

# On the Spectrum of Neutron Transport Equations with Reflecting Boundary Conditions

Degong Song

Dissertation submitted to the Faculty of the  
Virginia Polytechnic Institute and State University  
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy  
in  
Mathematics

William Greenberg, Chair  
George Hagedorn  
Martin Klaus  
Werner Kohler  
Peter Haskell

February 14, 2000  
Blacksburg, Virginia

Keywords: transport equation, spectrum, strongly continuous semigroup, stability  
Copyright 2000, Degong Song

# On the Spectrum of Neutron Transport Equations with Reflecting Boundary Conditions

Degong Song

(ABSTRACT)

This dissertation is devoted to investigating the time dependent neutron transport equations with reflecting boundary conditions. Two typical geometries — slab geometry and spherical geometry — are considered in the setting of  $L^p$  including  $L^1$ . Some aspects of the spectral properties of the transport operator  $A$  and the  $C_0$  semigroup  $T(t)$  generated by  $A$  are studied. It is shown under fairly general assumptions that the accumulation points of  $\text{Pas}(A) := \sigma(A) \cap \{\lambda : \text{Re}\lambda > -\lambda^*\}$ , if they exist, could only appear on the line  $\text{Re}\lambda = -\lambda^*$  (where  $\lambda^*$  is the essential infimum of the total collision frequency), and the spectrum of  $T(t)$  outside the disk  $\{\lambda : |\lambda| \leq \exp(-\lambda^*t)\}$  consists of isolated eigenvalues of  $T(t)$  with finite algebraic multiplicity, and the accumulation points of  $\sigma(T(t)) \cap \{\lambda : |\lambda| > \exp(-\lambda^*t)\}$ , if they exist, could only appear on the circle  $\{\lambda : |\lambda| = \exp(-\lambda^*t)\}$ . Consequently, the asymptotic behavior of the time dependent solution is obtained.

# Acknowledgments

I would first and foremost like to thank my advisor, Dr. William Greenberg, for his advice and much other help. I would also like to thank the other members of my Doctor's Degree Committee. Thanks also to Dr. Robert Rogers and Dr. Martin Day as well as Ms. Shugart for much help during my stay in Virginia Tech.

Also, I would like to thank the Department of Mathematics of Virginia Tech, where I could get financial support to finish my study. Thanks to everyone that helped me in these days.

# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>I</b>	<b>SLAB GEOMETRY</b>	<b>4</b>
<b>2</b>	<b>Spectral Properties of Transport Equations for Slab Geometry in <math>L^1</math></b>	<b>5</b>
2.1	Problem and Notations . . . . .	6
2.2	Preliminary Properties of $B$ and $A$ in $L^p(G)$ . . . . .	7
2.3	Spectral Properties of $A$ in $L^1(G)$ . . . . .	11
2.4	Spectral Properties of $T(t)$ in $L^1(G)$ . . . . .	25
<b>3</b>	<b>Spectral Properties of Transport Equations for Slab Geometry in <math>L^p</math></b>	<b>34</b>
3.1	Corresponding Problems in $L^\infty(G)$ . . . . .	34
3.2	Corresponding Problems in $L^p(G)$ . . . . .	39
3.3	Some Further Investigations in $L^p(G)$ . . . . .	50
3.4	Asymptotic Behavior of $\psi$ in $L^p(G)$ . . . . .	54
3.5	Some Remarks . . . . .	56
<b>II</b>	<b>SPHERICAL GEOMETRY</b>	<b>58</b>
<b>4</b>	<b>Spectral Properties of Transport Equations for Spherical Geometry in <math>L^2</math></b>	<b>59</b>
4.1	Problem and Notations . . . . .	59
4.2	Spectral Properties of $A$ in $L^2$ . . . . .	61

4.3	Spectral Properties of $T(t)$ in $L^2$ . . . . .	74
4.4	Some Remarks and Proof of Eqs. (4.10), (4.11) . . . . .	77
<b>5</b>	<b>Spectral Properties of Transport Equations for Spherical Geometry in <math>L^1</math></b>	<b>80</b>
5.1	Problem and Notations . . . . .	80
5.2	Spectral Properties of $A$ in $L^1$ . . . . .	82
5.3	Spectral Properties of $T(t)$ in $L^1$ . . . . .	103

# Chapter 1

## Introduction

As is well known, the stability of a given transport system is a quite important and interesting topic in transport theory. Various methods have been developed to investigate the asymptotic behavior of the time dependent solution (cf. [8, 14, 15, 18, 19, 26, 27, etc.]). One of the effective methods to treat the linear transport equation is the spectral method (e.g., [55, 8, 15, 52, etc.]).

The spectrum of the neutron transport operator was first studied by J. Lehner and G. M. Wing [19] for the one-speed equation. Subsequent studies by various authors considered the more realistic energy-dependent operator, but mostly with non-reentrant boundary conditions. It was not until 1970 that many authors began the study of transport equations with reentrant boundary conditions.

When using the spectral method, one usually selects the setting of  $L^p$  ( $1 \leq p < +\infty$ ), in which the corresponding transport operator  $A$  generates a  $C_0$  semigroup  $T(t)$ , and the asymptotic behavior of the time dependent solution can be obtained by investigating the spectral properties of  $T(t)$ . Since  $A$  is an unbounded nonself-adjoint operator, the discussion on the spectrum of  $A$  is usually rather difficult. Especially, because the “spectral mapping theorem” does not hold for general  $C_0$  semigroups, the investigation of the spectrum of  $T(t)$  can be complicated.

The transport operator  $A$  is the summation of two operators  $B$  and  $K$ , where  $B$  is the streaming operator, and  $K$  a collision operator defined by a scattering–fission kernel  $k$ . One way to study the spectrum of  $T(t)$  is to consider  $T(t)$  as a perturbation of the more tractable semigroup  $S(t)$  generated by  $B$  alone. Under some conditions, it has been shown that the spectral values of  $T(t)$  outside the spectral disk of  $S(t)$  are eigenvalues of finite multiplicity. This is very important in analyzing the large time behavior of solutions of the transport equation. The first conclusion of this kind was given by K. Jörgens [14] for a particular case, and then was analyzed in a general abstract setting by I. Vidav [38]. In recent years, Vidav’s analysis was refined and improved by J. Voigt [39, 41], G. Greiner [10], L. W. Weis [43] and

M. Mokhtar–Kharroubi [22] etc.

The study of the asymptotic behavior of neutron transport equations can be traced back to J. Lehner, G. M. Wing’s work in 1956 [19] and K. Jörgens’ work in 1958 [14]. Lehner and Wing investigated the most simple mono-energetic slab model with non-reentrant boundary conditions, while Jörgens considered a more general model — bounded convex body with energy bounded away from zero and with non-reentrant boundary conditions. After that, the most impressive work on this topic might be J. Voigt’s work in 1985 [41] and K. Latrach’s recent work [49, 48, 51]. J. Voigt investigated the bounded convex body model with lower energy being zero, which generalized Jörgens’ work. K. Latrach considered the mono-energy homogeneous slab geometry under very general boundary conditions. The transport operator  $A$  he considered can be described by

$$A\psi = -\mu \frac{\partial \psi}{\partial x} - \sigma(\mu)\psi + \int_{-1}^1 k(\mu, \mu')\psi(x, \mu')d\mu',$$

with the boundary conditions modeled by  $\psi|_{\Gamma_-} = H\psi|_{\Gamma_+}$ , where  $\psi|_{\Gamma_-}$  [resp.  $\psi|_{\Gamma_+}$ ] is the incoming [resp. outgoing] trace mapping and  $H$  is a weakly compact operator in the trace space.

Under the “no incoming particle” boundary conditions, there have been many good results. But few results of this kind have ever been given when reentrant particles exist. From the view of practice, the case with reentry boundary conditions is of particular interest and hence is worthy to be investigated.

The transport equations with reentry boundary conditions have been considered by many investigators since 1970. The slab model with non-reentrant boundary conditions was first studied by Morante under periodic boundary conditions. After that, much work has been done with various different boundary conditions like reflecting boundary conditions, integral boundary conditions or Maxwell boundary conditions (e.g., [28, 7, 31, 48, 45, etc.]).

The study of spherical geometry began with Norton’s work in 1962 [25] with one-speed non-reentrant boundary conditions. And some work has been done for reentrant boundary conditions [20, 42, 53, 46].

It is worthy to mention here that many other models have been proposed and considered, and some of them are very general models (e.g., [57, 47, etc.]). Since usually these models are more complicated, few further results were obtained.

Most of these papers (that deal with the reentry boundary conditions) investigate the spectrum of the transport operator  $A$ , not the semigroup  $T(t)$  generated by  $A$ . Since the spectral mapping theorem  $\sigma(T(t)) \setminus \{0\} = \{e^{\lambda t} : \lambda \in \sigma(A)\}$  does not hold for strongly continuous semigroups without any further conditions on the semigroups such as analyticity or norm continuity (cf. [23]), the information contained in  $\sigma(A)$  can not be directly transferred to  $\sigma(T(t))$ , and therefore the asymptotic behavior of the time dependent solution is still an unknown question. Although some conclusions about the asymptotic behavior of the time dependent

solution can be obtained by virtue of the inversion of Laplace transform (cf. [22, 3, 31, 19, 21, etc.]), it is always required that the initial distribution  $\psi_0 \in D(A^2)$ .

It has been shown that the restriction  $\psi_0 \in D(A^2)$  is unnecessary in case of no-reentry boundary conditions (cf. [14, 38, 41, 10, 43, 22, etc.]) and under certain conditions for slab geometry with generalized boundary conditions (cf. [48, etc.]), and it is one of the motivations of this paper to eliminate such conditions imposed on the initial distribution for time dependent transport equation with reflecting boundary conditions under very general assumptions.

The analyses given in [14, 38, 41, 10, 43, 22, etc.] concerning the spectral properties of  $T(t)$  depend on the “no incoming particle” boundary conditions, in which case the strongly continuous semigroup  $S(t)$  generated by the streaming operator  $B$  can be given by an exact expression, i.e.,

$$S(t)f(x, v) = \chi_D(x - vt) \exp \left[ - \int_0^t \sigma(x - vs, v) ds \right] f(x - vt, v)$$

(cf. Eq. (4.4) on page 14 of [43]). In case of reentry boundary conditions, we do not know the exact expression of  $S(t)$ , and thereby the semigroup perturbation method developed in [14, 38, 39, 41, 10, 43, 22] could not be applied to transport equation with reentry boundary conditions.

In this paper, the neutron transport equations with reflecting boundary conditions are considered for both slab geometry and spherical geometry under very general assumptions. A different method is developed to investigate the spectral properties of  $T(t)$ . First, the spectral properties of the transport operator  $A$  are studied, and it is shown that for any constant  $\beta_1 > -\lambda^*$  (where  $\lambda^*$  is the essential infimum of the total collision frequency) the set  $\{\lambda \in \sigma(A) : \operatorname{Re} \lambda \geq \beta_1\}$  contains only finitely many elements, each of which is an eigenvalue of  $A$  with finite algebraic multiplicity. Perturbation methods are used in this part to estimate the asymptotic behavior of the norm of powers of  $K(\lambda I - B)^{-1}$  as the imaginary part of  $\lambda$  tends to infinity. Then, by virtue of the theory of strongly continuous semigroups, the spectral properties of  $T(t)$  are investigated. It is finally shown that the spectrum of  $T(t)$  outside the disk  $\{\lambda : |\lambda| \leq \exp(-\lambda^*t)\}$  consists of isolated eigenvalues of  $T(t)$  with finite algebraic multiplicity, and the accumulation points of  $\sigma(T(t)) \cap \{\lambda : |\lambda| > \exp(-\lambda^*t)\}$  could only appear on the circle  $\{\lambda : |\lambda| = \exp(-\lambda^*t)\}$ .

This thesis is composed of two part. Part 1 deals with the slab geometry. It consists of two chapters, in which the problems are investigated in the setting of  $L^1$  and  $L^p$ . Part 2 deals with the spherical geometry. It consists of two chapters, in which the problems are investigated in the setting of  $L^2$  and  $L^1$ .

# Part I

## SLAB GEOMETRY

## Chapter 2

# Spectral Properties of Transport Equations for Slab Geometry in $L^1$

This part is devoted to investigating the time dependent neutron transport equation in a nonhomogeneous slab with generalized boundary conditions. In the setting of  $L^p$ , some aspects of the spectral properties of the transport operator  $A$  and the  $C_0$  semigroup  $T(t)$  generated by  $A$  are studied. It is shown under fairly general assumptions that the accumulation points of  $\text{Pas}(A) := \sigma(A) \cap \{\lambda : \text{Re}\lambda > -\lambda^*\}$ , if they exist, could only appear on the line  $\text{Re}\lambda = -\lambda^*$ , where  $\lambda^*$  is the essential infimum of the total collision frequency. In the setting of  $L^2$ , it is proved that the spectrum of  $T(t)$  outside the disk  $\{\lambda : |\lambda| \leq \exp(-\lambda^*t)\}$  consists of isolated eigenvalues of  $T(t)$  with finite algebraic multiplicity, and the accumulation points of  $\sigma(T(t)) \cap \{\lambda : |\lambda| > \exp(-\lambda^*t)\}$ , if they exist, could only appear on the circle  $\{\lambda : |\lambda| = \exp(-\lambda^*t)\}$ . In the setting of  $L^p$  ( $p \neq 2$ ), the same conclusions concerning  $T(t)$  still hold if some additional assumptions are added. Finally, the asymptotic behavior of the time dependent solution is given under the assumption that the dominant eigenvalue  $\beta_0$  of  $A$  exists and  $\beta_0 > -\lambda^*$ .

This part consists of two chapters. Chapter 2 (§2.1 ~ §2.4) mainly deals with the transport equation in the setting of  $L^1$ , and Chapter 3 (§3.1 ~ §3.5) investigates the equation in the setting of  $L^p$  ( $1 < p \leq +\infty$ ).

## 2.1 Problem and Notations

Consider the following transport equation in the setting of  $L^p$  ( $1 \leq p \leq +\infty$ )

$$(I) \quad \begin{cases} \frac{\partial \psi(x, v, \mu, t)}{\partial t} = -v\mu \frac{\partial \psi(x, v, \mu, t)}{\partial x} - \sigma(x, v, \mu)\psi(x, v, \mu, t) \\ \quad + \int_D \int_V k(x, v, v', \mu, \mu')\psi(x, v', \mu', t)dv'd\mu', \\ \psi(-a, v, \mu, t) = \alpha(v, \mu)\psi(-a, v, -\mu, t), \quad 0 < \mu \leq 1, \\ \psi(a, v, -\mu, t) = \gamma(v, \mu)\psi(a, v, \mu, t), \quad 0 < \mu \leq 1, \\ \psi(x, v, \mu, 0) = \psi_0(x, v, \mu), \\ x \in Q := [-a, a], v \in V := (0, v_M], \mu \in D := [-1, 1], t > 0, \end{cases}$$

where  $a > 0$ ,  $0 < v_M < +\infty$ ,  $\alpha(v, \mu)$  and  $\gamma(v, \mu)$  are reflection coefficients,  $\psi(x, v, \mu, t)$  is the particle density within the slab of thickness  $2a$ ,  $\sigma(x, v, \mu)$  is the total collision frequency,  $k(x, v, v', \mu, \mu')$  is the scattering fission kernel, and  $\psi_0(x, v, \mu)$  is the initial distribution.

Throughout this part, it is assumed that

$$(H1). \quad \alpha(v, \mu) \text{ and } \gamma(v, \mu) \text{ are measurable functions satisfying } 0 \leq \alpha(v, \mu) = \alpha(v, -\mu) \leq 1, \quad 0 \leq \gamma(v, \mu) = \gamma(v, -\mu) \leq 1.$$

$$(H2). \quad \sigma(x, v, \mu) \text{ is an essentially bounded real measurable function, and the essential infimum of } \sigma(x, v, \mu) \text{ is denoted by } \lambda^*.$$

$$(H3). \quad k(x, v, v', \mu, \mu') \text{ is a real measurable function satisfying}$$

$$|k(x, v, v', \mu, \mu')| \leq C(v|\mu|)^{-\delta_1}(v'|\mu'|)^{-\delta_2}, \quad (2.1)$$

where  $C$ ,  $\delta_1$  and  $\delta_2$  are nonnegative constants,  $0 \leq \delta_1 < p^{-1}$ ,  $0 \leq \delta_2 < 1 - p^{-1}$  (if  $p = 1$ , then  $0 \leq \delta_1 < 1$ ,  $\delta_2 = 0$ ; if  $p = +\infty$ , then  $\delta_1 = 0$ ,  $0 \leq \delta_2 < 1$ ).

**Remark 2.1.1.** If we define

$$\tilde{k}(x, v, v', \mu, \mu') := (v|\mu|)^{\delta_1}(v'|\mu'|)^{\delta_2}k(x, v, v', \mu, \mu'),$$

then  $\tilde{k}(x, v, v', \mu, \mu')$  is bounded measurable. (H3) is obviously true if  $k(x, v, v', \mu, \mu')$  is bounded measurable.

Set  $G = Q \times V \times D$ , and let  $L^p(G)$  ( $1 \leq p \leq \infty$ ) represent the Banach space composed of all  $p$ -integrable complex functions defined on  $G$ , with the norm  $\|\cdot\|_p$  given by

$$\|\psi\|_p = \left[ \iiint_G |\psi(x, v, \mu)|^p dx dv d\mu \right]^{1/p}.$$

Define operators on  $L^p(G)$  as follows:

$$\begin{aligned} B\psi &= -v\mu \frac{\partial \psi}{\partial x} - \sigma(x, v, \mu)\psi, \\ K\psi &= \int_D \int_V k(x, v, v', \mu, \mu') \psi(x, v', \mu') dv' d\mu', \\ A\psi &= B\psi + K\psi, \end{aligned}$$

with  $D(B) = D(A) = \{\psi \in L^p(G) : B\psi \in L^p(G); \psi(-a, v, \mu) = \alpha(v, \mu)\psi(-a, v, -\mu) \text{ and } \psi(a, v, -\mu) = \gamma(v, \mu)\psi(a, v, \mu) \text{ for every } \mu \in (0, 1]\}$ ,  $D(K) = L^p(G)$ . Then Eq. (I) can be written as

$$\frac{d\psi(t)}{dt} = A\psi(t), \quad \psi(0) = \psi_0.$$

## 2.2 Preliminary Properties of $B$ and $A$ in $L^p(G)$

For any  $\psi \in L^p(G)$  and  $\lambda$  with  $\text{Re}\lambda > -\lambda^*$ , after some manipulations from  $(\lambda I - B)\varphi = \psi$ , we have (cf. [28, 31])

$$(\lambda I - B)^{-1}\psi(x, v, \mu) = \frac{P_1\psi + \alpha(v, \mu)P_2\psi + \alpha(v, \mu)\gamma(v, \mu)P_3\psi}{1 - \alpha(v, \mu)\gamma(v, \mu) \exp\left[-\frac{2}{v\mu} \int_{-a}^a \Delta(\sigma) ds\right]} \quad (2.2)$$

for  $\mu$  positive, and

$$(\lambda I - B)^{-1}\psi(x, v, \mu) = \frac{P_4\psi + \gamma(v, \mu)P_5\psi + \alpha(v, \mu)\gamma(v, \mu)P_6\psi}{1 - \alpha(v, \mu)\gamma(v, \mu) \exp\left[\frac{2}{v\mu} \int_{-a}^a \Delta(\sigma) ds\right]} \quad (2.3)$$

for  $\mu$  negative, where  $\Delta(\sigma) = \lambda + \sigma(s, v, \mu)$ ,  $P_1, P_2, \dots, P_6$  are operators on  $L^p(G)$  given by (cf. [31, page 105])

$$P_1\psi(x, v, \mu) = \frac{1}{v\mu} \int_{-a}^x \psi(x', v, \mu) \exp\left[-\frac{1}{v\mu} \int_{x'}^x \Delta(\sigma) ds\right] dx', \quad (2.4)$$

$$P_2\psi(x, v, \mu) = \frac{1}{v\mu} \int_Q \psi(x', v, -\mu) \exp \left[ -\frac{1}{v\mu} \left( \int_{-a}^x \Delta(\sigma) ds + \int_{-a}^{x'} \Delta(\sigma) ds \right) \right] dx', \quad (2.5)$$

$$P_3\psi(x, v, \mu) = \frac{1}{v\mu} \exp \left[ -\frac{2}{v\mu} \int_Q \Delta(\sigma) ds \right] \int_x^a \psi(x', v, \mu) \exp \left[ -\frac{1}{v\mu} \int_{x'}^x \Delta(\sigma) ds \right] dx', \quad (2.6)$$

$$P_4\psi(x, v, \mu) = -\frac{1}{v\mu} \int_x^a \psi(x', v, \mu) \exp \left[ \frac{1}{v\mu} \int_x^{x'} \Delta(\sigma) ds \right] dx', \quad (2.7)$$

$$P_5\psi(x, v, \mu) = -\frac{1}{v\mu} \int_Q \psi(x', v, -\mu) \exp \left[ \frac{1}{v\mu} \left( \int_x^a \Delta(\sigma) ds + \int_{x'}^a \Delta(\sigma) ds \right) \right] dx', \quad (2.8)$$

$$P_6\psi(x, v, \mu) = -\frac{1}{v\mu} \exp \left[ \frac{2}{v\mu} \int_Q \Delta(\sigma) ds \right] \int_{-a}^x \psi(x', v, \mu) \exp \left[ \frac{1}{v\mu} \int_x^{x'} \Delta(\sigma) ds \right] dx'. \quad (2.9)$$

**Lemma 2.2.1.** *In the setting of  $L^p(G)$  ( $1 \leq p \leq +\infty$ ),  $\{\lambda : \operatorname{Re}\lambda > -\lambda^*\} \subset \rho(B)$ ,  $R(\lambda I - B) = L^p(G)$ , and  $\|(\lambda I - B)^{-1}\|_p \leq (\operatorname{Re}\lambda + \lambda^*)^{-1}$  for every  $\lambda$  with  $\operatorname{Re}\lambda > -\lambda^*$ .*

**Proof.** First, we prove this lemma for  $p = 1$ . For the sake of simplicity, set  $c = \operatorname{Re}\lambda + \lambda^*$ ,  $W = (v\mu)^{-1}$ . For any  $\psi \in L^1(G)$ , it follows from Eqs. (2.2) – (2.9) that

$$\begin{aligned} & \|(\lambda I - B)^{-1}\psi\|_1 \\ & \leq \int_V dv \int_0^1 d\mu \frac{W}{1 - e^{-4acW}} \int_Q dx \int_Q e^{-(2a+x+x')cW} |\psi(x', v, -\mu)| dx' \\ & \quad + \int_V dv \int_{-1}^0 d\mu \frac{-W}{1 - e^{-4acW}} \int_Q dx \int_Q e^{(2a-x-x')cW} |\psi(x', v, -\mu)| dx' \\ & \quad + \int_V dv \int_0^1 d\mu \frac{W}{1 - e^{-4acW}} \int_Q dx \int_{-a}^x e^{-(x-x')cW} |\psi(x', v, \mu)| dx' \\ & \quad + \int_V dv \int_{-1}^0 d\mu \frac{-W}{1 - e^{-4acW}} \int_Q dx \int_x^a e^{(x'-x)cW} |\psi(x', v, \mu)| dx' \\ & \quad + \int_V dv \int_0^1 d\mu \frac{W}{1 - e^{-4acW}} \int_Q dx \int_x^a e^{-(4a+x-x')cW} |\psi(x', v, \mu)| dx' \\ & \quad + \int_V dv \int_{-1}^0 d\mu \frac{-W}{1 - e^{-4acW}} \int_Q dx \int_{-a}^x e^{(4a+x'-x)cW} |\psi(x', v, \mu)| dx'. \end{aligned}$$

By exchanging the order of integration of  $x$  and  $x'$  in the above equation, we get

$$\|(\lambda I - B)^{-1}\psi\|_1$$

$$\begin{aligned}
&\leq \frac{1}{c} \int_V dv \int_0^1 d\mu \frac{1}{1 - e^{-4acW}} \int_Q \left[ e^{-(a+x')cW} - e^{-(3a+x')cW} \right] |\psi(x', v, -\mu)| dx' \\
&\quad + \frac{1}{c} \int_V dv \int_{-1}^0 d\mu \frac{1}{1 - e^{4acW}} \int_Q \left[ e^{(a-x')cW} - e^{(3a-x')cW} \right] |\psi(x', v, -\mu)| dx' \\
&\quad + \frac{1}{c} \int_V dv \int_0^1 d\mu \frac{1}{1 - e^{-4acW}} \int_Q \left[ 1 - e^{-(a-x')cW} \right] |\psi(x', v, \mu)| dx' \\
&\quad + \frac{1}{c} \int_V dv \int_{-1}^0 d\mu \frac{1}{1 - e^{4acW}} \int_Q \left[ 1 - e^{(x'+a)cW} \right] |\psi(x', v, \mu)| dx' \\
&\quad + \frac{1}{c} \int_V dv \int_0^1 d\mu \frac{1}{1 - e^{-4acW}} \int_Q \left[ e^{-(3a-x')cW} - e^{-4acW} \right] |\psi(x', v, \mu)| dx' \\
&\quad + \frac{1}{c} \int_V dv \int_{-1}^0 d\mu \frac{1}{1 - e^{4acW}} \int_Q \left[ e^{(3a+x')cW} - e^{4acW} \right] |\psi(x', v, \mu)| dx'.
\end{aligned}$$

Applying the transform  $\eta = -\mu$  to the first and the second terms of the right hand side of the above equation, we get

$$\|(\lambda I - B)^{-1}\psi\|_1 \leq c^{-1}\|\psi\|_1, \quad \forall \psi \in L^1(G), \quad (2.10)$$

which implies the conclusion when  $p = 1$ .

In case of  $p = +\infty$ , by virtue of Eqs. (2.2) – (2.9) and the technique of exchanging the integration orders of  $x$  and  $x'$ , it is easy to directly verify that for every  $\lambda$  with  $\operatorname{Re}\lambda > -\lambda^*$ ,

$$\|(\lambda I - B)^{-1}\psi\|_\infty \leq c^{-1}\|\psi\|_\infty, \quad \forall \psi \in L^\infty(G). \quad (2.11)$$

From Eqs. (2.10) and (2.11), by virtue of the Riesz–Thorin interpolation theorem (cf. [4, page 22]), we get

$$\|(\lambda I - B)^{-1}\psi\|_p \leq c^{-1}\|\psi\|_p, \quad \forall \psi \in L^p(G) \text{ and } p \in (1, +\infty).$$

This completes the proof. Q. E. D.

**Lemma 2.2.2.** (i)  $B$  generates a positive  $C_0$  semigroup  $S_p(t)$  in  $L^p(G)$  ( $1 \leq p < +\infty$ ). (ii) For any  $\psi \in L^{p_1}(G) \cap L^{p_2}(G)$  ( $1 \leq p_1, p_2 \leq +\infty$ ),  $S_{p_1}(t)\psi = S_{p_2}(t)\psi$ . (iii) Denote by  $\omega_p(S)$  the growth bound of  $S_p(t)$ ; then  $\omega_p(S) \leq -\lambda^*$ .

**Proof.** It is not difficult to see  $\overline{D(B)} = L^p(G)$  ( $1 \leq p < +\infty$ ), and thus the first conclusion is easily obtained from Lemma 2.2.1. The second conclusion can be seen from the relation (cf. [29, page 33])

$$S_p(t)\psi = \lim_{n \rightarrow \infty} \left[ \frac{n}{t} \left( \frac{n}{t} - B \right)^{-1} \right]^n \psi, \quad \forall \psi \in L^p(G)$$

and the expression for  $(\lambda I - B)^{-1}$  (cf. Eqs. (2.2) – (2.9)).

The third assertion follows from Lemma 2.2.1. Q. E. D.

From (H3), it is easy to see that  $K$  is a bounded operator on  $L^p(G)$ , and thus we have the following well known results.

**Lemma 2.2.3.** *In the setting of  $L^p(G)$  ( $1 \leq p \leq +\infty$ ),  $\{\lambda : \operatorname{Re}\lambda > \|K\|_p - \lambda^*\} \subset \rho(A)$ , and  $R(\lambda I - A) = L^p(G)$ ,  $\|(\lambda I - A)^{-1}\|_p \leq (\operatorname{Re}\lambda + \lambda^* - \|K\|_p)^{-1}$  for every  $\lambda$  with  $\operatorname{Re}\lambda > \|K\|_p - \lambda^*$ .*

**Lemma 2.2.4.** *(i)  $A$  generates a  $C_0$  semigroup  $T_p(t)$  in  $L^p(G)$  ( $1 \leq p < +\infty$ ). (ii) If  $K$  is a bounded operator in both  $L^{p_1}(G)$  and  $L^{p_2}(G)$  ( $1 \leq p_1, p_2 < +\infty$ ), then for any  $\psi \in L^{p_1}(G) \cap L^{p_2}(G)$ ,  $T_{p_1}(t)\psi = T_{p_2}(t)\psi$ . (iii) For every  $\psi_0 \in D(A)$ , the solution  $\psi(t)$  of Eq. (I) exists and is uniquely given by  $\psi(t) = T_p(t)\psi_0$ .*

From Lemma 2.2.2 (ii) and Lemma 2.2.4 (ii), we can denote the semigroups  $S_p(t)$  and  $T_p(t)$  in  $L^p(G)$  simply by  $S(t)$ ,  $T(t)$ .

Since the solution of Eq. (I) is  $\psi(t) = T(t)\psi_0$ , the asymptotic behavior of  $\psi(t)$  can be determined by the spectral properties of  $T(t)$ . The following sections will be devoted to the investigation of the spectral properties of  $T(t)$ .

Unlike the transport equation with “no incoming particle boundary conditions” (in which case the semigroup  $S(t)$  generated by  $B$  can be given by an exact expression, cf. [43, page 14, Eq. (4.4)]), it seems rather difficult to consider the spectral properties of  $T(t)$  directly. So, we will first consider the spectral properties of  $A$  (esp., in the half plane  $\operatorname{Re}\lambda > -\lambda^*$ ), and then transfer the spectral properties of  $A$  to  $T(t)$ .

## 2.3 Spectral Properties of $A$ in $L^1(G)$

This section is devoted to discussing the spectral properties of  $A$  in the strip  $\{\lambda : -\lambda^* < \operatorname{Re}\lambda \leq \|K\| - \lambda^*\}$  in the setting of  $L^1$ . Setting  $\operatorname{Pas}(A) := \{\lambda : \operatorname{Re}\lambda > -\lambda^*, \lambda \in \sigma(A)\}$ , it will be shown that  $\operatorname{Pas}(A)$  contains at most countable isolated points which are eigenvalues of  $A$  with finite algebraic multiplicity, and the accumulation points of  $\operatorname{Pas}(A)$ , if they exist, could only appear on the line  $\operatorname{Re}\lambda = -\lambda^*$ .

