormal completion.

To see the proof of 1: Using Theorem 2.11 and Corollary 2.9, we see that the sequence a_0, a_1, a_2 has a subnormal completion when and only when there exists a positive number a such that

$$\begin{cases} (a^2a_0^2)^2 \leq a^4a_0^2a_1^2 \leq a^2a_0^2, \\ \dfrac{(a^4a_0^2a_1^2)^2}{a^2a_0^2} \leq a^6a_0^2a_1^2a_2^2 \leq a^4a_0^2a_1^2 - \dfrac{(a^2a_0^2 - a^4a_0^2a_1^2)^2}{1 - a^2a_0^2}. \end{cases}$$

That is, for some $a > 0$, we have

$$\begin{cases} a_0 \leq a_1, \\ aa_1 \leq 1, \\ a_1 \leq a_2, \\ a^2a_2^2 \leq 1 - \dfrac{a_0^2}{a_1^2}\dfrac{(1 - a^2a_1^2)^2}{1 - a^2a_0^2}. \end{cases}$$

Since a_0, a_1, a_2 are distinct, the necessity is clear. As for the sufficiency we find a small positive number a so that $aa_1 \leq 1$ and $a^2a_2^2 \leq 1 - \frac{a_0^2}{a_1^2}\frac{(1 - a^2a_1^2)^2}{1 - a^2a_0^2}$. The proof for the first part is complete.

To see the proof of 2 and 3: Using Theorem 2.11 and Corollary 2.9, we glean that the sequence of four numbers a_0, a_1, a_2, a_3 has a subnormal completion when and only when there exists a positive number a so that

$$\begin{cases} a_0 \leq a_1, \\ aa_1 \leq 1, \\ a_1 \leq a_2, \\ a^2 a_2^2 \leq 1 - \dfrac{a_0^2}{a_1^2} \dfrac{(1 - a^2 a_1^2)^2}{1 - a^2 a_0^2}, \\ \dfrac{a_1^2}{a_2^2} \cdot \dfrac{a_2^4 - 2a_0^2 a_2^2 + a_0^2 a_1^2}{a_1^2 - a_0^2} \leq a_3^2, \\ a^2 a_3^2 \leq 1 - \dfrac{a_1^2}{a_2^2} \dfrac{(1 - a^2 a_2^2)^2}{1 - a^2 a_1^2}. \end{cases}$$

The rest of the proof of the theorem proceeds in the obvious fashion. □

Remark. Part 1 is due to Stampfli [170, Theorem 5 p373].

If a sequence has a subnormal completion, then the last proof implicitly exhibits the smallest norm among those subnormal completions. In fact, this norm equals $\frac{1}{a_{\max}}$, where $a_{\max}$ is the biggest a that makes the appropriate inequalities hold. (cf [170, p373 Theorem 5] and [46, p35 Theorem 3.9].)

Examples. Let us look at some interesting facts about the initial weights of Bergman shift.

1. The sequence of three numbers $\sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}$ has a subnormal completion of minimal norm $\dfrac{1}{\sqrt{3 - \sqrt{3}}}$ ($\cong 0.8881$).

   Reason: According to the previous remark, the minimal norm can be found by solving the system of inequalities for the biggest a:

$$\begin{cases} a^2 \leq \frac{3}{2}, \\ a^4 - 6a^2 + 6 \geq 0. \end{cases}$$

2. The sequence of four numbers $\sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}$ has a subnormal completion of minimal norm $\dfrac{1}{\sqrt{2 - \sqrt{\frac{2}{3}}}}$ ($\cong 0.9192$).

Reason: The answer is yielded by solving the following system for the biggest a:

$$\begin{cases} a^2 \leq \frac{3}{2}, \\ a^4 - 6a^2 + 6 \geq 0, \\ a^4 - 4a^2 + \frac{10}{3} \geq 0. \end{cases}$$

3. The sequence of four numbers $\sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, a_3$ has a subnormal completion if and only if $a_3 \geq \frac{\sqrt{7}}{3}$.

   The claim follows immediately from Theorem 2.12. In particular, this shows that $a_0 < a_1 < a_2 < a_3$ is not enough to guarantee that the corresponding sequence has a subnormal completion as noted in [170, p372 and p377]. See also Gellar-Wallen [66].

