On the Computation of Invariants in non-Normal, non-Pure Cubic Fields and in Their Normal Closures

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ABSTRACT

Let $K = \mathbb{Q}(\theta)$ be the algebraic number field formed by adjoining θ to the rationals where θ is a real root of an irreducible monic cubic polynomial $f_K(x) \in \mathbb{Z}[x]$. If θ is not the cube root of a rational integer, we call the field K a non-pure cubic field, and if K doesn't contain the conjugates of θ, we call K a non-normal cubic field. A method described by Martinet and Payan [12] allows us to construct such fields from elements of a quadratic field. In this work, we examine such non-normal, non-pure cubic fields and their normal closures, using algorithms in Mathematica to compute various invariants of these fields. In addition, we prove general results relating the ranks of the ideal class groups of the rings of integers of these cubic fields to those of their normal closures.

Dedication

To my parents without whose support I could have accomplished none of this.

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Contents

Chapter 1

Introduction

The study of the pure cubic fields is rather extensive. However, non-pure cubic fields have, for the most part, received much less attention. Here by a pure cubic field, we mean an extention of the rationals obtained by adjoining the cube root of a rational integer, for example $\mathbb{Q}(\sqrt[3]{17})$. By a non-pure cubic field on the other hand we mean an extension of the rationals obtained by adjoining a root θ of some irreducible monic cubic polynomial $f(x) = x^3 + ax^2 + bx + c$ over $\mathbb{Z}[x]$, where c and either a or b are nonzero.

In this work, we describe methods to compute various invariants of certain non-normal, non-pure cubic fields and their normal closures. These cubic fields were originally described in a paper by Martinet and Payan [12]. The cubic fields are constructed using the method of essential ideals from an element of a related quadratic field. However, this quadratic field, though related to the cubic field and its normal closure, is not a subfield of the normal closure as one might expect. Here we restrict our attention to cubic fields which have one real and two imaginary conjugate fields. The third chapter of this work is primarily a description of Martinet and Payan's method for constructing these cubic fields and finding their defining polynomials and their integral bases. The work of others is also referenced here, particularly Voronoi [15], Williams, Cormack, and Seah [16], Cremona [3], Barrucand and Cohn [1], and Parry [13]. Their work is referenced in the methods of computing the fundamental units of both these cubic fields and their normal closures.

In the following chapter, we refer to the work of Gerth [5], [6] and Lemmermeyer [10] in proving new results about the ideal class group structure of the rings of integers of cubic fields, their normal closures, and the relation between the two. In the next chapter, we apply these results to the fields constructed through Martinet and Payan's method [12]. Finally, we include results of the pre-existing and new algorithms mentioned above employed through programs in Mathematica to compute these invariants in a variety of these types of fields.

Chapter 2

Notation

The following notation will be used throughout the remainder of this paper.

General Notation

F: Any finite algebraic extension of $\mathbb{Q}$.

O_F: The ring of integers of F.

H_F: The ideal class group of O_F.

h_F: The order of H_F, the class number of O_F.

$H_{p,F}$: The p-Sylow subgroup of H_F, which we'll call the p-class group of O_F, for any rational prime p.

$r_{p,F}$: The rank of the p-Sylow subgroup of H_F, for any rational prime p.

$h_{p,F}$: The order of $H_{p,F}$, which we'll call the p-class number of O_F.

M/F: Any finite extension of number fields.

$N = N_{M/F}$: The relative norm function of the extension M/F.

$T = T_{M/F}$: The relative trace function of the extension M/F.

ρ: A ring homomorphism.

$\mathrm{Ker}(\rho)$: The kernel of the homomorphism ρ.

$\mathrm{Im}(\rho)$: The image of the homomorphism ρ.

$\psi = \psi_{M/F}$: The homomorphism induced by $N_{M/F}$ from H_M to H_F.

$\psi_{p,M/F}$: The restriction of ψ to $H_{p,M}$.

$H_{M/F}$: The kernel of ψ.

$H_{p,M/F}$: The kernel of the restriction to $H_{p,F}$ of ψ.

$f_F(x)$: A polynomial, one of whose roots defines the field F, so if θ is a root of $f_F(x)$, then $F = \mathbb{Q}(\theta)$.

d_F: The discriminant of the field F.

ϵ_F: The fundamental unit of the field F, if F has a single fundamental unit. In the case where the field F has multiple fundamental units, they will, where possible, be represented in terms of fundamental units of subfields of F. If this is not possible, they will be written as $\epsilon_{F,1}, \epsilon_{F,2}$, and so on as necessary.

U_F: The group of units of the field F.

B_F: An integral basis of the ring of integers of the field F.

$\mathfrak{f}_{M/F}$: The conductor of the field extension M/F. Any prime dividing $\mathfrak{f}_{M/F}$ totally ramifies in the extension.

ω, ω^2: The primitive cube roots of unity, where $\omega = (\frac{-1+\sqrt{-3}}{2})$ and $\omega^2 = (\frac{-1-\sqrt{-3}}{2})$.

The Non-Normal, Non-Pure Cubic Fields and Related Fields

$K = K_1, K_i$: A non-normal, non-pure, real cubic field.

$K' = K_1', K_i', K'' = K_1'', K_i''$: The two (imaginary) conjugate fields of K (K_i, respectively).

$L = L_1, L_i$: The normal closure of K (K_i, respectively), an extension of degree 2.

$k = k_1 = \mathbb{Q}(\sqrt{m_1})$: The quadratic subfield of L.

$k_0 = k_1(\sqrt{-3}) = k_1(\omega)$: The biquadratic extension of k_1 obtained by adjoining $\sqrt{-3}$ to k_1.

$k_2 = \mathbb{Q}(\sqrt{m_2})$: Another quadratic subfield of k_0. Here $m_2 = -\frac{m_1}{3}$, if 3 divides m_1, or $m_2 = -3m_1$ if 3 doesn't didvide m_1.

$k_3 = \mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\omega)$: The third cyclotomic field and the final quadratic subfield of k_0.

$L(\omega)$: An extension of L of degree 2 containing k_0.

For the remainder of this work, we will use the term "cubic fields" to refer to non-normal, non-pure cubic fields. The term will also refer (unless otherwise stated) to fields one of whose conjugate fields is real while the other two are imaginary.

The Galois Group of $L(\omega)$ and its Subfields

$G_{L/k_1} = \mathrm{Gal}(L/k_1)$: The Galois group of the extension L/k_1. Let $G_{L/k_1} =< \sigma >$, where $\sigma^3 = 1$.

$G_{L/K} = \mathrm{Gal}(L/K)$: The Galois group of the extension L/K. Let $G_{L/K} =< \tau >$, where $\tau^2 = 1$.

$G_{L(\omega)/L} = \mathrm{Gal}(L(\omega)/L)$: The Galois group of the extension $L(\omega)/L$. Let $G_{L(\omega)/L} =< \phi >$, where $\phi^2 = 1$.

$G_{L/\mathbb{Q}} = \mathrm{Gal}(L/\mathbb{Q})$: The Galois group of the extension $L/\mathbb{Q}$. Here $G_{L/\mathbb{Q}} =< \sigma, \tau >$.

$G = \mathrm{Gal}(L(\omega)/\mathbb{Q})$: The Galois group of the extension $L(\omega)/\mathbb{Q}$. Here $G =< \phi, \tau, \sigma >$. Here ϕ fixes all of L, τ fixes K, and σ fixes k_1. Here $\sigma : K \to K'$ and $\sigma : K' \to K''$. Likewise, $\sigma^2 : K \to K''$ and $\sigma^2 : K' \to K$.

Elements and Ideals of $L(\omega)$ and its Subfields

$\theta, \theta' = \theta^\sigma, \theta'' = \theta^{\sigma^2}$: A primitive element of K and its conjugates. Here $N_{K/\mathbb{Q}}(\theta) = \theta\theta'\theta''$ and $T_{K/\mathbb{Q}}(\theta) = \theta + \theta' + \theta''$.

$B_K = \{\theta_0 = 1, \theta_1 = \theta, \theta_2\}$: An integral basis of the ring of integers of K.

$\alpha, \beta, \delta, \lambda, \alpha' = \alpha^{\tau_i}, \beta' = \beta^{\tau_i}, \delta' = \delta^{\tau_i}, \lambda' = \lambda^{\tau_i}$: Primitive elements of one of the quadratic fields k_i and their conjugates. Here τ_i is the element of G which conjugates elements of k_i and $N_{k_i/\mathbb{Q}}(\alpha) = \alpha\alpha'$ and $T_{k_i/\mathbb{Q}}(\alpha) = \alpha + \alpha'$.

$B_k = \{\lambda_0 = 1, \lambda_1 = \lambda\}$: An integral basis of the ring of integers of k. Here if $k = \mathbb{Q}(\sqrt{m})$, we note that then $\lambda = \sqrt{m}$ if $m \equiv 2$ or $3 \pmod 4$, while $\lambda = \frac{1+\sqrt{m}}{2}$ if $m \equiv 1 \pmod 4$.

μ, ν: Primitive elements of L.

p, q: Rational primes.

$\mathfrak{p}_i, \mathfrak{q}_j$: Prime ideals of k lying over p and q.

P_i, Q_j: Prime ideals of K lying over p and q.

$\mathfrak{P}_i, \mathfrak{Q}_j$: Prime ideals of L lying over $\mathfrak{p}_i$ and $\mathfrak{q}_j$.

$\mathfrak{l}_i$: A prime ideal divisor of 3 in k.

$\mathfrak{L}_i$: A prime ideal divisor of 3 in L lying over $\mathfrak{l}_i$.

$\mathfrak{a}, \mathfrak{b}, \mathfrak{i}, \mathfrak{j}$: Ideals of k.

A, B, I, J: Ideals of K.

$\mathfrak{A}, \mathfrak{B}, \mathfrak{I}, \mathfrak{J}$: Ideals of L.

$\mathfrak{c}_i$: Ideal classes of H_k.

C_i: Ideal classes of H_K.

$\mathfrak{C}_i$: Ideal classes of H_L.

Chapter 3

Construction of Non-Normal, Non-Pure Cubic Fields

Generation of the Cubic Fields and their Defining Polynomials

In order to apply the algorithms used to compute the invariants of the non-normal, non-pure cubic fields, we must first use a method to define many such cubic fields. We will examine both these cubic fields and their normal closures. We use a method of Martinet and Payan [12], described below, to construct these fields.

First, let θ, θ', and θ'' be primitive conjugate elements of a non-normal, non-pure cubic field, K. Define γ_1 and γ_2 as below.

$$\gamma_1 = \theta + \omega\theta' + \omega^2\theta''$$

$$\gamma_2 = \theta + \omega^2\theta' + \omega\theta''$$

Then we turn to a Theorem of Hasse [7] to show how we can use the quadratic field k_2 to determine K.

Theorem 3.1 (H. Hasse (1947)) [7] *If we define γ_1 and γ_2 as above, $\alpha = \gamma_1^3$ and $\alpha' = \gamma_2^3$ are primitive conjugate elements in k_2, not in k_2^3. Further, $\frac{\gamma_1^2}{\gamma_2} \in k_2$ and $\gamma_1\gamma_2 \in \mathbb{Q}$.*

Conversely, if α and α' are two primitive conjugate elements of k_2 but not of k_2^3 such that $\frac{\alpha^2}{\alpha'} \in k_2^3$, there exists a triple of primitive conjugate elements θ, θ', and θ'' of the non-normal cubic extensions of $\mathbb{Q}$ so that $\gamma_1^3 = \alpha$ and $\gamma_2^3 = \alpha'$ with γ_1 and γ_2 defined as above.

Hence, if α and α' are primitive conjugates in k_2 such that $\frac{\alpha^2}{\alpha'} \in k_2^3$, we are guaranteed that there are primitive conjugate elements β and β' such that $\beta^3 = \frac{\alpha^2}{\alpha'}$ and $\beta'^3 = \frac{\alpha'^2}{\alpha}$. Thus, $\beta^6 = \frac{\alpha^4}{\alpha'^2}$ and if we multiply this by the expression for β'^3, we find that $\beta^6\beta'^3 = \frac{\alpha^4}{\alpha'^2}\frac{\alpha'^2}{\alpha} = \alpha^3$ and so $\alpha = \beta^2\beta'$. Similarly, we find that $\alpha' = \beta'^2\beta$. Conversely, for any two primitive conjugate elements β and β' of k_2 so that $\beta^2\beta'$ is not in k_2^3, the numbers $\alpha = \beta^2\beta'$ and $\alpha' = \beta'^2\beta$ satisfy the Theorem of Hasse above.

Suppose a conjugate pair α and α' of k_2 define a triple of primitive conjugate elements θ, θ', and θ''. Then we write $f(x) = \mathrm{Irr}(\theta)$ as the irreducible polynomial of which θ is a root. Define β and β' as before, so $\alpha = \beta^2\beta'$ and $\alpha' = \beta'^2\beta$. Let $S = T_{K/\mathbb{Q}}(\theta) = \theta + \theta' + \theta''$ be the trace of θ. Then if we recall our definitions of γ_1 and γ_2, and note that $\omega + \omega^2 + 1 = 0$, we find that $\gamma_1 + \gamma_2 + S = 3\theta$ and hence, that $3\theta - S = \gamma_1 + \gamma_2$. Then, $(3\theta - S)^3 = \gamma_1^3 + 3\gamma_1^2\gamma_2 + 3\gamma_1\gamma_2^2 + \gamma_2^3$. But $\alpha\alpha' = \beta^3\beta'^3$ and since $\alpha = \gamma_1^3$ and $\alpha' = \gamma_2^3$, then $\beta^3\beta'^3 = \gamma_1^3\gamma_2^3$ and hence $\beta\beta' = \gamma_1\gamma_2$. Therefore, $(3\theta - S)^3 = \gamma_1^3 + \gamma_2^3 + 3\gamma_1\gamma_2(\gamma_1 + \gamma_2) = \alpha + \alpha' + 3\beta\beta'(3\theta - S) = \beta^2\beta' + \beta'^2\beta + 3\beta\beta'(3\theta - S)$. So we find

$$f(x) = x^3 - Sx^2 - \frac{\beta\beta' - S^2}{3}x - \frac{S^3 - 3S\beta\beta' + \beta\beta'(\beta + \beta')}{27}$$

Occasionally we will need to find whether or not two polynomials definethe same cubic field. In order to do this, we turn to a Theorem of Martinet and Payan [12] describing when two pairs (α, α') and (α_1, α_1') of primitive conjugate numbers of the same quadratic field k define the same cubic field K.

Theorem 3.2 (J. Martinet and J.-J. Payan (1967)) [12] *The two pairs (α, α') and (α_1, α_1') of primitive conjugate numbers of the same quadratic field k define the same cubic field K if and only if one of the following conditions holds:*

There exists a $\lambda \in k$ such that $\alpha_1 = \lambda^3\alpha$ and $\alpha_1' = \lambda'^3\alpha'$.

There exists a $\mu \in k$ such that $\alpha_1 = \mu^3\alpha'$ and $\alpha_1' = \mu'^3\alpha$.

Now that we have constructed the defining polynomial for our cubic field K, we will examine a theorem of Reichardt [14] which will determine how many distinct cubic fields we can construct from a given conjugate pair α and α'. We will refer to these cubic fields alternately as being associated with, constructed from, or defined by α. In order to examine this theorem, first, we must define the essential ideals of a quadratic field k in accordance with the definition of Châtelet [2]. Then we must examine a theorem of Martinet and Payan [12] dealing with these essential ideals.

Definition 3.1 *A principal ideal A of a quadratic field k is called an essential ideal if there exists* *a principal ideal B of k so that $A^2A' = B^3$ where A' is the conjugate ideal of A in k.*

Theorem 3.3 (J. Martinet and J.-J. Payan (1967)) [12] *An essential ideal $\mathfrak{a}$ of k can be uniquely written in the form $\mathfrak{a} = (a^3)\mathfrak{j}^3\mathfrak{b}^2\mathfrak{b}'$. Here a is a positive rational number, $\mathfrak{j}$ is an ideal of k without rational factors, and $\mathfrak{b}$ is an ideal of k whose norm is without square factor. The ideal $\mathfrak{b}'$ belongs to the same ideal class as does $\mathfrak{j}^3$.*

Conversely, if an ideal $\mathfrak{a}$ of k can be written as $\mathfrak{a} = (a^3)\mathfrak{j}^3\mathfrak{b}^2\mathfrak{b}'$ where $\mathfrak{b}'$ belongs to the same ideal class as $\mathfrak{j}^3$, $\mathfrak{a}$ is an essential ideal.

We refer to $\mathfrak{b}'$ as the first canonical factor, $\mathfrak{j}$ as the second canonical factor, and a as the trivial factor. If we let α be an element of k_2 defining the cubic field K, (α) is an essential ideal of k_2. We can choose a number α_1 defining the same cubic field K so that (α_1) is without trivial factor and the second canonical factor is prime with any given ideal $\mathfrak{i}$ of k_2.

In order to further examine these cubic fields in their relation to these essential ideals, we now turn to a theorem of Reichardt [14]. This theorem first describes under which conditions an ideal $\mathfrak{b}$ of k can bring forth a canonical factor $\mathfrak{b}^2\mathfrak{b}'$ which will define a cubic field. Secondly, the theorem describes how many cubic fields exist having a given canonical factor $\mathfrak{b}^2\mathfrak{b}'$.

Theorem 3.4 (H. Reichardt (1933)) [14] *Let h_3 be the number of ideal classes of k whose cubes are principal. Let $\mathfrak{b}$ be an ideal of k without square factor whose norm is relatively prime to d_k. $\mathfrak{b}^2\mathfrak{b}'$ is only a canonical factor for a cubic field K if $\mathfrak{b}$ belongs to the cube of an ideal class of k.*

If this condition holds, the number of cubic fields having $\mathfrak{b}^2\mathfrak{b}'$ as a canonical factor is given by:

h_3	*if k is imaginary and $\mathfrak{b} \neq (1)$*
$3h_3$	*if k is real and $\mathfrak{b} \neq (1)$*
$\frac{h_3-1}{2}$	*if k is imaginary and $\mathfrak{b} = (1)$*
$\frac{3h_3-1}{2}$	*if k is real and $\mathfrak{b} = (1)$*

Here we note that if we examine cubic fields, K_i, constructed via the method above from a pair of conjugate elements of k_2, then the quadratic subfield of L_i is not k_2, but rather k_1. We will now examine a couple of examples, herein restricting ourselves to the case where k_2 is a real quadratic field.

Examples

We will examine a couple of examples in which a given element β of a quadratic field can be used to construct cubic fields. We will use the work above to construct the defining polynomials of these cubic fields, and herein we will restrict our examination to real quadratic fields. In these early examples, we will further restrict ourselves to the case in which h_{k_2} is relatively prime to 3.

Let $k_1 = \mathbb{Q}(\sqrt{-6})$ and $k_2 = \mathbb{Q}(\sqrt{2})$. We have here chosen a real field k_2 so that its class number is relatively prime to 3, so $h_3 = 1$. Then in accordance with Theorem 1.4, there is only one cubic field having $\mathfrak{b}^2\mathfrak{b}'$ as a canonical factor if $\mathfrak{b} = (1)$. On the other hand, if $\mathfrak{b} \neq (1)$, there are three cubic fields having $\mathfrak{b}^2\mathfrak{b}'$ as a canonical factor.

Example 1: First, we will construct a cubic field by the method above which we'll call K_0. We will again call a primitive element of this field θ, herein with $S = Tr_{K_0/\mathbb{Q}}(\theta) = 0$. Here we examine the case $\mathfrak{b} = (\beta) = (1)$. Hence by Theorem 1.4 above, there is only $\frac{3h_3-1}{2} = 1$ cubic field having $\mathfrak{b}^2\mathfrak{b}'$ as a canonical factor.

In this case, we must choose β to be a unit of k_2. Here we let $\beta = 1 + \sqrt{2}$ and $\beta' = 1 - \sqrt{2}$. Thus $\beta\beta' = -1$ and $\beta + \beta' = 2$. Hence, we find that the irreducible polynomial of which θ is a root is, in this case, $f(x) = x^3 + \frac{1}{3}x + \frac{2}{27}$. If we instead want a defining polynomial for the same field K_0 with integer coefficients, we note that if in the case above, when θ is a root of $f(x)$, then 3θ is a root of the polynomial $f_3(x) = x^3 + 3x + 2$.

Example 2: If, on the other hand, we want to construct cubic fields from an ideal $\mathfrak{b}$ of k_2 so that $\mathfrak{b} = (\beta) \neq (1)$, then we will have $3h_3 = 3$ cubic fields having $\mathfrak{b}^2\mathfrak{b}'$ as a canonical factor. Here we will call these fields K_1, K_2, and K_3. We must choose a β so that the norm $N_{k_2/\mathbb{Q}}(\beta)$ is relatively prime to the discriminant of the quadratic field k_2 (which here is $d_{k_2} = 8$) and so that if a rational prime $p = N_{k_2/\mathbb{Q}}(\beta)$, the Jacobi symbol $(\frac{2}{p}) = 1$ in order that $x^2 - 2y^2 = \pm p$ will have solutions (so here $p \equiv \pm 1 (\text{mod } 8)$). Here we will choose a $\mathfrak{b} = (\beta)$ whose norm $N_{k_2/\mathbb{Q}}(\mathfrak{b}) = 79$.

Example 2.a: For the first of these fields, K_1, we let $\beta = 9 + \sqrt{2}$ and $\beta' = 9 - \sqrt{2}$. Here we have $\beta\beta' = 79$ and $\beta + \beta' = 18$. Hence, this cubic field is defined by a root of the polynomial $f(x) = x^3 - \frac{79}{3}x - \frac{79 \cdot 18}{27} = x^3 - \frac{79}{3}x - \frac{1422}{27}$, or equivalently by a root of $f(x) = x^3 - 237x - 1422$. We can also define the same field we define by a root of the polynomial we generate by the pair of primitive conjugate elements (α, α') by a second pair of primitive conjugate elements (α_1, α_1') and the polynomial they generate, where, as in Theorem 1.2, $\alpha_1 = (-\sqrt{2})^3\alpha$ and $\alpha_1' = (\sqrt{2})^3\alpha$. Here, note that since $\alpha = \beta^2\beta'$ and $\alpha' = \beta'^2\beta$, we can define α_1 and α_1' similarly with $\alpha_1 = \beta_1^2\beta_1'$ and $\alpha' = \beta_1'^2\beta_1$, where $\beta_1 = (\sqrt{2})\beta$ and $\beta_1' = (-\sqrt{2})\beta'$. Hence, $\beta_1 = 2+9\sqrt{2}$ and $\beta_1' = 2-9\sqrt{2}$. We then have $\beta_1\beta_1' = -158$ and $\beta_1 + \beta_1' = 4$. Hence this cubic field is also defined by a root of the polynomial $f(x) = x^3 + \frac{158}{3}x + \frac{158\cdot 4}{27} = x^3 - \frac{158}{3}x - \frac{632}{27}$, or equivalently by a root of $f_3(x) = x^3 + 474x + 632$.

Example 2.b: For the second of these fields, K_2, we let $\beta = (9+\sqrt{2})(1+\sqrt{2}) = 11 + 10\sqrt{2}$ and $\beta' = (9-\sqrt{2})(1-\sqrt{2}) = 11 - 10\sqrt{2}$. Here we have $\beta\beta' = -79$ and $\beta + \beta' = 22$. Hence this cubic field is defined by a root of the polynomial $f(x) = x^3 + \frac{79}{3}x + \frac{79\cdot 22}{27} = x^3 + \frac{79}{3}x + \frac{1738}{27}$, or equivalently by a root of $f_3(x) = x^3 + 237x + 1738$.

Example 2.c: For the third of these fields, K_3, we let $\beta = (9+\sqrt{2})(1-\sqrt{2}) = 7 - 8\sqrt{2}$ and $\beta' = (9-\sqrt{2})(1+\sqrt{2}) = 7 + 8\sqrt{2}$. Here we have $\beta\beta' = -79$ and $\beta + \beta' = 14$. Hence this cubic field is defined by a root of the polynomial $f(x) = x^3 + \frac{79}{3}x + \frac{79\cdot 14}{27} = x^3 + \frac{79}{3}x + \frac{1106}{27}$, or equivalently by a root of $f_3(x) = x^3 + 237x + 1106$.

We will continue our examination here with examples in which a given element β of a quadratic field can be used to construct more cubic fields. We will retain our restriction to real quadratic fields but herein allow 3 to divide h_{k_2}.

Now let $k_1 = \mathbb{Q}(\sqrt{-237})$ and $k_2 = \mathbb{Q}(\sqrt{79})$. We have here chosen a real field k_2 so that its class number is equal to 3, so $h_3 = 3$. Here, then, in accordance with Theorem 1.4, there are four cubic fields having $\mathfrak{b}^2\mathfrak{b}'$ as a canonical factor if $\mathfrak{b} = (1)$. On the other hand, if $\mathfrak{b} \neq (1)$, there are nine cubic fields having $\mathfrak{b}^2\mathfrak{b}'$ as a canonical factor.

Example 3: We will begin by constructing four cubic fields by the method given above which we'll call $K_{0,0}, K_{0,1}, K_{0,2}$, and $K_{0,3}$. As before, call a primitive element of this field θ, herein with $S = Tr_{K_0/\mathbb{Q}}(\theta) = 0$. Here we examine the case $\mathfrak{b} = (\beta) = (1)$. Hence, here we have, by Theorem 1.4, $\frac{3h_3-1}{2} = 4$ cubic fields having $\mathfrak{b}^2\mathfrak{b}'$ as a canonical factor.

Example 3.a: In our first field, $K_{0,0}$, we must choose β to be a unit of k_2. We let $\beta = 80 + 9\sqrt{79}$ and $\beta' = 80 - 9\sqrt{79}$. Thus $\beta\beta' = 1$ and $\beta + \beta' = 160$. Hence, we find that the irreducible polynomial of which θ is a root is, in this case, $f(x) = x^3 - \frac{1}{3}x + \frac{160}{27}$. If we instead want a defining polynomial for the same field K_0 with integer coefficients, we note that if in the case above, when θ is a root of $f(x)$, then 3θ is a root of the polynomial $f_3(x) = x^3 + 3x + 160$.

Example 3.b: In the next field, $K_{0,1}$, we instead construct a field from an element of k_2 whose norm is a cube. Here, let $\beta = 21 + 2\sqrt{79}$ and $\beta' = 21 - 2\sqrt{79}$, where, hence, β and β' are cubes of the non-principal prime ideal divisors of 5, in accordance with Theorem 3.3. Hence $\beta\beta' = 125$ and $\beta + \beta' = 42$. Thus, this next cubic field is defined by a root of the polynomial $f(x) = x^3 - \frac{125}{3}x + \frac{125\cdot 42}{27} = x^3 - \frac{125}{3}x + \frac{5250}{27}$, or equivalently by a root of $f_3(x) = x^3 - 375x + 5250$.

Example 3.c: For the third of these fields, $K_{0,2}$, we let $\beta = (21 + 2\sqrt{79})(80 + 9\sqrt{79}) = 3102 + 349\sqrt{79}$ and $\beta = (21 - 2\sqrt{79})(80 - 9\sqrt{79}) = 3102 - 349\sqrt{79}$. Hence $\beta\beta' = 125$ and $\beta + \beta' = 6204$. Therefore, this cubic field is defined by a root of the polynomial $f(x) = x^3 - \frac{125}{3}x + \frac{125\cdot 6204}{27} = x^3 - \frac{125}{3}x + \frac{775500}{27}$, or equivalently by a root of $f_3(x) = x^3 - 375x + 775500$.

Example 3.d: For the fourth of these fields, $K_{0,3}$, we let $\beta = (21+2\sqrt{79})(80-9\sqrt{79}) = 258-29\sqrt{79}$ and $\beta = (21 - 2\sqrt{79})(80 + 9\sqrt{79}) = 258 + 29\sqrt{79}$. Hence $\beta\beta' = 125$ and $\beta + \beta' = 516$. Therefore, this cubic field is defined by a root of the polynomial $f(x) = x^3 - \frac{125}{3}x + \frac{125\cdot 516}{27} = x^3 - \frac{125}{3}x + \frac{64500}{27}$, or equivalently by a root of $f_3(x) = x^3 - 375x + 64500$.

Example 4: However, if we want again to construct cubic fields from an ideal $\mathfrak{b}$ of k_2 so that $\mathfrak{b} = (\beta) \neq (1)$, then we will have $3h_3 = 9$ cubic fields having $\mathfrak{b}^2\mathfrak{b}'$ as a canonical factor. Here we will call these fields $K_{1,1}, K_{1,2}, K_{1,3}, K_{2,1}, K_{2,2}, K_{2,3}, K_{3,1}, K_{3,2}$, and $K_{3,3}$. We must choose a β so that the norm $N_{k_2/\mathbb{Q}}(\beta)$ is relatively prime to the discriminant of the quadratic field k_2 (which here is $d_{k_2} = 3$) and so that if a rational prime $p = N_{k_2/\mathbb{Q}}(\beta)$, the Jacobi symbol $(\frac{79}{p}) = 1$ in order that $x^2 - 79y^2 = \pm p$ will have solutions. We will herein choose a $\mathfrak{b} = (\beta)$ whose norm $N_{k_2/\mathbb{Q}}(\mathfrak{b}) = 43$.

Example 4.a: For the first of these fields, $K_{1,1}$, we let $\beta = 6 + \sqrt{79}$ and $\beta' = 6 - \sqrt{79}$. Here we have $\beta\beta' = -43$ and $\beta + \beta' = 12$. Hence this cubic field is defined by a root of the polynomial $f(x) = x^3 + \frac{43}{3}x + \frac{43\cdot 12}{27} = x^3 + \frac{43}{3}x + \frac{516}{27}$, or equivalently by a root of $f_3(x) = x^3 + 129x + 516$.

Example 4.a.1: For the next of these fields, $K_{1,2}$, we let $\beta = (6+\sqrt{79})(21+2\sqrt{79}) = 284+33\sqrt{79}$

and $\beta' = (6-\sqrt{79})(21-2\sqrt{79}) = 284-33\sqrt{79}$, again using the cubes of the non-principal prime ideal divisors of 5, in accordance with Theorem 3.3. Here we have $\beta\beta' = -5375$ and $\beta+\beta' = 568$. Hence this cubic field is defined by a root of the polynomial $f(x) = x^3 + \frac{5375}{3}x + \frac{5375\cdot 568}{27} = x^3 + \frac{5375}{3}x + \frac{3053000}{27}$, or equivalently by a root of $f_3(x) = x^3 + 16125x + 3053000$.

Example 4.a.2: For the next of these fields, $K_{1,3}$, we let $\beta = (6+\sqrt{79})(21-2\sqrt{79}) = -32+9\sqrt{79}$ and $\beta' = (6-\sqrt{79})(21+2\sqrt{79}) = -32-9\sqrt{79}$. Here we have $\beta\beta' = -5375$ and $\beta+\beta' = -64$. Hence this cubic field is defined by a root of the polynomial $f(x) = x^3+\frac{5375}{3}x-\frac{5375\cdot 64}{27} = x^3+\frac{5375}{3}x+\frac{3053000}{27}$, or equivalently by a root of $f_3(x) = x^3 + 16125x + 344000$.

Example 4.b: For the next of these fields, $K_{2,1}$, we let $\beta = (6+\sqrt{79})(80+9\sqrt{79}) = 1191+134\sqrt{79}$ and $\beta' = (6-\sqrt{79})(80-9\sqrt{79}) = 1191-134\sqrt{79}$. Here we have $\beta\beta' = -43$ and $\beta+\beta' = 2382$. Hence this cubic field is defined by a root of the polynomial $f(x) = x^3 + \frac{43}{3}x + \frac{43\cdot 2382}{27} = x^3 + \frac{43}{3}x + \frac{102426}{27}$, or equivalently by a root of $f_3(x) = x^3 + 129x + 102426$.