For every  $\lambda$  with  $\operatorname{Re}\lambda > -\lambda^*$ , it follows from the expressions for  $(\lambda I - B)^{-1}$  (cf. Eqs. (2.2) – (2.9)) and  $K$  that for every  $\psi \in L^1(G)$ ,

$$K(\lambda I - B)^{-1}K\psi = \sum_{i=1}^6 T_i(\lambda, k, \sigma, \alpha, \gamma, k)\psi, \quad (2.12)$$

where

$$\begin{aligned} & T_1(\lambda, k, \sigma, \alpha, \gamma, k)\psi(x, v, \mu) \\ &= \int_{-a}^x \int_V \int_D \int_V \int_0^1 \frac{\Omega(\lambda, \alpha, \gamma, \sigma)}{w\xi} k(x, v, w, \mu, \xi) k(x', w, v', \xi, \mu') \\ & \quad \times \exp\left[-\frac{1}{w\xi} \int_{x'}^x \Delta(\sigma) ds\right] \psi(x', v', \mu') d\xi dw d\mu' dv' dx', \end{aligned} \quad (2.13)$$

$$\begin{aligned} & T_2(\lambda, k, \sigma, \alpha, \gamma, k)\psi(x, v, \mu) \\ &= \int_x^a \int_V \int_D \int_V \int_0^1 \frac{\Omega(\lambda, \alpha, \gamma, \sigma)}{w\xi} k(x, v, w, \mu, -\xi) k(x', w, v', -\xi, \mu') \\ & \quad \times \exp\left[-\frac{1}{w\xi} \int_x^{x'} \Delta(\sigma) ds\right] \psi(x', v', \mu') d\xi dw d\mu' dv' dx', \end{aligned} \quad (2.14)$$

$$\begin{aligned} & T_3(\lambda, k, \sigma, \alpha, \gamma, k)\psi(x, v, \mu) \\ &= \int_Q \int_V \int_D \int_V \int_0^1 \frac{\Omega(\lambda, \alpha, \gamma, \sigma)}{w\xi} \alpha(w, \xi) k(x, v, w, \mu, \xi) k(x', w, v', -\xi, \mu') \\ & \quad \times \exp\left[-\frac{1}{w\xi} \left(\int_{-a}^x \Delta(\sigma) ds + \int_{-a}^{x'} \Delta(\sigma) ds\right)\right] \psi(x', v', \mu') d\xi dw d\mu' dv' dx', \end{aligned} \quad (2.15)$$

$$\begin{aligned} & T_4(\lambda, k, \sigma, \alpha, \gamma, k)\psi(x, v, \mu) \\ &= \int_Q \int_V \int_D \int_V \int_0^1 \frac{\Omega(\lambda, \alpha, \gamma, \sigma)}{w\xi} \gamma(w, \xi) k(x, v, w, \mu, -\xi) k(x', w, v', \xi, \mu') \\ & \quad \times \exp\left[-\frac{1}{w\xi} \left(\int_x^a \Delta(\sigma) ds + \int_{x'}^a \Delta(\sigma) ds\right)\right] \psi(x', v', \mu') d\xi dw d\mu' dv' dx', \end{aligned} \quad (2.16)$$

$$\begin{aligned}
& T_5(\lambda, k, \sigma, \alpha, \gamma, k)\psi(x, v, \mu) \\
&= \int_{-a}^x \int_V \int_D \int_V \int_0^1 \frac{\Omega(\lambda, \alpha, \gamma, \sigma)}{w\xi} \alpha(w, \xi) \gamma(w, \xi) k(x, v, w, \mu, -\xi) k(x', w, v', -\xi, \mu') \\
&\quad \times \exp \left[ -\frac{1}{w\xi} \left( 2 \int_Q \Delta(\sigma) ds - \int_{x'}^x \Delta(\sigma) ds \right) \right] \psi(x', v', \mu') d\xi dw d\mu' dv' dx', \quad (2.17)
\end{aligned}$$

$$\begin{aligned}
& T_6(\lambda, k, \sigma, \alpha, \gamma, k)\psi(x, v, \mu) \\
&= \int_x^a \int_V \int_D \int_V \int_0^1 \frac{\Omega(\lambda, \alpha, \gamma, \sigma)}{w\xi} \alpha(w, \xi) \gamma(w, \xi) k(x, v, w, \mu, \xi) k(x', w, v', \xi, \mu') \\
&\quad \times \exp \left[ -\frac{1}{w\xi} \left( 2 \int_Q \Delta(\sigma) ds - \int_x^{x'} \Delta(\sigma) ds \right) \right] \psi(x', v', \mu') d\xi dw d\mu' dv' dx',
\end{aligned}$$

with  $\Delta(\sigma) = \lambda + \sigma(s, w, \xi)$ , and

$$\Omega(\lambda, \alpha, \gamma, \sigma) = \left\{ 1 - \alpha(w, \xi) \gamma(w, \xi) \exp \left[ -\frac{2}{w\xi} \int_Q \Delta(\sigma) ds \right] \right\}^{-1}. \quad (2.18)$$

Noting that

$$\Omega(\lambda, \alpha, \gamma, \sigma) = \sum_{n=0}^{\infty} \alpha^n(w, \xi) \gamma^n(w, \xi) \exp \left[ -\frac{2n}{w\xi} \int_Q \Delta(\sigma) ds \right],$$

we have

$$K(\lambda I - B)^{-1} K\psi = \sum_{i=1}^6 \sum_{n=0}^{\infty} T_{n,i}(\lambda, k, \sigma, \alpha, \gamma, k)\psi, \quad (2.19)$$

where  $T_{n,i}(\lambda, k, \sigma, \alpha, \gamma, k)\psi$  is the same as that of  $T_i(\lambda, k, \sigma, \alpha, \gamma, k)\psi$  except that the term  $\Omega(\lambda, \alpha, \gamma, \sigma)$  in  $T_i(\lambda, k, \sigma, \alpha, \gamma, k)\psi$  is replaced by

$$\Omega_n(\lambda, \alpha, \gamma, \sigma) = \alpha^n(w, \xi) \gamma^n(w, \xi) \exp \left[ -\frac{2n}{w\xi} \int_Q \Delta(\sigma) ds \right]. \quad (2.20)$$

**Lemma 2.3.1.** *Let  $\beta_1 > -\lambda^*$  be any constant. If the following additional condition is satisfied:*

(H4).  $\alpha(v, \mu), \gamma(v, \mu), \sigma(r, v, \mu)$  and

$$\tilde{k}(x, v, v', \mu, \mu') := (v|\mu|)^{\delta_1} (v'|\mu'|)^{\delta_2} k(x, v, v', \mu, \mu'),$$

(cf. (H3)) are partially differentiable with respect to  $\mu, \mu' \in D$  a. e., and the corresponding partial derivatives  $\frac{\partial \alpha}{\partial \mu}, \frac{\partial \gamma}{\partial \mu}, \frac{\partial \sigma}{\partial \mu}, \frac{\partial \tilde{k}}{\partial \mu}$  and  $\frac{\partial \tilde{k}}{\partial \mu'}$  are essentially bounded,

then there exist positive constants  $C_0$  and  $\tilde{\tau}$  independent of  $\beta \in [\beta_1, +\infty)$ , such that

$$\|K(\lambda I - B)^{-1}K\|_1 \leq C_0|\beta_1 + \lambda^* + i\tau|^{\delta_1-1} \log |\beta_1 + \lambda^* + i\tau| \quad (2.21)$$

uniformly in  $\{\lambda = \beta + i\tau : \beta \geq \beta_1, |\tau| \geq \tilde{\tau}\}$ .

**Proof.** First, we consider the operator  $T_{n,1}(\lambda, k, \sigma, \alpha, \gamma, k)$ , which is given by

$$\begin{aligned} & T_{n,1}(\lambda, k, \sigma, \alpha, \gamma, k)\psi(x, v, \mu) \\ &= \int_{-a}^x dx' \int_V dv' \int_D d\mu' \psi(x', v', \mu') \int_V \frac{dw}{w} \int_0^1 \frac{d\xi}{\xi} \\ & \quad \times k(x, v, w, \mu, \xi) k(x', w, v', \xi, \mu') \alpha^n(w, \xi) \gamma^n(w, \xi) \\ & \quad \times \exp \left[ -\frac{2n}{w\xi} \int_Q (\lambda + \sigma(s, w, \xi)) ds - \frac{1}{w\xi} \int_{x'}^x (\lambda + \sigma(s, w, \xi)) ds \right]. \end{aligned} \quad (2.22)$$

Define

$$\begin{aligned} & E(\lambda, n, x, x', \dots) \\ &= \int_0^1 \frac{d\xi}{\xi} k(x, v, w, \mu, \xi) k(x', w, v', \xi, \mu') \alpha^n(w, \xi) \gamma^n(w, \xi) \\ & \quad \times \exp \left[ -\frac{2n}{w\xi} \int_Q (\lambda + \sigma(s, w, \xi)) ds - \frac{1}{w\xi} \int_{x'}^x (\lambda + \sigma(s, w, \xi)) ds \right] \\ &= \int_1^\infty \frac{dt}{t} k(x, v, w, \mu, t^{-1}) k(x', w, v', t^{-1}, \mu') \alpha^n(w, t^{-1}) \gamma^n(v, t^{-1}) \\ & \quad \times \exp \left[ -\frac{2nt}{w} \int_Q (\lambda + \sigma(s, w, \xi)) ds - \frac{t}{w} \int_{x'}^x (\lambda + \sigma(s, w, \xi)) ds \right], \end{aligned} \quad (2.23)$$

then

$$\begin{aligned} & E(\lambda, n, x, x', \dots) \\ &= \frac{wi}{(4an + x - x')|\mu|^{\delta_1} v^{\delta_1} w^{\delta_1} \tau} \int_1^\infty \frac{\partial}{\partial t} \left\{ \exp \left[ -\frac{4an + x - x'}{w} i\tau t \right] \right\} \\ & \quad \times t^{\delta_1-1} \tilde{k}(x, v, w, \mu, t^{-1}) \tilde{k}(x', w, v', t^{-1}, \mu') \alpha^n(w, t^{-1}) \gamma^n(v, t^{-1}) \\ & \quad \times \exp \left[ -\frac{2nt}{w} \int_Q (\beta + \sigma(s, w, \xi)) ds - \frac{t}{w} \int_{x'}^x (\beta + \sigma(s, w, \xi)) ds \right] dt. \end{aligned}$$

By applying the technique of integration by parts to the above expression, we obtain the following estimation of  $E(\lambda, n, x, x', \dots)$ :

$$|E(\lambda, n, x, x', \dots)| \leq \frac{C_1 w^{1-\delta_1}}{(x - x')|\mu|^{\delta_1} v^{\delta_1} |\tau|} \exp(-4abn), \quad (2.24)$$

where  $b = (\beta_1 + \lambda^*)/v_M$ ,  $C_1$ , as well as all the notations  $C_i$  ( $i = 2, 3, \dots$ ) arising in the following, are positive constants.

On the other hand, from Eq. (2.23), we get

$$\begin{aligned} & |E(\lambda, n, x, x', \dots)| \\ & \leq C_2 |\mu|^{-\delta_1} v^{-\delta_1} w^{-\delta_1} \exp(-4abn) \int_1^\infty \frac{\exp[-t(\beta_1 + \lambda^*)(x - x')w^{-1}]}{t^{1-\delta_1}} dt. \end{aligned} \quad (2.25)$$

By use of the Hölder inequality (set  $p = (1 - \delta_1)^{-1}$ ,  $q = \delta_1^{-1}$ ), we have

$$\begin{aligned} & \int_1^\infty \frac{\exp[-t(\beta_1 + \lambda^*)(x - x')w^{-1}]}{t^{1-\delta_1}} dt \\ & = \int_1^\infty \frac{\exp[-t(\beta_1 + \lambda^*)(1 - \delta_1)(x - x')w^{-1}]}{t^{1-\delta_1}} \times \exp[-t(\beta_1 + \lambda^*)\delta_1(x - x')w^{-1}] dt \\ & \leq \left\{ \int_1^\infty \frac{\exp[-t(\beta_1 + \lambda^*)(x - x')w^{-1}]}{t} dt \right\}^{1-\delta_1} \\ & \quad \times \left\{ \int_1^\infty \exp[-t(\beta_1 + \lambda^*)(x - x')w^{-1}] dt \right\}^{\delta_1}. \end{aligned} \quad (2.26)$$

Since the function  $\frac{1}{\log u} \int_1^\infty \frac{e^{-ut}}{t} dt$  is analytic in the domain  $\{u \in \mathbb{C} : 0 \leq \text{Re} u < 1\}$ , if  $|\beta_1 + \lambda^* + i\tau|(x - x')w^{-1} \leq 1/2$ , which implies  $(\beta_1 + \lambda^*)(x - x')w^{-1} \leq 1/2$ , then from Eq. (2.26), we get

$$\begin{aligned} & \int_1^\infty \frac{\exp[-t(\beta_1 + \lambda^*)(x - x')w^{-1}]}{t^{1-\delta_1}} dt \\ & \leq C_3 w^{\delta_1} (x - x')^{-\delta_1} |\log[(\beta_1 + \lambda^*)(x - x')w^{-1}]|^{1-\delta_1} \\ & \leq C_3 w^{\delta_1} (x - x')^{-\delta_1} |\log[|\beta_1 + \lambda^* + i\tau|(x - x')w^{-1} \cdot (\beta_1 + \lambda^*)|\beta_1 + \lambda^* + i\tau|^{-1}]|^{1-\delta_1} \\ & \leq C_3 w^{\delta_1} (x - x')^{-\delta_1} \{C_4 |\log[|\beta_1 + \lambda^* + i\tau|(x - x')w^{-1}]| \\ & \quad + C_5 |\log |\beta_1 + \lambda^* + i\tau|| + C_6\}. \end{aligned} \quad (2.27)$$

From Eqs. (2.25) and (2.27), we have

$$\begin{aligned} & |E(\lambda, n, x, x', \dots)| \\ & \leq C_2 |\mu|^{-\delta_1} v^{-\delta_1} (x - x')^{-\delta_1} \exp(-4abn) \\ & \quad \times \{C_4 |\log[|\beta_1 + \lambda^* + i\tau|(x - x')w^{-1}]| + C_5 |\log |\beta_1 + \lambda^* + i\tau|| + C_6\}. \end{aligned} \quad (2.28)$$

Let  $Q_1 = \{x : |\beta_1 + \lambda^* + i\tau|(x - x')w^{-1} \leq 1/2\}$ ,  $Q_2 = \{x : |\beta_1 + \lambda^* + i\tau|(x - x')w^{-1} > 1/2\}$ ; then Eq. (2.28) holds in  $Q_1$  and Eq. (2.24) holds in  $Q_2$ . From Eqs. (2.22) – (2.24) and (2.28), by exchanging the integration orders of  $x$  and  $x'$ , we have

$$\begin{aligned}
& \|T_{n,1}(\lambda, k, \sigma, \alpha, \gamma, k)\psi\|_1 \\
& \leq \int_Q dx' \int_V dv' \int_D d\mu' |\psi(x', v', \mu')| \int_V dv \int_D d\mu \int_V \frac{dw}{w} \int_{x'}^a |E(\lambda, n, x, x', \dots)| dx \\
& = \int_Q dx' \int_V dv' \int_D d\mu' |\psi(x', v', \mu')| \int_V dv \int_D d\mu \int_V \frac{dw}{w} \\
& \quad \times \left\{ \int_{Q_2 \cap (x', a]} |E(\lambda, n, x, x', \dots)| dx + \int_{Q_1 \cap (x', a]} |E(\lambda, n, x, x', \dots)| dx \right\} \\
& \leq \int_Q dx' \int_V dv' \int_D d\mu' |\psi(x', v', \mu')| \int_V dv \int_D d\mu \int_V \frac{dw}{w} \\
& \quad \times \int_{Q_2 \cap (x', a]} \frac{C_1 w^{1-\delta_1}}{(x - x') |\mu|^{\delta_1} v^{\delta_1} |\tau|} \exp(-4abn) dx \\
& \quad + \int_Q dx' \int_V dv' \int_D d\mu' |\psi(x', v', \mu')| \int_V dv \int_D d\mu \int_V \frac{dw}{w} \\
& \quad \times \int_{Q_1 \cap (x', a]} C_2 (x - x')^{-\delta_1} |\mu|^{-\delta_1} v^{-\delta_1} \exp(-4abn) \\
& \quad \times \{C_4 |\log[|\beta_1 + \lambda^* + i\tau|(x - x')w^{-1}]| + C_5 |\log|\beta_1 + \lambda^* + i\tau|| + C_6\} dx.
\end{aligned} \tag{2.29}$$

If  $\delta_1 = 0$ , then by virtue of the transformation

$$y(x) = |\beta_1 + \lambda^* + i\tau|(x - x')w^{-1}, \tag{2.30}$$

we get

$$\begin{aligned}
& \int_{Q_2 \cap (x', a]} (x - x')^{-1} dx \\
& \leq \int_{1/2}^{2a|\beta_1 + \lambda^* + i\tau|/w} \frac{|\beta_1 + \lambda^* + i\tau|}{wy} \times \frac{w}{|\beta_1 + \lambda^* + i\tau|} dy \\
& = \log(2a|\beta_1 + \lambda^* + i\tau|/w) - \log(1/2) \\
& \leq C_7 |\log|\beta_1 + \lambda^* + i\tau|| + C_8 \log w + C_9.
\end{aligned} \tag{2.31}$$

If  $0 < \delta_1 < 1$ , then there exists a constant  $\delta$  such that  $\max\{\delta_1, 1 - \delta_1\} < \delta < 1$ . By virtue of the Hölder inequality (set  $p = \delta^{-1}$ ,  $q = (1 - \delta)^{-1}$ ) and the transformation (2.30), we have

the following estimation when  $|\beta_1 + \lambda^* + i\tau| > 1$ :

$$\begin{aligned}
& \int_{Q_2 \cap (x', a]} (x - x')^{-1} dx \\
& \leq C_{10} \left[ \int_{Q_2 \cap (x', a]} (x - x')^{-1/\delta} dx \right]^\delta \\
& \leq C_{10} \left[ \int_{1/2}^\infty \frac{|\beta_1 + \lambda^* + i\tau|^{1/\delta}}{w^{1/\delta} y^{1/\delta}} \cdot \frac{w}{|\beta_1 + \lambda^* + i\tau|} dy \right]^\delta \\
& \leq C_{11} w^{\delta-1} |\beta_1 + \lambda^* + i\tau|^{1-\delta} \\
& \leq C_{12} w^{\delta-1} |\beta_1 + \lambda^* + i\tau|^{\delta_1}. \tag{2.32}
\end{aligned}$$

As for the second term on the right hand side of Eq. (2.29), from (2.30) we have

$$\begin{aligned}
& \int_{Q_1 \cap (x', a]} \frac{C_4 |\log[|\beta_1 + \lambda^* + i\tau|(x - x')w^{-1}]| + C_5 |\log |\beta_1 + \lambda^* + i\tau|| + C_6}{(x - x')^{\delta_1}} dx \\
& \leq \int_0^{1/2} \frac{C_4 \log y + C_5 |\log |\beta_1 + \lambda^* + i\tau|| + C_6}{|\beta_1 + \lambda^* + i\tau|^{-\delta_1} y^{\delta_1} w^{\delta_1}} \cdot \frac{w}{|\beta_1 + \lambda^* + i\tau|} dy \\
& \leq w^{1-\delta_1} |\beta_1 + \lambda^* + i\tau|^{\delta_1-1} (C_{13} |\log |\beta_1 + \lambda^* + i\tau|| + C_{14}). \tag{2.33}
\end{aligned}$$

From Eqs. (2.29), (2.31), (2.32) and (2.33), it is easy to see

$$\begin{aligned}
& \|T_{n,1}(\lambda, k, \sigma, \alpha, \gamma, k)\|_1 \\
& \leq C_{15} \exp(-4abn) |\beta_1 + \lambda^* + i\tau|^{\delta_1-1} \log |\beta_1 + \lambda^* + i\tau| \tag{2.34}
\end{aligned}$$

when  $|\tau|$  is sufficiently large.

Similarly, it can be shown that  $\|T_{n,i}(\lambda, k, \sigma, \alpha, \gamma, k)\|_1$  ( $i = 2, 3, \dots, 6$ ) has the same estimation. These relations together with Eq. (2.19) imply the conclusion. Q. E. D.

**Lemma 2.3.2.** *Let  $\beta_1 > -\lambda^*$  be any given constant. Then for every integer  $n$ , there exist infinite order smooth functions  $\alpha_n(v, \mu)$  and  $\gamma_n(v, \mu)$  defined on  $V \times D$  and valued in  $(-1/2, \exp[av_M^{-1}(\beta_1 + \lambda^*)])$  such that  $\alpha_n(v, \mu) = \alpha_n(v, -\mu)$ ,  $\gamma_n(v, \mu) = \gamma_n(v, -\mu)$ ,  $\|\alpha - \alpha_n\|_{q_0} < 1/n$  and  $\|\gamma - \gamma_n\|_{q_0} < 1/n$ , where  $\|\cdot\|_{q_0}$  is the norm of  $L^{q_0}(V \times D)$ ,  $q_0 = 3(1 - \delta_1)^{-1}$ .*

This assertion can be easily seen from the theory of real analysis.

Similar to that of Lemma 2.3.2, for any given constant  $\beta_1 > -\lambda^*$ , there exists a sequence  $\{\sigma_n(x, v, \mu)\}$  composed of infinite order smooth functions such that  $\|\sigma_n - \sigma\|_{q_0} < 1/n$  with  $q_0 = 3(1 - \delta_1)^{-1}$ , and

$$\lambda^* - \frac{\beta_1 + \lambda^*}{2} \leq \sigma_n(x, v, \mu) \leq \text{ess sup}_{(x,v,\mu) \in G} \sigma(x, v, \mu) + 1.$$

Let  $\delta\sigma_n(x, v, \mu) = \sigma_n(x, v, \mu) - \sigma(x, v, \mu)$ , and define operators on  $L^1(G)$  as follows:

$$B_{n,n}\psi = -v\mu \frac{\partial\psi}{\partial x} - \sigma_n(x, v, \mu)\psi,$$

with  $D(B_{n,n}) = \{\psi \in L^1(G) : B_{n,n}\psi \in L^1(G); \psi(-a, v, \mu) = \alpha_n(v, \mu)\psi(-a, v, -\mu)$  and  $\psi(a, v, -\mu) = \gamma_n(v, \mu)\psi(a, v, \mu)$  for every  $\mu \in (0, 1]\}$ ;

$$B_{\sigma_n}\psi = -v\mu \frac{\partial\psi}{\partial x} - \sigma_n(x, v, \mu)\psi,$$

with  $D(B_{\sigma_n}) = \{\psi \in L^1(G) : B_{\sigma_n}\psi \in L^1(G); \psi(-a, v, \mu) = \alpha(v, \mu)\psi(-a, v, -\mu)$  and  $\psi(a, v, -\mu) = \gamma(v, \mu)\psi(a, v, \mu)$  for every  $\mu \in (0, 1]\}$ ;

$$B_{\alpha_n, \gamma_n}\psi = -v\mu \frac{\partial\psi}{\partial x} - \sigma(x, v, \mu)\psi,$$

with  $D(B_{\alpha_n, \gamma_n}) = \{\psi \in L^1(G) : B_{\alpha_n, \gamma_n}\psi \in L^1(G); \psi(-a, v, \mu) = \alpha_n(v, \mu)\psi(-a, v, -\mu)$  and  $\psi(a, v, -\mu) = \gamma_n(v, \mu)\psi(a, v, \mu)$  for every  $\mu \in (0, 1]\}$ ;

$$\delta F_n\psi = \delta\sigma_n(x, v, \mu)\psi,$$

with  $D(\delta F_n) = L^1(G)$ .

By a procedure similar to that for proving Lemma 2.2.1, it is not difficult to verify that  $\{\lambda : \text{Re}\lambda \geq \beta_1\} \subset \rho(B_{n,n}) \cap \rho(B_{\sigma_n}) \cap \rho(B_{\alpha_n, \gamma_n})$ , and

$$\|(\lambda I - B_{n,n})^{-1}\|_1 \leq c_1(\beta_1 + \lambda^*)^{-1},$$

$$\|(\lambda I - B_{\sigma_n})^{-1}\|_1 \leq c_2(\beta_1 + \lambda^*)^{-1},$$

$$\|(\lambda I - B_{\alpha_n, \gamma_n})^{-1}\|_1 \leq c_3(\beta_1 + \lambda^*)^{-1}$$

for every  $\lambda$  with  $\text{Re}\lambda \geq \beta_1$ , where  $c_1, c_2$  and  $c_3$  are positive constants independent of  $\lambda$  and  $n$ .

**Lemma 2.3.3.** For any constant  $\beta_1 > -\lambda^*$ ,

$$\lim_{n \rightarrow \infty} \|\delta F_n(\lambda I - B)^{-1}K\|_1 = 0$$

uniformly in  $\{\lambda : \operatorname{Re}\lambda \geq \beta_1\}$ .

**Proof.** For any  $\psi \in L^1(G)$ , it is not difficult to verify that

$$\delta F_n(\lambda I - B)^{-1}K\psi(x, v, \mu) = \sum_{i=1}^3 S_i(\lambda, \alpha, \gamma, \sigma, k)\psi(x, v, \mu)$$

for  $\mu > 0$ , with

$$\begin{aligned} & S_1(\lambda, \alpha, \gamma, \sigma, k)\psi(x, v, \mu) \\ &= \frac{\delta\sigma_n(x, v, \mu)\tilde{\Omega}(\lambda, \alpha, \gamma, \sigma)}{v\mu} \int_{-a}^x \exp\left[-\frac{1}{v\mu} \int_{x'}^x \tilde{\Delta}(\sigma)ds\right] \\ & \quad \times \int_V \int_D k(x', v, v', \mu, \mu')\psi(x', v', \mu')dx'dv'd\mu', \\ & S_2(\lambda, \alpha, \gamma, \sigma, k)\psi(x, v, \mu) \\ &= \frac{\delta\sigma_n(x, v, \mu)\alpha(v, \mu)\tilde{\Omega}(\lambda, \alpha, \gamma, \sigma)}{v\mu} \int_Q \exp\left[-\frac{1}{v\mu} \int_{-a}^x \tilde{\Delta}(\sigma)ds - \frac{1}{v\mu} \int_{-a}^{x'} \tilde{\Delta}(\sigma)ds\right] \\ & \quad \times \int_V \int_D k(x', v, v', -\mu, \mu')\psi(x', v', \mu')dx'dv'd\mu', \\ & S_3(\lambda, \alpha, \gamma, \sigma, k)\psi(x, v, \mu) \\ &= \frac{\delta\sigma_n(x, v, \mu)\alpha(v, \mu)\gamma(v, \mu)\tilde{\Omega}(\lambda, \alpha, \gamma, \sigma)}{v\mu} \int_x^a \exp\left[-\frac{2}{v\mu} \int_Q \tilde{\Delta}(\sigma)ds - \frac{1}{v\mu} \int_{x'}^x \tilde{\Delta}(\sigma)ds\right] \\ & \quad \times \int_V \int_D k(x', v, v', \mu, \mu')\psi(x', v', \mu')dx'dv'd\mu', \end{aligned}$$

$$\tilde{\Delta}(\sigma) = \lambda + \sigma(s, v, \mu),$$

$$\tilde{\Omega}(\lambda, \alpha, \gamma, \sigma) = \left\{ 1 - \alpha(v, \mu)\gamma(v, \mu) \exp\left[-\frac{2}{v|\mu|} \int_Q \tilde{\Delta}(\sigma)ds\right] \right\}^{-1},$$

and

$$\delta F_n(\lambda I - B)^{-1}K\psi(x, v, \mu) = \sum_{i=4}^6 S_i(\lambda, \alpha, \gamma, \sigma, k)\psi(x, v, \mu)$$

for  $\mu < 0$ , with

$$\begin{aligned}
& S_4(\lambda, \alpha, \gamma, \sigma, k)\psi(x, v, \mu) \\
&= -\frac{\delta\sigma_n(x, v, \mu)\tilde{\Omega}(\lambda, \alpha, \gamma, \sigma)}{v\mu} \int_x^a \exp\left[\frac{1}{v\mu} \int_x^{x'} \tilde{\Delta}(\sigma)ds\right] \\
&\quad \times \int_V \int_D k(x', v, v', \mu, \mu')\psi(x', v', \mu')dx' dv' d\mu', \\
& S_5(\lambda, \alpha, \gamma, \sigma, k)\psi(x, v, \mu) \\
&= -\frac{\delta\sigma_n(x, v, \mu)\gamma(v, \mu)\tilde{\Omega}(\lambda, \alpha, \gamma, \sigma)}{v\mu} \int_Q \exp\left[\frac{1}{v\mu} \int_x^a \tilde{\Delta}(\sigma)ds + \frac{1}{v\mu} \int_{x'}^a \tilde{\Delta}(\sigma)ds\right] \\
&\quad \times \int_V \int_D k(x', v, v', -\mu, \mu')\psi(x', v', \mu')dx' dv' d\mu', \\
& S_6(\lambda, \alpha, \gamma, \sigma, k)\psi(x, v, \mu) \\
&= \frac{\delta\sigma_n(x, v, \mu)\alpha(v, \mu)\gamma(v, \mu)\tilde{\Omega}(\lambda, \alpha, \gamma, \sigma)}{v\mu} \int_{-a}^x \exp\left[\frac{2}{v\mu} \int_Q \tilde{\Delta}(\sigma)ds + \frac{1}{v\mu} \int_x^{x'} \tilde{\Delta}(\sigma)ds\right] \\
&\quad \times \int_V \int_D k(x', v, v', \mu, \mu')\psi(x', v', \mu')dx' dv' d\mu'.
\end{aligned}$$

Denoting by  $\|\cdot\|_1^+$  and  $\|\cdot\|_1^-$  the norms of  $L^1(Q \times V \times (0, 1])$  and  $L^1(Q \times V \times [-1, 0))$  respectively, then

$$\begin{aligned}
& \|S_1(\lambda, \alpha, \gamma, \sigma, k)\psi\|_1^+ \\
&\leq C_{16} \int_Q dx \int_V dv \int_0^1 d\mu \frac{|\delta\sigma_n(x, v, \mu)|}{(v\mu)^{1+\delta_1}} \int_{-a}^x \exp\left[-\frac{(\beta_1 + \lambda^*)(x - x')}{v\mu}\right] \\
&\quad \times \int_V \int_D |\psi(x', v', \mu')| dx' dv' d\mu'.
\end{aligned}$$

By exchanging the integration orders of  $x$  and  $x'$ , we get

$$\begin{aligned}
& \|S_1(\lambda, \alpha, \gamma, \sigma, k)\psi\|_1^+ \\
&\leq C_{16} \int_Q dx' \int_V dv' \int_D d\mu' |\psi(x', v', \mu')| \int_{x'}^a dx \int_V dv \int_0^1 d\mu |\delta\sigma_n(x, v, \mu)| \\
&\quad \times \frac{1}{(v\mu)^{(1+\delta_1)}} \exp\left[-\frac{(\beta_1 + \lambda^*)(x - x')}{v\mu}\right].
\end{aligned}$$

Using the Hölder inequality (set  $p_0 = 3(2 + \delta_1)^{-1}$ ,  $q_0 = 3(1 - \delta_1)^{-1}$ ), we have

$$\begin{aligned}
& \|S_1(\lambda, \alpha, \gamma, \sigma, k)\psi\|_1^+ \\
& \leq C_{16} \int_Q dx' \int_V dv' \int_D d\mu' |\psi(x', v', \mu')| \left\{ \int_{x'}^a dx \int_V dv \int_0^1 d\mu |\delta\sigma_n(x, v, \mu)|^{q_0} \right\}^{1/q_0} \\
& \quad \times \left\{ \int_{x'}^a dx \int_V dv \int_0^1 d\mu \frac{1}{(v\mu)^{p_0(1+\delta_1)}} \exp \left[ -\frac{p_0(\beta_1 + \lambda^*)(x - x')}{v\mu} \right] \right\}^{1/p_0} \\
& \leq C_{17} \|\psi\|_1 \cdot \|\delta\sigma\|_{q_0}.
\end{aligned}$$

Similarly, we have

$$\|S_i(\lambda, \alpha, \gamma, \sigma, k)\psi\|_1^+ \leq C_{18} \|\psi\|_1 \cdot \|\delta\sigma\|_{q_0}, \quad i = 2, 3,$$

$$\|S_i(\lambda, \alpha, \gamma, \sigma, k)\psi\|_1^- \leq C_{19} \|\psi\|_1 \cdot \|\delta\sigma\|_{q_0}, \quad i = 4, 5, 6.$$

So

$$\|\delta F_n(\lambda I - B)^{-1}K\|_1 \leq C_{20} \|\delta\sigma\|_{q_0},$$

which completes the proof. Q. E. D.

**Corollary 2.3.4.**  $\lim_{n \rightarrow \infty} \|(\lambda I - B)^{-1}K - (\lambda I - B_{\sigma_n})^{-1}K\|_1 = 0$  uniformly in  $\{\lambda : \operatorname{Re}\lambda \geq \beta_1\}$ .

This can be easily obtained from Lemma 2.3.3 and the relation

$$(\lambda I - B)^{-1}K - (\lambda I - B_{\sigma_n})^{-1}K = (\lambda I - B_{\sigma_n})^{-1}\delta F_n(\lambda I - B)^{-1}K.$$

**Lemma 2.3.5.** For any given constant  $\beta_1 > -\lambda^*$ ,

$$\lim_{n \rightarrow \infty} \|K(\lambda I - B_{\sigma_n})^{-1}K - K(\lambda I - B_{n,n})^{-1}K\|_1 = 0$$

uniformly in  $\{\lambda : \operatorname{Re}\lambda \geq \beta_1\}$ .