Chapter 3

COMPLETELY POSITIVE FUNCTIONS

In this chapter we will consider the question: when does a continuous function on [0, 1] map each moment sequence into another one? A function possessing this property we shall call *completely positive*. A special case of interest is whether x^α is completely positive for each $\alpha > 0$. This seemingly distinguished case holds the key to understanding all the completely positive functions. We will show that the power function x^p for $0 < p < 1$ is not completely positive. A formula for completely positive functions is given. In the end we will consider the product of measures that makes $M([0,1])$ a Banach algebra.

3.1 Representation of Completely Positive Functions

Definition 3.1 *We call a continuous real function $\phi(t)$ on [0,1]* completely positive *if $\phi(1) = 1$ and whenever the sequence $\{x_k\}_0^\infty$ is a Hausdorff moment sequence and $x_0 = 1$, so is the sequence $\{\phi(x_k)\}_0^\infty$.*

Fix a probability measure μ and a completely positive function ϕ on [0, 1]. We define the probability measure $\phi(\mu)$ by

$$\int_0^1 t^k d\phi(\mu)(t) = \phi(\int_0^1 t^k d\mu(t)),$$

where $k = 0, 1, \cdots$.

Example. For any positive integer n, the power function x^n is a completely positive function. This follows immediately from Corollary 2.5. We will use the notation μ^n to denote the measure $\phi(\mu)$ for $\phi(x) = x^n$.

We enumerate several facts about completely positive functions.

Corollary 3.2

1. *Let ϕ be completely positive, and $(x_1, \cdots, x_n) \in \mathrm{co}(\Gamma_n)$. Then*

$$(\phi(x_1), \cdots, \phi(x_n)) \in \mathrm{co}(\Gamma_n).$$

2. *If $\{\phi_j\}_1^\infty$ is a sequence of completely positive functions and $\sum_1^\infty a_j = 1$ for non-negative numbers a_j's, then $\sum_1^\infty a_j\phi_j$ is completely positive.*

3. *If ϕ is completely positive, then the function is non-decreasing.*

Proof. The first statement follows easily from Corollary 2.4 and the definition of complete positivity. The second follows from the fact that if μ is a probability measure, then so is the measure

$$\sum_1^\infty a_j\phi_j(\mu).$$

Let us prove the last statement. First note that the sequence $\{\phi(t^k)\}_0^\infty$ is a Hausdorff moment sequence for $t \in [0,1]$, so $\phi(t^k) \leq \phi(t)$ for $k = 1, 2, \cdots$. Now suppose $0 < y < x < 1$, we need only to show that $\phi(y) \leq \phi(x)$ since the cases where $y = 0$ and $x = 1$ follow from the continuity of the function ϕ. Assume that $x^{2^{k+1}} \leq y < x^{2^k}$ for some non-negative integer k. We have that

$$\begin{aligned} \phi(y) &\leq \phi(x^{2^k}) \\ &\leq \phi(x). \end{aligned}$$

□

Remark. We do not know whether a completely positive function is convex. Neither do we know for a completely positive function ϕ that the measure $\phi(\mu)$ is supported on $[0, x]$ whenever the probability measure μ is supported on the interval $[0, x]$ for $0 < x < 1$.

The integer power functions and all convex combinations (indicated in the second part of the corollary) of them exhaust our supply of examples of completely positive functions.

The next theorem says that any completely positive function is a "convex combination" of x^{α}'s for $\alpha \geq 0$. Thus, the question of describing the class of completely positive functions depends on which power functions x^{α} are completely positive, where $\alpha > 0$. The latter problem will be briefly considered after the following theorem.

Let us denote the one point compactification of the locally compact half line $[0, \infty)$ by $[0, \infty]$, and use the notation x^t for fixed $x \in (0,1)$ to denote the continuous function $f_x(t)$ on the compact Hausdorff space $[0, \infty]$ such that $f_x(0) = 1, f_x(\infty) = 0$ and $f_x(t) = x^t$ for $0 < t < \infty$.

Theorem 3.3 *If ϕ is completely positive on the interval [0, 1], then there exists a probability measure μ on the compact space $[0, \infty]$ so that for $x \in (0,1)$,*

$$\phi(x) = \int_0^{\infty} x^t \, d\mu(t).$$

Proof. Fix the point $x \in (0,1)$. By the definition of complete positivity of ϕ, the sequence $\{\phi(x^k)\}_0^{\infty}$ is a Hausdorff moment sequence of a probability measure, say ν_x, on the interval [0, 1]. So we get that

$$\phi(x^k) \quad = \quad \int_0^1 t^k d\nu_x(t), \tag{3.1}$$

for $k = 0, 1, 2, \cdots$.

