A Classification of some Quadratic Algebras

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Dissertation submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
in
Mathematics

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July 16, 1997
Blacksburg, Virginia

Keywords: Koszul algebra, quadratic algebra, monomial generators

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(ABSTRACT)

In this paper, for a select group of quadratic algebras, we investigate restrictions necessary on the generators of the ideal for the resulting algebra to be Koszul. Techniques include the use of Gröbner bases and development of Koszul resolutions. When the quadratic algebra is Koszul, we provide the associated linear resolution of the field. When not Koszul, we describe the maps of the resolution up to the instance of nonlinearity.

Dedication

I wish to dedicate this work to my family: my father, Ed Davis, for his understanding and sound council; my mother, Bertha Davis, for her boundless strength and love; and my husband, Jim Dobberfuhl, for the joy and laughter he brings to my life.

Acknowledgments

I wish to thank my advisor, Dr. Green, for his guidance and support, as well as for directing me into such a demanding and wildly rewarding field of mathematics. Through him, I have gained a greater understanding of mathematics than I had dared to hope.

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Chapter 1

Introduction and Background

This paper aims to further understand for which quadratic generators of an ideal the resulting algebra will be Koszul. Additionally, we are interested in developing the associated Koszul resolutions for the algebras which are Koszul, as well as the nonlinear resolutions for the cases when the algebra is not Koszul. We begin with some fundamental definitions and concepts.

Given an ideal, I, and a free algebra, $K\langle X_1, \ldots, X_t\rangle$, we say the quotient

$$\Lambda = K\langle X_1, \ldots, X_t\rangle / I$$

is a quadratic algebra if I is generated by relations $\rho = \sum_{i=1}^{n} c_i y_i z_i$ where $y_i, z_i \in \{X_1, \ldots, X_t\}$ and $c_i \in K$ for all i. That is, I is generated by linear combinations of quadratic monomials. Some quadratic algebras are additionally Koszul algebras. A quadratic algebra, Λ, is Koszul if a linear resolution of the field can be obtained [8]. We say a projective resolution of a graded Λ-module, M, $\cdots \to P_n \stackrel{\varphi_n}{\to} P_{n-1} \stackrel{\varphi_{n-1}}{\to} \cdots \stackrel{\varphi_2}{\to} P_1 \stackrel{\varphi_1}{\to} P_0 \stackrel{\varphi_0}{\to} M \to 0$ is linear if each P_n is generated in degree n and $\ker \varphi_n$ is generated in degree $n+1$ for all $n \geq 0$.

For this specific class of algebras, a projective resolution of the field, K, which will be of the form $\ldots \stackrel{\varphi_3}{\to} \Lambda^{n_2} \stackrel{\varphi_2}{\to} \Lambda^{n_1} \stackrel{\varphi_1}{\to} \Lambda^{n_0} \stackrel{\varphi_0}{\to} K \to 0$, is said to be linear provided that φ_i, when regarded as an $(n_{i-1} \times n_i)$ matrix acting on an n_i-tuple, has linear entries, for all i. If such a resolution exists, then Λ will be Koszul. We note here that the above characterization of Koszul is not the classic definition as introduced by Priddy [12]. For a review of basic properties of Koszul algebras and modules, including alternate definitions of Koszul, see Backelin and Fröberg [1] and Green and Martinez-Villa [8].

Koszul algebras naturally arise in areas of mathematics such as algebraic topology, Lie theory, quantum groups, and algebraic geometry [2] [3] [4] [9] [11]. Considering the importance of these algebras in such varied topics, it is desirable to definitively isolate necessary and sufficient conditions on the ideals for the resulting quadratic algebra to be Koszul. However, in practice, the resolutions obtained are difficult to classify and describe. As a result, not much is known about what makes an algebra Koszul.

We do know that if the ideal, which is generated by linear combinations of quadratic monomials, further has a quadratic (noncommutative) Gröbner basis, then the resulting quadratic algebra will be Koszul. This is not a necessary condition, however. A Gröbner basis is a generating set of relations that also satisfies further criterion. In order to understand the additional conditions for a generating set to be a Gröbner basis, we must first impose an order on the monomials of the free algebra. More specifically, we need an admissible ordering. We now give some common examples of admissible orders.

(left)length-lexicographic order For length lexicographic order, we arbitrarily order the variables of the free algebra: $X_1 < X_2 < \cdots < X_t$. Then, if $M = X_{j_1}X_{j_2}\cdots X_{j_m}$ and $N = X_{k_1}X_{k_2}\cdots X_{k_n}$ are monomials of the free algebra, $M < N$ if $m < n$. If $m = n$, then $M < N$ if for some $1 \leq i \leq m$, $X_{j_l} = X_{k_l}$ for all $l < i$ and $X_{j_i} < X_{k_i}$. [7]

(left)weight-lexicographic order For weight lexicographic order, we start with a set map $w : \{X_1, \ldots, X_t\} \longrightarrow \{1, 2, \ldots\}$. We extend to $w : \{\text{monomials}\} \longrightarrow \{1, 2, \ldots\}$ by $w(X_{j_1}X_{j_2}\cdots X_{j_m}) = \sum_{i=1}^{m} w(X_{j_i})$. Then, the variables are ordered by $X_i < X_j$ if $w(X_i) < w(X_j)$. Finally, if $M = X_{j_1}X_{j_2}\cdots X_{j_m}$ and $N = X_{k_1}X_{k_2}\cdots X_{k_n}$ are monomials of the free algebra, $M < N$ if $w(M) < w(N)$. If $w(M) = w(N)$, then $M < N$ if $M < N$ under length lexicographical order. [7]

Once an admissible order is choosen, for any polynomial in the noncommutative polynomial ring in t-variables, we can identify the leading term of the polynomial; this term is also known as the "tip" of the polynomial. A generating set, $\mathcal{G}$, of an ideal, I, is a noncommutative Gröbner basis for I provided that the ideal generated by the set of leading terms of $\mathcal{G}$ is the same ideal generated by the set of leading terms of I;

$$\text{i.e.}\langle \text{Tip}(\mathcal{G}) \rangle = \langle \text{Tip}(I) \rangle$$

This means that if p is a polynomial in the ideal, then there is an element of the Gröbner basis whose leading term divides the leading term of p.

Not every generating set for an ideal is also a Gröbner basis for the ideal. However, a Gröbner basis may be obtained from any generating set via the Buchberger Algorithm [5], in the commutative case, and the Buchberger-Mora-Farkas-Green Algorithm [7] in the noncommutative case. In the commutative case, the algorithm always terminates in a finite number of steps, resulting in a finite Gröbner basis. However, the noncommutative algorithm does not always terminate or result in a finite Gröbner basis. Hence, in the noncommutative case, while the Gröbner basis of an ideal always exists, it is not necessarily finite.

So, if, from the set of quadratic relations of the ideal, we obtain a (finite) quadratic Gröbner basis, then the algebra is Koszul [9]. We wish to emphasize again that this is not a necessary condition. That is, conversely, if, using the Buchberger-Mora-Farkas-Green Algorithm, we obtain a nonquadratic term to add to the Gröbner basis, then there is inconclusive evidence as to the Koszul nature of the algebra. In this case, we must employ some other method of determining if the algebra is Koszul.

It has been shown by E. Green and D. Zacharia [10] that if the ideal is generated by quadratic monomials, then it will have a quadratic Gröbner basis, and hence, the resulting algebra will be Koszul [8]. Thus, it is certainly true that an algebra which is the quotient of an ideal generated by perfect square monomials is Koszul. We wish to understand the situation arising when, to the above ideal, we add a nonmonomial generator. This creates a class of algebras of the form

$$K\langle X_1, \cdots, X_t\rangle / \langle m_1, \cdots, m_n, p + cq\rangle$$

where $m_i = X_{j_i}^2$ is a perfect square for all i, p and q are monomials and $c \in K$. For this class, the Gröbner basis of the ideal is not always quadratic. It is this class of algebras, the Gröbner bases of the ideals, and the projective resolutions of the field that we explore in this paper. Additionally, we investigate the class of the form

$$K\langle X_1, \cdots, X_t\rangle / \langle m_1, \cdots, m_n, \sum c_1^i p_1^i, \cdots, \sum c_z^i p_z^i\rangle$$

where $c_j^i \in K$, m_i, p_j^i are perfect squares for all i, j.

Lastly, once the prior algebras are explored, we then apply dual basis theory to the Koszul algebras, expanding the known classes of quadratic Koszul algebras.

Chapter 2

Results

We began this work with the knowledge that an algebra of the form

$$K\langle X_1, \cdots, X_t\rangle / \langle m_1, \cdots, m_n\rangle$$

where each $m_i = X_{j_i}^2$ is a perfect square, will be Koszul since the Gröbner basis of the ideal is always quadratic. We found that adding a single nonmonomial term to the generating set of the previous ideal produces ideals and algebras of varied properties. Of the investigated algebras of the form

$$K\langle X_1, \cdots, X_t\rangle / \langle m_1, \cdots, m_n, p + cq\rangle$$

where $m_i = X_{j_i}^2$ is a perfect square for all i, p and q are monomials and $c \in K$, not all of the Gröbner bases of the ideals are quadratic, nor are all of the resulting algebras Koszul. However, those algebras that are Koszul, regardless of the status of their Gröbner bases, can be divided into two distinct, but similar, classes. The first can be characterized by having resolutions of the form :

$$\ldots \xrightarrow{\varphi} \Lambda^{n+1} \xrightarrow{\varphi} \Lambda^{n+1} \xrightarrow{\varphi_2} \Lambda^t \xrightarrow{\begin{pmatrix} X_1 & \cdots & X_t \end{pmatrix}} \Lambda \to K \to 0$$

This includes the algebras

$$\Lambda = K\langle X_1, \cdots, X_t\rangle / \langle m_1, \cdots, m_n, X_\gamma X_{j_i} + cX_\beta X_{j_i}\rangle$$

$$\Lambda = K\langle X_1, \cdots, X_t\rangle / \langle m_1, \cdots, m_n, X_{j_i} X_\gamma + cX_{j_i} X_\beta\rangle$$

$$\Lambda = K\langle X_1, \cdots, X_t\rangle / \langle m_1, \cdots, m_n, X_{j_i} X_{j_k} + cX_{j_i} X_{j_l}\rangle$$

and

$$\Lambda = K\langle X_1, \cdots, X_t\rangle / \langle m_1, \cdots, m_n, X_{j_k} X_{j_i} + cX_{j_l} X_{j_i}\rangle$$

where $\gamma \neq \beta, j_v$ for all v and i, k, l are all distinct, as described in Theorems 3.1, 3.2, 4.1, and 4.2. Each resolution consists of the same modules, and, while the algebras produce different maps, in each case the maps and modules begin to repeat at the first appearance of the Λ-module Λ^{n+1}.

Additionally, if we look at the maps associated with the algebras

$$\Lambda = K\langle X_1, \cdots, X_t\rangle / \langle m_1, \cdots, m_n, X_\gamma X_{j_i} + cX_\beta X_{j_i}\rangle$$

and

$$\Lambda = K\langle X_1, \cdots, X_t\rangle / \langle m_1, \cdots, m_n, X_{j_k} X_{j_i} + cX_{j_l} X_{j_i}\rangle$$

where $\gamma \neq \beta, j_v$ for all v and i, k, l all distinct, as described in Theorems 3.1 and 4.2, only one of which has a quadratic Gröbner basis, we find they are remarkably similar. They differ only in the placement of a single column in each matrix. However, of even more interest is the identical nature of the maps associated with the algebras

$$\Lambda = K\langle X_1, \cdots, X_t\rangle / \langle m_1, \cdots, m_n, X_{j_i} X_\gamma + cX_{j_i} X_\beta\rangle$$

and

$$\Lambda = K\langle X_1, \cdots, X_t\rangle / \langle m_1, \cdots, m_n, X_{j_i} X_{j_k} + cX_{j_i} X_{j_l}\rangle$$

where $\gamma \neq \beta, j_v$ for all v and i, k, l are all distinct, from Theorems 3.2 and 4.1. Again, only one of which has a quadratic Gröbner basis.

The second class consists of the algebras

$$\Lambda = K\langle X_1, \cdots, X_t\rangle / \langle m_1, \cdots, m_n, X_\beta X_\gamma + cX_\alpha X_\theta\rangle$$

and

$$\Lambda = K\langle X_1, \cdots, X_t\rangle / \langle m_1, \cdots, m_n, X_s^2 + cX_u^2\rangle \tag{2.1}$$

where $\beta, \gamma \neq j_v$ for all v, $\beta \neq \gamma$ and $s \neq u$, as described in Theorems 3.3 and 4.3. We note that 2.1 is obtained by reducing the algebra described in Theorem 4.3 down to the simplest nontrivial case. For this class, we again have for both algebras resolutions of the form:

$$\ldots \xrightarrow{\varphi} \Lambda^n \xrightarrow{\varphi} \Lambda^n \xrightarrow{\varphi_3} \Lambda^{n+1} \xrightarrow{\varphi_2} \Lambda^t \xrightarrow{(\, X_1 \ \cdots \ X_t \,)} \Lambda \to K \to 0$$

with repetition now occuring at the first appearance of the Λ-module Λ^n. In fact, for both algebras the resolution is made up of the same modules, and, as above in the case of the common term on the left, maps of identical form.

For the class of algebras found not to be Koszul, namely

$$\Lambda = K\langle X_1, \cdots, X_t\rangle / \langle m_1, \cdots, m_n, X_{j_i} X_\beta + cX_\xi X_{j_i}\rangle$$

where $\beta \neq \xi$ and $\beta, \xi \neq j_i$, we found that the resolution of the field bore a marked resemblence to the first class of Koszul algebras above, up to the first appearance of the Λ-module Λ^{n+1}. The difference is that instead of repetition beginning at this location in the resolution, as occurs for the first type of Koszul algebras, we get the first nonlinear map. In both cases, however, it is at the first appearance of the Λ-module, Λ^{n+1}, that something of import occurs.

Chapter 3

Koszul Algebras - Part I

Throughout this chapter, $I = \langle m_1, \dots, m_n, p + cq \rangle$ for $m_1 = X_{j_1}^2, \dots, m_n = X_{j_n}^2$ perfect square monomials and p and q monomials in $K\langle X_1, \dots, X_t \rangle$, $c \in K$. In addition, $\Lambda = K\langle X_1, \dots, X_t \rangle \,/\, \langle m_1, \dots, m_n, p + cq \rangle$.

Lemma 3.1 *If* $p+cq = \begin{cases} X_{j_i}X_\gamma + cX_{j_i}X_\beta \\ X_\gamma X_{j_i} + cX_\beta X_{j_i} \end{cases}$ *, where* $\gamma \neq \beta, j_k$ *for all k, then I has a quadratic Gröbner basis.*

Proof WLOG, let $p + cq = X_{j_i}X_\gamma + cX_{j_i}X_\beta$, where $X_\gamma, X_\beta \in \{X_1, \dots, X_t\}$. Then, $X_\gamma, X_\beta \neq X_{j_i}$ and $X_\gamma \neq X_{j_k}$ for all k. Choose an order so that tip$(p + cq)$= $X_{j_i}X_\gamma$. Since $X_\gamma \neq X_{j_i}$, $p + cq$ has no self overlap. Hence, the only overlap involving $p + cq$ occurs with m_i:

$$o(p + cq, m_i) = X_{j_i}(X_{j_i}X_\gamma + cX_{j_i}X_\beta) - (X_{j_i}^2)X_\gamma = cX_{j_i}^2X_\beta \overset{m_i}{\longmapsto} 0$$

Thus, $\{m_1, \dots, m_n, p + cq\}$ is a quadratic Gröbner basis for I.

□

Rephrasing, if supp(p)$\bigcap$supp(q)$\bigcap$ supp$(m_i) \neq \phi$, for some i, X_l^2 does not divide pq, qp $\forall l$ and supp(p)$\bigcap$supp(m_j)= ϕ for all $j \neq i$, then I has a quadratic Gröbner basis. We remark that if $\beta = \gamma$ in the above description of $p + cq$, then I is generated by quadratic monomials and hence has a quadratic Gröbner basis by remarks in Chapter 1.

Corollary 3.1 $\Lambda = K\langle X_1, \dots, X_t \rangle \,/\, \langle m_1, \dots, m_n, p + cq \rangle$*, for* $p + cq$ *as defined in Lemma 3.1, is Koszul.*

Lemma 3.2 *If* $p + cq = X_{j_i}X_\beta + cX_\beta X_{j_i}$ *with* $\beta \neq j_i$ *and* $c \in K$*, then*

$$I = \langle m_1, \dots, m_n, p + cq \rangle$$

has a quadratic Gröbner basis.

Proof Choose an order so that tip $(p+cq) = X_{j_i}X_\beta = p$. Since p is not a perfect square, $p+cq$ has no selfoverlap. And,

$$o(p+cq, m_i) = X_{j_i}(X_{j_i}X_\beta + cX_\beta X_{j_i}) - (X_{j_i}^2)X_\beta = cX_{j_i}X_\beta X_{j_i}$$

$$cX_{j_i}X_\beta X_{j_i} - c(X_{j_i}X_\beta + cX_\beta X_{j_i})X_{j_i} = -c^2X_\beta X_{j_i}^2 \stackrel{m_i}{\longmapsto} 0$$

Now, suppose $X_\beta \mid m_k$ for some k (i.e., $X_\beta = X_{j_k}$). Then,

$$o(p+cq, m_k) = (X_{j_i}X_\beta + cX_\beta X_{j_i})X_{j_k} - X_{j_i}X_{j_k}^2 = cX_\beta X_{j_i}X_\beta$$

$$cX_\beta X_{j_i}X_\beta - cX_\beta(X_{j_i}X_\beta + cX_\beta X_{j_i}) = -c^2X_\beta^2 X_{j_i} \stackrel{m_k}{\longmapsto} 0$$

Thus, $\{m_1, \ldots, m_n, p+cq\}$ is a quadratic Gröbner basis for I.

□

That is, if neither p nor q is a perfect square, $supp(p) = supp(q)$, $supp(p) \bigcap supp(m_i) \neq \phi$ for some i, and $p \neq q$, then I has a quadratic Gröbner basis.

Corollary 3.2 *$\Lambda = K\langle X_1, \ldots, X_t\rangle \,/\langle m_1, \ldots, m_n, X_{j_i}X_\beta + cX_\beta X_{j_i}\rangle$ is Koszul.*

Lemma 3.3 *If $p + cq = X_\beta X_\gamma + X_\alpha X_\theta$, for $\beta, \gamma \neq j_l$ for all l and $\beta \neq \gamma$, then $I = \langle m_1, \ldots, m_n, p+cq\rangle$ has a quadratic Gröbner basis.*

Proof Choose an order so that tip$(p+cq) = X_\beta X_\gamma$. Now, since $\beta, \gamma \neq j_l$ for all l, $p+cq$ has no overlaps with the m_i's. And, since p is not a perfect square, $p+cq$ has no selfoverlaps. Thus, $\{m_1, \ldots, m_n, p+cq\}$ is a quadratic Gröbner basis for I.

□

That is, if p is not a perfect square and supp(p)$\bigcap$supp(m_i)$= \phi$ for all i, then I has a quadratic Gröbner basis.

Corollary 3.3 *$\Lambda = K\langle X_1, \ldots, X_t\rangle \,/\langle m_1, \ldots, m_n, p+cq\rangle$, for $p+cq$ as defined in Lemma 3.3, is Koszul.*

Since the algebras in Corollaries 3.1 and 3.3 are Koszul, we know that there exist linear resolutions of the field, k, for each algebra. Showing the ideal has a quadratic Gröbner basis, while one of the simplest ways of proving Koszul, lacks the ability to illuminate the underlying structure of the algebra. Addressing this, we now give descriptions, without proofs, of the matrices associated with the linear resolutions of the field.

Theorem 3.1 *For*

$$\Lambda = K\langle X_1, \ldots, X_t\rangle \ /\langle m_1, \ldots, m_n, X_\gamma X_{j_i} + cX_\beta X_{j_i}\rangle$$

where $X_{j_1} < \cdots < X_{j_k} < X_\beta < X_{j_{k+1}} < \cdots < X_{j_l} < X_\gamma < X_{j_{l+1}} < \cdots < X_{j_n}$, *we have the following linear resolution of the field:*

$$\cdots \xrightarrow{\varphi} \Lambda^{n+1} \xrightarrow{\varphi} \Lambda^{n+1} \xrightarrow{\varphi_2} \Lambda^t \xrightarrow{(\ X_1 \ \cdots \ X_t\ )} \Lambda \longrightarrow k \longrightarrow 0$$

where $\varphi_2 = (a_{rs})$ *and* $\varphi = (b_{rs})$ *for*

$$a_{rs} = \begin{cases} X_r & \text{if } r = j_s \text{ and } 1 \leq s \leq l \\ X_r & \text{if } r = j_{s-1} \text{ and } l+2 \leq s \leq n+1 \\ cX_{j_i} & \text{if } r = \beta \text{ and } s = l+1 \\ X_{j_i} & \text{if } r = \gamma \text{ and } s = l+1 \\ 0 & \text{if otherwise} \end{cases}$$

(See Diagram 3.1)

$$b_{rs} = \begin{cases} X_{j_r} & \text{if } r = s \text{ and } 1 \leq r \leq l \\ X_{j_i} & \text{if } r = s = l+1 \\ X_{j_{r-1}} & \text{if } r = s \text{ and } l+2 \leq r \leq n+1 \\ 0 & \text{if otherwise} \end{cases}$$

(See Diagram 3.1)

Theorem 3.2 *For*

$$\Lambda = K\langle X_1, \ldots, X_t\rangle \ /\langle m_1, \ldots, m_n, X_{j_i} X_\gamma + cX_{j_i} X_\beta\rangle$$

we have the following linear resolution of the field:

$$\cdots \xrightarrow{\varphi} \Lambda^{n+1} \xrightarrow{\varphi} \Lambda^{n+1} \xrightarrow{\varphi_2} \Lambda^t \xrightarrow{(\ X_1 \ \cdots \ X_t\ )} \Lambda \longrightarrow k \longrightarrow 0$$

where $\varphi_2 = (a_{rs})$ *and* $\varphi = (b_{rs})$ *for*

$$a_{rs} = \begin{cases} X_r & \text{if } 1 \leq s \leq i \text{ and } r = j_s \\ X_\gamma + cX_\beta & \text{if } s = i+1 \text{ and } r = j_i \\ X_r & \text{if } i+2 \leq s \leq n+1 \text{ and } r = j_{s-1} \\ 0 & \text{if otherwise} \end{cases}$$

(See Diagram 3.2)

$$b_{rs} = \begin{cases} X_{j_r} & \text{if } r = s \text{ and } 1 \le r \le i \\ X_\gamma + cX_\beta & \text{if } r = i \text{ and } s = i+1 \\ X_{j_{r-1}} & \text{if } r = s \text{ and } i+2 \le r \le n+1 \\ 0 & \text{if otherwise} \end{cases}$$

(See Diagram 3.2)

Theorem 3.3 *For*

$$\Lambda = K\langle X_1, \ldots, X_t\rangle \,/\langle m_1, \ldots, m_n, X_\beta X_\gamma + X_\alpha X_\theta\rangle$$

where $\beta, \gamma \neq j_l$ *for all* l, $\beta \neq \gamma$ *and* $X_{j_1} < \ldots < X_{j_k} < X_\beta < X_{j_{k+1}} < \ldots < X_{j_n}$, *we have the following linear resolution of the field:*

$$\cdots \xrightarrow{\varphi} \Lambda^n \xrightarrow{\varphi} \Lambda^n \xrightarrow{\varphi_3} \Lambda^{n+1} \xrightarrow{\varphi_2} \Lambda^t \xrightarrow{\left(\begin{array}{ccc} X_1 & \cdots & X_t \end{array}\right)} \Lambda \longrightarrow k \longrightarrow 0$$

where $\varphi_2 = (a_{rs})$, $\varphi_3 = (b_{rs})$ *and* $\varphi = (c_{rs})$ *for*

$$a_{rs} = \begin{cases} X_r & \text{if } 1 \le s \le k \text{ and } r = j_s \\ X_\theta & \text{if } r = \alpha \text{ and } s = k+1 \\ X_\gamma & \text{if } s = k+1 \text{ and } r = \beta \\ X_r & \text{if } k+2 \le s \le n+1 \text{ and } r = j_{s-1} \\ 0 & \text{if otherwise} \end{cases}$$

(See Diagram 3.3)

$$b_{rs} = \begin{cases} X_{j_r} & \text{if } r = s \text{ and } 1 \le r \le k \\ X_{j_{r-1}} & \text{if } r = s+1 \text{ and } k+2 \le r \le n+1 \\ 0 & \text{if otherwise} \end{cases}$$

(See Diagram 3.3)

$$c_{rs} = \begin{cases} X_{j_r} & \text{if } r = s \\ 0 & \text{if otherwise} \end{cases}$$

(See Diagram 3.3)

Chapter 4

Koszul Algebras - Part II

In this chapter, for some results, we expand the generators of the ideal from those of Chapter 3. We will state when results are for the expanded generating set. However, throughout, $m_1 = X_{j_1}^2, \ldots, m_n = X_{j_n}^2$ are perfect square monomials in $K\langle X_1, \ldots, X_t\rangle$ and $\Lambda = K\langle X_1, \ldots, X_t\rangle / I$. Additionally, $\overline{\gamma}$ means $\gamma + I$.