**Proof.** For any  $\psi \in L^1(G)$ , it follows from Eq. (2.12) that

$$\begin{aligned}
& K(\lambda I - B_{\sigma_n})^{-1}K - K(\lambda I - B_{n,n})^{-1}K\psi \\
& = \sum_{i=1}^6 [T_i(\lambda, k, \sigma_n, \alpha, \gamma, k)\psi - T_i(\lambda, k, \sigma_n, \alpha_n, \gamma_n, k)\psi].
\end{aligned} \tag{2.35}$$

Consider  $T_1(\lambda, k, \sigma_n, \alpha, \gamma, k)\psi - T_1(\lambda, k, \sigma_n, \alpha_n, \gamma_n, k)\psi$ . Set

$$\delta\alpha_n(v, \mu) = \alpha(v, \mu) - \alpha_n(v, \mu), \quad \delta\gamma_n(v, \mu) = \gamma(v, \mu) - \gamma_n(v, \mu).$$

Noting that (cf. Eq. (2.18))

$$\begin{aligned} & \Omega(\lambda, \alpha, \gamma, \sigma_n) - \Omega(\lambda, \alpha_n, \gamma_n, \sigma_n) \\ &= \Omega(\lambda, \alpha, \gamma, \sigma_n)\Omega(\lambda, \alpha_n, \gamma_n, \sigma_n) [\alpha(w, \xi)\delta\gamma_n(w, \xi) + \delta\alpha_n(w, \xi)\gamma_n(w, \xi) \\ & \quad - \delta\alpha_n(w, \xi)\delta\gamma_n(w, \xi)] \exp \left[ -\frac{2}{w\xi} \int_Q \Delta(\sigma_n) ds \right], \end{aligned}$$

where  $\Delta(\sigma_n) = \lambda + \sigma_n(s, w, \xi)$ , we get the following expression from Eq. (2.13)

$$\begin{aligned} & T_1(\lambda, k, \sigma_n, \alpha, \gamma, k)\psi - T_1(\lambda, k, \sigma_n, \alpha_n, \gamma_n, k)\psi \\ &= \int_{-a}^x \int_V \int_D \int_V \int_0^1 \frac{\Omega(\lambda, \alpha, \gamma, \sigma_n)\Omega(\lambda, \alpha_n, \gamma_n, \sigma_n)}{w\xi} \\ & \quad \times [\alpha(w, \xi)\delta\gamma_n(w, \xi) + \delta\alpha_n(w, \xi)\gamma_n(w, \xi) - \delta\alpha_n(w, \xi)\delta\gamma_n(w, \xi)] \\ & \quad \times \exp \left[ -\frac{2}{w\xi} \int_Q \Delta(\sigma_n) ds \right] k(x, v, w, \mu, \xi)k(x', w, v', \xi, \mu') \\ & \quad \times \exp \left[ -\frac{1}{w\xi} \int_{x'}^x \Delta(\sigma_n) ds \right] \psi(x', v', \mu') d\xi dw d\mu' dv' dx'. \end{aligned}$$

Therefore, from (H3) and the above equation, we have

$$\begin{aligned} & \|T_1(\lambda, k, \sigma_n, \alpha, \gamma, k)\psi - T_1(\lambda, k, \sigma_n, \alpha_n, \gamma_n, k)\psi\|_1 \\ &\leq C_{21} \int_Q \int_V \int_D \int_{-a}^x \int_V \int_D \int_V \int_0^1 \frac{|k(x, v, w, \mu, \xi)k(x', w, v', \xi, \mu')|}{w\xi} \\ & \quad \times [|\delta\gamma_n(w, \xi)| + |\delta\alpha_n(w, \xi)| + |\delta\alpha_n(w, \xi)|] \\ & \quad \times \exp \left[ -\frac{(\beta_1 + \lambda^*)(x - x')}{2w\xi} \right] |\psi(x', v', \mu')| d\xi dw d\mu' dv' dx' d\mu dv dx \\ &\leq C_{22} \int_Q \int_V \int_D \int_{-a}^x \int_V \int_D \int_V \int_0^1 (v|\mu|)^{-\delta_1} (w\xi)^{-(1+\delta_1)} [|\delta\alpha_n(w, \xi)| + |\delta\gamma_n(w, \xi)|] \\ & \quad \times \exp \left[ -\frac{(\beta_1 + \lambda^*)(x - x')}{2w\xi} \right] |\psi(x', v', \mu')| d\xi dw d\mu' dv' dx' d\mu dv dx. \end{aligned}$$

By exchanging the integration orders of  $x$  and  $x'$ , we finally get

$$\begin{aligned} & \|T_1(\lambda, k, \sigma_n, \alpha, \gamma, k)\psi - T_1(\lambda, k, \sigma_n, \alpha_n, \gamma_n, k)\psi\|_1 \\ &\leq C_{23} \|\psi\|_1 \left[ \int_V \int_0^1 (w\xi)^{-\delta_1} |\delta\alpha_n(w, \xi)| d\xi dw + \int_V \int_0^1 (w\xi)^{-\delta_1} |\delta\gamma_n(w, \xi)| d\xi dw \right]. \end{aligned}$$

Using the Hölder inequality (set  $p_0 = 3(2 + \delta_1)^{-1}$ ,  $q_0 = 3(1 - \delta_1)^{-1}$ ), we obtain

$$\|T_1(\lambda, k, \sigma_n, \alpha, \gamma, k) - T_1(\lambda, k, \sigma_n, \alpha_n, \gamma_n, k)\|_1 \leq C_{24} (\|\delta\alpha_n\|_{q_0} + \|\delta\gamma_n\|_{q_0}).$$

Similarly, we have

$$\|T_i(\lambda, k, \sigma_n, \alpha, \gamma, k) - T_i(\lambda, k, \sigma_n, \alpha_n, \gamma_n, k)\|_1 \leq C_{25} (\|\delta\alpha_n\|_{q_0} + \|\delta\gamma_n\|_{q_0})$$

for  $i = 2, 3, \dots, 6$ . This completes the proof from Eq. (2.35).

**Corollary 2.3.6.**  $\lim_{n \rightarrow \infty} \|K(\lambda I - B)^{-1}K - K(\lambda I - B_{n,n})^{-1}K\|_1 = 0$  uniformly in  $\{\lambda : \operatorname{Re}\lambda \geq \beta_1\}$ .

This can be easily seen from Corollary 2.3.4 and Lemma 2.3.5.

From (H3), it is seen that  $\tilde{k}(x, v, v', \mu, \mu') := (v|\mu|)^{\delta_1}k(x, v, v', \mu, \mu')$  is bounded measurable. From the theory of real analysis, there exists a sequence  $\{\tilde{k}_n(x, v, v', \mu, \mu')\}$  composed of uniformly bounded polynomial functions such that

$$\lim_{n \rightarrow \infty} \|\tilde{k}_n - \tilde{k}\|_{q_0} = 0 \tag{2.36}$$

where  $q_0 = 3(1 - \delta_1)^{-1}$ . Let

$$\widetilde{\delta k}_n(x, v, v', \mu, \mu') = \tilde{k}(x, v, v', \mu, \mu') - \tilde{k}_n(x, v, v', \mu, \mu'),$$

$$k_n(x, v, v', \mu, \mu') = (v|\mu|)^{-\delta_1}\tilde{k}_n(x, v, v', \mu, \mu'),$$

$$\delta k_n(x, v, v', \mu, \mu') = (v|\mu|)^{-\delta_1}\widetilde{\delta k}_n(x, v, v', \mu, \mu'),$$

and define operators  $K_n, \delta K_n$  on  $L^1(G)$  as follows:

$$K_n\psi = \int_D \int_V k_n(x, v, v', \mu, \mu')\psi(x, v', \mu')dv'd\mu',$$

$$\delta K_n\psi = \int_D \int_V \delta k_n(x, v, v', \mu, \mu')\psi(x, v', \mu')dv'd\mu'.$$

Then we have the following results.

**Lemma 2.3.7.** Let  $\beta_1 > -\lambda^*$  be any given constant; then

$$\lim_{n \rightarrow \infty} \|\delta K_n(\lambda I - B_{n,n})^{-1}K\|_1 = 0$$

uniformly in  $\{\lambda : \operatorname{Re}\lambda \geq \beta_1\}$ .

**Proof.** For any  $\psi \in L^1(G)$ , it follows from Eq. (2.12) that

$$\delta K_n(\lambda I - B_{n,n})^{-1} K \psi = \sum_{i=1}^6 T_i(\lambda, \delta k_n, \sigma_n, \alpha_n, \gamma_n, k) \psi.$$

First, we consider  $T_1(\lambda, \delta k_n, \sigma_n, \alpha_n, \gamma_n, k) \psi$ . It is easy to see

$$\begin{aligned} & \|T_1(\lambda, \delta k_n, \sigma_n, \alpha_n, \gamma_n, k) \psi\|_1 \\ & \leq C_{26} \int_Q \int_V \int_D \int_{-a}^x \int_V \int_D \int_V \int_0^1 \frac{|\widetilde{\delta k_n}(x, v, w, \mu, \xi)|}{(w\xi)^{1+\delta_1} (v|\mu|^{\delta_1})} \\ & \quad \times \exp \left[ -\frac{(\beta_1 + \lambda^*)(x - x')}{2w\xi} \right] |\psi(x', v', \mu')| d\xi dw d\mu' dv' dx' d\mu dv dx. \end{aligned}$$

Again, by virtue of the technique of exchanging the integration orders of  $x$  and  $x'$ , we get

$$\begin{aligned} & \|T_1(\lambda, \delta k_n, \sigma_n, \alpha_n, \gamma_n, k) \psi\|_1 \\ & \leq C_{26} \int_Q dx' \int_V dv' \int_D d\mu' |\psi(x', v', \mu')| \int_{x'}^a \int_V \int_D \int_V \int_0^1 |\widetilde{\delta k_n}(x, v, w, \mu, \xi)| \\ & \quad \times (w\xi)^{-(1+\delta_1)} (v|\mu|^{\delta_1})^{-\delta_1} \exp \left[ -\frac{(\beta_1 + \lambda^*)(x - x')}{2w\xi} \right] d\xi dw d\mu dv dx. \end{aligned}$$

By the Hölder inequality (set  $p_0 = 3(2 + \delta_1)^{-1}$ ,  $q_0 = 3(1 - \delta_1)^{-1}$ ), we have

$$\begin{aligned} & \|T_1(\lambda, \delta k_n, \sigma_n, \alpha_n, \gamma_n, k) \psi\|_1 \\ & \leq C_{26} \int_Q dx' \int_V dv' \int_D d\mu' |\psi(x', v', \mu')| \\ & \quad \times \left\{ \int_{x'}^a \int_V \int_D \int_V \int_0^1 |\widetilde{\delta k_n}(x, v, w, \mu, \xi)|^{q_0} d\xi dw d\mu dv dx \right\}^{1/q_0} \\ & \quad \times \left\{ \int_{x'}^a \int_V \int_D \int_V \int_0^1 (w\xi)^{-p_0(1+\delta_1)} (v|\mu|^{\delta_1})^{-p_0\delta_1} \right. \\ & \quad \left. \times \exp \left[ -\frac{p_0(\beta_1 + \lambda^*)(x - x')}{2w\xi} \right] d\xi dw d\mu dv dx \right\}^{1/q_0} \\ & \leq C_{27} \|\psi\|_1 \cdot \|\widetilde{\delta k_n}\|_{q_0}. \end{aligned}$$

Similarly, we have

$$\|T_i(\lambda, \delta k_n, \sigma_n, \alpha_n, \gamma_n, k) \psi\|_1 \leq C_{28} \|\psi\|_1 \cdot \|\widetilde{\delta k_n}\|_{q_0}$$

for  $i = 2, 3, \dots, 6$ . This together with Eq. 2.36 completes the proof. Q. E. D.

**Lemma 2.3.8.** *Let  $\beta_1 > -\lambda^*$  be any given constant; then*

$$\lim_{n \rightarrow \infty} \|\delta K_n (\lambda I - B)^{-1} K\|_1 = 0$$

*uniformly in  $\{\lambda : \operatorname{Re} \lambda \geq \beta_1\}$ .*

This can be easily seen from the procedure for proving Lemma 2.3.7.

**Theorem 2.3.9.** *For every  $\lambda$  with  $\operatorname{Re} \lambda > -\lambda^*$ ,  $[K(\lambda I - B)^{-1}]^2 K$  is a compact operator on  $L^1(G)$ .*

**Proof:** It is easy to see

$$\begin{aligned} & K(\lambda I - B)^{-1} K(\lambda I - B)^{-1} K \\ = & [K(\lambda I - B)^{-1} K - K(\lambda I - B_{n,n})^{-1} K] (\lambda I - B)^{-1} K \\ & + (K_n + \delta K_n)(\lambda I - B_{n,n})^{-1} (K_n + \delta K_n)(\lambda I - B)^{-1} K \\ = & [K(\lambda I - B)^{-1} K - K(\lambda I - B_{n,n})^{-1} K] (\lambda I - B)^{-1} K \\ & + \delta K_n (\lambda I - B_{n,n})^{-1} K(\lambda I - B)^{-1} K + K_n (\lambda I - B_{n,n})^{-1} K_n (\lambda I - B)^{-1} K \\ & + K_n (\lambda I - B_{n,n})^{-1} \delta K_n (\lambda I - B)^{-1} K, \end{aligned} \tag{2.37}$$

where  $B_{n,n}$ ,  $K_n$  and  $\delta K_n$  are operators in Lemma 2.3.5 – Lemma 2.3.8.

Since  $\alpha_n(v, \mu)$ ,  $\gamma_n(v, \mu)$ ,  $\sigma_n(x, v, \mu)$  and  $\tilde{k}_n(x, v, v', \mu, \mu')$  are smooth functions, it can be shown that for every integer  $n$ ,  $K_n(\lambda I - B_{n,n})^{-1} K_n$  is a compact operator on  $L^1(G)$  (cf. [18]).

Thus, we get the conclusion from Eq. (2.37) and Lemma 2.3.3 – Lemma 2.3.7. Q. E. D.

**Theorem 2.3.10.** *Suppose (H1) – (H3) are satisfied, and  $\beta_1 > -\lambda^*$  is an arbitrarily given constant. Then for every  $\varepsilon > 0$ , there exists a positive constant  $\tilde{\tau}$  independent of  $\beta \in [\beta_1, +\infty)$ , such that*

$$\|[K(\lambda I - B)^{-1}]^2 K\|_1 < \varepsilon$$

*uniformly in  $\{\lambda = \beta + i\tau : \beta \geq \beta_1, |\tau| \geq \tilde{\tau}\}$ .*

**Proof.** For any  $\tilde{\varepsilon} > 0$ , it follows from Corollary 2.3.6 and Lemmas 2.3.7, 2.3.8 that there exists an integer  $n_0$  independent of  $\lambda \in \{\lambda : \operatorname{Re} \lambda \geq \beta_1\}$  such that

$$\|K(\lambda I - B)^{-1} K - K(\lambda I - B_{n_0, n_0})^{-1} K\|_1 < \tilde{\varepsilon}, \tag{2.38}$$

$$\|\delta K_{n_0}(\lambda I - B_{n_0, n_0})^{-1}K\|_1 < \tilde{\varepsilon}, \quad (2.39)$$

$$\|\delta K_{n_0}(\lambda I - B)^{-1}K\|_1 < \tilde{\varepsilon}. \quad (2.40)$$

Noting that  $\alpha_{n_0}(v, \mu)$ ,  $\gamma_{n_0}(v, \mu)$ ,  $\sigma_{n_0}(x, v, \mu)$  and  $\widetilde{k}_{n_0}(x, v, v', \mu, \mu')$  are smooth functions, we see from Lemma 2.3.1 that there exists a positive constant  $\tilde{\tau}$  independent of  $\beta \in [\beta_1, +\infty)$ , such that

$$\|K_{n_0}(\lambda I - B_{n_0, n_0})^{-1}K_{n_0}\|_1 < \tilde{\varepsilon} \quad (2.41)$$

uniformly in  $\{\lambda = \beta + i\tau : \beta \geq \beta_1, |\tau| \geq \tilde{\tau}\}$ .

From Eqs. (2.38) – (2.41) and Eq. (2.37) (with  $n = n_0$ ), we get the conclusion. Q. E. D.

From Theorem 2.3.9, 2.3.10 and the well known Gohberg's theorem (cf. [15, Cor. 11.6, page 259]), we have the following results about  $\sigma(A)$  (cf. [22, Lemma 1.1]).

**Theorem 2.3.11.** *Under the conditions (H1) – (H3), the following assertions hold for any constant  $\beta_1 > -\lambda^*$  in the setting of  $L^1(G)$ :*

(i).  *$\text{Pas}(A) := \sigma(A) \cap \{\lambda : \text{Re}\lambda > -\lambda^*\}$  contains at most countable isolated elements, each of which is an eigenvalue of  $A$  with finite algebraic multiplicity.*

(ii). *The set  $\sigma(A) \cap \{\lambda : \text{Re}\lambda \geq \beta_1\}$  contains at most finite elements.*

(iii). *There exists a positive constant  $\tilde{\tau}$  independent of  $\beta \in [\beta_1, +\infty)$ , such that  $\|(\lambda I - A)^{-1}\|_1$  is uniformly bounded in  $\{\lambda = \beta + i\tau : \beta \geq \beta_1, |\tau| \geq \tilde{\tau}\}$ .*

## 2.4 Spectral Properties of $T(t)$ in $L^1(G)$

From Theorem 2.3.11 and [22, page 44, Theorem 1.1], the asymptotic expansion of the solution  $\psi(t)$  of Eq. (I) can be obtained if  $\psi_0 \in D(A^2)$ . As previously mentioned, it is one of the motivations of this paper to eliminate the stern condition  $\psi_0 \in D(A^2)$ . So this section is devoted to discussing some aspects of the spectral properties of  $T(t)$ , and these spectral properties are closely related to the asymptotic behavior of  $\psi(t)$ .

From Theorem 2.3.11, the eigenvalues of  $A$  lying in the half-plane  $\text{Re}\lambda > -\lambda^*$  can be ordered in such a way that the real part decreases. Suppose  $\lambda_1, \lambda_2, \dots, \lambda_m, \lambda_{m+1}, \dots$  are eigenvalues

of  $A$ ,  $\operatorname{Re}\lambda_1 \geq \operatorname{Re}\lambda_2 \geq \cdots \geq \operatorname{Re}\lambda_m > \operatorname{Re}\lambda_{m+1} \geq \cdots > -\lambda^*$ , and  $\{\lambda : \operatorname{Re}\lambda > -\lambda^*\} \setminus \{\lambda_n : n = 1, 2, \dots\} \subset \rho(A)$ . For convenience' sake, the eigenvalues of  $A$  are ordered repeatedly according to their algebraic multiplicities (cf. [16, page 108]), i.e.,  $\lambda_1, \lambda_2, \dots, \lambda_m, \lambda_{m+1}, \dots$  are repeated eigenvalues of  $A$  (see [16, page 42]).

**Theorem 2.4.1.** *Let  $\lambda_1, \lambda_2, \dots, \lambda_m, \lambda_{m+1}, \dots$  be repeated eigenvalues of  $A$ ,  $\operatorname{Re}\lambda_1 \geq \operatorname{Re}\lambda_2 \geq \cdots \geq \operatorname{Re}\lambda_m > \operatorname{Re}\lambda_{m+1} \geq \cdots > -\lambda^*$ ,  $\{\lambda : \operatorname{Re}\lambda > -\lambda^*\} \setminus \{\lambda_n : n = 1, 2, \dots\} \subset \rho(A)$ , and let  $P_m$  be the projection operator of  $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$  corresponding to  $A$ . If (H1) – (H4) are satisfied, then for every  $\varepsilon > 0$ , there exists a positive constant  $M_1$  such that*

$$\|T(t)(I - P_m)\|_1 \leq M_1 \exp\{(\operatorname{Re}\lambda_{m+1} + \varepsilon)t\}.$$

**Proof.** From Lemma 2.2.2,  $B$  generates a positive  $C_0$  semigroup  $S(t)$  in  $L^1(G)$ , and the growth bound of  $S(t)$  is less than or equal to  $-\lambda^*$ . Thus, for every  $\varepsilon > 0$ , there exists a positive constant  $\widetilde{M}$  such that

$$\|S(t)\|_1 \leq \widetilde{M} \exp[(-\lambda^* + \varepsilon)t]. \quad (2.42)$$

For every integer  $m$  satisfying  $\operatorname{Re}\lambda_m > \operatorname{Re}\lambda_{m+1}$ , let  $\sigma_1 = \{\lambda_1, \lambda_2, \dots, \lambda_m\}$ ,  $\sigma_2 = \sigma(A) \setminus \sigma_1$ . Since  $\sigma_1$  is a compact set, it follows from [23, page 70] that there exists a unique spectral decomposition  $L^1(G) = E_1 \oplus E_2$  such that  $T^{(i)}(t)$ , the part of  $T(t)$  in  $E_i$  ( $i = 1, 2$ ), is a  $C_0$  semigroup. Furthermore, the spectral set of  $A^{(i)}$  (where  $A^{(i)}$  is the generator of  $T^{(i)}(t)$ ) is equal to  $\sigma_i$ , i.e.,  $\sigma(A^{(i)}) = \sigma_i$ ,  $i = 1, 2$ , and  $A^{(1)}$  is a bounded operator on  $E_1$ . Denoting by  $P_m$  the projection operator of  $\sigma_1$  corresponding to  $A$ , then (cf. [23, page 70])

$$\begin{aligned} T^{(1)}(t) &= T(t)P_m, \quad T^{(2)}(t) = T(t)(I - P_m), \\ (\lambda I - A^{(1)})^{-1} &= (\lambda I - A)^{-1}P_m, \quad (\lambda I - A^{(2)})^{-1} = (\lambda I - A)^{-1}(I - P_m), \\ \sigma(T(t)) &= \sigma(T^{(1)}(t)) \cup \sigma(T^{(2)}(t)). \end{aligned} \quad (2.43)$$

Since  $A^{(1)}$  is a bounded operator on  $E_1$ , we have

$$\sigma(T^{(1)}(t)) = \{\exp(\lambda_i t) : i = 1, 2, \dots, m\}. \quad (2.44)$$

Let  $V^{(0)}(t) = S(t)$  (the  $C_0$  semigroup generated by  $B$ ), and define operator  $V^{(k)}(t)$  on  $L^1(G)$  inductively by

$$V^{(k+1)}(t)\psi = \int_0^t V^{(0)}(t-s)KV^{(k)}(s)\psi ds, \quad \psi \in L^1(G), \quad k \geq 0.$$

Then from [12, 29] we have  $T(t) = \sum_{k=0}^{\infty} V^{(k)}(t)$ , and

$$\int_0^{\infty} e^{-\lambda t} V^{(k)}(t)\psi dt = (\lambda I - B)^{-1} [K(\lambda I - B)^{-1}]^k \psi, \quad k \geq 0, \quad (2.45)$$

$$\|V^{(k)}(t)\|_1 \leq \exp[(-\lambda^* + \varepsilon)t] \widetilde{M}^{k+1} \|K\|_1^k t^k / k!, \quad k \geq 0. \quad (2.46)$$

Suppose  $k_0$  is an integer satisfying  $(1 - \delta_1)k_0 > 1$ . Set

$$W(t) = T(t)(I - P_m) - \sum_{k=0}^{2k_0} V^{(k)}(t).$$

It is easy to see that  $t \longrightarrow W(t)$  is strongly continuous for  $t \geq 0$ . For every  $\psi \in L^1(G)$ , we have

$$W(t)(I - P_m)\psi = T(t)(I - P_m)\psi - \sum_{k=0}^{2k_0} V^{(k)}(t)(I - P_m)\psi. \quad (2.47)$$

Applying the Laplace transform to the above equality, we get (cf. Eq. (2.45))

$$\begin{aligned} & \int_0^{\infty} e^{-\lambda t} W(t)(I - P_m)\psi dt \\ &= (\lambda I - A)^{-1}(I - P_m)\psi - \sum_{k=0}^{2k_0} (\lambda I - B)^{-1} [K(\lambda I - B)^{-1}]^k (I - P_m)\psi, \end{aligned}$$

which, by virtue of the relation

$$(\lambda I - A)^{-1} = \sum_{k=0}^{\infty} (\lambda I - B)^{-1} [K(\lambda I - B)^{-1}]^k,$$

implies

$$\begin{aligned} & \int_0^{\infty} e^{-\lambda t} W(t)(I - P_m)\psi dt \\ &= \sum_{k=2k_0+1}^{\infty} (\lambda I - B)^{-1} [K(\lambda I - B)^{-1}]^k (I - P_m)\psi \\ &= (\lambda I - B)^{-1} [K(\lambda I - B)^{-1}]^{2k_0} K(\lambda I - A)^{-1}(I - P_m)\psi. \end{aligned} \quad (2.48)$$

For any given constant  $\beta_1 > \operatorname{Re}\lambda_{m+1}$ , it is easy to know from Theorem 2.3.11 (iii) that  $\|(\lambda I - A)^{-1}(I - P_m)\|_1$  is uniformly bounded in  $\{\lambda = \beta + i\tau : \beta \geq \beta_1, |\tau| \geq \tilde{\tau}\}$ , where  $\tilde{\tau}$  is a positive constant independent of  $\beta \in [\beta_1, +\infty)$ . On the other hand, since  $\sigma(A^{(2)}) = \sigma(A) \setminus \sigma_1$ , it is seen that  $\|(\lambda I - A_2)^{-1}\|_1$ , which is continuous with respect to  $\lambda \in \rho(A^{(2)})$ , is uniformly bounded in the bounded closed domain  $\{\lambda = \beta + i\tau : \beta_1 \leq \beta \leq \|K\|_1 + 1 - \lambda^*, |\tau| \leq \tilde{\tau}\}$ . From the relation

$$\begin{aligned} & \|(\lambda I - A)^{-1}(I - P_m)\|_1 = \|(\lambda I - A)^{-1}(I - P_m)(I - P_m)\|_1 \\ & \leq \|(\lambda I - A_2)^{-1}(I - P_m)\|_1 \leq \|(\lambda I - A_2)^{-1}\|_1, \end{aligned}$$

we see that  $\|(\lambda I - A)^{-1}(I - P_m)\|_1$  is also uniformly bounded in  $\{\lambda = \beta + i\tau : \beta_1 \leq \beta \leq \|K\|_1 + 1 - \lambda^*, |\tau| \leq \tilde{\tau}\}$ . From Lemma 2.2.3, it follows that  $\|(\lambda I - A)^{-1}(I - P_m)\|_1 \leq \|(\lambda I - A)^{-1}\|_1 \leq 1$  in  $\{\lambda = \beta + i\tau : \beta \geq \|K\|_1 + 1 - \lambda^*\}$ . These assertions indicate that  $\|(\lambda I - A)^{-1}(I - P_m)\|_1$  is uniformly bounded in  $\{\lambda : \operatorname{Re}\lambda \geq \beta_1\}$ .

For every  $\lambda$  with  $\operatorname{Re}\lambda \geq \beta_1$ , define

$$s(\lambda) := (\lambda I - B)^{-1} [K(\lambda I - B)^{-1}]^{2k_0} K(\lambda I - A)^{-1}(I - P_m)\psi. \quad (2.49)$$

Then from Eq. (2.48), we get

$$\int_0^\infty e^{-\lambda t} W(t)(I - P_m)\psi dt = s(\lambda). \quad (2.50)$$

From Eq. (2.49) and Lemma 2.3.1, noting that  $\|(\lambda I - A)^{-1}(I - P_m)\|_1$  is uniformly bounded in  $\{\lambda : \operatorname{Re}\lambda \geq \beta_1\}$ , we can easily verify that  $s(\lambda) \in H_1(\alpha, L^1(G))$ , where  $\alpha = \max\{0, \beta_1\}$ . (For the definition of  $H_1(\alpha, L^1(G))$ , see [12, Chap. 6]). It follows from [12, Theorem 6.6.1 and the remark] that there exists a continuous function

$$\alpha(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} s(\lambda) d\lambda, \quad \gamma > \alpha, \quad t \geq 0 \quad (2.51)$$

such that

$$\int_0^\infty e^{-\lambda t} \alpha(t) dt = s(\lambda). \quad (2.52)$$

By virtue of the uniqueness of the Laplace integral, Eqs. (2.50) and (2.52) imply

$$W(t)(I - P_m)\psi = \alpha(t).$$

Since  $s(\lambda)$  is analytic in the region  $\{\lambda : \operatorname{Re}\lambda \geq \beta_1\}$ , the integral path in the right-hand side of Eq. (2.51) can be shifted to  $\operatorname{Re}\lambda = \beta_1$ , i.e.,

$$\begin{aligned} \alpha(t) = & \frac{1}{2\pi i} \lim_{M \rightarrow \infty} \left\{ \int_{\beta_1 - iM}^{\beta_1 + iM} e^{\lambda t} s(\lambda) d\lambda \right. \\ & \left. + \int_{\beta_1}^{\gamma} e^{\lambda t} s(\beta + iM) d\beta + \int_{\gamma}^{\beta_1} e^{\lambda t} s(\beta - iM) d\beta \right\}. \end{aligned}$$

From Eq. (2.49) and Lemma 2.3.1, the second term and the third term of the above equation tend to zero, so

$$\begin{aligned} \alpha(t) &= \frac{1}{2\pi i} \int_{\beta_1 - i\infty}^{\beta_1 + i\infty} e^{\lambda t} s(\lambda) d\lambda \\ &= \frac{1}{2\pi i} \int_{\beta_1 - i\infty}^{\beta_1 + i\infty} e^{\lambda t} (\lambda I - B)^{-1} [K(\lambda I - B)^{-1}]^{2k_0} K(\lambda I - A)^{-1} (I - P_m) \psi d\lambda. \end{aligned}$$

And thereby

$$\|\alpha(t)\|_1 \leq C_{29} e^{\beta_1 t} \int_{-\infty}^{+\infty} \left\| [K(\lambda I - B)^{-1}]^2 \right\|_1^{k_0} d\lambda \|\psi\|_1 \leq C_{30} e^{\beta_1 t} \|\psi\|_1,$$

where  $C_{29}$  and  $C_{30}$  are positive constants. Thus, we get

$$\|W(t)(I - P_m)\| \leq C_{30} e^{\beta_1 t}. \quad (2.53)$$

Thereby, from Eqs. (2.46), (2.47) and (2.53), we have

$$\begin{aligned} \|T(t)(I - P_m)\|_1 &\leq \|W(t)(I - P_m)\|_1 + \sum_{k=0}^{2k_0} \|V^{(k)}(t)(I - P_m)\|_1 \\ &\leq C_{30} e^{\beta_1 t} + \sum_{k=0}^{2k_0} e^{(-\lambda^* + \varepsilon)t} \widetilde{M}^{k+1} \|K\|_1^{k+1} t^k / k! \\ &\leq C_{31} e^{\beta_1 t}, \end{aligned} \quad (2.54)$$

where  $C_{31} = \sup_{t \geq 0} \left[ C_{30} + \sum_{k=0}^{2k_0} e^{(-\lambda^* + \varepsilon - \beta_1)t} \widetilde{M}^{k+1} \|K\|_1^{k+1} t^k / k! \right] < +\infty$ .

Since  $\beta_1$  is any constant greater than  $\operatorname{Re}\lambda_{m+1}$ , we have completed the proof of Theorem 2.4.1.

Theorem 2.4.1 indicates that the growth bound of  $T^{(2)}(t)$  is  $\operatorname{Re}\lambda_{m+1}$ , and thus the spectral radius of  $T^{(2)}(t)$  is  $\exp(\operatorname{Re}\lambda_{m+1}t)$  (cf. [23, page 60]). Noting that  $m$  can be selected arbitrarily, we get the following conclusion from Eqs. (2.43) and (2.44).