By changing variables via the substitution $t = x^s$, we get from the equation (3.1) a probability measure on $[0, \infty]$, say μ_x, such that

$$\phi(x^k) = \int_0^{\infty} x^{ks} \, d\mu_x(s), \tag{3.2}$$

for $k = 0, 1, 2, \cdots$.

Now we choose $x = x_n = \frac{1}{\sqrt[n]{2}}$ for any positive integer n, and set $\mu_n = \mu_{x_n}$. There is a subsequence $\{\mu_{n_j}\}$ which converges to a measure μ in the weak star topology in the dual space $M(X)$ of the Banach space $\mathbf{C}(X)$ according to Helly's Selection Theorem 0.5.

Let r be any positive number, we choose for each index j the positive integer k_j such that

$$\frac{k_j - 1}{n_j} < r \leq \frac{k_j}{n_j}.$$

Taking the limit as $j \to \infty$ of the following:

$$\phi(2^{-\frac{k_j}{n_j}}) = \int_0^\infty (2^{-\frac{k_j}{n_j}})^s \, d\mu_{n_j}(s),$$

which follows from the equation (3.2), we obtain that

$$\begin{aligned}
& |\phi(2^{-r}) - \int_0^\infty 2^{-rs} \, d\mu(s)| \\
= \; & |\lim_{j\to\infty} \int_0^\infty (2^{-\frac{k_j}{n_j}s} - 2^{-rs}) \, d\mu_{n_j}(s)| \\
\leq \; & \lim_{j\to\infty} \int_0^\infty |2^{-\frac{k_j}{n_j}s} - 2^{-rs}| \, d\mu_{n_j}(s)| \\
\leq \; & \lim_{j\to\infty} \int_0^\infty \frac{|\frac{k_j}{n_j} - r|}{er} \, d\mu_{n_j}(s) \\
\leq \; & \lim_{j\to\infty} \frac{1}{ern_j} \\
= \; & 0.
\end{aligned}$$

Hence, for any positive number r, we have

$$\phi(2^{-r}) = \int_0^\infty 2^{-rs} \, d\mu(s).$$

The conclusion is now obvious. □

Remarks. The idea of choosing a special sequence of points x_n's in the last proof was inspired from reading Bernstein's paper [18, p20]. Also consult [1, p205].

The proof is valid for x^p for $0 < p < 1$, but as we shall see that these particular functions are not completely positive. So this representation in the theorem is not the one that always generates completely positive functions. Consult the remark of Corollary 3.2.

Corollary 3.4 *If the function ϕ is completely positive on the interval [0,1], then*

1. $\phi(x)\,\phi(y) \leq \phi(x\,y)$ *for all* $x,\ y \in (0,1)$.
2. $\phi^p(x) \leq \phi(x^p)$ *for any* $x \in (0,1), \quad p > 1$.
3. *The function* $\ln\ \phi(e^x)$ *is convex on* $(-\infty, 0)$.

Proof. We shall only prove 1. The rest follows easily from Hölder's inequality. By Theorem 3.3, we have a probability measure μ on the space $[0, \infty]$ so that

$$\phi(x) = \int x^t \, d\mu(t).$$

Thus, by Fubini's Theorem, we see that

$$\begin{aligned}
& \phi(x\, y) - \phi(x)\, \phi(y) \\
= & \int d\mu(t) \int [(xy)^s - x^t y^s] d\mu(s) \\
= & \int d\mu(t) \int_0^t (x^s - x^t) y^s d\mu(s) + \int d\mu(t) \int_t^\infty (x^s - x^t) y^s d\mu(s) \\
= & \int d\mu(t) \int_0^t (x^s - x^t) y^s d\mu(s) + \int d\mu(t) \int_0^t (x^t - x^s) y^t d\mu(s) \\
= & \int d\mu(t) \int_0^t (x^s - x^t)(y^s - y^t) d\mu(s) \\
\geq & \ 0.
\end{aligned}$$

□

The next proposition gives a modest answer to the problem of which power functions may be completely positive.