Example 4.b.1: For the next of these fields, $K_{2,2}$, we let $\beta = (1191+134\sqrt{79})(21+2\sqrt{79}) = 46183+5196\sqrt{79}$ and $\beta' = (1191-134\sqrt{79})(21-2\sqrt{79}) = 46183-5196\sqrt{79}$. Here we have $\beta\beta' = -5375$ and $\beta+\beta' = 7678$. Hence this cubic field is defined by a root of the polynomial $f(x) = x^3 + \frac{5375}{3}x + \frac{5375\cdot 92366}{27} = x^3 + \frac{5375}{3}x + \frac{496467250}{27}$, or equivalently by a root of $f_3(x) = x^3 + 16125x + 496467250$.

Example 4.b.2: For the next of these fields, $K_{2,3}$, we let $\beta = (1191+134\sqrt{79})(21-2\sqrt{79}) = 3839+432\sqrt{79}$ and $\beta' = (1191-134\sqrt{79})(21+2\sqrt{79}) = 3839-432\sqrt{79}$. Here we have $\beta\beta' = -5375$ and $\beta+\beta' = 7678$. Hence this cubic field is defined by a root of the polynomial $f(x) = x^3 + \frac{5375}{3}x + \frac{5375\cdot 7678}{27} = x^3 + \frac{5375}{3}x + \frac{41269250}{27}$, or equivalently by a root of $f_3(x) = x^3 + 16125x + 41269250$.

Example 4.c: For the next of these fields, $K_{3,1}$, we let $\beta = (6+\sqrt{79})(80-9\sqrt{79}) = -231+26\sqrt{79}$ and $\beta' = (6-\sqrt{79})(80+9\sqrt{79}) = -231-26\sqrt{79}$. Here we have $\beta\beta' = -43$ and $\beta+\beta' = -462$. Hence this cubic field is defined by a root of the polynomial $f(x) = x^3 + \frac{43}{3}x - \frac{43\cdot 462}{27} = x^3 + \frac{43}{3}x - \frac{19866}{27}$, or equivalently by a root of $f_3(x) = x^3 + 129x - 19866$.

Example 4.c.1: For the next of these fields, $K_{3,2}$, we let $\beta = (-231+26\sqrt{79})(21+2\sqrt{79}) =$

$-743+84\sqrt{79}$ and $\beta' = (-231-26\sqrt{79})(21-2\sqrt{79}) = -743-84\sqrt{79}$. Here we have $\beta\beta' = -5375$ and $\beta+\beta' = -1486$. Hence this cubic field is defined by a root of the polynomial $f(x) = x^3+\frac{5375}{3}x-\frac{5375\cdot1486}{27} = x^3+\frac{5375}{3}x-\frac{7987250}{27}$, or equivalently by a root of $f_3(x) = x^3+16125x-7987250$.

Example 4.c.2: For the next of these fields, $K_{3,3}$, we let $\beta = (-231+26\sqrt{79})(21-2\sqrt{79}) = -8959+1008\sqrt{79}$ and $\beta' = (-231-26\sqrt{79})(21+2\sqrt{79}) = -8959-1008\sqrt{79}$. Here we have $\beta\beta' = -5375$ and $\beta+\beta' = -17918$. Hence this cubic field is defined by a root of the polynomial $f(x) = x^3+\frac{5375}{3}x-\frac{5375\cdot17918}{27} = x^3+\frac{5375}{3}x-\frac{96309250}{27}$, or equivalently by a root of $f_3(x) = x^3+16125x-96309250$.

Discriminants and Ramified Primes of the Cubic Fields

Martinet and Payan [12] further examine these fields relating the discriminant of the cubic field K to that of the quadratic subfield k. In so doing they prove the following theorem.

Theorem 3.5 (J.Martinet and J.-J. Payan (1967)) [12] *The discriminant of a non-normal cubic field of degree* 3 *constructed from a quadratic field* $k_2 = \mathbb{Q}(\sqrt{m})$ *by the method above is of the form*

$$d_K = d_k \mathfrak{f}_{L/k}^2$$

where $\mathfrak{f}_{L/k}$ *is a product of* 3^n *and a product of primes* p_i *different from* 3 *without square factor each having Jacobi symbol* $(\frac{m}{p_i}) = 1$ *and where*

$n = 0$ *or* 1 *if* $d_k \equiv 3 \pmod 9$

$n = 0,\ 1,$ *or* 2 *if* $d_k \equiv -3 \pmod 9$

$n = 0$ *or* 2 *if* $d_k \equiv \pm1 \pmod 3$

Further, a theorem following the theorem above relates the number $\mathfrak{f}_{L/k}$ above with the rational primes p which totally ramify in the extension $K/\mathbb{Q}$. We will make a great deal of use of this theorem in chapter 5.

Theorem 3.6 (J.Martinet and J.-J. Payan (1967)) [12] *For* p *to be the cube of a prime ideal* P *of* K, *it is necessary and sufficient that* p *divides* $\mathfrak{f}_{L/k}$.

We also note here that Martinet and Payan [12], in their work, further clarify which primes divide this conductor. In particular, they show that

$$\mathfrak{f}_{L/k} = 3^n N_{k_2/\mathbb{Q}}(\mathfrak{b}) = N_{k_2/\mathbb{Q}}(\beta).$$

Here, n is as defined above in Theorem 3.5. This allows us to construct cubic fields from a quadratic field k_2 in which both 3 and some given prime p totally ramify by choosing β so that $N_{k_2/\mathbb{Q}}(\beta) = \pm p$. This result is summarized in the theorem below.

Theorem 3.7 (J.Martinet and J.-J. Payan (1967)) [12] *Let $\mathfrak{L}$ is the divisor of 3 in the biquadratic field k_0. Let α and β be defined as before. If 3 doesn't divide an element $\alpha \in k_2$ described above, then the discriminant of the cubic field K defined by α is determined as follows:*

$d_K = d_k N_{k_2/\mathbb{Q}}(\beta)^2$ *if $d_k \equiv \pm 3 \pmod 9$ and $\alpha \equiv \xi^3 \pmod 9$ for some $\xi \in k_2$*

$d_K = 9 d_k N_{k_2/\mathbb{Q}}(\beta)^2$ *if $d_k \equiv \pm 3 \pmod 9$ and $\alpha \not\equiv \xi^3 \pmod 9$ for some $\xi \in k_2$*

$d_K = d_k N_{k_2/\mathbb{Q}}(\beta)^2$ *if $d_k \equiv \pm 1 \pmod 3$ and $\alpha \equiv \xi^3 \pmod{3\mathfrak{L}}$ for some $\xi \in k_2$*

$d_K = 81 d_k N_{k_2/\mathbb{Q}}(\beta)^2$ *if $d_k \equiv \pm 1 \pmod 3$ and $\alpha \not\equiv \xi^3 \pmod{3\mathfrak{L}}$ for some $\xi \in k_2$*

If, on the other hand, 3 divides α, then $d_K = 9 d_k N_{k_2/\mathbb{Q}}(\beta)^2$.

Integral Bases of the Cubic Fields

We now wish to find an integral basis for $\mathbb{A}_K$, the ring of integers of K. We want to find a basis of the form $B_K = \{1, \theta_1 = \theta, \theta_2\}$, where again θ is the real root of $f_K(x)$, the defining polynomial of K we have described above, and where θ_2 is in terms of θ. Here we begin to construct an integral basis for our field with another theorem of Martinet and Payan [12].

Theorem 3.8 (J.Martinet and J.-J. Payan (1967)) [12] *Let $\mathfrak{L}$ is the divisor of 3 in the biquadratic field k_0. Let α and β be defined as before. Let $B_{k_2} = \{1, \lambda\}$ be an integral basis of the ring of integers of the quadratic field k_2. If 3 doesn't divide an element $\alpha \in k_2$ described above, then the elements γ_1 and γ_1^* defined by α are determined depending on α as in the following cases:*

Case A. for 3 not dividing α, $d_k \equiv \pm 3 \pmod 9$, and $\alpha \equiv \xi^3 \pmod 9$ for some $\xi \in k_2$

Case B. for 3 not dividing α, $d_k \equiv \pm 3 \pmod 9$, and $\alpha \not\equiv \xi^3 \pmod 9$ for some $\xi \in k_2$

Case C. for 3 not dividing α, $d_k \equiv \pm 1 \pmod 3$, and $\alpha \equiv \xi^3 \pmod{3\mathfrak{L}}$ for some $\xi \in k_2$

Case D. for 3 not dividing α, $d_k \equiv \pm 1 \pmod 3$, and $\alpha \not\equiv \xi^3 \pmod{3\mathfrak{L}}$ for some $\xi \in k_2$

Case E. for 3 divides α

Hence we find the following definitions of γ_1 and γ_1^* in each of the above cases as follows:

Case A. $\gamma_1^3 = \alpha$ *and* $(\gamma_1^*)^3 = (3\lambda)^3\alpha$

Case B. $\gamma_1^3 = 3^3\alpha$ *and* $(\gamma_1^*)^3 = (3\lambda)^3\alpha$

Case C. $\gamma_1^3 = \alpha$ *and* $(\gamma_1^*)^3 = (\lambda)^3\alpha$

Case D. $\gamma_1^3 = 3^3\alpha$ *and* $(\gamma_1^*)^3 = (3\lambda)^3\alpha$

Case E. $\gamma_1^3 = 3^3\alpha$ *and* $(\gamma_1^*)^3 = (3\lambda)^3\alpha$

Then γ_2 *and* γ_2^* *are defined analogously by replacing* α *with* α' *(and likewise* λ *with* λ'*) in the definitions of* γ_1 *and* γ_1^* *above. Then we have* $B_K = \{1, \theta, \theta^*\}$*, where*

$$\theta = \frac{1}{3}(\gamma_1 + \gamma_2 + S)$$

and

$$\theta^* = \frac{1}{3}(\gamma_1^* + \gamma_2^* + S^*)$$

and where S *and* S^* *are the traces of* θ *and* θ^**, respectively. The numbers* $B_K = \{1, \theta, \theta^*\}$ *form an integral basis for* K.

Note that in their paper, Martinet and Payan give results describing how to find S and S^* at least up to congruence (mod 3). In Cases B, D, and E, $S \equiv S^* \equiv 0 \pmod 3$. On the other hand, in Case A, $S^* \equiv 0 \pmod 3$, but S is determined up to its congruence (mod 3) by their Lemma 8 ([12] page 25). Finally, in Case C, S and S^* are both determined up to congruence (mod 3) by their Lemma 4 ([12] page 24).

Now we note from the theorem above, we have

$$\gamma_1^3 = 3^{3n}\alpha,$$
$$\gamma_2^3 = 3^{3n}\alpha',$$
$$(\gamma_1^*)^3 = (3^{n_*}\lambda)^3\alpha, \text{ and}$$
$$(\gamma_2^*)^3 = (3^{n_*}\lambda')^3\alpha'.$$

Here $n = 0$ or 1 and $n_* = 0$ or 1 and $n_* \geq n$. Hence, we can rewrite the above equations as

$$\gamma_1 = 3^n\sqrt[3]{\alpha},$$
$$\gamma_2 = 3^n\sqrt[3]{\alpha'},$$
$$\gamma_1^* = 3^{n_*}\lambda\sqrt[3]{\alpha}, \text{ and}$$
$$\gamma_2^* = 3^{n_*}\lambda'\sqrt[3]{\alpha'}$$

Since $B_{k_2} = \{1, \lambda\}$ is an integral basis of $\mathbb{A}_{k_2}$, and since we herein assume k_2 to be a real quadratic field, say $k_2 = \mathbb{Q}(\sqrt{m})$ for some $m > 0$ in $\mathbb{Z}$, we can define λ in terms of $\sqrt{m}$ in one of two ways. Either $m \equiv 1 \pmod 4$, or $m \equiv 2$ or $3 \pmod 4$. In the first case, then we can write $\lambda = \frac{(1+\sqrt{m})}{2}$ and hence $\lambda' = \frac{(1-\sqrt{m})}{2} = 1 - \lambda$. In the second case, we can write $\lambda = \sqrt{m}$ and so then $\lambda' = -\lambda$. We will first deal with the second case, where $m \equiv 2$ or $3 \pmod 4$ and hence $\lambda' = -\lambda$. With the theorem above, we then find

$$\theta = \frac{1}{3}(3^n(\sqrt[3]{\alpha} + \sqrt[3]{\alpha'}) + S)$$

and

$$\theta^* = \frac{1}{3}(3^{n_*}\lambda(\sqrt[3]{\alpha} - \sqrt[3]{\alpha'}) + S^*)$$

Further, we recall that $\alpha = \beta^2\beta'$ and $\alpha' = \beta\beta'^2$. Additionally, since β and β' are in k_2, of which $\{1, \lambda\}$ is an integral basis, we can write $\beta = u + v\lambda$ and $\beta' = u - v\lambda$ for some $u, v \in \mathbb{Z}$. We then define $\mathfrak{N} = N_{k_2/\mathbb{Q}}(\beta) = \beta\beta'$.

Using our notes on α and β above with our formulae for θ and θ^*, we find the following representation of θ^2:

$$\theta^2 = \frac{1}{9}(3^{2n}(\sqrt[3]{\alpha^2} + 2\sqrt[3]{\alpha\alpha'} + \sqrt[3]{\alpha'^2}) + 2 \cdot 3^n(\sqrt[3]{\alpha} + \sqrt[3]{\alpha'}) + S^2)$$

which becomes

$$\theta^2 = \frac{1}{9}(3^{2n}(\beta\sqrt[3]{\alpha'} + \beta'\sqrt[3]{\alpha} + 2\mathfrak{N}) + 2 \cdot 3^n(\sqrt[3]{\alpha} + \sqrt[3]{\alpha'}) + S^2)$$

which, in turn, becomes

$$\theta^2 = \frac{1}{9}(3^{2n}(u(\sqrt[3]{\alpha} + \sqrt[3]{\alpha'}) - v\lambda(\sqrt[3]{\alpha} - \sqrt[3]{\alpha'}) + 2\mathfrak{N}) + 2 \cdot 3^n(\sqrt[3]{\alpha} + \sqrt[3]{\alpha'}) + S^2)$$

If we recall now that $3\theta - S = 3^n(\sqrt[3]{\alpha} + \sqrt[3]{\alpha'})$ and note that similarly, $3\theta_* - S_* = 3^{n_*}\lambda(\sqrt[3]{\alpha} - \sqrt[3]{\alpha'})$, then we can again rewrite the above as

$$\theta^2 = \frac{1}{9}(3^n u(3\theta - S) - 3^{2n-n_*}v(3\theta_* - S_*) + 2 \cdot 3^{2n}\mathfrak{N} + 2(3\theta - S) + S^2)$$

or equivalently as

$$9\theta^2 = 3(3^n u + 2)\theta - 3^{2n-n_*+1} v\theta_* - (3^n u + 2 + S)S + 3^{2n-n_*} vS^* + 2 \cdot 3^{2n}\mathfrak{N}$$

If we solve for $-\theta^*$ (which we will use as our third basis element θ_2), we find

$$\theta_2 = -\theta^* = \frac{9\theta^2 - 3(3^n u + 2)\theta + (3^n u + 2 + S)S - 3^{2n-n_*} vS^* - 2 \cdot 3^{2n}\mathfrak{N}}{3^{2n-n_*+1} v}$$

our third basis element in terms of our second when $m \equiv 2$ or $3 \pmod 4$.

If, on the other hand, $m \equiv 1 \pmod 4$, then $\lambda' = 1 - \lambda$. Herein we will write $\lambda = \frac{1}{2} + \frac{\sqrt{m}}{2}$, $\lambda' = \frac{1}{2} - \frac{\sqrt{m}}{2}$, $\beta = u(\frac{1}{2}) + v(\frac{\sqrt{m}}{2})$, and $\beta' = u(\frac{1}{2}) - v(\frac{\sqrt{m}}{2})$ for some $u, v \in \mathbb{Z}$. With all other notation as above, we then find that

$$\theta = \frac{1}{3}(3^n(\sqrt[3]{\alpha} + \sqrt[3]{\alpha'}) + S)$$

and

$$\theta^* = \frac{1}{3}(3^{n_*}(\frac{1}{2})((\sqrt[3]{\alpha} + \sqrt[3]{\alpha'}) + \sqrt{m}(\sqrt[3]{\alpha} - \sqrt[3]{\alpha'})) + S^*)$$

Through similar operations to those used above, we then calculate

$$\theta^2 = \frac{1}{9}(3^{2n}(\sqrt[3]{\alpha^2} + 2\sqrt[3]{\alpha\alpha'} + \sqrt[3]{\alpha'^2}) + 2 \cdot 3^n(\sqrt[3]{\alpha} + \sqrt[3]{\alpha'}) + S^2)$$

which in this case leads us to

$$\theta_2 = \theta^* = \frac{18\theta^2 - (3^{n+1}(u - v) + 12)\theta + (3^n u + 3^{2n-n_*} v + 4 - S)S + 3^{2n-n_*} vS^* - 4 \cdot 3^{2n}\mathfrak{N}}{2 \cdot 3^{2n-n_*+1} v}$$

our third basis element in terms of our second when $m \equiv 1 \pmod 4$.

In constructing integral bases for our cubic fields, we turn next to a method of Cremona [3] of form reduction. We use this method to find an alternate polynomial defining the same field, in order to write our integral basis in terms of a root of our new defining polynomial, giving our third basis element a smaller denominator.

First, we find an index form for our polynomial $f_K(\theta)$; this form indicates the size of the denominator of the third basis element in terms of a root θ of the polynomial $f_K(\theta)$. If we suppose that $B_K = \{1, \theta_1 = \theta, \theta_2\}$ is a root of $f_K(\theta)$, we obtain this index form by finding a polynomial of which $x\theta_1 + y\theta_2$ is a root. If we call this new polynomial (usually found through a Tschirnhausen transformation) $f_2(x, y, \theta)$, we then find the index form as follows. First, we divide the discriminant of $f_2(x, y, \theta)$ by the discriminant of the cubic field. The index form is then found by taking the square root of this quotient. We'll call this index form $g(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$.

In order to reduce the index form, and hence the size of the denominator of our basis for the field, we must first examine some properties of the index form. We first note that if x is a root of the polynomial $g(x, 1) = ax^3 + bx^2 + cx + d$, and if we multiply $g(x, 1)$ by a^2, then we find $a^2g(x, 1) = (a^3x^3) + b(a^2x^2) + ac(ax) + a^2d$. If we rewrite this polynomial as a function $g_{x,1}$ of ax, we find that ax is a root of the monic polynomial $g_{x,1}(ax) = (ax)^3 + b(ax)^2 + ac(ax) + a^2d$ or equivalently of $g_{x,1}(x) = x^3 + bx^2 + acx + a^2d$. Hence, we obtain the same cubic fields by adjoining a root of $g(x, 1)$, or $g_{x,1}(x)$, to $\mathbb{Q}$.

In addition, we see that if y is a root of the polynomial $g(1, y) = dy^3 + cy^2 + bx + a$, and if we multiply $g(1, y)$ by d^2, then we find $d^2g(1, y) = (d^3y^3) + c(d^2y^2) + bd(dy) + ad^2$. If we rewrite this polynomial as a function $g_{1,y}$ of dy, we find that dy is a root of the monic polynomial $g_{1,y}(dy) = (dy)^3 + c(dy)^2 + bd(dy) + ad^2$ or equivalently of $g_{1,y}(y) = y^3 + by^2 + acy + a^2d$. Hence, we obtain the same cubic fields by adjoining a root of $g(1, y)$ or $g_{1,y}(y)$ to $\mathbb{Q}$.

Further, we note that if we divide $g(x, 1) = 0$ by x^3 we find $a + b(\frac{1}{x}) + c(\frac{1}{x^2}) + d(\frac{1}{x^3}) = 0$ and so $\frac{1}{x}$ is a root of $g(1, y)$. With an appropriate choice of name, we can set $y = \frac{1}{x}$ and then, if we write $ax^3 + bx^2 + cx = -d$ for our root x of $g(x, 1)$ and divide by $-dx$ we find then that

$$y = \frac{1}{x} = -(\frac{ax^2}{d} + \frac{bx}{d} + \frac{c}{d}).$$

Alternately, writing a root y of $g_{1,y}$ in terms of a root x of $g_{x,1}$ we find similarly that

$$y = -(\frac{x^2}{a} + \frac{bx}{a} + c).$$

Work by Delone and Fadeev [4] guarantees $\{1, x, y\}$ is an integral basis of the field K. Further, we see that the denominator of our basis is the coefficient a of $g(x, y)$. Similarly, we can write a basis in terms of y in which our denominator is the coefficient d of $g(x, y)$.

Similarly, we can see that if we replace x by $x + n$ (for any integer n) in $g(x, 1)$ we find $a(x + n)^3 + b(x + n)^2 + c(x + n) + d$ which defines the same field as does $g(x, 1)$. Also, with appropriate replacements of $-x$ for x and a constant multiple of ± 1, both $-ax^3 - bx^2 - cx - d$, and $-ax^3 + bx^2 - cx + d$ define the same field as $g(x, 1)$. Furthermore, we can again find an integral basis for the field in terms of a root of any of these polynomials. In order to find a reduced form for a given polynomial (notably here the index form g(x,y) described by Martinet and Payan [12] earlier in the section), we implement an algorithm of Cremona's [3]. Only an outline of that algorithm is supplied here.

First, we define the following expressions in terms of a root, x of our index form $g(x,1) = ax^3 + bx^2 + cx + d$

$$h_0 = 9a^2x^2 + 6abx + 6ac - b^2$$

$$h_1 = 6abx^2 + 6(b^2 - ac)x + 2bc$$

$$h_2 = 3acx^2 + 3(bc - 3ad)x + 2c^2 - 3bd$$

Then, to find a reduced cubic polynomial equivalent to $f(x)$, we first find the nearest integer n to $-\frac{h_1}{2h_0}$, and replace $g(x,1)$ with $g(x+n,1)$, recomputing h_0, h_1, and h_2, using a root of our new polynomial. If $h_0 > h_2$, then replace $g(x,1)$ with $x^3g(-\frac{1}{x},1)$, (i.e replacing $g(x,1)$ with $g(1,y)$) again recomputing h_0, h_1, and h_2 using a root of the new polynomial and then begin the algorithm again. If, on the other hand $h_0 \leq h_2$, the algorithm terminates and the current polynomial $g(x,1)$ is our reduced polynomial. Afterward, we replace $g(x,1)$ by a monic polynomial defining the same field as described above.

Fundamental Units of the Cubic Fields and Their Normal Closures

Finally, we turn to a computation of the fundamental units of our cubic field K and of its normal closure L. We use a variant of the Voronoi unit algorithm [15] by Williams, Cormack, and Seah [16] to calculate the fundamental units of our cubic fields. Following this computation, we use work of Barrucand and Cohn [1] and Parry [13] to compute from ϵ_K, the unit of the cubic field, a unit of its normal closure L, which, taken with ϵ_K, gives us a system of the fundamental units of L. Here, note that we need two units to form this system since the group of units of L has rank 2.

First we note that Barrucand and Cohn's work [1] on principal factors in a cubic field describes the relation between the fundamental unit ϵ_K of a cubic field K with an element of its normal closure L. Their theorem stated below uses Hilbert's well-known Theorem 90 [9] to describe this relation.

Theorem 3.9 (P.Barrucand and H. Cohn (1971)) [1] *The fundamental unit ϵ_K of a cubic field K satisfies*

$$\epsilon_K = \frac{\mu}{\mu^\sigma}$$

for some unique (except for unit factors), primitive integer μ of its normal closure L.

Here we obtain μ according to the usual procedure of Hilbert's Theorem 90 [9], finding an element ν of L by

$$\nu = 1 + \epsilon_K + \epsilon_K \epsilon'_K$$

and then finding μ so that $\nu = \delta\mu$ for some $\delta \in k$ and where (μ) is made up of totally ramified primes of L or their squares. [1]

Barrucand and Cohn further describe the relation of this element μ to the fundamental unit ϵ_K of K and a system of fundamental units of L in a theorem classifying these fields into four types, depending on the value of the norm $N_{L/k}(\mu)$ [1]. Parry, in his work [13] further refines this classification by proving that only three of Barrucand and Cohn's types of fields actually exist. The remaining types of fields are described in Parry's theorem below.

Theorem 3.10 (C.Parry (1977)) [13] *A cubic field K and its normal closure L can be classified into three types depending only on μ.*

Type I. $N_{L/k}(\mu) = 1$.

Type III. $N_{L/k}(\mu)$ *is not a unit.*

Type IV. $N_{L/k}(\mu) = \omega$ *or* ω^2.

Here we concern ourselves only with cubic fields K for which the quadratic subfield k of the normal closure L is not $\mathbb{Q}(\sqrt{-3})$. Hence, we will never encounter fields of Type IV in our studies, as $\omega \notin k$. Further, Barrucand and Cohn [1], in their original classification define a unit index Q_0 which we will find useful in the next chapter and relate to the type of the cubic field K.

We let U_L denote the unit group of L as stated previously, and likewise $U_K, U_{K'}, U_{K''}$, and U_k are the unit groups of the fields K, K', K'', and k, respectively. Then we define $U_0 = U_K \times U_{K'} \times U_{K''} \times U_k$, and define the unit index $Q_0 = [U_L : U_0]$. The theorem below relates Q_0 to the type of the cubic field.

Theorem 3.11 (P.Barrucand and H.Cohn (1971)) [1] *The unit index Q_0 of the normal closure L of a cubic field K depends only on the type of the field defined previously as described below.*

Type I. $Q_0 = 3$.

Type III. $Q_0 = 1$.

Type IV. $Q_0 = 3$.

The notion of the type of a cubic field further leads us to a way to compute a system of fundamental units of L from the fundamental unit ϵ_K of K. Barrucand and Cohn [1] note that in a Type I cubic field we can find a primitive $\mu \in L$ (again using Hilbert's Theorem 90 [9]) so that

$$\epsilon_K = \frac{\mu}{\mu^\sigma}$$

and hence

$$\epsilon_K^\sigma = \frac{\mu^\sigma}{\mu^{\sigma^2}}$$

so then

$$\frac{\epsilon_K^\sigma}{\epsilon_K} = \frac{\mu^\sigma \mu^\sigma}{\mu\mu^{\sigma^2}} = \frac{\mu^\sigma \mu^\sigma \mu^\sigma}{\mu\mu^\sigma \mu^{\sigma^2}} = \frac{(\mu^\sigma)^3}{N_{L/k}(\mu)}$$

And hence we can find in Type I cubic fields, a system of fundamental units of the normal closure to be

$$\{\epsilon_K, \mu^\sigma\}$$

On the other hand, in a Type III cubic field, since the element $\mu \in L$ is not a unit, instead a system of fundamental units of the normal closure is

$$\{\epsilon_K, \epsilon_K^\sigma\}$$

Here, then, the units of L are completely determined by the units of its subfields.

Chapter 4

Results on the Class Ranks of a Cubic Field and That of Its Normal Closure

Cubic Fields Whose 3-Class Groups are Cyclic

In this chapter we discuss a method for determining the relation between the 3-class rank of a cubic field and that of its normal closure. These results depend on the work of Gerth [5], [6] and Lemmermeyer [10] and require an additional restriction that the class number of k_1 be relatively prime to 3. Otherwise, these results can be applied to any real cubic field with two nonreal conjugate fields, not merely those studied by Martinet and Payan [12], on which the rest of this paper focuses. We begin by examining the relation between the class group stucture of a cubic field and its normal closure when the cubic field's 3-class group is cyclic.

Theorem 4.1 *If $H_{3,K}$ is cyclic with $h_{3,K} = 3^n$, i.e., $H_{3,K} \simeq \mathbb{Z}_{3^n}$, then $H_{3,L} \simeq \mathbb{Z}_{3^n} \times \mathbb{Z}_{3^n}, H_{3,L} \simeq \mathbb{Z}_{3^{n-1}} \times \mathbb{Z}_{3^n}$, or $H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^{n-1}} \times \mathbb{Z}_{3^n}$.*

Proof Suppose $H_{3,K} \simeq \mathbb{Z}_{3^n}$. Then it follows from the well-known class number formula

$$h_L = \frac{1}{3} Q_0 h_K^2 h_k,$$

where Q_0 is a unit index that is either 1 or 3, that $h_{3,L} = 3^{2n}$ or 3^{2n-1}.

A result of Lemmermeyer [10] states that if 3 doesn't divide $[L : K]$, then $H_{3,L} \simeq H_{3,L/K} \times H_{3,K}$. Since $h_{3,L} = 3^{2n}$ or 3^{2n-1}, the order of $H_{3,L/K}$ is either 3^n or 3^{n-1}.

Now define a homomorphism $\rho : H_{3,K} \times H_{3,K'} \to H_{3,L}$ by $\rho(A, B) = AB$ and note that $H_{3,K} \simeq H_{3,K'}$. Since $\mathrm{Im}(\rho) \simeq \frac{H_{3,K} \times H_{3,K'}}{Ker(\rho)}$, in order to examine $\mathrm{Im}(\rho)$ we must first examine $\mathrm{Ker}(\rho)$.

Suppose $(A, B) \in \text{Ker}(\rho)$. Then $AB \sim 1$ and hence $A \sim B^{-1}$. Since $A \in K, A$ is fixed by τ and since $B \in K', B$ is fixed by $\sigma^2\tau$. Hence $A^\tau = A$ and $B^{\sigma^2\tau} = B$ and since $A \sim B^{-1}$, the ideal class of A is also fixed by $\sigma^2\tau$ and hence by σ, so $A^\sigma \sim A$. Similarly, $B^\tau \sim B$ and $B^\sigma \sim B$. Thus, both A and B are ambiguous ideal classes for L/k, so $A \sim \mathfrak{P}_1\mathfrak{P}_2 \ldots \mathfrak{P}_r$ for some $\mathfrak{P}_1, \mathfrak{P}_2, \ldots \mathfrak{P}_r$ ramified over k. Also, since $A^\tau = A$, $A = A^\tau \sim \mathfrak{P}_1^\tau\mathfrak{P}_2^\tau \ldots \mathfrak{P}_r^\tau$, so $A^2 \sim \mathfrak{P}_1\mathfrak{P}_1^\tau\mathfrak{P}_2\mathfrak{P}_2^\tau \ldots \mathfrak{P}_r\mathfrak{P}_r^\tau \sim P_1P_2 \ldots P_r$ where the $P_i = \mathfrak{P}_i\mathfrak{P}_i^\tau$ are prime ideals of K. Hence $B \sim A^{-1} \sim A^2 \sim P_1P_2 \ldots P_r$. Thus, $A \sim A^4 \sim P_1^2P_2^2 \ldots P_r^2$, and $A^3 \sim B^3 \sim 1$. Hence we see that $\text{Ker}(\rho) \subseteq \langle (P_i, P_i^2) : p = P_i^3 \text{ is totally ramified in } K\rangle$. Clearly also $\langle (P_i, P_i^2) : p = P_i^3 \text{ is totally ramified in } K\rangle \subseteq \text{Ker}(\rho)$, and hence $\text{Ker}(\rho) = \langle (P_i, P_i^2) : p = P_i^3$ is totally ramified in $K\rangle$.