**Theorem 2.4.2.** *In the setting of  $L^1(G)$ , if (H1) – (H4) are satisfied, then the spectrum of  $T(t)$  outside the disk  $\{\lambda : |\lambda| \leq \exp(-\lambda^*t)\}$  consists of isolated eigenvalues of  $T(t)$  with finite algebraic multiplicity, and the accumulation points of the set  $\sigma(T(t)) \cap \{\lambda : |\lambda| > \exp(-\lambda^*t)\}$ , if they exist, could only appear on the circle  $\{\lambda : |\lambda| = \exp(-\lambda^*t)\}$ .*

If  $\text{Pas}(A)$  is not empty, then the asymptotic behavior of the solution  $\psi(t)$  of Eq. (I) can be easily derived from Theorem 2.4.1 and Theorem 2.4.2.

Noting that the differentiability assumption made in (H4) is rather strict for some practical cases, we will discuss in the sequel the replacement of (H4) with the following continuity assumption:

(H5).  $\alpha(v, \mu), \gamma(v, \mu), \sigma(x, v, \mu)$  and  $\tilde{k}(x, v, v', \mu, \mu')$  are piecewise continuous functions. Furthermore, there exist positive constants  $\alpha_0, \tilde{\alpha}_0, \gamma_0$  and  $\tilde{\gamma}_0$  such that  $0 < \alpha_0 \leq \alpha(v, \mu) \leq \tilde{\alpha}_0 < 1$  and  $0 < \gamma_0 \leq \gamma(v, \mu) \leq \tilde{\gamma}_0 < 1$ .

**Note:** A function  $f(x)$  is said to be piecewise continuous with respect to  $x \in G$ , if there exist domains  $G_i \subset G, i = 1, 2, \dots, k, \bigcup_{i=1}^m \overline{G_i} = \overline{G}, G_i \cap G_j = \Phi (i \neq j)$  such that  $f(x)$  is continuous on  $G_i (i = 1, 2, \dots, k)$ , and for any sequence  $\{x_n\} \subset \overline{G}$  with  $x_n \rightarrow x_0$ ,  $\lim_{n \rightarrow \infty} f(x_n)$  exists.

Since a continuous function defined on a bounded closed set can be uniformly approximated by a sequence composed of polynomial functions, a piecewise continuous function can be uniformly approximated by a sequence composed of piecewise polynomial functions. Thus, if (H5) is satisfied, then there exist sequences  $\{\alpha_n(v, \mu)\}, \{\gamma_n(v, \mu)\}, \{\sigma_n(x, v, \mu)\}$  and  $\{\tilde{k}_n(x, v, v', \mu, \mu')\}$  such that  $\alpha_n, \gamma_n, \sigma_n$  and  $k_n$  are all piecewise polynomial functions, and

$$\begin{aligned}
0 \leq \alpha_n(v, \mu) = \alpha_n(v, -\mu) \leq 1, \quad 0 \leq \gamma_n(v, \mu) = \gamma_n(v, -\mu) \leq 1, \\
\max_{(v, \mu) \in V \times D} |\alpha_n(v, \mu) - \alpha(v, \mu)| < \frac{1}{n}, \quad \max_{(v, \mu) \in V \times D} |\gamma_n(v, \mu) - \gamma(v, \mu)| < \frac{1}{n}, \\
\max_{(x, v, \mu) \in G} |\sigma_n(x, v, \mu) - \sigma(x, v, \mu)| < \frac{1}{n}, \\
\max_{(x, v, v', \mu, \mu') \in Q \times V \times V \times D \times D} \left| \tilde{k}_n(x, v, v', \mu, \mu') - \tilde{k}(x, v, v', \mu, \mu') \right| < \frac{1}{n}.
\end{aligned} \tag{2.55}$$

From Eq. (2.55), we have

$$\sigma_n(x, v, \mu) > \lambda^* - \frac{1}{n}. \quad (2.56)$$

For each  $\alpha_n$ ,  $\gamma_n$ ,  $\sigma_n$  and  $k_n$ , we have the corresponding operators  $B_{n,n}$ ,  $K_n$ ,  $\delta F_n$  and  $\delta K_n$  which are defined in Sec. 2.3. Furthermore, we define

$$A_n = B_{n,n} + K_n, \quad D(A_n) = D(B_{n,n}).$$

It is easy to see that  $\{\lambda : \operatorname{Re}\lambda > -\lambda^* + n^{-1}\} \subset \rho(B_{n,n})$ , and  $A_n$  generates a  $C_0$  semigroup  $T_n(t)$  in  $L^1(G)$ . From the relations

$$(\lambda I - A)^{-1} = [I - (\lambda I - B)^{-1}K]^{-1}(\lambda I - B)^{-1} \quad (2.57)$$

$$(\lambda I - A_n)^{-1} = [I - (\lambda I - B_{n,n})^{-1}K_n]^{-1}(\lambda I - B_{n,n})^{-1} \quad (2.58)$$

$$(\lambda I - B)^{-1} - (\lambda I - B_{\sigma_n})^{-1} = (\lambda I - B)^{-1}\delta F_n(\lambda I - B_{\sigma_n})^{-1}$$

and the expressions for  $(\lambda I - B_{\sigma_n})^{-1}$ ,  $(\lambda I - B_{n,n})^{-1}$  (cf. Eqs. (2.2) and (2.3)), it is seen that for sufficiently large  $\lambda$ ,

$$\lim_{n \rightarrow \infty} \|(\lambda I - A_n)^{-1} - (\lambda I - A)^{-1}\|_1 = 0. \quad (2.59)$$

So  $A_n$  converges to  $A$  in the generalized sense (cf. [16, page 206]). We denote this by  $\hat{\delta}(A_n, A) \rightarrow 0$ .

From the procedure for proving Lemma 2.3.1 and Theorem 2.3.11, we have

**Lemma 2.4.3.** *For every fixed  $n$  and every given constant  $\beta_1^{(n)} > -\lambda_n^*$  (where  $\lambda_n^* := \lambda^* - n^{-1}$ ), there exist positive constants  $C(n)$  and  $\tilde{\tau}(n)$  independent of  $\beta \in [\beta_1^{(n)}, +\infty)$ , such that*

$$\|K_n(\lambda I - B_{n,n})^{-1}K_n\|_1 \leq C(n)|\beta_1^{(n)} + \lambda_n^* + i\tau|^{\delta_1 - 1} \log |\beta_1^{(n)} + \lambda_n^* + i\tau| \quad (2.60)$$

*uniformly in  $\{\lambda = \beta + i\tau : \beta \geq \beta_1^{(n)}, |\tau| \geq \tilde{\tau}(n)\}$ .*

**Lemma 2.4.4.** (i).  *$\operatorname{Pas}(A_n) := \sigma(A_n) \cap \{\lambda : \operatorname{Re}\lambda > -\lambda_n^*\}$  contains at most countable isolated elements which are eigenvalues of  $A_n$  with finite algebraic multiplicity.*

(ii). *For any constant  $\beta_1^{(n)} > -\lambda_n^*$ , the set  $\sigma(A_n) \cap \{\lambda : \operatorname{Re}\lambda \geq \beta_1^{(n)}\}$  is finite.*

(iii). *There exists a positive constant  $\tilde{\tau}(n)$  independent of  $\beta \in [\beta_1^{(n)}, +\infty)$ , such that  $\|(\lambda I - A_n)^{-1}\|_1$  is uniformly bounded in  $\{\lambda = \beta + i\tau : \beta \geq \beta_1^{(n)}, |\tau| \geq \tilde{\tau}(n)\}$ .*

Lemma 2.4.4 indicates that all the elements in  $\text{Pas}(A_n)$  can be ordered in such a way that the real part decreases. Suppose  $\lambda_1^{(n)}, \lambda_2^{(n)}, \dots, \lambda_m^{(n)}, \lambda_{m+1}^{(n)}, \dots$  are eigenvalues of  $A_n$ ,  $\text{Re}\lambda_1^{(n)} \geq \text{Re}\lambda_2^{(n)} \geq \dots \geq \text{Re}\lambda_m^{(n)} > \text{Re}\lambda_{m+1}^{(n)} \geq \dots > -\lambda_n^*$ , and  $\{\lambda : \text{Re}\lambda > -\lambda_n^*\} \setminus \{\lambda_j^{(n)} : j = 1, 2, \dots\} \subset \rho(A_n)$ . (Again,  $\lambda_1^{(n)}, \lambda_2^{(n)}, \dots$  are repeated eigenvalues of  $A_n$ ).

Similar to that for proving Theorem 2.4.1, we have the following lemma.

**Lemma 2.4.5.** *Suppose  $\lambda_1^{(n)}, \lambda_2^{(n)}, \dots, \lambda_m^{(n)}, \lambda_{m+1}^{(n)}, \dots$  are repeated eigenvalues of  $A_n$ ,  $\text{Re}\lambda_1^{(n)} \geq \text{Re}\lambda_2^{(n)} \geq \dots \geq \text{Re}\lambda_m^{(n)} > \text{Re}\lambda_{m+1}^{(n)} \geq \dots > -\lambda_n^*$ ,  $\{\lambda : \text{Re}\lambda > -\lambda_n^*\} \setminus \{\lambda_j^{(n)} : j = 1, 2, \dots\} \subset \rho(A_n)$ , and let  $P_m^{(n)}$  represent the projection operator of  $\{\lambda_1^{(n)}, \lambda_2^{(n)}, \dots, \lambda_m^{(n)}\}$  corresponding to  $A_n$ . Then for every  $\varepsilon > 0$ , there exists a positive constant  $M_2$  such that*

$$\|T_n(t)(I - P_m^{(n)})\|_1 \leq M_2 \exp \left\{ \left( \text{Re}\lambda_{m+1}^{(n)} + \varepsilon \right) t \right\}.$$

Let  $\sigma_1 = \{\lambda_1, \lambda_2, \dots, \lambda_m\}$  with  $\text{Re}\lambda_m > \text{Re}\lambda_{m+1} > -\lambda^*$ ,  $\Gamma$  a simple closed curve separating  $\sigma(A)$  into two parts  $\sigma_1$  and  $\sigma(A) \setminus \sigma_1$ . Since  $\hat{\delta}(A_n, A) \rightarrow 0$ , we know from [16, Chap. 4] that there exists an eigenvalue set  $\sigma_1^{(n)} = \{\lambda_1^{(n)}, \lambda_2^{(n)}, \dots, \lambda_m^{(n)}\} \subset \text{Pas}(A_n)$  such that  $\sigma_1^{(n)}$  “converges” to  $\sigma_1$ , and

$$\lim_{n \rightarrow \infty} \|P_m^{(n)} - P_m\|_1 = 0, \quad (2.61)$$

(cf. [16, page 212–214]), where  $P_m^{(n)}$  is the projection operator of  $\sigma_1^{(n)}$  corresponding to  $A_n$ .

From Eqs. (2.59), (2.61) and Trotter’s approximation theorem (cf. [29, page 85]), we see that for every  $\psi \in L^1(G)$ ,

$$\begin{aligned} \lim_{n \rightarrow \infty} \|T_n(t)\psi - T(t)\psi\|_1 &= 0, \\ \lim_{n \rightarrow \infty} \|T_n(t)(I - P_m^{(n)})\psi - T(t)(I - P_m)\psi\|_1 &= 0. \end{aligned}$$

From the above discussion, we get the following results under the hypothesis (H5).

**Theorem 2.4.6.** *In the setting of  $L^1(G)$ , if (H1) – (H3) and (H5) are satisfied, then there exists a sequence  $\{T_n(t)\}$  composed of  $C_0$  semigroups on  $L^1(G)$  such that*

(i).  $A_n$ , the generator of  $T_n(t)$ , converges to  $A$  in the generalized sense. Moreover,  $A_n$  has similar spectral properties as that of  $A$  in the right half plane.

(ii). For every  $\psi \in L^1(G)$ ,  $\lim_{n \rightarrow \infty} \|T_n(t)\psi - T(t)\psi\|_1 = 0$ , and

$$\lim_{n \rightarrow \infty} \|P_m^{(n)} - P_m\|_1 = 0,$$

$$\lim_{n \rightarrow \infty} \|T_n(t)(I - P_m^{(n)})\psi - T(t)(I - P_m)\psi\|_1 = 0,$$

where  $P_m^{(n)}$  is the projection operator of  $\{\lambda_1^{(n)}, \lambda_2^{(n)}, \dots, \lambda_m^{(n)}\}$  corresponding to  $A_n$  with  $\operatorname{Re}\lambda_m > \operatorname{Re}\lambda_{m+1} > -\lambda^*$ .

(iii). For each  $n$ , the spectrum of  $T_n(t)$  outside the disk  $\{\lambda : |\lambda| \leq \exp(-\lambda^*t + n^{-1}t)\}$  consists of isolated eigenvalues of  $T_n(t)$  with finite algebraic multiplicity, and the accumulation points of the set  $\sigma(T_n(t)) \cap \{\lambda : |\lambda| > \exp(-\lambda^*t + n^{-1}t)\}$ , if they exist, could only appear on the circle  $\{\lambda : |\lambda| = \exp(-\lambda^*t + n^{-1}t)\}$ .

(iv). For every constant  $\beta_1 > \operatorname{Re}\lambda_{m+1}$ , the set  $\sigma(T_n(t)) \cap \{\lambda : |\lambda| > \exp(\beta_1 t)\}$  converges to  $\{\exp(\lambda_i t) : i = 1, 2, \dots, m\}$  as  $n \rightarrow +\infty$ .

**Remark 2.4.7.** For any  $\psi_0 \in L^1(G)$ , the asymptotic behavior of the solution  $\psi(t)$  of Eq. (I) is reflected to some extent by Theorem 2.4.6 if  $\operatorname{Pas}(A)$  is not empty.

# Chapter 3

## Spectral Properties of Transport Equations for Slab Geometry in $L^p$

### 3.1 Corresponding Problems in $L^\infty(G)$

This section will be devoted to discussing similar problems as that of §2.3, §2.4 in the setting of  $L^\infty(G)$ . We will see in the following sections that some results in  $L^\infty(G)$  are very helpful in dealing with the corresponding problems in  $L^p(G)$  ( $1 < p < \infty$ ).

First, define operators  $\tilde{B}$ ,  $\tilde{K}$  and  $\tilde{A}$  on  $L^1(G)$  as follows :

$$\begin{aligned}\tilde{B}\psi &= v\mu\frac{\partial\psi}{\partial x} - \sigma(x, v, \mu)\psi, \\ \tilde{K}\psi &= \int_D \int_V k(x, v', v, \mu', \mu)\psi(x, v', \mu')dv'd\mu', \\ \tilde{A}\psi &= \tilde{B}\psi + \tilde{K}\psi,\end{aligned}$$

with  $D(\tilde{B}) = D(\tilde{A}) = \{\psi \in L^1(G) : \tilde{B}\psi \in L^1(G); \psi(-a, v, \mu) = \alpha(v, \mu)\psi(-a, v, -\mu)$  and  $\psi(a, v, -\mu) = \gamma(v, \mu)\psi(a, v, \mu)$  for every  $\mu \in [-1, 0]\}$ ,  $D(\tilde{K}) = L^1(G)$ .

It is easy to see that the adjoint operators of  $\tilde{B}$ ,  $\tilde{K}$  and  $\tilde{A}$  are just the operators  $B$ ,  $K$  and  $A$  in  $L^\infty(G)$ .

By a procedure similar to that of §2.3, §2.4, we have the following conclusions.

**Lemma 3.1.1.** *Suppose (H1) – (H3) are satisfied; then*

(i).  $\{\lambda : \operatorname{Re}\lambda > -\lambda^*\} \subset \rho(\tilde{B})$ ,  $\{\lambda : \operatorname{Re}\lambda > \|\tilde{K}\|_1 - \lambda^*\} \subset \rho(\tilde{A})$ , and  $\|(\lambda I - \tilde{A})^{-1}\|_1 \leq (\operatorname{Re}\lambda + \lambda^* - \|\tilde{K}\|_1)^{-1}$  for every  $\lambda$  with  $\operatorname{Re}\lambda > \|\tilde{K}\|_1 - \lambda^*$ .

(ii).  $[\tilde{K}(\lambda I - \tilde{B})^{-1}]^2 \tilde{K}$  is a compact operator on  $L^1(G)$  for every  $\lambda$  with  $\operatorname{Re}\lambda > -\lambda^*$ .

(iii). Let  $\beta_1 > -\lambda^*$  be any constant,  $\tilde{k}(x, v, v', \mu, \mu') := (v|\mu|)^{\delta_1} (v'|\mu'|)^{\delta_2} k(x, v, v', \mu, \mu')$  (where  $\delta_1 = 0$  and  $0 \leq \delta_2 < 1$  since  $p = \infty$ , cf. (H3)). If the hypothesis (H4) is additionally satisfied, then there exist positive constants  $C_0$  and  $\tilde{\tau}$  independent of  $\beta \in [\beta_1, +\infty)$ , such that

$$\|\tilde{K}(\lambda I - \tilde{B})^{-1} \tilde{K}\|_1 \leq C_0 |\beta_1 + \lambda^* + i\tau|^{\delta_2 - 1} \log |\beta_1 + \lambda^* + i\tau|$$

uniformly in  $\{\lambda = \beta + i\tau : \beta \geq \beta_1, |\tau| \geq \tilde{\tau}\}$ .

(iv). For any given constant  $\beta_1 > -\lambda^*$  and  $\varepsilon > 0$ , there exists a positive constant  $\tilde{\tau}$  independent of  $\beta \in [\beta_1, +\infty)$ , such that

$$\|[\tilde{K}(\lambda I - \tilde{B})^2 \tilde{K}\|_1 < \varepsilon$$

uniformly in  $\{\lambda = \beta + i\tau : \beta \geq \beta_1, |\tau| \geq \tilde{\tau}\}$ .

(v).  $\operatorname{Pas}(\tilde{A}) := \sigma(\tilde{A}) \cap \{\lambda : \operatorname{Re}\lambda > -\lambda^*\}$  consists of purely isolated eigenvalues of  $\tilde{A}$  with finite algebraic multiplicity, and the accumulation points of  $\operatorname{Pas}(\tilde{A})$  could only appear on the line  $\operatorname{Re}\lambda = -\lambda^*$ .

(vi).  $\tilde{B}$  generates a positive  $C_0$  semigroup  $\tilde{S}(t)$  in  $L^1(G)$ , and the growth bound of  $\tilde{S}(t)$  is less than or equal to  $-\lambda^*$ .

(vii).  $\tilde{A}$  generates a  $C_0$  semigroup  $\tilde{T}(t)$  in  $L^1(G)$ . Suppose  $\lambda_1, \lambda_2, \dots, \lambda_m, \lambda_{m+1}, \dots$  are eigenvalues of  $\tilde{A}$ ,  $\operatorname{Re}\lambda_1 \geq \operatorname{Re}\lambda_2 \geq \dots \geq \operatorname{Re}\lambda_m > \operatorname{Re}\lambda_{m+1} \geq \dots > -\lambda^*$ ,  $\{\lambda : \operatorname{Re}\lambda > -\lambda^*\} \setminus \{\lambda_j : j = 1, 2, \dots\} \subset \rho(\tilde{A})$ , and let  $\tilde{P}_m$  represent the projection operator of  $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$  corresponding to  $\tilde{A}$ . If the hypothesis (H4) is additionally satisfied with  $\tilde{k}(x, v, v', \mu, \mu') = (v'|\mu'|)^{\delta_2} k(x, v, v', \mu, \mu')$ , then, for every  $\varepsilon > 0$ , there exists a positive constant  $M_3$  such that

$$\|\tilde{T}(t)(I - \tilde{P}_m)\|_1 \leq M_3 \exp\{(\operatorname{Re}\lambda_{m+1} + \varepsilon)t\}. \quad (3.1)$$

From Lemma 3.1.1, by virtue of the theory of adjoint operators, we get the corresponding properties of  $A$  in  $L^\infty(G)$ , i.e.,

**Lemma 3.1.2.** *Suppose (H1) – (H3) are satisfied; then the following conclusions hold in the setting of  $L^\infty(G)$ :*

(i).  $\{\lambda : \operatorname{Re}\lambda > -\lambda^*\} \subset \rho(B)$ ,  $\{\lambda : \operatorname{Re}\lambda > \|K\|_\infty - \lambda^*\} \subset \rho(A)$ , and  $\|(\lambda I - A)^{-1}\|_\infty \leq (\operatorname{Re}\lambda + \lambda^* - \|K\|_\infty)^{-1}$  for every  $\lambda$  with  $\operatorname{Re}\lambda > \|K\|_\infty - \lambda^*$ .

(ii).  $[K(\lambda I - B)^{-1}]^2 K$  is a compact operator on  $L^\infty(G)$  for every  $\lambda$  with  $\operatorname{Re}\lambda > -\lambda^*$ .

(iii). Let  $\beta_1 > -\lambda^*$  be any constant,  $\tilde{k}(x, v, v', \mu, \mu') := (v|\mu|^{\delta_1}(v'|\mu')^{\delta_2}k(x, v, v', \mu, \mu')$  (where  $\delta_1 = 0$  and  $0 \leq \delta_2 < 1$  since  $p = \infty$ , cf. (H3)). If the hypothesis (H4) is additionally satisfied, then there exist positive constants  $C_0$  and  $\tilde{\tau}$  independent of  $\beta \in [\beta_1, +\infty)$ , such that

$$\|K(\lambda I - B)^{-1}K\|_\infty \leq C_0|\beta_1 + \lambda^* + i\tau|^{\delta_2-1} \log |\beta_1 + \lambda^* + i\tau|$$

uniformly in  $\{\lambda = \beta + i\tau : \beta \geq \beta_1, |\tau| \geq \tilde{\tau}\}$ .

(iv). For any given constant  $\beta_1 > -\lambda^*$  and  $\varepsilon > 0$ , there exists a positive constant  $\tilde{\tau}$  independent of  $\beta \in [\beta_1, +\infty)$ , such that

$$\|[K(\lambda I - B)^{-1}]^2 K\|_\infty < \varepsilon$$

uniformly in  $\{\lambda = \beta + i\tau : \beta \geq \beta_1, |\tau| \geq \tilde{\tau}\}$ .

(v).  $\operatorname{Pas}(A) := \sigma(A) \cap \{\lambda : \operatorname{Re}\lambda > -\lambda^*\}$  consists of purely isolated eigenvalues of  $A$  with finite algebraic multiplicity, and the accumulation points of  $\operatorname{Pas}(A)$  could only appear on the line  $\operatorname{Re}\lambda = -\lambda^*$ .

(vi).  $\operatorname{Pas}(A) = \operatorname{Pas}(\tilde{A})$ , and  $\tilde{P}_m^* = P_m$ , where  $\tilde{P}_m$  ( $P_m$ ) is the projection operator of  $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$  corresponding to  $\tilde{A}$  ( $A$ ), and  $\tilde{P}_m^*$  is the adjoint operator of  $\tilde{P}_m$ .

Since  $D(B)$  and  $D(A)$  are not dense sets in  $L^\infty(G)$ ,  $B$  and  $A$  can not generate  $C_0$  semigroups in  $L^\infty(G)$ . However, some results can be obtained by use of the recently developed integrated semigroup theory (cf. [2, 17, 24, etc.]).

**Lemma 3.1.3.** [2, 17] *Let  $E$  be a Banach space, and  $H$  a closed linear operator. Then the following assertions are equivalent:*

(a).  $H$  is the generator of a locally Lipschitz continuous integrated semigroup.

(b). There exist real constants  $M$  and  $\omega$  such that  $(\omega, +\infty) \subset \rho(H)$ , and  $\|(\lambda I - H)^{-n}\| \leq M(\lambda - \omega)^{-n}$  for all  $\lambda > \omega$  and  $n = 1, 2, \dots$ .

Furthermore, when either of (a) and (b) is satisfied, the part  $H_F$  of  $H$  in  $F := \overline{D(H)}$  is the generator of a  $C_0$  semigroup on  $F$ .

From Lemma 3.1.2(i) and Lemma 3.1.3, we get

**Lemma 3.1.4.** *In the setting of  $L^p(G)$ , set  $F := \overline{D(A)}$  and let  $A_F$  be the part of  $A$  in  $F$  with its definition domain  $D(A_F) = \{\psi \in D(A) : A\psi \in F\}$ . Then  $A_F$  generates a  $C_0$  semigroup  $T_F(t)$  in  $F$ .*

Now we can consider the spectral properties of  $T_F(t)$ .

**Lemma 3.1.5.** *In the space  $F$ ,  $Pas(A_F) := \sigma(A_F) \cap \{\lambda : \operatorname{Re}\lambda > -\lambda^*\}$  is equal to  $Pas(A)$ , and  $P_{Fm} = P_m$ , where  $P_{Fm}$  ( $P_m$ ) is the projection operator of  $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$  corresponding to  $A_F$  ( $A$ ).*

**Proof.** Firstly, it is easily seen that  $P_\sigma(A)$ , the set composed of all the eigenvalues of  $A$  in  $L^\infty(G)$ , is the same as  $P_\sigma(A_F)$ , which is the set of all the eigenvalues of  $A_F$  in  $F$ .

Secondly, it will be shown that  $\rho(A) \subset \rho(A_F)$ . For every  $\lambda \in \rho(A)$  and  $\psi \in F$ , set  $\varphi = (\lambda I - A)^{-1}\psi$ ; then  $\varphi \in D(A_F)$  and  $(\lambda I - A_F)\varphi = (\lambda I - A)\varphi = \psi$ , so  $R(\lambda I - A_F) = F$ . If  $(\lambda I - A_F)\varphi = 0$ , then  $(\lambda I - A)\varphi = 0$ , and  $\varphi = 0$  because of  $\lambda \in \rho(A)$ . This indicates that for every  $\lambda \in \rho(A)$ ,  $\lambda I - A_F$  is invertible, and

$$(\lambda I - A_F)^{-1}\psi = (\lambda I - A)^{-1}\psi \quad (3.2)$$

for every  $\psi \in F$ . So  $\lambda \in \rho(A_F)$ , and  $\rho(A) \subset \rho(A_F)$ . From Lemma 3.1.2 (iv) and the above discussion, we get the conclusion. Q. E. D.

**Theorem 3.1.6.** *Let  $\lambda_1, \lambda_2, \dots, \lambda_m, \lambda_{m+1}, \dots$  be eigenvalues of  $A$  in  $L^\infty(G)$ ,  $\operatorname{Re}\lambda_1 \geq \operatorname{Re}\lambda_2 \geq \dots \geq \operatorname{Re}\lambda_m > \operatorname{Re}\lambda_{m+1} \geq \dots > -\lambda^*$ ,  $\{\lambda : \operatorname{Re}\lambda > -\lambda^*\} \setminus \{\lambda_n : n = 1, 2, \dots\} \subset \rho(A)$ , and let  $P_m$  be the projection operator of  $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$  corresponding to  $A$ . If (H1) - (H4)*

are satisfied, then for every  $\varepsilon > 0$ , there exists a positive constant  $M_3$  such that for every  $\psi \in F := \overline{D(A)}$ ,

$$\|T_F(t)(I - P_m)\psi\|_\infty \leq M_3 \exp\{(\operatorname{Re}\lambda_{m+1} + \varepsilon)t\} \|\psi\|_\infty.$$

**Proof.** From Eq. (3.1), Lemma 3.1.2 (vi) and the theory of adjoint operators, we have

$$\|\tilde{T}^*(t)(I - P_m)\psi\|_\infty \leq M_3 \exp\{(\operatorname{Re}\lambda_{m+1} + \varepsilon)t\} \|\psi\|_\infty. \quad (3.3)$$

From Eq. (3.2) and the relations

$$\begin{aligned} T_F(t)\psi &= \lim_{n \rightarrow \infty} \left[ \frac{n}{t} \left( \frac{n}{t} - A_F \right)^{-1} \right]^n \psi, \quad \psi \in F, \\ \tilde{T}(t)\psi &= \lim_{n \rightarrow \infty} \left[ \frac{n}{t} \left( \frac{n}{t} - \tilde{A} \right)^{-1} \right]^n \psi, \quad \psi \in L^1(G), \\ \tilde{T}^*(t)\psi &= \lim_{n \rightarrow \infty} \left[ \frac{n}{t} \left( \frac{n}{t} - A \right)^{-1} \right]^n \psi, \quad \psi \in L^\infty(G), \end{aligned}$$

(cf. [29, page 33]), it is easy to see

$$\tilde{T}^*(t)\psi = T_F(t)\psi \quad \text{for every } \psi \in F.$$

This together with Eq. (3.3) implies the conclusion. Q. E. D.

By virtue of Theorem 3.1.6, it is easy to get the following conclusion by a procedure similar to that for proving Theorem 2.4.2.

**Theorem 3.1.7.** *In the setting of  $L^\infty(G)$ , if (H1) – (H4) are satisfied with  $\tilde{k}(x, v, v', \mu, \mu') = (v'|\mu')^{\delta_2}k(x, v, v', \mu, \mu')$ , then the spectrum of  $T_F(t)$  outside the disk  $\{\lambda : |\lambda| \leq \exp(-\lambda^*t)\}$  consists of isolated eigenvalues of  $T_F(t)$  with finite algebraic multiplicity, and the accumulation points of the set  $\sigma(T_F(t)) \cap \{\lambda : |\lambda| > \exp(-\lambda^*t)\}$  could only appear on the circle  $\{\lambda : |\lambda| = \exp(-\lambda^*t)\}$ .*

If (H4) is not satisfied, then similar results as that of Theorem 2.4.6 can be obtained when (H1) – (H3) and (H5) are satisfied.

## 3.2 Corresponding Problems in $L^p(G)$

In this section, we will discuss the same problems as that of §2.3, §2.4 in the setting of  $L^p(G)$  ( $1 < p < \infty$ ). It is assumed that (H1) – (H3) are satisfied throughout this section.

**Lemma 3.2.1.** *If  $\delta_1 + \delta_2 < \max\{1/p, 1/q\}$  (where  $q = (1 - p^{-1})^{-1}$ ), then  $K(\lambda I - B)^{-1}K$  is a compact operator on  $L^p(G)$  for every  $\lambda$  with  $\operatorname{Re}\lambda > -\lambda^*$ , .*

**Proof:** First, we prove the conclusion in case of  $\delta_1 + \delta_2 < 1/q$ . For any  $\psi \in L^p(G)$ , the expression of  $K(\lambda I - B)^{-1}K\psi$  is given by Eqs. (2.12) – (2.18). From Eq. (2.13), (H3) and the Hölder inequality, it is easy to see

$$\begin{aligned}
& |T_1(\lambda, k, \sigma, \alpha, \gamma, k)\psi(x, v, \mu)| \\
& \leq \int_{-a}^x dx' \int_V dv' \int_D d\mu' \int_V dw \int_0^1 d\xi (w\xi)^{-(1+\delta_1+\delta_2)} (v|\mu|)^{-\delta_1} (v'|\mu'|)^{-\delta_2} \\
& \quad \times \exp\left[-\frac{(\operatorname{Re}\lambda + \lambda^*)(x - x')}{w\xi}\right] |\psi(x', v', \mu')| \\
& \leq \left\{ \int_{-a}^x dx' \int_V dv' \int_D d\mu' |\psi(x', v', \mu')|^p \right\}^{1/p} \\
& \quad \times \left\{ \int_{-a}^x dx' \int_V dv' \int_D d\mu' \left[ \int_V dw \int_0^1 d\xi (w\xi)^{-(1+\delta_1+\delta_2)} (v|\mu|)^{-\delta_1} (v'|\mu'|)^{-\delta_2} \right. \right. \\
& \quad \left. \left. \times \exp\left(-\frac{(\operatorname{Re}\lambda + \lambda^*)(x - x')}{w\xi}\right) \right]^q \right\}^{1/q} \\
& \leq C_{32} (v|\mu|)^{-\delta_1} \|\psi\|_p \left\{ \int_{-a}^x F(x, x', w, \xi) dx' \right\}^{1/q},
\end{aligned}$$

where

$$F(x, x', w, \xi) = \left[ \int_V dw \int_0^1 d\xi (w\xi)^{-(1+\delta_1+\delta_2)} \exp\left(-\frac{(\operatorname{Re}\lambda + \lambda^*)(x - x')}{w\xi}\right) \right]^q.$$

Thus

$$\|T_1(\lambda, k, \sigma, \alpha, \gamma, k)\psi\|_p^p \leq C_{33} \|\psi\|_p^p \int_Q \left\{ \int_{-a}^x F(x, x', w, \xi) dx' \right\}^{pq^{-1}} dx. \quad (3.4)$$

Select four positive constants  $\delta_3, \varepsilon_1, \varepsilon_2$  and  $\varepsilon_3$  such that

$$\varepsilon_1 < \varepsilon_2 < \varepsilon_3, \quad \delta_3 < p^{-1},$$

$$\delta_3 + \varepsilon_i > p^{-1}, \quad q(\delta_1 + \delta_2 + \varepsilon_i) < 1, \quad i = 1, 2, 3.$$

Obviously, such constants exist since  $\delta_1 < p^{-1}$ ,  $\delta_2 < 1 - p^{-1} = q^{-1}$  and  $q(\delta_1 + \delta_2) < 1$ .