Lemma 4.1 *If* $p + cq = \begin{cases} X_{j_i}X_{j_k} + cX_{j_i}X_{j_l} \\ X_{j_k}X_{j_i} + cX_{j_l}X_{j_i} \end{cases}$ *, with* i*,* k*, and* l *distinct and* $c \in K$*, then* $I = \langle m_1, \ldots, m_n, p + cq\rangle$ *has no quadratic Gröbner basis.*

Proof Let $p + cq = X_{j_i}X_{j_k} + cX_{j_i}X_{j_l}$ and suppose tip$(p + cq)$= $X_{j_i}X_{j_k}$ ($X_{j_k}X_{j_i} + cX_{j_l}X_{j_i}$ is analogous). Then, $p + cq$ overlaps with m_k. And,

$$\begin{aligned} o(p + cq, m_k) = & \\ & (X_{j_i}X_{j_k} + cX_{j_i}X_{j_l})X_{j_k} - X_{j_i}(X_{j_k}^2) \\ = \; & cX_{j_i}X_{j_l}X_{j_k} \end{aligned}$$

Now, $cX_{j_i}X_{j_l}X_{j_k}$ is irreducible since $i \neq j$ and $j \neq k$. Thus, we have an irreducible cubic term to add to the Gröbner basis for I.

□

That is, if, for some distinct i,j, and k, $supp(p) \bigcap supp(m_k) \neq \phi$, $supp(q) \bigcap supp(m_l) \neq \phi$, $supp(p) \bigcap supp(q) \bigcap supp(m_i) \neq \phi$, and X_j^2 does not divide pq or qp for all j, then $I = \langle m_1, \ldots, m_n, p + cq\rangle$ has no quadratic Gröbner basis.

Lemma 4.2 *If $p_1^1, \ldots, p_z^{s_z}$ are distinct perfect square monomials in $K\langle X_1, \ldots, X_t\rangle$ with $p_i^j = X_{k_j^i}^2 \neq m_v$ for all i,j,k, v, then,*

$$I = \left\langle m_1, \ldots, m_n, \sum_{i=1}^{s_1} c_1^i p_1^i, \ldots, \sum_{i=1}^{s_z} c_z^i p_z^i \right\rangle$$

has no quadratic Gröbner basis.

Proof Fix i. WLOG, we may assume that tip($\sum c_j^i p_j^i$)=p_j^1.

Consider

$$\begin{aligned} o(\sum c_1^i p_1^i, \sum c_1^i p_1^i) &= X_{k_1^1}(\sum c_1^i p_1^i) - (\sum c_1^i p_1^i) X_{k_1^1} \\ &= \sum c_1^i X_{k_1^1} p_1^i - \sum c_1^i p_1^i X_{k_1^1} \\ &= \sum_{i=2}^{s_1} c_1^i X_{k_1^1} p_1^i - \sum_{i=2}^{s_1} c_1^i p_1^i X_{k_1^1} \end{aligned}$$

Now, since each p_1^i is distinct, there is no cancellation between terms of the sums. Since p_1^i is a perfect square for all i, we can't reduce using p_1^1. Also, no m_i can reduce the overlap as the p_j^i's are distinct from the m_i's. We can't reduce using another sum since each p_i^j is distinct. Hence, we have irreducible cubic terms to add to the Gröbner basis for I. Thus, $I = \langle m_1, \ldots, m_n, \sum_{i=1}^{s_1} c_1^i p_1^i, \ldots, \sum_{i=1}^{s_z} c_z^i p_z^i\rangle$ has no quadratic Gröbner basis.

□

Thus, we see that for ideals with generators as described in Lemma 4.1 and Lemma 4.2, the resulting algebras do not have the ready implication of Koszul. However, as remarked in Chapter 1, the existence of a quadratic Gröbner basis, while a sufficient condition for the algebra to be Koszul, is not a necessary one. As we will see, the quadratic algebras generated from the ideals of Lemma 4.1 and Lemma 4.2 exemplify this fact.

For the ideal $I = \langle m_1, \ldots, m_n, X_{j_i}X_{j_k} + cX_{j_i}X_{j_l}\rangle$, we begin with a proposition clarifying ideal membership.

Proposition 4.1 *Let $I = \langle m_1, \ldots, m_n, p + cq\rangle$ be an ideal in $K\langle X_1, \ldots, X_t\rangle$ where $m_s = X_{j_s}^2$ for all s and $p + cq = X_{j_i}X_{j_k} + cX_{j_i}X_{j_l}$. If $X_s\gamma \in I$, then $\overline{\gamma} \equiv \overline{X_s\delta + (X_{j_k} + cX_{j_l})\nu}$ for some δ, ν.*

Proof Since $X_s\gamma \in I$, we can write

$$X_s\gamma = \sum_{a=1}^{n} \beta_a m_a \lambda_a + \beta(p + cq)\lambda$$

for some polynomials $\beta_a, \lambda_a, \beta, \lambda$. Rewriting,

$$X_s\gamma = \sum_{a=1}^{n_1} \mu_a^1 m_1 \rho_a^1 + \sum_{a=1}^{n_2} \mu_a^2 m_2 \rho_a^2 + \cdots + \sum_{a=1}^{n_n} \mu_a^n m_n \rho_a^n$$

$$+ \sum_{a=1}^{n_{sum}} \mu_a (p+cq)\rho_a$$

where $\mu_a^j, \rho_a^j, \mu_a, \rho_a$ are all monomials. Note that for fixed a, X_s is a prefix of $\mu_a(p+cq)\rho_a$ even if $\mu_a \in K \setminus 0$. Now, since X_s divides the lefthand side of the equation, X_s must divide a subsum of the right hand side. That is,

$$X_s | \sum_{\text{subsum}} \mu_a^1 m_1 \rho_a^1 + \cdots + \sum_{\text{subsum}} \mu_a^n m_n \rho_a^n + \sum_{\text{subsum}} \mu_a(p+cq)\rho_a$$

So,

$$X_s\gamma = \underbrace{\left(\sum_{\text{subsum}} \mu_a^1 m_1 \rho_a^1 + \cdots + \sum_{\text{subsum}} \mu_a^n m_n \rho_a^n + \sum_{\text{subsum}} \mu_a(p+cq)\rho_a \right)}_{\text{divisible by } X_s} + \underbrace{\Delta}_{X_s \not| \Delta}$$

Since any term in the leftover part (in Δ) does not begin with X_s, it must cancel with another term in Δ. Hence, the leftover terms sum to zero. That is, $\Delta = 0$. Thus,

$$X_s\gamma = \underbrace{\sum_{\text{subsum}} \mu_a^1 m_1 \rho_a^1 + \cdots + \sum_{\text{subsum}} \mu_a^n m_n \rho_a^n + \sum_{\text{subsum}} \mu_a(p+cq)\rho_a}_{\text{all divisible by } X_s}$$

First note that if $s \neq j_v$ for some v, then $\mu_a^j, \mu_a \notin K \setminus 0$ for all a and j. In this case, $\mu_a^j = X_s \sigma_a^j$ and $\mu_a = X_s\sigma_a$ for some monomials σ_a^j and σ_a. So,

$$X_s\gamma = \sum_{\text{subsum}} X_s \sigma_a^1 m_1 \rho_a^1 + \cdots + \sum_{\text{subsum}} X_s \sigma_a^n m_n \rho_a^n + \sum_{\text{subsum}} X_s \sigma_a (p+cq)\rho_a$$

Thus, for $\delta, \nu = 0$,

$$\overline{\gamma} \equiv \overline{X_s\delta + (X_{j_k} + cX_{j_l})\nu}$$

Now suppose $s = j_v$ for some $v \in \{1, \ldots, n\} \setminus \{i\}$ In this case, for all $j \neq v$, $\mu_a, \mu_a^j \notin K \setminus 0$. So, $\mu_a = X_{j_v}\sigma_a$ and $\mu_a^j = X_{j_v}\sigma_a^j$ for some monomials σ_a and σ_a^j.

$$X_{j_v}\gamma = \sum_{j \neq v} \left(\sum_{\text{subsum}} X_{j_v} \sigma_a^j m_j \rho_a^j \right) + \sum_{\text{subsum}} \mu_a^v m_v \rho_a^v + \sum_{\text{subsum}} X_{j_v} \sigma_a (p+cq)\rho_a$$

If $\mu_1^v, \ldots, \mu_\beta^v \in K \setminus 0$ and $\mu_{\beta+1}^v$, etc $\notin K \setminus 0$, then

$$X_{j_v}\gamma = \sum_{j \neq v} \left(\sum_{\text{subsum}} X_{j_v} \sigma_a^j m_j \rho_a^j \right) + \sum_{a=1}^{\beta} m_v(\mu_a^v \rho_a^v) + \sum_{a \geq \beta+1} X_{j_v} \sigma_a^v m_v \rho_a^v + \sum_{\text{subsum}} X_{j_v} \sigma_a (p+cq)\rho_a$$

Thus, for $\delta = \sum_{a=1}^{\beta} \mu_a^v \rho_a^v$ and $\nu = 0$,

$$\overline{\gamma} \equiv \overline{X_s\delta + (X_{j_k} + cX_{j_l})\nu}$$

Lastly, if $s = j_i$, we again have that for $j \neq i$, $\mu_a^j = X_{j_i}\sigma_a^j$ for some monomials σ_a^j. Suppose $\mu_1^i, \ldots, \mu_\beta^i, \mu_1, \ldots, \mu_\mu \in K \setminus 0$ and $\mu_{\beta+1}^i$, etc., $\mu_{\mu+1}$, etc. $\notin K \setminus 0$. Then, we have,

$$\begin{aligned} X_{j_i}\gamma &= \sum_{j\neq i}\left(\sum_{\text{subsum}} X_{j_i}\sigma_a^j m_j \rho_a^j\right) + \sum_{a+1}^{\beta} m_i(\mu_a^i\rho_a^i) + \sum_{a\geq\beta+1} X_{j_i}\sigma_a^i m_i \rho_a^i + \\ &\quad \sum_{a=1}^{\mu}(p+cq)\mu_a\rho_a + \sum_{a\geq\mu+1} X_{j_i}\sigma_a(p+cq)\rho_a \end{aligned}$$

Hence, we get, for $\delta = \sum_{a=1}^{\beta} \mu_a^i \rho_a^i$ and $\nu = \sum_{a=1}^{\mu} \mu_a\rho_a$,

$$\overline{\gamma} \equiv \overline{X_s\delta + (X_{j_k} + cX_{j_l})\nu}$$

□

Now that we have a better idea of the elemental structure of the ideal, we begin describing the linear maps of the field resolution.

Proposition 4.2 *For $I = \langle m_1, \ldots, m_n, p + cq\rangle$ an ideal in $K\langle X_1, \ldots, X_t\rangle$ with $m_s = X_{j_s}^2$ for all s, and $p + cq = X_{j_i}X_{j_k} + cX_{j_i}X_{j_l}$, let $\varphi_1 : \Lambda^t \longrightarrow \Lambda$ be defined by $\varphi_1 = \begin{pmatrix} X_1 & \cdots & X_t \end{pmatrix}$. If $\begin{pmatrix} \overline{\gamma_1} \\ \overline{\gamma_2} \\ \vdots \\ \overline{\gamma_t} \end{pmatrix} \in \ker\varphi_1$, then for some δ_s, δ, ν,*

$$\overline{\gamma_s} \equiv \begin{cases} \overline{0} & \text{if } s \in \{1, \ldots, t\} \setminus \{j_1, \ldots, j_n\} \\ \overline{X_s\delta_s} & \text{if } s \in \{j_1, \ldots, j_n\} \setminus \{j_i\} \\ \overline{X_{j_i}\delta + (X_{j_k} + cX_{j_l})\nu} & \text{if } s = j_i \end{cases}$$

<u>Proof</u> Let $\begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_t} \end{pmatrix} \in \ker\varphi_1$. Then, $\varphi_1\begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_t} \end{pmatrix} \equiv \overline{0}$. Thus, $\sum_{a=1}^{t} X_a\gamma_a \in I$, and

$$\sum_{a=1}^{t} X_a\gamma_a = \sum_{a=1}^{n} \beta_a m_a \lambda_a + \beta(X_{j_i}X_{j_k} + cX_{j_i}X_{j_l})\lambda$$

for some polynomials $\beta_a, \lambda_a, \beta, \lambda$. Rewriting:

$$\sum_{a=1}^{t} X_a\gamma_a = \sum_{a=1}^{n_1} \mu_a^1 m_1 \rho_a^1 + \sum_{a=1}^{n_2} \mu_a^2 m_2 \rho_a^2 + \ldots + \sum_{a=1}^{n_n} \mu_a^n m_n \rho_a^n + \sum_{a=1}^{n'} \mu_a(X_{j_i}X_{j_k} + cX_{j_i}X_{j_l})\rho_a$$

for some monomials $\mu_a^j, \rho_a^j, \mu_a, \rho_a$. Now, since for all s such that $\gamma_s \neq 0$, X_s divides terms on the lefthand side of the equation, X_s must divide terms on the righthand side on the equation:

$$X_s \left| \left(\sum_{\substack{\text{subsum}\\ \text{over a}}} \mu_a^1 m_1 \rho_a^1 + \ldots + \sum_{\substack{\text{subsum}\\ \text{over a}}} \mu_a^n m_n \rho_a^n + \sum_{\substack{\text{subsum}\\ \text{over a}}} \mu_a (X_{j_i} X_{j_k} + c X_{j_i} X_{j_l}) \rho_a \right) \right.$$

So,

$$X_s \gamma_s = \sum_{j=1}^{n} \left(\sum_{\substack{\text{subsum}\\ \text{over a}}} \mu_a^j m_j \rho_a^j \right) + \sum_{\substack{\text{subsum}\\ \text{over a}}} \mu_a (X_{j_i} X_{j_k} + c X_{j_i} X_{j_l}) \rho_a$$

Thus, $X_s \gamma_s \in I$, and so, by Proposition 4.1,

$$\overline{\gamma_s} \equiv \overline{X_s \delta + (X_{j_k} + c X_{j_l}) \rho}$$

where $\delta, \rho = 0$ if $s \in \{1, \ldots, t\} \setminus \{j_1, \ldots, j_n\}$ and $\rho = 0$ if $s \in \{j_1, \ldots, j_n\} \setminus \{j_i\}$

□

Proposition 4.3 *For* $I = \langle m_1, \ldots, m_n, p + cq \rangle$ *an ideal in* $K\langle X_1, \ldots, X_t \rangle$ *with* $m_s = X_{j_s}^2$ *for all* $1 \leq s \leq n$ *and* $p + cq = X_{j_i} X_{j_k} + c X_{j_i} X_{j_l}$, *let* $\varphi_2 : \Lambda^{n+1} \longrightarrow \Lambda_t$ *be defined by* $\varphi_2 = (a_{rs})_{t \times (n+1)}$ *where*

$$a_{rs} = \begin{cases} X_{j_s} & \text{if } r = j_s \text{ and } 1 \leq s \leq i \\ X_{j_k} + c X_{j_l} & \text{if } r = j_i \text{ and } s = i+1 \\ X_{j_{s-1}} & \text{if } r = j_{s-1} \text{ and } i+2 \leq s \leq n+1 \\ 0 & \text{otherwise} \end{cases}$$

(See Diagram 4.1)

If $\begin{pmatrix} \overline{\gamma_1} \\ \overline{\gamma_2} \\ \vdots \\ \overline{\gamma_{n+1}} \end{pmatrix} \in \ker \varphi_2$, *then for some polynomials* δ_s, δ, ξ,

$$\overline{\gamma_s} \equiv \begin{cases} \overline{X_{j_s} \delta_s} & \text{if } 1 \leq s \leq i-1 \\ \overline{X_{j_i} \delta + (X_{j_k} + c X_{j_l}) \xi} & \text{if } s = i \\ \overline{0} & \text{if } s = i+1 \\ \overline{X_{j_{s-1}} \delta_{s-1}} & \text{if } i+2 \leq s \leq n+1 \end{cases}$$

<u>Proof</u> Let $\begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_{n+1}} \end{pmatrix} \in \ker \varphi_2$. Then, $\varphi_2 \begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_{n+1}} \end{pmatrix} \equiv (\overline{0})_{t\times 1}$ Thus, $\left(\sum_{a=1}^{n+1} a_{rs}\gamma_s\right) \equiv (\overline{0})_{t\times 1}$.
For $\nu = 1, \ldots, i-1$,

$$X_{j_\nu}\gamma_\nu = \sum_{s=1}^{n+1} a_{j_\nu s}\gamma_s \in I$$

So, by proposition 1, $\overline{\gamma_\nu} \equiv \overline{X_{j_\nu}\delta_\nu}$ for some δ_ν. And, for $\nu = i+1, \ldots, n$,

$$X_{j_\nu}\gamma_{\nu+1} = \sum_{s=1}^{n+1} a_{j_\nu s}\gamma_s \in I$$

So, $\overline{\gamma_{\nu+1}} \equiv \overline{X_{j_\nu}\delta_{\nu+1}}$ for some $\delta_{\nu+1}$. Lastly, we also have that

$$X_{j_i}\gamma_i + (X_{j_k} + cX_{j_l})\gamma_{i+1} = \sum_{s=1}^{n+1} a_{j_i s}\gamma_s \in I$$

Hence,

$$X_{j_i}\gamma_i + X_{j_k}\gamma_{i+1} + cX_{j_l}\gamma_{i+1} = \sum_{s=1}^{n}\left(\sum_{a=1}^{n_s} \mu_a^s m_s \rho_a^s\right) + \sum_{a=1}^{n'} \mu_a(X_{j_i}X_{j_k} + cX_{j_i}X_{j_l})\rho_a$$

for some monomials $\mu_a^s, \rho_a^s, \mu_a, \rho_a$. Thus, as in the proof of Proposition 4.2, we can write

$$X_{j_i}\gamma_i = \sum_{s=1}^{n}\left(\sum_{\substack{\text{subsum} \\ \text{over a}}} \mu_a^s m_s \rho_a^s\right) + \sum_{\substack{\text{subsum} \\ \text{over a}}} \mu_a(X_{j_i}X_{j_k} + cX_{j_i}X_{j_l})\rho_a$$

So, $X_{j_i}\gamma_i \in I$. So, $\overline{\gamma_i} \equiv \overline{X_{j_i}\delta + (X_{j_k} + cX_{j_l})\mu}$ for some δ, μ. Similarly, $X_{j_k}\gamma_{i+1}, X_{j_l}\gamma_{i+1} \in I$ which means $\overline{\gamma_{i+1}} \equiv \overline{X_{j_k}\delta'}and\overline{\gamma_{i+1}} \equiv \overline{X_{j_l}\delta''}$ for some δ', δ''. The only way this can occur is if $\overline{X_{j_k}\delta'} \equiv \overline{0} \equiv \overline{X_{j_l}\delta''}$ Hence, $\overline{\gamma_{i+1}} \equiv \overline{0}$

□

Proposition 4.4 *For $I = \langle m_1, \ldots, m_n, p + cq\rangle$ an ideal in $K\langle X_1, \ldots, X_t\rangle$ with $m_s = X_{j_s}^2$ for all s and $p + cq = X_{j_i}X_{j_k} + cX_{j_i}X_{j_l}$, let $\varphi : \Lambda^{n+1} \longrightarrow \Lambda^{n+1}$ be defined by $\varphi = (b_{rs})_{(n+1)\times(n+1)}$ where*

$$b_{rs} = \begin{cases} X_{j_r} & \text{if } r = s \text{ and } 1 \le r \le i \\ X_{j_k} + cX_{j_l} & \text{if } r = i \text{ and } s = i+1 \\ X_{j_{r-1}} & \text{if } r = s \text{ and } i+2 \le r \le n+1 \\ 0 & \text{otherwise} \end{cases}$$

(See Diagram 4.1)

If $\begin{pmatrix} \overline{\gamma_1} \\ \overline{\gamma_2} \\ \vdots \\ \overline{\gamma_{n+1}} \end{pmatrix} \in \ker\varphi$, *then for some* δ_s, δ, ν

$$\overline{\gamma_s} \equiv \begin{cases} \overline{X_{j_s}\delta_s} & \text{if } 1 \le s \le i-1 \\ \overline{X_{j_i}\delta + (X_{j_k} + cX_{j_l})\nu} & \text{if } s = i \\ \overline{0} & \text{if } s = i+1 \\ \overline{X_{j_{s-1}}\delta_s} & \text{if } i+2 \le s \le n+1 \end{cases}$$

<u>Proof</u> Let $\begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_{n+1}} \end{pmatrix} \in \ker\varphi$. Then, $\varphi\begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_{n+1}} \end{pmatrix} \equiv (\overline{0})_{(n+1)\times 1}$. So, $\sum_{a=1}^{n+1} b_{rs}\gamma_s \in I$. For $s = 1, \ldots, i-1$,

$$X_{j_s}\gamma_s = \sum_{a=1}^{n+1} b_{sa}\gamma_a \in I$$

So, by Proposition 4.1, $\overline{\gamma_s} \equiv \overline{X_{j_s}\delta_s}$ for some δ_s. And, for $s = i+2, \ldots, n+1$,

$$X_{j_{s-1}}\gamma_s = \sum_{a=1}^{n+1} b_{sa}\gamma_a \in I$$

So, $\overline{\gamma_s} \equiv \overline{X_{j_{s-1}}\delta_s}$ for some δ_s. Lastly, we also have that

$$X_{j_i}\gamma_i + (X_{j_k} + cX_{j_l})\gamma_{i+1} = \sum_{a=1}^{n+1} b_{ia}\gamma_a \in I$$

So, $X_{j_i}\gamma_i \in I$ and $\overline{\gamma_i} \equiv \overline{X_{j_i}\delta + (X_{j_k} + cX_{j_l})\mu}$ for some δ, μ. And, $X_{j_k}\gamma_{i+1}, X_{j_l}\gamma_{i+1} \in I$. Thus, $\overline{\gamma_{i+1}} \equiv \overline{X_{j_k}\delta'}and\overline{\gamma_{i+1}} \equiv \overline{X_{j_l}\delta''}$ for some δ', δ''. Hence,

$$\overline{X_{j_k}\delta'} \equiv \overline{0} \equiv \overline{X_{j_l}\delta''}$$

Hence, $\overline{\gamma_{i+1}} \equiv \overline{0}$

□

Theorem 4.1 *Let* $I = \langle m_1, \ldots, m_n, p + cq\rangle$ *be an ideal in* $K\langle X_1, \ldots, X_t\rangle$ *where* $m_s = X_{j_s}^2$ *for all* s, *and* $p + cq = X_{j_i}X_{j_k} + cX_{j_i}X_{j_l}$ *for some distinct* $i, k, l \in \{1, \ldots, n\}$. *Then,* Λ *has the following linear resolution:*

$$(*) \cdots \xrightarrow{\varphi} \Lambda^{n+1} \xrightarrow{\varphi} \Lambda^{n+1} \xrightarrow{\varphi} \Lambda^{n+1} \xrightarrow{\varphi_2} \Lambda^t \xrightarrow{\varphi_1} \Lambda \to k \to 0$$

where φ, φ_1, *and* φ_2 *are as defined in Propositions 4.2, 4.3, 4.4.*

<u>Proof</u> First look at $\varphi_1 \circ \varphi_2 = \left(\sum_{r=1}^{t} X_r a_{rs}\right)$

For $1 \leq s \leq i$, $\sum_{r=1}^{t} X_r a_{rs} = \overline{X_{j_s}^2} \equiv \overline{0}$.

For $s = i+1$, $\sum_{r=1}^{t} X_r a_{rs} = X_{j_i}(\overline{X_{j_k} + cX_{j_l}}) \equiv \overline{X_{j_i}X_{j_k} + cX_{j_i}X_{j_l}} \equiv \overline{0}$.

For $i+2 \leq s \leq n+1$, $\sum_{r=1}^{t} X_r a_{rs} = \overline{X_{j_{s-1}}^2} \equiv \overline{0}$.

Hence, $\varphi_1 \circ \varphi_2 = (0)_{1\times(n+1)}$ and $Im\varphi_2 \subseteq \ker \varphi_1$.

Let $\begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_t} \end{pmatrix} \in \ker \varphi_1$. Then, by Proposition 4.2,

$$\overline{\gamma_j} = \begin{cases} \overline{0} & \text{if } j \in \{1, \dots, t\} \setminus \{j_1, \dots, j_n\} \\ \overline{X_j \delta_j} & \text{if } j \in \{j_1, \dots, j_n\} \setminus j_i\} \\ \overline{X_{j_i}\delta + (X_{j_k} + cX_{j_l})\nu} & \text{if } j = j_i \end{cases}$$

for some δ_j, δ, ν. Then,

$$\varphi_2 \begin{pmatrix} \overline{\delta_{j_1}} \\ \vdots \\ \overline{\delta_{j_{i-1}}} \\ \overline{\delta} \\ \overline{\nu} \\ \overline{\delta_{j_{i+1}}} \\ \vdots \\ \overline{\delta_{j_n}} \end{pmatrix} = \begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_t} \end{pmatrix}$$

So, $\ker \varphi_1 = Im\varphi_2$.