Proposition 3.5 *Given $\alpha > 0$. For the power function x^α to be completely positive it is necessary and sufficient that for $n \geq 2$ and $(x_1, \cdots, x_n) \in \mathrm{co}(\Gamma_n)$,*

$$\begin{cases} V_-(x_1^\alpha, \cdots, x_n^\alpha) \leq V_-^\alpha(x_1, \cdots, x_n), \\ V_+^\alpha(x_1, \cdots, x_n) \leq V_+(x_1^\alpha, \cdots, x_n^\alpha). \end{cases}$$

Proof. It follows from Corollary 2.4 and Theorem 2.7. □

The following corollary answers Question 2 at the end of [108] negatively.

Corollary 3.6 *The power functions x^p for $0 < p < 1$ is not completely positive.*

Proof. Suppose for some positive number $p < 1$, the power functions x^p is completely positive. Using Proposition 3.5, we have

$$V_+^p(x_1, x_2) \leq V_+(x_1^p, x_2^p)$$

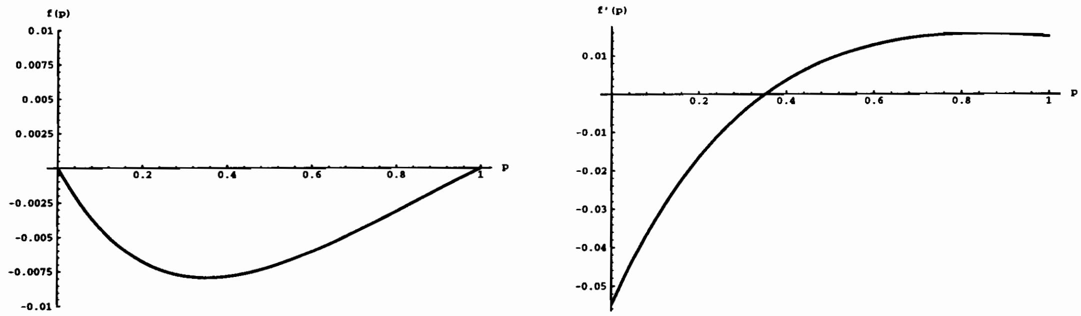

Figure 3.1: Graph of $f(p)$ and $f'(p)$

for $x_1^2 < x_2 < x_1$. In particular, we obtain that

$$(5/18)^p \leq V_+(2^{-p}, 3^{-p}).$$

Define $f(p) = 3^{-p} - (2^{-p} - 3^{-p})^2/(1 - 2^{-p}) - (5/18)^p$ on the interval $[0, 1]$, it is easy to check that $f(0) = f(1) = 0$. It suffices to show that $f(p) < 0$ for $0 < p < 1$. This can be seen clearly by plotting both $f(p)$ and $f'(p)$ on the interval $[0, 1]$ using *Mathematica* [181] or any other mathematical software package (consult Figure 3.1). (We can certainly prove this elementary fact, but it would be tedious.) That completes the proof. $\square$

Remark. We do not know whether the set of functions x^p for $p > 1$ is always completely positive.

3.2 $M([0,1])$ as a Banach Algebra

To finish the thesis we will look at the product of complex regular Borel measures on the interval $[0, 1]$, the set $M([0, 1])$ of all such measures is the dual Banach space of $\mathbf{C}([0, 1])$. Under this product, $M([0, 1])$ is also a unital Banach algebra.

We will also determine the measure μ^n for measure $\mu \in M([0, 1])$ and positive integer n in hope that the result may shed light on how to compute the measure μ^α when the positive number α is no longer an integer.

Definition 3.7 *Let μ, ν be two regular complex Borel measures on the interval [0, 1]. We define their* product measure, *denoted by $\mu\,\nu$, by*

$$\int f(x)\, d\mu\, \nu(x) = \int d\mu(s) \int f(st)\, d\nu(t),$$

for any function $f \in \mathbf{C}[0,1]$.

Remark. Since the map $f \to \int d\mu(s) \int f(st)\, d\nu(t)$ is a bounded linear functional on the Banach space $\mathbf{C}([0,1])$, using Riesz's representation theorem, we see that the product measure is well-defined.