By the Hölder inequality, we have

$$\begin{aligned} & F(x, x', w, \xi) \\ &= \left[ \int_V dw w^{-\delta_3} \cdot w^{\delta_3-1-\delta_1-\delta_2} \int_0^1 \xi^{-(1+\delta_1+\delta_2)} \exp\left(-\frac{(\operatorname{Re}\lambda + \lambda^*)(x-x')}{w\xi}\right) d\xi \right]^q \\ &\leq \left( \int_V w^{-\delta_3 p} dw \right)^{1/p} \cdot \int_V w^{q(\delta_3-1-\delta_1-\delta_2)} \\ &\quad \times \left[ \int_0^1 \xi^{-(1+\delta_1+\delta_2)} \exp\left(-\frac{(\operatorname{Re}\lambda + \lambda^*)(x-x')}{w\xi}\right) d\xi \right]^q dw. \end{aligned} \quad (3.5)$$

Since  $\xi^{-(1+\delta_1+\delta_2)} < \xi^{-(1+\delta_1+\delta_2+\varepsilon_2)}$  for all  $\xi \in (0, 1)$ , we get the following estimation by a procedure similar to that of Eqs. (2.26) and (2.27):

$$\begin{aligned} & \int_0^1 \xi^{-(1+\delta_1+\delta_2)} \exp\left(-\frac{(\operatorname{Re}\lambda + \lambda^*)(x-x')}{w\xi}\right) d\xi \\ &\leq \int_0^1 \xi^{-(1+\delta_1+\delta_2+\varepsilon_2)} \exp\left(-\frac{(\operatorname{Re}\lambda + \lambda^*)(x-x')}{w\xi}\right) d\xi \\ &\leq \int_1^\infty \frac{\exp[-w^{-1}t(\operatorname{Re}\lambda + \lambda^*)(x-x')]}{t^{1-(\delta_1+\delta_2+\varepsilon_2)}} dt \\ &\leq \left\{ \int_1^\infty \frac{\exp[-w^{-1}t(\operatorname{Re}\lambda + \lambda^*)(x-x')]}{t} dt \right\}^{1-(\delta_1+\delta_2+\varepsilon_2)} \\ &\quad \times \left\{ \int_1^\infty \exp[-w^{-1}t(\operatorname{Re}\lambda + \lambda^*)(x-x')] dt \right\}^{\delta_1+\delta_2+\varepsilon_2} \\ &\leq C_{34}[w(x-x')^{-1}]^{\delta_1+\delta_2+\varepsilon_2} [\log w + \log(x-x') + C_{35}] \\ &\leq C_{36} w^{\delta_1+\delta_2+\varepsilon_1} (x-x')^{-(\delta_1+\delta_2+\varepsilon_3)}. \end{aligned} \quad (3.6)$$

Applying Eq. (3.6) to Eq. (3.5) yields

$$\begin{aligned} & F(x, x', w, \xi) \\ &\leq C_{37} (x-x')^{-q(\delta_1+\delta_2+\varepsilon_3)} \int_Q w^{q(\delta_3-1-\delta_1-\delta_2)} \cdot w^{q(\delta_1+\delta_2+\varepsilon_1)} dw \\ &\leq C_{38} (x-x')^{-q(\delta_1+\delta_2+\varepsilon_3)}. \end{aligned}$$

From Eq. (3.4), we have

$$\|T_1(\lambda, k, \sigma, \alpha, \gamma, k)\|_p \leq C_{39},$$

which indicates that  $T_1(\lambda, k, \sigma, \alpha, \gamma, k)$  is a compact operator on  $L^p(G)$  since  $T_1(\lambda, k, \sigma, \alpha, \gamma, k)$  is in fact an integral operator (cf. Eq. (2.13)).

Similarly, it can be shown that  $T_i(\lambda, k, \sigma, \alpha, \gamma, k)$  ( $i = 2, 3, \dots, 6$ ) is also a compact operator. This implies the conclusion when  $\delta_1 + \delta_2 < q^{-1}$ .

If  $\delta_1 + \delta_2 < \max\{p^{-1}, q^{-1}\}$  but not  $\delta_1 + \delta_2 < q^{-1}$ , then we can consider in  $L^q(G)$  the adjoint operator  $K^*(\lambda I - B^*)^{-1}K^*$  of  $K(\lambda I - B)^{-1}K$  in  $L^p(G)$ , where

$$B^*\psi = v\mu \frac{\partial \psi}{\partial x} - \sigma(x, v, \mu)\psi,$$

$$K^*\psi = \int_D \int_V k(x, v', v, \mu', \mu)\psi(x, v', \mu')dv'd\mu',$$

with  $D(B^*) = \{\psi \in L^q(G) : B^*\psi \in L^q(G); \psi(-a, v, \mu) = \alpha(v, \mu)\psi(-a, v, -\mu), \psi(a, v, -\mu) = \gamma(v, \mu)\psi(a, v, \mu) \text{ for every } \mu \in [-1, 0)\}$ ,  $D(K^*) = L^q(G)$ .

Similar to that of  $(\lambda I - B)^{-1}$ , for every  $\psi \in L^q(G)$ , an expression for  $(\lambda I - B^*)^{-1}\psi$  is given by

$$(\lambda I - B^*)^{-1}\psi(x, v, \mu) = \frac{H_1\psi + \gamma(v, \mu)H_2\psi + \alpha(v, \mu)\gamma(v, \mu)H_3\psi}{1 - \alpha(v, \mu)\gamma(v, \mu) \exp\left[-\frac{2}{v\mu} \int_{-a}^a \Delta(\sigma)ds\right]}$$

for  $\mu$  positive, and

$$(\lambda I - B^*)^{-1}\psi(x, v, \mu) = \frac{H_4\psi + \alpha(v, \mu)H_5\psi + \alpha(v, \mu)\gamma(v, \mu)H_6\psi}{1 - \alpha(v, \mu)\gamma(v, \mu) \exp\left[\frac{2}{v\mu} \int_{-a}^a \Delta(\sigma)ds\right]}$$

for  $\mu$  negative, where  $\Delta(\sigma) = \lambda + \sigma(s, v, \mu)$ ,  $H_1, H_2, \dots, H_6$  are operators on  $L^q(G)$  given by

$$H_1\psi(x, v, \mu) = \frac{1}{v\mu} \int_x^a \psi(x', v, \mu) \exp\left[-\frac{1}{v\mu} \int_x^{x'} \Delta(\sigma)ds\right] dx',$$

$$H_2\psi(x, v, \mu) = \frac{1}{v\mu} \int_Q \psi(x', v, -\mu) \exp\left[-\frac{1}{v\mu} \left(\int_x^a \Delta(\sigma)ds + \int_{x'}^a \Delta(\sigma)ds\right)\right] dx',$$

$$H_3\psi(x, v, \mu) = \frac{1}{v\mu} \exp\left[-\frac{2}{v\mu} \int_Q \Delta(\sigma)ds\right] \int_{-a}^x \psi(x', v, \mu) \exp\left[-\frac{1}{v\mu} \int_x^{x'} \Delta(\sigma)ds\right] dx',$$

$$H_4\psi(x, v, \mu) = -\frac{1}{v\mu} \int_{-a}^x \psi(x', v, \mu) \exp\left[\frac{1}{v\mu} \int_{x'}^x \Delta(\sigma)ds\right] dx',$$

$$H_5\psi(x, v, \mu) = -\frac{1}{v\mu} \int_Q \psi(x', v, -\mu) \exp\left[\frac{1}{v\mu} \left(\int_{-a}^x \Delta(\sigma)ds + \int_{-a}^{x'} \Delta(\sigma)ds\right)\right] dx',$$

$$H_6\psi(x, v, \mu) = -\frac{1}{v\mu} \exp \left[ \frac{2}{v\mu} \int_Q \Delta(\sigma) ds \right] \int_x^a \psi(x', v, \mu) \exp \left[ \frac{1}{v\mu} \int_{x'}^x \Delta(\sigma) ds \right] dx'.$$

By a procedure similar to that of the case  $\delta_1 + \delta_2 < q^{-1}$ , it can be shown that if  $\delta_1 + \delta_2 < p^{-1}$ , then  $K^*(\lambda I - B^*)^{-1}K^*$  is a compact operator in  $L^q(G)$  for every  $\lambda$  with  $\operatorname{Re}\lambda > -\lambda^*$ . Again, by virtue of the theory of adjoint operators, we see that  $K(\lambda I - B)^{-1}K$  is a compact operator in  $L^p(G)$ . This completes the proof of Lemma 3.2.1. Q. E. D.

**Corollary 3.2.2.** *If  $\delta_1 + \delta_2 < \max\{p^{-1}, q^{-1}\}$ , then  $\operatorname{Pas}(A) := \sigma(A) \cap \{\lambda : \operatorname{Re}\lambda > -\lambda^*\}$  consists of purely isolated eigenvalues of  $A$  with finite algebraic multiplicity.*

If  $p = 2$ , then we can get the following lemma.

**Lemma 3.2.3.** *Suppose  $\delta_1 + \delta_2 < 1/2$  and let  $\beta_1 > -\lambda^*$  be any constant. If the hypothesis (H4) is satisfied with  $\tilde{k}(x, v, v', \mu, \mu') := (v|\mu|)^{\delta_1}(v'|\mu'|)^{\delta_2}k(x, v, v', \mu, \mu')$ , then there exist positive constants  $C_0$  and  $\tilde{\tau}$  independent of  $\beta \in [\beta_1, +\infty)$ , such that*

$$\|K(\lambda I - B)^{-1}K\|_2 \leq C_0|\beta_1 + \lambda^* + i\tau|^{\delta_1 + \delta_2 - \frac{1}{2}} \log |\beta_1 + \lambda^* + i\tau|$$

uniformly in  $\{\lambda = \beta + i\tau : \beta \geq \beta_1, |\tau| \geq \tilde{\tau}\}$ .

**Proof.** The proof of this lemma is quite similar to that of Lemma 2.3.1. First, we consider the operator  $T_{n,1}(\lambda, k, \sigma, \alpha, \gamma, k)$ , which is given by Eq. (2.22) (see the proof of Lemma 2.3.1). It is easy to see

$$\begin{aligned} & \|T_{n,1}(\lambda, k, \sigma, \alpha, \gamma, k)\|_2^2 \\ & \leq \int dx \int_V dv \int_D d\mu \int_{-a}^x dx' \int_V dv' \int_D d\mu' \left[ \int_V \frac{dw}{w} |E(\lambda, n, x, x', \dots)| \right]^2, \end{aligned} \quad (3.7)$$

where  $E(\lambda, n, x, x', \dots)$  is given by Eq. (2.23).

Similar to that for proving Eq. (2.24) and Eq. (2.28), it can be easily shown that

$$|E(\lambda, n, x, x', \dots)| \leq \frac{C_{40}w^{1-\delta_1-\delta_2}}{(x-x')|\mu|^{\delta_1}v^{\delta_1}(v'|\mu'|)^{\delta_2}|\tau|} \exp(-4abn), \quad (3.8)$$

and, when  $|\beta_1 + \lambda^* + i\tau|(x-x')w^{-1} \leq 1/2$ ,

$$|E(\lambda, n, x, x', \dots)|$$

$$\begin{aligned}
&\leq C_{41} |\mu|^{-\delta_1} v^{-\delta_1} (x - x')^{-(\delta_1 + \delta_2)} (v' |\mu'|)^{-\delta_2} \exp(-4abn) \\
&\quad \times \{C_4 |\log[|\beta_1 + \lambda^* + i\tau|(x - x')w^{-1}]| + C_5 |\log|\beta_1 + \lambda^* + i\tau|| + C_6\}.
\end{aligned} \tag{3.9}$$

For convenience' sake, we combine Eqs. (3.8) and (3.9) as

$$|E(\lambda, n, x, x', \dots)| \leq \frac{C_{42} \exp(-4abn)}{(v|\mu|)^{\delta_1} (v'|\mu'|)^{\delta_2}} G(x, x', w, \tau), \tag{3.10}$$

where

$$G(x, x', w, \tau) \leq \frac{w^{1-\delta_1-\delta_2}}{(x - x')|\tau|}, \tag{3.11}$$

and, when  $|\beta_1 + \lambda^* + i\tau|(x - x')w^{-1} \leq 1/2$ ,

$$\begin{aligned}
&G(x, x', w, \tau) \\
&\leq \frac{C_4 |\log[|\beta_1 + \lambda^* + i\tau|(x - x')w^{-1}]| + C_5 |\log|\beta_1 + \lambda^* + i\tau|| + C_6}{(x - x')^{\delta_1 + \delta_2}}.
\end{aligned} \tag{3.12}$$

Select a constant  $t \in (1/2, 1 - \delta_1 - \delta_2)$ , and let  $s = 1 - t$ . Obviously,  $2s < 1$ . From Eqs. (3.7) and (3.10), we have

$$\begin{aligned}
&\|T_{n,1}(\lambda, k, \sigma, \alpha, \gamma, k)\|_2^2 \\
&\leq C_{43} \exp(-8abn) \int_V dv \int_D d\mu \frac{1}{(v|\mu|)^{2\delta_1}} \int_V dv' \int_D d\mu' \frac{1}{(v'|\mu'|)^{2\delta_2}} \\
&\quad \times \int_Q dx \int_{-a}^x dx' \left[ \int_0^{v_M} \frac{1}{w^s} \cdot \frac{G(x, x', w, \tau)}{w^t} dw \right]^2 \\
&\leq C_{44} \exp(-8abn) \int_0^{v_M} \frac{1}{w^{2s}} dw \cdot \int_Q dx \int_{-a}^x dx' \int_0^{v_M} \frac{G^2(x, x', w, \tau)}{w^{2t}} dw \\
&\leq C_{45} \exp(-8abn) \int_0^{v_M} \frac{dw}{w^{2t}} \int_Q dx' \int_{x'}^a G^2(x, x', w, \tau) dx \\
&\leq C_{45} \exp(-8abn) \int_0^{v_M} \frac{dw}{w^{2t}} \int_Q dx' \left\{ \int_{Q_2 \cap (x', a]} G^2(x, x', w, \tau) dx \right. \\
&\quad \left. + \int_{Q_1 \cap (x', a]} G^2(x, x', w, \tau) dx \right\},
\end{aligned}$$

where  $Q_1$  and  $Q_2$  are the same as that in the proof of Lemma 2.3.1.

From Eqs. (3.11) and (3.12), by virtue of the transformation (2.30), we get

$$\begin{aligned}
& \|T_{n,1}(\lambda, k, \sigma, \alpha, \gamma, k)\|_2^2 \\
\leq & C_{46} \exp(-8abn) \int_0^{v_M} \frac{dw}{w^{2t}} \int_Q dx' \left\{ \int_{Q_2 \cap (x', a]} \frac{w^{2(1-\delta_1-\delta_2)}}{(x-x')^2 |\tau|^2} dx \right. \\
& \left. + \int_{Q_1 \cap (x', a]} \frac{\log^2[|\beta_1 + \lambda^* + i\tau|(x-x')w^{-1}] + \log^2|\beta_1 + \lambda^* + i\tau| + 1}{(x-x')^{2(\delta_1+\delta_2)}} dx \right\} \\
\leq & C_{46} \exp(-8abn) \int_0^{v_M} \frac{dw}{w^{2t}} \int_Q dx' \left\{ \int_{1/2}^\infty \frac{w^{2(1-\delta_1-\delta_2)} |\beta_1 + \lambda^* + i\tau|}{y^2 w |\tau|^2} dy \right. \\
& \left. + \int_0^{1/2} \frac{\log^2 y + \log^2 |\beta_1 + \lambda^* + i\tau| + 1}{y^{2(\delta_1+\delta_2)} w^{2(\delta_1+\delta_2)}} \cdot |\beta_1 + \lambda^* + i\tau|^{2(\delta_1+\delta_2)-1} w dy \right\} \\
\leq & C_{47} \exp(-8abn) \int_0^{v_M} \frac{dw}{w^{2t+2(\delta_1+\delta_2)-1}} \\
& \times \left\{ \frac{|\beta_1 + \lambda^* + i\tau|}{|\tau|^2} + [\log^2 |\beta_1 + \lambda^* + i\tau| + C_{48}] |\beta_1 + \lambda^* + i\tau|^{2(\delta_1+\delta_2)-1} \right\}
\end{aligned}$$

When  $|\tau|$  is sufficiently large, we have

$$\begin{aligned}
& \|T_{n,1}(\lambda, k, \sigma, \alpha, \gamma, k)\|_2 \\
\leq & C_{49} \exp(-4abn) |\beta_1 + \lambda^* + i\tau|^{\delta_1+\delta_2-\frac{1}{2}} \log |\beta_1 + \lambda^* + i\tau|.
\end{aligned}$$

Similarly, it can be shown that  $\|T_{n,i}(\lambda, k, \sigma, \alpha, \gamma, k)\|_2$  ( $i = 2, 3, \dots, 6$ ) has the same estimation. These relations together with Eq. (2.19) imply the conclusion. Q. E. D.

Let  $\beta_1 > -\lambda^*$  be any given constant. Since  $\alpha(v, \mu), \gamma(v, \mu), \sigma(x, v, \mu)$  and  $\tilde{k}(x, v, v', \mu, \mu')$  are bounded measurable, there exist four sequences  $\{\alpha_n(v, \mu)\}, \{\gamma_n(v, \mu)\}, \{\sigma_n(x, v, \mu)\}$  and  $\{\tilde{k}_n(x, v, v', \mu, \mu')\}$  such that (cf. Lemma 2.3.2)

(a). for every  $n$ ,  $\alpha_n, \gamma_n, \sigma_n$  and  $k_n$  are all smooth functions of infinite order; moreover,

$$\begin{aligned}
& -1 < \alpha_n(v, \mu) < \exp[av_M^{-1}(\beta_1 + \lambda^*)], \\
& -1 < \gamma_n(v, \mu) < \exp[av_M^{-1}(\beta_1 + \lambda^*)], \\
& \lambda^* - \frac{\beta_1 + \lambda^*}{2} \leq \sigma_n(x, v, \mu) \leq \text{ess sup } \sigma(x, v, \mu) + 1, \\
& \left| \tilde{k}_n(x, v, v', \mu, \mu') \right| \leq \text{ess sup } \left| \tilde{k}(x, v, v', \mu, \mu') \right| + 1;
\end{aligned}$$

(b).  $\alpha_n(v, \mu)$  converges to  $\alpha(v, \mu)$  almost everywhere, and so do  $\gamma_n(v, \mu)$ ,  $\sigma_n(x, v, \mu)$  and  $\widetilde{k}_n(x, v, v', \mu, \mu')$ .

Let

$$\begin{aligned}\widetilde{\delta k}_n(x, v, v', \mu, \mu') &= \widetilde{k}(x, v, v', \mu, \mu') - \widetilde{k}_n(x, v, v', \mu, \mu'), \\ k_n(x, v, v', \mu, \mu') &= (v|\mu|)^{-\delta_1} (v'|\mu'|)^{-\delta_2} \widetilde{k}_n(x, v, v', \mu, \mu'), \\ \delta k_n(x, v, v', \mu, \mu') &= (v|\mu|)^{-\delta_1} (v'|\mu'|)^{-\delta_2} \widetilde{\delta k}_n(x, v, v', \mu, \mu'), \\ \delta \sigma_n(x, v, \mu) &= \sigma_n(x, v, \mu) - \sigma(x, v, \mu),\end{aligned}$$

and define operators  $B_{n,n}$ ,  $B_{\sigma_n}$ ,  $B_{\alpha_n, \gamma_n}$ ,  $\delta F_n$ ,  $K_n$  and  $\delta K_n$  in  $L^p(G)$  similarly to that of §2.3.

By virtue of Lebesgue's dominated convergence theorem and the norm expression of the integral operator defined on  $L^p(G)$ , we can prove the following results (The proof are omitted since they are lengthy and tedious).

**Lemma 3.2.4.** *Suppose  $\delta_1 + \delta_2 < \max\{p^{-1}, q^{-1}\}$ , and  $\beta_1 > -\lambda^*$  is a given constant. Then*

$$\begin{aligned}\lim_{n \rightarrow \infty} \|\delta K_n(\lambda I - B_{n,n})^{-1} K\|_p &= 0, \\ \lim_{n \rightarrow \infty} \|K_n(\lambda I - B_{n,n})^{-1} \delta K_n\|_p &= 0, \\ \lim_{n \rightarrow \infty} \|K(\lambda I - B)^{-1} K - K(\lambda I - B_{n,n})^{-1} K\|_p &= 0\end{aligned}$$

*uniformly in  $\{\lambda : \operatorname{Re} \lambda \geq \beta_1\}$ .*

Similar to that for proving Theorem 2.3.10, we get the following result from Lemma 3.2.3 and Lemma 3.2.4.

**Lemma 3.2.5.** *Suppose (H1) – (H3) are satisfied, and  $\beta_1 > -\lambda^*$  is an arbitrarily given constant. If  $\delta_1 + \delta_2 < 1/2$ , then for every  $\varepsilon > 0$ , there exists a positive constant  $\widetilde{\tau}$  independent of  $\beta \in [\beta_1, +\infty)$ , such that  $\|K(\lambda I - B)^{-1} K\|_2 < \varepsilon$  uniformly in  $\{\lambda = \beta + i\tau : \beta \geq \beta_1, |\tau| \geq \widetilde{\tau}\}$ .*

From Lemma 3.2.3, Lemma 3.2.5 and the Riesz–Thorin interpolation theorem, we get

**Lemma 3.2.6.** *Suppose (H1) – (H3) are satisfied, and  $\beta_1 > -\lambda^*$  is an arbitrarily given constant. If  $\delta_1 < \min\{1/p, 1/2\}$ ,  $\delta_2 < \min\{1/q, 1/2\}$  and  $\delta_1 + \delta_2 < 1/2$ , then for every  $\varepsilon > 0$ , there exists a positive constant  $\tilde{\tau}$  independent of  $\beta \in [\beta_1, +\infty)$ , such that*

$$\|K(\lambda I - B)^{-1}K\|_p < \varepsilon$$

*uniformly in  $\{\lambda = \beta + i\tau : \beta \geq \beta_1, |\tau| \geq \tilde{\tau}\}$ . Furthermore, if the hypothesis (H4) is additionally satisfied with  $\tilde{k}(x, v, v', \mu, \mu') := (v|\mu|^{\delta_1}(v'|\mu'|)^{\delta_2}k(x, v, v', \mu, \mu'))$ , then there exist positive constants  $r \in (0, 1)$ ,  $C_0$  and  $\tilde{\tau}$  independent of  $\beta \in [\beta_1, +\infty)$ , such that*

$$\|K(\lambda I - B)^{-1}K\|_p \leq C_0|\beta_1 + \lambda^* + i\tau|^{r(\delta_1 + \delta_2 - \frac{1}{2})} \log^r |\beta_1 + \lambda^* + i\tau|$$

*uniformly in  $\{\lambda = \beta + i\tau : \beta \geq \beta_1, |\tau| \geq \tilde{\tau}\}$ .*

Lemma 3.2.1 and Lemma 3.2.6 imply the following conclusion (cf. [22, page 43]).

**Theorem 3.2.7.** *Suppose (H1) – (H3) are satisfied,  $\delta_1 < \min\{1/p, 1/2\}$ ,  $\delta_2 < \min\{1/q, 1/2\}$  and  $\delta_1 + \delta_2 < 1/2$ . Then the following assertions hold for any constant  $\beta_1 > -\lambda^*$  in the setting of  $L^p(G)$ :*

(i).  $\sigma(A) \cap \{\lambda : \operatorname{Re}\lambda \geq \beta_1\}$  contains at most finite elements;

(ii). there exists a positive constant  $\tilde{\tau}$  independent of  $\beta \in [\beta_1, +\infty)$ , such that  $\|(\lambda I - A)^{-1}\|_p$  is uniformly bounded in  $\{\lambda = \beta + i\tau : \beta \geq \beta_1, |\tau| \geq \tilde{\tau}\}$ .

By a procedure similar to that for proving Theorem 2.4.1 and Theorem 2.4.2, we get the following main results of this section.

**Theorem 3.2.8.** *In the setting of  $L^p(G)$  ( $1 < p < +\infty$ ), suppose (H1) – (H4) are satisfied,  $\delta_1 < \min\{1/p, 1/2\}$ ,  $\delta_2 < \min\{1/q, 1/2\}$  and  $\delta_1 + \delta_2 < 1/2$ . Then the spectrum of  $T(t)$  outside the disk  $\{\lambda : |\lambda| \leq \exp(-\lambda^*t)\}$  consists of isolated eigenvalues of  $T(t)$  with finite algebraic multiplicity, and the accumulation points of the set  $\sigma(T(t)) \cap \{\lambda : |\lambda| > \exp(-\lambda^*t)\}$  could only appear on the circle  $\{\lambda : |\lambda| = \exp(-\lambda^*t)\}$ . In particular, for every  $\varepsilon > 0$ , there exists a positive constant  $M_4$  such that*

$$\|T(t)(I - P_m)\|_p \leq M_4 \exp\{(\operatorname{Re}\lambda_{m+1} + \varepsilon)t\},$$

where  $\lambda_1, \lambda_2, \dots, \lambda_m, \lambda_{m+1}, \dots$  and  $P_m$  have similar meanings as that of Theorem 2.4.1.

**Theorem 3.2.9.** *In the setting of  $L^p(G)$  ( $1 < p < +\infty$ ), if (H1) – (H3) and (H5) are satisfied, then similar results as that of Theorem 2.4.6 still hold.*

This theorem can be easily proved by a process similar to that of Theorem 2.4.6.

In case of  $p = 2$ , the conclusions in Theorem 3.2.8 still hold without the hypothesis (H4).

To prove this result, we need the following lemma.

**Lemma 3.2.10.** [13, 30, 44] *Let  $U(t)$  be the  $C_0$  semigroup generated by the operator  $H$  in a Hilbert space  $X$ , with  $\omega_0(H)$  the growth bound of  $U(t)$ , and  $s_0(H)$  the spectral bound of  $H$ , i.e.,*

$$\omega_0(H) = \lim_{t \rightarrow +\infty} t^{-1} \log \|U(t)\|, \quad s_0(H) = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(H)\}.$$

*Then  $\omega_0(H) = s_0(H)$  if and only if for every  $\varepsilon > 0$ , there exists a positive constant  $M_\varepsilon$  such that  $\|(\lambda I - H)^{-1}\| \leq M_\varepsilon$  uniformly in  $\{\lambda : \operatorname{Re}\lambda \geq s_0(H) + \varepsilon\}$ .*

**Theorem 3.2.11.** *In the setting of  $L^2(G)$ , suppose (H1) – (H3) are satisfied and  $\delta_1 + \delta_2 < 1/2$ . Then the spectrum of  $T(t)$  outside the disk  $\{\lambda : |\lambda| \leq \exp(-\lambda^*t)\}$  consists of isolated eigenvalues of  $T(t)$  with finite algebraic multiplicity, and the accumulation points of the set  $\sigma(T(t)) \cap \{\lambda : |\lambda| > \exp(-\lambda^*t)\}$  could only appear on the circle  $\{\lambda : |\lambda| = \exp(-\lambda^*t)\}$ . In particular, for every  $\varepsilon > 0$ , there exists a positive constant  $M_5$  such that*

$$\|T(t)(I - P_m)\|_2 \leq M_5 \exp\{(\operatorname{Re}\lambda_{m+1} + \varepsilon)t\}, \quad (3.13)$$

*where  $\lambda_1, \lambda_2, \dots, \lambda_m, \lambda_{m+1}, \dots$  and  $P_m$  have similar meanings as that of Theorem 2.4.1.*

**Proof.** For every integer  $m$  satisfying  $\operatorname{Re}\lambda_m > \operatorname{Re}\lambda_{m+1}$ , let

$$\sigma_1 := \{\lambda_1, \lambda_2, \dots, \lambda_m\}, \quad \sigma_2 := \sigma(A) \setminus \sigma_1.$$

Since  $\sigma_1$  is a compact set, it follows from [23, page 70] that there exists a unique spectral decomposition  $L^2(G) = E_1 \oplus E_2$  such that  $T_i(t)$ , the part of  $T(t)$  in  $E_i$  ( $i = 1, 2$ ), is a  $C_0$  semigroup. Furthermore, the spectral set of  $A_i$  (where  $A_i$  is the generator of  $T_i(t)$ ) is equal to  $\sigma_i$ , i.e.,  $\sigma(A_i) = \sigma_i$ ,  $i = 1, 2$ , and  $A_1$  is a bounded operator on  $E_1$ . Denoting by  $P_m$  the projection operator of  $\sigma_1$  corresponding to  $A$ , then (cf. [23, page 70])

$$T_1(t) = T(t)P_m, \quad T_2(t) = T(t)(I - P_m),$$

$$\begin{aligned}
(\lambda I - A_1)^{-1} &= (\lambda I - A)^{-1}P_m, \quad (\lambda I - A_2)^{-1} = (\lambda I - A)^{-1}(I - P_m), \\
\sigma(T(t)) &= \sigma(T_1(t)) \cup \sigma(T_2(t)).
\end{aligned} \tag{3.14}$$

Since  $A_1$  is a bounded operator on  $E_1$ , we have

$$\sigma(T_1(t)) = \{\exp(\lambda_n t) : n = 1, 2, \dots, m\}. \tag{3.15}$$

From Theorem 3.2.7(ii), it is easy to see that for every  $\varepsilon > 0$ , there exists a positive constant  $M_{m,\varepsilon}$  such that  $\|(\lambda I - A_2)^{-1}\| \leq M_{m,\varepsilon}$  uniformly in  $\{\lambda : \operatorname{Re}\lambda \geq \operatorname{Re}\lambda_{m+1} + \varepsilon\}$ . Thus, Lemma 3.2.10 (with  $X = E_2$ ,  $U(t) = T_2(t)$ ,  $H = A_2$ ) implies that the growth bound of  $T_2(t)$  is  $\operatorname{Re}\lambda_{m+1}$ , and thereby the spectral radius of  $T_2(t)$  is  $\exp(\operatorname{Re}\lambda_{m+1}t)$ . This implies Eq. (3.13).

Noting that  $m$  can be selected arbitrarily, by virtue of Eqs. (3.8) and (3.9), we complete the proof of Theorem 3.2.11.

Theorem 3.2.11 indicates that the hypothesis (H4) can be completely dropped if  $p = 2$ . Then, what is the situation for  $p \neq 2$  if (H4) is not satisfied?

In case of  $\psi_0 \in D(A^2)$ , by using Laplace–inversion formula representing the semigroup in terms of resolvent integrals (cf. [19, 21, 22, etc.]), it is easy to see that for every  $\varepsilon > 0$ ,

$$\|T(t)(I - P_m)\psi_0\|_p = o(\exp\{(\operatorname{Re}\lambda_{m+1} + \varepsilon)t\}). \tag{3.16}$$

Since for any  $\psi_0 \in D(A)$ , the (classical) solution  $\psi(t)$  of Eq. (I) exists and is unique (cf. [29]), and Eq. (3.16) does not give the asymptotic behavior of  $\psi(t)$  for all  $\psi_0 \in D(A) \setminus D(A^2)$ , so the condition  $\psi_0 \in D(A^2)$  is rather strict.

We will see in the sequel that in the setting of  $L^p(G)$  ( $1 < p < \infty, p \neq 2$ ), the condition  $\psi_0 \in D(A^2)$  can be replaced by  $\psi_0 \in D(A)$ .

