AIP | Physics of Fluids

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Citation: Physics of Fluids (1958-1988) **18**, 945 (1975); doi: 10.1063/1.861266
View online: http://dx.doi.org/10.1063/1.861266
View Table of Contents: http://scitation.aip.org/content/aip/journal/pof1/18/8?ver=pdfcov
Published by the AIP Publishing

Nonparallel stability of boundary-layer flows

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(Received 10 February 1975)

The spatial stability of two-dimensional incompressible boundary-layer flows is analyzed using the method of multiple scales. The analysis takes into account the streamwise variations of the mean flow, the disturbance amplitude, and the wavenumber. The theory is applied to the Blasius and the Falkner–Skan flows. For the Blasius flow, the nonparallel analytical results are in good agreement with the experimental data. The results show that the nonparallel effects increase as the pressure gradient decreases.

I. INTRODUCTION

In this article, we consider the linear stability of nonparallel boundary-layer flows. Recently, considerable attention has been given to linear stability theories of shear flows that account for nonparallel flow effects because classical linear stability theories which treat the primary flows as quasi-parallel flows have not produced satisfactory results. For a flow past a flat plate (Blasius flow), the critical Reynolds number predicted by parallel stability theories is about 30% above the experimental observations of Schubauer and Skramstad[1] and Ross *et al.*[2] A summary of parallel stability calculations is given by Jordinson.[3]

Progress in the development of stability theories for nonparallel flows has been impeded because the governing equations are nonseparable. Some attempts to account for nonparallel flow effects involved the retention of the normal component of velocity and some of the streamwise derivatives of the primary flow. In this model, the governing equations are separable and they reduce to a modified Orr–Sommerfeld equation (see e.g., Barry and Ross[4] and Boehman[5]). The results of Barry and Ross overpredict by 25% the critical Reynolds number for the Blasius flow. Wazzan *et al.*[6] used the model of Barry and Ross[4] to calculate the stability of the Falkner–Skan flows.

Lanchon and Eckhaus[7] considered the stability of nonparallel flows by using a straightforward expansion about the parallel flow solution. They did not present quantitative results because they found that their formal solutions for boundary-layer flows reduces, for large Reynolds numbers, to the parallel flow problem. However, this conclusion is a consequence of the use of a straightforward expansion.

Volodin[8] presented an expansion procedure which accounts for all nonparallel effects. However, he assumed the lowest-order problem to be separable. Consequently, he overpredicted the critical Reynolds number for the Blasius flow by 18%. Ling and Reynolds[9] also accounted for all nonparallel effects. However, they considered the temporal rather than the spatial stability problem (note that the experiments were conducted by introducing disturbances of fixed frequencies and observing their spatial behavior) and they determined an expansion that is valid for short distances by expanding the mean flow, the eigenfunctions, and the eigenvalues in power series about a given streamwise location. Consequently, they obtained only a 0.1% reduction in the critical Reynolds number calculated from parallel flow stability for the Blasius flow.

Bouthier[10,11], Nayfeh *et al.*[12] and Gaster[13] determined expansions that account for all nonparallel flow effects. Whereas Nayfeh *et al.*[12] carried out their expansion in rectangular coordinates, Bouthier and Gaster carried out their expansions in similarity variables. These theories were applied to the Blasius flow. Levchenko and Soloveyv[14] extended the analysis of Ref. 12 to the case of boundary-layer flows over wavy walls. In this article, we follow Nayfeh *et al.* and present a consistent nonparallel stability theory for boundary-layer flows and apply it to the Falkner-Skan flows.

II. PROBLEM FORMULATION AND METHOD OF SOLUTION

We consider the stability of a two-dimensional, steady, incompressible flow over a flat surface described by the stream function $\Psi(x, y)$. The x coordinate is in the plane of the flat surface and parallel to the free stream velocity and the y coordinate is normal to the flat surface. We introduce dimensionless quantities using a suitable reference length δ_r and a suitable reference velocity U_r so that the Reynolds number is given by $R = U_r \delta_r / \nu$, where ν is the kinematic viscosity which is assumed to be constant.