Now, since $H_{3,K} \simeq \mathbb{Z}_{3^n}$ and since the orders of all elements of $\text{Ker}(\rho)$ divide 3, $\text{Ker}(\rho)$ is cyclic of order 3 or 1, so $\text{Ker}(\rho) \simeq \mathbb{Z}_3$ or $\text{Ker}(\rho) \simeq \{0\}$. Therefore,

$$\begin{aligned}\text{Im}(\rho) \simeq \frac{H_{3,K} \times H_{3,K'}}{\text{Ker}(\rho)} &\simeq \frac{H_{3,K} \times H_{3,K'}}{\{0\}} \\ &\simeq \frac{\mathbb{Z}_{3^n} \times \mathbb{Z}_{3^n}}{\{0\}} \\ &\simeq \mathbb{Z}_{3^n} \times \mathbb{Z}_{3^n}\end{aligned}$$

or

$$\begin{aligned}\text{Im}(\rho) \simeq \frac{H_{3,K} \times H_{3,K'}}{\text{Ker}(\rho)} &\simeq \frac{H_{3,K} \times H_{3,K'}}{\mathbb{Z}_3} \\ &\simeq \frac{\mathbb{Z}_{3^n} \times \mathbb{Z}_{3^n}}{\mathbb{Z}_3} \\ &\simeq \mathbb{Z}_{3^{n-1}} \times \mathbb{Z}_{3^n},\end{aligned}$$

where $\text{Im}(\rho)$ must be a subgroup of $H_{3,L}$. Further, since $H_{3,L} \simeq H_{3,K} \times H_{3,L/K}$ as we saw before, $\mathbb{Z}_{3^n}$ must be a factor of $H_{3,L}$. Hence, if $h_{3,L} = 3^{2n-1}$, $H_{3,L} \simeq \text{Im}(\rho) \simeq \mathbb{Z}_{3^{n-1}} \times \mathbb{Z}_{3^n}$. Otherwise, if $h_{3,L} = 3^{2n}$, either $H_{3,L} \simeq \mathbb{Z}_{3^n} \times \mathbb{Z}_{3^n}$ or $\mathbb{Z}_3 \times \mathbb{Z}_{3^{n-1}} \times \mathbb{Z}_{3^n}$.

We will use this Theorem and some Theorems of Gerth [5], [6] to prove several related results. These Theorems are stated below as Lemmas 4.2 and 4.3.

Lemma 4.2 (F. Gerth (1976)) [5] *Let* $S_A = \{\mathfrak{C} \in H_{3,L} : \mathfrak{C}^\sigma = \mathfrak{C}$ *and* $\psi_{3,L/k}(\mathfrak{C}) = 1\}$ *and* $S_A^- = \{\mathfrak{C} \in S_A : \mathfrak{C}^\tau = \mathfrak{C}^{-1}\}$. *Then*

$$\text{rank } H_{3,K} = \text{rank } S_A - \text{rank } \frac{S_A^-}{(S_A^- \cap H_{3,L}^{(1-\sigma)})}$$

Now let r *denote the number of totally ramified primes in* L/k *and let* $t = r - 1$. *If* $r = 0$ *or* 1, *rank* $\frac{S_A^-}{(S_A^- \cap H_{3,L}^{(1-\sigma)})} = 0$. *If* $r \geq 2$, *let* $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r$ *be the prime ideals of* k *which ramify in* L *and let* $\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_{s^*}$ *be ideals of* L *whose ideal classes generate* S_A^-. *Let* $N_{L/k}(\mathfrak{A}_i) = \mathfrak{a}_i = (x_i)$, *where* $x_i \in k$ *for* $1 \leq i \leq s^*$. *Let* E_k *be the group of units of* k. *Let* $\rho_j : k^\times \to$ *Gal(L/k) be defined by* $\rho_j(z) = (\frac{z,L/k}{\mathfrak{p}_j})$ *where* $k^\times = k - \{0\}$ *and* $(\frac{z,L/k}{\mathfrak{p}_j})$ *denotes the norm residue symbol, and let* $\rho : k^\times \to$ *Gal(L/k)*$\times$ *Gal(L/k)* $\times \ldots t$ *copies* $\ldots \times$ *Gal(L/k) be defined by* $\rho(z) = (\rho_1(z), \rho_2(z), \ldots, \rho_t(z)$. *Then*

$$s_1 = \text{rank } \frac{S_A^-}{(S_A^- \cap H_{3,L}^{(1-\sigma)})} = \text{rank } \{[(\frac{x_i, L/k}{\mathfrak{p}_j})] \bmod \rho(E_k)\}$$

where $[(\frac{x_i,L/k}{\mathfrak{p}_i})]$ *is the* $t \times s^*$ *matrix (over the finite field of three elements isomorphic to* $\mathbb{Z}_3$*) whose* i, j*-th element is the norm residue symbol* $(\frac{x_i,L/k}{\mathfrak{p}_i})$.

Here we note that ideals of $H_{3,L}$ whose ideal classes are in S_A all must satisfy

$$\mathfrak{A}^\sigma \sim \mathfrak{A}$$

and

$$N_{L/k}(\mathfrak{A}) = \mathfrak{A}\mathfrak{A}^\sigma\mathfrak{A}^{\sigma^2} = 1.$$

Further, those ideals whose classes are in S_A^- must satisfy the further restriction $\mathfrak{A}\mathfrak{A}^\tau \sim 1$ which yields

$$\mathfrak{A}^2\mathfrak{A}^\tau \sim \mathfrak{A}.$$

Hence, the $\mathfrak{a}_i$ in k lying under the ideals $\mathfrak{A}_i$ of L whose ideal classes generate S_A^- as described in Lemma 4.2 above can be decomposed into prime ideals $\mathfrak{p}_{i,l}$ of k which ramify totally in L. Further, since $\mathfrak{A}_i \sim {\mathfrak{A}_i}^2{\mathfrak{A}_i}^\tau$, as we saw above, then $\mathfrak{p}_{i,l} \sim \mathfrak{p}_{i,l}^2\mathfrak{p}_{i,l}^\tau$. Since every $\mathfrak{a}_i$ can be so decomposed, we can restate the result of the above lemma with

$$s_1 = \text{rank } \{[(\frac{\mathfrak{p}_i^2\mathfrak{p}_i^\tau, L/k}{\mathfrak{p}_j})] \bmod \rho(E_k)\}$$

where $\mathfrak{p}_i$ and $\mathfrak{p}_i^\tau$ are conjugate ideals. Finally, since S_A is the group of ambiguous ideal classes, rank $S_A = r - 1 = t$ and hence,

$$\text{rank } H_{3,K} = t - s_1.$$

Corollary 4.2.1 *$\frac{(t-1)}{2} \leq t - s_1 \leq t$. Also, $t - s_1 \leq t \leq 2(t - s_1) + 1$*

Proof Since $0 \leq s_1 \leq \frac{(t+1)}{2}, -\frac{(t+1)}{2} \leq -s_1 \leq 0$, and so $t - \frac{(t+1)}{2} \leq t - s_1 \leq t - 0$ or $\frac{(t-1)}{2} \leq t - s_1 \leq t$.

Further, since $\frac{(t-1)}{2} \leq t - s_1, t - 1 \leq 2(t - s_1)$, and so $t \leq 2(t - s_1) + 1$.

Lemma 4.3 (F. Gerth (1976)) [6] *Let L be a cyclic cubic extension of a (finite) number field k. Let σ be a generator of Gal(L/k). Let $H_{3,L}$ (respectively $H_{3,k}$) denote the 3-class group of L (respectively k). Suppose $H_{3,k} = \{(1)\}$. Let t denote the rank of ambiguous ideal classes $H_{3,L}^{(\sigma)}$ in $H_{3,L}$, and let $s =$ rank $\frac{(H_{3,L}^{\sigma} \cdot H_{3,L}^{1-\sigma})}{H_{3,L}^{1-\sigma}}$. Then rank $H_{3,L} = 2t - s$ and $H_{3,L}$ is isomorphic to the direct product of an abelian 3-group of rank $2(t - s)$ and an elementary abelian 3-group of rank s, where each element of the elementary abelian 3-group of rank s is an ambigous ideal class.*

First, we note that from Lemma 4.3 the rank of $H_{3,L}$ equals $2t - s$, where t is the rank of the group of ambiguous ideal classes in $H_{3,L}$ (i.e. one less than the number of primes in k that totally ramify in L) and s is the rank of the genus matrix $\frac{(H_{3,L}^{\sigma} \cdot H_{3,L}^{1-\sigma})}{H_{3,L}^{1-\sigma}}$. By a similar argument to that given in the proof of Lemma 4.2 [5], we can write

$$s = \text{rank } \{[(\frac{\mathfrak{p}_i, L/k}{\mathfrak{p}_j})] \text{ mod } \rho(E_k)\}$$

where the $\mathfrak{p}_i$ are ideals of k that ramify totally in L and ρ and E_k are defined as in Lemma 4.2.

Corollary 4.3.1 *$t \leq 2t - s \leq 2t$.*

Proof Since $0 \leq s \leq t, -t \leq -s \leq 0$, and so $2t - t \leq 2t - s \leq 2t - 0$ or $t \leq 2t - s \leq 2t$.

Corollary 4.3.2 *If $H_{3,K}$ and $H_{3,L}$ both have rank 1, $t = s = 1$ and $H_{3,K} = H_{3,L} \simeq \mathbb{Z}_3$, i.e. in the terms of Theorem 4.1, $n = 1$.*

Proof Suppose $H_{3,K}$ and $H_{3,L}$ both have rank 1. Hence, since from Lemma 4.3, $H_{3,L}$ has rank $2t - s = 1$, and by Corollary 4.3.1, $t \leq 2t - s \leq 2t$. Hence, if $2t - s = 1, t \leq 1 \leq 2t$, which can only hold for integer values of t if $t = 1$. So, since $2t - s = 1$ and $t = 1, s = 1$. Therefore, since $H_{3,L}$ has at least $s = 1$ factors of $\mathbb{Z}_3$, and since $H_{3,L}$ has rank 1, $H_{3,L} \simeq \mathbb{Z}_3$. Hence, $n = 1$.

Corollary 4.3.3 *If $H_{3,K}$ has rank 1 and $H_{3,L}$ has rank 2, either $t = 1$ and $s = 0$ or $t = s = 2$, in which case $H_{3,K} \simeq \mathbb{Z}_3$ and $H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$, i.e. in the terms of Theorem 4.1, $n = 1$.*

Proof By Corollary 4.3.1, $t \leq 2t - s \leq 2t$. Then we find that $t \leq 2 \leq 2t$, which holds for integer values of t only if $t = 1$ or $t = 2$.

If $t = 1$, then $2t - s = 2$, so $s = 0$.

If $t = 2$, then $2t - s = 2$, so $s = 2$. In this case, Lemma 4.3 tells us that $H_{3,L}$ has at least $s = 2$ factors of $\mathbb{Z}_3$, and since $H_{3,L}$ has rank 2, $H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$. Hence $n = 1$.

Corollary 4.3.4 *If $H_{3,K}$ has rank 1 and $H_{3,L}$ has rank 3, then $t = 2$ and $s = 1$ and $H_{3,L} \simeq \mathbb{Z}_{3^n} \times \mathbb{Z}_{3^{n-1}} \times \mathbb{Z}_3$, where, in the terms of Theorem 4.1, $n \geq 2$.*

Proof By Corollary 4.3.1, $t \leq 2t - s \leq 2t$. Then we find here that $t \leq 3 \leq 2t$, which holds for integer values of t only if $t = 2$ or $t = 3$.

If $t = 2$, then $2t - s = 3$, so $s = 1$. In this case, Lemma 4.3 tells us that $H_{3,L}$ has at least $s = 1$ factors of $\mathbb{Z}_3$, and since $H_{3,L}$ has rank 3, by Theorem 4.1, $H_{3,L} \simeq \mathbb{Z}_{3^n} \times \mathbb{Z}_{3^{n-1}} \times \mathbb{Z}_3$, where $n \geq 2$, as if $n < 2, H_{3,L}$ would only have rank 2.

If $t = 3$, then $2t - s = 3$, so $s = 3$. Here, Lemma 4.3 requires that $H_{3,L}$ have at least $s = 3$ factors of $\mathbb{Z}_3$. But Theorem 4.1 only allows $H_{3,L}$ to have rank 3 if $H_{3,L} \simeq \mathbb{Z}_{3^n} \times \mathbb{Z}_{3^{n-1}} \times \mathbb{Z}_3$. However, this would mean that $\mathbb{Z}_{3^n} \simeq \mathbb{Z}_{3^{n-1}} \simeq \mathbb{Z}_3$, which requires $n - 1 = n = 1$, and so is impossible.

Lemma 4.4 *If we define a homomorphism $\rho : H_{3,K} \times H_{3,K'} \to H_{3,L}$ by $\rho(A, B) = AB$ as above, then $H_{3,L}^3 \subseteq Im(\rho)$.*

Proof Suppose $\mathfrak{C}$ is an ideal class in $H_{3,L}$. Note then that if $C_1 = \mathfrak{C}\mathfrak{C}^{\tau}, C_2 = \mathfrak{C}\mathfrak{C}^{\sigma\tau}$, and $C_3 = \mathfrak{C}\mathfrak{C}^{\sigma^2\tau}$ are classes in $H_{3,K}, H_{3,K'}$, and $H_{3,K''}$, respectively, then $C_1C_2C_3 = (\mathfrak{C}\mathfrak{C}^{\tau})(\mathfrak{C}\mathfrak{C}^{\sigma\tau})(\mathfrak{C}\mathfrak{C}^{\sigma^2\tau}) = \mathfrak{C}^3(\mathfrak{C}^{1+\sigma+\sigma^2})^{\tau}$. Now since $\mathfrak{c} = \mathfrak{C}^{1+\sigma+\sigma^2}$ is a class in $H_{3,k}$, and since $h_{3,k}$ is not divisible by 3, $\mathfrak{c} = \mathfrak{C}^{1+\sigma+\sigma^2} \sim 1$. Therefore, $C_1C_2C_3 = \mathfrak{C}^3(\mathfrak{C}^{1+\sigma+\sigma^2})^{\tau} \sim \mathfrak{C}^3$.

If we note that $C_3C_3^{\sigma}C_3^{\sigma^2} \sim (\mathfrak{C}\mathfrak{C}^{\sigma^2\tau})(\mathfrak{C}\mathfrak{C}^{\sigma^2\tau})^{\sigma}(\mathfrak{C}\mathfrak{C}^{\sigma^2\tau})^{\sigma^2} \sim (\mathfrak{C}\mathfrak{C}^{\sigma^2\tau})^{1+\sigma+\sigma^2} \sim 1$, then $C_3 \sim C_3^{-\sigma}C_3^{-\sigma^2}$. Since $\sigma : K'' \to K$ and $\sigma^2 : K'' \to K'$, then $C_3 \sim C_3^{-\sigma}C_3^{-\sigma^2} = B_1B_2$, where $B_1 = C_3^{-\sigma} \in H_{3,K}$ and $B_2 = C_3^{-\sigma^2} \in H_{3,K'}$. So $\mathfrak{C}^3 \sim C_1C_2C_3 \sim C_1C_2B_1B_2 = (B_1C_1)(B_2C_2)$. Therefore, $\rho(B_1C_1, B_2C_2) = (B_1C_1)(B_2C_2) \sim C^3$, and so $H_{3,L}^3 \subseteq \mathrm{Im}(\rho)$.

Cubic Fields Whose 3-Class Groups Have Rank 2

We continue by examining the relation between the class group stucture of a cubic field and its normal closure when the cubic field's 3-class group has rank 2. In order to do so, we will use an argument analogous to that given in Theorem 4.1 and apply the Lemmas above. After proving the Theorem below, we will prove some related results that describe particular relations between the class group structures of the cubic field and its normal closure when their 3-class groups are of specific ranks.

Theorem 4.5 *If $H_{3,K}$ has rank 2 with $h_{3,K} = 3^{n_1+n_2}$,i.e., $H_{3,K} \simeq \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, then*
$H_{3,L} \simeq \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$,
$H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$,
$H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2+1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$,
$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$,
$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, *or*
$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$.

Proof Suppose $H_{3,K} \simeq \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$. As with Theorem 4.1, then it follows from the well-known class number formula

$$h_L = \frac{1}{3} Q_0 h_K^2 h_k,$$

where Q_0 is a unit index that is either 1 or 3, that $h_{3,L} = 3^{2(n_1+n_2)}$ or $3^{2(n_1+n_2)-1}$.

Again, the result of Lemmermeyer [10] used in Theorem 4.1 states that since 3 doesn't divide $[L : K]$, $H_{3,L} \simeq H_{3,L/K} \times H_{3,K}$. Since $h_{3,L} = 3^{2(n_1+n_2)}$ or $3^{2(n_1+n_2)-1}$, the order of $H_{3,L/K}$ is either $3^{n_1+n_2}$ or $3^{n_1+n_2-1}$.

As in Theorem 4.1, define a homomorphism $\rho : H_{3,K} \times H_{3,K'} \to H_{3,L}$ by $\rho(A, B) = AB$ and note that $H_{3,K} \simeq H_{3,K'}$. Again note that $\mathrm{Im}(\rho) \simeq \frac{H_{3,K} \times H_{3,K'}}{Ker(\rho)}$. As in Theorem 4.1, $\mathrm{Ker}(\rho) = \langle (P_i, P_i^2) : p = P_i^3$ is totally ramified in $K\rangle$.

Now, since $H_{3,K} \simeq \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, and since the orders of all elements of $\mathrm{Ker}(\rho)$ divide 3, $\mathrm{Ker}(\rho)$ is isomorphic to an elementary 3-group of rank 0, 1, or 2, so $\mathrm{Ker}(\rho) \simeq \{0\}$, $\mathrm{Ker}(\rho) \simeq \mathbb{Z}_3$ or $\mathrm{Ker}(\rho) \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$. Therefore, without loss of generality (by renaming n_1 and n_2 if necessary),

$$\mathrm{Im}(\rho) \simeq \frac{H_{3,K} \times H_{3,K'}}{\mathrm{Ker}(\rho)} \simeq \frac{H_{3,K} \times H_{3,K'}}{\{0\}}$$

$$\simeq \frac{\mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}}{\{0\}}$$

$$\simeq \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}},$$

or

$$\mathrm{Im}(\rho) \simeq \frac{H_{3,K} \times H_{3,K'}}{\mathrm{Ker}(\rho)} \simeq \frac{H_{3,K} \times H_{3,K'}}{\mathbb{Z}_3}$$

$$\simeq \frac{\mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}}{\mathbb{Z}_3}$$

$$\simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}},$$

or

$$\mathrm{Im}(\rho) \simeq \frac{H_{3,K} \times H_{3,K'}}{\mathrm{Ker}(\rho)} \simeq \frac{H_{3,K} \times H_{3,K'}}{\mathbb{Z}_3 \times \mathbb{Z}_3}$$

$$\simeq \frac{\mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}}{\mathbb{Z}_3 \times \mathbb{Z}_3}$$

$$\simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}},$$

where $\mathrm{Im}(\rho)$ must be a subgroup of $H_{3,L}$. Further, since $H_{3,L} \simeq H_{3,K} \times H_{3,L/K}$ as we saw before, $\mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$ must be a factor of $H_{3,L}$. In addition, from Lemma 4.4, since $H_{3,L}^3 \subseteq \mathrm{Im}(\rho)$, if we know that $\mathrm{Im}(\rho) \simeq \mathbb{Z}_{3^{m_1}} \times \mathbb{Z}_{3^{m_2}} \times \ldots \mathbb{Z}_{3^{m_r}}$, then $H_{3,L} \subseteq \mathbb{Z}_3 \times \mathbb{Z}_3 \times \ldots \times \mathbb{Z}_3 \times \mathbb{Z}_{3^{m_1+1}} \times \mathbb{Z}_{3^{m_2+1}} \times \ldots \mathbb{Z}_{3^{m_r+1}}$.

Hence, without loss of generality (again renaming n_1 and n_2 if necessary), if $h_{3,L} = 3^{2(n_1+n_2)}$ and $\mathrm{Im}(\rho) \simeq \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, then

$H_{3,L} \simeq \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$.

Further, if $h_{3,L} = 3^{2(n_1+n_2)}$ and $\mathrm{Im}(\rho) \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, then

$H_{3,L} \simeq \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$,

$H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2+1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, or

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$.

Finally, if $h_{3,L} = 3^{2(n_1+n_2)}$ and $\mathrm{Im}(\rho) \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, then

$H_{3,L} \simeq \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$,

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, or

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$.

If, on the other hand, $h_{3,L} = 3^{2(n_1+n_2)-1}$ and $\mathrm{Im}(\rho) \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, then

$H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$.

Further, if $h_{3,L} = 3^{2(n_1+n_2)-1}$ and $\mathrm{Im}(\rho) \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, then

$H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$ or

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$.

We will again use this Theorem along with Lemmas 4.2 and 4.3 to prove some related results. In these results, we will be particularly concerned with fields K and L for which $t \leq 3$ (although the Theorem above applies to fields in which $t > 3$ as well), as we will be primarily examining these fields in later sections.

Corollary 4.5.1 *In the situation described in Theorem* 4.5 *above,* $2 \leq t \leq 5$.

Proof In Theorem 4.5, since $H_{3,K}$ has rank 2, we know that by Lemma 4.2, $t - s_1 = 2$. Then by Corollary 4.2.1, we have that $(t - s_1) \leq t \leq 2(t - s_1) + 1$, or here $2 \leq t \leq 5$.

In the following Corollaries, we will only consider the cases where $t = 2$ or $t = 3$. Here we should first examine the possible ranks of $H_{3,L}$ when $t = 2$ or $t = 3$.

First, note that if $t = 2$, since $H_{3,K}$ has rank $t - s_1 = 2$, then $s_1 = 0$. Further, since $0 \leq s_1 \leq s \leq t$, then $0 \leq s \leq 2$. Hence, since we know that the rank of $H_{3,L}$ is $2t - s$, these possible values for s give us the following as the only possible ranks for $H_{3,L}$ when $t = 2$.

If $s = 0$, the rank of $H_{3,L}$ is $2t - s = 4$.

If $s = 1$, the rank of $H_{3,L}$ is $2t - s = 3$.

If $s = 2$, the rank of $H_{3,L}$ is $2t - s = 2$.

Secondly, note that if $t = 3$, since $H_{3,K}$ has rank $t - s_1 = 2$, then $s_1 = 1$. Further, since $0 \leq s_1 \leq s \leq t$, then $1 \leq s \leq 3$. Hence, since we know that the rank of $H_{3,L}$ is $2t - s$, these possible values for s give us the following as the only possible ranks for $H_{3,L}$ when $t = 3$.

If $s = 1$, the rank of $H_{3,L}$ is $2t - s = 5$.

If $s = 2$, the rank of $H_{3,L}$ is $2t - s = 4$.

If $s = 3$, the rank of $H_{3,L}$ is $2t - s = 3$.

Corollary 4.5.2 *If $H_{3,K}$ has rank 2, $H_{3,L}$ cannot also have rank 2.*

Proof Since $H_{3,K}$ has rank 2, clearly both n_1 and n_2 (in the terms of Theorem 4.5) must be greater than zero. Therefore factors of $\mathbb{Z}_3$, $\mathbb{Z}_{3^{n_1}}$, $\mathbb{Z}_{3^{n_2}}$, and $\mathbb{Z}_{3^{n_2+1}}$ in $H_{3,L}$ must each contribute to its rank (as they are not isomorphic to $\{0\}$). Therefore, by Theorem 4.5, the rank of $H_{3,L}$ must be at least 3.

Corollary 4.5.3 *If $H_{3,K}$ has rank 2 and $H_{3,L}$ has rank 3 (and $t \leq 3$), then either $t = 2$ and $s = 1$ or $t = s = 3$. In either case, $H_{3,L}$ must satisfy one of the following:*

$H_{3,L} \simeq \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_2}}$ or

$H_{3,L} \simeq \mathbb{Z}_{3^{n_2+1}} \times \mathbb{Z}_3 \times \mathbb{Z}_{3^{n2}}$, where, in the terms of Theorem 2.5, $n_1 = 1$.

Further, if $t = s = 3$, we know further that $H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, where, in the terms of Theorem 4.5, $n_1 = n_2 = 1$.

Proof Again, since $H_{3,K}$ has rank 2, clearly both n_1 and n_2 (in the terms of Theorem 4.5) must be greater than zero. So, again, factors of $\mathbb{Z}_3$, $\mathbb{Z}_{3^{n_1}}$, $\mathbb{Z}_{3^{n_2}}$, and $\mathbb{Z}_{3^{n_2+1}}$ in $H_{3,L}$ must each contribute to its rank (as they are not isomorphic to $\{0\}$). Hence, we find that $H_{3,L}$ can only have rank 3 if

$H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, where $n_1 = 1$,

$H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2+1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, where $n_1 = 1$, or

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, where $n_1 = n_2 = 1$, which duplicates the first possibility with the additional condition $n_2 = 1$.

Furthermore, if $t = 2$ and $s = 1$, Lemma 4.3 tells us that we must have at least $s = 1$ factors of $\mathbb{Z}_3$, which is satisfied in any of the above cases, since $n_1 = 1$. However, if $t = s = 3$, Lemma 4.3 tells us that we must have at least $s = 3$ factors of $\mathbb{Z}_3$, which requires that $n_1 = n_2 = 1$ and hence $H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, so $H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

Corollary 4.5.4 *If $H_{3,K}$ has rank 2 and $H_{3,L}$ has rank 4 (and $t \leq 3$), then either $t = 2$ and $s = 0$ or $t = 3$ and $s = 2$. In either case, $H_{3,L}$ must satisfy one of the following:*

$H_{3,L} \simeq \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$,

$H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, where $n_1 \geq 2$, or

$H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2+1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, where $n_1 \geq 2$.

Further, if $t = 3$ and $s = 2$, then either $H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_2}}$, where, in the terms of Theorem 4.5, $n_1 = 1$, or $H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_3$, where, in the terms of Theorem 4.5, $n_1 \geq 2$ and $n_2 = 1$.

Proof Again, since $H_{3,K}$ has rank 2, clearly both n_1 and n_2 (in the terms of Theorem 4.5) must be greater than zero. So, again, factors of $\mathbb{Z}_3$, $\mathbb{Z}_{3^{n_1}}$, $\mathbb{Z}_{3^{n_2}}$, and $\mathbb{Z}_{3^{n_2+1}}$ in $H_{3,L}$ must each contribute to its rank (as they are not isomorphic to $\{0\}$). Hence, we find that $H_{3,L}$ can only have rank 4 if

$H_{3,L} \simeq \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$,

$H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, where $n_1 \geq 2$,

$H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2+1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, where $n_1 \geq 2$,

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, where $n_1 = 1$, noting that this possibility duplicates the first with the additional condition $n_1 = 1$,

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, where $n_1 \geq 2$ and $n_2 = 1$, noting that this possibility duplicates the second with the additional condition $n_2 = 1$, or

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, where $n_1 = n_2 = 1$, noting that this possibility duplicates the first with the additional conditions $n_1 = n_2 = 1$.

Hence we can see that $H_{3,L}$ only has rank 4 if either $H_{3,L} \simeq \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, $H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, where $n_1 \geq 2$, or $H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2+1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, where $n_1 \geq 2$.

If $t = 3$ and $s = 2$, then we also know from Lemma 4.3 that $H_{3,L}$ must have $s = 2$ factors of $\mathbb{Z}_3$. Hence, $H_{3,L}$ can only have rank 4 with $t = 3$ and $s = 2$ if either $H_{3,L} \simeq \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, where $n_1 = 1$, $H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, where $n_1 \geq 2$ and $n_2 = 1$, or $H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2+1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, where $n_1 = 2$ and $n_2 = 1$, which, if we rename n_1 and n_2, duplicates the first possibility, with the additional condition $n_1 = 2$.

Corollary 4.5.5 *If $H_{3,K}$ has rank 2 and $H_{3,L}$ has rank 5 (and $t \leq 3$), then $t = 3$ and $s = 1$, and $H_{3,L}$ must satisfy one of the following:*

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, where $n_1 \geq 2$, or

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, where $n_1 \geq 2$ and $n_2 \geq 2$.

Proof Again, since $H_{3,K}$ has rank 2, clearly both n_1 and n_2 (in the terms of Theorem 4.5) must be greater than zero. So, again, factors of $\mathbb{Z}_3$, $\mathbb{Z}_{3^{n_1}}$, $\mathbb{Z}_{3^{n_2}}$, and $\mathbb{Z}_{3^{n_2+1}}$ in $H_{3,L}$ must each contribute to its rank (as they are not isomorphic to $\{0\}$). Hence, we find that $H_{3,L}$ can only have rank 5 if

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, where $n_1 \geq 2$,

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, where $n_1 \geq 2$ and $n_2 \geq 2$, or

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, where $n_1 \geq 2$ and $n_2 = 1$, noting that this possibility duplicates the first with the additional condition $n_2 = 1$.

Hence we can see that $H_{3,L}$ only has rank 5 if either $H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, where $n_1 \geq 2$ or $H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, where $n_1 \geq 2$ and $n_2 \geq 2$.

Further, since $t = 3$ and $s = 1$, then we also know from Lemma 4.3 that $H_{3,L}$ must have $s = 1$ factors of $\mathbb{Z}_3$. Since each of the above possibilities for $H_{3,L}$ with rank 5 already has at least one factor of $\mathbb{Z}_3$, there are no further conditions imposed because $s = 2$.

Cubic Fields Whose 3-Class Groups Have Rank 3

We end this chapter with an examination of the relation between the class group stucture of a cubic field and its normal closure when the cubic field's 3-class group has rank 3. We will again use an argument analogous to that given in Theorem 4.1 and apply Lemmas 4.2, 4.3, and 4.4. Finally, after proving this Theorem , we will prove Corollaries that describe further relations between the class group structures of the cubic field and its normal closure in particular cases.

Theorem 4.6 *If $H_{3,K}$ has rank 3 with $h_{3,K} = 3^{n_1+n_2+n_3}$, i.e., $H_{3,K} \simeq \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$ then*
$H_{3,L} \simeq \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$,
$H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$,
$H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2+1}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$,
$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$,
$H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3+1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$,
$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$,
$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3+1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$,
$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$,
$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3-1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, *or*
$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3-1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$.

Proof Suppose $H_{3,K} \simeq \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$. As with Theorem 4.1, then it follows from the well-known class number formula

$$h_L = \frac{1}{3} Q_0 h_K^2 h_k,$$

where Q_0 is a unit index that is either 1 or 3, that $h_{3,L} = 3^{2(n_1+n_2+n_3)}$ or $3^{2(n_1+n_2+n_3)-1}$.

Again, the result of Lemmermeyer [10] used in Theorem 4.1 states that since 3 doesn't divide $[L : K]$, $H_{3,L} \simeq H_{3,L/K} \times H_{3,K}$. Since $h_{3,L} = 3^{2(n_1+n_2+n_3)}$ or $3^{2(n_1+n_2+n_3)-1}$, the order of $H_{3,L/K}$ is either $3^{n_1+n_2+n_3}$ or $3^{n_1+n_2+n_3-1}$.

As in Theorem 4.1, define a homomorphism $\rho : H_{3,K} \times H_{3,K'} \to H_{3,L}$ by $\rho(A, B) = AB$ and note that $H_{3,K} \simeq H_{3,K'}$. Again note that $\mathrm{Im}(\rho) \simeq \frac{H_{3,K} \times H_{3,K'}}{Ker(\rho)}$. As in Theorem 4.1, $\mathrm{Ker}(\rho) = \langle (P_i, P_i^2) : p = P_i^3 \text{ is totally ramified in } K\rangle$.