**Lemma 3.2.12.** [44] *Let  $U(t)$  be a  $C_0$  semigroup in a  $B$ -convex Banach space  $(X, \|\cdot\|)$  with generator  $H$ ; then  $\omega_1(H) \leq s_0(H)$ , where*

$$s_0(H) = \inf\{s > s(H) : \|(\beta + \tau i - H)^{-1}\| = O(1) \text{ as } |\tau| \rightarrow +\infty \text{ for } \beta \geq s\},$$

$$s(H) = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(H)\},$$

$$\omega_1(H) = \inf\{\omega \in R : \sup_{t \geq 0} \|e^{-\omega t}U(t)(\lambda_0 I - H)^{-1}\| < \infty\}.$$

**Theorem 3.2.13.** *In the setting of  $L^p(G)$  ( $1 < p < \infty$ ), suppose (H1) – (H3) are satisfied,  $\delta_1 < \min\{1/p, 1/2\}$ ,  $\delta_2 < \min\{1/q, 1/2\}$  and  $\delta_1 + \delta_2 < 1/2$ . Furthermore, let  $\lambda_0$  be any element of  $\rho(A)$ . Then for every  $\varepsilon > 0$ , there exists a positive constant  $M_6$  such that*

$$\|T(t)(I - P_m)\psi\|_p \leq M_6 \exp\{(\operatorname{Re}\lambda_{m+1} + \varepsilon)t\} \|(\lambda_0 I - A)\psi\|_p$$

for every  $\psi \in D(A)$ . Here  $\lambda_1, \lambda_2, \dots, \lambda_m, \lambda_{m+1}, \dots$  and  $P_m$  have similar meanings as that of Theorem 2.4.1.

**Proof.** Similar to the proof of Theorem 3.2.11, it follows from [23, page 70] that there exists a unique spectral decomposition  $L^p(G) = E_1 \oplus E_2$  such that  $T_i(t)$ , the part of  $T(t)$  in  $E_i$  ( $i = 1, 2$ ), is a  $C_0$  semigroup. Furthermore,  $\sigma(A_i) = \sigma_i$ , where  $A_i$  is the generator of  $T_i(t)$  ( $i = 1, 2$ ), and

$$\begin{aligned} T_1(t) &= T(t)P_m, & T_2(t) &= T(t)(I - P_m), \\ (\lambda I - A_1)^{-1} &= (\lambda I - A)^{-1}P_m, & (\lambda I - A_2)^{-1} &= (\lambda I - A)^{-1}(I - P_m). \end{aligned}$$

Consider Lemma 3.2.12 with  $X = E_2, U(t) = T_2(t)$  and  $H = A_2$ . Obviously, the spectral bound of  $A_2$  is  $\operatorname{Re}\lambda_{m+1}$ . Moreover, from Theorem 3.2.7(ii), it is easy to see that  $s_0(A_2) = \operatorname{Re}\lambda_{m+1}$ . Thus, by virtue of Lemma 3.2.12, it is known that for every  $\varepsilon > 0$ , there exists a positive constant  $M_7$  such that for all  $\psi \in E_2$ ,

$$\|T_2(t)(\lambda_0 - A_2)^{-1}\psi\|_p \leq M_7 \exp\{(\operatorname{Re}\lambda_{m+1} + \varepsilon)t\} \|\psi\|_p. \quad (3.17)$$

For any  $\psi \in E_2$  and  $\lambda_0 \in \rho(A)$ , we have

$$(\lambda_0 I - A_2)^{-1}\psi = (\lambda_0 I - A)^{-1}(I - P_m)\psi.$$

Noting that  $(I - P_m)\psi \in E_2$  for every  $\psi \in L^p(G)$ , we get

$$(\lambda_0 I - A_2)^{-1}(I - P_m)\psi = (\lambda_0 I - A)^{-1}(I - P_m)\psi \quad (3.18)$$

for every  $\psi \in L^p(G)$ . Thus, for every  $\psi \in L^p(G)$ ,  $(\lambda_0 I - A)^{-1}(I - P_m)\psi \in D(A_2)$ .

For any  $\psi \in D(A)$  and  $\lambda_0 \in \rho(A)$ , from Eq. (3.18), we have

$$T(t)(I - P_m)\psi = T(t)(I - P_m)(\lambda_0 I - A)^{-1}(\lambda_0 I - A)\psi$$

$$\begin{aligned}
&= T(t)(I - P_m)(I - P_m)(\lambda_0 I - A)^{-1}(\lambda_0 I - A)\psi \\
&= T(t)(I - P_m)(\lambda_0 I - A)^{-1}(I - P_m)(\lambda_0 I - A)\psi \\
&= T_2(t)(\lambda_0 I - A_2)^{-1}(I - P_m)(\lambda_0 I - A)\psi.
\end{aligned}$$

Applying Eq. (3.17) to the above equation yields

$$\begin{aligned}
\|T(t)(I - P_m)\psi\|_p &\leq M_7 \exp\{(\operatorname{Re}\lambda_{m+1} + \varepsilon)t\} \|(I - P_m)(\lambda_0 I - A)\psi\|_p \\
&\leq M_7 \exp\{(\operatorname{Re}\lambda_{m+1} + \varepsilon)t\} \|(\lambda_0 I - A)\psi\|_p.
\end{aligned} \tag{3.19}$$

This completes the proof. Q. E. D.

Theorem 3.2.11 and Theorem 3.2.13 indicate that under the hypotheses (H1) – (H3), the well known condition  $\psi \in D(A^2)$  imposed on the initial distribution  $\psi_0$  (cf. [19, 21, 22, etc.]) can be completely dropped in the setting of  $L^2(G)$  and can be weakened to  $\psi \in D(A)$  in the setting of  $L^p(G)$  ( $1 < p < +\infty$ ,  $p \neq 2$ ). In §3.4 of this paper, we will show (to some extent) that the condition  $\psi \in D(A)$  can be completely eliminated if we just consider the asymptotic behavior of the solution  $\psi(t)$  of Eq. (I).

### 3.3 Some Further Investigations in $L^p(G)$

In the discussion made in §3.2, it is usually assumed that  $\delta_1 + \delta_2 < 1/2$ . However, the constants  $\delta_1 = 2/5$  and  $\delta_2 = 1/5$  in the setting of  $L^2(G)$  do not satisfy  $\delta_1 + \delta_2 < 1/2$ , and thus this case is not treated in §3.2.

This section is devoted to weakening some conditions imposed on  $\delta_1, \delta_2 < 1/2$  and  $\tilde{k}(x, v, v', \mu, \mu')$ . Similar results as that of §3.2 are obtained if the conditions “ $\delta_1 < \min\{1/p, 1/2\}$ ,  $\delta_2 < \min\{1/q, 1/2\}$  and  $\delta_1 + \delta_2 < 1/2$ ” made in §3.2 are replaced by “ $\delta_1 < \min\{1/p, 1/4\}$ ,  $\delta_2 < \min\{1/q, 1/2\}$ ” or “ $\delta_1 < \min\{1/p, 1/2\}$ ,  $\delta_2 < \min\{1/q, 1/4\}$ ”, and the hypothesis (H4) is replaced by

( $\widetilde{\text{H4}}$ ).  $\alpha(v, \mu), \gamma(v, \mu), \sigma(r, v, \mu)$  and

$$\tilde{k}(x, v, v', \mu, \mu') := (v|\mu)^{\delta_1}(v'|\mu')^{\delta_2}k(x, v, v', \mu, \mu'),$$

(cf. (H3)) are partially differentiable with respect to  $\mu \in D$  a. e., and the corresponding partial derivatives  $\frac{\partial \alpha}{\partial \mu}$ ,  $\frac{\partial \gamma}{\partial \mu}$ ,  $\frac{\partial \sigma}{\partial \mu}$  and  $\frac{\partial \tilde{k}}{\partial \mu}$  are essentially bounded.

**Note:** In  $(\widetilde{H4})$ , the differentiability properties of  $\tilde{k}(x, v, v', \mu, \mu')$  with respect to  $\mu'$  are completely eliminated.

**Lemma 3.3.1.** [33] *Let  $H$  be a bounded linear operator in  $L^2(G)$ , and  $H^*$  the adjoint operator of  $H$ . Then for every  $\lambda$  with  $\operatorname{Re} \lambda > -\lambda^*$ ,*

$$\|H(\lambda I - B)^{-1}\|_2 \leq [(\operatorname{Re} \lambda + \lambda^*)^{-1} \|H(\lambda I - B)^{-1} H^*\|_2]^{1/2}, \quad (3.20)$$

$$\|(\lambda I - B)^{-1} H\|_2 \leq [(\operatorname{Re} \lambda + \lambda^*)^{-1} \|H^*(\lambda I - B)^{-1} H\|_2]^{1/2}. \quad (3.21)$$

It is worthy to mention that many results in §3.2 are obtained under the condition  $\delta_1 + \delta_2 < 1/2$ , which indicates  $\delta_1 < 1/4$  or  $\delta_2 < 1/4$  since both  $\delta_1$  and  $\delta_2$  are supposed to be nonnegative. We will see in the sequel that fine results can be obtained for  $\delta_1 < 1/4$  or  $\delta_2 < 1/4$ .

First, we consider the case  $\delta_1 < 1/4$ . Similar to that for proving Lemma 3.2.3 and Lemma 3.2.5, we can show that all the conclusions in Lemma 3.2.3, Lemma 3.2.5 still hold if the condition “ $\delta_1 + \delta_2 < 1/2$ ” is replaced by “ $\delta_1 < 1/4$  and  $\delta_2 < 1/2$ ” and the term  $\|K(\lambda I - B)^{-1} K\|_2$  is replaced by  $\|K^*(\lambda I - B)^{-1} K\|_2$ . This together with Eq.(3.21) implies that similar conclusions also hold for  $\|(\lambda I - B)^{-1} K\|_2$ . By virtue of the Riesz–Thorin interpolation theorem. we see that similar results also hold for  $\|(\lambda I - B)^{-1} K\|_p$  if the condition “ $\delta_1 < \min\{1/p, 1/4\}$ ,  $\delta_2 < \min\{1/q, 1/2\}$ ” is satisfied. And consequently, Theorem 3.2.7 and Theorem 3.2.8 still hold if the condition “ $\delta_1 < \min\{1/p, 1/2\}$ ,  $\delta_2 < \min\{1/q, 1/2\}$  and  $\delta_1 + \delta_2 < 1/2$ ” made in §3.2 is replaced by “ $\delta_1 < \min\{1/p, 1/4\}$ ,  $\delta_2 < \min\{1/q, 1/2\}$ ”

**Lemma 3.3.2.** *Suppose (H1) – (H3) are satisfied and let  $\delta K_n$  be the operator given in §3.2. If  $\delta_1 < \min\{1/p, 1/4\}$ ,  $\delta_2 < \min\{1/q, 1/2\}$ , then  $\lim_{n \rightarrow \infty} \|(\lambda I - B)^{-1} \delta K_n\|_p = 0$  uniformly in  $\{\lambda : \operatorname{Re} \lambda \geq \beta_1\}$ .*

**Proof.** Similar to that for proving Lemma 3.2.4, we have

$$\lim_{n \rightarrow \infty} \|\delta K_n^*(\lambda I - B)^{-1} \delta K_n\|_2 = 0$$

uniformly in  $\{\lambda : \operatorname{Re}\lambda \geq \beta_1\}$ . From Eq. (3.21), we get

$$\lim_{n \rightarrow \infty} \|(\lambda I - B)^{-1} \delta K_n\|_2 = 0,$$

which completes the proof by virtue of the Riesz–Thorin interpolation theorem.

**Lemma 3.3.3.** *In addition to (H1) – (H3), suppose  $\alpha(v, \mu)$ ,  $\gamma(v, \mu)$  and  $\sigma(x, v, \mu)$  are piecewise continuous. If  $\delta_1 < \min\{1/p, 1/4\}$ ,  $\delta_2 < \min\{1/q, 1/2\}$ , then*

$$\lim_{n \rightarrow \infty} \|(\lambda I - A_n)^{-1} - (\lambda I - A)^{-1}\|_p = 0$$

uniformly in  $\{\lambda : \operatorname{Re}\lambda \geq \tilde{\beta}\}$ , where  $\tilde{\beta}$  is a sufficiently large constant,  $A_n = B_{n,n} + K_n$ ,  $B_{n,n}$  and  $K_n$  are operators previously defined.

**Proof.** Since  $\alpha(v, \mu)$ ,  $\gamma(v, \mu)$  and  $\sigma(x, v, \mu)$  are piecewise continuous, they can be uniformly approximated by sequences composed of piecewise smooth functions, and therefore

$$\lim_{n \rightarrow \infty} \|(\lambda I - B_{n,n})^{-1} - (\lambda I - B)^{-1}\|_p = 0$$

for every  $\lambda$  with  $\operatorname{Re}\lambda > -\lambda^*$ . From the relation

$$\begin{aligned} & (\lambda I - A_n)^{-1} - (\lambda I - A)^{-1} \\ &= [I - (\lambda I - B_{n,n})^{-1}(K - \delta K_n)]^{-1} (\lambda I - B_{n,n})^{-1} \\ & \quad - [I - (\lambda I - B)^{-1}K]^{-1} (\lambda I - B)^{-1}, \end{aligned}$$

and Lemma 3.3.2, it is easy to get the conclusion. Q. E. D.

The case  $\delta_2 < 1/4$  can be similarly treated. That is, we can first show that all the conclusions in Lemma 3.2.3 and Lemma 3.2.5 still hold if the condition “ $\delta_1 + \delta_2 < 1/2$ ” is replaced by “ $\delta_1 < 1/2$  and  $\delta_2 < 1/4$ ” and the term  $\|K(\lambda I - B)^{-1}K\|_2$  is replaced by  $\|K(\lambda I - B)^{-1}K^*\|_2$ . Then from Eq. (3.20) and the Riesz–Thorin interpolation theorem, we get similar conclusions for  $\|K(\lambda I - B)^{-1}\|_2$  and  $\|K(\lambda I - B)^{-1}\|_p$  when the condition “ $\delta_1 < \min\{1/p, 1/2\}$ ,  $\delta_2 < \min\{1/q, 1/4\}$ ” is satisfied. Thus, the conclusions in Theorem 3.2.7 and Theorem 3.2.8 are obtained under the condition “ $\delta_1 < \min\{1/p, 1/2\}$ ,  $\delta_2 < \min\{1/q, 1/4\}$ ”.

Similar to that of Lemma 3.3.2, we get

$$\lim_{n \rightarrow \infty} \|\delta K_n (\lambda I - B)^{-1}\|_p = 0$$

uniformly in  $\{\lambda : \operatorname{Re}\lambda \geq \beta_1\}$ , and from the relation

$$\begin{aligned} & (\lambda I - A_n)^{-1} - (\lambda I - A)^{-1} \\ &= (\lambda I - B_{n,n})^{-1} [I - (\lambda I - B_{n,n})^{-1}(K - \delta K_n)]^{-1} \\ & \quad - (\lambda I - B)^{-1} [I - (\lambda I - B)^{-1}K]^{-1}, \end{aligned}$$

we see that Lemma 3.3.3 also holds in this case.

Noting that the assertion “ $\|(\lambda I - A_n)^{-1} - (\lambda I - A)^{-1}\|_p \rightarrow 0$ ” plays a key role in the proof of Theorem 3.2.9,

**Theorem 3.3.4.** *The conclusions obtained in §3.2 still hold if the conditions “ $\delta_1 < \min\{1/p, 1/2\}$ ,  $\delta_2 < \min\{1/q, 1/2\}$  and  $\delta_1 + \delta_2 < 1/2$ ” made in §3.2 are replaced by “ $\delta_1 < \min\{1/p, 1/4\}$ ,  $\delta_2 < \min\{1/q, 1/2\}$ ” or  $\delta_1 < \min\{1/p, 1/2\}$ ,  $\delta_2 < \min\{1/q, 1/4\}$ ”, the hypothesis (H4) is replaced by  $(\widetilde{H4})$ , and the the hypothesis (H5) is replaced by*

$(\widetilde{H5})$ .  $\alpha(v, \mu), \gamma(v, \mu)$  and  $\sigma(r, v, \mu)$  are piecewise continuous functions. Furthermore, there exist positive constants  $\alpha_0, \widetilde{\alpha}_0, \gamma_0$  and  $\widetilde{\gamma}_0$  such that  $0 < \alpha_0 \leq \alpha(v, \mu) \leq \widetilde{\alpha}_0 < 1$  and  $0 < \gamma_0 \leq \gamma(v, \mu) \leq \widetilde{\gamma}_0 < 1$ .

Since the conditions “ $\delta_1 < \min\{1/p, 1/2\}$ ,  $\delta_2 < \min\{1/q, 1/2\}$  and  $\delta_1 + \delta_2 < 1/2$ ” imply “ $\delta_1 < \min\{1/p, 1/4\}$ ,  $\delta_2 < \min\{1/q, 1/2\}$ ” or “ $\delta_1 < \min\{1/p, 1/2\}$ ,  $\delta_2 < \min\{1/q, 1/4\}$ ”, we see from Theorem 3.3.4 that the conditions imposed on  $\delta_1, \delta_2$  and especially on  $\widetilde{k}(x, v, v', \mu, \mu') = (v|\mu)^{\delta_1}(v'|\mu')^{\delta_2}k(x, v, v', \mu, \mu')$  in §3.2 have been effectively eliminated. Furthermore, some conclusions (such as the estimation of  $\|K(\lambda I - B)^{-1}\|_p$  and  $\|(\lambda I - B)^{-1}K\|_p$  as  $|\operatorname{Im}\lambda| \rightarrow +\infty$ ) which are not given in §3.2 are also obtained.

We complete this section with the following assertion concerning the compactness of  $K(\lambda I - B)^{-1}$  or  $(\lambda I - B)^{-1}K$ .

**Theorem 3.3.5.** *Under the conditions (H1) – (H3), if  $\delta_1 < \min\{1/p, 1/4\}$ ,  $\delta_2 < \min\{1/q, 1/2\}$ , then  $(\lambda I - B)^{-1}K$  is a compact operator on  $L^p(G)$  for every  $\lambda$  with  $\operatorname{Re}\lambda > -\lambda^*$ . If  $\delta_1 < \min\{1/p, 1/2\}$ ,  $\delta_2 < \min\{1/q, 1/4\}$ , then  $K(\lambda I - B)^{-1}$  is a compact operator on  $L^p(G)$  for every  $\lambda$  with  $\operatorname{Re}\lambda > -\lambda^*$ .*

**Proof.** First, we consider the case  $p = 2$ . For any  $\psi \in L^2(G)$ , applying  $f = (\lambda I - B)^{-1}K\psi \in D(B)$  to the well known relation

$$\operatorname{Re}((\lambda I - B)f, f) \geq (\operatorname{Re}\lambda + \lambda^*)\|f\|_2^2 \quad \text{for } f \in D(B)$$

yields

$$\|(\lambda I - B)^{-1}K\psi\|_2^2 \leq (\operatorname{Re}\lambda + \lambda^*)^{-1}\|\psi\|_2\|K^*(\lambda I - B)^{-1}K\psi\|_2.$$

Thus, for any sequence  $\{\psi_n\} \subset L^2(G)$ , we get

$$\begin{aligned} & \|(\lambda I - B)^{-1}K(\psi_m - \psi_n)\|_2^2 \\ & \leq (\operatorname{Re}\lambda + \lambda^*)^{-1}\|\psi_m - \psi_n\|_2\|K^*(\lambda I - B)^{-1}K(\psi_m - \psi_n)\|_2. \end{aligned} \quad (3.22)$$

If  $\delta_1 < 1/4$ , then similar to that of Lemma 3.2.1, it is known that  $K^*(\lambda I - B)^{-1}K$  is a compact operator on  $L^2(G)$ . From Eq. (3.22), we see that  $(\lambda I - B)^{-1}K$  is also a compact operator on  $L^2(G)$ .

If  $\delta_1 < \min\{1/p, 1/4\}$ ,  $\delta_2 < \min\{1/q, 1/2\}$ , then from the Riesz–Thorin interpolation theorem, we know that  $(\lambda I - B)^{-1}K$  is a compact operator on  $L^p(G)$ .

If  $\delta_1 < \min\{1/p, 1/2\}$ ,  $\delta_2 < \min\{1/q, 1/4\}$ , then we can first show that  $K(\lambda I - B)^{-1}K^*$  is compact on  $L^2(G)$ , and then obtain the compactness of  $K(\lambda I - B)^{-1}$  on  $L^2(G)$  from the relation

$$\operatorname{Re}((\lambda I - B^*)f, f) \geq (\operatorname{Re}\lambda + \lambda^*)\|f\|_2^2 \quad \text{for } f \in D(B^*),$$

and then prove the compactness of  $K(\lambda I - B)^{-1}$  on  $L^p(G)$  by virtue of interpolation theorem. Q. E. D.

### 3.4 Asymptotic Behavior of $\psi$ in $L^p(G)$

In addition to (H1), (H2), it is assumed that  $k(x, v, v', \mu, \mu')$  is nonnegative and essentially bounded throughout this section. In this case,  $A$  generates a positive  $C_0$  semigroup  $T(t)$ ,  $\operatorname{Pas}(A)$  and the algebraic multiplicity of every point  $\lambda_i \in \operatorname{Pas}(A)$  as well as the projection subspace corresponding to  $\lambda_i$ , do not change with respect to  $p \in [1, +\infty)$ . With some

additional assumptions, the dominant eigenvalue  $\beta_0$  of  $A$  exists and  $\beta_0 > -\lambda^*$  (cf. [31, 40, etc.]). Furthermore, the projection operator  $P_0$  corresponding to  $\beta_0$  satisfies

$$T(t)P_0 \cdot = \exp(\beta_0 t)(\cdot, \psi_{\beta_0}^*)(\psi_{\beta_0}, \psi_{\beta_0}^*)^{-1} \psi_{\beta_0}, \quad (3.23)$$

where  $\psi_{\beta_0}$  is the positive eigenfunction of  $\beta_0$  corresponding to  $A$ ,  $\psi_{\beta_0}^*$  is the positive eigenfunction of  $\beta_0$  corresponding to  $A^*$  (the adjoint operator of  $A$ ).

From Theorem 3.2.7(i),  $\beta_0$  is strictly dominant, so  $d := \beta_0 - \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A), \lambda \neq \beta_0\} > 0$ . For any  $\varepsilon \in (0, d)$ , it follows from Theorem 3.2.11 that

$$\|T(t) - T(t)P_0\|_2 \leq M_5 \exp\{(\beta_0 - \varepsilon)t\}. \quad (3.24)$$

Since  $A$  generates a positive  $C_0$  semigroup  $T(t)$  in  $L^1(G)$ , it follows from [6] that the growth bound of  $T(t)$  in  $L^1(G)$  is equal to  $\beta_0$ . So for every  $\varepsilon > 0$ , there exists a positive constant  $M_7$  such that

$$\|T(t)\|_1 \leq M_7 \exp\{(\beta_0 + \varepsilon)t\}. \quad (3.25)$$

On the other hand, from the discussions made in §3.1, it is not difficult to obtain

$$\|T(t)|_{L^\infty(G)}\|_\infty = \|\tilde{T}^*(t)\|_\infty = \|\tilde{T}(t)\|_1 \leq M_7 \exp\{(\beta_0 + \varepsilon)t\}. \quad (3.26)$$

for every  $\varepsilon > 0$ , where  $\tilde{T}(t)$  is the positive  $C_0$  semigroup generated by  $\tilde{A}$  in  $L^1(G)$ .

From Eqs. (3.25), (3.26) and the Riesz–Thorin interpolation theorem, we have

$$\|T(t)\|_p \leq M_8 \exp\{(\beta_0 + \varepsilon)t\}. \quad (3.27)$$

for every  $p \in [1, +\infty)$  and  $\varepsilon > 0$ .

Eq. (3.27) indicates that the growth bound of  $T(t)$  in any space  $L^p(G)$  is equal to  $\beta_0$ .

It follows from Eqs. (3.23) and (3.27) that

$$\|T(t) - T(t)P_0\|_p \leq M_9 \exp\{(\beta_0 + \tilde{\varepsilon})t\} \quad (3.28)$$

for any  $p \in [1, +\infty)$  and  $\tilde{\varepsilon} > 0$ . Again, by virtue of the Riesz–Thorin interpolation theorem, we get the following theorem from Eqs. (3.24) and (3.28).

**Theorem 3.4.1.** *In addition to (H1), (H2), suppose  $k(x, v, v', \mu, \mu')$  is nonnegative essentially bounded, and  $\beta_0 > -\lambda^*$  is the dominant eigenvalue of  $A$ . Then for any initial distribution  $\psi_0 \in L^p(G)$  ( $1 < p < +\infty$ ), the “solution”  $\psi(t)$  of Eq. (I) is given by  $\psi(t) = T(t)\psi_0$ , and*

$$\|T(t)\psi_0 - \exp(\beta_0 t)(\psi_0, \psi_{\beta_0}^*)(\psi_{\beta_0}, \psi_{\beta_0}^*)^{-1}\psi_{\beta_0}\|_p \leq M_{10} \exp\{(\beta_0 - \varepsilon)t\}\|\psi_0\|_p,$$

where  $\varepsilon \in (0, 2dp^{-1})$  (if  $p \geq 2$ ) or  $\varepsilon \in (0, 2d(1 - p^{-1}))$  (if  $p < 2$ ),  $d = \beta_0 - \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A), \lambda \neq \beta_0\} > 0$ . As  $t \rightarrow +\infty$ , the solution  $\psi(t)$  of Eq. (I) tends to 0,  $+\infty$ , or  $(\psi_0, \psi_{\beta_0}^*)(\psi_{\beta_0}, \psi_{\beta_0}^*)^{-1}\psi_{\beta_0}$ , depending on  $\beta_0 < 0$ ,  $\beta_0 > 0$  and  $\beta_0 = 0$  respectively.

Remark: It is worthy to mention that no conditions such as partial differentiability are imposed in Theorem 3.4.1. It is only required that each of the physical parameters  $\alpha(v, \mu)$ ,  $\gamma(v, \mu)$ ,  $\sigma(x, v, \mu)$  and  $k(x, v, v', \mu, \mu')$  is nonnegative bounded measurable.

## 3.5 Some Remarks

**Remark 3.5.1.** All the conclusions obtained in the previous sections still hold if the velocity domain  $V = (0, v_M]$  is replaced by  $V = [v_m, v_M]$  with  $0 < v_m < v_M < +\infty$ . All the conclusions also hold for the mono-energetic transport problem.

**Remark 3.5.2.** For reactor problems,  $\sigma(x, v, \mu)$  is in fact independent of  $\mu$ , so all the conditions imposed on  $\sigma(x, v, \mu)$  such as the partial differentiability properties can be completely eliminated. It is only required that  $\sigma(x, v)$  is bounded measurable.

**Remark 3.5.3.** The positivity of  $k(x, v, v', \mu, \mu')$  as well as  $\sigma(x, v, \mu)$  is not needed in the discussions made in this paper except §3.4.

**Remark 3.5.4.** Noting that it is still okay if we exchange the positions of  $w$  and  $\xi$  in the proof of Lemma 2.3.1 and some other conclusions, we see that all the results obtained in this paper still hold if the hypothesis (H4) is replaced by

(H4').  $\alpha(v, \mu)$ ,  $\gamma(v, \mu)$ ,  $\sigma(r, v, \mu)$  and

$$\tilde{k}(x, v, v', \mu, \mu') := (v|\mu|)^{\delta_1}(v'|\mu'|)^{\delta_2}k(x, v, v', \mu, \mu'),$$

(cf. (H3)) are partially differentiable with respect to  $v, v' \in V$  a. e., and the corresponding partial derivatives  $\frac{\partial \alpha}{\partial v}, \frac{\partial \gamma}{\partial v}, \frac{\partial \sigma}{\partial v}, \frac{\partial \tilde{k}}{\partial v}$  and  $\frac{\partial \tilde{k}}{\partial v'}$  are essentially bounded.

**Remark 3.5.5.** If  $k(x, v, v', \mu, \mu')$  is bounded measurable (i.e.,  $\delta_1 = \delta_2 = 0$ ), then many conclusions in  $L^p(G)$  (cf. §3.2, §3.3) can be easily obtained from the corresponding results in  $L^1(G)$  and  $L^\infty(G)$  by virtue of the Riesz–Thorin interpolation theorem. Also, it is worthy to mention that many conclusions in  $L^\infty(G)$  can be derived from the corresponding results in  $L^1(G)$  from the theory of adjoint operators.

**Remark 3.5.6.** Sometimes the order of  $|\operatorname{Im}\lambda|$  concerning the estimation of  $\|K(\lambda I - B)^{-1}K\|_p$  or similar things as  $|\operatorname{Im}\lambda| \rightarrow +\infty$  is quite important. By use of the technique described in Remark 3.5.5, we have the following result concerning the estimation of  $\|K(\lambda I - B)^{-1}K\|_p$ , which is much better than that of Lemma 3.2.3 and Lemma 3.2.6.

**Proposition 3.5.7.** *In addition to (H1), (H2) and (H4) (or (H4')), if  $k(x, v, v', \mu, \mu')$  is bounded measurable (i.e.,  $\delta_1 = \delta_2 = 0$ ), then for any constants  $\beta_1 > -\lambda^*$  and  $\varepsilon > 0$ , there exist positive constants  $C_0$  and  $\tilde{\tau}$  independent of  $\beta \in [\beta_1, +\infty)$ , such that*

$$\|K(\lambda I - B)^{-1}K\|_p \leq C_0 |\beta_1 + \lambda^* + i\tau|^{-1} \log |\beta_1 + \lambda^* + i\tau|$$

*uniformly in  $\{\lambda = \beta + i\tau : \beta \geq \beta_1, |\tau| \geq \tilde{\tau}\}$ .*

**Proof.** From Lemma 2.3.1 and Lemma 3.1.2 (iii), this assertion is obviously true for  $p = 1$  and  $p = \infty$ . By virtue of the Riesz–Thorin interpolation theorem, the conclusion is easily obtained.

## Part II

# SPHERICAL GEOMETRY

# Chapter 4

## Spectral Properties of Transport Equations for Spherical Geometry in $L^2$

The time dependent transport equation in a sphere with reflecting boundary conditions will be discussed in the setting of  $L^2$ . Some aspects of the spectral properties of the strongly continuous semigroup  $T(t)$  generated by the corresponding transport operator  $A$  are studied, and it is shown that the spectrum of  $T(t)$  outside the disk  $\{\lambda : |\lambda| \leq \exp(-\lambda^*t)\}$ , where  $\lambda^*$  is the essential infimum of the total collision frequency  $\sigma(r, v)$ , or  $\lambda^* = \text{ess inf}_r \lim_{v \rightarrow 0^+} \sigma(r, v)$ , consists of isolated eigenvalues of  $T(t)$  with finite algebraic multiplicity, and the accumulation points of  $\sigma(T(t)) \cap \{\lambda : |\lambda| > \exp(-\lambda^*t)\}$  could only appear on the circle  $\{\lambda : |\lambda| = \exp(-\lambda^*t)\}$ . Consequently, the asymptotic behavior of the time dependent solution is obtained.

### 4.1 Problem and Notations

Consider the following neutron transport equation (cf. [20, 42, 25])

$$(I) \quad \left\{ \begin{array}{l} \frac{\partial f(r, v, \mu, t)}{\partial t} = -v\mu \frac{\partial f(r, v, \mu, t)}{\partial r} - v \frac{1 - \mu^2}{r} \frac{\partial f(r, v, \mu, t)}{\partial \mu} \\ \quad - \sigma(r, v) f(r, v, \mu, t) + \frac{1}{2} \int_0^{v_M} \int_{-1}^1 k(r, v, v') f(r, v', \mu', t) dv' d\mu', \\ r \in V := [0, R], \quad v \in E := (0, v_M], \quad \mu \in \Omega := [-1, 1], \quad t > 0, \\ f(R, v, \mu, t) = \alpha(v, \mu) f(R, v, -\mu, t) \text{ for every } \mu \in [-1, 0) \text{ and } v \in E, \\ f(r, v, \mu, 0) = f_0(r, v, \mu), \end{array} \right.$$

where the region occupied by the reactor media is a sphere of radius  $R > 0$ ,  $r$  is the distance from the center of the sphere,  $v$  is the velocity,  $0 < v_M < +\infty$ ,  $\mu$  is the cosine of the angle the neutron velocity makes with the radius vector,  $f(r, v, \mu, t)$  is the neutron distribution at time  $t$ ,  $\sigma(r, v)$  is the total collision frequency,  $k(r, v, v')$  is the scattering fission kernel,  $\alpha(v, \mu)$  is the boundary reflection coefficient, and  $f_0(r, v, \mu)$  is the initial distribution.