To examine the stability of flows with $\Psi(x, y)$ presumed known, we superpose small disturbances and examine their growth or decay. Thus, the stream function $\tilde{\psi}(x, y, t)$ of the disturbed flow is taken to have the form

$$\tilde{\psi}(x, y, t) = \Psi(x, y) + \psi(x, y, t) , \tag{1}$$

where ψ is small compared with Ψ. Substituting Eq. (1) into the Navier–Stokes equations, subtracting the mean flow quantities, and keeping linear terms in ψ, we obtain

$$\frac{\partial}{\partial t}(\nabla^2\psi) + \frac{\partial \Psi}{\partial y}\frac{\partial}{\partial x}(\nabla^2\psi) + \frac{\partial}{\partial x}(\nabla^2\Psi)\frac{\partial \psi}{\partial y} - \frac{\partial \Psi}{\partial x}\frac{\partial}{\partial y}(\nabla^2\psi) - \frac{\partial}{\partial y}(\nabla^2\Psi)\frac{\partial\psi}{\partial x} = \frac{1}{R}\nabla^4\psi . \tag{2}$$

To complete the problem formulation, Eq. (2) needs to be supplemented by the appropriate boundary conditions. For boundary-layer flows, the boundary conditions are

$$\psi = \partial\psi/\partial y = 0 \quad \text{at} \quad y = 0 , \tag{3}$$

$$\psi \to 0 \quad \text{as} \quad y \to \infty . \tag{4}$$

In what follows, we restrict our analysis to primary flows which are slightly nonparallel so that $\Psi(x, y)$ is a slowly varying function of x. To express this slow variation, we introduce the scale

$$x_1 = \epsilon x , \tag{5}$$

where ϵ is a small dimensionless parameter characterizing the nonparallelism of the primary flow; $\epsilon = 0$ for truly parallel flows (note that ϵ and R may be related). Using x_1, we express the stream function of the primary flow as

$$\Psi = \Psi(x_1, y) . \tag{6}$$

Thus, the velocity components of the primary flow can be written as

$$\frac{\partial\Psi}{\partial y} = U = U_0(x_1, y) + \epsilon U_1(x_1, y) + \dots , \tag{7a}$$

$$-\frac{\partial\Psi}{\partial x} = -\epsilon \frac{\partial\Psi}{\partial x_1} = \epsilon V(x_1, y) + \dots , \tag{7b}$$

where ϵU_1 is due to higher-order boundary-layer effects such as free stream vorticity. Note that for the flows considered here the effects of displacement thickness are $O(\epsilon^2)$.

To determine an approximate solution to Eqs. (2)–(4), we follow Nayfeh *et al.*[12] by using the method of multiple scales[15] and let

$$\psi = \phi(x_1, y; \epsilon) \exp(i\theta) , \tag{8}$$

where

$$\frac{\partial\theta}{\partial x} = k_0(x_1) , \quad \frac{\partial\theta}{\partial t} = -\omega , \tag{9}$$

with ω being real. Here, ω is the dimensionless frequency of the disturbance, the real part of k_0 is the wavenumber, and the imaginary part of k_0 is the spatial growth rate α_0. For the case of parallel mean flows, ϕ and k_0 are independent of x_1. In terms of x_1 and θ, the temporal and spatial derivatives transform according to

$$\frac{\partial}{\partial t} = -\omega \frac{\partial}{\partial\theta} , \tag{10a}$$

$$\frac{\partial}{\partial x} = k_0 \frac{\partial}{\partial\theta} + \epsilon \frac{\partial}{\partial x_1} . \tag{10b}$$

Since ϵ is assumed to be small, we expand ϕ as

$$\phi(x_1, y) = \phi_0(x_1, y) + \epsilon\phi_1(x_1, y) + \dots . \tag{11}$$

In contrast with the present approach, Ling and Reynolds[9] considered the temporal stability case. Moreover, they expanded the mean flow, the amplification rate, and the disturbance amplitude in power series around a given streamwise location. As a result, their expansion is valid only for short distances and they obtained only a 0.1% reduction in the critical Reynolds number calculated from parallel-flow stability for the Blasius flow.

Substituting Eqs. (8)–(11) into Eqs. (2)–(4), using Eq. (7), and equating coefficients of like powers of ϵ, we obtain

Order ϵ^0:

$$\mathcal{L}(\phi_0) \equiv \left(\frac{\partial^2}{\partial y^2} - k_0^2\right)^2 \phi_0 - ik_0 R\left[\left(U_0 - \frac{\omega}{k_0}\right)\left(\frac{\partial^2\phi_0}{\partial y^2} - k_0^2\phi_0\right) - \frac{\partial^2 U_0}{\partial y^2}\phi_0\right] = 0 , \tag{12}$$

$$\phi_0 = \frac{\partial\phi_0}{\partial y} = 0 \quad \text{at} \quad y = 0 , \tag{13}$$

$$\phi_0 \to 0 \quad \text{as} \quad y \to \infty , \tag{14}$$

Order ϵ:

$$\begin{aligned}\mathcal{L}(\phi_1) = {} & \left[R\left(2k_0\omega - 3U_0k_0^2 - \frac{\partial^2 U_0}{\partial y^2}\right) + 4ik_0^3\right]\frac{\partial\phi_0}{\partial x_1} + (RU_0 - 4ik_0) \\ & \times \frac{\partial^3\phi_0}{\partial y^2\partial x_1} + RV\left(\frac{\partial^3\phi_0}{\partial y^3} - k_0^2\frac{\partial\phi_0}{\partial y}\right) - R\frac{\partial^2 V}{\partial y^2}\frac{\partial\phi_0}{\partial y} \\ & + \left[-2i\frac{\partial^2\phi_0}{\partial y^2} + (R\omega - 3Rk_0U_0 + 6ik_0^2)\phi_0\right]\frac{dk_0}{dx_1} \\ & + ik_0 R\left[U_1\left(\frac{\partial^2\phi_0}{\partial y^2} - k_0^2\phi_0\right) - \frac{\partial^2 U_1}{\partial y^2}\phi_0\right] ,\end{aligned} \tag{15}$$

$$\phi_1 = \partial\phi_1/\partial y = 0 \quad \text{at} \quad y = 0 , \tag{16}$$

$$\phi_1 \to 0 \quad \text{as} \quad y \to \infty . \tag{17}$$

The first two terms on the right-hand side of Eq. (15) represent the effects of the streamwise variation of the disturbance amplitude; the third and fourth terms represent the effects of the transverse mean-velocity component; the fifth term represents the effects of the streamwise variation of the wavenumber, and the last term represents the effects of higher-order boundary-layer theory. Barry and Ross,[4] Boehman,[5] and Wazzan *et al.*[6] included the transverse mean-velocity component and the streamwise variation of U_0 but neglected the effects of the first, second, fifth, and sixth terms on the right-hand side of Eq. (15). Thus, they solved locally Eq. (12)–(14) with Eq. (12) modified by including the third and fourth terms on the right-hand side of Eq. (15). To account for blowing or suction, Chen and coworkers[16] neglected the streamwise variations of the mean flow, disturbance amplitude, and wavenumber, but included the effects of the transverse mean velocity component by solving a modified Orr–Sommerfeld problem that is essentially modified to include the third and fourth terms on the right-hand side of Eq. (15).

III. SOLUTION

The scale x_1 appears implicitly in Eq. (12). If U_0 is independent of x_1 (i.e., parallel mean flow), Eqs. (12)–(14) reduce to the familiar Orr–Sommerfeld problem which is used extensively in the literature to analyze the linear stability of parallel boundary-layer flows. For a given ω and $U_0(x_1, y)$, the eigenvalue problem, Eqs. (12)–(14), can be solved numerically to determine the eigenvalue $k_0(x_1)$ and the eigenfunction $\phi_0(x_1, y)$. The solution can be expressed as

$$\phi_0 = A(x_1)\,\zeta(y;\, x_1)\;, \tag{18}$$

where $A(x_1)$ is still an undetermined function at this level of approximation. It will be determined at the next level of approximation. In the analysis of Barry and Ross,[4] Boehman,[5] and Wazzan *et al.*,[6] A is a constant. On the other hand, Volodin[8] considered A to be a function of x, but he assumed the variables to be separable in Eq. (12) so that ζ and k_0 are independent of x. As a result, he overpredicted by 18% the critical Reynolds for the Blasius flow.

Substituting for ϕ_0 from Eq. (18) into Eq. (15), we have

$$\mathfrak{L}(\phi_1) = R\,\mathfrak{M}(A,\, \zeta,\, U_0,\, U_1,\, V,\, k_0)\;, \tag{19}$$

$$\mathfrak{M} = \left[B_1\zeta + B_2\,\frac{\partial^2\zeta}{\partial y^2}\right]\frac{dA}{dx_1} + \left[B_1\,\frac{\partial\zeta}{\partial x_1} + B_2\,\frac{\partial^3\zeta}{\partial y^2\partial x_1} + \left(B_3\,\zeta + B_4\,\frac{\partial^2\zeta}{\partial y^2}\right)\frac{dk_0}{dx_1} + B_5\zeta + B_6\,\frac{\partial\zeta}{\partial y} + B_7\,\frac{\partial^2\zeta}{\partial y^2} + B_8\,\frac{\partial^3\zeta}{\partial y^3}\right]A\;, \tag{20}$$

where the B's are known function of x_1 and y and are defined in the appendix. The inhomogeneous problem consisting of Eq. (19) and the boundary conditions (16) and (17) has a solution if, and only if, the inhomogeneous terms are orthogonal to every solution of the adjoint homogeneous problem; that is