Next, look at $\varphi_2 \circ \varphi = (a_{rs})(b_{rs}) = \left(\sum_{s=1}^{n+1} a_{rs}b_{sw}\right)$.

For $r = j_v$, $v \neq i$ and $w = v$, $\sum a_{rs}b_{sw} = a_{j_v v}b_{vv} = \overline{X_{j_v}^2} \equiv \overline{0}$.

For $r = j_i$ and $w = i$, $\sum a_{rs}b_{sw} = a_{j_i i}b_{ii} = \overline{X_{j_i}^2} \equiv \overline{0}$.

For $r = j_i$ and $w = i+1$, $\sum a_{rs}b_{sw} = a_{j_i i}b_{i(i+1)} = \overline{X_{j_i}(X_{j_k} + cX_{j_l})} \equiv \overline{0}$.

For all other r and w, $\sum a_{rs}b_{sw} = 0$.

So, $\varphi_2 \circ \varphi \equiv (0)_{t\times(n+1)}$ and $Im\varphi_2 \subseteq \ker \varphi$.

Let $\begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_{n+1}} \end{pmatrix} \in \ker \varphi_2$. Then, by Proposition 4.3,

$$\overline{\gamma_s} = \begin{cases} \overline{X_{j_s}\delta_s} & \text{if } 1 \leq s \leq i-1 \\ \overline{X_{j_i}\delta + (X_{j_k} + cX_{j_l})\nu} & \text{if } s = i \\ \overline{0} & \text{if } s = i+1 \\ \overline{X_{j_{s-1}}\delta_{s-1}} & \text{if } i+2 \leq s \leq n+1 \end{cases}$$

for some δ_s, δ, ν. And,

$$\varphi \begin{pmatrix} \overline{\delta_1} \\ \vdots \\ \overline{\delta_{i-1}} \\ \overline{\delta} \\ \overline{\nu} \\ \overline{\delta_{i+1}} \\ \vdots \\ \overline{\delta_n} \end{pmatrix} = \begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_{n+1}} \end{pmatrix}$$

Hence, $\ker \varphi_2 = Im\varphi$.

Lastly, $\varphi \circ \varphi = \left(\sum_{v=1}^{n+1} b_{rv} b_{vs}\right)$.

For $r \neq i, i+1$ and $s = r$,

$$\sum b_{rv} b_{vs} = b_{rr} b_{rr} = \left\{ \begin{array}{ll} \overline{X_{j_r}^2} & \text{if } 1 \leq r \leq i-1 \\ \overline{X_{j_{r-1}}^2} & \text{if } i+2 \leq r \leq n+1 \end{array} \right\} \equiv \overline{0}$$

For $r, s = i$, $\sum b_{rv} b_{vs} = b_{ii} b_{ii} = \overline{X_{j_i}^2} \equiv \overline{0}$.

For $r = i$ and $s = i+1$, $\sum b_{rv} b_{vs} = b_{ii} b_{i(i+1)} = \overline{X_{j_i}(X_{j_k} + cX_{j_l})} \equiv \overline{0}$.

For all other r and s, $\sum b_{rv} b_{vs} = 0$.

Hence, $\varphi \circ \varphi \equiv (0)_{(n+1)\times(n+1)}$ and $Im\varphi \subseteq \ker \varphi$.

Let $\begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_{n+1}} \end{pmatrix} \in \ker \varphi$. Then, by Proposition 4.4,

$$\overline{\gamma_s} = \begin{cases} \overline{X_{j_s}\delta_s} & \text{if } 1 \leq s \leq i-1 \\ \overline{X_{j_i}\delta + (X_{j_k} + cX_{j_l})\nu} & \text{if } s = i \\ \overline{0} & \text{if } s = i+1 \\ \overline{X_{j_{s-1}}\delta_s} & \text{if } i+2 \leq s \leq n+1 \end{cases}$$

for some δ_s, δ, ν. Then,

$$\varphi \begin{pmatrix} \overline{\delta_1} \\ \vdots \\ \overline{\delta i-1} \\ \overline{\delta} \\ \overline{\nu} \\ \overline{\delta_{i+1}} \\ \vdots \\ \overline{\delta_{n+1}} \end{pmatrix} = \begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_{n+1}} \end{pmatrix}$$

Thus, $\ker\varphi = Im\varphi$.

Hence, $(*)$ is exact.

□

So, here we see our first example of a quadratic algebra having no quadratic Gröbner basis which is nonetheless a Koszul algebra. The symmetric case, $I = \langle m_1, \ldots, m_n, X_{j_k}X_{j_i} + cX_{j_l}X_{j_i}\rangle$, while similar in nature, has its own set of idiosyncracies. As before, we start with an investigation of ideal membership.

Proposition 4.5 *Let $I = \langle m_1, \ldots, m_n, p + cq\rangle$ be an ideal in $K\langle X_1, \ldots, X_t\rangle$, with $p + cq = X_{j_k}X_{j_i} + cX_{j_l}X_{j_i}$ for $k \neq i, k \neq l, i \neq l$, If $X_{j_v}\gamma \in I$, $v \neq k, l$, then $\overline{\gamma} \equiv \overline{X_{j_v}\delta}$ for some δ. Futhermore, if $X_s\gamma \in I$,for $s \neq j_v$ for all v, then $\gamma \in I$.*

<u>Proof</u> Suppose $X_{j_v}\gamma \in I$ for some $v \neq k, l$. Then, for some monomials $\mu_j^a, \rho_j^a, \mu_j, \rho_j$,

$$X_{j_v}\gamma = \sum_{a=1}^{n}\left(\sum_{j=1}^{n_a}\mu_j^a m_a \rho_j^a\right) + \sum_{j=1}^{n_{sum}}\mu_j(X_{j_k}X_{j_i} + cX_{j_l}X_{j_i})\rho_j$$

Since $v \neq k, l$, we have that $\mu_j \notin K$ $\forall j = 1, \ldots, n_{sum}$, and so, $\mu_j = X_{j_k}\sigma_j$ for some monomial σ_j. Similarly, for all j and $a \neq v$, $\mu_j^a = X_{j_k}\sigma_j^a$ for some monomial σ_j^a.

$$\text{Thus, } X_{j_v}\gamma = \sum_{a\neq v}\left(\sum_{j=1}^{n_a}X_{j_v}\sigma_j^a m_a \rho_j^a\right) + \sum_{\substack{\text{subsum}\\ \text{over j}}} X_{j_v}\sigma_j^v m_v \rho_j^v$$

$$+ \sum_{\substack{\text{subsum}\\ \text{over j}}} \mu_j^v X_{j_v}^2 \rho_j^v + \sum_{j=1}^{n_{sum}} X_{j_v}\sigma_j(X_{j_k}X_{j_i} + cX_{j_l}X_{j_i})\rho_j$$

$$\text{So, } \overline{\gamma} = \overline{X_{j_v}\delta} \text{ for } \delta = \sum \mu_j^v \rho_j^v$$

And, if $X_s\gamma \in I, s \neq j_v$ for all v, then, for some monomials $\mu_j^a, \rho_j^a, \mu_j, \rho_j$,

$$X_s\gamma = \sum_{a=1}^{n}\left(\sum_{j=1}^{n_a}\mu_j^a m_a \rho_j^a\right) + \sum_{j=1}^{n_{sum}}\mu_j(X_{j_k}X_{j_i} + cX_{j_l}X_{j_i})\rho_j$$

But, since $s \neq j_v\ \forall v$, $\mu_j^a = X_s\sigma_j^a\ \forall a, j$, and $\mu_j = X_s\sigma_j\ \forall j$ for some monomials σ_j^a, σ_j. Hence, $\overline{\gamma} \equiv \overline{0}$.

□

We now begin delineating the maps of the linear resolution of the field.

Proposition 4.6 *For $p + cq = X_{j_k}X_{j_i} + cX_{j_l}X_{j_i}$, where i, j, k are distinct, let $\varphi_1 : \Lambda^t \to \Lambda$ be defined by $\varphi_1 = (X_1 \cdots X_t)$. Then,*

$$ker\varphi_1 = \left\{ \begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_t} \end{pmatrix} \mid \overline{\gamma_s} \equiv \begin{cases} \overline{0} & \text{if } s \in \{1, \ldots, t\} \setminus \{j_1, \ldots, j_n\} \\ \overline{X_s\delta_s} & \text{if } s \in \{j_1, \ldots, j_n\} \setminus \{j_k, j_l\} \\ \overline{X_{j_k}\delta_{j_k} + X_{j_i}\nu} & \text{if } s = j_k \\ \overline{X_{j_l}\delta_{j_l} + cX_{j_i}\nu} & \text{if } s = j_l \end{cases} \right\}$$

for some $\delta_1, \ldots, \delta_t, \nu$.

<u>Proof</u> Let $\begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_t} \end{pmatrix} \in \ker\varphi_1$. Then, $\varphi_1 \begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_t} \end{pmatrix} = \sum_{s=1}^{t} X_s\gamma_s \in I$.

$$\text{And, } \sum_{s=1}^{t} X_s\gamma_s = \sum_{a=1}^{n}\left(\sum_{j=1}^{n_a}\mu_j^a m_a \rho_j^a\right) + \sum_{j=1}^{n_{sum}}\mu_j(X_{j_k}X_{j_i} + cX_{j_l}X_{j_i})\rho_j$$

for some monomials $\mu_j^a, \mu_j, \rho_j^a, \rho_j$. So, for $s \neq j_v\ \forall v$,

$$X_s\gamma_s = \underbrace{\sum_{a=1}^{n}\left(\sum_{\substack{\text{subsum} \\ \text{over j}}}\mu_j^a m_a \rho_j^a\right) + \sum_{\substack{\text{subsum} \\ \text{over j}}}\mu_j(X_{j_k}X_{j_i} + cX_{j_l}X_{j_i})\rho_j}_{\text{divisible by } X_s}$$

Thus, $X_s\gamma_s \in I$and $\overline{\gamma_s} \equiv \overline{0}$. For $s = j_v$, $v \neq k, l$,

$$X_{j_v}\gamma_{j_v} = \underbrace{\sum_{a=1}^{n}\left(\sum_{\substack{\text{subsum}\\ \text{over j}}} \mu_j^a m_a \rho_j^a\right) + \sum_{\substack{\text{subsum}\\ \text{over j}}} \mu_j(X_{j_k}X_{j_i} + cX_{j_l}X_{j_i})\rho_j}_{\text{divisible by } X_{j_v}}$$

Then, $X_{j_v}\gamma_{j_v} \in I$ and $\overline{\gamma_{j_v}} \equiv \overline{X_{j_v}\delta_{j_v}}$ for some δ_{j_v} And,

$$X_{j_k}\gamma_{j_k} = \sum_{a=1}^{n}\left(\sum_{\substack{\text{subsum}\\ \text{over j}}} \mu_j^a m_a \rho_j^a\right) + \sum_{\substack{\text{subsum}\\ \text{over j}}} \overbrace{\mu_j}^{\notin K}(X_{j_k}X_{j_i} + cX_{j_l}X_{j_i})\rho_j$$

$$+ \sum_{\substack{\text{subsum}\\ \text{over a}}} \overbrace{\mu_j}^{\in K} X_{j_k}X_{j_i}\rho_j$$

$$X_{j_l}\gamma_{j_l} = \sum_{a=1}^{n}\left(\sum_{\substack{\text{subsum}\\ \text{over j}}} \mu_j^a m_a \rho_j^a\right) + \sum_{\substack{\text{subsum}\\ \text{over j}}} \overbrace{\mu_j}^{\notin K}(X_{j_k}X_{j_i} + cX_{j_l}X_{j_i})\rho_j$$

$$+ \sum_{\substack{\text{subsum}\\ \text{over j}}} \overbrace{\mu_j}^{\in K} cX_{j_l}X_{j_i}\rho_j$$

Now, if $\mu_j X_{j_k}X_{j_i}\rho_j$ is a term in $\sum \mu_j X_{j_k}X_{j_i}\rho_j$, then its pair, $c\mu_j X_{j_l}X_{j_i}\rho_j$, either appears as a term in $\sum \mu_j cX_{j_l}X_{j_i}\rho_j$ in $X_{j_l}\gamma_{j_l}$, or it cancels with a $\mu_{j'}^a m_a \rho_{j'}^a$ term in the original sum. Hence, either $a = i$ and $\rho_j = X_{j_i}\rho_{j'}^a$ or $m_a \mid \rho_j$. Similarly, if $\mu_j cX_{j_l}X_{j_i}\rho_j$ is a term in $\sum \mu_j cX_{j_l}X_{j_i}\rho_j$, then its pair, $\mu_j X_{j_k}X_{j_i}\rho_j$, either appears as a term in $\sum \mu_j X_{j_k}X_{j_i}\rho_j$ in $X_{j_k}\gamma_{j_k}$, or it cancels with a $\mu_{j''}^a m_a \rho_{j''}^a$ term in the original sum. Thus, either $a = i$ and $\rho_j = X_{j_i}\rho_{j''}^a$ or $m_a \mid \rho_j$. Hence,

$$X_{j_k}\gamma_{j_k} = \sum_{a\neq k}\left(\sum_{\substack{\text{subsum}\\ \text{over j}}} X_{j_k}\sigma_j^a m_a \rho_j^a\right) + \sum_{\substack{\text{subsum}\\ \text{over j}}} X_{j_k}\sigma_j^k m_k \rho_j^k$$

$$+ \sum_{\substack{\text{subsum}\\ \text{over j}}} \overbrace{\mu_j^k}^{\in K} X_{j_k}^2 \rho_j^k + \sum_{\substack{\text{subsum}\\ \text{over j}}} X_{j_k}\sigma_j(X_{j_k}X_{j_i} + cX_{j_l}X_{j_i})\rho_j$$

$$+ \sum_{m_a|\rho_j} \mu_j X_{j_k}X_{j_i}\rho_j + \sum \mu_j X_{j_k}X_{j_i}^2\rho_j^i + \sum_{\text{pair in } X_{j_l}\gamma_{j_l}} \mu_j X_{j_k}X_{j_i}\rho_j$$

And,

$$X_{j_l}\gamma_{j_l} = \sum_{a\neq l}\left(\sum_{\substack{\text{subsum}\\ \text{over j}}} X_{j_l}\sigma_j^a m_a \rho_j^a \right) + \sum_{\substack{\text{subsum}\\ \text{over j}}} X_{j_l}\sigma_j^l m_l \rho_j^l$$

$$+ \sum_{\substack{\text{subsum}\\ \text{over j}}} \overbrace{\mu_j^l}^{\in K} X_{j_l}^2 \rho_j^l + \sum_{\substack{\text{subsum}\\ \text{over j}}} X_{j_l}\sigma_j(X_{j_k}X_{j_i} + cX_{j_l}X_{j_i})\rho_j$$

$$+ \sum_{m_a|\rho_j} \mu_j cX_{j_l}X_{j_i}\rho_j + \sum \mu_j cX_{j_l}X_{j_i}^2\rho_j^i + \sum_{\text{pair in } X_{j_k}\gamma_{j_k}} \mu_j cX_{j_l}X_{j_i}\rho_j$$

So,

$$\overline{\gamma_{j_k}} \equiv \overline{X_{j_k}\delta_{j_k} + X_{j_i}\nu}$$
$$\overline{\gamma_{j_l}} \equiv \overline{X_{j_l}\delta_{j_l} + cX_{j_i}\nu}$$

for $\delta_{j_k} = \sum \mu_j^k \rho_j^k$, $\delta_{j_l} = \sum \mu_j^l \rho_j^l$, and $\nu = \sum_{\text{over pairs}} \mu_j \rho_j$.

□

Proposition 4.7 *For $I = \langle m_1, \ldots, m_n, p + cq\rangle$ an ideal in $K\langle X_1, \ldots, X_t\rangle$ with $m_s = X_{j_s}^2$ for all s and $p + cq = X_{j_k}X_{j_i} + cX_{j_l}X_{j_i}$, let $\varphi_2 : \Lambda^{n+1} \to \Lambda^t$ be defined by $\varphi_2 = (a_{rs})$ where*

$$a_{rs} = \begin{cases} X_{j_s} & \text{if } r = j_s \text{ and } 1 \leq s \leq n \\ cX_{j_i} & \text{if } r = j_l \text{ and } s = n+1 \\ X_{j_i} & \text{if } r = j_k \text{ and } s = n+1 \\ 0 & \text{if otherwise} \end{cases}$$

(See Diagram 4.2)

Then,

$$ker\varphi_2 = \left\{ \begin{pmatrix} \overline{X_{j_1}\delta_1} \\ \vdots \\ \overline{X_{j_n}\delta_n} \\ \overline{X_{j_i}\delta_{n+1}} \end{pmatrix} \mid \delta_s \in K\langle X_1, \ldots, X_t\rangle \text{ for } 1 \leq s \leq n+1 \right\}$$

<u>Proof</u> Let $\begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_{n+1}} \end{pmatrix} \in \ker\varphi_2$ Then, $\varphi_2 \begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_{n+1}} \end{pmatrix} = \left(\sum_{s=1}^{n+1} a_{rs}\overline{\gamma_s}\right) \equiv (\overline{0})$. And,

$$\sum_{s=1}^{n+1} a_{rs}\gamma_s \in I$$

forall r. So, for $r = j_v, v \neq k, l$,

$$\sum_{s=1}^{n+1} a_{j_v s}\gamma_s = a_{j_v}\gamma_v = X_{j_v}\gamma_v \in I \text{ and } \overline{\gamma_v} \equiv \overline{X_{j_v}\delta_v} \text{ for some } \delta_v$$

And, for $r = j_k, j_l$,

$$\sum_{s=1}^{n+1} a_{j_v s}\gamma_s = a_{j_k k}\gamma_k + a_{j_k n+1}\gamma_{n+1} = X_{j_k}\gamma_k + X_{j_i}\gamma_{n+1} \in I$$

$$\text{Thus, } \sum_{s=1}^{n+1} a_{j_l s}\gamma_s = a_{j_l l}\gamma_l + a_{j_l n+1}\gamma_{n+1} = X_{j_l}\gamma_l + cX_{j_i}\gamma_{n+1} \in I$$

Now, $X_{j_k}\gamma_k + X_{j_i}\gamma_{n+1} \in I$

$$\text{So, } X_{j_k}\gamma_k + X_{j_i}\gamma_{n+1} = \sum_{a=1}^{n}\left(\sum_{j=1}^{n_a}\mu_j^a m_a \rho_j^a\right) + \sum_{j=1}^{n_{sum}}\mu_j(X_{j_k}X_{j_i} + cX_{j_l}X_{j_i})\rho_j$$

for some monomials $\mu_j^a, \rho_j^a, \mu_j, \rho_j$.

$$\text{And, } X_{j_k}\gamma_k = \sum_{a=1}^{n}\left(\sum_{\substack{\text{subsum}\\ \text{over j}}}\mu_j^a m_a \rho_j^a\right) + \sum_{\substack{\text{subsum}\\ \text{over j}}}\overbrace{\mu_j}^{\notin K}(X_{j_k}X_{j_i} + cX_{j_l}X_{j_i})\rho_j$$

$$+ \sum_{\substack{\text{subsum}\\ \text{over j}}}\overbrace{\mu_j}^{\in K}X_{j_k}X_{j_i}\rho_j$$

As in the proof of Proposition 4.6 we get,

$$X_{j_k}\gamma_k = \sum_{a\neq k}\left(\sum X_{j_k}\sigma_j^a m_a \rho_j^a\right) + \sum X_{j_k}\sigma_j^k m_k \rho_j^k + \sum \overbrace{\mu_j^k}^{\in K} X_{j_k}^2 \rho_j^k$$

$$+ \sum X_{j_k}\sigma_j(X_{j_k}X_{j_i} + cX_{j_l}X_{j_i})\rho_j + \sum_{m_a|\rho_j \text{for some a}}\mu_j X_{j_k}X_{j_i}\rho_j$$

$$+\sum \mu_j X_{j_k} X_{j_i}^2 \rho_{j'}^a$$

$$\text{Thus, } \overline{\gamma_k} \equiv \overline{X_{j_k}\delta_k} \text{for } \delta_k = \sum \mu_j^k \rho_j^k$$

And,

$$X_{j_i}\gamma_{n+1} \in I \text{ which implies } \overline{\gamma_{n+1}} \equiv \overline{X_{j_i}\delta_{n+1}} \text{ for some } \delta_{n+1}$$

Similarly,

$$X_{j_l}\gamma_l + cX_{j_i}\gamma_{n+1} \in I$$

$$\text{and } \overline{\gamma_l} \equiv \overline{X_{j_l}\delta_l} \text{ for } \delta_l = \sum \mu_j^l \rho_j^l$$

□

Proposition 4.8 *For $I = \langle m_1, \ldots, m_n, p + cq\rangle$ an ideal in $K\langle X_1, \ldots, X_t\rangle$ with $m_s = X_{j_s}^2$ for all s and $p + cq = X_{j_k}X_{j_i} + cX_{j_l}X_{j_i}$, let $\varphi : \Lambda^{n+1} \to \Lambda^{n+1}$ be defined by $\varphi = (b_{rs})$ where*

$$b_{rs} = \begin{cases} X_{j_r} & \text{if } r = s \text{ and } 1 \le s \le n \\ X_{j_i} & \text{if } r = s = n+1 \\ 0 & \text{if otherwise} \end{cases}$$

(See Diagram 4.2)

Then,

$$ker\varphi = \left\{ \begin{pmatrix} \overline{X_{j_1}\delta_1} \\ \vdots \\ \overline{X_{j_n}\delta_n} \\ \overline{X_{j_i}\delta_{n+1}} \end{pmatrix} \mid \delta_s \in \langle X_1, \ldots, X_t\rangle \text{ for } s = 1, \ldots, n+1 \right\}$$

<u>Proof</u> Let $\begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_{n+1}} \end{pmatrix} \in \ker\varphi$

Then, $\varphi \begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_{n+1}} \end{pmatrix} = \left(\sum_{s=1}^{n+1} b_{rs}\overline{\gamma_s}\right) \equiv (\overline{0})$. Hence, $\sum_{s=1}^{n+1} b_{rs}\gamma_s \in I \ \forall r$

So, for $1 \le r \le n$, $r \ne k, l$,

$$\sum_{s=1}^{n+1} b_{rs}\gamma_s = b_{rr}\gamma_r = X_{j_r}\gamma_r \in I \text{ and } \forall r \ne k, l, \overline{\gamma_r} \equiv \overline{X_{j_r}\delta_r} \text{ for some } \delta_r$$

And, for $r = n + 1$,

$$\sum_{s=1}^{n+1} b_{(n+1)s}\gamma_s = b_{(n+1)(n+1)}\gamma_{n+1} = X_{j_i}\gamma_{n+1} \in I.$$

$$\text{So, } \overline{\gamma_{n+1}} \equiv \overline{X_{j_i}\delta_{n+1}}$$

for some δ_{n+1}.

Now, since $X_{j_k}\gamma_k \in I$,

$$X_{j_k}\gamma_k = \sum_{a=1}^{n}\left(\sum_{j=1}^{n_a} \mu_j^a m_a \rho_j^a\right) + \sum_{j=1}^{n_{sum}} \mu_j (X_{j_k}X_{j_i} + cX_{j_l}X_{j_i})\rho_j$$

for some monomials $\mu_j^a, \rho_j^a, \mu_j, \rho_j$.

$$\text{Hence, } X_{j_k}\gamma_k = \sum_{a=1}^{n}\left(\sum_{\substack{\text{subsum}\\ \text{over j}}} \mu_j^a m_a \rho_j^a\right) + \sum_{\substack{\text{subsum}\\ \text{over j}}} \overbrace{\mu_j}^{\notin K}(X_{j_k}X_{j_i} + cX_{j_l}X_{j_i})\rho_j$$

$$+ \sum_{\substack{\text{subsum}\\ \text{over j}}} \overbrace{\mu_j}^{\in K} X_{j_k}X_{j_i}\rho_j$$

As in the proof of Proposition 4.6,

$$X_{j_k}\gamma_k = \sum_{a\neq k}\left(\sum X_{j_k}\sigma_j^a m_a \rho_j^a\right) + \sum X_{j_k}\sigma_j^k m_k \rho_j^k$$

$$+\sum \overbrace{\mu_j^k}^{\in K} X_{j_k}^2 \rho_j^k + \sum X_{j_k}\sigma_j (X_{j_k}X_{j_i} + cX_{j_l}X_{j_i})\rho_j$$
$$+ \sum_{m_a|\rho_j \text{for some a}} \mu_j X_{j_k}X_{j_i}\rho_j + \sum \mu_j X_{j_k}X_{j_i}^2\rho_{j'}^a$$

$$\text{Thus, } \overline{\gamma_k} \equiv \overline{X_{j_k}\delta_k} \text{for } \delta_k = \sum \mu_j^k \rho_j^k$$

Similarly, $X_{j_l}\gamma_l \in I$. And, $\overline{\gamma_l} \equiv \overline{X_{j_l}\delta_l}$ for $\delta_l = \sum \mu_j^l \rho_j^l$

□

Theorem 4.2 *Let $I = \langle m_1, \ldots, m_n, p + cq\rangle$ be an ideal in $K\langle X_1, \ldots, X_t\rangle$ where $m_s = X_{j_s}^2$ for all s, and $p + cq = X_{j_k}X_{j_i} + cX_{j_l}X_{j_i}$ for some distinct $i, k, l \in \{1, \ldots, n\}$. Then, $\Lambda = K\langle X_1, \ldots, X_t\rangle/I$ has the following linear resolution:*

$$(*) \cdots \xrightarrow{\varphi} \Lambda^{n+1} \xrightarrow{\varphi} \Lambda^{n+1} \xrightarrow{\varphi} \Lambda^{n+1} \xrightarrow{\varphi_2} \Lambda^t \xrightarrow{\varphi_1} \Lambda \to k \to 0$$

where φ, φ_1, and φ_2 are as defined in Propositions 4.6, 4.7, 4.8.