Throughout this part, it is assumed that

(H1).  $\sigma(r, v)$  is a real bounded measurable function.

(H2).  $\alpha(v, \mu)$  is bounded measurable, and  $0 \leq \alpha(v, \mu) \leq \alpha_0 < 1$ , where  $\alpha_0$  is a constant.

(H3).  $k(r, v, v')$  is a real measurable function, and there exist positive constants  $\delta < 1/2$  and  $M$  such that

$$|k(r, v, v')| \leq Mv^{-\delta}, \quad (4.1)$$

or

$$|k(r, v, v')| \leq Mv'^{-\delta}. \quad (4.2)$$

Remark: Obviously, (H3) is satisfied if  $k(r, v, v')$  is bounded measurable.

By virtue of the transform (cf. [42, 25])

$$x = r\mu,$$

$$y = r\sqrt{1 - \mu^2},$$

$$\psi(x, y, v, t) = f(r(x, y), \mu(x, y), v, t),$$

Eq. (I) can be equivalently written as

$$(II) \quad \left\{ \begin{array}{l} \frac{\partial \psi(x, y, v, t)}{\partial t} = -v \frac{\partial \psi(x, y, v, t)}{\partial x} - \sigma(r, v) \psi(x, y, v, t) \\ \quad + \frac{1}{2r} \int_0^{v_M} \int_{-r}^r k(r, v, v') \psi(z, \sqrt{r^2 - z^2}, v', t) dz, \\ y \in V, 0 \leq r = \sqrt{x^2 + y^2} \leq R, v \in E, t > 0, \\ \psi(-\sqrt{R^2 - y^2}, y, v, t) = \alpha \left( v, -\sqrt{1 - R^{-2}y^2} \right) \psi(\sqrt{R^2 - y^2}, y, v, t), y \in V, \\ \psi(x, y, v, 0) = \psi_0(x, y, v). \end{array} \right.$$

Set  $D = \{(x, y) : y \geq 0, 0 \leq \sqrt{x^2 + y^2} \leq R\}$ ,  $G = D \times E$ , and let  $L^2(G)$  represent the Hilbert space composed of all measurable complex functions defined and square integrable over  $G$ , with the inner product  $(\cdot, \cdot)$  and the norm  $\|\cdot\|$  given by

$$(\psi, \varphi) = \frac{1}{2} \int_0^{v_M} dv \int_0^R y dy \int_{-\sqrt{R^2 - y^2}}^{\sqrt{R^2 - y^2}} \psi(x, y, v) \overline{\varphi(x, y, v)} dx,$$

$$\|\psi\| = \sqrt{(\psi, \psi)}.$$

Define operators on  $L^2(G)$  as follows:

$$B\psi = -v \frac{\partial \psi}{\partial x} - \sigma(r, v) \psi,$$

$$K\psi = \frac{1}{2r} \int_0^{v_M} dv' \int_{-r}^r k(r, v, v') \psi(z, \sqrt{r^2 - z^2}, v') dz,$$

$$A\psi = B\psi + K\psi,$$

with  $D(B) = D(A) = \{\psi \in L^2(G) : B\psi \in L^2(G) \text{ and } \psi(-\sqrt{R^2 - y^2}, y, v) = \alpha \left( v, -\sqrt{1 - R^{-2}y^2} \right) \psi(\sqrt{R^2 - y^2}, y, v) \text{ for every } y \in V \text{ and } v \in E\}$ ,  $D(K) = L^2(G)$ . Then Eq. (II) can be written as

$$\frac{d\psi(t)}{dt} = A\psi(t), \quad \psi(0) = \psi_0.$$

## 4.2 Spectral Properties of $A$ in $L^2$

From the hypothesis (H3), it is easy to see that  $K$  is a bounded operator.

Set  $\lambda^* = \text{ess inf}_{(r,v) \in V \times E} \sigma(r, v)$ . (Most of the conclusions given in this part still hold if  $\lambda^*$  is replaced by  $\lambda_0^* = \text{ess inf}_{r \in V} \lim_{v \rightarrow 0^+} \sigma(r, v)$ , see Remark 4.4.2 in Sec. 4.4).

From [42], we have the following conclusions.

**Lemma 4.2.1.** [42]  $B$  is a densely defined operator,  $\{\lambda : \operatorname{Re}\lambda > -\lambda^*\} \subset \rho(B)$ , and  $\|(\lambda I - B)^{-1}\| \leq (\operatorname{Re}\lambda + \lambda^*)^{-1}$  for every  $\lambda$  with  $\operatorname{Re}\lambda > -\lambda^*$ .

**Lemma 4.2.2.** [42]  $\{\lambda : \operatorname{Re}\lambda > \|K\| - \lambda^*\} \subset \rho(A)$ . For every  $\lambda$  with  $\operatorname{Re}\lambda > -\lambda^*$ ,  $\lambda \in P_\sigma(A)$  if and only if  $1 \in P_\sigma(K(\lambda I - B)^{-1})$ . The set

$$\operatorname{Pas}(A) := \{\lambda : -\lambda^* < \operatorname{Re}\lambda \leq \|K\| - \lambda^*, \lambda \in \sigma(A)\}$$

contains at most countable isolated elements, each of which is an eigenvalue of  $A$  with finite algebraic multiplicity.

**Lemma 4.2.3.** For every  $\lambda$  with  $\operatorname{Re}\lambda > -\lambda^*$ ,  $K(\lambda I - B)^{-1}K$  is an integral operator defined on  $L^2(G)$ . For every  $\psi \in L^2(G)$ ,

$$\begin{aligned} & K(\lambda I - B)^{-1}K\psi(x, y, v) \\ &= \iiint_G k(\lambda, x, x', y, y', v, v') dx' dy' dv', \end{aligned} \quad (4.3)$$

where

$$\begin{aligned} & k(\lambda, x, x', y, y', v, v') \\ &= \frac{y'}{4r\rho} \int_0^{v_M} dv_1 \frac{k(r, v, v_1)k(\rho, v_1, v')}{v_1} \int_{|r-\rho|}^{r+\rho} \frac{1}{s} \\ & \cdot \frac{\exp[-\Sigma_3(\lambda, \sigma, \dots)] + \alpha(v_1, \sqrt{\cdot}) \exp[-\Sigma_4(\lambda, \sigma, \dots) + \Sigma_3(\lambda, \sigma, \dots)]}{1 - \alpha(v_1, \sqrt{\cdot}) \exp[-\Sigma_4(\lambda, \sigma, \dots)]} ds, \end{aligned} \quad (4.4)$$

with

$$\begin{aligned} r &= \sqrt{x^2 + y^2}, \quad \rho = \sqrt{x'^2 + y'^2}, \\ \alpha(v_1, \sqrt{\cdot}) &= \alpha\left(v_1, \sqrt{r^2 - \Delta_2^2(r, \rho, s)}\right), \\ \Sigma_3(\lambda, \sigma, \dots) &= \int_{\Delta_1(r, \rho, s)}^{\Delta_2(r, \rho, s)} \frac{\lambda + \sigma(\sqrt{t^2 + r^2 - \Delta_2^2(r, \rho, s)}, v_1)}{v_1} dt, \\ \Sigma_4(\lambda, \sigma, \dots) &= \int_{-\Delta(r, \rho, s)}^{\Delta(r, \rho, s)} \frac{\lambda + \sigma(\sqrt{t^2 + r^2 - \Delta_2^2(r, \rho, s)}, v_1)}{v_1} dt, \\ \Delta_1(r, \rho, s) &= -\frac{\rho^2 - r^2}{2s} - \frac{s}{2}, \quad \Delta_2(r, \rho, s) = -\frac{\rho^2 - r^2}{2s} + \frac{s}{2}, \\ \Delta(r, \rho, s) &= \sqrt{R^2 + \frac{1}{4s^2}(\rho^2 - r^2)^2 + \frac{s^2}{4} - \frac{1}{2}(\rho^2 + r^2)}. \end{aligned}$$

**Proof.** Define operators  $J$  and  $H$  by (cf.[42])

$$J : L^2(D \times E) \longrightarrow L^2(V \times E)$$

$$J\psi(r, v) = \frac{1}{2} \int_{-r}^r \psi(z, \sqrt{r^2 - z^2}, v) dz, \quad r = \sqrt{x^2 + y^2},$$

$$H : L^2(V \times E) \longrightarrow L^2(D \times E)$$

$$H\varphi(x, y, v) = \frac{1}{\sqrt{x^2 + y^2}} \int_0^{v_M} k(\sqrt{x^2 + y^2}, v, v') \varphi(\sqrt{x^2 + y^2}, v') dv'.$$

Then  $K = HJ$ . From [42, page 14] it is known that

$$J(\lambda I - B)^{-1} H\psi(r, v) = \int_0^{v_M} dv' \int_0^R d\rho h(\lambda, v, v', r, \rho) \psi(\rho, v'),$$

where

$$\begin{aligned} & h(\lambda, v, v', r, \rho) \\ = & \frac{k(\rho, v, v')}{2v} \int_{|r-\rho|}^{r+\rho} \frac{1}{s} \\ & \cdot \frac{\exp[-\tilde{\Sigma}_3(\lambda, \sigma, \dots)] + \alpha(v, \sqrt{\cdot}) \exp[-\tilde{\Sigma}_4(\lambda, \sigma, \dots) + \tilde{\Sigma}_3(\lambda, \sigma, \dots)]}{1 - \alpha(v, \sqrt{\cdot}) \exp[-\tilde{\Sigma}_4(\lambda, \sigma, \dots)]} ds. \end{aligned}$$

with

$$\begin{aligned} \tilde{\Sigma}_3(\lambda, \sigma, \dots) &= \int_{\Delta_1(r, \rho, s)}^{\Delta_2(r, \rho, s)} \frac{\lambda + \sigma(\sqrt{t^2 + r^2 - \Delta_2^2(r, \rho, s)}, v)}{v} dt, \\ \tilde{\Sigma}_4(\lambda, \sigma, \dots) &= \int_{-\Delta(r, \rho, s)}^{\Delta(r, \rho, s)} \frac{\lambda + \sigma(\sqrt{t^2 + r^2 - \Delta_2^2(r, \rho, s)}, v)}{v} dt. \end{aligned}$$

Thus,

$$\begin{aligned} & J(\lambda I - B)^{-1} HJ\psi(r, v) \\ = & \int_0^{v_M} dv' \int_0^R d\rho h(\lambda, v, v', r, \rho) \cdot \frac{1}{2} \int_{-\rho}^{\rho} \psi(z, \sqrt{\rho^2 - z^2}, v') dz, \end{aligned}$$

and

$$\begin{aligned} & HJ(\lambda I - B)^{-1} HJ\psi(x, y, v) \\ = & \frac{1}{\sqrt{x^2 + y^2}} \int_0^{v_M} dv_1 k(\sqrt{x^2 + y^2}, v, v_1) \end{aligned}$$

$$\begin{aligned}
& \cdot \int_0^{v_M} dv' \int_0^R d\rho h(\lambda, v_1, v', r, \rho) \cdot \frac{1}{2} \int_{-\rho}^{\rho} \psi(z, \sqrt{\rho^2 - z^2}, v') dz \\
= & \int_0^{v_M} dv' \int_0^R d\rho \int_{-\rho}^{\rho} \psi(z, \sqrt{\rho^2 - z^2}, v') dz \\
& \cdot \frac{1}{2\sqrt{x^2 + y^2}} \int_0^{v_M} k(\sqrt{x^2 + y^2}, v, v_1) h(\lambda, v_1, v', r, \rho) dv_1.
\end{aligned}$$

By virtue of the transform

$$x' = z, \quad y' = \sqrt{\rho^2 - z^2},$$

we get

$$\begin{aligned}
& K(\lambda I - B)^{-1} K \psi(x, y, v) \\
= & HJ(\lambda I - B)^{-1} HJ \psi(x, y, v) \\
= & \int_0^{v_M} dv' \iint_D dx' dy' \psi(x', y', v') \frac{y'}{\sqrt{x'^2 + y'^2}} \\
& \cdot \frac{1}{2\sqrt{x^2 + y^2}} \int_0^{v_M} k(\sqrt{x^2 + y^2}, v, v_1) h(\lambda, v_1, v', r, \rho) dv_1 \\
= & \int_0^{v_M} dv' \iint_D dx' dy' k(\lambda, x, x', y, y', v, v') \psi(x', y', v'),
\end{aligned}$$

where  $k(\lambda, x, x', y, y', v, v')$  is just the expression given by Eq. (4.4). This completes the proof.

Q. E. D.

**Lemma 4.2.4.** *Let  $H$  be an integral operator on  $L^2(G)$  with kernel  $h(x, x', y, y', v, v')$ ; then*

$$\|H\|^2 \leq \iiint_G \iiint_G \frac{y}{y'} |h(x, x', y, y', v, v')|^2 dx' dy' dv' dx dy dv.$$

This lemma can be easily verified.

**Theorem 4.2.5.** *Set*

$$\tilde{k}(r, v, v') = v^\delta k(r, v, v')$$

*if Eq. (4.1) in (H3) is satisfied, or*

$$\tilde{k}(r, v, v') = v'^\delta k(r, v, v')$$

*if Eq. (4.2) in (H3) is satisfied. Let  $\lambda = \beta + i\tau$ ,  $\beta \in [\beta_1, \beta_2]$ ,  $\beta_2 > \beta_1 > -\lambda^*$ , and assume the following conditions are satisfied:*

(H4).  $\alpha(v, \mu), \sigma(r, v)$  and  $\tilde{k}(r, v, v')$  are partially differentiable with respect to  $v, v'$ , and the corresponding partial derivatives  $\frac{\partial \alpha}{\partial v}, \frac{\partial \sigma}{\partial v}, \frac{\partial \tilde{k}}{\partial v}$  and  $\frac{\partial \tilde{k}}{\partial v'}$  are uniformly bounded.

Then for every  $\varepsilon > 0$ , there exists a positive constant  $\tilde{\tau}$  independent of  $\beta \in [\beta_1, \beta_2]$ , such that

$$\|K(\lambda I - B)^{-1}K\| \leq \varepsilon$$

uniformly in  $\{\lambda = \beta + i\tau : \beta_1 \leq \beta \leq \beta_2, |\tau| \geq \tilde{\tau}\}$ .

**Proof.** Without loss of any generality, we assume  $\tilde{k}(r, v, v') = v^\delta k(r, v, v')$  in the following proof. A similar procedure can be applied if  $\tilde{k}(r, v, v') = v'^\delta k(r, v, v')$ .

Since  $0 \leq \alpha(v_1, \sqrt{\cdot}) \leq \alpha_0 < 1$ , we have

$$\frac{1}{1 - \alpha(v_1, \sqrt{\cdot}) \exp[-\Sigma_4(\lambda, \sigma, \dots)]} = \sum_{n=0}^{\infty} \alpha^n(v_1, \sqrt{\cdot}) \exp[-n\Sigma_4(\lambda, \sigma, \dots)]. \quad (4.5)$$

For every  $n = 0, 1, 2, \dots$ , set

$$\begin{aligned} & g_{n,1}(\lambda, x, x', y, y', v, v') \\ &= \frac{y'}{4r\rho} \int_0^{v_M} dv_1 \frac{k(r, v, v_1)k(\rho, v_1, v')}{v_1} \\ & \quad \cdot \int_{|r-\rho|}^{r+\rho} \alpha^n(v_1, \sqrt{\cdot}) \exp[-n\Sigma_4(\lambda, \sigma, \dots) - \Sigma_3(\lambda, \sigma, \dots)] s^{-1} ds, \\ & g_{n,2}(\lambda, x, x', y, y', v, v') \\ &= \frac{y'}{4r\rho} \int_0^{v_M} dv_1 \frac{k(r, v, v_1)k(\rho, v_1, v')}{v_1} \\ & \quad \cdot \int_{|r-\rho|}^{r+\rho} \alpha^{n+1}(v_1, \sqrt{\cdot}) \exp[-(n+1)\Sigma_4(\lambda, \sigma, \dots) + \Sigma_3(\lambda, \sigma, \dots)] s^{-1} ds, \end{aligned} \quad (4.6)$$

and define operators  $G_{n,1}, G_{n,2}$  on  $L^2(G)$  by

$$G_{n,j}\psi = \iiint_G g_{n,j}(\lambda, x, x', y, y', v, v')\psi(x', y', v')dx'dy'dv', \quad n = 0, 1, \dots, \quad j = 1, 2.$$

Then from Eqs. (4.3) – (4.5), we get

$$K(\lambda I - B)^{-1}K = \sum_{n=0}^{\infty} G_{n,1} + \sum_{n=0}^{\infty} G_{n,2}. \quad (4.7)$$

First, we consider  $G_{n,2}$ . From Lemma 4.2.4 and Eq. (4.6), we have

$$\begin{aligned} \|G_{n,2}\|^2 &\leq \frac{1}{16} \iint_D dx dy \iint_D dx' dy' \int_E dv \int_E dv' \frac{y}{y'} \\ &\quad \cdot \frac{y'^2}{r^2 \rho^2} \left| \int_0^{v_M} dv_1 \frac{k(r, v, v_1) k(\rho, v_1, v')}{v_1} \right. \\ &\quad \left. \cdot \int_{|r-\rho|}^{r+\rho} \alpha^{n+1}(v_1, \sqrt{\cdot}) \exp[-(n+1)\Sigma_4(\lambda, \sigma, \dots) + \Sigma_3(\lambda, \sigma, \dots)] s^{-1} ds \right|^2. \end{aligned}$$

By virtue of the transform

$$x = z, \quad y = \sqrt{r^2 - z^2}; \quad x' = z', \quad y' = \sqrt{\rho^2 - z'^2},$$

we get

$$\begin{aligned} \|G_{n,2}\|^2 &\leq \frac{1}{16} \int_0^R dr \int_{-r}^r \frac{r}{y} dz \int_0^R d\rho \int_{-\rho}^{\rho} \frac{\rho}{y'} dz' \int_E dv \int_E dv' \frac{y}{y'} \\ &\quad \cdot \frac{y'^2}{r^2 \rho^2} \left| \int_0^{v_M} dv_1 \frac{k(r, v, v_1) k(\rho, v_1, v')}{v_1} \right. \\ &\quad \left. \cdot \int_{|r-\rho|}^{r+\rho} \alpha^{n+1}(v_1, \sqrt{\cdot}) \exp[-(n+1)\Sigma_4(\lambda, \sigma, \dots) + \Sigma_3(\lambda, \sigma, \dots)] s^{-1} ds \right|^2 \\ &= \frac{1}{4} \int_E dv \int_E dv' \int_0^R dr \int_0^R d\rho \left| \int_{|r-\rho|}^{r+\rho} F(n, \lambda, r, \rho, \dots) s^{-1} ds \right|^2, \end{aligned} \quad (4.8)$$

where

$$\begin{aligned} &F(n, \lambda, r, \rho, \dots) \\ &= \int_0^{v_M} \frac{k(r, v, v_1) k(\rho, v_1, v')}{v_1} \\ &\quad \cdot \alpha^{n+1}(v_1, \sqrt{\cdot}) \exp[-(n+1)\Sigma_4(\lambda, \sigma, \dots) + \Sigma_3(\lambda, \sigma, \dots)] dv_1. \end{aligned} \quad (4.9)$$

By virtue of the technique of integration by parts, we obtain the following estimation of  $F(n, \lambda, r, \rho, \dots)$  from the above equation:

$$|F(n, \lambda, r, \rho, \dots)| \leq \frac{C_1 \alpha_0^n}{|\tau| \cdot |r - \rho| v^\delta}, \quad (4.10)$$

where  $C_1$  as well as all the notations  $C_i$  ( $i = 2, 3, \dots$ ) arising in the following are positive constants.

On the other hand, using the fact that the function  $\frac{1}{\log u} \int_1^\infty \frac{e^{-ut}}{t} dt$  is analytic in  $\{u \in \mathbb{C} : 0 \leq \operatorname{Re} u < 1\}$ , we get the following estimation when  $|\lambda + \lambda^*| \cdot |r - \rho| \leq 1/2$ .

$$\begin{aligned} & |F(n, \lambda, r, \rho, \dots)| \\ & \leq C_2 \alpha_0^n (\operatorname{Re} \lambda + \lambda^*)^{-\delta} |r - \rho|^{-\delta} v^{-\delta} \\ & \quad \cdot \{|\log(|\lambda + \lambda^*| \cdot |r - \rho|)| + |\log |\lambda + \lambda^*|| + |\log(\operatorname{Re} \lambda + \lambda^*)| + 1\}. \end{aligned} \quad (4.11)$$

The proof of Eqs. (4.10) and (4.11) will be given in Sec. 4.4.

From Eqs. (4.10) and (4.11), we see that  $|F(n, \lambda, r, \rho, \dots)|$  is dominated by a function  $P$  independent of  $s$ , i.e.,

$$|F(n, \lambda, r, \rho, \dots)| \leq P(n, \lambda, |r - \rho|, v). \quad (4.12)$$

From Eq. (4.8), we know

$$\begin{aligned} & \|G_{n,2}\|^2 \\ & \leq \frac{1}{4} \int_E dv \int_E dv' \int_0^R dr \int_0^R d\rho \left[ \int_{|r-\rho|}^{r+\rho} s^{-1} P(n, \lambda, |r - \rho|, v) ds \right]^2 \\ & \leq \frac{1}{4} \int_E dv \int_E dv' \int_0^R dr \int_0^R d\rho \left[ \int_{|r-\rho|}^{2R} s^{-1} P(n, \lambda, |r - \rho|, v) ds \right]^2 \\ & \leq \frac{1}{4} \int_E dv \int_E dv' \int_0^R dr \int_0^R d\rho [\ln(2R) - \ln|r - \rho|]^2 P^2(n, \lambda, |r - \rho|, v) \\ & \leq C_3 \int_E dv \int_E dv' \int_0^R dr \int_0^R d\rho [C_4 + \log^2 |r - \rho|] P^2(n, \lambda, |r - \rho|, v). \end{aligned}$$

Consider the transformation

$$u = r + \rho, \quad w = r - \rho.$$

The domain  $\{(r, \rho) : r, \rho \in [0, R]\}$  is then transformed into

$$S := \{(u, w) : 0 \leq u - w \leq 2R, 0 \leq u + w \leq 2R\},$$

and

$$\begin{aligned} & \|G_{n,2}\|^2 \\ & \leq \frac{C_3}{2} \int_E dv \int_E dv' \iint_S dudw (C_4 + \log^2 |w|) P^2(n, \lambda, |w|, v) \end{aligned}$$

$$\begin{aligned}
&= C_3 \int_E dv \int_E dv' \iint_{S \cap \{(u,w): w \geq 0\}} dudw (C_4 + \log^2 w) P^2(n, \lambda, w, v) \\
&= C_3 \int_E dv \int_E dv' \int_0^R dw \int_w^{2R-w} du (C_4 + \log^2 w) P^2(n, \lambda, w, v) \\
&\leq C_5 \int_E dv \int_E dv' \int_0^R (C_4 + \log^2 w) P^2(n, \lambda, w, v) dw. \tag{4.13}
\end{aligned}$$

Let  $W_1 = \{w \in [0, R] : |\lambda + \lambda^*|w \leq 1/2\}$ ,  $W_2 = \{w \in [0, R] : |\lambda + \lambda^*|w > 1/2\}$ . Since

$$\begin{aligned}
\log^2 w &= \log^2(|\lambda + \lambda^*|w \cdot |\lambda + \lambda^*|^{-1}) \\
&\leq 2 \log^2(|\lambda + \lambda^*|w) + 2 \log^2|\lambda + \lambda^*|,
\end{aligned}$$

it follows from Eq. (4.13) and Eqs. (4.10) – (4.12) that

$$\begin{aligned}
&\|G_{n,2}\|^2 \\
&\leq 2C_5 \int_E dv \int_E dv' \int_0^R [\log^2(|\lambda + \lambda^*|w) + \log^2|\lambda + \lambda^*| + C_4] P^2(n, \lambda, w, v) dw \\
&\leq C_6 \alpha_0^{2n} (\beta + \lambda^*)^{-2\delta} \int_E v^{-2\delta} dv \int_E dv' \int_{W_1} [\log^2(|\lambda + \lambda^*|w) + \log^2|\lambda + \lambda^*| + C_4] \\
&\quad \cdot [4 \log^2(|\lambda + \lambda^*|w) + 4 \log^2|\lambda + \lambda^*| + 4 \log^2(\beta + \lambda^*) + 4] w^{-2\delta} dw \\
&\quad + C_7 \alpha_0^{2n} |\tau|^{-2} \int_E v^{-2\delta} dv \int_E dv' \int_{W_2} [\log^2(|\lambda + \lambda^*|w) + \log^2|\lambda + \lambda^*| + C_4] w^{-2} dw.
\end{aligned}$$

Applying the transformation  $z(w) = |\lambda + \lambda^*|w$  to the right-hand side of the above equation, we get

$$\begin{aligned}
&\|G_{n,2}\|^2 \\
&\leq C_8 \alpha_0^{2n} (\beta + \lambda^*)^{-2\delta} \int_0^{1/2} [\log^2 z + \log^2|\beta + \lambda^* + i\tau| + C_4] \\
&\quad \cdot [\log^2 z + \log^2|\beta + \lambda^* + i\tau| + \log^2(\beta + \lambda^*) + 1] |\beta + \lambda^* + i\tau|^{2\delta-1} z^{-2\delta} dz \\
&\quad + C_9 \alpha_0^{2n} |\tau|^{-2} \int_{1/2}^\infty [\log^2 z + \log^2|\beta + \lambda^* + i\tau| + C_4] |\beta + \lambda^* + i\tau| z^{-2} dz. \tag{4.14}
\end{aligned}$$

By a similar procedure, we can get a similar estimation about  $\|G_{n,1}\|$ . From Eqs. (4.7) and (4.14), the proof is completed. Q. E. D.

**Lemma 4.2.6.** *Suppose  $f \in L^2(G)$  and  $m - f_1 \leq f \leq M - f_2$ , where  $f_1, f_2 \in L^2(G)$ , and  $m$  and  $M$  are constants. Then for every positive constant  $\eta$ , there exists a sequence  $\{g_n\}$  composed of polynomial functions such that  $g_n$  converges to  $f$  almost everywhere and*

$$m - \eta - f_1 \leq g_n \leq M + \eta - f_2 \text{ on } G.$$

**Proof.** For any  $\varepsilon > 0$ , there exists a polynomial function  $w$  such that  $\|w - f\| < \varepsilon/2$ . Defining a function on  $\overline{G}$  as

$$w_1 = \begin{cases} w, & \text{if } m - f_1 \leq w \leq M - f_2, \\ m - f_1, & \text{if } w < m - f_1, \\ M - f_2, & \text{if } w > M - f_2, \end{cases}$$

then  $\|w_1 - f\| \leq \|w - f\| < \varepsilon/2$ . It is easy to see that  $w_1$  is continuous on  $\overline{G}$ , hence a polynomial function  $g$  exists such that

$$\max |g - w_1| < \max\{\eta, C\varepsilon\},$$

where  $C$  is a suitable constant such that  $\|g - w_1\| < \varepsilon/2$ . Therefore, for any  $\varepsilon > 0$ ,

$$\|g - f\| \leq \|g - w_1\| + \|w_1 - f\| < \varepsilon.$$

From the above discussion, we know that for  $\varepsilon_n = 1/n$ ,  $n = 1, 2, \dots$ , there exists a polynomial function  $g_n$  such that  $\|g_n - f\| < 1/n$  and

$$\max |g_n - w_1| < \eta,$$

which together with the definition of  $w_1$  implies

$$m - \eta - f_1 \leq g_n \leq M + \eta - f_2 \text{ on } G.$$

Since  $\lim_{n \rightarrow \infty} \|g_n - f\| = 0$ , it is known from the theory of real analysis that a subsequence of  $\{g_n\}$  exists such that it converges to  $f$  almost everywhere in the domain  $G$ . This completes the proof. Q. E. D.

**Theorem 4.2.7.** *The conclusion given in Theorem 4.2.5 still holds if the hypothesis (H4) is not satisfied.*

**Proof.** Since  $\alpha(v, \mu)$ ,  $\sigma(r, v)$  and  $\tilde{k}(r, v, v')$  are bounded measurable, it follows from Lemma 4.2.6 that (set  $f_1 = f_2 = 0$ ) there exist three sequences  $\{\alpha_n(v, \mu)\}$ ,  $\{\sigma_n(r, v)\}$  and  $\{\tilde{k}_n(r, v, v')\}$  such that

- (a). for every  $n$ ,  $\alpha_n(v, \mu)$ ,  $\sigma_n(r, v)$  and  $\tilde{k}_n(r, v, v')$  are polynomial functions;
- (b).  $|\alpha_n(v, \mu)| \leq \alpha_0 + (1 - \alpha_0)/2 < 1$ , and  $\alpha_n(v, \mu)$  converges to  $\alpha(v, \mu)$  almost everywhere in  $E \times \Omega$ ;
- (c).  $(\lambda^* - \beta_1)/2 \leq \sigma_n(r, v) \leq \text{ess sup}_{(r,v)} \sigma(r, v) + (\beta_1 + \lambda^*)/2$ , and  $\sigma_n(r, v)$  converges to  $\sigma(r, v)$  almost everywhere in  $V \times E$ ;
- (d).  $|\tilde{k}_n(r, v, v')| \leq \text{ess sup}_{(r,v,v')} \tilde{k}(r, v, v') + 1$ , and  $\tilde{k}_n(r, v, v')$  converges to  $\tilde{k}(r, v, v')$  almost everywhere in  $V \times E \times E$ .

Set

$$k_n(r, v, v') = \begin{cases} v^{-\delta} \tilde{k}_n(r, v, v'), & \text{if } \tilde{k}(r, v, v') = v^\delta k(r, v, v'), \\ v'^{-\delta} \tilde{k}_n(r, v, v'), & \text{if } \tilde{k}(r, v, v') = v'^\delta k(r, v, v'), \end{cases}$$

$$\delta k_n(r, v, v') = k(r, v, v') - k_n(r, v, v'),$$

and define operators on  $L^2(G)$  as follows:

$$B_n \psi = -v \frac{\partial \psi}{\partial x} - \sigma_n(r, v) \psi,$$

$$K_n \psi = \frac{1}{2r} \int_E dv' \int_{-r}^r k_n(r, v, v') \psi(z, \sqrt{r^2 - z^2}, v') dz,$$

$$\delta K_n \psi = \frac{1}{2r} \int_E dv' \int_{-r}^r \delta k_n(r, v, v') \psi(z, \sqrt{r^2 - z^2}, v') dz,$$

with  $D(K_n) = D(\delta K_n) = L^2(G)$ ,  $D(B_n) = \{\psi \in L^2(G) : B_n \psi \in L^2(G) \text{ and } \psi(-\sqrt{R^2 - y^2}, y, v) = \alpha_n(v, -\sqrt{1 - R^{-2}y^2}) \psi(\sqrt{R^2 - y^2}, y, v) \text{ for every } y \in V \text{ and } v \in E\}$ . Obviously,  $K = K_n + \delta K_n$ .

Similar to that of Lemma 4.2.1, it can be shown that  $\{\lambda : \text{Re} \lambda \geq \beta_1\} \subset \rho(B_n)$ , and  $\|(\lambda I - B_n)^{-1}\| \leq 2(\beta_1 + \lambda^*)^{-1}$  for every  $\lambda$  with  $\text{Re} \lambda \geq \beta_1$ .