$$\int_0^\infty \mathfrak{M}\zeta^*\,dy = 0\;, \tag{21}$$

where $\zeta^*(y;\, x_1)$ is the eigenfunction corresponding to the eigenvalue k_0 of the adjoint homogeneous problem

$$\mathfrak{L}^*(\zeta^*) \equiv \left(\frac{\partial^2}{\partial y^2} - k_0^2\right)\zeta^* - ik_0 R\left[\left(U_0 - \frac{\omega}{k_0}\right)\left(\frac{\partial^2\zeta^*}{\partial y^2} - k_0^2\,\zeta^*\right) + 2\,\frac{\partial U_0}{\partial y}\,\frac{\partial\zeta^*}{\partial y}\right] = 0\;, \tag{22}$$

$$\zeta^* = \partial\zeta^*/\partial y = 0 \quad \text{at} \quad y = 0\;, \tag{23}$$

$$\zeta^* \to 0 \quad \text{as} \quad y \to \infty\;. \tag{24}$$

Substituting for $\mathfrak{M}$ from Eq. (20) into Eq. (21), we obtain the following equation for the evolution of A,

$$\frac{dA}{dx_1} = ik_1(x_1)A\;, \tag{25}$$

where

$$ik_1 = g_2(x_1)/g_1(x_1)\;, \tag{26a}$$

$$g_1(x_1) = -\int_0^\infty\left(B_1\zeta + B_2\,\frac{\partial^2\zeta}{\partial y^2}\right)\zeta^*\,dy\;, \tag{26b}$$

$$g_2(x_1) = \int_0^\infty\left[\left(B_1 + \frac{\partial^2 B_2}{\partial y^2}\right)\zeta^* + 2\,\frac{\partial B_2}{\partial y}\,\frac{\partial\zeta^*}{\partial y} + B_2\,\frac{\partial^2\zeta^*}{\partial y^2}\right]\frac{\partial\zeta}{\partial x_1}\,dy + \int_0^\infty\left[\left(B_3\zeta + B_4\frac{\partial^2\zeta}{\partial y^2}\right)\frac{dk_0}{dx_1} + B_5\zeta + B_6\,\frac{\partial\zeta}{\partial y} + B_7\,\frac{\partial^2\zeta}{\partial y^2} + B_8\,\frac{\partial^3\zeta}{\partial y^3}\right]\zeta^*\,dy\;. \tag{26c}$$

The first term in Eq. (26c) was obtained by integration by parts. The solution of Eq. (25) is

$$A = A_0\exp\left[i\int k_1(x_1)\,dx_1\right] = A_0\exp\left[i\epsilon\int k_1(x_1)\,dx\right]\;, \tag{27}$$

where A_0 is a constant of integration. Hence, to a first approximation,

$$\psi = A_0\zeta(y;\, x_1)\exp\left[i\int(k_0 + \epsilon k_1)\,dx - i\omega t\right]\;, \tag{28}$$

where ζ and k_0 are calculated at each axial location as if the mean flow were parallel, and k_1 contains the effects of the streamwise derivatives of the mean flow, the eigenfunction ζ, and the eigenvalue k_0.

For a parallel mean flow, ζ is a function of y only, $k_1 = 0$, and k_0 is a constant. Hence, the streamwise behavior of any disturbance quantity, such as the velocity, the pressure, and the kinetic energy, is governed by the exponent k_0. In particular, the amplification and attenuation rates of any disturbance are given uniquely by α_0, the imaginary part of k_0. On the other hand, the effects of nonparallelism are to make k_0 be a function of x_1, to produce a correction $\epsilon k_1(x_1)$ to k_0, and to make the mode shape ζ vary in the streamwise direction. Hence, the streamwise variation of each flow quantity depends on its distance from the wall. Moreover, at each distance from the wall, the different flow quantities vary differently in the streamwise direction. Thus, unless these factors are taken into account in conducting the stability experiments, meaningful comparisons cannot be made between theoretical and experimental results. Therefore, extreme care must be made in comparing nonparallel stability theories such as the one presented here and available experimental data.