<u>Proof</u> First look at $\varphi_1 \circ \varphi_2 = \left(\sum_{r=1}^{t} X_r a_{rs}\right)$.

For $1 \leq s \leq n$, $\sum_{r=1}^{t} X_r a_{rs} = \overline{X_{j_s}^2} \equiv \overline{0}$.

For $s = n+1$, $\sum_{r=1}^{t} X_r a_{rs} = X_{j_l}(cX_{j_i}) + X_{j_k}X_{j_i} \equiv \overline{0}$.

Hence, $Im\varphi_2 \subseteq \ker\varphi_1$.

Let $\begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_t} \end{pmatrix} \in \ker\varphi_1$. Then, by Proposition 4.6,

$$\overline{\gamma_s} = \begin{cases} \overline{0} & \text{if } s \in \{1, \ldots, t\} \setminus \{j_1, \ldots, j_n\} \\ \overline{X_s\delta_s} & \text{if } s \in \{j_1, \ldots, j_n\} \setminus j_k, j_l\} \\ \overline{X_s\delta_s + X_{j_i}\nu} & \text{if } s = j_k \\ \overline{X_s\delta_s + cX_{j_i}\nu} & \text{if } s = j_l \end{cases}$$

for some δ_s, ν. Then,

$$\varphi_2 \begin{pmatrix} \overline{\delta_{j_1}} \\ \vdots \\ \overline{\delta_{j_n}} \\ \overline{\nu} \end{pmatrix} = \begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_t} \end{pmatrix}$$

So, $\ker\varphi_1 = Im\varphi_2$.

Next, $\varphi_2 \circ \varphi = (a_{rs})(b_{rs}) = \left(\sum_{v=1}^{n+1} a_{rv}b_{vs}\right)$.

For $r = j_w$, $w \neq l, k$ and $s = w$, $\sum a_{rs}b_{sw} = a_{j_w w}b_{ww} = \overline{X_{j_w}^2} \equiv \overline{0}$.

For $r = j_l$ and $s = l$, $\sum a_{rv}b_{vs} = a_{j_l l}bll = \overline{X_{j_l}^2} \equiv \overline{0}$.

For $r = j_l$ and $s = n+1$, $\sum a_{rv}b_{vs} = a_{j_l(n+1)}b_{(n+1)(n+1)} = \overline{cX_{j_i}X_{j_i}} \equiv \overline{0}$

For $r = j_k, s = k$, $\sum a_{rv}b_{vs} = a_{j_k k}bkk = \overline{X_{j_k}^2} \equiv \overline{0}$.

For $r = j_k, s = n+1$, $\sum a_{rv}b_{vs} = a_{j_k(n+1)}b_{(n+1)(n+1)} = \overline{X_{j_i}X_{j_i}} \equiv \overline{0}$.

For all other r, s, $\sum a_{rv}b_{vs} = 0$.

So, $Im\varphi \subseteq \ker\varphi_2$.

Let $\begin{pmatrix} \overline{X_{j_1}\delta_1} \\ \vdots \\ \overline{X_{j_n}\delta_n} \\ \overline{X_{j_i}\delta_{n+1}} \end{pmatrix} \in \ker \varphi_2$.

Then,

$$\varphi \begin{pmatrix} \overline{\delta_1} \\ \vdots \\ \overline{\delta_n} \\ \overline{\delta_{n+1}} \end{pmatrix} = \begin{pmatrix} \overline{X_{j_1}\delta_1} \\ \vdots \\ \overline{X_{j_n}\delta_n} \\ \overline{X_{j_i}\delta_{n+1}} \end{pmatrix}$$

Hence, $\ker \varphi_2 = Im\varphi$.

Lastly, look at $\varphi \circ \varphi = \left(\sum_{v=1}^{n+1} b_{rv} b_{vs} \right)$.

For $r = s$ and $1 \leq r \leq n$, $\sum b_{rv} b_{vs} = b_{rr} b_{rr} = \overline{X_{j_r}^2} \equiv \overline{0}$.

For $r, s = n+1$, $\sum b_{rv} b_{vs} = b_{(n+1)(n+1)} b_{(n+1)(n+1)} = \overline{X_{j_i}^2} \equiv \overline{0}$

For all other r and s, $\sum b_{rv} b_{vs} = 0$.

Hence, $Im\varphi \subseteq \ker \varphi$.

Let $\begin{pmatrix} \overline{X_{j_1}\delta_1} \\ \vdots \\ \overline{X_{j_n}\delta_n} \\ \overline{X_{j_i}\delta_{n+1}} \end{pmatrix} \in \ker \varphi$.

Then,

$$\varphi \begin{pmatrix} \overline{\delta_1} \\ \vdots \\ \overline{\delta_n} \\ \overline{\delta_{n+1}} \end{pmatrix} = \begin{pmatrix} \overline{X_{j_1}\delta_1} \\ \vdots \\ \overline{X_{j_n}\delta_n} \\ \overline{X_{j_i}\delta_{n+1}} \end{pmatrix} \in \ker \varphi$$

Thus, $\ker \varphi = Im\varphi$.

Hence, $(*)$ is exact.

□

For our third, and final class of Koszul algebras without a quadratic Gröbner basis, we look at the ideal with $n + z$ generators,

$$I = \left\langle m_1, \ldots, m_n, \sum_{i=1}^{s_1} c_1^i p_1^i, \ldots, \sum_{i=1}^{s_z} c_z^i p_z^i \right\rangle$$

as described in Lemma 4.2 with $m_1 < \cdots < m_{c_1} < p_1^1 < m_{c_1+1} < \cdots < m_{c_2} < p_1^2 < m_{c_2+1} < \cdots < m_{c_z} < p_1^z < m_{c_z+1} < \cdots < m_n$.

We first need to understand the structure of certain ideal elements.

Proposition 4.9 *Let* $I = \left\langle m_1, \ldots, m_n, \sum_{j=1}^{s_1} c_j^1 p_j^1, \ldots, \sum_{j=1}^{s_z} c_j^z p_j^z \right\rangle$ *where* $m_i = X_{j_i}^2$ *for all* i, *and* $p_j^i = X_{k_j^i}^2$ *for all* i, j, *be an ideal of* $K\langle X_1, \ldots, X_t\rangle$. *Suppose that each* m_i *and* p_j^i *are distinct for all* i. *In addition, suppose each* m_i *is distinct from each* p_j^i. *Then, if* $X_{j_v}\gamma \in I$, *then* $\overline{\gamma} \equiv \overline{X_{j_v}\delta}$ *for some* δ. *Futhermore, if* $X_v\gamma \in I$, *for* $v \neq j_k, k_i^j$ *for all* i, j, k, *then* $\overline{\gamma} \equiv \overline{0}$.

<u>Proof</u> Suppose $X_{j_v}\gamma \in I$. Then,

$$X_{j_v}\gamma = \sum_{a=1}^{n}\left(\sum_{j=1}^{n_a} \mu_j^a m_a \rho_j^a\right) + \sum_{a=1}^{n_{sum}} \mu_a \left(\sum_{j=1}^{s_a} c_j^a p_j^a\right)\rho_a$$

for some monomials $\mu_j^a, \rho_j^a, \mu_a, \rho_a$. And,

$$X_{j_v}\gamma =$$

$$\underbrace{\sum_{a=1}^{n}\left(\sum_{\text{subsum}} \mu_j^a m_a \rho_j^a\right) + \sum_{\text{subsum}} \mu_a \left(\sum_{j=1}^{s_a} c_j^a p_j^a\right)\rho_a}_{\text{divisible by } x_{j_v}}$$

Now, since X_{j_v} does not divide p_j^a for all a, j, for all a, $\mu_a = X_{j_v}\sigma_a$ for some monomial σ_a. Similarly, since X_{j_v} does not divide m_a for all $a \neq v$, for all $a \neq v$, $\mu_j^a = X_{j_v}\sigma_j^a$ for some monomial σ_j^a. Suppose that $\mu_1^v, \ldots, \mu_l^v \in K$, and $\mu_{l+1}^v, \ldots, \mu_{l_v}^v \notin K$. Then, $\mu_{l+1}^v = X_{j_v}\sigma_{l+1}^v, \ldots, \mu_{l_v}^v = X_{j_v}\sigma_{l_v}^v$ for some monomials $\sigma_{l+1}^v, \ldots, \sigma_{l_v}^v$.

Thus $X_{j_v}\gamma =$

$$\sum_{a\neq v}\left(\sum_{\text{subsum}} X_{j_v}\sigma_j^a m_a \rho_j^a\right) + \sum_{j=1}^{l} \mu_j^v X_{j_v}^2 \rho_j^v + \sum_{j=l+1}^{l_v} X_{j_v}\sigma_j^v m_v \rho_j^v$$

$$+ \sum_{\text{subsum}} X_{j_v}\sigma_a \left(\sum_{j=1}^{s_a} c_j^a p_j^a\right)\rho_a$$

Thus, $\overline{\gamma} \equiv \overline{X_{j_v}\delta}$ for $\delta = \sum_1^l \mu_j^v \rho_j^v$.

Now, suppose X_v does not divide m_j, p_j^i for all i, j. If

$$X_v\gamma = \sum_{a=1}^{n}\left(\sum_{j=1}^{n_a} \mu_j^a m_a \rho_j^a\right) + \sum_{a=1}^{n_{sum}} \mu_a \left(\sum_{j=1}^{s_a} c_j^a p_j^a\right)\rho_a$$

for some monomials $\mu_j^a, \rho_j^a, \mu_a, \rho_a$, then $\mu_j^a = X_v\sigma_j^a$ and $\mu_a = X_v\sigma_a$ for all a, j, since X_v does not divide m_a, p_j^i for all a,i, j. Thus, $\gamma \in I$.

□

Using Proposition 4.9, we can now define all linear maps appearing in the linear resolution of the field.

Proposition 4.10 *Let $\varphi_1 : \Lambda^t \to \Lambda$ be defined by $\varphi_1 = (X_1 \cdots X_t)$. Then,*

$$ker\varphi_1 = \left\{ \begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_t} \end{pmatrix} \mid \overline{\gamma_s} \equiv \begin{cases} \overline{X_s\delta_s} & \text{if } s \in \{j_1, \ldots, j_n\} \\ \overline{c_i^l X_{k_i^l}\xi_l} & \text{if } s = k_i^l \text{ for some } l \\ \overline{0} & \text{if otherwise} \end{cases} \right\}$$

for some polynomial δ_s.

Proof Let $\begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_t} \end{pmatrix} \in \ker\varphi_1$. Then, $\varphi_1 \begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_t} \end{pmatrix} = \sum_{s=1}^t X_s\overline{\gamma_s} \equiv \overline{0}$. So, for some monomials $\mu_j^a, \rho_j^a, \mu_a, \rho_a$,

$$\sum_{s=1}^t X_s\gamma_s = \sum_{a=1}^n \left(\sum_{j=1}^{n_a} \mu_j^a m_a \rho_j^a \right) + \sum_{a=1}^{n_{sum}} \mu_a (\sum_{j=1}^{s_a} c_j^a p_j^a)\rho_a$$

So, for $s \neq j_v$ $\forall v$, and $s \neq k_j^i$ for all i,j

$$X_s\gamma_s = \sum_{a=1}^n \left(\sum_{\text{subsum}} \mu_j^a m_a \rho_j^a \right) + \sum_{\text{subsum}} \mu_a (\sum_{j=1}^{s_a} c_j^a p_j^a)\rho_a$$

for some monomials $\mu_j^a, \mu_a, \rho_a^j, \rho_a$. Hence, $X_s\gamma_s \in I$ and $\overline{\gamma_s} \equiv \overline{0}$. For $s = j_v$, (so $X_s^2 \neq p_j^i$ for all i,j)

$$X_{j_v}\gamma_{j_v} = \sum_{a=1}^n \left(\sum_{\text{subsum}} \mu_j^a m_a \rho_j^a \right) + \sum_{\text{subsum}} \mu_a (\sum_{j=1}^{s_a} c_j^a p_j^a)\rho_a \in I$$

Thus, $X_{j_v}\gamma_{j_v} \in I$ and $\overline{\gamma_{j_v}} \equiv \overline{X_{j_v}\delta_{j_v}}$ for some δ_{j_v}

Lastly, if $s = k_i^l$ for some i, l, then for some monomials μ_l, ρ_l,

$$\text{then, } X_s\gamma_s = \sum_{a=1}^n (\sum_{\text{subsum}} \mu_j^a m_a \rho_j^a) + \sum_{\text{subsum}} \mu_a (\sum c_j^a p_j^a)\rho_a + \mu_l c_i^l p_i^l \rho_l$$

where μ_j^a and μ_a are divisible by X_s.

$$\text{Hence, } \overline{\gamma_s} \equiv \overline{c_i^l X_s \xi_l}$$

for $\xi_l = \mu_l\rho_l$. Hence, for fixed l, $\overline{\gamma_{k_i^l}} \equiv \overline{c_i^l X_{k_i^l}\xi_l}$ for $\xi_l = \mu_l\rho_l$ for all $i = 1, \ldots, s_l$.

□

Proposition 4.11 *Let $\varphi_2 : \Lambda^{n+z} \to \Lambda^t$ be defined by $\varphi_2 = (a_{rs})$ where*

$$a_{rs} = \begin{cases} X_r & \text{if } 1 \leq s \leq c_1 \text{ and } r = j_s \\ X_{r-1} & \text{if } c_1+2 \leq s \leq c_2+1 \text{ and } r = j_{s-1} \\ \vdots & \vdots \\ X_{r-z} & \text{if } c_z+z+1 \leq s \leq n+z \text{ and } r = j_{s-z} \\ c_j^v X_{k_j^v} & \text{if } s = c_v + v \text{ and } r = k_j^v \\ 0 & \text{if otherwise} \end{cases}$$

(See Diagram 4.3)

Then,

$$ker\varphi_2 = \left\{ \begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_{n+z}} \end{pmatrix} \mid \gamma_v = \begin{cases} \overline{X_{j_v}\delta_v} & \text{if } 1 \leq v \leq c_1 \\ \overline{X_{j_{v-1}}\delta_{v-1}} & \text{if } c_1+2 \leq v \leq c_2+1 \\ \vdots & \vdots \\ \overline{X_{j_{v-z}}\delta_{v-z}} & \text{if } c_z+z+1 \leq v \leq n+z \\ \overline{0} & \text{if } v = c_k + k \text{ for some } k \end{cases} \right\}$$

for some polynomial δ_s.

<u>Proof</u> First note that we clearly have $\supseteq$. Let $\begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_{n+z}} \end{pmatrix} \in \ker\varphi_2$. Then, $\varphi_2 \begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_{n+z}} \end{pmatrix} =$ $(\sum_{v=1}^{n+z} a_{rv}\overline{\gamma_v}) \equiv (\overline{0})$.

Thus, for $r = j_k$ for some $1 \leq k \leq c_1$, we have $\sum_{v=1}^{n+z} a_{rv}\gamma_v = X_{j_k}\gamma_k \in I$ and $\overline{\gamma_k} \equiv \overline{X_{j_k}\delta_k}$ for some δ_k.

For $r = j_k$ for some $c_1 + 1 \leq k \leq c_2$, we have $\sum_{v=1}^{n+z} a_{rv}\gamma_v = X_{j_k}\gamma_{k+1} \in I$ and $\overline{\gamma_{k+1}} \equiv \overline{X_{j_k}\delta_k}$ for some δ_k.

$\vdots$

And, for $r = j_k$ for some $c_z + 1 \leq k \leq n$, $\sum_{v=1}^{n+z} a_{rv}\gamma_v = X_{j_k}\gamma_k \in I$. So, $\overline{\gamma_{k+z}} \equiv \overline{X_{j_k}\delta_k}$ for some δ_k. That is,

$$\text{If } 1 \leq v \leq c_1 \text{ , then } \overline{\gamma_v} \equiv \overline{X_{j_v}\delta_v}$$

$$\text{If } c_1 + 2 \leq v \leq c_2 + 1 \text{ , then } \overline{\gamma_v} \equiv \overline{X_{j_{v-1}}\delta_v}$$

$$\vdots$$

$$\text{If } c_z + z + 1 \leq v \leq n + z \text{ , then } \overline{\gamma_v} \equiv \overline{X_{j_{v-z}}\delta_v}$$

for some δ_v.
Now, fix j. If $r = k_i^j$ for some i, $1 \leq i \leq s_j$, then $\sum a_{rv}\gamma_v = c_i^j X_{k_i^j}\gamma_{c_j+j} \in I$. Thus, for some δ_j, $\overline{\gamma_{c_j+j}} \equiv \overline{X_{k_i^j}\delta_j}$ for all $1 \leq i \leq s_j$. Look at $X_{k_1^j}\delta_j - X_{k_2^j}\delta_j = \gamma_{c_j+j} - \gamma_{c_j+j} \in I$. Then, $X_{k_1^j}\delta_j - X_{k_2^j}\delta_j = \sum\sum \mu_j^a m_a \rho_j^a + \sum \mu_a(\sum C_j^a p_j^a)\rho_a$ for some monomials $\rho_a, \rho_j^a, \mu_a, \mu_j^a$.
As before, we get that $\overline{\delta_j} \equiv \overline{c_i^j X_{k_i^j}\xi_j}$ for all $i = 1, \dots, s_j$.
Thus, $\overline{\gamma_{c_j+j}} \equiv \overline{c_i^j p_i^j \xi_j}$ for all $i = 1, \dots s_j$ for some ξ_j.
Hence, $\sum_{i=1}^{s_j} \gamma_{c_j+j} = \sum_{i=1}^{s_j}(c_i^j p_i^j \xi_j) = (\sum_{i=1}^{s_j} c_i^j p_i^j)\xi_j \in I$.
Hence, $\gamma_{c_j+j} \in I$.

□

Proposition 4.12 *Let $\varphi_3 : \Lambda^n \to \Lambda^{n+z}$ be defined by $\varphi_3 = (b_{rs})$ where*

$$b_{rs} = \begin{cases} X_{j_r} & \textit{if } 1 \leq r \leq c_1 \textit{ and } r = s \\ X_{j_{r-1}} & \textit{if } c_1 + 2 \leq r \leq c_2 + 1 \textit{ and } s = r - 1 \\ \vdots & \vdots \\ X_{j_{r-z}} & \textit{if } c_z + z + 1 \leq r \leq n + z \textit{ and } s = r - z \\ 0 & \textit{if otherwise} \end{cases}$$

(See Diagram 4.3

Then,

$$ker\varphi_3 = \left\{ \begin{pmatrix} \overline{X_{j_1}\delta_1} \\ \vdots \\ \overline{X_{j_n}\delta_n} \end{pmatrix} \mid \delta_k \in K\langle X_1, \dots, X_t\rangle \forall k \right\}$$

<u>Proof</u> First note that we clearly have $\supseteq$. Let $\begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_n} \end{pmatrix} \in \ker\varphi_3$. Then, $\varphi_3 \begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_n} \end{pmatrix} =$ $(\sum_{v=1}^n b_{rv}\overline{\gamma_v}) \equiv (\overline{0})$. Thus, for $1 \leq r \leq c_1$, we have $\sum_{v=1}^n b_{rv}\gamma_v = X_{j_r}\gamma_r \in I$ and $\overline{\gamma_r} \equiv \overline{X_{j_r}\delta_r}$ for some δ_r. And, for $c_z+z+1 \leq r \leq n+z$, $\sum_{v=1}^n b_{rv}\gamma_v = X_{j_{r-z}}\gamma_{r-z} \in I$. So, $\overline{\gamma_{r-z}} \equiv \overline{X_{j_{r-z}}\delta_{r-z}}$ for some δ_{r-z}. And, in general, for $c_k + k + 1 \leq r \leq c_{k+1} + k$, $\sum_{v=1}^n b_{rv}\gamma_v = X_{j_{r-k}}\gamma_{r-k} \in I$ and $\overline{\gamma_{r-k}} \equiv \overline{X_{j_{r-k}}\delta_{r-k}}$ for some δ_{r-k}, for $1 \leq k \leq z - 1$.
Thus, for all r, $\overline{\gamma_r} \equiv \overline{X_{j_r}\delta_r}$ for some δ_r.

□

Proposition 4.13 *Let $\varphi : \Lambda^n \to \Lambda^n$ be defined by $\varphi = (c_{rs})$ where*

$$c_{rs} = \begin{cases} X_{j_r} & \textit{if } r = s \\ 0 & \textit{if otherwise} \end{cases}$$

(See Diagram 4.3)

Then,

$$ker\varphi = \left\{ \begin{pmatrix} \overline{X_{j_1}\delta_1} \\ \vdots \\ \overline{X_{j_n}\delta_n} \end{pmatrix} \mid \delta_k \right\}$$

Proof First note that we clearly have $\supseteq$. Let $\begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_n} \end{pmatrix} \in \ker\varphi$. Then, $\varphi \begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_n} \end{pmatrix} = (\sum_{v=1}^{n} c_{rv}\overline{\gamma_v}) \equiv (\overline{0})$. Thus, for $1 \leq r \leq n$, we have $\sum_{v=1}^{n} b_{rv}\gamma_v = X_{j_r}\gamma_r \in I$ and $\overline{\gamma_r} \equiv \overline{X_{j_r}\delta_r}$ for some δ_r.

$\square$

Theorem 4.3 *Let* $I = \left\langle m_1, \ldots, m_n, \sum_{j=1}^{s_1} c_j^1 p_j^1, \ldots, \sum_{j=1}^{s_z} c_j^z p_j^z \right\rangle$ *be an ideal in* $K\langle X_1, \ldots, X_t\rangle$ *where* $m_i = X_{j_i}^2$, $c_j^i \in K$, *and* $p_j^i = X_{k_j^i}^2$ *for all i,j. Suppose in addition that* $m_1 < m_2 < \ldots < m_{c_1} < p_1^1 = tip(\sum c_j^1) < m_{c_1+1} < \ldots < m_{c_2} < p_1^2 = tip(\sum c_j^2 p_j^2) < m_{c_2+1} < \ldots < m_{c_z} < p_1^z = tip(\sum c_j^z p_j^z) < m_{c_z+1} < \ldots < m_n$ *with each* m_i *distinct, each* m_i *distinct from the* p_j^i*'s and each* p_j^i *distinct. Then,* $\Lambda = K\langle X_1, \ldots, X_t\rangle\, /I$ *has the following linear resolution:*

$$(*) \cdots \xrightarrow{\varphi} \Lambda^n \xrightarrow{\varphi} \Lambda^n \xrightarrow{\varphi_3} \Lambda^{n+z} \xrightarrow{\varphi_2} \Lambda^t \xrightarrow{\varphi_1} \Lambda \to k \to 0$$

where $\varphi, \varphi_1, \varphi_2$, *and* φ_3 *are as defined in Propositions 4.10, 4.11, 4.12, 4.13.*

Proof First look at $\varphi_1 \circ \varphi_2 = \left(\sum_{r=1}^{t} X_r a_{rs}\right)$.

For $1 \leq s \leq c_1$, $\sum_{r=1}^{t} X_r a_{rs} = \overline{X_{j_s}^2} \equiv \overline{0}$

For $s = c_1 + 1$, $\sum_{r=1}^{t} X_r a_{rs} = \sum_{j=1}^{s_1} c_j^1 \overline{X_{k_j^1}^2} \equiv \overline{0}$

For $c_1 + 2 \leq s \leq c_2 + 1$, $\sum_{r=1}^{t} X_r a_{rs} = \overline{X_{j_{s-1}}^2} \equiv \overline{0}$

For $s = c_2 + 2$, $\sum_{r=1}^{t} X_r a_{rs} = \sum_{j=1}^{s_2} c_j^2 \overline{X_{k_j^2}^2} \equiv \overline{0}$

$$\vdots$$

For $s = c_z + z$, $\sum_{r=1}^{t} X_r a_{rs} = \sum_{j=1}^{s_z} c_j^z \overline{X_{k_j^z}^2} \equiv \overline{0}$

Hence, $\varphi_1 \circ \varphi_2 = (0)_{1\times(n)}$. Thus, $Im\varphi_2 \subseteq \ker\varphi_1$.