Now, since $H_{3,K} \simeq \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_2}}$, and since the orders of all elements of $\mathrm{Ker}(\rho)$ divide 3, $\mathrm{Ker}(\rho)$ is isomorphic to an elementary 3-group of rank $0, 1, 2$, or 3, so $\mathrm{Ker}(\rho) \simeq \{0\}$, $\mathrm{Ker}(\rho) \simeq \mathbb{Z}_3$, $\mathrm{Ker}(\rho) \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$, or $\mathrm{Ker}(\rho) \simeq \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. Therefore, without loss of generality (by renaming n_1, n_2, and n_3 if necessary),

$$Im(\rho) \simeq \frac{H_{3,K} \times H_{3,K'}}{Ker(\rho)} \simeq \frac{H_{3,K} \times H_{3,K'}}{\{0\}}$$

$$\simeq \frac{\mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}}{\{0\}}$$

$$\simeq \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_2}},$$

or

$$Im(\rho) \simeq \frac{H_{3,K} \times H_{3,K'}}{Ker(\rho)} \simeq \frac{H_{3,K} \times H_{3,K'}}{\mathbb{Z}_3}$$

$$\simeq \frac{\mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}}{\mathbb{Z}_3}$$

$$\simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}},$$

or

$$Im(\rho) \simeq \frac{H_{3,K} \times H_{3,K'}}{Ker(\rho)} \simeq \frac{H_{3,K} \times H_{3,K'}}{\mathbb{Z}_3 \times \mathbb{Z}_3}$$

$$\simeq \frac{\mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}}{\mathbb{Z}_3 \times \mathbb{Z}_3}$$

$$\simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}},$$

or

$$Im(\rho) \simeq \frac{H_{3,K} \times H_{3,K'}}{Ker(\rho)} \simeq \frac{H_{3,K} \times H_{3,K'}}{\mathbb{Z}_3 \times \mathbb{Z}_3}$$

$$\simeq \frac{\mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}}{\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3}$$

$$\simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3-1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}},$$

where Im(ρ) must be a subgroup of $H_{3,L}$. Further, since $H_{3,L} \simeq H_{3,K} \times H_{3,L/K}$ as we saw before, $\mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$ must be a factor of $H_{3,L}$. In addition, from Lemma 4.4, since $H_{3,L}^3 \subseteq$ Im(ρ), if we know that Im(ρ) $\simeq \mathbb{Z}_{3^{m_1}} \times \mathbb{Z}_{3^{m_2}} \times \ldots \mathbb{Z}_{3^{m_r}}$, then $H_{3,L} \subseteq \mathbb{Z}_3 \times \mathbb{Z}_3 \times \ldots \times \mathbb{Z}_3 \times \mathbb{Z}_{3^{m_1+1}} \times \mathbb{Z}_{3^{m_2+1}} \times \ldots \mathbb{Z}_{3^{m_r+1}}$.

Hence, without loss of generality (again renaming n_1, n_2, and n_3 if necessary), if $h_{3,L} = 3^{2(n_1+n_2+n_3)}$ and Im(ρ) $\simeq \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, then

$H_{3,L} \simeq \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$.

Further, if $h_{3,L} = 3^{2(n_1+n_2+n_3)}$ and Im(ρ) $\simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, then

$H_{3,L} \simeq \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$,

$H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2+1}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, or

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$.

Further, if $h_{3,L} = 3^{2(n_1+n_2+n_3)}$ and Im(ρ) $\simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, then

$H_{3,L} \simeq \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$,

$H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3+1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$ (or $H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2+1}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$ if we rename n_2 and n_3),

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$,

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3+1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, or

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$.

Finally, if $h_{3,L} = 3^{2(n_1+n_2+n_3)}$ and Im(ρ) $\simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3-1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, then

$H_{3,L} \simeq \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$,

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$,

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, or

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3-1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$.

If, on the other hand, $h_{3,L} = 3^{2(n_1+n_2+n_3)-1}$ and $\text{Im}(\rho) \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, then

$H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$.

Further, if $h_{3,L} = 3^{2(n_1+n_2+n_3)-1}$ and $\text{Im}(\rho) \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, then

$H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$,

$H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3+1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, or

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$.

Finally, if $h_{3,L} = 3^{2(n_1+n_2+n_3)-1}$ and $\text{Im}(\rho) \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3-1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, then

$H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$,

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, or

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3-1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$.

We will again use this Theorem along with Lemmas 4.2 and 4.3 to prove some related results. Again, we will focus on fields K and L for which $t \leq 3$.

Corollary 4.6.1 *In the situation described in Theorem* 4.6 *above,* $3 \leq t \leq 7$.

Proof In Theorem 4.6, since $H_{3,K}$ has rank 3, we know that by Lemma 4.2, $t - s_1 = 3$. Then by Corollary 4.2.1, we have that $(t - s_1) \leq t \leq 2(t - s_1) + 1$, or here $3 \leq t \leq 7$.

In the following Corollaries, we will only consider the case where $t = 3$. Note that if $t = 3$, since $H_{3,K}$ has rank $t - s_1 = 3$, then $s_1 = 0$. Further, since $0 \leq s_1 \leq s \leq t$, then $0 \leq s \leq 3$. Hence, since we know that the rank of $H_{3,L}$ is $2t - s$, these possible values for s give us the following as the only possible ranks for $H_{3,L}$ when $t = 3$.

If $s = 0$, the rank of $H_{3,L}$ is $2t - s = 6$.

If $s = 1$, the rank of $H_{3,L}$ is $2t - s = 5$.

If $s = 2$, the rank of $H_{3,L}$ is $2t - s = 4$.

If $s = 3$, the rank of $H_{3,L}$ is $2t - s = 3$.

Corollary 4.6.2 *If* $H_{3,K}$ *has rank* 3*,* $H_{3,L}$ *cannot also have rank* 3*.*

Proof Since $H_{3,K}$ has rank 3, clearly n_1, n_2, and n_3 (in the terms of Theorem 4.6) must be greater than zero. Therefore factors of $\mathbb{Z}_3$, $\mathbb{Z}_{3^{n_1}}$, $\mathbb{Z}_{3^{n_2}}$, $\mathbb{Z}_{3^{n_3}}$, $\mathbb{Z}_{3^{n_2+1}}$, and $\mathbb{Z}_{3^{n_3+1}}$ in $H_{3,L}$ must each contribute to its rank (as they are not isomorphic to $\{0\}$). Therefore, by Theorem 4.6, the rank of $H_{3,L}$ must be at least 4.

Corollary 4.6.3 *If $H_{3,K}$ has rank 3 and $H_{3,L}$ has rank 4 (and $t \leq 3$), then $t = 3$ and $s = 2$ and $H_{3,L} \simeq \mathbb{Z}_{3^{n_3+1}} \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_3}}$, where, in the terms of Theorem 4.6, $n_1 = n_2 = 1$.*

Proof Again, since $H_{3,K}$ has rank 3, clearly n_1, n_2, and n_3 (in the terms of Theorem 4.6) must be greater than zero. So, again, factors of $\mathbb{Z}_3$, $\mathbb{Z}_{3^{n_1}}$, $\mathbb{Z}_{3^{n_2}}$, $\mathbb{Z}_{3^{n_3}}$, $\mathbb{Z}_{3^{n_2+1}}$, and $\mathbb{Z}_{3^{n_3+1}}$ in $H_{3,L}$ must each contribute to its rank (as they are not isomorphic to $\{0\}$). Hence, we find that $H_{3,L}$ can only have rank 4 if $H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3+1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, where, in the terms of Theorem 4.6, $n_1 = n_2 = 1$. Hence, $H_{3,L} \simeq \mathbb{Z}_{3^{n_3+1}} \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_3}}$,

Further, since $t = 3$ and $s = 2$, then we also know from Lemma 4.3 that $H_{3,L}$ must have $s = 2$ factors of $\mathbb{Z}_3$. Since $H_{3,L}$ must already have two factors of $\mathbb{Z}_3$, as shown above, in order to have rank 4, there are no further conditions imposed because $s = 2$.

Corollary 4.6.4 *If $H_{3,K}$ has rank 3 and $H_{3,L}$ has rank 5 (and $t \leq 3$), then $t = 3$ and $s = 1$ and $H_{3,L}$ must satisfy one of the following:*

$H_{3,L} \simeq \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, where, in the terms of Theorem 4.6, $n_1 = 1$,

$H_{3,L} \simeq \mathbb{Z}_{3^{n_2+1}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, where, in the terms of Theorem 4.6, $n_1 = 1$, or

$H_{3,L} \simeq \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, where, in the terms of Theorem 4.6, $n_1 = 1$ and $n_2 \geq 2$.

Proof Again, since $H_{3,K}$ has rank 3, clearly n_1, n_2, and n_3 (in the terms of Theorem 4.6) must be greater than zero. So, again, factors of $\mathbb{Z}_3$, $\mathbb{Z}_{3^{n_1}}$, $\mathbb{Z}_{3^{n_2}}$, $\mathbb{Z}_{3^{n_3}}$, $\mathbb{Z}_{3^{n_2+1}}$, and $\mathbb{Z}_{3^{n_3+1}}$ in $H_{3,L}$ must each contribute to its rank (as they are not isomorphic to $\{0\}$). Hence, we find that $H_{3,L}$ can only have rank 5 if

$H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, where $n_1 = 1$,

$H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2+1}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, where $n_1 = 1$,

$H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3+1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, where $n_1 = 1$ and $n_2 \geq 2$,

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, where $n_1 = n_2 = 1$, noting that this possibility duplicates the first with the additional condition $n_2 = 1$,

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3+1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, where $n_1 = n_2 = 1$, noting that this possibility duplicates the second if we rename n_2 and n_3 and with the additional condition $n_2 = 1$, or

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3-1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, where $n_1 = n_2 = n_3 = 1$, noting that this possibility duplicates the first with the additional conditions $n_2 = n_3 = 1$.

Hence we can see that $H_{3,L}$ only has rank 5 if either $H_{3,L} \simeq \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, $H_{3,L} \simeq \mathbb{Z}_{3^{n_2+1}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, or $H_{3,L} \simeq \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, where $n_2 \geq 2$.

Since $t = 3$ and $s = 1$, then we also know from Lemma 4.3 that $H_{3,L}$ must have $s = 1$ factors of $\mathbb{Z}_3$. Since $H_{3,L}$ must already have one factor of $\mathbb{Z}_3$, as shown above, in order to have rank 5, there are no further conditions imposed because $s = 1$.

Corollary 4.6.5 *If $H_{3,K}$ has rank 3 and $H_{3,L}$ has rank 6 (and $t \leq 3$), then $t = 3$ and $s = 0$ and $H_{3,L}$ must satisfy one of the following:*

$H_{3,L} \simeq \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$,

$H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, *where* $n_1 \geq 2$,

$H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2+1}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, *where* $n_1 \geq 2$, *or*

$H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3+1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, *where both* $n_1 \geq 2$ *and* $n_2 \geq 2$.

Proof Again, since $H_{3,K}$ has rank 3, clearly n_1, n_2, and n_3 (in the terms of Theorem 4.6) must be greater than zero. So, again, factors of $\mathbb{Z}_3$, $\mathbb{Z}_{3^{n_1}}$, $\mathbb{Z}_{3^{n_2}}$, $\mathbb{Z}_{3^{n_3}}$, $\mathbb{Z}_{3^{n_2+1}}$, and $\mathbb{Z}_{3^{n_3+1}}$ in $H_{3,L}$ must each contribute to its rank (as they are not isomorphic to $\{0\}$). Hence, we find that $H_{3,L}$ can only have rank 6 if

$H_{3,L} \simeq \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$,

$H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, where $n_1 \geq 2$,

$H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2+1}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, where $n_1 \geq 2$,

$H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3+1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, where both $n_1 \geq 2$ and $n_2 \geq 2$,

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, where $n_1 = 1$, noting that this possibility duplicates the first with the additional condition $n_1 = 1$,

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, where $n_1 \geq 2$ and $n_2 = 1$, noting that this possibility duplicates the second with the additional condition $n_2 = 1$,

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3+1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, where $n_1 \geq 2$ and $n_2 = 1$, noting that this possibility duplicates the third if we rename n_2 and n_3 and with the additional condition $n_2 = 1$,

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, where $n_1 = n_2 = 1$, noting that this possibility duplicates the first with the additional conditions $n_1 = n_2 = 1$,

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3-1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, where $n_1 \geq 2$ and $n_2 = n_3 = 1$, noting that this possibility duplicates the second with the additional conditions $n_1 = n_2 = 1$, or

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3-1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, where $n_1 = n_2 = n_3 = 1$, noting that this possibility duplicates the first with the additional conditions $n_1 = n_2 = n_3 = 1$.

Hence we can see that $H_{3,L}$ only has rank 5 if either $H_{3,L} \simeq \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, $H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, where $n_1 \geq 2$, $H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2+1}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, where $n_1 \geq 2$, or $H_{3,L} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3+1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, where both $n_1 \geq 2$ and $n_2 \geq 2$.

Chapter 5

Application of Class Rank Results to Fields of Conductor 3, 9, p, 3p, and 9p

Prime Decomposition in the Quadratic and Cubic Fields

Let L_0, L_1, L_2, L_3 be the normal closures of the cubic fields K_0, K_1, K_2, K_3, defined above, with $p = \pm N_{k_2/\mathbb{Q}}(\beta) \neq \pm 3$ prime. Let $m > 1$ be a square-free integer with Jacobi symbol $(\frac{m}{p}) = 1$. Let $k = k_1 = \mathbb{Q}(\sqrt{-3m})$ or $\mathbb{Q}(\sqrt{-m/3})$ according as m is relatively prime to 3 or not; k is the quadratic subfield of the normal closures L_i. Let $k_2 = \mathbb{Q}(\sqrt{m})$ and $k_3 = \mathbb{Q}(\sqrt{-3})$. Let $k_0 = k(\sqrt{-3}) = k(\omega)$ be a biquadratic extension of k. Let ϵ denote the fundamental unit of k.

We know that both p and 3 ramify in the extensions L_1/k and L_2/k but p alone ramifies in the extension L_3/k. We first wish to describe the conditions in which p splits in k as opposed to those in which in p remains prime in k. We must examine the cases where m is relatively prime to 3 or not separately. First, if m is relatively prime to 3, p splits in k when the Jacobi symbol $(\frac{-3m}{p}) = 1$ but stays prime when $(\frac{-3m}{p}) = -1$. Therefore, in this case, since $(\frac{m}{p}) = 1$ and $(\frac{-3m}{p}) = (\frac{-3}{p})(\frac{m}{p})$, p splits in k when $(\frac{-3}{p}) = 1$ and p remains prime in k when $(\frac{-3}{p}) = -1$. Hence, since $(\frac{-3}{p}) = (\frac{-1}{p})(\frac{3}{p})$ and since $-1 \equiv 3 \pmod 4$, we must examine two possible cases, namely, whether $p \equiv 1$ or $p \equiv 3 \pmod 4$. If $p \equiv 1 \pmod 4$, then $(\frac{-1}{p}) = 1$ and $(\frac{3}{p}) = (\frac{p}{3})$, so then $(\frac{-3}{p}) = (\frac{-1}{p})(\frac{3}{p}) = (\frac{p}{3})$. Alternately, if $p \equiv 3 \pmod 4$, then $(\frac{-1}{p}) = -1$ and $(\frac{3}{p}) = -(\frac{p}{3})$, so again $(\frac{-3}{p}) = (\frac{-1}{p})(\frac{3}{p}) = (\frac{p}{3})$. Hence, whatever the congruence of $p \pmod 4$, p splits in k when $(\frac{p}{3}) = 1$, or equivalently when $p \equiv 1 \pmod 3$, and p remains prime in k when $(\frac{p}{3}) = -1$, or equivalently when $p \equiv 2 \pmod 3$.

If, on the other hand, 3 divides m, to discover the conditions under which p splits or remians prime in k, we need to examine the conditions under which the Jacobi symbol $(\frac{-m/3}{p}) = 1$ or $(\frac{-m/3}{p}) = -1$. Hence, in this case, since $(\frac{m}{p}) = 1$, then $(\frac{m}{p}) = (\frac{-3}{p})(\frac{-m/3}{p})$ and so since $(\frac{-3}{p}) = \pm 1$ and hence $(\frac{-3}{p})^2 = 1$, $(\frac{-m/3}{p}) = (\frac{-m/3}{p})(\frac{-3}{p})^2 = (\frac{m}{p})(\frac{-3}{p}) = (\frac{-3}{p})$. As we saw before, this shows that p splits in k when $(\frac{-3}{p}) = 1$ and p remains prime in k when $(\frac{-3}{p}) = -1$. This shows again that p splits in k when $(\frac{p}{3}) = 1$, or equivalently when $p \equiv 1 \pmod 3$, and that p remains prime in k when $(\frac{p}{3}) = -1$, or equivalently when $p \equiv 2 \pmod 3$.

We continue our examination of the conditions in which 3 splits in k as opposed to those in which in 3 does not split in k. We again examine the cases where m is relatively prime to 3 or not separately. First, we note that if m is relatively prime to 3, 3 does not split in k since the Jacobi symbol $(\frac{-3m}{p}) = 0$. If 3 divides m, we see that in order to discover the conditions under which 3 splits or not in k, we must examine when the Jacobi symbol $(\frac{-m/3}{3}) = 1$ or $(\frac{-m/3}{3}) = -1$. So we see then that 3 splits in k when $(\frac{-m/3}{3}) = 1$, that is when $-\frac{m}{3}$ is a quadratic residue (mod 3), so when $-\frac{m}{3} \equiv 1 \pmod 3$ or equivalently when $m \equiv -3 \pmod 9$. Likewise, we see that 3 does not split in k when $(\frac{-m/3}{3}) = -1$, that is when $-\frac{m}{3}$ is not a quadratic residue (mod 3), so when $-\frac{m}{3} \equiv 2 \pmod 3$ or equivalently when $m \equiv 3 \pmod 9$.

Quadratic Subfields in Which Neither p nor 3 Splits

In the theorems in this section we examine the ranks of the 3-class groups of the cubic fields K_1, K_2, and K_3 and their normal closures L_1, L_2, and L_3 of conductors $3p, 3p$, and p. In our first theorem, we examine their ranks when neither 3 nor p split in the quadratic subfield k, i.e. when m is relatively prime to 3 or $m \equiv 3 \pmod 9$ and $p \equiv 2 \pmod 3$.

Theorem 5.1 *Let m be relatively prime to 3 or $m \equiv 3$ (mod 9) and $p \equiv 2$ (mod 3). The ranks of the 3-class groups of the the fields L_1, L_2, L_3, K_1, K_2, and K_3 are determined as follows. The ranks of the 3-class groups of both cubic fields K_1 and K_2 are 1. The ranks of the 3-class groups of both normal closures L_1 and L_2 are 2. The rank of the 3-class groups of both the normal closure L_3 and the cubic field K_3 are 0.*

Proof Let $\mathfrak{p} = \mathfrak{P}_1\mathfrak{P}_2$ and $(3) = \mathfrak{L}^2$ in k_0. Let $(p) = \mathfrak{p}$ in k. Let $\mathfrak{l}$ be the prime divisor of 3 in k. Let x and y be defined by the norm residue symbols $(\frac{\mathfrak{l}, L_1/k}{\mathfrak{p}}) = \omega^x$ and $(\frac{\mathfrak{p}, L_1/k}{\mathfrak{l}}) = \omega^y$. First, by Hasse's results on the norm residue symbol, [8] $(\frac{\mathfrak{l}, L_1/k}{\mathfrak{p}}) = (\frac{\mathfrak{L}, f\beta}{\mathfrak{P}_1})(\frac{\mathfrak{L}, f\beta}{\mathfrak{P}_2})$ and $(\frac{\mathfrak{p}, L_1/k}{\mathfrak{l}}) = (\frac{\mathfrak{P}_1, f\beta}{\mathfrak{L}})$. Further, since 3 doesn't ramify in L_3, $(\frac{\mathfrak{p}, L_3/k}{\mathfrak{l}}) = 1$, but $(\frac{\mathfrak{p}, L_3/k}{\mathfrak{l}}) = (\frac{\mathfrak{P}_1, f\beta\epsilon^2}{\mathfrak{L}}) = (\frac{\mathfrak{P}_1, f\beta}{\mathfrak{L}})(\frac{\mathfrak{P}_1, \epsilon}{\mathfrak{L}})^2 = 1$, so $(\frac{\mathfrak{P}_1,f\beta}{\mathfrak{L}}) = (\frac{\mathfrak{P}_1,\epsilon}{\mathfrak{L}})$, and so $(\frac{\mathfrak{p},L_2/k}{\mathfrak{l}}) = (\frac{\mathfrak{P}_1,f\beta\epsilon}{\mathfrak{L}}) = (\frac{\mathfrak{P}_1,f\beta}{\mathfrak{L}})(\frac{\mathfrak{P}_1,\epsilon}{\mathfrak{L}}) = (\frac{\mathfrak{P}_1,f\beta}{\mathfrak{L}})^2 = (\frac{\mathfrak{p},L_1/k}{\mathfrak{l}})^2$. Also, since p doesn't ramify in $L_0, (\frac{\mathfrak{l},L_0/k}{\mathfrak{p}}) = 1$ and since $(\frac{\mathfrak{l},L_0/k}{\mathfrak{p}}) = (\frac{\mathfrak{L},\epsilon}{\mathfrak{P}_1})(\frac{\mathfrak{L},\epsilon}{\mathfrak{P}_2}), (\frac{\mathfrak{L},\epsilon}{\mathfrak{P}_1})(\frac{\mathfrak{L},\epsilon}{\mathfrak{P}_2}) = 1$. Then, $(\frac{\mathfrak{l},L_2/k}{\mathfrak{p}}) = (\frac{\mathfrak{L},f\beta\epsilon}{\mathfrak{P}_1})(\frac{\mathfrak{L},f\beta\epsilon}{\mathfrak{P}_2}) = (\frac{\mathfrak{L},f\beta}{\mathfrak{P}_1})(\frac{\mathfrak{L},f\beta}{\mathfrak{P}_2})(\frac{\mathfrak{L},\epsilon}{\mathfrak{P}_1})(\frac{\mathfrak{L},\epsilon}{\mathfrak{P}_2}) = (\frac{\mathfrak{L},f\beta}{\mathfrak{P}_1})(\frac{\mathfrak{L},f\beta}{\mathfrak{P}_2}) = (\frac{\mathfrak{l},L_1/k}{\mathfrak{p}})$.

Hence, we can construct exponent matrices for both L_1 and L_2 as follows. If $(\frac{\mathfrak{q}_i,L_1/k}{\mathfrak{q}_j}) = \omega^z$, then in our exponent matrix, the entry in the row corresponding to $\mathfrak{q}_i$ and the column corresponding to $\mathfrak{q}_j$ will be z. By the product formula, all rows must sum to 0 (mod 3). Hence we find that the exponent matrices for L_1 and L_2 are, respectively:

	$\mathfrak{l}$	$\mathfrak{p}$
$\mathfrak{l}$	$2x$	x
$\mathfrak{p}$	y	$2y$
$\mathfrak{l}$	$2x$	x
$\mathfrak{p}$	$2y$	y

The result of Gerth [6] stated previously as Lemma 4.3 states that the rank of the 3-class group of the normal closure of a cubic field is $2t - s$ where t is the number of ambiguous ideal classes of L over k (i.e. one less than the number of primes that ramify in the extension L/k) and s is the rank of the matrix above. Here $t = 1$.

Now we note, again from Hasse's work regarding the norm residue symbol [8], that $(\frac{\mathfrak{l},L/k}{\mathfrak{p}}) = (\frac{\mathfrak{L},f\beta}{\mathfrak{P}_1})(\frac{\mathfrak{L},f\beta}{\mathfrak{P}_2}) = (\frac{f\beta,\mathfrak{L}}{\mathfrak{P}_1})^{-1}(\frac{f\beta,\mathfrak{L}}{\mathfrak{P}_2})^{-1} = (\frac{\mathfrak{L}}{\mathfrak{P}_1})_3^{-b_1}(\frac{\mathfrak{L}}{\mathfrak{P}_2})_3^{-b_2}$ where we define b_1 and b_2 as the largest numbers so that $\mathfrak{P}_1^{b_1}$ and $\mathfrak{P}_2^{b_2}$ divide $f\beta$. Then $(\frac{\mathfrak{L}}{\mathfrak{P}_1})_3 \equiv \mathfrak{L}^{\frac{p^2-1}{3}}$ (mod $\mathfrak{P}_1$). So, since we know that $p \equiv 2$ (mod 3), then $\frac{p+1}{3}$ is an integer (mod 3). Therefore, since $\sqrt{-3}$ generates $\mathfrak{L}, (\frac{\mathfrak{L}}{\mathfrak{P}_1})_3 \equiv \sqrt{-3}^{(p-1)(\frac{p+1}{3})}$ (mod $\mathfrak{P}_1$). By Fermat's Little Theorem, $(\frac{\mathfrak{L}}{\mathfrak{P}_1})_3^2 \equiv 3^{(p-1)(\frac{p+1}{3})} \equiv 1$ (mod $\mathfrak{P}_1$). Similarly, $(\frac{\mathfrak{L}}{\mathfrak{P}_2})_3^2 \equiv 3^{(p-1)(\frac{p+1}{3})} \equiv 1$ (mod $\mathfrak{P}_2$). Hence, since $(\frac{\mathfrak{l},L/k}{\mathfrak{p}})^2 = 1$, and since these symbols are all powers of $\omega, (\frac{\mathfrak{l},L/k}{\mathfrak{p}}) = 1$. Therefore, $\omega^x = 1$ and so $x = 0$.

Recall now that $(\frac{\mathfrak{p},L_1/k}{\mathfrak{l}}) = (\frac{\mathfrak{P}_1,f\beta}{\mathfrak{L}}) = (\frac{\mathfrak{P}_1,\epsilon}{\mathfrak{L}})$. But by the product formula, $(\frac{\mathfrak{P}_1,\epsilon}{\mathfrak{L}})(\frac{\mathfrak{P}_1,\epsilon}{\mathfrak{P}_1}) = 1$, and so, $(\frac{\mathfrak{P}_1,\epsilon}{\mathfrak{L}}) = (\frac{\mathfrak{P}_1,\epsilon}{\mathfrak{P}_1})^2 = ((\frac{\epsilon}{\mathfrak{P}_1})_3^{-1})^2 = (\frac{\epsilon}{\mathfrak{P}_1})_3^{-2}$. Again, since $p \equiv 2$ (mod 3), $\frac{p+1}{3}$ is an integer (mod 3), by Fermat's Little Theorem, $(\frac{\epsilon}{\mathfrak{P}_1})_3 \equiv \epsilon^{\frac{p^2-1}{3}} \equiv (\epsilon^{p-1})^{\frac{p+1}{3}} \equiv 1$ (mod $\mathfrak{P}_1$). Hence, $(\frac{\mathfrak{p},L/k}{\mathfrak{l}}) = (\frac{\epsilon}{\mathfrak{P}_1})_3^{-2} \equiv 1^{-2} \equiv 1$ (mod $\mathfrak{P}_1$). Therefore $\omega^y = 1$ and so $y = 0$.

Since $x = y = 0$, the matrices above both have rank 0, so the ranks of the 3-class groups of both L_1 and L_2 are $2t - s = 2 - 0 = 2$.

The related result of Gerth [5] stated previously as Lemma 4.2 states that the rank of the 3-class group of a cubic field is $t - s_1$ where t is defined as above and s_1 is the rank of the exponent matrix

$$[(\frac{\mathfrak{q}_i^2\mathfrak{q}_i^\tau, L/k}{\mathfrak{q}_j})]$$

as described in Chapter 4 following Lemma 4.2. Here, since neither p nor 3 splits in k this matrix is the zero matrix and hence $s_1 = 0$, regardless of the values of x or y. Therefore, the ranks of the 3-class groups of both K_1 and K_2 are always $t - s_1 = 1 - 0 = 1$.

Similarly, in the case of L_3, only p is ramified in L_3 over k. Hence, since in this case $t = 0$ and the rows must sum to 0 (mod 3), our matrix is the 1×1 zero matrix, so $s = s_1 = 0$. Hence, here the ranks of the 3-class groups of both L_3 and K_3 are 0.

Quadratic Subfields in Which p Splits but 3 Does Not

We continue with an examination of the situation when p splits in k but 3 does not. Hence, in the theorems in this section we examine the ranks of the 3-class groups of the cubic fields K_1, K_2, and K_3 and their normal closures L_1, L_2, and L_3 of conductors $3p, 3p$, and p. Recall our earlier notes that 3 does not split in k when m is relatively prime to 3 or $m \equiv 3 \pmod 9$ and that p splits in k when $p \equiv 1 \pmod 3$. The theorem below examines the ranks of the cubic fields and the normal closures in this situation.

Theorem 5.2 *Let m be relatively prime to 3 or $m \equiv 3 \pmod 9$ and $p \equiv 1 \pmod 3$. The ranks of the 3-class groups of the the fields L_1, L_2, L_3, K_1, K_2, and K_3 are determined as follows. Let $(p) = \mathfrak{p}_1\mathfrak{p}_2$ in k. Let $\mathfrak{l}$ be the prime divisor of 3 in k. Let x, y, and z be defined by the norm residue symbols $(\frac{\mathfrak{l}, L_1/k}{\mathfrak{p}_1}) = \omega^x$, $(\frac{\mathfrak{p}_1, L_1/k}{\mathfrak{l}}) = \omega^y$ and $(\frac{\mathfrak{p}_1, L_1/k}{\mathfrak{p}_2}) = \omega^z$. The 3-class groups of K_1 and K_2 have rank equal to 2 or 1 according as $y = 0$ or not. If $y = 0$ then the rank of the 3-class groups of both L_1 and L_2 are 4 or 3 according as $z = x = 0$ or not. If $y \neq 0$, then the rank of the 3-class group of L_1 is 3 or 2 according as both $x = 0$ and $y = z$ or not and the rank of the 3-class group of L_2 is 3 or 2 according as both $x = 0$ and $y = 2z$ or not. The rank of the 3-class group of K_3 is 1. The rank of the 3-class group of L_3 is 2 or 1 according as $z = 0$ or not.*

Proof Let $\mathfrak{p}_1 = \mathfrak{P}_1\mathfrak{P}_2$, $\mathfrak{p}_2 = \mathfrak{P}_3\mathfrak{P}_4$ and $(3) = \mathfrak{L}^2$ in k_0. Then, since conjugation of ideals within one of the norm residue symbols yields conjugation of the value of the symbol itself, we know that $(\frac{\mathfrak{l},L_1/k}{\mathfrak{p}_2}) = \omega^{2x}$, $(\frac{\mathfrak{p}_2,L_1/k}{\mathfrak{l}}) = \omega^{2y}$, and $(\frac{\mathfrak{p}_2,L_1/k}{\mathfrak{p}_1}) = \omega^{2z}$. Further, $(\frac{\mathfrak{p}_1,L_2/k}{\mathfrak{p}_1}) = (\frac{\mathfrak{P}_1,f\beta\epsilon}{\mathfrak{P}_1})(\frac{\mathfrak{P}_1,f\beta\epsilon}{\mathfrak{P}_2}) = (\frac{\mathfrak{P}_1,f\beta}{\mathfrak{P}_1})(\frac{\mathfrak{P}_1,f\beta}{\mathfrak{P}_2})(\frac{\mathfrak{P}_1,\epsilon}{\mathfrak{P}_1})(\frac{\mathfrak{P}_1,\epsilon}{\mathfrak{P}_2}) = (\frac{\mathfrak{p}_1,L_1/k}{\mathfrak{p}_1})(\frac{\epsilon}{\mathfrak{P}_1})_3$, since $\mathfrak{P}_2$ doesn't divide $\mathfrak{P}_1$ and so, $(\frac{\mathfrak{P}_1,\epsilon}{\mathfrak{P}_2}) = (\frac{\epsilon}{\mathfrak{P}_1})_3^0$. Similarly, $(\frac{\mathfrak{p}_1,L_3/k}{\mathfrak{p}_1}) = (\frac{\mathfrak{p}_1,L_1/k}{\mathfrak{p}_1})(\frac{\epsilon}{\mathfrak{P}_1})_3^2$.