IV. COMPUTATIONAL PROCEDURE

To determine $g_2(x_1)$, we need to evaluate $\partial\zeta/\partial x_1$ and dk_0/dx_1. To do this, we replace ϕ_0 by ζ in Eqs. (12)–(14), differentiate the result with respect to x_1, and obtain

$$\mathfrak{L}\left(\frac{\partial\zeta}{\partial x_1}\right) = h_1 + h_2\,\frac{dk_0}{dx_1}\;, \tag{29}$$

$$\frac{\partial\zeta}{\partial x_1} = \frac{\partial}{\partial y}\left(\frac{\partial\zeta}{\partial x_1}\right) = 0 \quad \text{at} \quad y = 0\;, \tag{30}$$

$$\frac{\partial\zeta}{\partial x_1} \to 0 \quad \text{as} \quad y \to \infty\;, \tag{31}$$

where

$$h_1 = iR\left[k_0^3\zeta\,\frac{\partial V}{\partial y} + k_0\zeta\,\frac{\partial^3 V}{\partial y^3} - k_0\,\frac{\partial V}{\partial y}\,\frac{\partial^2\zeta}{\partial y^2}\right]\;, \tag{32a}$$

$$h_2 = 4k_0\left(\frac{\partial^2\zeta}{\partial y^2} - k_0^2\zeta\right) + iR\left(U_0\,\frac{\partial^2\zeta}{\partial y^2} - 3k_0^2\,U_0\,\zeta + 2k_0\omega\zeta - \zeta\,\frac{\partial^2 U_0}{\partial y}\right). \tag{32b}$$

Equations (29)–(31) have a solution if, and only if, the inhomogeneous terms are orthogonal to every solution of the adjoint homogeneous problem; that is, to ζ^*. This solvability condition yields the following equation for dk_0/dx_1:

$$dk_0/dx_1 = -\left[\int_0^\infty h_1\,\zeta^*\,dy\right]\bigg/\left[\int_0^\infty h_2\zeta^*\,dy\right]\;. \tag{33}$$

Condition (33) permits the solution of Eqs. (29)–(32) to determine $\partial\zeta/\partial x_1$.

The problems describing ζ and ζ^* are

$$\mathfrak{L}(\zeta)=0\ , \tag{34a}$$

$$\zeta=\partial\zeta/\partial y=0 \quad \text{at} \quad y=0\ , \tag{34b}$$

$$\zeta\to 0 \quad \text{as} \quad y\to\infty\ , \tag{34c}$$

and

$$\mathfrak{L}^*(\zeta^*)=0\ , \tag{35a}$$

$$\zeta^*=\partial\zeta^*/\partial y=0 \quad \text{at} \quad y=0\ , \tag{35b}$$

$$\zeta^*\to 0 \quad \text{as} \quad y\to\infty\ . \tag{35c}$$

The solution of the eigenvalue problem described by Eqs. (34) is carried out by first choosing a $y \geq y_m$ such that $U_0=1$ and $\partial^2 U_0/\partial y^2=0$. Then, for $y\geq y_m$, Eq. (34a) reduces to an equation with constant coefficients whose linearly independent solutions which satisfy Eq. (34c) are

$$\zeta_1=\exp(-k_0 y)\ , \tag{36a}$$

$$\zeta_2=\exp(-\tilde{k}_0 y), \quad \tilde{k}_0=[k_0^2+iR(k_0-\omega)]^{1/2}\ . \tag{36b}$$

These asymptotic solutions are used as starting conditions at $y=y_m$ and the integration scheme of Reynolds[17] is used to find the eigenvalues and eigenfunctions. Given ω and a guess for k_0, a fourth-order algorithm is used to integrate Eq. (34a) from $y=y_m$ to $y=0$ and the filtering technique of Kaplan[18] is used to keep ζ_1 and ζ_2 linearly independent. The linear combination

$$\zeta=\zeta_1-c(x_1)\,\zeta_2\ , \quad c(x_1)=\left[\frac{\partial\zeta_1}{\partial y}(0)\right]\Big/\left[\frac{\partial\zeta_2}{\partial y}(0)\right] \tag{37}$$

is formed to satisfy the condition $\partial\zeta/\partial y(0)=0$. If an eigenvalue k_0 has been used, the condition $\zeta(0)=0$ is satisfied; otherwise, k_0 is incremented by using a Newton–Raphson procedure and the procedure is repeated until $\zeta(0)=0$.

Equation (37) gives the desired eigenfunction to within a multiplicative function of x_1. In this article, we normalize the eigenfunctions by taking this function to be unity so that

$$\zeta\to\exp(-k_0 y)-c\exp(-\tilde{k}_0 y) \text{ for } y\geq y_m\ . \tag{38}$$

With the above calculated eigenvalue, a procedure similar to that above is used to solve Eqs. (35) to determine ζ^*. The only difference is that no iteration is needed because the adjoint problem has the same eigenvalues as the original problem. Hence, the calculation of ζ^* can be used as a check on the accuracy of the calculated eigenvalues.