Let $\begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_t} \end{pmatrix} \in \ker \varphi_1$. Then, by Proposition 4.10,

$$\overline{\gamma_s} \equiv \begin{cases} \overline{X_s \delta_s} & \text{if } s \in \{j_1, \ldots, j_n\} \\ \overline{c_i^j X_{k_i^j} \xi_j} & \text{if } s = k_i^j \text{ for some } i, j \\ \overline{0} & \text{if otherwise} \end{cases}$$

for some δ_s.

Then,

$$\varphi_2 \begin{pmatrix} \overline{\delta_{j_1}} \\ \vdots \\ \overline{\delta_{j_{c_1}}} \\ \overline{\xi_1} \\ \overline{\delta_{j_{c_1+1}}} \\ \vdots \\ \overline{\delta_{j_{c_2}}} \\ \overline{\xi_2} \\ \overline{\delta_{j_{c_2+1}}} \\ \vdots \\ \overline{\delta_{j_{c_z}}} \\ \overline{\xi_z} \\ \overline{\delta_{j_{c_z+1}}} \\ \vdots \\ \overline{\delta_{j_n}} \end{pmatrix} = \begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_t} \end{pmatrix}$$

Hence, $\ker \varphi_1 \subseteq \mathrm{im}\varphi_2$. So, $\ker \varphi_1 = Im\varphi_2$.

Next, $\varphi_2 \circ \varphi_3 = \left(\sum_{v=1}^{n+z} a_{rv} b_{vs}\right)$.

Now, for $r = j_w$ for some $1 \leq w \leq n$ $\sum_{v=1}^{n+z} a_{rv} b_{vs} = a_{j_w w} b_{ww} = \overline{X_{j_w}^2} \equiv \overline{0}$

And, for all other r, $\sum_{v=1}^{n+z} a_{rv} b_{vs} = 0$. Hence, $\mathrm{im}\varphi_3 \subseteq \ker \varphi_2$.

Let $\begin{pmatrix} \overline{X_{j_1}\delta_1} \\ \vdots \\ \overline{X_{j_{c_1}}\delta_{c_1}} \\ \overline{0} \\ \overline{X_{j_{c_1+1}}\delta_{c_1+1}} \\ \vdots \\ \overline{X_{j_{c_2}}\delta_{c_2}} \\ \overline{0} \\ \overline{X_{j_{c_2+1}}\delta_{c_2+1}} \\ \vdots \\ \overline{X_{j_{c_z}}\delta_{c_z}} \\ \overline{0} \\ \overline{X_{j_{c_z+1}}\delta_{c_z+1}} \\ \vdots \\ \overline{X_{j_n}\delta_n} \end{pmatrix} \in \ker\varphi_2$. Then, $\varphi_3\begin{pmatrix} \overline{\delta_1} \\ \vdots \\ \overline{\delta_n} \end{pmatrix} = \begin{pmatrix} \overline{X_{j_1}\delta_1} \\ \vdots \\ \overline{X_{j_{c_1}}\delta_{c_1}} \\ \overline{0} \\ \overline{X_{j_{c_1+1}}\delta_{c_1+1}} \\ \vdots \\ \overline{X_{j_{c_2}}\delta_{c_2}} \\ \overline{0} \\ \overline{X_{j_{c_2+1}}\delta_{c_2+1}} \\ \vdots \\ \overline{X_{j_{c_z}}\delta_{c_z}} \\ \overline{0} \\ \overline{X_{j_{c_z+1}}\delta_{c_z+1}} \\ \vdots \\ \overline{X_{j_n}\delta_n} \end{pmatrix}$

Hence, $\ker\varphi_2 = Im\varphi$.

$$\varphi_3 \circ \varphi = \left(\sum_{v=1}^{n} b_{rv}c_{vs}\right).$$

Now, for $r \neq c_i + i$, $1 \leq i \leq z$, $s = r$ for $1 \leq r \leq c_1$ and $s = r - i$ for $c_i + 1 \leq r \leq c_{i+1} + i$, $\sum_{v=1}^{n} b_{rv}c_{vs} = \overline{X_{j_r}^2} \equiv \overline{0}$.

And, for all other r, $\sum_{v=1}^{n} b_{rv}c_{vs} = 0$. Hence, $\varphi_3 \circ \varphi \equiv (0)_{(n+z)\times(n)}$ Thus, $Im\varphi \subseteq \ker\varphi_3$.

Let $\begin{pmatrix} \overline{X_{j_1}\delta_1} \\ \vdots \\ \overline{X_{j_n}\delta_n} \end{pmatrix} \in \ker\varphi_3$. Then, $\varphi\begin{pmatrix} \overline{\delta_1} \\ \vdots \\ \overline{\delta_n} \end{pmatrix} = \begin{pmatrix} \overline{X_{j_1}\delta_1} \\ \vdots \\ \overline{X_{j_n}\delta_n} \end{pmatrix}$. Thus, $\ker\varphi_3 = Im\varphi$.

And, finally, $\varphi \circ \varphi = (\sum_{v=1}^{n} c_{rv}c_{vs})$

where for $1 \leq r \leq n$ and $r = s$, $\sum_{v=1}^{n} c_{rv}c_{vs} = \overline{X_{j_r}^2} \equiv \overline{0}$. For all other r and s, $\sum_{v=1}^{n} c_{rv}c_{vs} = 0$.

Hence, im$\varphi \subseteq \ker\varphi$. Now, let $\begin{pmatrix} \overline{X_{j_1}\delta_1} \\ \vdots \\ \overline{X_{j_n}\delta_n} \end{pmatrix} \in \ker\varphi$. Then, $\varphi\begin{pmatrix} \overline{\delta_1} \\ \vdots \\ \overline{\delta_n} \end{pmatrix} = \begin{pmatrix} \overline{X_{j_1}\delta_1} \\ \vdots \\ \overline{X_{j_n}\delta_n} \end{pmatrix}$. So, im$\varphi = \ker\varphi$. Hence, $(*)$ is exact.

□

Chapter 5

A Non-Koszul Algebra

We now arrive at the last of our investigated classes of quotients of the non-commutative polynomial ring. Throughout this chapter, $m_1 = X_{j_1}^2, \ldots, m_n = X_{j_n}^2$ are perfect square monomials in $K\langle X_1, \ldots, X_t\rangle$ and $\Lambda = K\langle X_1, \ldots, X_t\rangle/I$. Additionally, by $\overline{\gamma}$ we mean $\gamma + I$.

Lemma 5.1 *If $p + cq = X_{j_i}X_\beta + cX_\xi X_{j_i}$ for some $1 \leq i \leq n$ and some $\beta \neq \xi$, $\beta, \xi \neq j_i$, then $I = \langle m_1, \ldots, m_n, p + cq\rangle$ has no quadratic Gröbner basis.*

Proof Let us suppose tip$(p+cq)$= $X_{j_i}X_\beta$ (tip$(p+cq)$= $X_\xi X_{j_i}$ is analogous) and consider

$$o(p + cq, m_i) = X_{j_i}(X_{j_i}X_\beta + cX_\xi X_{j_i}) - (X_{j_i}^2)X_\beta = cX_{j_i}X_\xi X_{j_i}$$

Now, since $X_\xi \neq X_{j_i}$ or X_β, $X_{j_i}X_\xi X_{j_i}$ is irreducible. Thus, we have an irreducible cubic term to add to the Gröbner basis for I.

□

That is, if neither p nor q is a perfect square, $supp(p) \neq supp(q)$, and if for some i, $supp(p) \bigcap supp(q) \bigcap supp(m_i) \neq \phi$ and $m_i \mid pq$ or qp, then $I = \langle m_1, \ldots, m_n, p + cq\rangle$ has no quadratic Gröbner basis.

As repeatedly emphasized, the existence of a quadratic Gröbner basis is not necessary for the resulting algebra to be Koszul. In fact, in this case, the resulting algebra is not Koszul. To show this, we will describe the maps up to the first non-linear map of the field resolution, as well as provide a description of that first non-linear map. We begin with a result concerning ideal membership.

Proposition 5.1 *Let $I = \langle m_1, \ldots, m_n, p + cq\rangle$ with $m_s = X_{j_s}^2$ for all s, and $p + cq = X_{j_i}X_\beta + cX_\xi X_{j_i}$ for some $1 \leq i \leq n$, $\beta \neq \xi$ and $j_i \neq \beta, \xi$. If $X_{j_v}\gamma \in I$ for some $v \neq i$ and $j_v \neq \xi$, then $\overline{\gamma} \equiv \overline{X_{j_v}\delta}$ for some δ. Futhermore, if $X_v\gamma \in I$ for $v \neq j_s$ for all s and $v \neq \beta, \xi$, then $\gamma \in I$.*

<u>Proof</u> Let $X_{j_v}\gamma \in I$. Then, for some monomials $\rho_j^a, \rho_j, \mu_j^a, \mu_j$,

$$X_{j_v}\gamma = \sum_{a=1}^{n}(\sum_{j=1}^{n_a}\mu_j^a m_a \rho_j^a) + \sum_{j=1}^{n_{sum}}\mu_j(X_{j_i}X_\beta + cX_\xi X_{j_i})\rho_j$$

Since $j_v \neq \xi$, j_i, $\mu_j \notin K \setminus 0$ for all $1 \leq j \leq n_{sum}$. Hence, for all j, $\mu_j = X_{j_v}\sigma_j$ for some monomial σ_j. Similarly, for all j and all $a \neq v$, $\mu_j^a = X_{j_v}\sigma_j^a$ for some monomials σ_j^a.

$$\text{Thus, } X_{j_v}\gamma = \sum_{a\neq v}(\sum^{n_a} X_{j_v}\sigma_j^a m_a \rho_j^a) + \sum^{n_v}\mu_j^v m_v \rho_j^v + \sum^{n_{sum}} X_{j_v}\sigma_j(X_{j_i}X_\beta + cX_\xi X_{j_i})\rho_j$$

Now, suppose $\mu_1^v, \ldots, \mu_{\lambda_v}^v \in K \setminus 0$ and $\mu_{\lambda_v+1}^v, \ldots, \mu_{n_v}^v \notin K \setminus 0$. Then, for all $\lambda_v + 1 \leq s \leq n_v$, $\mu_s^v = X_{j_v}\sigma_s^v$ for some σ_s^v.

$$\text{Hence } X_{j_v}\gamma = \sum_{a\neq v}(\sum^{n_a} X_{j_v}\sigma_j^a m_a \rho_j^a) + \sum^{\lambda_v}\mu_j^v X_{j_v}^2 \rho_j^v +$$

$$\sum_{\lambda_v+1}^{n_v} X_{j_v}\sigma_j^v m_v \rho_j^v + \sum^{n_{sum}} X_{j_v}\sigma_j(X_{j_i}X_\beta + cX_\xi X_{j_i})\rho_j$$

Hence, $\overline{\gamma} \equiv \overline{X_{j_v}\delta}$ for some δ. Futhermore, if for some monomials $\mu_j, \rho_j, \mu_j^a, \rho_j^a$,

$$X_v\gamma = \sum_{a=1}^{n}(\sum_{j=1}^{n_a}\mu_j^a m_a \rho_j^a) + \sum_{j=1}^{n_{sum}}\mu_j(X_{j_i}X_\beta + cX_\xi X_{j_i})\rho_j$$

where X_v does not appear as any m_a nor as X_β or X_ξ, then we must have $X_v|\mu_j$ and μ_j^a for all a, j. Hence, $\overline{\gamma} \equiv \overline{0}$.

□

We will show that Λ, for I defined in Proposition 5.1, is not Koszul. Thus, we will show that regardless of order, the field has no linear resolution. For Propositions 5.2, 5.3, we suppose $tip(p + cq) = X_{j_i}X_\beta$. The next result gives a description of the kernel of the last linear map in the resolution of the field supposing that $tip(p + cq) = X_{j_i}X_\beta$. This leads to Proposition 5.3, showing the resolution is exact at the point of non-linearity.

Proposition 5.2 *Let $\varphi_2 : \Lambda^{n+1} \longrightarrow \Lambda^t$ be defined by $\varphi_2 = (a_{rs})$ where*

$$a_{rs} = \begin{cases} X_{j_s} & \text{if } 1 \leq s \leq i, r = j_s \\ X_\beta & \text{if } s = i+1, r = j_i \\ cX_{j_i} & \text{if } s = i+1, r = \xi \\ X_{j_{s-1}} & \text{if } i+2 \leq s \leq n+1, r = j_{s-1} \\ 0 & \text{if otherwise} \end{cases}$$

(See Diagram 5.1, Case I)

Then,

$$\ker\varphi_2 = \left\{ \begin{pmatrix} \overline{X_{j_1}\delta_1} \\ \vdots \\ \overline{X_{j_{k-1}}\delta_{k-1}} \\ \overline{\nabla k} \\ \overline{X_{j_{k+1}}\delta_{k+1}} \\ \vdots \\ \overline{X_{j_{i-1}}\delta_{i-1}} \\ \overline{X_{j_i}\delta_i + X_\beta X_{j_i}\theta} \\ \overline{\nabla i+1} \\ \overline{X_{j_{i+1}}\delta_{i+1}} \\ \vdots \\ \overline{X_{j_n}\delta_n} \end{pmatrix} \middle| \delta_1,\ldots,\delta_{n+1},\theta \in K\langle X_1,\ldots,X_t\rangle \right\}$$

where

$$\nabla k = \begin{cases} X_{j_k}\delta_k & \text{if } X_\xi = X_{j_k} \text{ for some } k,\ X_\beta \neq X_{j_l} \text{ for all } l. \\ X_{j_k}\delta_k & \text{if } X_\beta = X_{j_l} \text{ for some } l,\ X_\xi \neq X_{j_k} \text{ for all } k. \\ X_{j_k}\delta_k + cX_{j_i}\mu & \text{if } X_\xi = X_{j_k} \text{ for some } k,\ X_\beta = X_{j_l} \text{ for some } l. \end{cases}$$

and

$$\nabla i+1 = \begin{cases} 0 & \text{if } X_\xi = X_{j_k} \text{ for some } k,\ X_\beta \neq X_{j_l} \text{ for all } l. \\ X_{j_l}X_{j_i}\mu & \text{if } X_\beta = X_{j_l} \text{ for some } l,\ X_\xi \neq X_{j_k} \text{ for all } k. \\ c^{-1}X_{j_l}\mu' & \text{if } X_\xi = X_{j_k} \text{ for some } k,\ X_\beta = X_{j_l} \text{ for some } l. \end{cases}$$

<u>Proof</u> <u>Case 1</u> Suppose$X_\xi = X_{j_k}$ for some k, $X_\beta \neq X_{j_l}$ for all l. Let $\begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_{n+1}} \end{pmatrix} \in \ker\varphi_2$.

Then $\varphi_2 \begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_{n+1}} \end{pmatrix} = \begin{pmatrix} \sum a_{1s}\overline{\gamma_s} \\ \vdots \\ \sum a_{ts}\overline{\gamma_s} \end{pmatrix} \equiv \begin{pmatrix} \overline{0} \\ \vdots \\ \overline{0} \end{pmatrix}$. That is, for all $1 \leq r \leq t$, $\sum a_{rs}\gamma_s \in I$.

For $r = j_v$, $1 \leq v \leq k-1$ and $k+1 \leq v \leq i-1$, $\sum a_{j_v s}\gamma_s = X_{j_v}\gamma_v \in I$ and $\overline{\gamma_v} \equiv \overline{X_{j_v}\delta_v}$ for some δ_v by Proposition 5.1.

For $r = j_v$, $i+1 \leq v \leq n$, $\sum a_{j_v s}\gamma_s = X_{j_v}\gamma_{v+1} \in I$ and $\overline{\gamma_{v+1}} \equiv \overline{X_{j_v}\delta_{v+1}}$ for some δ_{v+1} by Proposition 5.1.

For $r = j_k$, $\sum a_{j_k s}\gamma_s = X_{j_k}\gamma_k + cX_{j_i}\gamma_{i+1} \in I$. Thus, $X_{j_k}\gamma_k + cX_{j_i}\gamma_{i+1} = \sum_{a=1}^{n}\sum_{j=1}^{n_a}\mu_j^a m_a \rho_j^a + \sum_{j=1}^{n_{sum}}\mu_j(X_{j_i}X_\beta + cX_\xi X_{j_i})\rho_j$ for some monomials $\mu_j, \mu_j^a, \rho_j, \rho_j^a$. Hence,

$$X_{j_k}\gamma_k = \sum_{a=1}^{n}\sum_{\substack{\text{subsum}\\ \text{over j}}}\mu_j^a m_a \rho_j^a + \sum_{\substack{\text{subsum over j}\\ \text{such that } X_{j_k}|\mu_j}}\mu_j(X_{j_i}X_\beta + cX_\xi X_{j_i})\rho_j +$$

$$\sum_{\substack{\text{subsum over j}\\ \text{such that } \mu_j \in K\setminus 0}}\mu_j(cX_{j_k}X_{j_i})\rho_j$$

and

$$X_{j_i}\gamma_{i+1} = \sum_{a=1}^{n}\sum_{\substack{\text{subsum}\\ \text{over j}}}\mu_j^a m_a \rho_j^a + \sum_{\substack{\text{subsum over j}\\ \text{such that } X_{j_i}|\mu_j}}\mu_j(X_{j_i}X_\beta + cX_\xi X_{j_i})\rho_j +$$

$$\sum_{\substack{\text{subsum over j}\\ \text{such that } \mu_j \in K\setminus 0}}\mu_j(X_{j_i}X_\beta)\rho_j$$

Thus, $\overline{\gamma_k} \equiv \overline{X_{j_k}\delta_k + cX_{j_i}\mu}$ and $\overline{\gamma_{i+1}} \equiv \overline{X_{j_i}\delta_{i+1} + X_\beta\mu}$ for some $\delta_k, \delta_{i+1}, \mu$.

Lastly, for $r = j_i$, $\sum a_{j_i s}\gamma_s = X_{j_i}\gamma_i + X_\beta\gamma_{i+1} \in I$.

Thus, $X_{j_i}\gamma_i + X_\beta\gamma_{i+1} = \sum_{a=1}^{n}\sum_{j=1}^{n_a}\mu_j^a m_a\rho_j^a + \sum_{j=1}^{n_{sum}}\mu_j(X_{j_i}X_\beta + cX_\xi X_{j_i})\rho_j$ for some monomials $\mu_j, \mu_j^a, \rho_j, \rho_j^a$.

$$\text{Hence, } X_{j_i}\gamma_i = \sum_{a=1}^{n}\sum_{\substack{\text{subsum}\\ \text{over j}}}\mu_j^a m_a\rho_j^a +$$

$$\sum_{\substack{\text{subsum over j}\\ \text{such that } X_{j_i}|\mu_j}}\mu_j(X_{j_i}X_\beta + cX_\xi X_{j_i})\rho_j + \sum_{\mathcal{I}}\mu_j(X_{j_i}X_\beta)\rho_j$$

where $\mathcal{I} = \{j|\mu_j \in K\setminus 0$ and $\mu_j cX_{j_k}X_{j_i}\rho_j$ cancels with a $\mu_{j'}^{a'} m_{a'}\rho_{j'}^{a'}$ term $\}$

And,

$$X_\beta\gamma_{i+1} = \sum_{a=1}^{n}\sum_{\substack{\text{subsum}\\ \text{over j}}}\mu_j^a m_a\rho_j^a + \sum_{\substack{\text{subsum over j}\\ \text{such that } X_\beta|\mu_j}}\mu_j(X_{j_i}X_\beta + cX_\xi X_{j_i})\rho_j$$

Thus, $\overline{\gamma_i} \equiv \overline{X_{j_i}\delta_i + X_\beta X_{j_i}\theta}$ for some δ_i, θ and $\overline{\gamma_{i+1}} \equiv \overline{0}$.

So, we have two representations for γ_{i+1}: $\overline{0} \equiv \overline{\gamma_{i+1}} \equiv \overline{X_{j_i}\delta_{i+1} + X_\beta\mu}$. Thus, $X_{j_i}\delta_{i+1}+X_\beta\mu \in I$ As above, we get $\overline{\mu} \equiv \overline{0}$. Hence, $\overline{\gamma_k} \equiv \overline{X_{j_k}\delta_k}$ for some δ_k.

<u>Case 2</u> Suppose $X_\beta = X_{j_l}$ for some l, $X_\xi \neq X_{j_k}$ for all k. Let $\begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_{n+1}} \end{pmatrix} \in \ker \varphi_2$. Then $\varphi_2 \begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_{n+1}} \end{pmatrix} = \begin{pmatrix} \sum a_{1s}\overline{\gamma_s} \\ \vdots \\ \sum a_{ts}\overline{\gamma_s} \end{pmatrix} \equiv \begin{pmatrix} \overline{0} \\ \vdots \\ \overline{0} \end{pmatrix}$. That is, for all $1 \leq r \leq t$, $\sum a_{rs}\gamma_s \in I$.

For $r = j_v$, $1 \leq v \leq i-1$, $\sum a_{j_v s}\gamma_s = X_{j_v}\gamma_v \in I$ and $\overline{\gamma_v} \equiv \overline{X_{j_v}\delta_v}$ for some δ_v by Proposition 5.1.

For $r = j_v$, $i+1 \leq v \leq n$, $\sum a_{j_v s}\gamma_s = X_{j_v}\gamma_{v+1} \in I$ and $\overline{\gamma_{v+1}} \equiv \overline{X_{j_v}\delta_{v+1}}$ for some δ_{v+1} by Proposition 5.1.

For $r = \xi$, $\sum a_{\xi s}\gamma_s = cX_{j_i}\gamma_{i+1} \in I$. Thus, $cX_{j_i}\gamma_{i+1} = \sum_{a=1}^{n}\sum_{j=1}^{n_a}\mu_j^a m_a \rho_j^a + \sum_{j=1}^{n_{sum}}\mu_j(X_{j_i}X_\beta + cX_\xi X_{j_i})\rho_j$ for some monomials $\mu_j, \mu_j^a, \rho_j, \rho_j^a$.

Hence,

$$X_{j_i}\gamma_{i+1} = \sum_{a=1}^{n} \sum_{\substack{\text{subsum} \\ \text{over j}}} \mu_j^a m_a \rho_j^a +$$

$$\sum_{\substack{\text{subsum over j} \\ \text{such that } X_{j_i}|\mu_j}} \mu_j(X_{j_i}X_\beta + cX_\xi X_{j_i})\rho_j + \sum_{\mathcal{I}} \mu_j(X_{j_i}X_\beta)\rho_j$$

where $\mathcal{I} = \{j | \mu_j \in K \setminus 0$ and $\mu_j X_\xi X_{j_i}\rho_j$ cancels with a $\mu_{j'}^{a'}\rho_{j'}^{a'}$ term $\}$

So, $\overline{\gamma_{i+1}} \equiv \overline{X_{j_i}\delta_{i+1} + X_{j_l}X_{j_i}\mu}$ for some δ_{i+1}, μ.

Lastly, for $r = j_i$, $\sum a_{j_i s}\gamma_s = X_{j_i}\gamma_i + X_{j_l}\gamma_{i+1} \in I$.

Thus, $X_{j_i}\gamma_i + X_{j_l}\gamma_{i+1} = \sum_{a=1}^{n}\sum_{j=1}^{n_a}\mu_j^a m_a \rho_j^a + \sum_{j=1}^{n_{sum}}\mu_j(X_{j_i}X_\beta + cX_\xi X_{j_i})\rho_j$ for some monomials $\mu_j, \mu_j^a, \rho_j, \rho_j^a$.

$$\text{Thus, } X_{j_i}\gamma_i = \sum_{a=1}^{n} \sum_{\substack{\text{subsum} \\ \text{over j}}} \mu_j^a m_a \rho_j^a +$$

$$\sum_{\substack{\text{subsum over j} \\ \text{such that } X_{j_i}|\mu_j}} \mu_j(X_{j_i}X_\beta + cX_\xi X_{j_i})\rho_j + \sum_{\mathcal{I}} \mu_j(X_{j_i}X_\beta)\rho_j$$

where $\mathcal{I} = \{j | \mu_j \in K \setminus 0$ and $\mu_j cX_\xi X_{j_i}\rho_j$ cancels with a $\mu^{a'}_{j'} m_{a'} \rho^{a'}_{j'}$term $\}$

And,

$$X_{j_l}\gamma_{i+1} = \sum_{a=1}^{n} \sum_{\substack{\text{subsum} \\ \text{over j}}} \mu^a_j m_a \rho^a_j + \sum_{\substack{\text{subsum over j} \\ \text{such that } X_{j_l}|\mu_j}} \mu_j(X_{j_i}X_\beta + cX_\xi X_{j_i})\rho_j$$

Again, $\overline{\gamma_i} \equiv \overline{X_{j_i}\delta_i + X_{j_l}X_{j_i}\theta}$ for some δ_i, θ and $\overline{\gamma_{i+1}} \equiv \overline{X_{j_l}\delta}$ for some δ.