Now $(\frac{\mathfrak{p}_1 L_1/k}{\mathfrak{l}}) = (\frac{\mathfrak{P}_1,f\beta}{\mathfrak{L}}), (\frac{\mathfrak{p}_1,L_2/k}{\mathfrak{l}}) = (\frac{\mathfrak{P}_1,f\beta\epsilon}{\mathfrak{L}}) = (\frac{\mathfrak{P}_1,f\beta}{\mathfrak{L}})(\frac{\mathfrak{P}_1,\epsilon}{\mathfrak{L}}) = (\frac{\mathfrak{p}_1,L_1/k}{\mathfrak{l}})(\frac{\mathfrak{P}_1,\epsilon}{\mathfrak{L}})$. By the product formula $(\frac{\mathfrak{P}_1,\epsilon}{\mathfrak{P}_1})(\frac{\mathfrak{P}_1,\epsilon}{\mathfrak{L}}) = 1$, so $(\frac{\mathfrak{P}_1,\epsilon}{\mathfrak{L}}) = (\frac{\mathfrak{P}_1,\epsilon}{\mathfrak{P}_1})^2 = (\frac{\epsilon}{\mathfrak{P}_1})_3^2$. Thus $(\frac{\mathfrak{p}_1,L_2/k}{\mathfrak{l}}) = (\frac{\mathfrak{p}_1,L_1/k}{\mathfrak{l}})(\frac{\epsilon}{\mathfrak{P}_1})_3^2$.

Since $\mathfrak{l}$ is unramified in K_3/k, $1 = (\frac{\mathfrak{p}_1,L_3/k}{\mathfrak{l}}) = (\frac{\mathfrak{P}_1,f\beta\epsilon^2}{\mathfrak{L}}) = (\frac{\mathfrak{P}_1,f\beta}{\mathfrak{L}})(\frac{\mathfrak{P}_1,\epsilon}{\mathfrak{L}})^2 = (\frac{\mathfrak{p}_1,L_1/k}{\mathfrak{l}})(\frac{\epsilon}{\mathfrak{P}_1})_3$. So $\omega^y = (\frac{\mathfrak{p}_1,L_1/k}{\mathfrak{l}}) = (\frac{\epsilon}{\mathfrak{P}_1})_3^2$. Similarly, $(\frac{\mathfrak{p}_2,L_1/k}{\mathfrak{l}}) = (\frac{\epsilon}{P_1})_3$.

Hence, we find exponent martices for the fields L_1, L_2, and L_3 to be the following:

	$\mathfrak{l}$	$\mathfrak{p}_1$	$\mathfrak{p}_2$
$\mathfrak{l}$	0	x	$2x$
$\mathfrak{p}_1$	y	$2(y+z)$	z
$\mathfrak{p}_2$	$2y$	$2z$	$y+z$
$\mathfrak{l}$	0	x	$2x$
$\mathfrak{p}_1$	$2y$	$y+2z$	z
$\mathfrak{p}_2$	y	$2z$	$2y+z$
$\mathfrak{p}_1$		$2z$	z
$\mathfrak{p}_2$		$2z$	z

If we add the third rows to the second rows of the first two tables, then add the new second row to the third row, we find, for L_1 and L_2, respectively,

	$\mathfrak{l}$	$\mathfrak{p}_1$	$\mathfrak{p}_2$
$\mathfrak{l}$	0	x	$2x$
$\mathfrak{p}_1\mathfrak{p}_2$	0	$2y+z$	$2(2y+z)$
$\mathfrak{p}_1\mathfrak{p}_2^2$	$2y$	$2y$	$2y$
$\mathfrak{l}$	0	x	$2x$
$\mathfrak{p}_1\mathfrak{p}_2$	0	$y+z$	$2(y+z)$
$\mathfrak{p}_1\mathfrak{p}_2^2$	y	y	y

First suppose that $y \neq 0$. Then the last row of each of these matrices is nonzero, and hence the ranks of the matrices are at least one. The first and the second rows of the matrix for L_1 will both be zero only if $x = 0$ and $y = z$. In this case, the rank of the matrix will be $s = 1$ and hence the rank of H_{3,L_1} will be $2t - s = 4 - 1 = 3$. If, on the other hand, $x \neq 0, y \neq z$, or both, then whichever of these rows are nonzero will be linearly independent to the third row, as $y \neq 0$. However, if both the first and the second rows are nonzero, as the first entries of both are zero and the third entries of both are twice the second, then they will be linearly dependent. Hence, if $x \neq 0, y \neq z$, or both, then the rank of the matrix will be $s = 2$ and hence the rank of H_{3,L_1} will be $2t - s = 4 - 2 = 2$.

Similarly, if $y \neq 0, x = 0$, and $y = 2z$, the rank of the matrix will be $s = 1$ and hence the rank of H_{3,L_2} will be $2t - s = 4 - 1 = 3$. If, on the other hand, $y \neq 0$ but either $x \neq 0, y \neq z$, or both, the rank of the matrix will be $s = 2$ and hence the rank of H_{3,L_2} will be $2t - s = 4 - 2 = 2$.

However if we suppose that $y = 0$ the matrices for L_1 and L_2 become

	$\mathfrak{l}$	$\mathfrak{p}_1$	$\mathfrak{p}_2$
$\mathfrak{l}$	0	x	$2x$
$\mathfrak{p}_1\mathfrak{p}_2$	0	z	$2z$
$\mathfrak{p}_1\mathfrak{p}_2^2$	0	0	0
$\mathfrak{l}$	0	x	$2x$
$\mathfrak{p}_1\mathfrak{p}_2$	0	z	$2z$
$\mathfrak{p}_1\mathfrak{p}_2^2$	0	0	0

The ranks of these matrices are $s = 0$ only if $x = z = 0$, and hence the ranks of H_{3,L_1} and H_{3,L_2} are both $2t - s = 4 - 0 = 4$. If, on the other hand, $x \neq 0, z \neq 0$, or both we will have one or two nonzero rows. Further, if both are nonzero, they will be multiples of one another (mod 3) and hence the rank of the matrices above are $s = 1$. Therefore, the ranks of H_{3,L_1} and H_{3,L_2} are both $2t - s = 4 - 1 = 3$.

Furthermore, by adding the second row to twice the third, we find for K_1 and K_2, respectively,

	$\mathfrak{l}$	$\mathfrak{p}_1$	$\mathfrak{p}_2$
$\mathfrak{p}_1\mathfrak{p}_2^2$	$2y$	$2y$	$2y$
$\mathfrak{p}_1\mathfrak{p}_2^2$	y	y	y

Hence, the ranks of these matrices are $s_1 = 0$ if $y = 0$ and $s_1 = 1$ if $y \neq 0$. Therefore, if $y = 0$ the ranks of both H_{3,K_1} and H_{3,K_2} are $t - s_1 = 2 - 0 = 2$, while if $y \neq 0$ the ranks of both H_{3,K_1}

and H_{3,K_2} are $t - s_1 = 2 - 1 = 1$.

Recall that our exponent matrix for L_3 is

	$\mathfrak{p}_1$	$\mathfrak{p}_2$
$\mathfrak{p}_1$	$2z$	z
$\mathfrak{p}_2$	$2z$	z

Note then that the rank of this matrix is $s = 0$ if $z = 0$ and $s = 1$ if $z \neq 0$. Therefore, if $z = 0$, the rank of H_{3,L_3} will be $2t - s = 2 - 0 = 2$. However, if $z \neq 0$, the rank of the matrix will be $s = 1$ and hence the rank of H_{3,L_3} will be $2t - s = 2 - 1 = 1$.

Furthermore, by adding the second row to twice the third, we find for K_3,

	$\mathfrak{p}_1$	$\mathfrak{p}_2$
$\mathfrak{p}_1\mathfrak{p}_2^2$	0	0

Hence, since the rank of this matrix is $s_1 = 0$, the rank of H_{3,K_3} is $t - s1 = 1 - 0 = 1$.

Corollary 5.2.1 *$x = 0$ if and only if $(\frac{3}{\mathfrak{p}})_3 = 1$.*

$(\frac{\mathfrak{l}, L_1/k}{\mathfrak{p}_1}) = (\frac{\mathfrak{L}, f\beta}{\mathfrak{P}_1})(\frac{\mathfrak{L}, f\beta}{\mathfrak{P}_2}) = (\frac{\mathfrak{L}, f\beta}{\mathfrak{P}_1})^2$ since $\mathrm{Gal}(K(\sqrt{-3})/k)$ is abelian. But $\mathfrak{L} = (\sqrt{-3})$ so $(\frac{\mathfrak{L}, f\beta}{\mathfrak{P}_1})^2 = (\frac{\sqrt{-3}, f\beta}{\mathfrak{P}_1})^2 = (\frac{-3}{\mathfrak{P}_1}) = (\frac{3}{\mathfrak{P}_1})$. Hence $x = 0$ if and only if $(\frac{3}{\mathfrak{p}})_3 = 1$.

Corollary 5.2.2 *In the situation described in Theorem 5.2 above, if $y \neq 0, x = 0$, and $y = z$, $H_{3,K_1} \simeq \mathbb{Z}_{3^n}$ and $H_{3,L_1} \simeq \mathbb{Z}_{3^n} \times \mathbb{Z}_{3^{n-1}} \times \mathbb{Z}_3$, where $n \geq 2$. If $y \neq 0$, $x = 0$, and $y \neq z$, $H_{3,K_1} \simeq \mathbb{Z}_3$ and $H_{3,L_1} \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$.*

If $y \neq 0, x = 0$, and $y = 2z$, $H_{3,K_2} = \mathbb{Z}_{3^n}$ and $H_{3,L_2} \simeq \mathbb{Z}_{3^n} \times \mathbb{Z}_{3^{n-1}} \times \mathbb{Z}_3$, where $n \geq 2$. If $y \neq 0$, $x = 0$, and $y \neq 2z$, $H_{3,K_2} \simeq \mathbb{Z}_3$ and $H_{3,L_2} \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$.

Further, if $z \neq 0$, $H_{3,K_3} = H_{3,L_3} = \mathbb{Z}_3$.

Proof If $y \neq 0, x = 0$, and $y = z$, we know from Theorem 5.2 that H_{3,K_1} and H_{3,K_2} have rank 1 (both isomorphic to $\mathbb{Z}_{3^n}$) and H_{3,L_1} has rank 3. Then by Corollary 4.3.4, we know that $H_{3,L_1} \simeq \mathbb{Z}_{3^n} \times \mathbb{Z}_{3^{n-1}} \times \mathbb{Z}_3$ and $n \geq 2$. An identical result holds for H_{3,K_2} and H_{3,L_2} when $y \neq 0, x = 0$, and $y = 2z$.

If $y \neq 0, x = 0$, and $y \neq z$, we know from Theorem 5.2 that H_{3,K_1} and H_{3,K_2} have rank 1 and H_{3,L_1} has rank 2. Then by Corollary 4.3.3, we know that since $t = s = 2$, $H_{3,K_1} \simeq \mathbb{Z}_3$ and $H_{3,L_1} \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$. An identical result holds for H_{3,K_2} and H_{3,L_2} when $y \neq 0, x = 0$, and $y \neq 2z$.

Furthermore, if $z \neq 0$, we know from Theorem 5.2 that both H_{3,K_3} and H_{3,L_3} have rank 1. By Corollary 4.3.2, $n = 1$ and $H_{3,K_3} = H_{3,L_3} = \mathbb{Z}_3$.

Corollary 5.2.3 *In the situation described in Theorem 5.2 above, if $y = 0$, but either $x \neq 0$ or $z \neq 0$, then H_{3,K_1} and H_{3,K_2} both must be isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_{3^n}$ and each of H_{3,L_1} and H_{3,L_2} must satisfy one of the following (where $i = 1$ or 2):*

$H_{3,L_i} \simeq \mathbb{Z}_{3^n} \times \mathbb{Z}_3 \times \mathbb{Z}_{3^n}$ *or*

$H_{3,L_i} \simeq \mathbb{Z}_{3^{n+1}} \times \mathbb{Z}_3 \times \mathbb{Z}_{3^n}$.

Proof If $y = 0$, but either $x \neq 0$ or $z \neq 0$ we know from Theorem 5.2 that H_{3,K_1} and H_{3,K_2} have rank 2, both H_{3,L_1} and H_{3,L_2} have rank 3, $t = 2$, and $s = 1$. Then by Corollary 4.5.3, we know that for $i = 1$ or 2,

$H_{3,K_i} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^n}$ and either

$H_{3,L_i} \simeq \mathbb{Z}_{3^n} \times \mathbb{Z}_3 \times \mathbb{Z}_{3^n}$ or

$H_{3,L_i} \simeq \mathbb{Z}_{3^{n+1}} \times \mathbb{Z}_3 \times \mathbb{Z}_{3^n}$.

Corollary 5.2.4 *In the situation described in Theorem 5.2 above, if $x = y = z = 0$, then H_{3,K_1} and H_{3,K_2} both must be isomorphic to $\mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$ and both H_{3,L_1} and H_{3,L_2} must satisfy one of the following (where $i = 1$ or 2):*

$H_{3,L_i} \simeq \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$,

$H_{3,L_i} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, *where $n_1 \geq 2$, or*

$H_{3,L_i} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2+1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, *where $n_1 \geq 2$.*

Proof If $x = y = z = 0$, we know from Theorem 5.2 that H_{3,K_1} and H_{3,K_2} have rank 2, both H_{3,L_1} and H_{3,L_2} have rank 4, $t = 2$, and $s = 0$. Then by Corollary 4.5.4, we know that for $i = 1$ or 2,

$H_{3,L_i} \simeq \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$,

$H_{3,L_i} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, where $n_1 \geq 2$, or

$H_{3,L_i} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2+1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, where $n_1 \geq 2$.

Quadratic Subfields in Which p Remains Prime but 3 Splits

We continue with an examination of the situation when 3 splits in k but p remains prime. Hence, in the theorems in this section we examine the ranks of the 3-class groups of the cubic fields K_1, K_2, and K_3 and their normal closures L_1, L_2, and L_3 of conductors $9p, 9p$, and p. Recall again our earlier notes that 3 splits in k when $m \equiv -3 \pmod 9$ and that p remains prime in k when $p \equiv 2 \pmod 3$. The theorem below examines the ranks of the cubic fields and the normal closures in this situation.

Theorem 5.3 *Let $m \equiv -3$ (mod 9) and $p \equiv 2$ (mod 3). The ranks of the 3-class groups of the fields L_1, L_2, L_3, K_1, K_2, and K_3 are determined as follows. Let $\mathfrak{l}_1$ and $\mathfrak{l}_2$ be the prime divisors of 3 in k, and $\mathfrak{p}$ be the prime divisor of p in k. Let w, x, y, and z be defined by the norm residue symbols $(\frac{\mathfrak{l}_1, L_0/k}{\mathfrak{l}_2}) = \omega^w$, $(\frac{\mathfrak{l}_1, L_1/k}{\mathfrak{l}_2}) = \omega^x$, $(\frac{\mathfrak{l}_1, L_1/k}{\mathfrak{p}}) = \omega^y$ and $(\frac{\mathfrak{p}, L_1/k}{\mathfrak{l}_1}) = \omega^z$. K_1 and K_2 have the rank of their 3-class groups equal to 2 or 1 according as $y = 0$ or not. If $y = 0$ then the rank of the 3-class group of L_1 is 4 or 3 according as $x = z = 0$ or not and the rank of the 3-class group of L_2 is 4 or 3 according as $z = 0$ and $x = 2w$ or not. If $y \neq 0$, the rank of the 3-class group of L_1 is 3 or 2 according as $x + 2y = z = 0$ or not. If $y \neq 0$, the rank of the 3-class group of L_2 is 3 or 2 according as $x + 2y + w = z = 0$ or not. The ranks of the 3-class groups of the fields K_3 and L_3 are 0.*

Proof Let $\mathfrak{l}_1 = \mathfrak{L}_1^2$, $\mathfrak{l}_2 = \mathfrak{L}_2^2$ and $(p) = \mathfrak{P}_1\mathfrak{P}_2$ in k_0. Then $(\frac{\mathfrak{l}_1, L_0/k}{\mathfrak{l}_2}) = (\frac{\mathfrak{L}_1, \epsilon}{\mathfrak{L}_2})$, $(\frac{\mathfrak{l}_1, L_1/k}{\mathfrak{l}_2}) = (\frac{\mathfrak{L}_1, f\beta}{\mathfrak{L}_2})$, $(\frac{\mathfrak{l}_1, L_1/k}{\mathfrak{p}}) = (\frac{\mathfrak{L}_1, f\beta}{\mathfrak{P}_1})$, and $(\frac{\mathfrak{p}, L_1/k}{\mathfrak{l}_1}) = (\frac{\mathfrak{P}_1, f\beta}{\mathfrak{L}_1})$.

Since $\mathfrak{l}_1$ is unramified in L_3/k, $1 = (\frac{\mathfrak{p}, L_3/k}{\mathfrak{l}_1}) = (\frac{\mathfrak{P}_1, f\beta\epsilon^2}{\mathfrak{L}_1}) = (\frac{\mathfrak{P}_1, f\beta}{\mathfrak{L}_1})(\frac{\mathfrak{P}_1, \epsilon}{\mathfrak{L}_1})^2 = (\frac{\mathfrak{p}, L_1/k}{\mathfrak{l}_1})(\frac{\mathfrak{p}, L_0/k}{\mathfrak{l}_1})^2$. So $(\frac{p, L_1/k}{\mathfrak{l}_1}) = (\frac{\mathfrak{p}, L_0/k}{\mathfrak{l}_1})$. Similarly, $(\frac{\mathfrak{p}, L_1/k}{\mathfrak{l}_2}) = (\frac{\mathfrak{p}, L_0/k}{\mathfrak{l}_2})$. Hence, $(\frac{\mathfrak{p}, L_2/k}{\mathfrak{l}_1}) = (\frac{\mathfrak{P}_1, f\beta\epsilon}{\mathfrak{L}_1}) = (\frac{\mathfrak{P}_1, f\beta}{\mathfrak{L}_1})(\frac{\mathfrak{P}_1, \epsilon}{\mathfrak{L}_1}) = (\frac{\mathfrak{p}, L_1/k}{\mathfrak{l}_1})(\frac{\mathfrak{p}, L_0/k}{\mathfrak{l}_1}) = (\frac{\mathfrak{p}, L_1/k}{\mathfrak{l}_1})^2$. Similarly, $(\frac{\mathfrak{p}, L_2/k}{\mathfrak{l}_2}) = (\frac{\mathfrak{p}, L_1/k}{\mathfrak{l}_2})^2$.

Also, since p doesn't ramify in L_0, $(\frac{\mathfrak{l}_1, L_0/k}{\mathfrak{p}}) = 1$. So, by properties of the norm residue symbol, $(\frac{\mathfrak{l}_1, L_2/k}{\mathfrak{p}}) = (\frac{\mathfrak{L}_1, f\beta\epsilon}{\mathfrak{P}_1})(\frac{\mathfrak{L}_1, f\beta\epsilon}{\mathfrak{P}_2}) = (\frac{\mathfrak{L}_1, f\beta}{\mathfrak{P}_1})(\frac{\mathfrak{L}_1, \epsilon}{\mathfrak{P}_1})(\frac{\mathfrak{L}_1, f\beta}{\mathfrak{P}_2})(\frac{\mathfrak{L}_1, \epsilon}{\mathfrak{P}_2}) = (\frac{\mathfrak{L}_1, f\beta}{\mathfrak{P}_1})(\frac{\mathfrak{L}_1, f\beta}{\mathfrak{P}_2})(\frac{\mathfrak{L}_1, \epsilon}{\mathfrak{P}_1})(\frac{\mathfrak{L}_1, \epsilon}{\mathfrak{P}_2}) = (\frac{\mathfrak{l}_1, L_1/k}{\mathfrak{p}})(\frac{\mathfrak{l}_1, L_0/k}{\mathfrak{p}}) = (\frac{\mathfrak{l}_1, L_1/k}{\mathfrak{p}})$. Similarly, $(\frac{\mathfrak{l}_2, L_2/k}{\mathfrak{p}}) = (\frac{\mathfrak{l}_2, L_1/k}{\mathfrak{p}})$.

Furthermore, $(\frac{\mathfrak{l}_1, L_2/k}{\mathfrak{l}_2}) = (\frac{\mathfrak{L}_1, f\beta\epsilon}{\mathfrak{L}_2}) = (\frac{\mathfrak{L}_1, f\beta}{\mathfrak{L}_2})(\frac{\mathfrak{L}_1, \epsilon}{\mathfrak{L}_2}) = (\frac{\mathfrak{l}_1, L_1/k}{\mathfrak{l}_2})(\frac{\mathfrak{l}_1, L_0/k}{\mathfrak{l}_2})$. Similarly, $(\frac{\mathfrak{l}_2, L_2/k}{\mathfrak{l}_1}) = (\frac{\mathfrak{l}_2, L_1/k}{\mathfrak{l}_1})(\frac{\mathfrak{l}_2, L_0/k}{\mathfrak{l}_1})$.

So, in L_0 we have,

	$\mathfrak{l}_1$	$\mathfrak{l}_2$
$\mathfrak{l}_1$	$2w$	w
$\mathfrak{l}_2$	$2w$	w

and in L_1 and L_2 we have, respectively,

	$\mathfrak{l}_1$	$\mathfrak{l}_2$	$\mathfrak{p}$
$\mathfrak{l}_1$	$2(x+y)$	x	y
$\mathfrak{l}_2$	$2x$	$x+y$	$2y$
$\mathfrak{p}$	z	$2z$	0
$\mathfrak{l}_1$	$2(x+y+w)$	$x+w$	y
$\mathfrak{l}_2$	$2(x+w)$	$x+y+w$	$2y$
$\mathfrak{p}$	$2z$	z	0

First, suppose $y = 0$. Then in L_1 and L_2 we have respectively,

	$\mathfrak{l}_1$	$\mathfrak{l}_2$	$\mathfrak{p}$
$\mathfrak{l}_1$	$2x$	x	0
$\mathfrak{l}_2$	$2x$	x	0
$\mathfrak{p}$	z	$2z$	0
$\mathfrak{l}_1$	$2(x+w)$	$x+w$	0
$\mathfrak{l}_2$	$2(x+w)$	$x+w$	0
$\mathfrak{p}$	$2z$	z	0

Since the first and last rows are multiples of one another, the matrix for L_1 will have rank $s = 0$ or 1 accordinig as $x = z = 0$ or not and the matrix for L_2 will have rank $s = 0$ or 1 according as $x + w = z = 0$ or not. Hence, the rank of H_{3,L_1} is $2t - s = 4 - 0 = 4$ if $x = z = 0$ or $2t - s = 4 - 1 = 3$ if $x \neq 0$ or $z \neq 0$. Similarly, the rank of H_{3,L_2} is $2t - s = 4 - 0 = 4$ if $x = 2w$ and $z = 0$ or $2t - s = 4 - 1 = 3$ if $x \neq 2w$ or $z \neq 0$.

Secondly, suppose $y \neq 0$. Then, we find by adding the first to the second row of each matrix in L_1 and L_2, respectively,

	$\mathfrak{l}_1$	$\mathfrak{l}_2$	$\mathfrak{p}$
$\mathfrak{l}_1$	$2(x+y)$	x	y
$\mathfrak{l}_1\mathfrak{l}_2$	$x+2y$	$2(x+2y)$	0
$\mathfrak{p}$	z	$2z$	0
$\mathfrak{l}_1$	$2(x+w+y)$	$x+w$	y
$\mathfrak{l}_1\mathfrak{l}_2$	$2(2x+2w+y)$	$2x+2w+y$	0
$\mathfrak{p}$	$2z$	z	0

Since $y \neq 0$ and the last two rows are multiples of one another, the matrix for L_1 will have rank $s = 1$ or 2 according as $x + 2y = z = 0$ or not and the matrix for L_2 will have rank $s = 1$ or 2 according as $x + w + 2y = z = 0$ or not. Hence, the rank of H_{3,L_1} is $2t - s = 4 - 1 = 3$ if $x = y$ and $z = 0$ or $2t - s = 4 - 2 = 2$ if $x \neq y$ or $z \neq 0$. Similarly, the rank of H_{3,L_2} is $2t - s = 4 - 1 = 3$ if $x = y + 2w$ and $z = 0$ or $2t - s = 4 - 2 = 2$ if $x \neq y + 2w$ or $z \neq 0$.

Furthermore, by adding the first row to twice the second of the original matrices for both L_1 and L_2, we find that, in K_1 and K_2, we have respectively,

	$\mathfrak{l}_1$	$\mathfrak{l}_2$	$\mathfrak{p}$
$\mathfrak{l}_1\mathfrak{l}_2^2$	$2y$	$2y$	$2y$
$\mathfrak{l}_1\mathfrak{l}_2^2$	$2y$	$2y$	$2y$

Both of which have rank $s_1 = 1$ if $y \neq 0$ or rank $s_1 = 0$ if $y = 0$. Hence, the ranks of both H_{3,K_1} and H_{3,K_2} are $t - s_1 = 2 - 1 = 1$ if $y \neq 0$ or $t - s_1 = 2 - 0 = 2$ if $y = 0$

Again, in the case of L_3, only $\mathfrak{p}$ is ramified in L_3 over k. Hence, our matrix is the 1×1 zero matrix, so $t = s = s_1 = 0$. Hence, here the ranks of the 3-class groups of both L_3 and K_3 are 0.

Corollary 5.3.1 *In the situation described in Theorem* 5.3 *above, if* $y \neq 0$ *and* $x+2y = z = 0$ *then* $H_{3,K_1} \simeq \mathbb{Z}_{3^n}$ *and* $H_{3,L_1} \simeq \mathbb{Z}_{3^n} \times \mathbb{Z}_{3^{n-1}} \times \mathbb{Z}_3$*, where* $n \geq 2$*. Also, if* $y \neq 0$ *and* $x + 2y + w = z = 0$ *then* $H_{3,K_2} \simeq \mathbb{Z}_{3^n}$ *and* $H_{3,L_2} \simeq \mathbb{Z}_{3^n} \times \mathbb{Z}_{3^{n-1}} \times \mathbb{Z}_3$*, where* $n \geq 2$.

Further, if $y \neq 0$ *and either* $x + 2y \neq 0$ *or* $z \neq 0$ *then* $H_{3,K_1} \simeq \mathbb{Z}_3$ *and* $H_{3,L_1} \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$*. Further, if* $y \neq 0$ *and either* $x + 2y + w \neq 0$ *or* $z \neq 0$ *then* $H_{3,K_2} \simeq \mathbb{Z}_3$ *and* $H_{3,L_2} \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$.

Proof If $y \neq 0$ and $x + 2y = z = 0$ we know from Theorem 5.3 that H_{3,K_1} has rank 1 and H_{3,L_1} has rank 3. By Corollary 4.3.4, we know that $H_{3,L_1} \simeq \mathbb{Z}_{3^n} \times \mathbb{Z}_{3^{n-1}} \times \mathbb{Z}_3$ and $n \geq 2$. An identical result holds for H_{3,K_2} and H_{3,L_2} when $x + 2y + w = z = 0$.

Furthermore, if $y \neq 0$ and either $x + 2y \neq 0$ or $z \neq 0$, again from Theorem 5.3, we know that H_{3,K_1} has rank 1 and H_{3,L_1} has rank 2 with $t = s = 2$. Again, by Corollary 4.3.3, we know that $H_{3,K_1} \simeq \mathbb{Z}_3$ and $H_{3,L_1} \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$. An identical result holds for H_{3,K_2} and H_{3,L_2} when $y \neq 0$ and either $x + 2y + w \neq 0$ or $z \neq 0$.

Corollary 5.3.2 *In the situation described in Theorem 5.3 above, if $y = 0$, and either $x \neq 0$ or $z \neq 0$, then H_{3,K_1} must be isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_{3^n}$ and H_{3,L_1} must satisfy one of the following:*

$H_{3,L_1} \simeq \mathbb{Z}_{3^n} \times \mathbb{Z}_3 \times \mathbb{Z}_{3^n}$ *or*

$H_{3,L_1} \simeq \mathbb{Z}_{3^{n+1}} \times \mathbb{Z}_3 \times \mathbb{Z}_{3^n}$.

Further, if $y = 0$, and either $x \neq 2w$ or $z \neq 0$, then H_{3,K_2} must be isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_{3^n}$ and H_{3,L_2} must satisfy one of the following:

$H_{3,L_2} \simeq \mathbb{Z}_{3^n} \times \mathbb{Z}_3 \times \mathbb{Z}_{3^n}$ *or*

$H_{3,L_2} \simeq \mathbb{Z}_{3^{n+1}} \times \mathbb{Z}_3 \times \mathbb{Z}_{3^n}$.

Proof If $y = 0$, but either $x \neq 0$ or $z \neq 0$ we know from Theorem 5.3 that H_{3,K_1} and H_{3,K_2} have rank 2, H_{3,L_1} has rank 3, $t = 2$, and $s = 1$. Then by Corollary 4.5.3, we know that

$H_{3,K_1} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^n}$ and either

$H_{3,L_1} \simeq \mathbb{Z}_{3^n} \times \mathbb{Z}_3 \times \mathbb{Z}_{3^n}$ or

$H_{3,L_1} \simeq \mathbb{Z}_{3^{n+1}} \times \mathbb{Z}_3 \times \mathbb{Z}_{3^n}$.

An identical result holds for H_{3,K_2} and H_{3,L_2} when $y = 0$ and either $x \neq 2w$ or $z \neq 0$.

Corollary 5.3.3 *In the situation described in Theorem 5.3 above, if $x = y = z = 0$, then H_{3,K_1} must be isomorphic to $\mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$ and H_{3,L_1} must satisfy one of the following:*

$H_{3,L_1} \simeq \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$,

$H_{3,L_1} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, *where $n_1 \geq 2$, or*

$H_{3,L_1} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2+1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, *where $n_1 \geq 2$.*

Further, if $y = z = 0$, and $x = 2w$ or $z \neq 0$, then H_{3,K_2} must be isomorphic to $\mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$ and H_{3,L_2} must satisfy one of the following:

$H_{3,L_2} \simeq \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$,

$H_{3,L_2} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, *where $n_1 \geq 2$, or*

$H_{3,L_2} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2+1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, *where $n_1 \geq 2$.*

Proof If $x = y = z = 0$, we know from Theorem 5.3 that H_{3,K_1} has rank 2, both H_{3,L_1} has rank 4, $t = 2$, and $s = 0$. Then by Corollary 4.5.4, we know that

$H_{3,L_1} \simeq \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$,

$H_{3,L_1} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, where $n_1 \geq 2$, or

$H_{3,L_1} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2+1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, where $n_1 \geq 2$.

An identical result holds for H_{3,K_2} and H_{3,L_2} when $y = z = 0$ and $x = 2w$.

Quadratic Subfields in Which p and 3 Split

We finish our examination of these fields with a look at what happens when both p and 3 split in k. Hence, in the theorems in this section we examine the ranks of the 3-class groups of the cubic fields K_1, K_2, and K_3 and their normal closures L_1, L_2, and L_3 of conductors $9p, 9p$, and p. Recall our earlier notes that 3 splits in k when $m \equiv -3 \pmod 9$ and that p splits in k when $p \equiv 1 \pmod 3$. The theorem below examines the ranks of the cubic fields and the normal closures in this situation.