The calculations of ζ and ζ^* were checked by comparison with those of Jordinson,[3] Ling and Reynolds,[9] and Wazzan *et al.*[19]

With k_0, ζ, and ζ^* known, dk_0/dk_1 is calculated from Eqs. (32) and (33). To calculate $\partial\zeta/\partial x_1$, we differentiate ζ as given by Eq. (38) to determine the necessary initial conditions; that is,

$$\frac{\partial\zeta}{\partial x_1}=-\frac{dk_0}{dk_1}\,y\,e^{-k_0 y}+\left(c\,\frac{d\tilde{k}_0}{dx_1}\,y-\frac{dc}{dx_1}\right)e^{-\tilde{k}_0 y} \quad \text{for} \quad y\geq y_m\ . \tag{39}$$

Thus, by using the initial conditions

$$\frac{\partial\zeta_1}{\partial x_1}=-\frac{dk_0}{dx_1}\,y\,e^{-k_0 y} \quad \text{at} \quad y=y_m\ , \tag{40a}$$

$$\frac{\partial\zeta_2}{\partial x_1}=\left(c\,\frac{d\tilde{k}_0}{dx_1}\,y-\frac{dc}{dx_1}\right)e^{-\tilde{k}_0 y} \quad \text{at } y=y_m\ , \tag{40b}$$

we integrate Eqs. (29) from $y=y_m$ and use Kaplan's filtering technique to keep the two calculated solutions linearly independent. Then, $\partial\zeta/\partial x_1$ is the sum of these two solutions.

As a check on this procedure, we solved Eqs. (34a)–(34c) at three streamwise locations for constant ω and R and obtained $\zeta(y;\, x_1)$ and $k_0(x_1)$. Using central differences, we calculated $\Delta k_0/\Delta x_1$ and $\Delta\zeta/\Delta x_1$. The values of $\Delta k_0/\Delta x_1$ and $\Delta\zeta/\Delta x_1$ are in agreement with the values of dk_0/dx_1 and $\partial\zeta/\partial x_1$ to within computational accuracy; $\Delta\zeta/\Delta x_1$ and $\partial\zeta/\partial x_1$ are in agreement at every point in the boundary layer.

V. APPLICATION TO THE FALKNER-SKAN FLOWS

The stream function of the primary flow for a Falkner–Skan flow can be written as

$$\Psi(x,\ y)=(x_1)^{1/2}f(\eta)+O[\epsilon^2\ln\epsilon]\ , \quad \eta=y/(x_1)^{1/2}\ , \tag{41a}$$

where $\epsilon=R^{-1}$ and f satisfies[20]

$$f'''+ff''+\beta(f'^2-1)=0\ , \tag{41b}$$

subject to the boundary conditions

$$f(0)=f'(0)=0\ , \quad \lim_{\eta\to\infty} f'(\eta)-1=0\ . \tag{41c}$$

The parameter β is a measure of the pressure gradient; $\beta=0$ corresponds to the Blasius flow, $\beta=1.0$ corresponds to a stagnation flow, and $\beta=-0.1988$ corresponds to a separated flow. In Eqs. (41), velocities are made dimensionless by using the free-stream velocity U_∞ and lengths are made dimensionless by using the displacement thickness $\delta^*=\Delta(\nu L_0/U_\infty)^{1/2}$, where ν is the fluid kinetic viscosity, L_0 is a convenient reference length, and $\Delta=(2-\beta)^{1/2}\int_0^\infty[1-f'(\eta)]\,d\eta$. The values of Δ used in the computations are shown in Table I.

Wazzan *et al.*[19] calculated the parallel stability of the Falkner–Skan flows for a wide range of values of β.

Before presenting and discussing the results for the cases of pressure gradients, we present and discuss the results for the Blasius flow and compare them with available experimental data.

A. The Blasius flow

The available experimental stability studies rely almost conclusively on measurements made with hot-wire anemometers. Usually, the behavior of the streamwise component u of disturbance velocity is used to define the stability. In particular, the maxima and minima

TABLE I. Variation of Δ with β.

β	0.1	0.0	−0.1	−0.1988
Δ	1.489	1.720	2.091	3.498

of u are used to define the neutral stability curves separating stability from instability. Schubauer and Skramstad[1] measured the amplification rates by placing their probes at a fixed position from the plate below the point of maximum velocity fluctuations, while Ross *et al.*[2] performed their measurements by having their probes transverse curves with a constant mean-flow similarity parameter and making corrections to their results to account for the variation of the mode shape with streamwise position.

To remove the ambiguity in comparing theory and experiment, Bouthier[11] investigated the behavior of the streamwise u and transverse v velocity fluctuations and what he termed the local kinetic energy u^2+v^2. We note that local disturbance kinetic energy is proportional to $2U_0 u + O(\epsilon)$ rather than the one used by Bouthier. Then, he chose to define the stability by using the amplification rate α_e of $\bar{u}^2+\bar{v}^2$. Thus, he defined two neutral stability curves, one by determining the lowest Reynolds number where α_e becomes zero and the other by determining the maximum Reynolds number for which $\alpha_e>0$ at every point in the boundary layer. The curve based on the lowest Reynolds number is in good agreement with the experimental results.