So, we have two representations for γ_{i+1}: $\overline{X_{j_l}\delta} \equiv \overline{\gamma_{i+1}} \equiv \overline{X_{j_i}\delta_{i+1} + X_{j_l}X_{j_i}\mu}$ So, $X_{j_i}\delta_{i+1} + X_{j_l}(X_{j_i}\mu - \delta) = \sum_{a=1}^{n}\sum_{j=1}^{n_a} \mu^a_j m_a \rho^a_j + \sum_{j=1}^{n_{sum}} \mu_j(X_{j_i}X_\beta + cX_\xi X_{j_i})\rho_j$ for some monomials $\mu_j, \mu^a_j, \rho_j, \rho^a_j$.

As before, we get that $\overline{\delta_{i+1}} \equiv \overline{X_{j_i}\omega + X_{j_l}X_{j_i}\omega'}$ for some ω, ω'. Thus, $\overline{X_{j_i}\delta_{i+1}} \equiv \overline{0}$.

Hence, $\overline{\gamma_{i+1}} \equiv \overline{X_{j_l}X_{j_l}\mu}$.

<u>Case 3</u> Suppose $X_\xi = X_{j_k}$ for some k, $X_\beta = X_{j_l}$ for some l. Let $\begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_{n+1}} \end{pmatrix} \in \ker\varphi_2$. Then $\varphi_2\begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_{n+1}} \end{pmatrix} = \begin{pmatrix} \sum a_{1s}\overline{\gamma_s} \\ \vdots \\ \sum a_{ts}\overline{\gamma_s} \end{pmatrix} \equiv \begin{pmatrix} \overline{0} \\ \vdots \\ \overline{0} \end{pmatrix}$. That is, for all $1 \le r \le t$, $\sum a_{rs}\gamma_s \in I$.

For $r = j_v$, $1 \le v \le k-1$ and $k+1 \le v \le i-1$, $\sum a_{j_v s}\gamma_s = X_{j_v}\gamma_v \in I$ and $\overline{\gamma_v} \equiv \overline{X_{j_v}\delta_v}$ for some δ_v by Proposition 5.1.

For $r = j_v$, $i+1 \le v \le n$, $\sum a_{j_v s}\gamma_s = X_{j_v}\gamma_{v+1} \in I$ and $\overline{\gamma_{v+1}} \equiv \overline{X_{j_v}\delta_{v+1}}$ for some δ_{v+1} by Proposition 5.1.

For $r = j_k$, $\sum a_{j_k s}\gamma_s = X_{j_k}\gamma_k + cX_{j_i}\gamma_{i+1} \in I$. Thus, $X_{j_k}\gamma_k + cX_{j_i}\gamma_{i+1} = \sum_{a=1}^{n}\sum_{j=1}^{n_a} \mu^a_j m_a \rho^a_j + \sum_{j=1}^{n_{sum}} \mu_j(X_{j_i}X_\beta + cX_\xi X_{j_i})\rho_j$ for some monomials $\mu_j, \mu^a_j, \rho_j, \rho^a_j$.

Hence,

$$X_{j_k}\gamma_k = \sum_{a=1}^{n} \sum_{\substack{\text{subsum} \\ \text{over j}}} \mu^a_j m_a \rho^a_j + \sum_{\substack{\text{subsum over j} \\ \text{such that } X_{j_k}|\mu_j}} \mu_j(X_{j_i}X_\beta + cX_\xi X_{j_i})\rho_j +$$

$$\sum_{\substack{\text{subsum over j} \\ \text{such that } \mu_j \in K\setminus 0}} \mu_j(cX_{j_k}X_{j_i})\rho_j$$

and

$$cX_{j_i}\gamma_{i+1} = \sum_{a=1}^{n} \sum_{\substack{\text{subsum}\\ \text{over j}}} \mu_j^a m_a \rho_j^a + \sum_{\substack{\text{subsum over j}\\ \text{such that } X_{j_i}|\mu_j}} \mu_j(X_{j_i}X_\beta + cX_\xi X_{j_i})\rho_j +$$

$$\sum_{\substack{\text{subsum over j}\\ \text{such that } \mu_j \in K\setminus 0}} \mu_j(X_{j_i}X_\beta)\rho_j$$

So, $\overline{\gamma_k} \equiv \overline{X_{j_k}\delta_k + cX_{j_i}\mu}$ and $\overline{\gamma_{i+1}} \equiv \overline{c^{-1}X_{j_i}\delta_{i+1} + c^{-1}X_{j_l}\mu}$ for some $\delta_k, \delta i+1, \mu$.

Lastly, for $r = j_i$, $\sum a_{j_i s}\gamma_s = X_{j_i}\gamma_i + X_{j_l}\gamma_{i+1} \in I$.

Thus, $X_{j_i}\gamma_i + X_{j_l}\gamma_{i+1} = \sum_{a=1}^{n}\sum_{j=1}^{n_a} \mu_j^a m_a \rho_j^a + \sum_{j=1}^{n_{sum}} \mu_j(X_{j_i}X_\beta + cX_\xi X_{j_i})\rho_j$ for some monomials $\mu_j, \mu_j^a, \rho_j, \rho_j^a$.

$$\text{Hence, } X_{j_i}\gamma_i = \sum_{a=1}^{n} \sum_{\substack{\text{subsum}\\ \text{over j}}} \mu_j^a m_a \rho_j^a + \sum_{\substack{\text{subsum over j}\\ \text{such that } X_{j_i}|\mu_j}} \mu_j(X_{j_i}X_\beta + cX_\xi X_{j_i})\rho_j +$$

$$\sum_{\mathcal{I}} \mu_j(X_{j_i}X_\beta)\rho_j$$

where $\mathcal{I} = \{j | \mu_j \in K\setminus 0 \text{and } \mu_j cX_{j_k}X_{j_i}\rho_j \text{cancels with a } \mu_{j'}^{a'} m_{a'} \rho_{j'}^{a'} \text{term }\}$

And,

$$X_{j_l}\gamma_{i+1} = \sum_{a=1}^{n} \sum_{\substack{\text{subsum}\\ \text{over j}}} \mu_j^a m_a \rho_j^a + \sum_{\substack{\text{subsum over j}\\ \text{such that } X_\beta|\mu_j}} \mu_j(X_{j_i}X_\beta + cX_\xi X_{j_i})\rho_j$$

Thus, $\overline{\gamma_i} \equiv \overline{X_{j_i}\delta_i + X_{j_l}X_{j_i}\theta}$ for some δ_i, θ and $\overline{\gamma_{i+1}} \equiv \overline{X_{j_l}\rho}$ for some ρ.

So, we have two representations for γ_{i+1}: $\overline{X_{j_l}\rho} \equiv \overline{\gamma_{i+1}} \equiv \overline{c^{-1}X_{j_i}\delta_{i+1} + c^{-1}X_{j_l}\mu}$

Thus, $c^{-1}X_{j_i}\delta_{i+1} + X_{j_l}(c^{-1}\mu - \rho) \in I$. As above, we get that $\overline{\delta_{i+1}} \equiv \overline{X_{j_i}\omega + X_{j_l}X_{j_i}\omega'}$ for some ω, ω'. Which means, $\overline{X_{j_i}\delta_{i+1}} \equiv \overline{0}$.

Hence, $\overline{\gamma_{i+1}} \equiv \overline{c^{-1}X_{j_l}\mu}$.

□

Proposition 5.3 *Let φ_2 be as defined in Proposition 5.2. Let $\varphi_3 : \Lambda^{n+2} \longrightarrow \Lambda^{n+1}$ be defined by $\varphi_3 = (b_{rs})$ where*

Case 1: $X_\xi = X_{j_k}$ *for some* k, $X_\beta \neq X_{j_l}$ *for all* l

$$b_{rs} = \begin{cases} X_{j_r} & \text{if } 1 \leq r \leq i \text{ and } r = s \\ X_\beta X_{j_i} & \text{if } r = i,\ s = i+1 \\ X_{j_i}^2 & \text{if } r = i \text{ and } s = i+2 \\ X_{j_{r-1}} & \text{if } i+2 \leq r \leq n+1 \text{ and } s = r+1 \\ 0 & \text{if otherwise} \end{cases}$$

(See Diagram 5.1, Case I, Case 1)

Case 2: $X_\beta = X_{j_l}$ *for some* l *and* $X_\xi \neq X_{j_k}$ *for all* k.

$$b_{rs} = \begin{cases} X_{j_r} & \text{if } 1 \leq r \leq k \text{ and } r = s \\ X_\beta X_{j_i} & \text{if } r = i+1,\ s = k+1 \\ X_{j_r} & \text{if } k+1 \leq r \leq i \text{ and } s = r+1 \\ X_\beta X_{j_i} & \text{if } r = i \text{ and } s = i+2 \\ X_{j_{r-1}} & \text{if } i+2 \leq r \leq n+1 \text{ and } s = r+1 \\ 0 & \text{if otherwise} \end{cases}$$

(See Diagram 5.1, Case I, Case 2)

Case 3: $X_\xi = X_{j_k}$, $X_\beta = X_{j_l}$ *for some* k, l

$$b_{rs} = \begin{cases} X_{j_r} & \text{if } 1 \leq r \leq i \text{ and } r = s \\ c^2 X_{j_i} & \text{if } r = k,\ s = i+1 \\ X_\beta & \text{if } r = i+1 \text{ and } s = i+1 \\ X_\beta X_{j_i} & \text{if } r = i \text{ and } s = i+2 \\ X_{j_{r-1}} & \text{if } i+2 \leq r \leq n+1 \text{ and } s = r+1 \\ 0 & \text{if otherwise} \end{cases}$$

(See Diagram 5.1, Case I, Case 3)

Then, $\ker \varphi_2 = im\varphi_3$.

Proof First note that $\varphi_2 \circ \varphi_3 = \left(\sum_{w=1}^{n+1} a_{rw} b_{ws}\right)$

Case 1 $X_\xi = X_{j_k}$ for some k, $X_\beta \neq X_{j_l}$ for all l.

For $r = j_v$, $1 \leq v \leq i$ and $s = v$, $\sum a_{j_v w} b_{wv} = \overline{X_{j_v}^2} \equiv \overline{0}$.

For $r = j_i$, $s = i+1$, $\sum a_{j_i w} b_{w(i+1)} = \overline{X_{j_i} X_\beta X_{j_i}} \equiv \overline{0}$.

For $r = j_i$, $s = i+2$, $\sum a_{j_i w} b_{w(i+2)} = \overline{X_{j_i}^3} \equiv \overline{0}$.

For $r = j_v$, $i+1 \le v \le n$ and $s = v+2$, $\sum a_{j_v w} b_{w(v+2)} = \overline{X_{j_v}^2} \equiv \overline{0}$.

For all other r, s, $\sum a_{rv} b_{vs} = 0$. Thus, $\mathrm{im}\varphi_3 \subseteq \ker \varphi_2$.

By Proposition 5.2 , let $\begin{pmatrix} \overline{X_{j_1}\delta_1} \\ \vdots \\ \overline{X_{j_{k-1}}\delta_{k-1}} \\ \overline{X_{j_k}\delta_k} \\ \overline{X_{j_{k+1}}\delta_{k+1}} \\ \vdots \\ \overline{X_{j_{i-1}}\delta_{i-1}} \\ \overline{X_{j_i}\delta_i + X_\beta X_{j_i}\theta} \\ \overline{0} \\ \overline{X_{j_{i+1}}\delta_{i+2}} \\ \vdots \\ \overline{X_{j_n}\delta_{n+1}} \end{pmatrix} \in \ker \varphi_2$. Then,

$$\varphi_3 \begin{pmatrix} \overline{\delta_1} \\ \vdots \\ \overline{\delta_k} \\ \overline{\delta_{k+1}} \\ \vdots \\ \overline{\delta_{i-1}} \\ \overline{\delta_i} \\ \overline{\theta} \\ \overline{0} \\ \overline{\delta_{i+2}} \\ \vdots \\ \overline{\delta_{n+1}} \end{pmatrix} = \begin{pmatrix} \overline{X_{j_1}\delta_1} \\ \vdots \\ \overline{X_{j_{k-1}}\delta_{k-1}} \\ \overline{X_{j_k}\delta_k} \\ \overline{X_{j_{k+1}}\delta_{k+1}} \\ \vdots \\ \overline{X_{j_{i-1}}\delta_{i-1}} \\ \overline{X_{j_i}\delta_i + X_\beta X_{j_i}\theta} \\ \overline{0} \\ \overline{X_{j_{i+1}}\delta_{i+2}} \\ \vdots \\ \overline{X_{j_n}\delta_{n+1}} \end{pmatrix}$$

Hence, $\ker \varphi_2 \subseteq \mathrm{im}\varphi_3$.

<u>Case 2</u> $X_\beta = X_{j_l}$ for some l, and $X_{j_k} < X_\xi < X_{j_{k+1}}$.

For $r = j_v$, $1 \le v \le k$ and $s = v$, $\sum a_{j_v w} b_{wv} = \overline{X_{j_v}^2} \equiv \overline{0}$.

For $r = j_v$, $k+1 \le v \le i$ and $s = v+1$, $\sum a_{j_v w} b_{w(v+1)} = \overline{X_{j_v}^2} \equiv \overline{0}$.

For $r = j_i$, $s = k+1$, $\sum a_{j_i w} b_{w(k+1)} = \overline{X_\beta^2 X_{j_i}} \equiv \overline{0}$.

For $r = j_i$, $s = i+1$, $\sum a_{j_i w} b_{w(i+1)} = \overline{X_{j_i}^2} \equiv \overline{0}$.

For $r = \xi$, $s = k+1$, $\sum a_{\xi w} b_{w(k+1)} = \overline{c X_{j_i} X_\beta X_{j_i}} \equiv \overline{0}$.

For $r = j_i$, $s = i+2$, $\sum a_{j_i w} b_{w(i+2)} = \overline{X_{j_i} X_\beta X_{j_i}} \equiv \overline{0}$.

For $r = j_v$, $i+1 \leq v \leq n$ and $s = v+2$, $\sum a_{j_v w} b_{w(v+2)} = \overline{X_{j_v}^2} \equiv \overline{0}$.

For all other r, s, $\sum a_{rv} b_{vs} = 0$. Thus, $\operatorname{im}\varphi_3 \subseteq \ker \varphi_2$.

By Proposition 5.2 , let $\begin{pmatrix} \overline{X_{j_1}\delta_1} \\ \vdots \\ \overline{X_{j_{i-1}}\delta_{i-1}} \\ \overline{X_{j_i}\delta_i + X_\beta X_{j_i}\theta} \\ \overline{X_{j_l}X_{j_i}\mu} \\ \overline{X_{j_{i+1}}\delta_{i+2}} \\ \vdots \\ \overline{X_{j_n}\delta_{n+1}} \end{pmatrix} \in \ker \varphi_2$. Then,

$$\varphi_3 \begin{pmatrix} \overline{\delta_1} \\ \vdots \\ \overline{\delta_k} \\ \overline{\mu} \\ \overline{\delta_{k+1}} \\ \vdots \\ \overline{\delta_{i-1}} \\ \overline{\delta_i} \\ \overline{\theta} \\ \overline{\delta_{i+2}} \\ \vdots \\ \overline{\delta_{n+1}} \end{pmatrix} = \begin{pmatrix} \overline{X_{j_1}\delta_1} \\ \vdots \\ \overline{X_{j_{i-1}}\delta_{i-1}} \\ \overline{X_{j_i}\delta_i + X_\beta X_{j_i}\theta} \\ \overline{X_{j_l}X_{j_i}\mu} \\ \overline{X_{j_{i+1}}\delta_{i+2}} \\ \vdots \\ \overline{X_{j_n}\delta_{n+1}} \end{pmatrix}$$

Hence, $\ker \varphi_2 \subseteq \operatorname{im}\varphi_3$.

<u>Case 3</u> $X_\beta = X_{j_l}$ and $X_\xi = X_{j_k}$ for some k, l.

For $r = j_v$, $1 \leq v \leq i$ and $s = v$, $\sum a_{j_v w} b_{wv} = \overline{X_{j_v}^2} \equiv \overline{0}$.

For $r = j_i$, $s = i+1$, $\sum a_{j_i w} b_{w(i+1)} = \overline{X_\beta^2} \equiv \overline{0}$.

For $r = j_k$, $s = i+1$, $\sum a_{j_k w} b_{w(i+1)} = \overline{cX_{j_i}X_\beta + c^2 X_\xi X_{j_i}} \equiv \overline{0}$.

For $r = j_i$, $s = i+2$, $\sum a_{j_i w} b_{w(i+2)} = \overline{X_{j_i}X_\beta X_{j_i}} \equiv \overline{0}$.

For $r = j_v$, $i+1 \leq v \leq n$ and $s = v+2$, $\sum a_{j_v w} b_{w(v+2)} = \overline{X_{j_v}^2} \equiv \overline{0}$.

For all other r, s, $\sum a_{rv} b_{vs} = 0$. Thus, $\operatorname{im}\varphi_3 \subseteq \ker \varphi_2$.

By Proposition 5.2, let $\begin{pmatrix} \overline{X_{j_1}\delta_1} \\ \vdots \\ \overline{X_{j_{k-1}}\delta_{k-1}} \\ \overline{X_{j_k}\delta_k + cX_{j_i}\mu} \\ \overline{X_{j_{k+1}}\delta_{k+1}} \\ \vdots \\ \overline{X_{j_{i-1}}\delta_{i-1}} \\ \overline{X_{j_i}\delta_i + X_\beta X_{j_i}\theta} \\ \overline{c^{-1}X_{j_l}\mu} \\ \overline{X_{j_{i+1}}\delta_{i+1}} \\ \vdots \\ \overline{X_{j_n}\delta_n} \end{pmatrix} \in \ker\varphi_2$. Then,

$$\varphi_3 \begin{pmatrix} \overline{\delta_1} \\ \vdots \\ \overline{\delta_k} \\ \overline{\delta_{k+1}} \\ \vdots \\ \overline{\delta_{i-1}} \\ \overline{\delta_i} \\ \overline{c^{-1}\mu} \\ \overline{\theta} \\ \overline{\delta_{i+2}} \\ \vdots \\ \overline{\delta_{n+1}} \end{pmatrix} = \begin{pmatrix} \overline{X_{j_1}\delta_1} \\ \vdots \\ \overline{X_{j_{k-1}}\delta_{k-1}} \\ \overline{X_{j_k}\delta_k + cX_{j_i}\mu} \\ \overline{X_{j_{k+1}}\delta_{k+1}} \\ \vdots \\ \overline{X_{j_{i-1}}\delta_{i-1}} \\ \overline{X_{j_i}\delta_i + X_\beta X_{j_i}\theta} \\ \overline{c^{-1}X_{j_l}\mu} \\ \overline{X_{j_{i+1}}\delta_{i+1}} \\ \vdots \\ \overline{X_{j_n}\delta_n} \end{pmatrix}$$

Hence, $\ker\varphi_2 \subseteq \mathrm{im}\varphi_3$.

Thus, in all cases, $\mathrm{im}\varphi_3 = \ker\varphi_2$.

□

Now, we resolve the case when $tip(p + cq) = X_\xi X_{j_i}$. For Proposition 5.4, 5.5, suppose $tip(p + cq) = X_\xi X_{j_i}$. The following result gives a description of the kernel of the last linear map in the resolution of the field for this case. This is followed by Proposition 5.5, showing exactness where the first nonlinear map arises.

Proposition 5.4 *Suppose* $m_1 < \ldots < m_\gamma < tip(p+cq) = X_\xi X_{j_i} < m_{\gamma+1} < \ldots < m_n$. *Let* $\theta_2 : \Lambda^{n+1} \longrightarrow \Lambda^t$ *be defined by* $\theta_2 = (a_{rs})$ *where*

$$a_{rs} = \begin{cases} X_{j_s} & \text{if } 1 \leq s \leq \gamma, r = j_s \\ X_\beta & \text{if } s = \gamma+1, r = j_i \\ cX_{j_i} & \text{if } s = \gamma+1, r = \xi \\ X_{j_{s-1}} & \text{if } \gamma+2 \leq s \leq n+1, r = j_{s-1} \\ 0 & \text{if otherwise} \end{cases}$$

(See Diagram 5.1, Case II)

Then,

$$\ker\theta_2 = \left\{ \begin{pmatrix} \overline{X_{j_1}\delta_1} \\ \vdots \\ \overline{X_{j_{i-1}}\delta_{i-1}} \\ \overline{X_{j_i}\delta_i + X_\beta X_{j_i}\theta} \\ \overline{X_{j_{i+1}}\delta_{i+1}} \\ \vdots \\ \overline{X_{j_{k-1}}\delta_{k-1}} \\ \overline{\nabla_k} \\ \overline{\nabla_{k+1}} \\ \overline{X_{j_{k+1}}\delta_{k+1}} \\ \vdots \\ \overline{X_{j_n}\delta_n} \end{pmatrix} | \delta_1, \ldots, \delta_n, \theta \in K\langle X_1, \ldots, X_t \rangle \right\}$$

where

$$\nabla_k = \begin{cases} 0 & \text{if } X_\xi = X_{j_k} \text{ for some } k,\ X_\beta \neq X_{j_l} \text{ for all } l. \\ X_{j_k}\delta_k & \text{if } X_\beta = X_{j_l} \text{ for some } l,\ X_\xi \neq X_{j_k} \text{ for all } k. \\ X_\beta\theta' & \text{if } X_\xi = X_{j_k} \text{ for some } k,\ X_\beta = X_{j_l} \text{ for some } l. \end{cases}$$

and

$$\nabla_{k+1} = \begin{cases} X_{j_k}\delta_k & \text{if } X_\xi = X_{j_k} \text{ for some } k,\ X_\beta \neq X_{j_l} \text{ for all } l. \\ X_\beta X_{j_i}\mu & \text{if } X_\beta = X_{j_l} \text{ for some } l,\ X_\xi \neq X_{j_k} \text{ for all } k. \\ X_{j_k}\delta_k + c^{-1}X_{j_i}\theta' & \text{if } X_\xi = X_{j_k} \text{ for some } k,\ X_\beta = X_{j_l} \text{ for some } l. \end{cases}$$

Proof Case 1 Suppose$X_\xi = X_{j_k}$ for some k, $X_\beta \neq X_{j_l}$ for all l. Let $\begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_{n+1}} \end{pmatrix} \in \ker\theta_2$.

Then $\theta_2 \begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_{n+1}} \end{pmatrix} = \begin{pmatrix} \sum a_{1s}\overline{\gamma_s} \\ \vdots \\ \sum a_{ts}\overline{\gamma_s} \end{pmatrix} \equiv \begin{pmatrix} \overline{0} \\ \vdots \\ \overline{0} \end{pmatrix}$. That is, for all $1 \leq r \leq t$, $\sum a_{rs}\gamma_s \in I$.

For $r = j_v$, $1 \leq v \leq k-1$, $\sum a_{j_v s}\gamma_s = X_{j_v}\gamma_v \in I$ and $\overline{\gamma_v} \equiv \overline{X_{j_v}\delta_v}$ for some δ_v by Proposition 5.1.

For $r = j_v$, $k+1 \leq v \leq n$, $\sum a_{j_v s}\gamma_s = X_{j_v}\gamma_{v+1} \in I$ and $\overline{\gamma_{v+1}} \equiv \overline{X_{j_v}\delta_{v+1}}$ for some δ_{v+1} by Proposition 5.1.

For $r = j_k$, $\sum a_{j_k s}\gamma_s = X_{j_k}\gamma_{k+1} + X_{j_i}\gamma_k \in I$. Thus, $X_{j_k}\gamma_{k+1} + X_{j_i}\gamma_k = \sum_{a=1}^{n}\sum_{j=1}^{n_a}\mu_j^a m_a \rho_j^a + \sum_{j=1}^{n_{sum}}\mu_j(cX_{j_i}X_\beta + X_\xi X_{j_i})\rho_j$ for some monomials μ_j, μ_j^a, ρ_j, and ρ_j^a.

Hence,

$$X_{j_k}\gamma_{k+1} = \sum_{a=1}^{n} \sum_{\substack{\text{subsum} \\ \text{over j}}} \mu_j^a m_a \rho_j^a +$$

$$\sum_{\substack{\text{subsum over j} \\ \text{such that } X_{j_k}|\mu_j}} \mu_j(cX_{j_i}X_\beta + X_\xi X_{j_i})\rho_j + \sum_{\substack{\text{subsum over j} \\ \text{such that } \mu_j \in K\setminus 0}} \mu_j(X_{j_k}X_{j_i})\rho_j$$

and

$$X_{j_i}\gamma_k = \sum_{a=1}^{n} \sum_{\substack{\text{subsum} \\ \text{over j}}} \mu_j^a m_a \rho_j^a +$$

$$\sum_{\substack{\text{subsum over j} \\ \text{such that } X_{j_i}|\mu_j}} \mu_j(cX_{j_i}X_\beta + X_\xi X_{j_i})\rho_j + \sum_{\substack{\text{subsum over j} \\ \text{such that } \mu_j \in K\setminus 0}} c\mu_j(X_{j_i}X_\beta)\rho_j$$

So, $\overline{\gamma_{k+1}} \equiv \overline{X_{j_k}\delta_k + X_{j_i}\theta}$ and $\overline{\gamma_k} \equiv \overline{X_{j_i}\delta'_{i+1} + X_\beta X_{j_i}\xi' + cX_\beta\theta}$ for some $\delta_{k+1}, \delta_k, \xi, \theta, \xi'$.