Theorem 5.4 *Let $m \equiv -3$ (mod 9) and $p \equiv 1$ (mod 3). Let $(p) = \mathfrak{p}_1\mathfrak{p}_2$ in k. Let $\mathfrak{l}_1$ and $\mathfrak{l}_2$ be the prime divisors of 3 in k. Let x_1, x_2, x_3, x_4, and x_5 be defined by the norm residue symbols $(\frac{\mathfrak{l}_1, L_0/k}{\mathfrak{l}_2}) = \omega^{x_0}$, $(\frac{\mathfrak{l}_1, L_1/k}{\mathfrak{p}_1}) = \omega^{x_1}$, $(\frac{\mathfrak{l}_1, L_1/k}{\mathfrak{p}_2}) = \omega^{x_2}$, $(\frac{\mathfrak{p}_1, L_1/k}{\mathfrak{l}_1}) = \omega^{x_3}$, $(\frac{\mathfrak{p}_1, L_1/k}{\mathfrak{l}_2}) = \omega^{x_4}$, and $(\frac{\mathfrak{p}_1, L_1/k}{\mathfrak{p}_2}) = \omega^{x_5}$.*

The ranks of the 3-class groups of K_1 and K_2 are 3 or 2 according as $x_1 + x_2 = x_3 + x_4 = 0$ or not. The rank of the 3-class group of K_3 is 1.

If $x_1 + x_2 = x_3 + x_4 = 0$, the ranks of the 3-class groups of L_1 and L_2 are determined as follows. Both 3-class groups have rank 6 if $x_0 = x_1 = x_3 = x_5 = 0$. Both 3-class groups have rank 5 if either $x_0 = x_1 = 0$ or $x_3 = x_5 = 0$ but not both, or if $x_0 = x_3$ and $x_1 = x_5$, not all zero, or if $x_0 = 2x_3$ and $x_1 = 2x_5$, not all zero. Both 3-class groups have rank 4 otherwise.

If $x_1 + x_2 \neq 0$ or $x_3 + x_4 \neq 0$, the ranks of the 3-class groups of L_1 is determined as follows. The 3-class group has rank 5 if $x_0 = 2x_1 = 2x_2$ and $x_3 = x_4 = 2x_5$. The 3-class group has rank 4 if either $x_0 = 2x_1 = 2x_2$ or $x_3 = x_4 = 2x_5$ but not both, or if $x_0 = x_1 + x_2 + x_3 + 2x_4$ and $x_5 = x_1 + 2x_2 + x_3 + x_4$, or if $x_0 = x_1 + x_2 + 2x_4$ and $x_5 = 2x_1 + x_2 + 2x_3 + 2x_4$. The 3-class group has rank 3 otherwise.

If $x_1 + x_2 \neq 0$ or $x_3 + x_4 \neq 0$, the ranks of the 3-class groups of L_2 is determined as follows. The 3-class group has rank 5 if $x_0 = x_1 = x_2$ and $x_3 = x_4 = x_5$. The 3-class group has rank 4 if either $x_0 = x_1 = x_2$ or $x_3 = x_4 = x_5$ but not both, or if $x_0 = 2x_1 + 2x_2 + x_3 + 2x_4$ and $x_5 = x_1 + 2x_2 + 2x_3 + 2x_4$, or if $x_0 = 2x_1 + 2x_2 + 2x_3 + x_4$ and $x_5 = 2x_1 + x_2 + 2x_3 + 2x_4$. The 3-class group has rank 3 otherwise.

The rank of the 3-class group of L_3 is 2 or 1 according as $x_5 = 0$ or not.

Proof Let $\mathfrak{p}_1 = \mathfrak{P}_1\mathfrak{P}_2, \mathfrak{p}_2 = \mathfrak{P}_3\mathfrak{P}_4, \mathfrak{l}_1 = {\mathfrak{L}_1}^2$, and $\mathfrak{l}_2 = {\mathfrak{L}_2}^2$ in k_0. First note that since 3 doesn't ramify in L_3, $(\frac{\mathfrak{l}_1, L_3/k}{\mathfrak{l}_2}) = 1$, and $(\frac{\mathfrak{l}_1, L_3/k}{\mathfrak{l}_2}) = (\frac{\mathfrak{L}_1, f\beta\epsilon^2}{\mathfrak{L}_2}) = (\frac{\mathfrak{L}_1, f\beta}{\mathfrak{L}_2})(\frac{\mathfrak{L}_1, \epsilon}{\mathfrak{L}_2})^2 = (\frac{\mathfrak{l}_1, L_1/k}{\mathfrak{l}_2})(\frac{\mathfrak{l}_1, L_0/k}{\mathfrak{l}_2})^2$, so $(\frac{\mathfrak{l}_1, L_1/k}{\mathfrak{l}_2}) = (\frac{\mathfrak{l}_1, L_0/k}{\mathfrak{l}_2})$. Similarly, $(\frac{\mathfrak{q}_i, L_1/k}{\mathfrak{l}_j}) = (\frac{\mathfrak{q}_i, L_0/k}{\mathfrak{l}_j})$ for any $\mathfrak{q}_i \neq \mathfrak{l}_j$, for $\mathfrak{q}_i \in \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{l}_1, \mathfrak{l}_2\}$ and $j =$ 1 or 2. Also, $(\frac{\mathfrak{l}_1, L_2/k}{\mathfrak{l}_2}) = (\frac{\mathfrak{L}_1, f\beta\epsilon}{\mathfrak{L}_2}) = (\frac{\mathfrak{L}_1, f\beta}{\mathfrak{L}_2})(\frac{\mathfrak{L}_1, \epsilon}{\mathfrak{L}_2}) = (\frac{\mathfrak{l}_1, L_1/k}{\mathfrak{l}_2})(\frac{\mathfrak{l}_1, L_0/k}{\mathfrak{l}_2})$, and so, $(\frac{\mathfrak{l}_1, L_2/k}{\mathfrak{l}_2}) = (\frac{\mathfrak{l}_1, L_0/k}{\mathfrak{l}_2})^2$. Similarly, $(\frac{\mathfrak{q}_i, L_2/k}{\mathfrak{l}_j}) = (\frac{\mathfrak{q}_i, L_1/k}{\mathfrak{l}_j})^2$ for any $\mathfrak{q}_i \neq \mathfrak{l}_j$, for $\mathfrak{q}_i \in \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{l}_1, \mathfrak{l}_2\}$ and $j = 1$ or 2.

Next, note that since p doesn't ramify in K_0, $(\frac{\mathfrak{l}_1, L_0/k}{\mathfrak{p}_1}) = 1$, and $(\frac{\mathfrak{l}_1, L_2/k}{\mathfrak{p}_1}) = (\frac{\mathfrak{L}_1, f\beta\epsilon}{\mathfrak{P}_1})(\frac{\mathfrak{L}_1, f\beta\epsilon}{\mathfrak{P}_2}) = (\frac{\mathfrak{L}_1, f\beta}{\mathfrak{P}_1})(\frac{\mathfrak{L}_1, \epsilon}{\mathfrak{P}_1})(\frac{\mathfrak{L}_1, f\beta}{\mathfrak{P}_2})(\frac{\mathfrak{L}_1, \epsilon}{\mathfrak{P}_2}) = (\frac{\mathfrak{L}_1, f\beta}{\mathfrak{P}_1})(\frac{\mathfrak{L}_1, f\beta}{\mathfrak{P}_2})(\frac{\mathfrak{L}_1, \epsilon}{\mathfrak{P}_1})(\frac{\mathfrak{L}_1, \epsilon}{\mathfrak{P}_2}) = (\frac{\mathfrak{l}_1, L_1/k}{\mathfrak{p}_1})(\frac{\mathfrak{l}_1, L_0/k}{\mathfrak{p}_1}) = (\frac{\mathfrak{l}_1, L_1/k}{\mathfrak{p}_1})$. Similarly, $(\frac{\mathfrak{q}_i, L_2/k}{\mathfrak{p}_j}) = (\frac{\mathfrak{q}_i, L_1/k}{\mathfrak{p}_j})$ for any $\mathfrak{q}_i \neq \mathfrak{p}_j$ for $\mathfrak{q}_i \in \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{l}_1, \mathfrak{l}_2\}$ and $j = 1$ or 2. Also, $(\frac{\mathfrak{p}_1, L_0/k}{\mathfrak{p}_2}) = 1$, and $(\frac{\mathfrak{p}_1, L_3/k}{\mathfrak{p}_2}) = (\frac{\mathfrak{P}_1, f\beta\epsilon^2}{\mathfrak{P}_3})(\frac{\mathfrak{P}_1, f\beta\epsilon^2}{\mathfrak{P}_4}) = (\frac{\mathfrak{P}_1, f\beta}{\mathfrak{P}_3})(\frac{\mathfrak{P}_1, \epsilon}{\mathfrak{P}_3})^2(\frac{\mathfrak{P}_1, f\beta}{\mathfrak{P}_4})(\frac{\mathfrak{P}_1, \epsilon}{\mathfrak{P}_4})^2 = (\frac{\mathfrak{P}_1, f\beta}{\mathfrak{P}_3})(\frac{\mathfrak{P}_1, f\beta}{\mathfrak{P}_4})(\frac{\mathfrak{P}_1, \epsilon}{\mathfrak{P}_3})^2(\frac{\mathfrak{P}_1, \epsilon}{\mathfrak{P}_4})^2 = (\frac{\mathfrak{p}_1, L_1/k}{\mathfrak{p}_2})(\frac{\mathfrak{p}_1, L_0/k}{\mathfrak{p}_2}) = (\frac{\mathfrak{p}_1, L_1/k}{\mathfrak{p}_2})$. Similarly, $(\frac{\mathfrak{p}_2, L_3/k}{\mathfrak{p}_1}) = (\frac{\mathfrak{p}_2, L_1/k}{\mathfrak{p}_1})$.

In L_0 we have the exponent matrix,

	$\mathfrak{l}_1$	$\mathfrak{l}_2$
$\mathfrak{l}_1$	$2x_0$	x_0
$\mathfrak{l}_2$	$2x_0$	x_0

So, in L_1, L_2, and L_3 we have, respectively, the exponent matrices,

	$\mathfrak{l}_1$	$\mathfrak{l}_2$	$\mathfrak{p}_1$	$\mathfrak{p}_2$
$\mathfrak{l}_1$	$2(x_0+x_1+x_2)$	x_0	x_1	x_2
$\mathfrak{l}_2$	$2x_0$	$x_0+x_1+x_2$	$2x_2$	$2x_1$
$\mathfrak{p}_1$	x_3	x_4	$2(x_3+x_4+x_5)$	x_5
$\mathfrak{p}_2$	$2x_4$	$2x_3$	$2x_5$	$x_3+x_4+x_5$
$\mathfrak{l}_1$	$x_0+2x_1+2x_2$	$2x_0$	x_1	x_2
$\mathfrak{l}_2$	x_0	$2x_0+x_1+x_2$	$2x_2$	$2x_1$
$\mathfrak{p}_1$	$2x_3$	$2x_4$	$x_3+x_4+2x_5$	x_5
$\mathfrak{p}_2$	x_4	x_3	$2x_5$	$2x_3+2x_4+x_5$
$\mathfrak{p}_1$			$2x_5$	x_5
$\mathfrak{p}_2$			$2x_5$	x_5

Adding the first row to twice the second and the third row to twice the fourth for the matrices for L_1 and L_2, we obtain

	$\mathfrak{l}_1$	$\mathfrak{l}_2$	$\mathfrak{p}_1$	$\mathfrak{p}_2$
$\mathfrak{l}_1\mathfrak{l}_2^2$	$2(x_1+x_2)$	$2(x_1+x_2)$	x_1+x_2	x_1+x_2
$\mathfrak{p}_1\mathfrak{p}_2^2$	x_3+x_4	x_3+x_4	$2(x_3+x_4)$	$2(x_3+x_4)$
$\mathfrak{l}_1\mathfrak{l}_2^2$	$2(x_1+x_2)$	$2(x_1+x_2)$	x_1+x_2	x_1+x_2
$\mathfrak{p}_1\mathfrak{p}_2^2$	$2(x_3+x_4)$	$2(x_3+x_4)$	x_3+x_4	x_3+x_4

If $x_1+x_2 \neq 0$ and $x_3+x_4 \neq 0$, the rows can only differ by a multiple. Hence, the rank of both of these matrices depends entirely on whether $x_1+x_2 = x_3+x_4 = 0$ or not.

If $x_1+x_2 = x_3+x_4 = 0$, then the ranks of the matrices above are both $s_1 = 0$, and hence both H_{3,K_1} and H_{3,K_2} have rank $t - s_1 = 3$.

If, on the other hand, either $x_1+x_2 \neq 0$ or $x_3+x_4 \neq 0$, then the ranks of the matrices above are both $s_1 = 1$, and hence both H_{3,K_1} and H_{3,K_2} have rank $t - s_1 = 2$.

To examine the 3-class groups of L_1 and L_2, let us first define $a = x_1 + x_2$ and $b = x_3 + x_4$. Then $x_2 = 2x_1 + a$ and $x_4 = 2x_3 + b$, and our exponent matrices for L_1 and L_2 become

	$\mathfrak{l}_1$	$\mathfrak{l}_2$	$\mathfrak{p}_1$	$\mathfrak{p}_2$
$\mathfrak{l}_1$	$2x_0 + 2a$	x_0	x_1	$2x_1 + a$
$\mathfrak{l}_2$	$2x_0$	$x_0 + a$	$x_1 + 2a$	$2x_1$
$\mathfrak{p}_1$	x_3	$2x_3 + b$	$2x_5 + 2b$	x_5
$\mathfrak{p}_2$	$x_3 + 2b$	$2x_3$	$2x_5$	$x_5 + b$
$\mathfrak{l}_1$	$x_0 + 2a$	$2x_0$	x_1	$2x_1 + a$
$\mathfrak{l}_2$	x_0	$2x_0 + a$	$x_1 + 2a$	$2x_1$
$\mathfrak{p}_1$	$2x_3$	$x_3 + 2b$	$2x_5 + b$	x_5
$\mathfrak{p}_2$	$2x_3 + b$	x_3	$2x_5$	$x_5 + 2b$

Then, by adding twice the second row to the first and twice the third row to the fouth, we obtain, for L_1 and L_2 respectively,

	$\mathfrak{l}_1$	$\mathfrak{l}_2$	$\mathfrak{p}_1$	$\mathfrak{p}_2$
$\mathfrak{l}_1\mathfrak{l}_2^2$	$2a$	$2a$	a	a
$\mathfrak{l}_2$	$2x_0$	$x_0 + a$	$x_1 + 2a$	$2x_1$
$\mathfrak{p}_1$	x_3	$2x_3 + b$	$2x_5 + 2b$	x_5
$\mathfrak{p}_1^2\mathfrak{p}_2$	$2b$	$2b$	b	b
$\mathfrak{l}_1\mathfrak{l}_2^2$	$2a$	$2a$	a	a
$\mathfrak{l}_2$	x_0	$2x_0 + a$	$x_1 + 2a$	$2x_1$
$\mathfrak{p}_1$	$2x_3$	$x_3 + 2b$	$2x_5 + b$	x_5
$\mathfrak{p}_1^2\mathfrak{p}_2$	b	b	$2b$	$2b$

First, we will examine the case where $a = b = 0$. Then our matrices become

	$\mathfrak{l}_1$	$\mathfrak{l}_2$	$\mathfrak{p}_1$	$\mathfrak{p}_2$
$\mathfrak{l}_1\mathfrak{l}_2^2$	0	0	0	0
$\mathfrak{l}_2$	$2x_0$	x_0	x_1	$2x_1$
$\mathfrak{p}_1$	x_3	$2x_3$	$2x_5$	x_5
$\mathfrak{p}_1^2\mathfrak{p}_2$	0	0	0	0
$\mathfrak{l}_1\mathfrak{l}_2^2$	0	0	0	0
$\mathfrak{l}_2$	x_0	$2x_0$	x_1	$2x_1$
$\mathfrak{p}_1$	$2x_3$	x_3	$2x_5$	x_5
$\mathfrak{p}_1^2\mathfrak{p}_2$	0	0	0	0

which reduce to

	$\mathfrak{l}_1$	$\mathfrak{l}_2$	$\mathfrak{p}_1$	$\mathfrak{p}_2$
$\mathfrak{l}_2$	$2x_0$	x_0	x_1	$2x_1$
$\mathfrak{p}_1$	x_3	$2x_3$	$2x_5$	x_5
$\mathfrak{l}_2$	x_0	$2x_0$	x_1	$2x_1$
$\mathfrak{p}_1$	$2x_3$	x_3	$2x_5$	x_5

These matrices both have rank $s = 0$, and hence H_{3,L_1} and H_{3,L_2} both have rank $2t - s = 6 - 0 = 6$, when both rows are zero, i.e. when $x_0 = x_1 = x_3 = x_5 = 0$.

The matrices for L_1 and L_2 both have rank $s = 1$, and hence H_{3,L_1} and H_{3,L_2} both have rank $2t - s = 6 - 1 = 5$, when one row is zero and the other isn't or when both rows are linearly dependent and nonzero, i.e. when $x_0 = x_1 = 0$ but $x_3 \neq 0$ or $x_5 \neq 0$ or when $x_3 = x_5 = 0$ but $x_0 \neq 0$ or $x_1 \neq 0$ or when $x_0 = x_3$ and $x_1 = x_5$, not all zero, or when $x_0 = 2x_3$ and $x_1 = 2x_5$, not all zero.

The matrices for L_1 and L_2 have rank $s = 2$, and hence H_{3,L_1} and H_{3,L_2} both have rank $2t - s = 6 - 2 = 4$, when both rows are linearly independent, i.e. when $x_0 = 0$ but $x_1 \neq 0$ and $x_3 \neq 0$ or when $x_1 = 0$ but $x_0 \neq 0$ and $x_5 \neq 0$ or when $x_3 = 0$ but $x_0 \neq 0$ and $x_5 \neq 0$ or when $x_5 = 0$ but $x_1 \neq 0$ and $x_3 \neq 0$ or when $x_0 = x_3$ and $x_1 = 2x_5$, all nonzero, or when $x_0 = 2x_3$ and $x_1 = x_5$, all nonzero.

Now suppose that either $a \neq 0$ or $b \neq 0$. Then, since either the first or last row is nonzero and since, if both are nonzero one is a multiple of the other (mod 3), then we can reduce our matrices above to the matrices below, for L_1 and L_2, respectively.

	$\mathfrak{l}_1$	$\mathfrak{l}_2$	$\mathfrak{p}_1$	$\mathfrak{p}_2$
	1	1	2	2
$\mathfrak{l}_2$	$2x_0$	$x_0 + a$	$x_1 + 2a$	$2x_1$
$\mathfrak{p}_1$	x_3	$2x_3 + b$	$2x_5 + 2b$	x_5
	1	1	2	2
$\mathfrak{l}_2$	x_0	$2x_0 + a$	$x_1 + 2a$	$2x_1$
$\mathfrak{p}_1$	$2x_3$	$x_3 + 2b$	$2x_5 + b$	x_5

Where the first row is a multiple of the first or last row of the original matrix. Adding x_0 and $2x_0$ times the first rows to the second rows and $2x_3$ and x_3 times the first rows to the third rows of the matrices for L_1 and L_2, respectively, we obtain

$\mathfrak{l}_1$	$\mathfrak{l}_2$	$\mathfrak{p}_1$	$\mathfrak{p}_2$
1	1	2	2
0	$2x_0 + a$	$2x_0 + x_1 + 2a$	$2x_0 + 2x_1$
0	$x_3 + b$	$x_3 + 2x_5 + 2b$	$x_3 + x_5$
1	1	2	2
0	$x_0 + a$	$x_0 + x_1 + 2a$	$x_0 + 2x_1$
0	$2x_3 + 2b$	$2x_3 + 2x_5 + b$	$2x_3 + x_5$

Here the matrix for L_1 has rank $s = 1$, and hence H_{3,L_1} has rank $2t - s = 6 - 1 = 5$, when both the second and the third row of the matrix are zero, i.e. when $x_0 = a, x_1 = 2a, x_3 = 2b$, and $x_5 = b$. Similarly, H_{3,L_2} has rank 5 when $x_0 = 2a, x_1 = 2a, x_3 = 2b$, and $x_5 = 2b$.

The matrix for L_1 has rank $s = 2$, and hence H_{3,L_1} has rank $2t - s = 6 - 2 = 4$, when either the second or the third row of the matrix is zero, but the other isn't, or when the second row is a multiple of the third, i.e. when $x_0 = a$ and $x_1 = 2a$ but $x_3 \neq 2b$ or $x_5 \neq b$, or when $x_3 = 2b$ and $x_5 = b$ but $x_0 \neq a$ and $x_1 \neq 2a$, or when $x_0 + x_3 = 2x_1 + 2x_5 = a + 2b$, or when $x_0 + 2x_3 = 2x_1 + x_5 = a + 2b$. Similarly, the H_{3,L_2} has rank 4 when $x_0 = 2a$ and $x_1 = 2a$ but $x_3 \neq 2b$ or $x_5 \neq 2b$, or when

$x_3 = 2b$ and $x_5 = 2b$ but $x_0 \neq 2a$ and $x_1 \neq 2a$, or when $x_0 + x_3 = x_1 + x_5 = 2a + 2b$, or when $x_0 + 2x_3 = x_1 + 2x_5 = 2a + b$.

Finally, the matrix for L_1 has rank $s = 3$, and hence H_{3,L_1} has rank $2t - s = 6 - 3 = 3$, when both the second and the third row are linearly independent and nonzero, i.e. when either $x_0 \neq a$ or $x_1 \neq 2a$ and $x_3 \neq 2b$ or $x_5 \neq b$, and when $x_0 + x_3 \neq a + 2b$ or $2x_1 + 2x_5 \neq a + 2b$, and when $x_0 + 2x_3 \neq a + 2b$ or $2x_1 + x_5 \neq a + 2b$. Similarly, H_{3,L_2} has rank 3 when either $x_0 \neq 2a$ or $x_1 \neq 2a$ and $x_3 \neq 2b$ or $x_5 \neq 2b$, and when $x_0 + x_3 \neq 2a + 2b$ or $x_1 + x_5 \neq 2a + 2b$, and when $x_0 + 2x_3 \neq 2a + b$ or $x_1 + 2x_5 \neq 2a + b$.

Recall that our exponent matrix for L_3 is

	$\mathfrak{p}_1$	$\mathfrak{p}_2$
$\mathfrak{p}_1$	$2x_5$	x_5
$\mathfrak{p}_2$	$2x_5$	x_5

Note then that the rank of this matrix is $s = 0$ if $x_5 = 0$ and $s = 1$ if $x_5 \neq 0$. Therefore, if $x_5 = 0$, the rank of the matrix will be $s = 0$ and hence the rank of H_{3,L_3} will be $2t - s = 2 - 0 = 2$. However, if $x_5 \neq 0$, the rank of the matrix will be $s = 1$ and hence the rank of H_{3,L_3} will be $2t - s = 2 - 1 = 1$.

Furthermore, by adding the second row to twice the third, we find for K_3,

	$\mathfrak{p}_1$	$\mathfrak{p}_2$
$\mathfrak{p}_1\mathfrak{p}_2{}^2$	0	0

Hence, since the rank of this matrix is $s_1 = 0$, the rank of H_{3,K_3} is $t - s_1 = 1 - 0 = 1$.

Corollary 5.4.1 *In the situation described in Theorem* 5.4 *above, if* $x_5 \neq 0$*, then* $H_{3,K_3} = H_{3,L_3} = \mathbb{Z}_3$.

Proof By Theorem 5.4, the rank of H_{3,K_3} is always 1. Since $x_5 \neq 0$, the rank of H_{3,L_3} is also 1. By Corollary 4.3.2, $n = 1$ and $H_{3,K_3} = H_{3,L_3} = \mathbb{Z}_3$.

Corollary 5.4.2 *In the situation described in Theorem* 5.4 *above, if either* $x_1 + x_2 \neq 0$ *or* $x_3 + x_4 \neq 0$ *and both* $x_0 = 2x_1 = 2x_2$ *and* $x_3 = x_4 = 2x_5$*, then* H_{3,K_1} *must be isomorphic to* $\mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$ *and* H_{3,L_1} *must satisfy one of the following:*

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, where $n_1 \geq 2$, or

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, where $n_1 \geq 2$ and $n_2 \geq 2$.

Further, if either $x_1 + x_2 \neq 0$ or $x_3 + x_4 \neq 0$ and both $x_0 = x_1 = x_2$ and $x_3 = x_4 = x_5$, then H_{3,K_2} must be isomorphic to $\mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$ and H_{3,L_2} must satisfy one of the following:

$H_{3,L_2} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, where $n_1 \geq 2$, or

$H_{3,L_2} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, where $n_1 \geq 2$ and $n_2 \geq 2$.

Proof By Theorem 5.4, if either $x_1 + x_2 \neq 0$ or $x_3 + x_4 \neq 0$ and both $x_0 = 2x_1 = 2x_2$ and $x_3 = x_4 = 2x_5$, then H_{3,K_1} must have rank 2, H_{3,L_1} must have rank 5, $t = 3$, and $s = 1$. Then by Corollary 4.5.5, H_{3,K_1} must be isomorphic to $\mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$ and H_{3,L_1} must satisfy one of the following:

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, where $n_1 \geq 2$, or

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, where $n_1 \geq 2$ and $n_2 \geq 2$.

An identical result holds for H_{3,K_2} and H_{3,L_2} if either $x_1 + x_2 \neq 0$ or $x_3 + x_4 \neq 0$ and both $x_0 = x_1 = x_2$ and $x_3 = x_4 = x_5$.

Corollary 5.4.3 *In the situation described in Theorem 5.4 above, if $x_1 + x_2 \neq 0$ or $x_3 + x_4 \neq 0$ and either if $x_0 = 2x_1 = 2x_2$ or $x_3 = x_4 = 2x_5$ (but not both), if $x_0 = x_1 + x_2 + x_3 + 2x_4$ and $x_5 = x_1 + 2x_2 + x_3 + x_4$, or if $x_0 = x_1 + x_2 + 2x_4$ and $x_5 = 2x_1 + x_2 + 2x_3 + 2x_4$, then H_{3,K_1} must be isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_{3^n}$ and H_{3,L_1} must satisfy one of the following:*

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^n} \times \mathbb{Z}_3 \times \mathbb{Z}_{3^n}$ or

$H_{3,L} \simeq \mathbb{Z}_{3^{n-1}} \times \mathbb{Z}_3 \times \mathbb{Z}_{3^n} \times \mathbb{Z}_3$, where $n \geq 2$.

Further, if $x_1 + x_2 \neq 0$ or $x_3 + x_4 \neq 0$ and either if $x_0 = x_1 = x_2$ or $x_3 = x_4 = x_5$ (but not both), if $x_0 = 2x_1 + 2x_2 + x_3 + 2x_4$ and $x_5 = x_1 + 2x_2 + 2x_3 + 2x_4$, or if $x_0 = 2x_1 + 2x_2 + 2x_3 + x_4$ and $x_5 = 2x_1 + x_2 + 2x_3 + 2x_4$, then H_{3,K_2} must be isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_{3^n}$ and H_{3,L_2} must satisfy one of the following:

$H_{3,L_2} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^n} \times \mathbb{Z}_3 \times \mathbb{Z}_{3^n}$ or

$H_{3,L_2} \simeq \mathbb{Z}_{3^{n-1}} \times \mathbb{Z}_3 \times \mathbb{Z}_{3^n} \times \mathbb{Z}_3$, where $n \geq 2$.

Proof By Theorem 5.4, if $x_1 + x_2 \neq 0$ or $x_3 + x_4 \neq 0$ and either if $x_0 = 2x_1 = 2x_2$ or $x_3 = x_4 = 2x_5$ (but not both), if $x_0 = x_1 + x_2 + x_3 + 2x_4$ and $x_5 = x_1 + 2x_2 + x_3 + x_4$, or if $x_0 = x_1 + x_2 + 2x_4$ and $x_5 = 2x_1 + x_2 + 2x_3 + 2x_4$, then H_{3,K_1} must have rank 2, H_{3,L_1} must have rank 4, $t = 3$, and $s = 2$. Then by Corollary 4.5.4, H_{3,K_1} must be isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_{3^n}$ and H_{3,L_1} must satisfy one of the following:

$H_{3,L} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^n} \times \mathbb{Z}_3 \times \mathbb{Z}_{3^n}$ or

$H_{3,L} \simeq \mathbb{Z}_{3^{n-1}} \times \mathbb{Z}_3 \times \mathbb{Z}_{3^n} \times \mathbb{Z}_3$, where $n \geq 2$.

An identical result holds for H_{3,K_2} and H_{3,L_2} if $x_1 + x_2 \neq 0$ or $x_3 + x_4 \neq 0$ and either if $x_0 = x_1 = x_2$ or $x_3 = x_4 = x_5$ (but not both), if $x_0 = 2x_1 + 2x_2 + x_3 + 2x_4$ and $x_5 = x_1 + 2x_2 + 2x_3 + 2x_4$, or if $x_0 = 2x_1 + 2x_2 + 2x_3 + x_4$ and $x_5 = 2x_1 + x_2 + 2x_3 + 2x_4$.

Corollary 5.4.4 *In the situation described in Theorem* 5.4 *above, if* $x_1+x_2 \neq 0$ *or* $x_3+x_4 \neq 0$ *and if neither* $x_0 = 2x_1 = 2x_2$ *nor* $x_3 = x_4 = 2x_5$*, and if* $x_0 \neq x_1+x_2+x_3+2x_4$ *and* $x_5 \neq x_1+2x_2+x_3+x_4$*, and if* $x_0 \neq x_1+x_2+2x_4$ *and* $x_5 \neq 2x_1+x_2+2x_3+2x_4$*, then* $H_{3,K_1} \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$ *and* $H_{3,L_1} \simeq \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

Further, if $x_1 + x_2 \neq 0$ *or* $x_3 + x_4 \neq 0$ *and if neither* $x_0 = x_1 = x_2$ *nor* $x_3 = x_4 = x_5$*, and if* $x_0 \neq 2x_1 + 2x_2 + x_3 + 2x_4$ *and* $x_5 \neq x_1 + 2x_2 + 2x_3 + 2x_4$*, and if* $x_0 \neq 2x_1 + 2x_2 + 2x_3 + x_4$ *and* $x_5 \neq 2x_1 + x_2 + 2x_3 + 2x_4$*, then* $H_{3,K_2} \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$ *and* $H_{3,L_2} \simeq \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

Proof By Theorem 5.4, if $x_1 + x_2 \neq 0$ or $x_3 + x_4 \neq 0$ and if neither $x_0 = 2x_1 = 2x_2$ nor $x_3 = x_4 = 2x_5$, and if $x_0 \neq x_1 + x_2 + x_3 + 2x_4$ and $x_5 \neq x_1 + 2x_2 + x_3 + x_4$, and if $x_0 \neq x_1 + x_2 + 2x_4$ and $x_5 \neq 2x_1 + x_2 + 2x_3 + 2x_4$, then H_{3,K_1} must have rank 2, H_{3,L_1} must have rank 3, $t = 3$, and $s = 3$. Then by Corollary 4.5.3, $H_{3,K_1} \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$ and $H_{3,L_1} \simeq \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

An identical result holds for H_{3,K_2} and H_{3,L_2} if $x_1 + x_2 \neq 0$ or $x_3 + x_4 \neq 0$ and if neither $x_0 = x_1 = x_2$ nor $x_3 = x_4 = x_5$, and if $x_0 \neq 2x_1 + 2x_2 + x_3 + 2x_4$ and $x_5 \neq x_1 + 2x_2 + 2x_3 + 2x_4$, and if $x_0 \neq 2x_1 + 2x_2 + 2x_3 + x_4$ and $x_5 \neq 2x_1 + x_2 + 2x_3 + 2x_4$.