Gaster[13] calculated neutral stability curves based on (a) the "kinetic-energy" integral: $\int_0^\infty(\bar{u}^2+\bar{v}^2)\,dy$, (b) the integral $\int_0^\infty \bar{u}^2\,dy$, and (c) the u component of velocity. The curve based on the integral of $\bar{u}^2$ deviates only slightly from the parallel stability result, while the curve based on the "kinetic-energy" integral overpredicts the critical Reynolds number by about 15%. The curve based on calculating the amplification of u at $\eta=0.08$ overpredicts the critical Reynolds number determined by Ross *et al.* by about 22%, while the curve based on calculating the amplification rate of u at fixed values of y below the point of maximum u overpredicts the critical Reynolds number determined by Shubauer and Skramstad by about 20%.

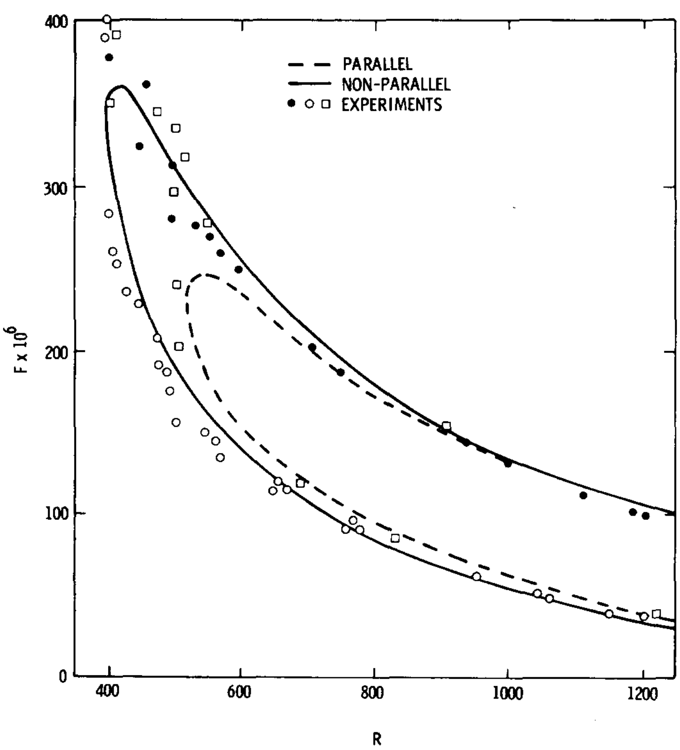

FIG. 1. Comparison between the neutral stability curves based on parallel and nonparallel stability theories and experimental data—□, data of Schubauer and Skramstad, ○, ●, data of Ross *et al.*

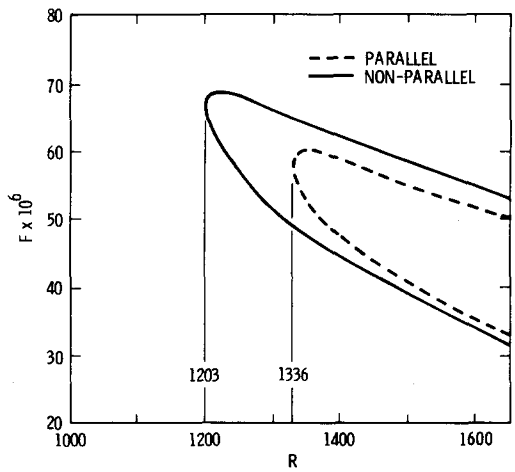

FIG. 2. Neutral stability curves for the Falkner–Skan flow with $\beta=0.1$.

In this article, we remove the ambiguity by defining the neutral curve by

$$\alpha_0+\epsilon\alpha_1=0\ ,$$

where α_j is the imaginary part of k_j when ζ is normalized outside the boundary layer as in Eq. (38). The resulting neutral stability curve is shown in Fig. 1, where the dimensionless frequency $F=\omega^*\delta^*\nu/U_\infty^3$ and the Reynolds number $R=U_\infty\delta^*/\nu$ with δ^* being the local displacement thickness and ω^* is the dimensionless frequency. Figure 1 shows good agreement between our neutral stability curve and the experimental results of Shubauer and Skramstad[1] and Ross *et al.*[2]