Lastly, for $r = j_i$, $\sum a_{j_i s}\gamma_s = X_{j_i}\gamma_i + cX_\beta\gamma_k \in I$.

Thus, $X_{j_i}\gamma_i + cX_\beta\gamma_k = \sum_{a=1}^{n}\sum_{j=1}^{n_a}\mu_j^a m_a\rho_j^a + \sum_{j=1}^{n_{sum}}\mu_j(cX_{j_i}X_\beta + X_\xi X_{j_i})\rho_j$ for some monomials $\mu_j, \mu_j^a, \rho_j, \rho_j^a$.

$$\text{And, } X_{j_i}\gamma_i = \sum_{a=1}^{n} \sum_{\substack{\text{subsum} \\ \text{over j}}} \mu_j^a m_a \rho_j^a +$$

$$\sum_{\substack{\text{subsum over j} \\ \text{such that } X_{j_i}|\mu_j}} \mu_j(cX_{j_i}X_\beta + X_\xi X_{j_i})\rho_j + \sum_{\mathcal{I}} \mu_j(X_{j_i}X_\beta)\rho_j$$

where $\mathcal{I} = \{j | \mu_j \in K\setminus 0$ and $\mu_j cX_{j_k}X_{j_i}\rho_j$ cancels with a $\mu_{j'}^{a'} m_{a'}\rho_{j'}^{a'}$ term $\}$

And,

$$cX_\beta\gamma_k = \sum_{a=1}^{n} \sum_{\substack{\text{subsum}\\ \text{over j}}} \mu_j^a m_a \rho_j^a + \sum_{\substack{\text{subsum over j}\\ \text{such that } X_\beta|\mu_j}} \mu_j(cX_{j_i}X_\beta + X_\xi X_{j_i})\rho_j$$

Thus, $\overline{\gamma_i} \equiv \overline{X_{j_i}\delta_i + X_\beta X_{j_i}\theta}$ for some δ_i, θ.

And, since $\beta \neq j_l$ for all l, $\overline{\gamma_k} \equiv \overline{0}$.

So, $\overline{\gamma_k} \equiv \overline{0} \equiv \overline{X_{j_i}\delta'_{i+1} + X_\beta X_{j_i}\xi' + cX_\beta\theta}$. Thus, $\overline{X_{j_i}\xi' + c\theta} \equiv \overline{0}$ which means that $cX_{j_i}\theta \in I$. Thus, $\overline{\gamma_{k+1}} \equiv \overline{X_{j_k}\delta_k}$.

<u>Case 2</u> Suppose $X_\beta = X_{j_l}$ for some l, $X_\xi \neq X_{j_k}$ for all k. Let $\begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_{n+1}} \end{pmatrix} \in \ker\theta_2$. Then $\theta_2 \begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_{n+1}} \end{pmatrix} = \begin{pmatrix} \sum a_{1s}\overline{\gamma_s} \\ \vdots \\ \sum a_{ts}\overline{\gamma_s} \end{pmatrix} \equiv \begin{pmatrix} \overline{0} \\ \vdots \\ \overline{0} \end{pmatrix}$. That is, for all $1 \leq r \leq t$, $\sum a_{rs}\gamma_s \in I$.

For $r = j_v$, $1 \leq v \leq k$, $v \neq i$, $\sum a_{j_v s}\gamma_s = X_{j_v}\gamma_v \in I$ and $\overline{\gamma_v} \equiv \overline{X_{j_v}\delta_v}$ for some δ_v by Proposition 5.1.

For $r = j_v$, $k+1 \leq v \leq n$, $\sum a_{j_v s}\gamma_s = X_{j_v}\gamma_{v+1} \in I$ and $\overline{\gamma_{v+1}} \equiv \overline{X_{j_v}\delta_{v+1}}$ for some δ_{v+1} by Proposition 5.1.

For $r = \xi$, $\sum a_{\xi s}\gamma_s = X_{j_i}\gamma_{k+1} \in I$. Thus, $X_{j_i}\gamma_{k+1} = \sum_{a=1}^{n}\sum_{j=1}^{n_a} \mu_j^a m_a \rho_j^a + \sum_{j=1}^{n_{sum}} \mu_j(cX_{j_i}X_\beta + X_\xi X_{j_i})\rho_j$ for some monomials $\mu_j, \mu_j^a, \rho_j, \rho_j^a$.

Hence,

$$X_{j_i}\gamma_{k+1} = \sum_{a=1}^{n} \sum_{\substack{\text{subsum}\\ \text{over j}}} \mu_j^a m_a \rho_j^a +$$

$$\sum_{\substack{\text{subsum over j}\\ \text{such that } X_{j_i}|\mu_j}} \mu_j(cX_{j_i}X_\beta + X_\xi X_{j_i})\rho_j + \sum_{\mathcal{I}} \mu_j(cX_{j_i}X_\beta)\rho_j$$

where $\mathcal{I} = \{j | \mu_j \in K \setminus 0$ and $\mu_j X_\xi X_{j_i}\rho_j$ cancels with a $\mu_{j'}^{a'}\rho_{j'}^{a'}$ term $\}$

And, $\overline{\gamma_{k+1}} \equiv \overline{X_{j_i}\delta' + X_\beta X_{j_i}\mu}$ for some δ', μ.

Lastly, for $r = j_i$, $\sum a_{j_i s}\gamma_s = X_{j_i}\gamma_i + cX_{j_l}\gamma_{k+1} \in I$.

Thus, $X_{j_i}\gamma_i + cX_{j_l}\gamma_{k+1} = \sum_{a=1}^{n}\sum_{j=1}^{n_a} \mu_j^a m_a \rho_j^a + \sum_{j=1}^{n_{sum}} \mu_j(cX_{j_i}X_\beta + X_\xi X_{j_i})\rho_j$ for some monomials $\mu_j, \mu_j^a, \rho_j, \rho_j^a$.

$$\text{So, } X_{j_i}\gamma_i = \sum_{a=1}^{n} \sum_{\substack{\text{subsum} \\ \text{over j}}} \mu_j^a m_a \rho_j^a +$$

$$\sum_{\substack{\text{subsum over j} \\ \text{such that } X_{j_i}|\mu_j}} \mu_j(cX_{j_i}X_\beta + X_\xi X_{j_i})\rho_j + \sum_{\mathcal{I}} \mu_j(cX_{j_i}X_\beta)\rho_j$$

where $\mathcal{I} = \{j|\mu_j \in K \setminus 0 \text{ and } \mu_j cX_\xi X_{j_i}\rho_j \text{ cancels with a } \mu_{j'}^{a'} m_{a'} \rho_{j'}^{a'} \text{term } \}$

And,

$$X_{j_l}\gamma_{k+1} = \sum_{a=1}^{n} \sum_{\substack{\text{subsum} \\ \text{over j}}} \mu_j^a m_a \rho_j^a + \sum_{\substack{\text{subsum over j} \\ \text{such that } X_{j_l}|\mu_j}} \mu_j(cX_{j_i}X_\beta + X_\xi X_{j_i})\rho_j$$

Again, $\overline{\gamma_i} \equiv \overline{X_{j_i}\delta_i + X_{j_l}X_{j_i}\theta}$ for some δ_i, θ and $\overline{\gamma_{k+1}} \equiv \overline{X_{j_l}\delta_{k+1}}$ for some δ_{k+1}.

So, we have two representations for γ_{k+1}: $\overline{X_{j_l}\delta_{k+1}} \equiv \overline{\gamma_{k+1}} \equiv \overline{X_{j_i}\delta' + X_{j_l}X_{j_i}\mu}$. Hence, $X_{j_i}\delta_{k+1} + X_{j_l}(X_{j_i}\mu - \delta') = \sum_{a=1}^{n}\sum_{j=1}^{n_a} \mu_j^a m_a \rho_j^a + \sum_{j=1}^{n_{sum}} \mu_j(cX_{j_i}X_\beta + X_\xi X_{j_i})\rho_j$ for some monomials $\mu_j, \mu_j^a, \rho_j, \rho_j^a$.

As before, we get that $\overline{\delta_{k+1}} \equiv \overline{X_{j_l}\theta' + X_{j_i}\alpha}$. Thus, $\overline{X_{j_l}\delta_{k+1}} \equiv \overline{X_\beta X_{j_i}\theta}$.

Hence, $\overline{\gamma_{k+1}} \equiv \overline{X_{j_l}X_{j_i}\theta}$.

Case 3 Suppose $X_\xi = X_{j_k}$ for some k, $X_\beta = X_{j_l}$ for some l. Let $\begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_{n+1}} \end{pmatrix} \in \ker\theta_2$. Then $\theta_2 \begin{pmatrix} \overline{\gamma_1} \\ \vdots \\ \overline{\gamma_{n+1}} \end{pmatrix} = \begin{pmatrix} \sum a_{1s}\overline{\gamma_s} \\ \vdots \\ \sum a_{ts}\overline{\gamma_s} \end{pmatrix} \equiv \begin{pmatrix} \overline{0} \\ \vdots \\ \overline{0} \end{pmatrix}$. That is, for all $1 \le r \le t$, $\sum a_{rs}\gamma_s \in I$.

For $r = j_v$, $1 \le v \le k-1$, $v \ne i$, $\sum a_{j_v s}\gamma_s = X_{j_v}\gamma_v \in I$ and $\overline{\gamma_v} \equiv \overline{X_{j_v}\delta_v}$ for some δ_v by Proposition 5.1.

For $r = j_v$, $k+1 \le v \le n$, $\sum a_{j_v s}\gamma_s = X_{j_v}\gamma_{v+1} \in I$ and $\overline{\gamma_{v+1}} \equiv \overline{X_{j_v}\delta_{v+1}}$ for some δ_{v+1} by Proposition 5.1.

For $r = j_k$, $\sum a_{j_k s}\gamma_s = X_{j_k}\gamma_{k+1} + X_{j_i}\gamma_k \in I$. Thus, $X_{j_k}\gamma_{k+1} + X_{j_i}\gamma_k = \sum_{a=1}^{n}\sum_{j=1}^{n_a} \mu_j^a m_a \rho_j^a + \sum_{j=1}^{n_{sum}} \mu_j(cX_{j_i}X_\beta + X_\xi X_{j_i})\rho_j$ for some monomials $\mu_j, \mu_j^a, \rho_j, \rho_j^a$.

Hence,

$$X_{j_k}\gamma_{k+1} = \sum_{a=1}^{n} \sum_{\substack{\text{subsum} \\ \text{over j}}} \mu_j^a m_a \rho_j^a +$$

$$\sum_{\substack{\text{subsum over j} \\ \text{such that } X_{j_k}|\mu_j}} \mu_j(cX_{j_i}X_\beta + X_\xi X_{j_i})\rho_j + \sum_{\substack{\text{subsum over j} \\ \text{such that } \mu_j \in K\setminus 0}} \mu_j(cX_{j_k}X_{j_i})\rho_j$$

and

$$X_{j_i}\gamma_k = \sum_{a=1}^{n} \sum_{\substack{\text{subsum} \\ \text{over j}}} \mu_j^a m_a \rho_j^a +$$

$$\sum_{\substack{\text{subsum over j} \\ \text{such that } X_{j_i}|\mu_j}} \mu_j(cX_{j_i}X_\beta + X_\xi X_{j_i})\rho_j + \sum_{\substack{\text{subsum over j} \\ \text{such that } \mu_j \in K\setminus 0}} \mu_j(X_{j_i}X_\beta)\rho_j$$

So, $\overline{\gamma_{k+1}} \equiv \overline{X_{j_k}\delta_{k+1} + X_{j_i}X_{j_l}\xi' + X_{j_i}\theta'}$ and $\overline{\gamma_k} \equiv \overline{X_{j_i}\delta + cX_{j_l}X_{j_i}\xi + cX_\beta\theta'}$ for some $\delta_k, \delta_{i+1}, \mu$.

Lastly, for $r = j_i$, $\sum a_{j_i s}\gamma_s = X_{j_i}\gamma_i + cX_{j_l}\gamma_k \in I$.

Thus, $X_{j_i}\gamma_i + X_{j_l}\gamma_k = \sum_{a=1}^{n}\sum_{j=1}^{n_a} \mu_j^a m_a \rho_j^a + \sum_{j=1}^{n_{sum}} \mu_j(cX_{j_i}X_\beta + X_\xi X_{j_i})\rho_j$ for some monomials $\mu_j, \mu_j^a, \rho_j, \rho_j^a$.

$$\text{Then, } X_{j_i}\gamma_i = \sum_{a=1}^{n} \sum_{\substack{\text{subsum} \\ \text{over j}}} \mu_j^a m_a \rho_j^a +$$

$$\sum_{\substack{\text{subsum over j} \\ \text{such that } X_{j_i}|\mu_j}} \mu_j(cX_{j_i}X_\beta + X_\xi X_{j_i})\rho_j + \sum_{\mathcal{I}} \mu_j(cX_{j_i}X_\beta)\rho_j$$

where $\mathcal{I} = \{j|\mu_j \in K\setminus 0$and $\mu_j X_{j_k}X_{j_i}\rho_j$ cancels with a $\mu_{j'}^{a'} m_{a'} \rho_{j'}^{a'}$term $\}$

And,

$$X_{j_l}\gamma_k = \sum_{a=1}^{n} \sum_{\substack{\text{subsum} \\ \text{over j}}} \mu_j^a m_a \rho_j^a + \sum_{\substack{\text{subsum over j} \\ \text{such that } X_\beta|\mu_j}} \mu_j(cX_{j_i}X_\beta + X_\xi X_{j_i})\rho_j$$

Thus, $\overline{\gamma_i} \equiv \overline{X_{j_i}\delta_i + X_{j_l}X_{j_i}\theta}$ for some δ_i, θ and $\overline{\gamma_k} \equiv \overline{X_{j_l}\delta'}$for some θ'.

So, we have two representations for γ_k and we thus obtain $\overline{X_{j_i}\delta} \equiv \overline{0}$.

Similarly, by looking at $X_{j_i}\gamma_k + X_{j_k}\gamma_{k+1} \in I$, we get that $\overline{X_{j_i}\delta'} \equiv \overline{cX_{j_i}\theta'}$. Lastly, setting $\delta_k = \delta_{k+1} + c^{-1}X_{j_i}\xi'$, we see that $\overline{\gamma_{k+1}} \equiv \overline{X_{j_k}\delta_k + c^{-1}X_{j_i}\delta'}$.

□

Finally, for $tip(p+cq) = X_\xi X_{j_i}$, we show the resolution is exact at the point of nonlinearity.

Proposition 5.5 *Let θ_2 be as defined in Proposition 5.4. Let $\theta_3 = (b_{rs})$ where*

Case 1: $X_\xi = X_{j_k}$ for some k, $X_\beta \neq X_{j_l}$ for all l

$$b_{rs} = \begin{cases} X_{j_r} & \text{if } 1 \leq r \leq i \text{ and } r = s \\ X_\beta X_{j_i} & \text{if } r = i,\ s = i+1 \\ X_\beta^2 & \text{if } r = i \text{ and } s = i+2 \\ X_{j_r} & \text{if } i+1 \leq r \leq k-1 \text{ and } s = r+2 \\ X_{j_{r-1}} & \text{if } k+1 \leq r \leq n+1 \text{ and } s = r+1 \\ 0 & \text{if otherwise} \end{cases}$$

(See Diagram 5.1, Case II, Case 1)

Case 2: $X_\beta = X_{j_l}$ for some l and $X_\xi \neq X_{j_k}$ for all k.

$$b_{rs} = \begin{cases} X_{j_r} & \text{if } 1 \leq r \leq i \text{ and } r = s \\ X_\beta X_{j_i} & \text{if } r = i,\ s = r+1 \\ X_{j_r} & \text{if } i+1 \leq r \leq k \text{ and } s = r+1 \\ X_\beta X_{j_i} & \text{if } r = k+1 \text{ and } s = r+1 \\ X_{j_{r-1}} & \text{if } k+2 \leq r \leq n+1 \text{ and } s = r+1 \\ 0 & \text{if otherwise} \end{cases}$$

(See Diagram 5.1, Case II, Case 2)

Case 3: $X_\xi = X_{j_k}$, $X_\beta = X_{j_l}$ for some k, l

$$b_{rs} = \begin{cases} X_{j_r} & \text{if } 1 \leq r \leq i \text{ and } r = s \\ X_\beta X_{j_i} & \text{if } r = i \text{ and } s = i+1 \\ X_{j_r} & \text{if } i+1 \leq r \leq k-1 \text{ and } s = r+1 \\ cX_\beta & \text{if } r = k \text{ and } s = k+1 \\ X_{j_i} & \text{if } r = k+1 = s \\ X_{j_{r-1}} & \text{if } k+1 \leq r \leq n+1 \text{ and } s = r+1 \\ 0 & \text{if otherwise} \end{cases}$$

(See Diagram 5.1, Case II, Case 3)

Then, $\ker\theta_2 = im\theta_3$.

Proof First note that $\theta_2 \circ \theta_3 = \left(\sum_{w=1}^{n+1} a_{rw} b_{ws}\right)$

Case 1 $X_\xi = X_{j_k}$ for some k, $X_\beta \neq X_{j_l}$ for all l.

For $r = j_v$, $1 \leq v \leq i$ and $s = v$, $\sum a_{j_v w} b_{wv} = \overline{X_{j_v}^2} \equiv \overline{0}$.

For $r = j_i$, $s = i+1$, $\sum a_{j_i w} b_{w(i+1)} = \overline{X_{j_i} X_\beta X_{j_i}} \equiv \overline{0}$.

For $r = j_i$, $s = i+2$, $\sum a_{j_i w} b_{w(i+2)} = \overline{X_{j_i} X_\beta^2} \equiv \overline{0}$.

For $r = j_v$, $i+1 \leq v \leq k-1$ and $s = v+2$, $\sum a_{j_v w} b_{w(v+2)} = \overline{X_{j_v}^2} \equiv \overline{0}$.

For $r = j_v$, $k \leq v \leq n$ and $s = v+2$, $\sum a_{j_v w} b_{w(v+2)} = \overline{X_{j_v}^2} \equiv \overline{0}$.

For all other r, s, $\sum a_{rv} b_{vs} = 0$. Thus, $\mathrm{im}\theta_3 \subseteq \ker\theta_2$.

By Proposition 5.4, let $\begin{pmatrix} \overline{X_{j_1}\delta_1} \\ \vdots \\ \overline{X_{j_{i-1}}\delta_{i-1}} \\ \overline{X_{j_i}\delta_i + X_\beta X_{j_i}\theta} \\ \overline{X_{j_{i+1}}\delta_{i+2}} \\ \vdots \\ \overline{X_{j_{k-1}}\delta_{k-1}} \\ \overline{0} \\ \overline{X_{j_k}\delta_k} \\ \overline{X_{j_{k+1}}\delta_{k+1}} \\ \vdots \\ \overline{X_{j_n}\delta_n} \end{pmatrix} \in \ker\theta_2$. Then,

$$\theta_3 \begin{pmatrix} \overline{\delta_1} \\ \vdots \\ \overline{\delta_{i-1}} \\ \overline{\delta_i} \\ \overline{\theta} \\ \overline{0} \\ \overline{\delta_{i+1}} \\ \vdots \\ \overline{\delta_k} \\ \overline{\delta_{k+1}} \\ \vdots \\ \overline{\delta_n} \end{pmatrix} = \begin{pmatrix} \overline{X_{j_1}\delta_1} \\ \vdots \\ \overline{X_{j_{i-1}}\delta_{i-1}} \\ \overline{X_{j_i}\delta_i + X_\beta X_{j_i}\theta} \\ \overline{X_{j_{i+1}}\delta_{i+2}} \\ \vdots \\ \overline{X_{j_{k-1}}\delta_{k-1}} \\ \overline{0} \\ \overline{X_{j_k}\delta_k} \\ \overline{X_{j_{k+1}}\delta_{k+1}} \\ \vdots \\ \overline{X_{j_n}\delta_n} \end{pmatrix}$$

Hence, $\ker\theta_2 \subseteq \mathrm{im}\theta_3$.

<u>Case 2</u> $X_\beta = X_{j_l}$ for some l, and $X_{j_k} < X_\xi < X_{j_{k+1}}$.

For $r = j_v$, $1 \le v \le i$ and $s = v$, $\sum a_{j_v w} b_{wv} = \overline{X_{j_v}^2} \equiv \overline{0}$.

For $r = j_v$, $i+1 \le v \le k$ and $s = v+1$, $\sum a_{j_v w} b_{w(v+1)} = \overline{X_{j_v}^2} \equiv \overline{0}$.

For $r = j_i$, $s = k+2$, $\sum a_{j_i w} b_{w(k+2)} = \overline{X_\beta^2 X_{j_i}} \equiv \overline{0}$.

For $r = j_i$, $s = i+1$, $\sum a_{j_i w} b_{w(i+1)} = \overline{X_{j_i} X_\beta X_{j_i}} \equiv \overline{0}$.

For $r = \xi$, $s = k+2$, $\sum a_{\xi w} b_{w(k+2)} = \overline{X_{j_i} X_\beta X_{j_i}} \equiv \overline{0}$.

For $r = j_v$, $k+1 \le v \le n$ and $s = v+2$, $\sum a_{j_v w} b_{w(v+2)} = \overline{X_{j_v}^2} \equiv \overline{0}$.

For all other r, s, $\sum a_{rv} b_{vs} = 0$. Thus, $\mathrm{im}\theta_3 \subseteq \ker\theta_2$.

By Proposition 5.4 , let $\begin{pmatrix} \overline{X_{j_1}\delta_1} \\ \vdots \\ \overline{X_{j_{i-1}}\delta_{i-1}} \\ \overline{X_{j_i}\delta_i + X_\beta X_{j_i}\theta} \\ \overline{X_{j_{i+1}}\delta_{i+1}} \\ \vdots \\ \overline{X_{j_{k-1}}\delta_{k-1}} \\ \overline{X_{j_k}\delta_k} \\ \overline{X_{j_l}X_{j_i}\mu} \\ \overline{X_{j_{k+1}}\delta_{k+1}} \\ \vdots \\ \overline{X_{j_n}\delta_n} \end{pmatrix} \in \ker\theta_2$. Then,

$$\theta_3 \begin{pmatrix} \overline{\delta_1} \\ \vdots \\ \overline{\delta_{i-1}} \\ \overline{\delta_i} \\ \overline{\theta} \\ \overline{\delta_{i+1}} \\ \vdots \\ \overline{\delta_k} \\ \overline{\mu} \\ \overline{\delta k+1} \\ \vdots \\ \overline{\delta_n} \end{pmatrix} = \begin{pmatrix} \overline{X_{j_1}\delta_1} \\ \vdots \\ \overline{X_{j_{i-1}}\delta_{i-1}} \\ \overline{X_{j_i}\delta_i + X_\beta X_{j_i}\theta} \\ \overline{X_{j_{i+1}}\delta_{i+1}} \\ \vdots \\ \overline{X_{j_{k-1}}\delta_{k-1}} \\ \overline{X_{j_k}\delta_k} \\ \overline{X_{j_l}X_{j_i}\mu} \\ \overline{X_{j_{k+1}}\delta_{k+1}} \\ \vdots \\ \overline{X_{j_n}\delta_n} \end{pmatrix}$$

Hence, $\ker\theta_2 \subseteq \mathrm{im}\theta_3$.

<u>Case 3</u> $X_\beta = X_{j_l}$ and $X_\xi = X_{j_k}$ for some k, l.

For $r = j_v$, $1 \le v \le i$ and $s = v$, $\sum a_{j_v w} b_{wv} = \overline{X_{j_v}^2} \equiv \overline{0}$.

For $r = j_i$, $s = i+1$, $\sum a_{j_i w} b_{w(i+1)} = \overline{X_{j_i} X_\beta X_{j_i}} \equiv \overline{0}$.

For $r = j_k$, $s = k+1$, $\sum a_{j_k w} b_{w(k+1)} = \overline{cX_{j_i} X_\beta + X_\xi X_{j_i}} \equiv \overline{0}$.

For $r = j_i$, $s = k+1$, $\sum a_{j_i w} b_{w(k+1)} = \overline{c^2 X_\beta^2} \equiv \overline{0}$.

For $r = j_v$, $i+1 \le v \le k-1$ and $s = v+1$, $\sum a_{j_v w} b_{w(v+1)} = \overline{X_{j_v}^2} \equiv \overline{0}$.