Corollary 5.4.5 *In the situation described in Theorem* 5.4 *above, if* $x_1 + x_2 = x_3 + x_4 = 0$ *and* $x_0 = x_1 = x_3 = x_5 = 0$*, then* H_{3,K_1} *and* H_{3,K_1} *must both be isomorphic to* $\mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$ *and* H_{3,L_1} *and* H_{3,L_2} *must satisfy one of the following (where* $i = 1$ *or* 2*):*

$H_{3,L_i} \simeq \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$,

$H_{3,L_i} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, *where* $n_1 \geq 2$,

$H_{3,L_i} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2+1}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, *where* $n_1 \geq 2$, *or*

$H_{3,L_i} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3+1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, *where both* $n_1 \geq 2$ *and* $n_2 \geq 2$.

Proof By Theorem 5.4, if $x_1 + x_2 = x_3 + x_4 = 0$ and $x_0 = x_1 = x_3 = x_5 = 0$, then the ranks of both H_{3,K_1} and H_{3,K_1} are 3 and the ranks of both H_{3,L_1} and H_{3,L_2} are 6. Then by Corollary 4.6.5, we know that

$H_{3,L_1} \simeq \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$,

$H_{3,L_1} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, where $n_1 \geq 2$,

$H_{3,L_1} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2+1}} \times \mathbb{Z}_{3^{n_3}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, where $n_1 \geq 2$, or

$H_{3,L_1} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2-1}} \times \mathbb{Z}_{3^{n_3+1}} \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_{3^{n_3}}$, where both $n_1 \geq 2$ and $n_2 \geq 2$.

An identical result holds for H_{3,L_2}.

Corollary 5.4.6 *In the situation described in Theorem* 5.4 *above, if $x_1 + x_2 = x_3 + x_4 = 0$ and if either $x_0 = x_1 = 0$ or $x_3 = x_5 = 0$ (but not both), or if $x_0 = x_1$ and $x_3 = x_5$ (not all zero), or if $x_0 = 2x_1$ and $x_3 = 2x_5$ (not all zero), then H_{3,K_1} and H_{3,K_1} must both be isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$ and H_{3,L_1} and H_{3,L_2} must satisfy one of the following (where $i = 1$ or 2):*

$H_{3,L_i} \simeq \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$,

$H_{3,L_i} \simeq \mathbb{Z}_{3^{n_1+1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, *or*

$H_{3,L_i} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, *where $n_1 \geq 2$.*

Proof By Theorem 5.4, if $x_1 + x_2 = x_3 + x_4 = 0$, if either $x_0 = x_1 = 0$ or $x_3 = x_5 = 0$ (but not both), or if $x_0 = x_1$ and $x_3 = x_5$ (not all zero), or if $x_0 = 2x_1$ and $x_3 = 2x_5$ (not all zero), then the ranks of both H_{3,K_1} and H_{3,K_1} are 3 and the ranks of both H_{3,L_1} and H_{3,L_2} are 6. Then by Corollary 4.6.4, we know that

$H_{3,L_1} \simeq \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$,

$H_{3,L_1} \simeq \mathbb{Z}_{3^{n_1+1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, or

$H_{3,L_1} \simeq \mathbb{Z}_{3^{n_1-1}} \times \mathbb{Z}_{3^{n_2}} \times \mathbb{Z}_3 \times \mathbb{Z}_{3^{n_1}} \times \mathbb{Z}_{3^{n_2}}$, where $n_1 \geq 2$.

An identical result holds for H_{3,L_2}.

Corollary 5.4.7 *In the situation described in Theorem* 5.4 *above, if $x_1 + x_2 = x_3 + x_4 = 0$ and if neither $x_0 = x_1 = 0$ or $x_3 = x_5 = 0$, and if $x_0 \neq x_1$ or $x_3 \neq x_5$, and if $x_0 \neq 2x_1$ or $x_3 \neq 2x_5$, then H_{3,K_1} and H_{3,K_1} must both be isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3^n}$ and both H_{3,L_1} and H_{3,L_2} must be isomorphic to $\mathbb{Z}_{3^{n+1}} \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3^n}$*

Proof By Theorem 5.4, if $x_1 + x_2 = x_3 + x_4 = 0$ and if neither $x_0 = x_1 = 0$ or $x_3 = x_5 = 0$, and if $x_0 \neq x_1$ or $x_3 \neq x_5$, and if $x_0 \neq 2x_1$ or $x_3 \neq 2x_5$, then the ranks of both H_{3,K_1} and H_{3,K_1} are 3 and the ranks of both H_{3,L_1} and H_{3,L_2} are 4. Then by Corollary 4.6.3, we know that $H_{3,L_1} \simeq \mathbb{Z}_{3^{n+1}} \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3^n}$.

An identical result holds for H_{3,L_2}.

Fields Defined by β Where $N(\beta) = \pm 3$

Now that we have proved results about the ranks of the 3-class groups of the fields L_1, L_2, L_3, K_1, K_2, and K_3 constructed by Martinet and Payan's method [12] from an element β of k_2 where $N_{k_2/\mathbb{Q}}(\beta) = \pm p$ for some prime $p \neq 3$, we will examine the case where $p = 3$. Here we let L_1, L_2, L_3 be the normal closures of the cubic fields K_1, K_2, K_3, defined as in section 1, each of conductor 9. Let $m > 1$ be a square-free integer relatively prime to 3, with Jacobi symbol $(\frac{m}{3}) = 1$. Further, if m is composite, say factoring into primes as $m = p_1 p_2 ... p_n$, then each p_i must have Jacobi symbol $(\frac{p_i}{3}) = 1$ for $i = 1, 2, ...n$. Let $k = k_1$, k_2, k_3, and k_0 be defined as before and again let ϵ denote the fundamental unit of k.

Theorem 5.5 *Let m be relatively prime to 3. Further, suppose that m factors as $m = p_1 p_2 ... p_n$ with each p_i prime and satisfying $p_i \equiv 1 \pmod 3$ for $i = 1, 2, ...n$. The ranks of the 3-class groups of the fields L_1, L_2, L_3, K_1, K_2, and K_3 are determined as follows. The ranks of the 3-class groups of the fields L_1, L_2, L_3, K_1, K_2, and K_3 are all 0.*

Proof Let $(3) = \mathfrak{L}^2$ in k_0. Let $\mathfrak{l}$ be the prime divisor of 3 in k. Since, by the product formula, all rows of our exponent matrices must sum to 0 (mod 3), $(\frac{\mathfrak{l}, L_1/k}{\mathfrak{l}}) = (\frac{\mathfrak{l}, L_2/k}{\mathfrak{l}}) = (\frac{\mathfrak{l}, L_3/k}{\mathfrak{l}}) = 1$. Hence, we find exponent matrices for L_1, L_2, and L_3 to all be the 1×1 zero matrix. Since here only one prime ramifies in the extension L/k, $t = 0$. Since our exponent matrices are zero matrices, we also have obviously $s = s_1 = 0$ for each of our fields here and hence, the ranks of the 3-class groups of L_1, L_2, and L_3 are all $2t - s = 0$ and the ranks of the 3-class groups of K_1, K_2, and K_3 are all $t - s_1 = 0$.

Fields Defined by β Where β is a Unit

Finally, we will turn our attention to those fields L_0 and K_0 constructed by Martinet and Payan's method [12] from a unit $\epsilon_{k_2} = \beta$ of k_2. Here we let L_0 be the normal closure of the cubic field K_0 defined as in section 1, of conductor 3 (or 9 if 3 splits in k). Let $m > 1$ be a square-free integer. Let $k = k_1$, k_2, k_3, and k_0 be defined as before and again let ϵ denote the fundamental unit of k.

Theorem 5.6 *Let m be relatively prime to 3 or $m \equiv 3 \pmod 9$. The ranks of the 3-class groups* *of the fields L_0 and K_0 are determined as follows. The ranks of the 3-class groups of both the cubic field K_0 and its normal closure L_0 are 0.*

Proof Let $(3) = \mathfrak{L}^2$ in k_0. Let $\mathfrak{l}$ be the prime divisor of 3 in k. Since, by the product formula, all rows of our exponent matrices must sum to 0 (mod 3), $(\frac{\mathfrak{l}, L_0/k}{\mathfrak{l}}) = 1$, so $w = 0$. Hence, we find the exponent matrix for L_0 to be the 1×1 zero matrix. Since here only one prime ideal totally ramifies in the extension L/k, $t = 0$. Further, since our exponent matrices are zero matrices, we also have obviously $s = s_1 = 0$ for each of our fields here and hence, the rank of the 3-class group of L_0 is $2t - s = 0$ and the rank of the 3-class group of K_0 is $t - s_1 = 0$.

Theorem 5.7 *Let $m \equiv -3$ (mod 9). The ranks of the 3-class groups of the fields L_0 and K_0 are determined as follows. Let $\mathfrak{l}_1$ and $\mathfrak{l}_2$ be the prime divisors of 3 in k. Let w be defined by the norm residue symbol $(\frac{\mathfrak{l}_1, L_0/k}{\mathfrak{l}_2}) = \omega^w$. The rank of the 3-class group of K_0 is 1. The rank of the 3-class group of L_0 is 2 or 1 according as $w = 0$ or not.*

Proof Let $\mathfrak{l}_1 = \mathfrak{L}_1{}^2$ and $\mathfrak{l}_2 = \mathfrak{L}_2{}^2$ in k_0. Then $(\frac{\mathfrak{l}_1, L_0/k}{\mathfrak{l}_2}) = (\frac{\mathfrak{L}_1, \epsilon}{\mathfrak{L}_2})$. Again, since conjugation of ideals within one of the norm residue symbols yields conjugation of the value of the symbol itself, $(\frac{\mathfrak{l}_2, L_0/k}{\mathfrak{l}_1}) = (\frac{\mathfrak{L}_2, \epsilon}{\mathfrak{L}_1}) = \omega^{2w}$. Finally, by the product formula, $(\frac{\mathfrak{L}_1, \epsilon}{\mathfrak{L}_1})(\frac{\mathfrak{L}_1, \epsilon}{\mathfrak{L}_2}) = 1$, so $(\frac{\mathfrak{L}_1, \epsilon}{\mathfrak{L}_1}) = (\frac{\mathfrak{L}_1, \epsilon}{\mathfrak{L}_2})^2$ and similarly, $(\frac{\mathfrak{L}_2, \epsilon}{\mathfrak{L}_2}) = (\frac{\mathfrak{L}_2, \epsilon}{\mathfrak{L}_1})^2$.

Hence we find the exponent matrix for L_0 to be

	$\mathfrak{l}_1$	$\mathfrak{l}_2$
$\mathfrak{l}_1$	$2w$	w
$\mathfrak{l}_2$	$2w$	w

If we add the first row of our matrix to twice the second, we find that

	$\mathfrak{l}_1$	$\mathfrak{l}_2$
$\mathfrak{l}_1$	$2w$	w
$\mathfrak{l}_1\mathfrak{l}_2{}^2$	0	0

Since 2 primes totally ramify in L_0, $t = 2 - 1 = 1$. If $w = 0$, we note that this matrix is the zero matrix, and so its rank is $s = 0$. Hence if $w = 0$, the rank of H_{3,L_0} is $2t - s = 2 - 0 = 2$. If on the other hand, $w \neq 0$, our first row is nonzero, so the rank of our matrix above is $s = 1$ and hence, the rank of H_{3,L_0} is $2t - s = 2 - 1 = 1$.

Recall that when we added the first row of our matrix to twice the second, we found that

	$\mathfrak{l}_1$	$\mathfrak{l}_2$
$\mathfrak{l}_1 \mathfrak{l}_2{}^2$	0	0

This matrix is the zero matrix and hence its rank is $s_1 = 0$, so the rank of H_{3,K_0} is $t - s_1 = 1 - 0 = 1$.

Corollary 5.7.1 *In the situation described in Theorem* 5.7 *above, if* $w \neq 0$*, then* $H_{3,K_3} = H_{3,L_3} = \mathbb{Z}_3$.

Proof By Theorem 5.7, the rank of H_{3,K_0} is always 1. Since $w \neq 0$, the rank of H_{3,L_0} is also 1. By Corollary 4.3.2, $n = 1$ and $H_{3,K_0} = H_{3,L_0} = \mathbb{Z}_3$.

Appendix A

Tables

Presented below are tables of invariants of the cubic fields described in chapter 3 and their normal closures. These invariants were generated by computer programs run in Mathematica , and include defining polynomials, class numbers and class group ranks. First, however, we will present a short list of fundamental units for certain of these fields. We only present a short selection of these units, as their coefficients quickly become too large for printing.

Cubic Fields Constructed from Elements of $\mathbb{Q}(\sqrt{2})$

We first examine fields generated by conjugate pairs (α, α') of the quadratic field $k_2 = \mathbb{Q}(\sqrt{2})$. In those cubic fields listed below, we first list a defining polynomial, $f(x)$, generated from α and α' as described in chapter 3. Then we list a basis, B_{K_i}, for the ring of integers of the cubic field where $B_{K_i} = \{\theta_0 = 1, \theta_1 = \theta, \theta_2\}$, θ is a root of the defining polynomial $f(x)$, and θ_2 is written in terms of θ. Then we give coefficients for the fundamental unit ϵ_{K_i} written in terms of these basis elements. Finally, we list a unit of the normal closure L_i in terms of θ that forms, along with ϵ_{K_i}, a system of fundamental units of L_i.

We begin with the cubic field defined by the unit of k_2, $(1+\sqrt{2})$. Here, $N_{k_2/\mathbb{Q}(\sqrt{2})}(\beta) = 1$.

$$f_{K_0}(x) = 2 + 3x + x^3$$

$$B_{K_0} = \{1, \theta, \theta^2\}$$

$$\epsilon_{K_0} = \{17, -3, 5\} \cdot B_{K_0}$$

$$\epsilon_{L_0} = \frac{1}{6}(-12 + 8\sqrt{-6} + 3\theta - \sqrt{-6}\theta - 3\theta^2 + 2\sqrt{-6}\theta^2)$$

Now we will examine the fields defined by elements of k_2 for which $N(\beta) = N_{k_2/\mathbb{Q}(\sqrt{2})}(\beta) = \pm 7$. These are listed below as K_1 for $\beta = (3+\sqrt{2})$, K_2 for $\beta = (5+4\sqrt{2})$, and K_3 for $\beta = (1-2\sqrt{2})$.

$$f_{K_1}(x) = -42 - 21x + x^3$$

$$B_{K_1} = \{1, \theta, \theta^2\}$$

$$\epsilon_{K_1} = \{4201, 2833, 537\} \cdot B_{K_1}$$

$$\epsilon_{L_1} = \frac{1}{2}(20 - 16\sqrt{-6} + 15\theta - 11\sqrt{-6}\theta + 3\theta^2 - 2\sqrt{-6}\theta^2)$$

$$f_{K_2}(x) = 36 - 9x + 6x^2 + x^3$$

$$B_{K_2} = \{1, \theta, \frac{\theta^2}{3}\}$$

$$\epsilon_{K_2} = \{85435, -32371, 55235\} \cdot B_{K_2}$$

$$\epsilon_{L_2} = \frac{1}{6}(150 + 162\sqrt{-6} - 57\theta - 62\sqrt{-6}\theta + 33\theta^2 + 35\sqrt{-6}\theta^2)$$

$$f_{K_3}(x) = 6 - 2x + x^2 + x^3$$

$$B_{K_3} = \{1, \theta, \theta^2\}$$

$$\epsilon_{K_3} = \{7, -5, 3\} \cdot B_{K_3}$$

$$\epsilon_{L_3} = \frac{1}{6}(102 - 9\sqrt{-6} + 15\theta + 23\sqrt{-6}\theta - 9\theta^2 + 10\sqrt{-6}\theta^2)$$

Finally we will examine the fields defined by elements of k_2 for which $N(\beta) = N_{k_2/\mathbb{Q}(\sqrt{2})}(\beta) = \pm 17$. These are listed below as K_1 for $\beta = (1-3\sqrt{2})$, K_2 for $\beta = (5+2\sqrt{2})$, and K_3 for $\beta = (9+7\sqrt{2})$.

$$f_{K_1}(x) = 12 + 6x + 18x^2 + x^3$$

$$B_{K_1} = \{1, \theta, \frac{\theta^2}{2}\}$$

$$\epsilon_{K_1} = \{55213, 24487, 162872\} \cdot B_{K_1}$$

$$\epsilon_{L_1} = \frac{1}{12}(48 - 48\sqrt{-6} + 18\theta - 24\sqrt{-6}\theta + 78\theta^2 - 73\sqrt{-6}\theta^2)$$

$$f_{K_2}(x) = 44 - 24x + 9x^2 + x^3$$

$$B_{K_2} = \{1, \theta, \frac{\theta}{2} + \frac{\theta^2}{2}\}$$

$$\epsilon_{K_2} = \{4056304382436678, -3621450504061723, 2108403250376723\} \cdot B_{K_2}$$

$$\epsilon_{L_2} = \frac{1}{12}(-113843190 + 12068294\sqrt{-6} + 720517747\theta - 7638064\sqrt{-6}\theta - 295869951\theta^2 + 3136454\sqrt{-6}\theta^2)$$

$$f_{K_3}(x) = 28 - 10x + 2x^2 + x^3$$

$$B_{K_3} = \{1, \theta, \frac{\theta^2}{2}\}$$

$$\epsilon_{K_3} = \{233850487239373, -129684056177665, 84610846485232\} \cdot B_{K_3}$$

$$\epsilon_{L_2} = \frac{1}{12}(52426512 - 14353720\sqrt{-6} - 290736300\theta + 7959996\sqrt{-6}\theta + 9484374\theta^2 - 2596703\sqrt{-6}\theta^2)$$

The tables below describe the cubic fields K_i constructed through Martinet and Payan's method for $k_2 = \mathbb{Q}(\sqrt{2})$. These tables list the primitive element β which defines K_i and the coefficients a, b, and c for a defining polynomial $f(x) = a + bx + cx^2 + x^3$ of K_i. Further, the table lists the element θ_2 of the integral basis B_{K_i} in terms of a root θ of the defining polynomial $f(x)$. Finally, the table also lists h_{K_i}, the class number of K_i, r_{3,K_i}, the rank of its 3-class group, h_{3,L_i}, the 3-class number of the normal closure L_i, and r_{3,L_i}, the rank of its 3-class group.

$N(\beta)$	i	β	a, b, c	θ_2	h_{K_i}	r_{3,K_i}	h_{3,L_i}	r_{3,L_i}
-1	0	$1+\sqrt{2}$	$2, 3, 0$	θ^2	1	0	1	0
7	1	$3+\sqrt{2}$	$-42, -21, 0$	θ^2	3	1	9	2
-7	2	$5+4\sqrt{2}$	$36, -9, 6$	$\frac{\theta^2}{3}$	3	1	9	2
-7	3	$1-2\sqrt{2}$	$6, -2, 1$	θ^2	3	1	3	1
-17	1	$1-3\sqrt{2}$	$12, 6, 18$	$\frac{\theta^2}{2}$	3	1	9	2
17	2	$5+2\sqrt{2}$	$44, -24, 9$	$\frac{\theta}{2}+\frac{\theta^2}{2}$	3	1	9	2
-17	3	$9+7\sqrt{2}$	$28, -10, 2$	$\frac{\theta^2}{2}$	1	0	1	0
-23	1	$7+6\sqrt{2}$	$252, 72, -3$	$\frac{\theta}{2}+\frac{\theta^2}{6}$	3	1	9	2
-23	2	$3-4\sqrt{2}$	$208, 72, 3$	$-\frac{\theta}{4}+\frac{\theta^2}{4}$	3	1	9	2
23	3	$5+\sqrt{2}$	$40, 8, -1$	$\frac{\theta}{2}+\frac{\theta^2}{2}$	1	0	1	0
-31	1	$1-4\sqrt{2}$	$16, 6, 24$	$\frac{\theta^2}{2}$	3	1	9	2
31	2	$7+3\sqrt{2}$	$126, -45, 12$	$\frac{\theta^2}{3}$	3	1	9	2
-31	3	$13+10\sqrt{2}$	$16, 26, -4$	$\frac{\theta^2}{2}$	3	1	3	1
-41	2	$3-5\sqrt{2}$	$150, 150, 9$	$-\frac{\theta}{5}+\frac{\theta^2}{5}$	3	1	9	2
41	2	$7+2\sqrt{2}$	$56, 24, 21$	$\frac{\theta}{2}+\frac{\theta^2}{2}$	3	1	9	2
-41	3	$11+9\sqrt{2}$	$24, 15, -2$	θ^2	1	0	1	0
-47	1	$5-6\sqrt{2}$	$612, 144, 3$	$\frac{\theta}{2}+\frac{\theta^2}{6}$	3	1	9	2
47	2	$7+\sqrt{2}$	$-658, -141, 0$	θ^2	3	1	9	2
-47	3	$9+8\sqrt{2}$	$44, 2, 10$	$\frac{\theta^2}{2}$	1	0	1	0
-71	1	$21+16\sqrt{2}$	$784, -105, 18$	$-\frac{3\theta}{7}+\frac{\theta^2}{7}$	3	1	9	2
-71	2	$1-6\sqrt{2}$	$24, 6, 36$	$\frac{\theta^2}{2}$	6	1	9	2
71	3	$11+5\sqrt{2}$	$360, -42, 4$	$-\frac{\theta}{3}+\frac{\theta^2}{6}$	1	0	1	0

$N(\beta)$	i	β	a, b, c	θ_2	h_{K_i}	r_{3,K_i}	h_{3,L_i}	r_{3,L_i}
-73	1	$13+11\sqrt{2}$	$1452, 231, -6$	$\frac{5\theta}{11}+\frac{\theta^2}{11}$	3	1	9	2
-73	2	$5-7\sqrt{2}$	$1176, 231, 6$	$-\frac{\theta}{7}+\frac{\theta^2}{7}$	9	1	81	3
73	3	$9+2\sqrt{2}$	$24, 54, 4$	$\frac{\theta^2}{2}$	3	1	3	1
-79	1	$11+10\sqrt{2}$	$1500, 240, -3$	$-\frac{3\theta}{10}+\frac{\theta^2}{10}$	27	2	729	3
-79	2	$7-8\sqrt{2}$	$1344, 240, 3$	$\frac{3\theta}{8}+\frac{\theta^2}{8}$	9	2	81	3
79	3	$9+\sqrt{2}$	$12, 54, 2$	$\frac{\theta^2}{2}$	9	1	27	2
-89	1	$19+15\sqrt{2}$	$1500, -120, 21$	$\frac{\theta}{10}+\frac{\theta^2}{10}$	15	1	9	2
-89	2	$3-7\sqrt{2}$	$252, 54, 42$	$\frac{\theta^2}{6}$	6	1	9	2
89	3	$11+4\sqrt{2}$	$234, 30, -1$	$-\frac{\theta}{3}+\frac{\theta^2}{3}$	1	0	1	0
97	1	$15+8\sqrt{2}$	$1216, -144, 21$	$-\frac{3\theta}{8}+\frac{\theta^2}{8}$	3	1	9	2
97	2	$-13+6\sqrt{2}$	$828, -144, 21$	$\frac{\theta}{2}+\frac{\theta^2}{6}$	15	1	9	2
-97	3	$1+7\sqrt{2}$	$504, -24, 5$	$-\frac{\theta}{6}+\frac{\theta^2}{6}$	6	1	3	1
103	1	$11+3\sqrt{2}$	$198, 54, 33$	$\frac{\theta^2}{3}$	27	2	243	4
-103	2	$17+14\sqrt{2}$	$2366, -117, 24$	$-\frac{2\theta}{13}+\frac{\theta^2}{13}$	9	2	81	4
-103	3	$5-8\sqrt{2}$	$810, -18, 7$	$-\frac{2\theta}{9}+\frac{\theta^2}{9}$	3	1	9	2
113	1	$11+2\sqrt{2}$	$88, 24, 33$	$\frac{\theta}{2}+\frac{\theta^2}{2}$	30	1	9	2
-113	2	$15+13\sqrt{2}$	$2704, 351, -6$	$-\frac{6\theta}{13}+\frac{\theta^2}{13}$	3	1	9	2
-113	3	$7-9\sqrt{2}$	$46, 38, -1$	θ^2	1	0	1	0

Note that we can also determine the conductor $\mathfrak{f}_{L/k}$ in the fields above by the norm of β (since $N_{k_2/\mathbb{Q}}(\beta) \neq \pm 3$) and the ordering of the fields, as we have denoted the fields in which 3 doesn't ramify by $i = 3$. The table below summarizes the value of the conductor in all of the fields above.

i	$\mathfrak{f}_{L/k}$
0	3
1	$3\lvert N_{k_2/\mathbb{Q}}(\beta)\rvert$
2	$3\lvert N_{k_2/\mathbb{Q}}(\beta)\rvert$
3	$\lvert N_{k_2/\mathbb{Q}}(\beta)\rvert$

Cubic Fields Constructed from Elements of $\mathbb{Q}(\sqrt{3})$

The tables below describe the cubic fields K_i constructed through Martinet and Payan's method for $k_2 = \mathbb{Q}(\sqrt{3})$. These tables list the same invariants for these fields as did the tables above for those fields constructed through Martinet and Payan's method for $k_2 = \mathbb{Q}(\sqrt{2})$.

$N(\beta)$	i	β	a, b, c	θ_2	h_{K_i}	r_{3,K_i}	h_{3,L_i}	r_{3,L_i}
1	0	$2+\sqrt{3}$	$-4,-3,0$	θ^2	1	0	1	0
-11	1	$1+2\sqrt{3}$	$12,36,3$	$\frac{\theta}{2}+\frac{\theta^2}{2}$	6	1	9	2
-11	2	$8+5\sqrt{3}$	$20,-18,12$	$\frac{\theta^2}{2}$	3	1	9	2
-11	3	$-4+3\sqrt{3}$	$4,9,4$	θ^2	1	0	1	0
13	1	$5-2\sqrt{3}$	$40,-12,9$	$\frac{\theta}{2}+\frac{\theta^2}{2}$	3	1	9	2
13	2	$4+\sqrt{3}$	$-104,-39,0$	θ^2	6	1	9	2
13	3	$11+6\sqrt{3}$	$-14,1,4$	$\frac{\theta}{2}+\frac{\theta^2}{2}$	3	1	3	1
-23	1	$5+4\sqrt{3}$	$160,72,-3$	$\frac{\theta}{4}+\frac{\theta^2}{4}$	6	1	9	2
-23	2	$22+13\sqrt{3}$	$32,-30,18$	$\frac{\theta^2}{2}$	3	1	9	2
-23	3	$-2+3\sqrt{3}$	$2,9,2$	θ^2	2	0	1	0
37	1	$7+2\sqrt{3}$	$88,-36,15$	$\frac{\theta}{2}+\frac{\theta^2}{2}$	3	1	9	2
37	2	$20+11\sqrt{3}$	$1100,270,12$	$\frac{\theta}{5}+\frac{\theta^2}{10}$	3	1	9	2
37	3	$8-3\sqrt{3}$	$-36,-7,4$	θ^2	6	1	3	1
-47	1	$-1+4\sqrt{3}$	$36,9,36$	$\frac{\theta^2}{3}$	12	1	9	2
-47	2	$10+7\sqrt{3}$	$490,168,-9$	$-\frac{2\theta}{7}+\frac{\theta^2}{7}$	3	1	9	2
-47	3	$41+24\sqrt{3}$	$12,-10,8$	$\frac{\theta^2}{2}$	1	0	1	0
-59	1	$32+19\sqrt{3}$	$56,-54,30$	$\frac{\theta^2}{2}$	3	1	9	2
-59	2	$-4+5\sqrt{3}$	$650,180,3$	$-\frac{2\theta}{5}+\frac{\theta^2}{5}$	12	1	9	2
-59	3	$7+6\sqrt{3}$	$44,21,2$	$\frac{\theta}{2}+\frac{\theta^2}{2}$	2	0	1	0
61	1	$8+\sqrt{3}$	$-976,-183,0$	θ^2	12	1	9	2
61	2	$19+10\sqrt{3}$	$1400,60,27$	$-\frac{3\theta}{10}+\frac{\theta^2}{10}$	18	1	81	3
61	3	$13-6\sqrt{3}$	$-72,-19,2$	$\frac{\theta}{2}+\frac{\theta^2}{2}$	3	1	3	1

$N(\beta)$	i	β	a,b,c	θ_2	h_{K_i}	r_{3,K_i}	h_{3,L_i}	r_{3,L_i}
-71	1	$-2+5\sqrt{3}$	$150,225,6$	$\frac{\theta}{5}+\frac{\theta^2}{5}$	6	1	9	2
-71	2	$11+8\sqrt{3}$	$896,240,-9$	$-\frac{\theta}{8}+\frac{\theta^2}{8}$	3	1	9	2
-71	3	$46+27\sqrt{3}$	$16,-14,10$	$\frac{\theta^2}{2}$	2	0	1	0
73	1	$29+16\sqrt{3}$	$3136,546,18$	$\frac{2\theta}{7}+\frac{\theta^2}{14}$	6	1	9	2
73	2	$11-4\sqrt{3}$	$416,-72,21$	$\frac{\theta}{4}+\frac{\theta^2}{4}$	18	1	81	3
73	3	$10+3\sqrt{3}$	$-70,-23,2$	θ^2	3	1	3	1
-83	1	$8+7\sqrt{3}$	$1078,252,-3$	$-\frac{3\theta}{7}+\frac{\theta^2}{7}$	3	1	9	2
-83	2	$37+22\sqrt{3}$	$68,-66,36$	$\frac{\theta^2}{2}$	3	1	9	2
-83	3	$-5+6\sqrt{3}$	$-12,29,2$	$\frac{\theta}{2}+\frac{\theta^2}{2}$	2	0	1	0
97	1	$17-8\sqrt{3}$	$1408,-48,27$	$\frac{3\theta}{8}+\frac{\theta^2}{8}$	18	2	27	3
97	2	$10+\sqrt{3}$	$-1940,-291,0$	θ^2	9	2	27	3
97	3	$23+12\sqrt{3}$	$-206,-27,4$	$\frac{1}{2}-\frac{\theta}{4}+\frac{\theta^2}{4}$	6	1	9	2
-107	1	$20+13\sqrt{3}$	$1300,-210,36$	$-\frac{2\theta}{5}+\frac{\theta^2}{5}$	3	1	9	2
-107	2	$-16+11\sqrt{3}$	$1694,396,-15$	$-\frac{4\theta}{11}+\frac{\theta^2}{11}$	6	1	9	2
-107	3	$1+6\sqrt{3}$	$2,1,18$	θ^2	4	0	1	0
109	1	$16-7\sqrt{3}$	$1274,-84,27$	$-\frac{\theta}{7}+\frac{\theta^2}{7}$	6	1	9	2
109	2	$11+2\sqrt{3}$	$132,36,33$	$\frac{\theta}{2}+\frac{\theta^2}{2}$	12	1	9	2
109	3	$28+15\sqrt{3}$	$-250,-35,2$	$\frac{2\theta}{5}+\frac{\theta^2}{5}$	3	1	3	1

We can again determine the conductor $\mathfrak{f}_{L/k}$ in the fields described above by the norm of β and the ordering of the fields, as summarized in the table for $k_2=\mathbb{Q}(\sqrt{2})$.

Cubic Fields Constructed from Elements of $\mathbb{Q}(\sqrt{5})$

The tables below describe the cubic fields K_i constructed through Martinet and Payan's method for $k_2 = \mathbb{Q}(\sqrt{5})$. These tables list the same invariants for these fields as did the tables above for those fields constructed through Martinet and Payan's method for $k_2 = \mathbb{Q}(\sqrt{2})$.