B. Pressure-gradient cases

Figure 2 shows a comparison between the neutral

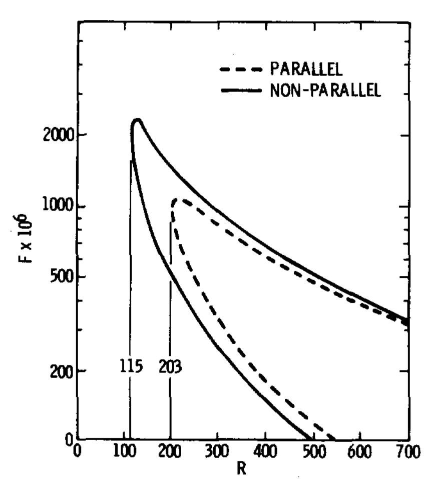

FIG. 3. Neutral stability curves for the Falkner–Skan flow with $\beta=-0.1$.

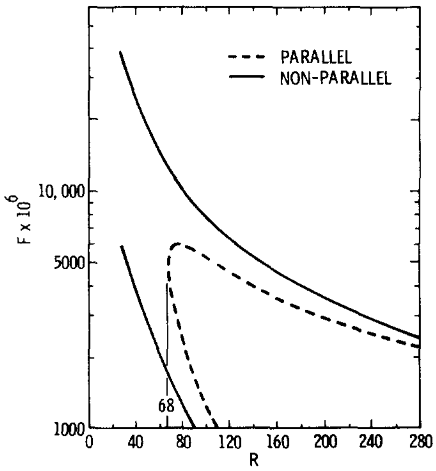

FIG. 4. Neutral stability curves for the Falkner–Skan flow with $\beta = -0.1988$.

curves based on the parallel and nonparallel stability theories for a favorable pressure gradient case, namely, $\beta = 0.1$. The effects of nonparallelism are not as large as in the case of $\beta = 0.0$, corresponding to the Blasius flow. Including the nonparallel effects decreases the critical Reynolds number from 1336 to 1203. As β increases (i.e., as the favorable pressure gradient increases), other numerical results show that the nonparallel effects decrease.

Figure 3 shows a comparison between the neutral stability curves based on the parallel and nonparallel stability theories for an adverse pressure gradient case, namely, $\beta = -0.1$. This figure shows that in this case the nonparallel effects are much more important than in the case of the Blasius flow. In fact, including the nonparallel effects shifts the critical Reynolds number from 203 to 115. Thus, the parallel stability theory overpredicts the critical Reynolds number by 77%.

Figure 4 shows the neutral stability curves based on the parallel and nonparallel stability theories for $\beta = -0.1988$, corresponding to a separated flow. The near-parallel stability theory does not predict a critical Reynolds number. We note that although the present theory may indicate the trends it is not expected to be accurate at such Reynolds numbers. Thus, a full two-dimensional stability theory must be used to predict the stability of separated or near separated flows. Figures 1–4 show that the nonparallel effects increase as β decreases.

VI. CONCLUDING REMARKS

The method of multiple scales is used to determine an approximate expression for the stream function of a sinusoidal single-frequency disturbance in boundary-layer flows, including the effects of nonparallelism. The nonparallel effects (a) make the quasi-parallel amplification rate be a function of the streamwise position, (b) change the quasi-parallel amplification rate, and (c) make the mode shape be a function of the streamwise position in addition to the distance from the wall.

For the Blasius flow, the neutral curve calculated from the present theory is in good agreement with those determined experimentally by Schubauer and Skramstad[1] and Ross *et al.*[2] The present results show that the effects of nonparallelism decrease for favorable pressure gradients and increase for adverse pressure gradients.

ACKNOWLEDGMENTS

This work was supported by the United States Atomic Energy Commission and the Fluid Dynamics Program of the Office of Naval Research.

APPENDIX

$$B_1 = 2k_0\,\omega - 3U_0\,k_0^2 - \frac{\partial^2 U_0}{\partial y} + 4ik_0^3/R\,, \tag{A1}$$

$$B_2 = U_0 - 4ik_0/R\,, \tag{A2}$$

$$B_3 = \omega - 3U_0\,k_0 + 6ik_0^2/R\,, \tag{A3}$$

$$B_4 = -2i/R\,, \tag{A4}$$

$$B_5 = -ik_0\left[\frac{\partial^2 U_1}{\partial y^2} + k_0^2\,U_1\right], \tag{A5}$$

$$B_6 = -\frac{\partial^2 V}{\partial y^2} - k_0^2 V\,, \tag{A6}$$

$$B_7 = ik_0\,U_1\,, \tag{A7}$$

$$B_8 = V\,. \tag{A8}$$

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