For $r = j_v$, $k \le v \le n$ and $s = v+2$, $\sum a_{j_v w} b_{w(v+2)} = \overline{X_{j_v}^2} \equiv \overline{0}$.

For all other r, s, $\sum a_{rv} b_{vs} = 0$. Thus, $\mathrm{im}\theta_3 \subseteq \ker\theta_2$.

By Proposition 5.4, let $\begin{pmatrix} \overline{X_{j_1}\delta_1} \\ \vdots \\ \overline{X_{j_{i-1}}\delta_{i-1}} \\ \overline{X_{j_i}\delta_i + X_\beta X_{j_i}\theta} \\ \overline{X_{j_{i+1}}\delta_{i+1}} \\ \vdots \\ \overline{X_{j_{k-1}}\delta_{k-1}} \\ \overline{X_\beta\theta'} \\ \overline{X_{j_k}\delta_k + c^{-1}X_{j_i}\theta'} \\ \overline{X_{j_{k+1}}\delta_{k+1}} \\ \vdots \\ \overline{X_{j_n}\delta_n} \end{pmatrix} \in \ker\theta_2$. Then,

$$\theta_3 \begin{pmatrix} \overline{\delta_1} \\ \vdots \\ \overline{\delta_{i-1}} \\ \overline{\delta_i} \\ \overline{\theta} \\ \overline{\delta_{i+1}} \\ \vdots \\ \overline{\delta_{k-1}} \\ \overline{c^{-1}\theta'} \\ \overline{\delta_k} \\ \overline{\delta_{k+1}} \\ \vdots \\ \overline{\delta_n} \end{pmatrix} = \begin{pmatrix} \overline{X_{j_1}\delta_1} \\ \vdots \\ \overline{X_{j_{i-1}}\delta_{i-1}} \\ \overline{X_{j_i}\delta_i + X_\beta X_{j_i}\theta} \\ \overline{X_{j_{i+1}}\delta_{i+1}} \\ \vdots \\ \overline{X_{j_{k-1}}\delta_{k-1}} \\ \overline{X_\beta\theta'} \\ \overline{X_{j_k}\delta_k + c^{-1}X_{j_i}\theta'} \\ \overline{X_{j_{k+1}}\delta_{k+1}} \\ \vdots \\ \overline{X_{j_n}\delta_n} \end{pmatrix}$$

Hence, $\ker\theta_2 \subseteq \mathrm{im}\theta_3$.

Thus, in all cases, $\mathrm{im}\theta_3 = \ker\theta_2$.

□

Theorem 5.1 *Let $I = \langle m_1, \ldots, m_n, p + cq\rangle$ be an ideal in $K\langle X_1, \ldots, X_t\rangle$ where $p + cq = X_{j_i}X_\beta + cX_\xi X_{j_i}$ for some $1 \leq i \leq n$ and $\beta \neq \xi$, $\beta, \xi \neq j_i$. Then, $\Lambda = K\langle X_1, \cdots, X_t\rangle / I$ is not Koszul.*

Proof

CASE I Suppose $\mathrm{tip}(X_{j_i}X_\beta + cX_\xi X_{j_i}) = X_{j_i}X_\beta$. Then, the field, k, has the following resolution:

$$\cdots \longrightarrow \Lambda^{n+2} \xrightarrow{\varphi_3} \Lambda^{n+1} \xrightarrow{\varphi_2} \Lambda^t \xrightarrow{(\,X_1 \;\cdots\; X_t\,)} \Lambda \longrightarrow k \longrightarrow 0$$

where φ_2, φ_3 are as defined in Proposition 5.2 and Proposition 5.3.

CASE II Suppose $\text{tip}(X_{j_i}X_\beta + cX_\xi X_{j_i}) = X_\xi X_{j_i}$. Then, the field k has the following resolution:

$$\cdots \longrightarrow \Lambda^{n+2} \xrightarrow{\theta_3} \Lambda^{n+1} \xrightarrow{\theta_2} \Lambda^t \xrightarrow{\begin{pmatrix} X_1 & \cdots & X_t \end{pmatrix}} \Lambda \longrightarrow k \longrightarrow 0$$

where θ_2, θ_3 are as defined in Proposition 5.4 and Proposition 5.5.

Thus, $\begin{pmatrix} \overline{0} \\ \vdots \\ \overline{0} \\ \overline{X_\beta X_{j_i}} \\ \overline{0} \\ \vdots \\ \overline{0} \end{pmatrix} \in \ker \varphi_2$ and $\ker \theta_2$. So,

$$\Lambda^{n+1} \xrightarrow{\varphi_2} \Lambda^t \xrightarrow{\begin{pmatrix} X_1 & \cdots & X_t \end{pmatrix}} \Lambda \longrightarrow k \longrightarrow 0$$

and

$$\Lambda^{n+1} \xrightarrow{\theta_2} \Lambda^t \xrightarrow{\begin{pmatrix} X_1 & \cdots & X_t \end{pmatrix}} \Lambda \longrightarrow k \longrightarrow 0$$

are linear resolutions of length 2. If we were to extend to length 3 by any map φ_3' or θ_3' and projective module P_3, then the image of this new map must equal $\ker \varphi_2$ or $\ker \theta_2$ and hence would be nonlinear.

□

Chapter 6

Additional Classes of Koszul Algebras

The duality between a ring and its opposite is well known. This duality allows much to be concluded about a ring's opposite since a ring and its opposite share many properties. In fact, the opposite algebra of a Koszul algebra is also Koszul. [6] By selecting suitably restricted ideals from the results of the previous chapters and looking at the opposite algebras, we easily obtain results for more general quadratic algebras.

Proposition 6.1 $\Lambda = K\langle X_1, \ldots, X_t\rangle \,/\langle X_iX_l + cX_iX_k, \hat{p_{\alpha\beta}}, \hat{m_1}, \ldots \hat{m_n}\rangle$ *for* i, j, *and* k *distinct,* $\hat{m_j} \neq X_i^2, X_k^2, X_l^2$ *a perfect square monomial for all* j, *and* $\hat{p_{\alpha\beta}} = X_\alpha X_\beta \neq X_iX_l, X_iX_k$ *with* $\alpha \neq \beta$ *ranging from* $1, \ldots, t$, *is Koszul.*

Proof We remark that for I described in Theorem 4.1, Λ^{op} has dual basis

$$\{(X_\gamma^2)^*, (X_\alpha X_\beta)^*, (X_{j_i}X_{j_l})^* - c(X_{j_i}X_{j_k})^* | \gamma \neq j_s \forall s, \alpha \neq \beta, \text{and } X_\alpha X_\beta \neq X_{j_i}X_{j_l}, X_{j_i}X_{j_k}\}$$

□

Proposition 6.2 $\Lambda = K\langle X_1, \ldots, X_t\rangle \,/\langle X_lX_i + cX_kX_i, \hat{p_{\alpha\beta}}, \hat{m_1}, \ldots \hat{m_n}\rangle$ *for* i, j, *and* k *distinct,* $\hat{m_j} \neq X_i^2, X_k^2, X_l^2$ *a perfect square monomial for all* j, *and* $\hat{p_{\alpha\beta}} = X_\alpha X_\beta \neq X_iX_l, X_iX_k$ *with* $\alpha \neq \beta$ *ranging from* $1, \ldots, t$, *is Koszul.*

Proof We remark that for I described in Theorem 4.2, Λ^{op} has dual basis

$$\{(X_\gamma^2)^*, (X_\alpha X_\beta)^*, (X_{j_l}X_{j_i})^* - c(X_{j_k}X_{j_i})^* | \gamma \neq j_s \forall s, \alpha \neq \beta, \text{and } X_\alpha X_\beta \neq X_{j_l}X_{j_i}, X_{j_k}X_{j_l}\}$$

□

Proposition 6.3 *$\Lambda = K\langle X_1, \ldots, X_t\rangle \ /\langle X_iX_l + cX_iX_k, \hat{p_{\alpha\beta}}, \hat{m}_1, \ldots \hat{m}_n\rangle$ for i, j, and k distinct, $\hat{m}_j \neq X_i^2, X_k^2$ a perfect square monomial for all j, and $\hat{p_{\alpha\beta}} = X_\alpha X_\beta \neq X_iX_l, X_iX_k$ with α and β ranging from $1, \ldots, t$, is Koszul.*

Proof We remark that for I described in Theorem 3.2, Λ^{op} has dual basis

$$\{(X_\xi^2)^*, (X_\alpha X_\theta)^*, (X_{j_i}X_\gamma)^* - c(X_{j_i}X_\beta)^* | \xi \neq j_s \forall s, \alpha \neq \theta, \text{and } X_\alpha X_\theta \neq X_{j_i}X_\gamma, X_{j_i}X_\beta\}$$

□

Proposition 6.4 *$\Lambda = K\langle X_1, \ldots, X_t\rangle \ /\langle X_lX_i + cX_kX_i, \hat{p_{\alpha\beta}}, \hat{m}_1, \ldots \hat{m}_n\rangle$ for i, j, and k distinct, $\hat{m}_j \neq X_i^2, X_k^2$ a perfect square monomial for all j, and $\hat{p_{\alpha\beta}} = X_\alpha X_\beta \neq X_iX_l, X_iX_k$ with α and β ranging from $1, \ldots, t$, is Koszul.*

Proof We remark that for I described in Theorem 3.1, Λ^{op} has dual basis

$$\{(X_\xi^2)^*, (X_\alpha X_\theta)^*, (X_\gamma X_{j_i})^* - c(X_\beta X_{j_i})^* | \xi \neq j_s \forall s, \alpha \neq \theta, \text{and } X_\alpha X_\theta \neq X_\gamma X_{j_i}, X_\beta X_{j_i}\}$$

□

Proposition 6.5 *$\Lambda = K\langle X_1, \ldots, X_t\rangle \ /\langle X_iX_\xi + cX_\xi X_i, \hat{p_{\alpha\beta}}, \hat{m}_1, \ldots \hat{m}_n\rangle$ for $i \neq \xi$, $\hat{m}_j \neq X_i^2$ a perfect square monomial for all j, and $\hat{p_{\alpha\beta}} = X_\alpha X_\beta \neq X_iX_\xi, X_\xi X_i$ with α and β ranging from $1, \ldots, t$, is Koszul.*

Proof We remark that for I described in Lemma 3.2, Λ^{op} has dual basis

$$\{(X_\xi^2)^*, (X_\alpha X_\theta)^*, (X_\beta X_{j_i})^* - c(X_{j_i}X_\beta)^* | \xi \neq j_s \forall s, \alpha \neq \theta, \text{and } X_\alpha X_\theta \neq X_{j_i}X_\beta, X_\beta X_{j_i}\}$$

□

Here we also note that we have a similar situation as above with nonKoszul algebras. That is, the duals of nonKoszul algebras produce nonKoszul algebras. Thus, we obtain the following class of algebras which are not Koszul:

Proposition 6.6 *$\Lambda = K\langle X_1, \ldots, X_t\rangle \ /\langle X_iX_l + cX_kX_i, \hat{p_{\alpha\beta}}, \hat{m}_1, \ldots \hat{m}_n\rangle$ for i, j, and k distinct, $\hat{m}_j \neq X_i^2$ a perfect square monomial for all j, and $\hat{p_{\alpha\beta}} = X_\alpha X_\beta \neq X_iX_l, X_iX_k$ with $\alpha \neq beta$ ranging from $1, \ldots, t$, is not Koszul.*

Proof We remark that for I described in Theorem 5.1, Λ^{op} has dual basis

$$\{(X_\gamma^2)^*, (X_\alpha X_\beta)^*, (X_\xi X_{j_i})^* - c(X_{j_i}X_\beta)^* | \gamma \neq j_s \forall s, \alpha \neq \beta, \text{and } X_\alpha X_\beta \neq X_\xi X_{j_i}, X_{j_i}X_{j_k}\}$$

□

Chapter 7

Matrix Diagrams

To elucidate the maps of the resolutions developed in Chapters 3, 4, and 5, we now provide diagrams of the associated matrices. The diagram numbers correspond to the theorem numbers from the previous chapters. That is, for example, Diagram 3.1 clarifies the matrices associated with the maps of the linear resolution of the field as defined in Theorem 3.1.

Diagram 3.1

$$\varphi = \begin{pmatrix} X_{j_1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & X_{j_2} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_{j_l} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & X_{j_i} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & X_{j_{l+1}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & X_{j_n} \end{pmatrix}$$

Diagram 3.1, cont.

$$\varphi_2 = \begin{array}{c} 1 \\ \vdots \\ j_1 \\ \vdots \\ j_2 \\ \vdots \\ j_k \\ \vdots \\ \beta \\ \vdots \\ j_{k+1} \\ \vdots \\ j_l \\ \vdots \\ \gamma \\ \vdots \\ j_{l+1} \\ \vdots \\ j_n \\ \vdots \\ t \end{array} \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ X_{j_1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & X_{j_2} & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_{j_k} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & cX_{j_i} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & X_{j_{k+1}} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & X_{j_l} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & X_{j_i} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & X_{j_{l+1}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & X_{j_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Diagram 3.2

$$\varphi_2 = \begin{matrix} 1 \\ \vdots \\ j_1 \\ \vdots \\ j_2 \\ \vdots \\ j_i \\ \vdots \\ j_{i+1} \\ \vdots \\ j_n \\ \vdots \\ t \end{matrix} \quad \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ X_{j_1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & X_{j_2} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_{j_i} & X_\gamma + cX_\beta & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & X_{j_{i+1}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & X_{j_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$\varphi = \begin{pmatrix} X_{j_1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & X_{j_2} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \cdots \\ 0 & 0 & \cdots & X_{j_i} & X_\gamma + cX_\beta & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & X_{j_{i+1}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & X_{j_n} \end{pmatrix}$$

Diagram 3.3

$$\varphi_2 = \begin{array}{c} 1 \\ \vdots \\ j_1 \\ \vdots \\ j_2 \\ \vdots \\ \alpha \\ \vdots \\ j_k \\ \vdots \\ \beta \\ \vdots \\ j_{k+1} \\ \vdots \\ j_n \\ \vdots \\ t \end{array} \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ X_{j_1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & X_{j_2} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & X_\theta & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_{j_k} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & X_\gamma & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & X_{j_{k+1}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & X_{j_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$\varphi_3 = \begin{pmatrix} X_{j_1} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \cdots \\ 0 & \cdots & X_{j_k} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & X_{j_{k+1}} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \cdots \\ 0 & \cdots & 0 & 0 & \cdots & X_{j_n} \end{pmatrix}$$

$$\varphi = \begin{pmatrix} X_{j_1} & 0 & \cdots & 0 \\ 0 & X_{j_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots \\ 0 & 0 & \cdots & X_{j_n} \end{pmatrix}$$

Diagram 4.1

$$\varphi_2 = \begin{array}{c} 1 \\ \vdots \\ j_1 \\ \vdots \\ j_2 \\ \vdots \\ j_i \\ \vdots \\ j_{i+1} \\ \vdots \\ j_n \\ \vdots \\ t \end{array} \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ X_{j_1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & X_{j_2} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_{j_i} & X_{j_k} + cX_{j_l} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & X_{j_{i+1}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & X_{j_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$\varphi = \begin{array}{c} 1 \\ 2 \\ \vdots \\ i \\ i+1 \\ i+2 \\ \vdots \\ n+1 \end{array} \begin{pmatrix} X_{j_1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & X_{j_2} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_{j_i} & X_{j_k} + cX_{j_l} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & X_{j_{i+1}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & X_{j_n} \end{pmatrix}$$

Diagram 4.2

$$\varphi_2 = \begin{array}{c} 1 \\ \vdots \\ j_1 \\ \vdots \\ j_2 \\ \vdots \\ j_l \\ \vdots \\ j_i \\ \vdots \\ j_k \\ \vdots \\ j_n \\ \vdots \\ t \end{array} \begin{pmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ X_{j_1} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots\vdots & \ddots & \vdots & \vdots & \\ 0 & X_{j_2} & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & X_{j_l} & \cdots & 0 & \cdots & 0 & \cdots & 0 & cX_{j_i} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & X_{j_i} & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & X_{j_k} & \cdots & 0 & X_{j_i} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & X_{j_n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 \end{pmatrix}$$

$$\varphi = \begin{pmatrix} X_{j_1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & X_{j_2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & X_{j_3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & X_{j_n} & 0 \\ 0 & 0 & 0 & \cdots & 0 & X_{j_i} \end{pmatrix}$$

Diagram 4.3

$$\varphi_2 = \begin{array}{c} 1 \\ \vdots \\ k^1_{s_1} \\ \vdots \\ j_1 \\ \vdots \\ k^1_2 \\ \vdots \\ j_2 \\ \vdots \\ k^z_{s_z} \\ \vdots \\ j_{c_1} \\ \vdots \\ k^1_1 \\ \vdots \\ j_{c_1+1} \\ \vdots \\ j_{c_z} \\ \vdots \\ k^1_z \\ \vdots \\ j_{c_z+1} \\ \vdots \\ j_n \\ \vdots \\ t \end{array} \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & c^1_{s_1}X_{k^1_{s_1}} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ X_{j_1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & c^1_2X_{k^1_2} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & X_{j_2} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & c^z_{s_z}X_{k^z_{s_z}} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_{j_{c_1}} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & X_{k^1_1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & X_{j_{c_1+1}} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & X_{j_{c_z}} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & X_{k^1_z} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & X_{j_{c_z+1}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots 0 & 0 & 0 & \cdots & X_{j_n} & \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Diagram 4.3, cont.

$$\varphi_3 = \begin{array}{l} 1 \\ 2 \\ \vdots \\ c_1 \\ c_1+1 \\ c_1+2 \\ \vdots \\ c_z+z-1 \\ c_z+z \\ c_z+1 \\ \vdots \\ n+z \end{array} \begin{pmatrix} X_{j_1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & X_{j_2} & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_{j_{c_1}} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & X_{j_{c_1+1}} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & X_{j_{c_z}} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & X_{j_{c_z+1}} \cdots & 0 & \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & X_{j_n} \end{pmatrix}$$

$$\varphi = \begin{pmatrix} X_{j_1} & 0 & 0 & \cdots & 0 \\ 0 & X_{j_2} & 0 & \cdots & 0 \\ 0 & 0 & X_{j_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & X_{j_n} \end{pmatrix}$$

Diagram 5.1

CASE I: Suppose $\text{tip}(X_{j_i}X_\beta + cX_\xi X_{j_i}) = X_{j_i}X_\beta$.

Case 1 Suppose$X_\xi = X_{j_k}$ for some k, $X_\beta \neq X_{j_l}$ for all l.

$$\varphi_2 = \begin{array}{l} 1 \\ \vdots \\ j_1 \\ \vdots \\ j_k = \xi \\ \vdots \\ j_{k+1} \\ \vdots \\ j_i \\ \vdots \\ j_{i+1} \\ \vdots \\ j_n \\ \vdots \\ t \end{array} \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ X_{j_1} & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & X_{j_k} & 0 & \cdots & 0 & cX_{j_i} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & X_{j_{k+1}} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & X_{j_i} & X_\beta & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & X_{j_{i+1}} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & X_{j_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$\varphi_3 = \begin{array}{l} 1 \\ 2 \\ \vdots \\ i \\ i+1 \\ i+2 \\ \vdots \\ n+1 \end{array} \begin{pmatrix} X_{j_1} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & X_{j_2} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_{j_i} & X_\beta X_{j_i} & X_{j_i}^2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & X_{j_{i+1}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & X_{j_n} \end{pmatrix}$$

Diagram 5.1, cont.

<u>Case 2</u> Suppose$X_\beta = X_{j_l}$ for some l, $X_\xi \neq X_{j_k}$ for all k.

$$\varphi_2 = \begin{array}{l} 1 \\ \vdots \\ j_1 \\ \vdots \\ j_k \\ \vdots \\ \xi \\ \vdots \\ j_{k+1} \\ \vdots \\ j_i \\ \vdots \\ j_{i+1} \\ \vdots \\ j_n \\ \vdots \\ t \end{array} \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ X_{j_1} & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & X_{j_k} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & cX_{j_i} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & X_{j_{k+1}} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & X_{j_i} & X_\beta & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & X_{j_{i+1}} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & X_{j_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$\varphi_3 =$$

$$\begin{array}{l} 1 \\ 2 \\ \vdots \\ k \\ k+1 \\ \vdots \\ i-1 \\ i \\ i+1 \\ i+2 \\ \vdots \\ n+1 \end{array} \begin{pmatrix} X_{j_1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & X_{j_2} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_{j_k} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & X_{j_{k+1}} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & X_{j_{i-1}} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & X_{j_i} & X_\beta X_{j_i} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & X_\beta X_{j_i} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & X_{j_{i+1}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & X_{j_n} \end{pmatrix}$$

Diagram 5.1, cont.

<u>Case 3</u> Suppose$X_\beta = X_{j_l}$ for some l, $X_\xi = X_{j_k}$ for some k.

φ_2 is the same as in Case 1.

$$\varphi_3 = \begin{matrix} 1 \\ 2 \\ \vdots \\ k \\ k+1 \\ \vdots \\ i \\ i+1 \\ i+2 \\ \vdots \\ n+1 \end{matrix} \begin{pmatrix} X_{j_1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & X_{j_2} & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_{j_k} & 0 & \cdots & 0 & c^2 X_{j_i} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & X_{j_{k+1}} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & X_{j_i} & 0 & X_\beta X_{j_i} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & X_\beta & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & X_{j_{i+1}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & X_{j_n} \end{pmatrix}$$

Diagram 5.1, cont.

<u>CASE II:</u> Suppose $\text{tip}(X_{j_i}X_\beta + cX_\xi X_{j_i}) = X_\xi X_{j_i}$.

<u>Case 1</u> Suppose$X_\xi = X_{j_k}$ for some k, $X_\beta \neq X_{j_l}$ for all l.

$$\theta_2 = \begin{array}{l} 1 \\ \vdots \\ j_1 \\ \vdots \\ j_{i-1} \\ \vdots \\ j_i \\ \vdots \\ j_{i+1} \\ \vdots \\ j_{k-1} \\ \vdots \\ j_k = \xi \\ \vdots \\ j_{k+1} \\ \vdots \\ j_n \\ \vdots \\ t \end{array} \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ X_{j_1} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & X_{j_{i-1}} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & X_{j_i} & 0 & \cdots & 0 & cX_\beta & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & X_{j_{i+1}} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & X_{j_{k-1}} & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & X_{j_i} & X_{j_k} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & X_{j_{k+1}} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & X_{j_n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$\theta_3 = \begin{array}{l} 1 \\ \vdots \\ i-1 \\ i \\ i+1 \\ \vdots \\ k-1 \\ k \\ k+1 \\ \vdots \\ n+1 \end{array} \begin{pmatrix} X_{j_1} & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & X_{j_{i-1}} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & X_{j_i} & X_\beta X_{j_i} & X_\beta^2 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & X_{j_{i+1}} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & X_{j_{k-1}} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & X_{j_k} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & X_{j_n} \end{pmatrix}$$

Diagram 5.1, cont.

Case 2 Suppose$X_\beta = X_{j_l}$ for some l, $X_\xi \neq X_{j_k}$ for all k.

$$\theta_2 = \begin{array}{l} 1 \\ \vdots \\ j_1 \\ \vdots \\ j_{i-1} \\ \vdots \\ j_i \\ \vdots \\ j_{i+1} \\ \vdots \\ j_k \\ \vdots \\ \xi \\ \vdots \\ j_{k+1} \\ \vdots \\ j_n \\ \vdots \\ t \end{array} \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ X_{j_1} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & X_{j_{i-1}} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & X_{j_i} & 0 & \cdots & 0 & cX_\beta & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & X_{j_{i+1}} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & X_{j_k} & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & X_{j_i} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & X_{j_{k+1}} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & X_{j_n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$\theta_3 = \begin{array}{l} 1 \\ \vdots \\ i-1 \\ i \\ i+1 \\ \vdots \\ k \\ k+1 \\ k+2 \\ \vdots \\ n+1 \end{array} \begin{pmatrix} X_{j_1} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & X_{j_{i-1}} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & X_{j_i} & X_\beta X_{j_i} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & X_{j_{i+1}} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & X_{j_k} & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & X_\beta X_{j_i} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & X_{j_{k+1}} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & X_{j_n} \end{pmatrix}$$

Diagram 5.1, cont.

<u>Case 3</u> Suppose$X_\beta = X_{j_l}$ for some l, $X_\xi = X_{j_k}$ for some k.

θ_2 is the same as in Case 1.

$$\theta_3 = \begin{array}{l} 1 \\ \vdots \\ i-1 \\ i \\ i+1 \\ \vdots \\ k-1 \\ k \\ k+1 \\ k+2 \\ \vdots \\ n+1 \end{array} \begin{pmatrix} X_{j_1} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & X_{j_{i-1}} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & X_{j_i} & X_\beta X_{j_i} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & X_{j_{i+1}} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & X_{j_{k-1}} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & cX_\beta & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & X_{j_i} & X_{j_k} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & X_{j_{k+1}} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & X_{j_n} \end{pmatrix}$$

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