$N(\beta)$	i	β	a, b, c	θ_2	h_{K_i}	r_{3,K_i}	h_{3,L_i}	r_{3,L_i}
-1	0	$\frac{1}{2} + \frac{\sqrt{5}}{2}$	$1, 3, 0$	θ^2	1	0	1	0
-11	1	$3 + 2\sqrt{5}$	$16, -6, 9$	$\frac{\theta}{2} + \frac{\theta^2}{2}$	3	1	9	2
-11	2	$\frac{1}{2} - \frac{3\sqrt{5}}{2}$	$45, 36, 3$	$\frac{\theta^2}{3}$	3	1	9	2
11	3	$\frac{7}{2} - \frac{\sqrt{5}}{2}$	$-9, 4, -1$	θ^2	1	0	1	0
-19	1	$1 + 2\sqrt{5}$	$96, 60, 3$	$-\frac{\theta}{4} + \frac{\theta^2}{4}$	3	1	9	2
19	2	$\frac{11}{2} - \frac{3\sqrt{5}}{2}$	$45, -9, 12$	$\frac{\theta^2}{3}$	3	1	9	2
19	3	$-\frac{9}{2} + \frac{\sqrt{5}}{2}$	$45, -6, -1$	$-\frac{\theta}{3} + \frac{\theta^2}{3}$	3	1	3	1
-29	1	$4 + 3\sqrt{5}$	$96, -12, 15$	$-\frac{\theta}{4} + \frac{\theta^2}{4}$	18	1	81	2
-29	2	$\frac{3}{2} - \frac{5\sqrt{5}}{2}$	$175, 90, 3$	$-\frac{2\theta}{5} + \frac{\theta^2}{5}$	9	1	81	2
29	3	$\frac{11}{2} + \frac{\sqrt{5}}{2}$	$15, 10, -1$	θ^2	1	0	1	0
31	1	$6 + \sqrt{5}$	$32, -18, 15$	$\frac{\theta}{2} + \frac{\theta^2}{2}$	3	1	9	2
-31	2	$\frac{11}{2} + \frac{7\sqrt{5}}{2}$	$63, -18, 15$	$\frac{\theta^2}{3}$	3	1	9	2
-31	3	$\frac{1}{2} - \frac{5\sqrt{5}}{2}$	$-23, -10, -1$	θ^2	3	1	9	2
-41	1	$2 - 3\sqrt{5}$	$288, 126, 3$	$\frac{\theta}{2} + \frac{\theta^2}{6}$	3	1	9	2
41	2	$\frac{13}{2} + \frac{\sqrt{5}}{2}$	$-533, -123, 0$	θ^2	3	1	9	2
-41	3	$\frac{9}{2} + \frac{7\sqrt{5}}{2}$	$99, 15, 2$	$-\frac{\theta}{3} + \frac{\theta^2}{3}$	1	0	1	0
-59	1	$\frac{3}{2} - \frac{7\sqrt{5}}{2}$	$539, 189, 6$	$-\frac{\theta}{7} + \frac{\theta^2}{7}$	15	1	9	2
59	2	$8 + \sqrt{5}$	$48, -30, 21$	$\frac{\theta}{2} + \frac{\theta^2}{2}$	12	1	9	2
-59	3	$\frac{13}{2} + \frac{9\sqrt{5}}{2}$	$15, 21, -2$	θ^2	1	0	1	0
-61	1	$\frac{19}{2} + \frac{11\sqrt{5}}{2}$	$99, -36, 21$	$\frac{\theta^2}{3}$	3	1	9	2
-61	2	$-\frac{1}{2} - \frac{7\sqrt{5}}{2}$	$637, 210, 9$	$\frac{2\theta}{7} + \frac{\theta^2}{7}$	9	1	81	3
61	3	$9 + 2\sqrt{5}$	$192, -20, -1$	$-\frac{\theta}{4} + \frac{\theta^2}{4}$	3	1	3	1

$N(\beta)$	i	β	a, b, c	θ_2	h_{K_i}	r_{3,K_i}	h_{3,L_i}	r_{3,L_i}
-71	1	$3-4\sqrt{5}$	$640, 216, 3$	$\frac{3\theta}{8}+\frac{\theta^2}{8}$	3	1	9	2
71	2	$\frac{17}{2}+\frac{\sqrt{5}}{2}$	$-1207, -213, 0$	θ^2	6	1	9	2
-71	3	$\frac{11}{2}+\frac{9\sqrt{5}}{2}$	$21, 24, -1$	θ^2	1	0	1	0
79	1	$\frac{19}{2}+\frac{3\sqrt{5}}{2}$	$117, -45, 24$	$\frac{\theta^2}{3}$	6	1	9	2
-79	2	$\frac{17}{2}+\frac{11\sqrt{5}}{2}$	$275, -45, 24$	$-\frac{\theta}{5}+\frac{\theta^2}{5}$	3	1	9	2
-79	3	$1-4\sqrt{5}$	$80, -10, 7$	$\frac{\theta}{2}+\frac{\theta^2}{2}$	3	1	3	1
89	1	$\frac{19}{2}+\frac{\sqrt{5}}{2}$	$-1691, -267, 0$	θ^2	3	1	9	2
-89	2	$6+5\sqrt{5}$	$640, -24, 27$	$\frac{3\theta}{8}+\frac{\theta^2}{8}$	3	1	9	2
-89	3	$\frac{7}{2}-\frac{9\sqrt{5}}{2}$	$3, 31, -2$	θ^2	1	0	1	0
101	1	$11+2\sqrt{5}$	$224, -60, 27$	$-\frac{\theta}{4}+\frac{\theta^2}{4}$	9	1	81	2
-101	2	$\frac{21}{2}+\frac{13\sqrt{5}}{2}$	$325, -60, 27$	$\frac{2\theta}{5}+\frac{\theta^2}{5}$	9	1	81	2
-101	3	$\frac{1}{2}-\frac{9\sqrt{5}}{2}$	$15, 34, 1$	θ^2	10	0	1	0
-109	1	$4-5\sqrt{5}$	$1200, 330, 3$	$\frac{3\theta}{10}+\frac{\theta^2}{10}$	3	1	9	2
109	2	$\frac{21}{2}+\frac{\sqrt{5}}{2}$	$-2289, -327, 0$	θ^2	6	1	9	2
-109	3	$\frac{13}{2}-\frac{11\sqrt{5}}{2}$	$33, -3, 10$	θ^2	3	1	3	1

We can yet again determine the conductor $\mathfrak{f}_{L/k}$ in the fields described above by the norm of β and the ordering of the fields, as summarized in the table for $k_2 = \mathbb{Q}(\sqrt{2})$.

Cubic Fields Constructed from Elements of $\mathbb{Q}(\sqrt{6})$

The tables below describe the cubic fields K_i constructed through Martinet and Payan's method for $k_2 = \mathbb{Q}(\sqrt{6})$. These tables list the same invariants for these fields as did the tables above for those fields constructed through Martinet and Payan's method for $k_2 = \mathbb{Q}(\sqrt{2})$.

$N(\beta)$	i	β	a, b, c	θ_2	h_{K_i}	r_{3,K_i}	h_{3,L_i}	r_{3,L_i}
1	0	$5+2\sqrt{6}$	$8, 3, 0$	$\frac{\theta}{2}+\frac{\theta^2}{2}$	3	1	3	1
−5	1	$1+\sqrt{6}$	$10, 15, 0$	θ^2	3	1	9	2
−5	2	$17+7\sqrt{6}$	$28, -18, 6$	$\frac{\theta^2}{2}$	3	1	9	2
−5	3	$-7+3\sqrt{6}$	$2, 7, 4$	θ^2	1	0	1	0
19	1	$13-5\sqrt{6}$	$350, -30, 9$	$-\frac{\theta}{5}+\frac{\theta^2}{5}$	9	2	81	4
19	2	$5+\sqrt{6}$	$-190, -57, 0$	θ^2	9	2	81	4
19	3	$37+15\sqrt{6}$	$8, 18, 4$	$\frac{\theta^2}{2}$	3	1	3	1
−23	1	$-19+8\sqrt{6}$	$640, 96, -9$	$-\frac{\theta}{8}+\frac{\theta^2}{8}$	3	1	9	2
−23	2	$1+2\sqrt{6}$	$24, 72, 3$	$\frac{\theta}{2}+\frac{\theta^2}{2}$	6	1	9	2
−23	3	$29+12\sqrt{6}$	$16, -10, 4$	$\frac{\theta^2}{2}$	1	0	1	0
−29	1	$11+5\sqrt{6}$	$80, 72, 39$	$-\frac{\theta}{4}+\frac{\theta^2}{4}$	3	1	9	2
−29	2	$115+47\sqrt{6}$	$48, 18, 24$	$\frac{\theta^2}{2}$	6	1	9	2
−29	3	$-5+3\sqrt{6}$	$-4, 11, 2$	θ^2	1	0	1	0
43	1	$7+\sqrt{6}$	$-602, -129, 0$	θ^2	27	2	243	4
43	2	$47+19\sqrt{6}$	$1900, 270, 6$	$-\frac{2\theta}{5}+\frac{\theta^2}{10}$	9	2	27	3
43	3	$23-9\sqrt{6}$	$-90, -9, 4$	$\frac{\theta}{3}+\frac{\theta^2}{3}$	3	1	3	1
−47	1	$137+56\sqrt{6}$	$60, 18, 30$	$\frac{\theta^2}{2}$	3	1	9	2
−47	2	$-7+4\sqrt{6}$	$208, 168, -9$	$-\frac{\theta}{4}+\frac{\theta^2}{4}$	3	1	9	2
−47	3	$13+6\sqrt{6}$	$2, 7, 16$	θ^2	1	0	1	0
−53	1	$41+17\sqrt{6}$	$1700, -210, 18$	$-\frac{\theta}{5}+\frac{\theta^2}{10}$	9	1	81	3
−53	2	$-31+13\sqrt{6}$	$2366, 234, -15$	$-\frac{2\theta}{13}+\frac{\theta^2}{13}$	3	1	9	2
−53	3	$1+3\sqrt{6}$	$16, 19, 2$	θ^2	2	0	1	0

$N(\beta)$	i	β	a, b, c	θ_2	h_{K_i}	r_{3,K_i}	h_{3,L_i}	r_{3,L_i}
67	1	$19 + 7\sqrt{6}$	$1666, -126, 15$	$\frac{\theta}{7} + \frac{\theta^2}{7}$	27	2	729	4
67	2	$179 + 73\sqrt{6}$	$360, 72, 45$	$\frac{\theta}{2} + \frac{\theta^2}{6}$	9	2	27	3
67	3	$11 - 3\sqrt{6}$	$-82, -17, 4$	θ^2	9	1	27	2
-71	1	$-5 + 4\sqrt{6}$	$496, 216, -3$	$\frac{\theta}{4} + \frac{\theta^2}{4}$	6	1	9	2
-71	2	$23 + 10\sqrt{6}$	$2600, 240, -9$	$\frac{\theta}{10} + \frac{\theta^2}{10}$	9	1	81	3
-71	3	$235 + 96\sqrt{6}$	$28, 18, 14$	$\frac{\theta^2}{2}$	1	0	1	0
73	1	$13 + 4\sqrt{6}$	$656, 24, 27$	$-\frac{\theta}{4} + \frac{\theta^2}{4}$	18	2	27	3
73	2	$113 + 46\sqrt{6}$	$90, 18, 45$	$\frac{\theta^2}{3}$	27	2	729	5
73	3	$17 - 6\sqrt{6}$	$-122, -19, 4$	$\frac{\theta}{2} + \frac{\theta^2}{2}$	3	1	3	1
97	1	$11 + 2\sqrt{6}$	$244, -48, 27$	$\frac{\theta}{2} + \frac{\theta^2}{2}$	9	2	27	3
97	2	$79 + 32\sqrt{6}$	$6272, 630, 12$	$-\frac{\theta}{7} + \frac{\theta^2}{14}$	9	2	27	3
97	3	$31 - 12\sqrt{6}$	$-244, -31, 2$	$-\frac{\theta}{4} + \frac{\theta^2}{4}$	3	1	3	1
-101	1	$-7 + 5\sqrt{6}$	$800, 315, -6$	$-\frac{\theta}{5} + \frac{\theta^2}{5}$	3	1	9	2
-101	2	$25 + 11\sqrt{6}$	$4114, 330, -9$	$\frac{2\theta}{11} + \frac{\theta^2}{11}$	3	1	9	2
-101	3	$257 + 105\sqrt{6}$	$32, 18, 16$	$\frac{\theta^2}{2}$	2	0	1	0

We can again determine the conductor $\mathfrak{f}_{L/k}$ in the fields described above by the norm of β and the ordering of the fields. However, in the case of $k_2 = \mathbb{Q}(\sqrt{6})$, the conductor is different as 3 splits in $k_1 = \mathbb{Q}(\sqrt{-2})$. The table below summarizes the value of the conductor in all of the fields described above.

i	$\mathfrak{f}_{L/k}$
0	9
1	$9\lvert N_{k_2/\mathbb{Q}}(\beta)\rvert$
2	$9\lvert N_{k_2/\mathbb{Q}}(\beta)\rvert$
3	$\lvert N_{k_2/\mathbb{Q}}(\beta)\rvert$

Cubic Fields Constructed from Elements of $\mathbb{Q}(\sqrt{7})$

The tables below describe the cubic fields K_i constructed through Martinet and Payan's method for $k_2 = \mathbb{Q}(\sqrt{7})$. These tables list the same invariants for these fields as did the tables above for those fields constructed through Martinet and Payan's method for $k_2 = \mathbb{Q}(\sqrt{2})$.

$N(\beta)$	i	β	a, b, c	θ_2	h_{K_i}	r_{3,K_i}	h_{3,L_i}	r_{3,L_i}
1	0	$8+3\sqrt{7}$	$8, 6, 6$	$\frac{\theta^2}{2}$	1	0	1	0
-3	1	$2+1\sqrt{7}$	$12, 9, 0$	θ^2	1	0	1	0
-3	2	$37+14\sqrt{7}$	$12, 18, 0$	$\frac{\theta^2}{2}$	1	0	1	0
-3	3	$-5+2\sqrt{7}$	$20, 12, -3$	$\frac{\theta}{2}+\frac{\theta^2}{2}$	1	0	1	0
-19	1	$291+110\sqrt{7}$	$96, -24, 21$	$\frac{\theta}{4}+\frac{\theta^2}{4}$	3	1	9	2
-19	2	$-3+2\sqrt{7}$	$56, 60, -3$	$\frac{\theta}{2}+\frac{\theta^2}{2}$	3	1	9	2
-19	3	$18+7\sqrt{7}$	$56, 14, -2$	$\frac{\theta^2}{2}$	3	1	3	1
29	1	$6+\sqrt{7}$	$-348, -87, 0$	θ^2	3	1	9	2
29	2	$69+26\sqrt{7}$	$396, 18, 24$	$\frac{\theta^2}{6}$	3	1	9	2
29	3	$27-10\sqrt{7}$	$84, -14, 4$	$\frac{\theta^2}{2}$	1	0	1	0
-31	1	$12+5\sqrt{7}$	$550, 105, -6$	$-\frac{\theta}{5}+\frac{\theta^2}{5}$	3	1	9	2
-31	2	$201+76\sqrt{7}$	$504, 198, 6$	$\frac{\theta^2}{6}$	3	1	9	2
-31	3	$-9+4\sqrt{7}$	$252, 42, -8$	$-\frac{\theta}{3}+\frac{\theta^2}{6}$	3	1	3	1
37	1	$10-3\sqrt{7}$	$360, -63, 12$	$\frac{\theta^2}{3}$	3	1	9	2
37	2	$17+6\sqrt{7}$	$828, -36, 15$	$\frac{\theta}{2}+\frac{\theta^2}{6}$	3	1	9	2
37	3	$262+99\sqrt{7}$	$396, -12, -1$	$-\frac{\theta}{6}+\frac{\theta^2}{6}$	9	1	27	2
-47	1	$-4+3\sqrt{7}$	$234, 144, -3$	$\frac{\theta^2}{3}$	6	1	9	2
-47	2	$31+12\sqrt{7}$	$1452, -231, 24$	$\frac{2\theta}{11}+\frac{\theta^2}{11}$	3	1	9	2
-47	3	$500+189\sqrt{7}$	$336, 16, 1$	$\frac{\theta}{4}+\frac{\theta^2}{4}$	1	0	1	0
53	1	$114+43\sqrt{7}$	$576, -18, 30$	$\frac{\theta^2}{6}$	9	1	81	2
53	2	$30-11\sqrt{7}$	$2420, 33, 24$	$\frac{2\theta}{11}+\frac{\theta^2}{11}$	9	1	81	2
53	3	$9+2\sqrt{7}$	$-954, -159, 0$	$\frac{\theta}{2}+\frac{\theta^2}{6}$	1	0	1	0

$N(\beta)$	i	β	a,b,c	θ_2	h_{K_i}	r_{3,K_i}	h_{3,L_i}	r_{3,L_i}
-59	1	$2+3\sqrt{7}$	$126, 189, 6$	$\frac{\theta^2}{3}$	3	1	9	2
-59	2	$79+30\sqrt{7}$	$14, -24, 39$	θ^2	3	1	9	2
-59	3	$-47+18\sqrt{7}$	$168, 28, -5$	$\frac{\theta}{2}+\frac{\theta^2}{2}$	1	0	1	0
-83	1	$365+138\sqrt{7}$	$2000, 510, 6$	$-\frac{2\theta}{5}+\frac{\theta^2}{10}$	3	1	9	2
-83	2	$-13+6\sqrt{7}$	$24, -42, 66$	$\frac{\theta^2}{2}$	3	1	9	2
-83	3	$22+9\sqrt{7}$	$96, 33, -4$	θ^2	1	0	1	0
-103	1	$-60+23\sqrt{7}$	$2300, -420, 39$	$-\frac{\theta}{10}+\frac{\theta^2}{10}$	3	1	9	2
-103	2	$3+4\sqrt{7}$	$928, 312, 3$	$-\frac{\theta}{4}+\frac{\theta^2}{4}$	6	1	9	2
-103	3	$108+41\sqrt{7}$	$196, 74, -4$	$\frac{\theta^2}{2}$	3	1	9	2
109	1	$397+150\sqrt{7}$	$128, -66, 42$	$\frac{\theta^2}{2}$	3	1	9	2
109	2	$19-6\sqrt{7}$	$2196, -180, 21$	$\frac{\theta}{2}+\frac{\theta^2}{6}$	3	1	9	2
109	3	$26+9\sqrt{7}$	$132, -15, 8$	θ^2	3	1	3	1
113	1	$15+4\sqrt{7}$	$992, 24, 33$	$\frac{\theta}{4}+\frac{\theta^2}{4}$	3	1	9	2
113	2	$204+77\sqrt{7}$	$936, -90, 42$	$\frac{\theta^2}{6}$	3	1	9	2
113	3	$36-13\sqrt{7}$	$1008, -42, 10$	$-\frac{\theta}{3}+\frac{\theta^2}{6}$	1	0	1	0

We can yet again determine the conductor $\mathfrak{f}_{L/k}$ in the fields described above by the norm of β and the ordering of the fields, as summarized in the table for $k_2=\mathbb{Q}(\sqrt{2})$, when $N_{k_2/\mathbb{Q}}(\beta)\neq\pm 3$. However, in this case, where $k_2=\mathbb{Q}(\sqrt{7})$, 3 splits in k_2 and hence, if the norm of β is ± 3, then all three fields have conductor $\mathfrak{f}_{L_1/k}=\mathfrak{f}_{L_2/k}=\mathfrak{f}_{L_3/k}=9$.

Cubic Fields Constructed from Elements of $\mathbb{Q}(\sqrt{10})$

The tables below describe the cubic fields K_i constructed through Martinet and Payan's method for $k_2 = \mathbb{Q}(\sqrt{10})$. These tables list the same invariants for these fields as did the tables above for those fields constructed through Martinet and Payan's method for $k_2 = \mathbb{Q}(\sqrt{2})$.

$N(\beta)$	i	β	a, b, c	θ_2	h_{K_i}	r_{3,K_i}	h_{3,L_i}	r_{3,L_i}
-1	0	$3+\sqrt{10}$	$6, 3, 0$	θ^2	1	0	1	0
31	1	$-11+3\sqrt{10}$	$342, -18, 15$	$\frac{\theta^2}{3}$	30	1	9	2
-31	2	$3+2\sqrt{10}$	$92, 96, -3$	$\frac{\theta}{2}+\frac{\theta^2}{2}$	6	1	9	2
31	3	$29+9\sqrt{10}$	$60, -9, 2$	θ^2	9	1	27	2
-41	1	$47+15\sqrt{10}$	$2940, 714, 66$	$-\frac{2\theta}{7}+\frac{\theta^2}{14}$	6	1	9	2
-41	2	$7-3\sqrt{10}$	$12, -24, 57$	$\frac{\theta}{2}+\frac{\theta^2}{2}$	3	1	9	2
41	3	$9+2\sqrt{10}$	$-738, -123, 0$	$\frac{\theta}{2}+\frac{\theta^2}{6}$	1	0	1	0
-71	1	$37+12\sqrt{10}$	$48, -78, 60$	$\frac{\theta^2}{2}$	15	1	9	2
-71	2	$17-6\sqrt{10}$	$1224, 288, -15$	$\frac{\theta}{2}+\frac{\theta^2}{6}$	3	1	9	2
71	3	$9+\sqrt{10}$	$-1278, -213, 0$	$\frac{\theta^2}{3}$	1	0	1	0
79	1	$13+3\sqrt{10}$	$738, -90, 21$	$\frac{\theta^2}{3}$	15	1	9	2
-79	2	$69+22\sqrt{10}$	$9702, -189, 12$	$-\frac{3\theta}{7}+\frac{\theta^2}{21}$	15	1	9	2
-79	3	$9-4\sqrt{10}$	$1200, 270, 40$	$\frac{\theta^2}{10}$	3	1	9	2
-89	1	$1+3\sqrt{10}$	$90, 270, 3$	$\frac{\theta^2}{3}$	6	1	9	2
89	2	$33+10\sqrt{10}$	$5100, -240, 9$	$-\frac{\theta}{10}+\frac{\theta^2}{10}$	6	1	9	2
89	3	$-27+8\sqrt{10}$	$28, 62, 50$	$\frac{\theta^2}{2}$	1	0	1	0

We can again determine the conductor $\mathfrak{f}_{L/k}$ in the fields described above by the norm of β and the ordering of the fields, as summarized in the table for $k_2 = \mathbb{Q}(\sqrt{2})$.

Cubic Fields Constructed from Elements of $\mathbb{Q}(\sqrt{11})$

The tables below describe the cubic fields K_i constructed through Martinet and Payan's method for $k_2 = \mathbb{Q}(\sqrt{11})$. These tables list the same invariants for these fields as did the tables above for those fields constructed through Martinet and Payan's method for $k_2 = \mathbb{Q}(\sqrt{2})$.

$N(\beta)$	i	β	a, b, c	θ_2	h_{K_i}	r_{3,K_i}	h_{3,L_i}	r_{3,L_i}
1	0	$10+3\sqrt{11}$	$12, 6, 0$	$\frac{\theta^2}{2}$	1	0	1	0
5	1	$7+2\sqrt{11}$	$56, -12, 3$	$\frac{\theta}{2}+\frac{\theta^2}{2}$	3	1	9	2
5	2	$136+41\sqrt{11}$	$20, 30, 0$	$\frac{\theta^2}{2}$	3	1	9	2
5	3	$4-\sqrt{11}$	$12, 12, -1$	$\frac{\theta}{2}+\frac{\theta^2}{2}$	1	0	1	0
-7	1	$-2+\sqrt{11}$	$-28, 21, 0$	θ^2	3	1	9	2
-7	2	$13+4\sqrt{11}$	$36, 45, 24$	$\frac{\theta^2}{3}$	3	1	9	2
-7	3	$262+79\sqrt{11}$	$6, 12, 1$	θ^2	9	1	27	2
-19	1	$5+2\sqrt{11}$	$132, 60, -3$	$\frac{\theta}{2}+\frac{\theta^2}{2}$	9	2	27	3
-19	2	$116+35\sqrt{11}$	$64, -6, 18$	$\frac{\theta^2}{2}$	9	2	27	3
-19	3	$-16+5\sqrt{11}$	$96, 40, -5$	$-\frac{\theta}{4}+\frac{\theta^2}{4}$	3	1	3	1
37	1	$156+47\sqrt{11}$	$288, 234, -6$	$\frac{\theta^2}{6}$	3	1	9	2
37	2	$24-7\sqrt{11}$	$1470, -84, 9$	$\frac{2\theta}{7}+\frac{\theta^2}{7}$	3	1	9	2
37	3	$9+2\sqrt{11}$	$-666, -111, 0$	$\frac{\theta}{2}+\frac{\theta^2}{6}$	3	1	3	1
-43	1	$76+23\sqrt{11}$	$2300, -210, 12$	$\frac{\theta}{5}+\frac{\theta^2}{10}$	3	1	9	2
-43	2	$-65+17\sqrt{11}$	$3332, -210, 12$	$-\frac{\theta}{7}+\frac{\theta^2}{14}$	3	1	9	2
-43	3	$1+2\sqrt{11}$	$216, 132, 3$	$\frac{\theta}{2}+\frac{\theta^2}{6}$	3	1	9	2
53	1	$8+\sqrt{11}$	$-848, -159, 0$	θ^2	6	1	9	2
53	2	$113+34\sqrt{11}$	$90, 72, 51$	$\frac{\theta^2}{3}$	3	1	9	2
53	3	$47-14\sqrt{11}$	$20, 14, 28$	$\frac{\theta^2}{2}$	1	0	1	0
-79	1	$-25+8\sqrt{11}$	$3712, 240, -3$	$-\frac{3\theta}{8}+\frac{\theta^2}{8}$	6	1	9	2
-79	2	$14+5\sqrt{11}$	$1200, 285, -12$	$-\frac{2\theta}{5}+\frac{\theta^2}{5}$	3	1	9	2
-79	3	$305+92\sqrt{11}$	$828, 132, -1$	$-\frac{\theta}{6}+\frac{\theta^2}{6}$	3	1	9	2

$N(\beta)$	i	β	a, b, c	θ_2	h_{K_i}	r_{3,K_i}	h_{3,L_i}	r_{3,L_i}
-83	1	$4+3\sqrt{11}$	$414, 252, -3$	$\frac{\theta^2}{3}$	6	1	9	2
-83	2	$139+42\sqrt{11}$	$4200, -390, 18$	$-\frac{\theta}{5}+\frac{\theta^2}{10}$	3	1	9	2
-83	3	$-59+18\sqrt{11}$	$312, 36, -5$	$\frac{\theta}{2}+\frac{\theta^2}{2}$	1	0	1	0
89	1	$67-20\sqrt{11}$	$10400, -120, 21$	$\frac{\theta}{20}+\frac{\theta^2}{20}$	3	1	9	2
89	2	$10+\sqrt{11}$	$-1780, -267, 0$	θ^2	3	1	9	2
89	3	$133+40\sqrt{11}$	$1694, 209, -2$	$-\frac{2\theta}{11}+\frac{\theta^2}{11}$	1	0	1	0
97	1	$41-12\sqrt{11}$	$6624, -216, 15$	$\frac{\theta}{4}+\frac{\theta^2}{12}$	3	1	9	2
97	2	$14+3\sqrt{11}$	$900, -99, 24$	$\frac{\theta^2}{3}$	6	1	9	2
97	3	$239+72\sqrt{11}$	$1728, -32, -1$	$-\frac{\theta}{8}+\frac{\theta^2}{8}$	3	1	3	1
-107	1	$577+174\sqrt{11}$	$176, -54, 42$	$\frac{\theta^2}{2}$	9	1	81	2
-107	2	$-17+6\sqrt{11}$	$1908, 396, -15$	$\frac{\theta}{2}+\frac{\theta^2}{6}$	9	1	81	2
-107	3	$28+9\sqrt{11}$	$172, 41, -4$	θ^2	1	0	1	0
113	1	$38-11\sqrt{11}$	$7018, -264, 15$	$\frac{4\theta}{11}+\frac{\theta^2}{11}$	3	1	9	2
113	2	$17+4\sqrt{11}$	$1520, -96, 27$	$-\frac{\theta}{4}+\frac{\theta^2}{4}$	3	1	9	2
113	3	$302+91\sqrt{11}$	$120, -12, 23$	$\frac{\theta}{2}+\frac{\theta^2}{2}$	5	0	1	0

We can again determine the conductor $\mathfrak{f}_{L/k}$ in the fields described above by the norm of β and the ordering of the fields, as summarized in the table for $k_2 = \mathbb{Q}(\sqrt{2})$.

Appendix B

Further Research

The studies of non-normal, non-pure cubic fields are by no means complete. Though this work is a good beginning to an examination of the ideal class group structure of such fields, there is a great deal more that can be done. In this work, we have focused entirely on those cubic fields K constructed by Martinet and Payan's [12] method from elements (α, α') and (β, β') of a real quadratic field $k_2 = \mathbb{Q}(\sqrt{m})$. Further work can be done in the case of cubic fields constructed by the same method from elements of an imaginary quadratic field, which allows us to construct a real cubic field whose conjugate fields are also real and each of which has a system of two fundamental units. Though the cases are not the same, they are similar in many ways and it would not be difficult to extend this work (much of which would still apply) to those cubic fields.

Other extensions of Martinet and Payan's methods [12] for constructing cubic fields would yield still other further research to be accomplished. We further restricted ourselves in this work to fields defined by elements (α, α') and (β, β') of the quadratic field $k_2 = \mathbb{Q}(\sqrt{m})$ for which $N_{k_2/\mathbb{Q}}(\beta) = \pm p$ for some prime p. We can extend these works to cases in which the cubic field K is defined by elements of the quadratic k_2 where $N_{k_2/\mathbb{Q}}(\beta) = \pm n$ for any square free integers $n \in \mathbb{Z}$ for which $x^2 - my^2 \equiv \pm n$ has solutions. Further, except for the most cursory examination, we have not considered the case in which the class number of k_2 is divisible by 3.

Even further, the work of Parry [13] could perhaps be used to extend this research to bicubic fields whose cubic subfields are the non-normal, non-pure cubic fields described by Martinet and Payan in their work [12]. This would require greater study than the relatively simpler examinations described above, but the bicubic fields have received less study than the cubic. Therefore, the study of bicubic fields whose cubic subfields are neither normal nor pure could prove very fruitful.

Further studies in the area could, on the other hand, be narrowed rather than extended. More in-depth studies of non-normal, non-pure cubic fields constructed from elements of a given quadratic field could also prove valuable. Preliminary examinations (some of which resulted in the tables in Appendix A) have been made for cubic fields defined from elements of certain quadratics and have given conditions on the value of $N_{k_2/\mathbb{Q}}(\beta) = \pm p$ in which certain class group structures occur. In particular, the studies of cubic fields K defined by elements of $k_2 = \mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{3}), \mathbb{Q}(\sqrt{5})$, and $\mathbb{Q}(\sqrt{6})$ have proven valuable to our studies herein and have greatly contributed to the general results proved in Chapters 4 and 5.

Studies could further be made more specific to the normal closures of these cubic fields and perhaps could generalize results for more general sextic fields. The class group structure of one of these normal closures (or again, possibly a general sextic field) could be examined in relation to that of its quadratic and cubic subfields. Perhaps this could prove results parallel to those in Chapter 4 relating the 2-class group of a sextic field with that of its quadratic subfield. Further, the normal closure in the bicubic case described above would be very different and might warrant further study on its own.

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