Due to the fact that $\int_0^1 t^k\, d\mu\, \nu(t) = \int_0^1 t^k\, d\mu(t) \int_0^1 t^k\, d\nu(t)$ for $k = 0, 1, 2, \cdots$, we may write $\mu\,\mu$ as μ^2, which is consistent with our previous notation in the case where the measure μ is a probability measure.

Let E be a subset of the closed interval $[0,1]$. For convenience, we use the set notation $\frac{E}{t}$ to denote the set $\{\frac{1}{t} \cdot e : e \in E\}$, and the notation $\frac{E}{0}$ to denote the set $[0,1]$ if $0 \in E$ and the empty set $\emptyset$ if $0 \notin E$.

Proposition 3.8 *Let μ, ν be regular complex Borel measures on [0, 1], and let E be any measurable subset of [0,1]. We have the product measure formula*

$$\mu\,\nu(E) = \int_0^1 \mu(\frac{E}{t})\, d\nu(t).$$

Consequently we also have

$$\mu^{n+1}(E) = \int_0^1 d\mu(t_1) \int_0^1 d\mu(t_2) \cdots \int_0^1 \mu(\frac{E}{t_1 t_2 \cdots t_n})\, d\mu(t_n),$$

where $n = 1, 2, \cdots$.

Proof. All we need to show is that when both measures are positive the first equality is valid. To this end, it is enough to establish the equality for any compact subset K of the interval [0,1].

Let u_n be a sequence of continuous functions so that

$$\begin{cases} 0 \le u_{n+1} \le u_n \le 1, \\ u_n \to \chi_K \quad \textit{everywhere} \text{ on } [0,\ 1]. \end{cases}$$

First note for any integer n that

$$\int_0^1 u_n(t)\, d\mu\, \nu(t) = \int_0^1 d\nu(t) \int_0^1 u_n(s\, t)\, d\mu(s).$$

Now taking the limit of the last equality and using Lebesgue dominant convergence theorem, we get that

$$\begin{aligned} \mu\, \nu(K) &= \int_0^1 d\nu(t) \int_0^1 \chi_K(s\, t)\, d\mu(s) \\ &= \chi_K(0)\, \mu([0,1])\, \nu(\{0\}) + \int_{(0,1]} \mu(\frac{K}{t})\, d\nu(t) \\ &= \int_{[0,1]} \mu(\frac{K}{t})\, d\nu(t). \end{aligned}$$

This completes the proof. □

Corollary 3.9 *Let μ, ν be regular complex Borel measures on [0,1], and let a, b be two points in the interval $[0,1]$. We have the following:*

1. $\mu\, \nu([0,1]) = \mu([0,1])\, \nu([0,1])$.

2. $\delta_a\, \mu(E) = \mu(\frac{E}{a})$, *for any measurable subset E of the interval [0, 1].*

3. $\delta_a\, \delta_b = \delta_{ab}$.

As a consequence, the Banach space $M([0,1])$ of all regular complex Borel measures is also a unital Banach algebra with unit δ_1.

Proof. It consists of some routine computations. We omit the details. □

Example. Let μ be the measure $\delta_0 + i\delta_1$ on the interval $[0,1]$. Then $\overline{\mu} = \delta_0 - i\delta_1$, so

$$\|\mu\, \overline{\mu}\| = \|\delta_0 + \delta_1\| = 2.$$

Whereas we have $\|\mu\|^2 = 4$. Thus the Banach algebra $M([0,1])$ is *not* a C^*-algebra under the natural convolution $\mu \longrightarrow \overline{\mu}$. It partially answers Question 3 at the end of [108].

This example is suggested by Dr. James Thomson.

Even though the power function x^p for $0 < p < 1$ is not completely positive, we can give an interesting example to illustrate that all power functions *do* map the Hausdorff moment sequence of Lebesgue measure m on the interval [0, 1] into a Hausdorff moment sequence. The latter fact implies that the multiplication operators on Dirichlet spaces are subnormal. Hence, the claim made in the first paragraph on page 154 in [139] is established.

Example. Let m be the Lebesgue measure on $[0, 1]$. We want to compute the measure m^{n+1} for $n = 0, 1, \cdots$.

Claim. $d\, m^{n+1}(x) = \frac{1}{n!} \ln^n(\frac{1}{x})\, dx.$

It is true for $n = 0$. We assume by induction that it is true also for $n < k$. In light of Proposition 3.8, we see for $0 < a < b < 1$ that

$$
\begin{aligned}
& m^{k+1}([a,b]) \\
= \; & \int_0^1 m(\frac{[a,b]}{x})\, dm^k(x) \\
= \; & \int_a^b (1 - \frac{a}{x})\, dm^k(x) + \int_b^1 \frac{b-a}{x}\, dm^k(x) \\
= \; & \int_a^b dm^k(x) + \int_b^1 \frac{b}{x}\, dm^k(x) - \int_a^1 \frac{a}{x}\, dm^k(x) \\
= \; & \int_a^b dm^k(x) - \int_b^1 d\frac{b}{k!}(\ln\frac{1}{x})^k + \int_a^1 d\frac{a}{k!}(\ln\frac{1}{x})^k \\
= \; & \int_a^b dm^k(x) + \frac{x}{k!}(\ln\frac{1}{x})^k \Big|_a^b \\
= \; & \int_a^b \frac{1}{(k-1)!}(\ln\frac{1}{x})^{k-1}\, dx + \int_a^b \frac{d\frac{x}{k!}(\ln\frac{1}{x})^k}{dx}\, dx \\
= \; & \int_a^b \frac{1}{k!}\, (\ln\frac{1}{x})^k\, dx.
\end{aligned}
$$

Our claim now follows by a standard argument.

Now it is natural to define

$$dm^\alpha(x) = \frac{1}{\Gamma(\alpha)}(\ln\frac{1}{x})^{\alpha-1}\, dx,$$

where $\alpha > 0$. It is not difficult to check that

$$\int_0^1 t^n \, dm^\alpha(t) \;=\; \left(\frac{1}{1+n}\right)^\alpha, \tag{3.3}$$

for $n = 0, 1, \cdots$.

As a consequence, the multiplication operator M_z on the Dirichlet space $D_{-\alpha} = \{f(z) \in \mathbf{H}(D) : \sum_0^\infty (1+n)^{-\alpha} |\hat{f}(n)|^2 < \infty\}$ (consult [139, p151]), where $\mathbf{H}(D)$ is the set of all analytic functions on the open unit disk $\mathbf{D}$ and $\hat{f}(n)$ is the n-th Taylor coefficient, is a cyclic subnormal operator. Because $\{(1+n)^{\alpha/2} z^n\}_{n=0}^\infty$ is an orthonormal basis for $D_{-\alpha}$, and M_z is a unilateral weighted shift with weight sequence $\{(\frac{1+n}{2+n})^{\alpha/2}\}_0^\infty$ under this basis, the subnormality follows immediately from Berger's theorem 0.8 and the computation (3.3).

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Education

Ph. D. in Mathematics, Virginia Tech, in August 1994.

M. S. in Mathematics, Sichuan University, China, in July 1986.

B. S. in Mathematics, Jilin University, China, in July 1983.

Experience

1993–1994

Taught Multivariable Calculus with Professor Robert F. Olin by incorporating **Mathematica** into the course. This is a renewal of the Calculus effort at Virginia Tech.

1992–1994

Developed a strong working knowledge of **Mathematica**, **Matlab**, **C**, **UNIX** operating systems, and **DOS** operating system.

1991–1994

Taught various calculus courses, such as Algebra with trigonometry I and II, Calculus with matrices I and II, at Virginia Tech.

1986–1989

Taught advanced calculus, real analysis, and functional analysis at People's University of China.

Teaching evaluation: 3.0 — 3.2.

Research and Publication

Thesis title: *Moment sequences and their applications.*

Thesis advisor: Dr. Robert F. Olin.

Area of research: Functional analysis, moment theory, complex analysis, operator theory, and harmonic analysis.

Xiaoguang Li, **On positive moment sequences**, to appear in Houston Journal of Mathematics.

Professional Lectures

Presented a 50-minute lecture at Old-Dominion Analysis Conference in Richmond, VA, April 1994.

Presented a 20-minute lecture at South-Eastern Analysis Meeting (SEAM IX) in Memphis, TN, March 1993.

Delivered a lecture series on "Weak star generators of H^∞" in the Functional Analysis seminar at Virginia Tech, 1992.

Delivered a lecture series on "Removable sets of analytic functions satisfying a Lipschitz condition" in the Complex Analysis Seminar at Virginia Tech, 1991.

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