For every  $\lambda$  with  $\text{Re} \lambda > -\lambda^*$ , similar to the expression of  $K(\lambda I - B)^{-1}K$  (cf. Eqs. (4.3) and (4.4)), it can be shown that:

(a).  $\delta K_n(\lambda I - B_n)^{-1}K$  is an integral operator on  $L^2(G)$  with the integral kernel given by

$$\begin{aligned} & h_{n,1}(\lambda, x, x', y, y', v, v') \\ = & \frac{y'}{4r\rho} \int_0^{v_M} dv_1 \frac{\delta k_n(r, v, v_1)k(\rho, v_1, v')}{v_1} \int_{|r-\rho|}^{r+\rho} \frac{1}{s} \\ & \cdot \frac{\exp[-\Sigma_3(\lambda, \sigma_n, \dots)] + \alpha_n(v_1, \sqrt{\cdot}) \exp[-\Sigma_4(\lambda, \sigma_n, \dots) + \Sigma_3(\lambda, \sigma_n, \dots)]}{1 - \alpha_n(v_1, \sqrt{\cdot}) \exp[-\Sigma_4(\lambda, \sigma_n, \dots)]} ds, \end{aligned}$$

where

$$\begin{aligned} \alpha_n(v_1, \sqrt{\cdot}) &= \alpha_n \left( v_1, \sqrt{r^2 - \Delta_2^2(r, \rho, s)} \right), \\ \Sigma_3(\lambda, \sigma_n, \dots) &= \int_{\Delta_1(r, \rho, s)}^{\Delta_2(r, \rho, s)} \frac{\lambda + \sigma_n(\sqrt{t^2 + r^2 - \Delta_2^2(r, \rho, s)}, v_1)}{v_1} dt, \\ \Sigma_4(\lambda, \sigma_n, \dots) &= \int_{-\Delta(r, \rho, s)}^{\Delta(r, \rho, s)} \frac{\lambda + \sigma_n(\sqrt{t^2 + r^2 - \Delta_2^2(r, \rho, s)}, v_1)}{v_1} dt. \end{aligned}$$

(b).  $K_n(\lambda I - B_n)^{-1}\delta K_n$  is an integral operator on  $L^2(G)$  with the kernel

$$\begin{aligned} & h_{n,2}(\lambda, x, x', y, y', v, v') \\ = & \frac{y'}{4r\rho} \int_0^{v_M} dv_1 \frac{k_n(r, v, v_1)\delta k_n(\rho, v_1, v')}{v_1} \int_{|r-\rho|}^{r+\rho} \frac{1}{s} \\ & \cdot \frac{\exp[-\Sigma_3(\lambda, \sigma_n, \dots)] + \alpha_n(v_1, \sqrt{\cdot}) \exp[-\Sigma_4(\lambda, \sigma_n, \dots) + \Sigma_3(\lambda, \sigma_n, \dots)]}{1 - \alpha_n(v_1, \sqrt{\cdot}) \exp[-\Sigma_4(\lambda, \sigma_n, \dots)]} ds. \end{aligned}$$

(c).  $K(\lambda I - B)^{-1}K - K(\lambda I - B_n)^{-1}K$  is an integral operator on  $L^2(G)$  with the kernel

$$\begin{aligned} & h_{n,3}(\lambda, x, x', y, y', v, v') \\ = & \frac{y'}{4r\rho} \int_0^{v_M} dv_1 \frac{k(r, v, v_1)k(\rho, v_1, v')}{v_1} \int_{|r-\rho|}^{r+\rho} \frac{1}{s} \\ & \cdot \left\{ \frac{\exp[-\Sigma_3(\lambda, \sigma, \dots)] + \alpha(v_1, \sqrt{\cdot}) \exp[-\Sigma_4(\lambda, \sigma, \dots) + \Sigma_3(\lambda, \sigma, \dots)]}{1 - \alpha(v_1, \sqrt{\cdot}) \exp[-\Sigma_4(\lambda, \sigma, \dots)]} \right. \\ & \left. - \frac{\exp[-\Sigma_3(\lambda, \sigma_n, \dots)] + \alpha_n(v_1, \sqrt{\cdot}) \exp[-\Sigma_4(\lambda, \sigma_n, \dots) + \Sigma_3(\lambda, \sigma_n, \dots)]}{1 - \alpha_n(v_1, \sqrt{\cdot}) \exp[-\Sigma_4(\lambda, \sigma_n, \dots)]} \right\} ds. \end{aligned}$$

From Lemma 4.2.4, by virtue of the transform

$$x = z, \quad y = \sqrt{r^2 - z^2}; \quad x' = z', \quad y' = \sqrt{\rho^2 - z'^2}$$

and Lebesgue's dominated convergence theorem, it is not difficult to see that  $\|\delta K_n(\lambda I - B_n)^{-1}K\|$ ,  $\|K_n(\lambda I - B_n)^{-1}\delta K_n\|$  and  $\|K(\lambda I - B)^{-1}K - K(\lambda I - B_n)^{-1}K\|$  converge to 0 uniformly in  $\{\lambda : \operatorname{Re}\lambda \geq \beta_1\}$ . (The proof is tedious and hence is omitted). Thus, for any  $\varepsilon > 0$ , there exists an integer  $n_0$  such that

$$\|\delta K_{n_0}(\lambda I - B_{n_0})^{-1}K\| < \varepsilon/4, \quad (4.15)$$

$$\|K_{n_0}(\lambda I - B_{n_0})^{-1}\delta K_{n_0}\| < \varepsilon/4, \quad (4.16)$$

$$\|K(\lambda I - B)^{-1}K - K(\lambda I - B_{n_0})^{-1}K\| < \varepsilon/4. \quad (4.17)$$

Since  $\alpha_{n_0}(v, \mu)$ ,  $\sigma_{n_0}(r, v)$  and  $\tilde{k}_{n_0}(r, v, v')$  are polynomial functions, it follows from Theorem 4.2.5 that there exists a positive constant  $\tilde{\tau}$  independent of  $\beta \in [\beta_1, \beta_2]$ , such that

$$\|K_{n_0}(\lambda I - B_{n_0})^{-1}K_{n_0}\| < \varepsilon/4 \quad (4.18)$$

uniformly in  $\{\lambda = \beta + i\tau : \beta_1 \leq \beta \leq \beta_2, |\tau| \geq \tilde{\tau}\}$ .

From the relation

$$\begin{aligned} & K(\lambda I - B)^{-1}K \\ = & [K(\lambda I - B)^{-1}K - K(\lambda I - B_{n_0})^{-1}K] \\ & + (K_{n_0} + \delta K_{n_0})(\lambda I - B_{n_0})^{-1}(K_{n_0} + \delta K_{n_0}) \\ = & [K(\lambda I - B)^{-1}K - K(\lambda I - B_{n_0})^{-1}K] + K_{n_0}(\lambda I - B_{n_0})^{-1}K_{n_0} \\ & + \delta K_{n_0}(\lambda I - B_{n_0})^{-1}K + K_{n_0}(\lambda I - B_{n_0})^{-1}\delta K_{n_0} \end{aligned}$$

and Eqs. (4.15) – (4.18), the proof is completed. Q. E. D.

In fact, some better results than that of Theorem 4.2.7 can be obtained, i.e.,

**Theorem 4.2.8.** *Suppose (H1) – (H3) are satisfied and let  $\beta_2 > \beta_1 > -\lambda^*$  be two constants. If Eq. (4.1) in (H3) holds, then for every  $\varepsilon > 0$ , there exists a positive constant  $\tilde{\tau}$  independent of  $\beta \in [\beta_1, \beta_2]$  such that*

$$\|K(\lambda I - B)^{-1}\| \leq \varepsilon \quad (4.19)$$

*uniformly in the domain  $D_{\tilde{\tau}} := \{\lambda = \beta + i\tau : \beta_1 \leq \beta \leq \beta_2, |\tau| \geq \tilde{\tau}\}$ . If Eq. (4.2) in (H3) holds, then*

$$\|(\lambda I - B)^{-1}K\| \leq \varepsilon \quad (4.20)$$

uniformly in  $D_{\tilde{\tau}}$ . If the constant  $\delta$  in Eq. (4.1) or Eq. (4.2) is less than  $1/4$ , then

$$\|K(\lambda I - B)^{-1}\| \leq \varepsilon \quad \text{and} \quad \|(\lambda I - B)^{-1}K\| \leq \varepsilon \quad (4.21)$$

uniformly in  $D_{\tilde{\tau}}$ .

**Proof.** Similar to that of [33, page 713 and 719], it is easy to show

$$\|K(\lambda I - B)^{-1}\| \leq [(\operatorname{Re}\lambda + \lambda^*)^{-1}\|K(\lambda I - B)^{-1}K^*\|]^{1/2} \quad (4.22)$$

$$\|(\lambda I - B)^{-1}K\| \leq [(\operatorname{Re}\lambda + \lambda^*)^{-1}\|K^*(\lambda I - B)^{-1}K\|]^{1/2} \quad (4.23)$$

for every  $\lambda$  with  $\operatorname{Re}\lambda > -\lambda^*$ , where  $K^*$  is the adjoint operator of  $K$  given by

$$K^*\psi = \frac{1}{2r} \int_0^{v_M} dv' \int_{-r}^r k(r, v', v)\psi(z, \sqrt{r^2 - z^2}, v')dz.$$

If Eq. (4.1) in (H3) is satisfied, then by a procedure similar to that of Theorem 4.2.5 and Theorem 4.2.7, it can be shown that for every  $\tilde{\varepsilon} > 0$ , there exists  $\tilde{\tau} > 0$  independent of  $\beta$  such that

$$\|K(\lambda I - B)^{-1}K^*\| < \tilde{\varepsilon} \quad (4.24)$$

uniformly in  $D_{\tilde{\tau}} := \{\lambda = \beta + i\tau : \beta_1 \leq \beta \leq \beta_2, |\tau| \geq \tilde{\tau}\}$ . This together with Eq. (4.22) implies Eq. (4.19).

If Eq. (4.2) in (H3) is satisfied, then we consider  $K^*(\lambda I - B)^{-1}K$ , and it can be shown that for every  $\tilde{\varepsilon} > 0$ , there exists  $\tilde{\tau} > 0$  independent of  $\beta$  such that

$$\|K^*(\lambda I - B)^{-1}K\| < \tilde{\varepsilon} \quad (4.25)$$

uniformly in  $D_{\tilde{\tau}}$ . This together with Eq. (4.23) implies Eq. (4.20).

If the constant  $\delta$  in Eq. (4.1) or Eq. (4.2) is less than  $1/4$ , then it can be shown that both Eqs. (4.24) and (4.25) hold in this case, and so do Eqs. (4.19) and (4.20). This completes the proof. Q. E. D.

By virtue of Weierstrass's accumulation principle, we get the main result of this section from Lemma 4.2.2 and Theorem 4.2.7, i.e.,

**Theorem 4.2.9.**  $\sigma(A) \cap \{\lambda : \beta_1 \leq \operatorname{Re}\lambda \leq \beta_2\}$  contains at most finitely many elements, each of which is an eigenvalue of  $A$  with finite algebraic multiplicity. The accumulation points of  $\operatorname{Pas}(A)$  could only appear on the line  $\operatorname{Re}\lambda = -\lambda^*$ .

### 4.3 Spectral Properties of $T(t)$ in $L^2$

From [42], it is known that both  $B$  and  $A$  generate strongly continuous semigroups in  $L^2(G)$ , which are denoted by  $S(t)$  and  $T(t)$  respectively. This section will be devoted to discussing some aspects of the spectral properties of  $T(t)$ , and these spectral properties are closely linked to the asymptotic behavior of the solution  $\psi(t)$  of Eq. (I).

**Lemma 4.3.1.** [30, 13] *Let  $T(t)$  be the strongly continuous semigroup generated by the operator  $A$  in a Hilbert space  $H$ , with  $\omega_0(A)$  the growth bound of  $T(t)$ , and  $s_0(A)$  the spectral bound of  $A$ , i.e.,*

$$\omega_0(A) = \lim_{t \rightarrow +\infty} t^{-1} \log \|T(t)\|, \quad s_0(A) = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}.$$

*Then  $\omega_0(A) = s_0(A)$  if and only if for every  $\varepsilon > 0$ , there exists a positive constant  $M_\varepsilon$  such that  $\|(\lambda I - A)^{-1}\| \leq M_\varepsilon$  uniformly in  $\{\lambda : \operatorname{Re}\lambda \geq s_0(A) + \varepsilon\}$ .*

From Theorem 4.2.9, the eigenvalues of  $A$  lying in the half-plane  $\operatorname{Re}\lambda > -\lambda^*$  can be ordered in such a way that the real part decreases. Suppose  $\lambda_1, \lambda_2, \dots, \lambda_m, \lambda_{m+1}, \dots$  are eigenvalues of  $A$ ,  $\operatorname{Re}\lambda_1 \geq \operatorname{Re}\lambda_2 \geq \dots \geq \operatorname{Re}\lambda_m > \operatorname{Re}\lambda_{m+1} \geq \dots > -\lambda^*$ , and  $\{\lambda : \operatorname{Re}\lambda > -\lambda^*\} \setminus \{\lambda_n : n = 1, 2, \dots\} \subset \rho(A)$ .

For every integer  $m$  satisfying  $\operatorname{Re}\lambda_m > \operatorname{Re}\lambda_{m+1}$ , let

$$\sigma_1 := \{\lambda_1, \lambda_2, \dots, \lambda_m\}, \quad \sigma_2 := \sigma(A) \setminus \sigma_1.$$

Since  $\sigma_1$  is a compact set, it follows from [23, page 70] that there exists a unique spectral decomposition  $L^2(G) = H_1 \oplus H_2$  such that  $T_i(t)$ , the part of  $T(t)$  in  $H_i$  ( $i = 1, 2$ ), is a strongly continuous semigroup. Furthermore, the spectral set of  $A_i$  (where  $A_i$  is the generator of  $T_i(t)$ ) is equal to  $\sigma_i$ , i.e.,  $\sigma(A_i) = \sigma_i$ ,  $i = 1, 2$ , and  $A_1$  is a bounded operator on  $H_1$ . Denoting by  $P$  the projection operator of  $\sigma_1$  corresponding to  $A$ , then (cf. [23, page 70])

$$\begin{aligned} T_1(t) &= T(t)P, \quad T_2(t) = T(t)(I - P), \\ (\lambda I - A_1)^{-1} &= (\lambda I - A)^{-1}P, \quad (\lambda I - A_2)^{-1} = (\lambda I - A)^{-1}(I - P), \\ \sigma(T(t)) &= \sigma(T_1(t)) \cup \sigma(T_2(t)). \end{aligned} \tag{4.26}$$

Since  $A_1$  is a bounded operator on  $H_1$ , we have

$$\sigma(T_1(t)) = \{\exp(\lambda_n t) : n = 1, 2, \dots, m\}. \quad (4.27)$$

**Lemma 4.3.2.** *For every  $\varepsilon > 0$ , there exists a positive constant  $M_{m,\varepsilon}$  such that  $\|(\lambda I - A_2)^{-1}\| \leq M_{m,\varepsilon}$  uniformly in  $\{\lambda : \operatorname{Re}\lambda \geq \operatorname{Re}\lambda_{m+1} + \varepsilon\}$ .*

**Proof.** From  $\sigma(A_2) = \sigma_2$  and the definition of  $\sigma_2$ , it is seen that the spectral bound of  $A_2$  is  $\operatorname{Re}\lambda_{m+1}$ . Thus,  $(\lambda I - A_2)^{-1}$  is a bounded operator for every  $\lambda$  with  $\operatorname{Re}\lambda > \operatorname{Re}\lambda_{m+1}$ .

For every  $\varepsilon > 0$ , let  $\beta_1 = \operatorname{Re}\lambda_{m+1} + \varepsilon$ ,  $\beta_2 = \|K\| - \lambda^* + 1$ ,  $K_\lambda := (\lambda I - B)^{-1}K$ . From Theorem 4.2.7, there exists  $\tilde{\tau} > 0$  independent of  $\beta \in [\beta_1, \beta_2]$  such that  $\|K_\lambda^2\| < 1/2$  uniformly in the domain  $D_1 := \{\lambda = \beta + i\tau : \beta_1 \leq \beta \leq \beta_2, |\tau| \geq \tilde{\tau}\}$ .

Obviously,  $(I - K_\lambda^2)^{-1}$  exists in  $D_1$ , and

$$(I - K_\lambda^2)^{-1} = \sum_{n=0}^{\infty} K_\lambda^{2n}, \quad (4.28)$$

$$\|(I - K_\lambda^2)^{-1}\| \leq \sum_{n=0}^{\infty} \|K_\lambda^2\|^n < 2. \quad (4.29)$$

From the relation  $(I + K_\lambda)(I - K_\lambda^2)^{-1} = (I - K_\lambda^2)^{-1}(I + K_\lambda)$  (which can be verified from Eq. (4.28)) and  $(I - K_\lambda)(I + K_\lambda) = I - K_\lambda^2$ , we have

$$(I - K_\lambda)^{-1} = (I - K_\lambda^2)^{-1}(I + K_\lambda). \quad (4.30)$$

Therefore,

$$(\lambda I - A)^{-1} = (\lambda I - B - K)^{-1} = [(\lambda I - B)(I - K_\lambda)]^{-1} = (I - K_\lambda)^{-1}(\lambda I - B)^{-1}. \quad (4.31)$$

From Eqs. (4.29) – (4.31), it follows that  $\|(\lambda I - A_2)^{-1}\| = \|(\lambda I - A)^{-1}(I - P)\|$  is uniformly bounded in  $D_1$ .

Set  $D_2 := \{\lambda = \beta + i\tau : \beta_1 \leq \beta \leq \beta_2, |\tau| \leq \tilde{\tau}\}$ ; then  $D_2 \subset \rho(A_2)$  and  $\|(\lambda I - A_2)^{-1}\|$  is continuous with respect to  $\lambda \in D_2$ . Noting that  $D_2$  is a compact set, we know that  $\|(\lambda I - A_2)^{-1}\|$  is uniformly bounded in  $D_2$ . Therefore,  $\|(\lambda I - A_2)^{-1}\|$  is uniformly bounded in  $D_1 \cup D_2 = \{\lambda : \beta_1 \leq \operatorname{Re}\lambda \leq \beta_2\}$ .

It is easy to show  $\|(\lambda I - A)^{-1}\| \leq (\operatorname{Re}\lambda + \lambda^* - \|K\|)^{-1} \leq 1$  for every  $\lambda \in \{\lambda : \operatorname{Re}\lambda \geq \beta_2\}$ , and so is  $\|(\lambda I - A_2)^{-1}\|$ . This completes the proof. Q. E. D.

From Lemma 4.3.1 and Lemma 4.3.2, it is known that the growth bound of  $T_2(t)$  is  $\operatorname{Re}\lambda_{m+1}$ , and thereby the spectral radius of  $T_2(t)$  is  $\operatorname{Re}\lambda_{m+1}$ . Since  $m$  can be selected arbitrarily, we get the following conclusion.

**Theorem 4.3.3.** *The spectrum of  $T(t)$  outside the disk  $\{\lambda : |\lambda| \leq \exp(-\lambda^*t)\}$  consists of isolated eigenvalues of  $T(t)$  with finite algebraic multiplicity, and the accumulation points of the set  $\sigma(T(t)) \cap \{\lambda : |\lambda| > \exp(-\lambda^*t)\}$  could only appear on the circle  $\{\lambda : |\lambda| = \exp(-\lambda^*t)\}$ .*

**Corollary 4.3.4.** *Suppose  $\lambda_1, \lambda_2, \dots, \lambda_m, \lambda_{m+1}, \dots$  are eigenvalues of  $A$ ,  $\operatorname{Re}\lambda_1 \geq \operatorname{Re}\lambda_2 \geq \dots \geq \operatorname{Re}\lambda_m > \operatorname{Re}\lambda_{m+1} \geq \dots > -\lambda^*$ , and  $\{\lambda : \operatorname{Re}\lambda > -\lambda^*\} \setminus \{\lambda_n : n = 1, 2, \dots\} \subset \rho(A)$ . Furthermore, let  $P_n(t)$  be the projection operator associated with  $\lambda_n$  to  $A$ , i.e.,*

$$P_n(t) = \sum_{j=1}^{k_n} \frac{t^{j-1} B_j^{(\lambda_n)}}{(j-1)!},$$

where  $k_n$  is the algebraic multiplicity of  $\lambda_n$ ,

$$B_1^{(\lambda_n)} = \frac{1}{2\pi i} \oint_{C_{\lambda_n}} (\lambda I - A)^{-1} d\lambda, \quad C_{\lambda_n} = \{\lambda : |\lambda - \lambda_n| = r_{\lambda_n}\},$$

$$B_{j+1}^{(\lambda_n)} = (A - \lambda_n I) B_j^{(\lambda_n)}, \quad j = 1, 2, \dots, k_n - 1,$$

$r_{\lambda_n}$  is sufficiently small such that  $\{\lambda : |\lambda - \lambda_n| \leq r_{\lambda_n}\} \cap \sigma(A) = \{\lambda_n\}$ . Then for every  $b \in (\operatorname{Re}\lambda_{m+1}, \operatorname{Re}\lambda_m)$  and  $\psi_0 \in L^2(G)$ ,

$$\left\| T(t)\psi_0 - \sum_{n=0}^m \exp(\lambda_n t) P_n(t)\psi_0 \right\| \leq \exp(bt)\|\psi_0\|.$$

With additional assumptions, the dominant eigenvalue  $\beta_0$  of  $A$  exists and  $\beta_0 > -\lambda^*$  (cf. [42]).

In this case, we have

**Corollary 4.3.5.** *Suppose  $\beta_0 > -\lambda^*$  is the dominant eigenvalue of  $A$ ; then for every  $b < \beta_0$  and  $\psi_0 \in L^2(G)$ ,*

$$\|T(t)\psi_0 - \exp(\beta_0 t)(\psi_0, \psi_{\beta_0}^*)(\psi_{\beta_0}, \psi_{\beta_0}^*)^{-1}\psi_{\beta_0}\| \leq \exp(bt)\|\psi_0\|.$$

As  $t \rightarrow +\infty$ , the solution  $\psi(t)$  of Eq. (I) tends to 0,  $+\infty$ , or  $(\psi_0, \psi_{\beta_0}^*)(\psi_{\beta_0}, \psi_{\beta_0}^*)^{-1}\psi_{\beta_0}$ , depending on  $\beta_0 < 0$ ,  $\beta_0 > 0$  and  $\beta_0 = 0$  respectively.

## 4.4 Some Remarks and Proof of Eqs. (4.10), (4.11)

**Remark 4.4.1.** All the conclusions obtained in the previous sections still hold if the velocity domain  $E = (0, v_M]$  is replaced by  $E = [v_m, v_M]$ , where  $0 < v_m < v_M < +\infty$ .

**Remark 4.4.2.** Suppose there exists a constant  $c \geq 0$  such that

$$\sigma(r, v) \geq \lambda_0^* - cv,$$

where  $\lambda_0^* = \text{ess inf}_{r \in V} \lim_{v \rightarrow 0^+} \sigma(r, v)$  (cf. [18, page 1988]). Then all the conclusions except Theorem 4.2.8 and Lemma 4.2.1 obtained in the previous sections still hold if  $\lambda^*$  (the infimum of  $\sigma(r, v)$  on  $V \times E$ ) is replaced by  $\lambda_0^*$ .

This conclusion can be obtained by making some amendments in the proof of Theorem 4.2.5 and Theorem 4.2.7. We think that Remark 4.4.2 is of quite importance.

**Remark 4.4.3.** With some additional assumptions, it is known that the spectral bound of  $B$  is  $-\lambda^*$  (cf. [42]). Since  $S(t)$  (the strongly continuous semigroup generated by  $B$ ) is positive, it follows from [23, 9] that the growth bound of  $S(t)$  is also  $-\lambda^*$ , and the spectral disk of  $S(t)$  is  $\{\lambda : |\lambda| \leq \exp(-\lambda^*t)\}$ . Thus, the conclusions obtained in [14, 38, 39, 41, 10, 43, 22] are extended to transport equations with reentry boundary conditions.

The following part of this section will be devoted to the proof of Eqs. (4.10) and (4.11). From the process, it is easy to see that these inequalities still hold if the condition  $\delta < 1/2$  is replaced by  $\delta < 1$  (This fact will be used when we discuss the problem in the setting of  $L^1$ ).

*Proof of Eq. (4.10):* From Eq. (4.9), we have

$$\begin{aligned} & F(n, \lambda, r, \rho, \dots) \\ = & \int_0^{v_M} \exp\{i\tau v_1^{-1}[-2(n+1)\Delta(r, \rho, s) + s]\} k(r, v, v_1) k(\rho, v_1, v') \\ & \cdot \alpha^{n+1}(v_1, \sqrt{\cdot}) \exp[-(n+1)\Sigma_4(\beta, \sigma, \dots) + \Sigma_3(\beta, \sigma, \dots)] v_1^{-1} dv_1 \\ = & \int_0^{v_M} \frac{\partial \exp\{i\tau v_1^{-1}[-2(n+1)\Delta(r, \rho, s) + s]\}}{\partial v_1} \\ & \cdot \{i\tau[2(n+1)\Delta(r, \rho, s) - s]\}^{-1} v_1^2 v^{-\delta} \tilde{k}(r, v, v_1) v_1^{-\delta} \tilde{k}(\rho, v_1, v') \\ & \cdot \alpha^{n+1}(v_1, \sqrt{\cdot}) \exp[-(n+1)\Sigma_4(\beta, \sigma, \dots) + \Sigma_3(\beta, \sigma, \dots)] v_1^{-1} dv_1. \end{aligned}$$

By virtue of the technique of integration by parts, we get

$$\begin{aligned}
& F(n, \lambda, r, \rho, \dots) \\
&= \{i\tau[2(n+1)\Delta(r, \rho, s) - s]\}^{-1} \{\exp[i\tau v_1^{-1}(-2(n+1)\Delta(r, \rho, s) + s)] \\
&\quad \cdot v_1^{1-\delta} v^{-\delta} \tilde{k}(r, v, v_1) \tilde{k}(\rho, v_1, v') \alpha^{n+1}(v_1, \sqrt{\cdot}) \\
&\quad \cdot \exp[-(n+1)\Sigma_4(\beta, \sigma, \dots) + \Sigma_3(\beta, \sigma, \dots)]\} \Big|_{v_1=0}^{v_1=v_M} \\
&\quad - \{i\tau[2(n+1)\Delta(r, \rho, s) - s]\}^{-1} \int_0^{v_M} \exp\{i\tau v_1^{-1}[-2(n+1)\Delta(r, \rho, s) + s]\} \\
&\quad \cdot \frac{\partial}{\partial v_1} \{v_1^{1-\delta} v^{-\delta} \tilde{k}(r, v, v_1) \tilde{k}(\rho, v_1, v') \alpha^{n+1}(v_1, \sqrt{\cdot}) \\
&\quad \cdot \exp[-(n+1)\Sigma_4(\beta, \sigma, \dots) + \Sigma_3(\beta, \sigma, \dots)]\} dv_1.
\end{aligned}$$

Expanding the term  $\frac{\partial}{\partial v_1}\{\dots\}$  in the above equation, it is not difficult to verify

$$\begin{aligned}
& |F(n, \lambda, r, \rho, \dots)| \\
&\leq \tau^{-1}[2(n+1)\Delta(r, \rho, s) - s]^{-1} v^{-\delta} [C_{10}\alpha_0^{n+1} + C_{11}(n+1)\alpha_0^n] \\
&\leq C_{12}(n+1)\alpha_0^n \tau^{-1} v^{-\delta} [2(n+1)\Delta(r, \rho, s) - s]^{-1}.
\end{aligned}$$

From the definition of  $\Delta(r, \rho, s)$ , it is easy to see (cf. [42])

$$2\Delta(r, \rho, s) \geq |r - \rho| + s. \quad (4.32)$$

Therefore, we get

$$\begin{aligned}
& F(n, \lambda, r, \rho, \dots) \\
&\leq C_{12}(n+1)\alpha_0^n \tau^{-1} v^{-\delta} [(n+1)|r - \rho| + ns]^{-1} \\
&\leq C_{13}\alpha_0^n \tau^{-1} v^{-\delta} |r - \rho|^{-1},
\end{aligned}$$

and thus Eq. (4.10) is proved.

*Proof of Eq. (4.11):* From Eq. (4.9), Eq. (4.1) and the expressions for  $\Sigma_3(\lambda, \sigma, \dots)$ ,  $\Sigma_4(\lambda, \sigma, \dots)$  as well as Eq. (4.32), we have

$$\begin{aligned}
& |F(n, \lambda, r, \rho, \dots)| \\
&\leq C_{14}\alpha_0^n v^{-\delta} \int_0^{v_M} v_1^{-(1+\delta)} \exp\left\{\frac{[-2(n+1)\Delta(r, \rho, s) + s](\beta + \lambda^*)}{v_1}\right\} dv_1
\end{aligned}$$

$$\begin{aligned}
&\leq C_{14}\alpha_0^n v^{-\delta} \int_0^{v_M} v_1^{-(1+\delta)} \exp\left\{-\frac{(\beta + \lambda^*)|r - \rho|}{v_1}\right\} dv_1 \\
&= C_{14}\alpha_0^n v^{-\delta} \int_{t_0}^{+\infty} \frac{\exp[-(\beta + \lambda^*)|r - \rho|t]}{t^{1-\delta}} dt,
\end{aligned}$$

where  $t_0 = v_M^{-1}$ . By use of the Hölder inequality (set  $p = (1 - \delta)^{-1}$ ,  $q = \delta^{-1}$ ), we have

$$\begin{aligned}
&|F(n, \lambda, r, \rho, \dots)| \\
&\leq C_{14}\alpha_0^n v^{-\delta} \int_{t_0}^{+\infty} \frac{\exp[-(\beta + \lambda^*)|r - \rho|(1 - \delta)t]}{t^{1-\delta}} \exp[-(\beta + \lambda^*)|r - \rho|\delta t] dt \\
&\leq C_{14}\alpha_0^n v^{-\delta} \left\{ \int_{t_0}^{+\infty} \frac{\exp[-(\beta + \lambda^*)|r - \rho|t]}{t} dt \right\}^{1-\delta} \\
&\quad \cdot \left\{ \int_{t_0}^{+\infty} \exp[-(\beta + \lambda^*)|r - \rho|t] dt \right\}^{\delta}. \tag{4.33}
\end{aligned}$$

Since the function  $\frac{1}{\log u} \int_{t_0}^{\infty} \frac{e^{-ut}}{t} dt$  is analytic in  $\{u \in C : 0 \leq \operatorname{Re} u < 1\}$ , if  $|\lambda + \lambda^*| \cdot |r - \rho| \leq 1/2$  (which implies  $(\beta + \lambda^*)|r - \rho| \leq 1/2$ ), then from Eq. (4.33), it follows

$$\begin{aligned}
&|F(n, \lambda, r, \rho, \dots)| \\
&\leq C_{15}\alpha_0^n v^{-\delta} (\beta + \lambda^*)^{-\delta} |r - \rho|^{-\delta} |\log [(\beta + \lambda^*)|r - \rho]|^{1-\delta} \\
&= C_{15}\alpha_0^n v^{-\delta} (\beta + \lambda^*)^{-\delta} |r - \rho|^{-\delta} \\
&\quad \cdot |\log [|\lambda + \lambda^*| \cdot |r - \rho| \cdot (\beta + \lambda^*)|\lambda + \lambda^*|^{-1}]|^{1-\delta} \\
&\leq C_{15}\alpha_0^n v^{-\delta} (\beta + \lambda^*)^{-\delta} |r - \rho|^{-\delta} \{ |\log [|\lambda + \lambda^*| \cdot |r - \rho]| | \\
&\quad + |\log |\lambda + \lambda^*| | + |\log(\beta + \lambda^*)| + 1 \}.
\end{aligned}$$

This completes the proof of Eq. (4.11).

# Chapter 5

## Spectral Properties of Transport Equations for Spherical Geometry in $L^1$

The time dependent transport equation in a sphere with reflecting boundary conditions will be discussed in the setting of  $L^1$ . Some aspects of the spectral properties of the strongly continuous semigroup  $T(t)$  generated by the corresponding transport operator  $A$  are studied, and it is shown that the spectrum of  $T(t)$  outside the disk  $\{\lambda : |\lambda| \leq \exp(-\lambda^*t)\}$ , where  $\lambda^*$  is the essential infimum of the total collision frequency  $\sigma(r, v)$ , or  $\lambda^* = \text{ess inf}_r \lim_{v \rightarrow 0^+} \sigma(r, v)$ , consists of isolated eigenvalues of  $T(t)$  with finite algebraic multiplicity, and the accumulation points of  $\sigma(T(t)) \cap \{\lambda : |\lambda| > \exp(-\lambda^*t)\}$  could only appear on the circle  $\{\lambda : |\lambda| = \exp(-\lambda^*t)\}$ . Consequently, the asymptotic behavior of the time dependent solution is obtained.

### 5.1 Problem and Notations

In the following part, we are going to investigate the neutron transport equation in the setting of  $L^1$ .

Throughout this chapter, the hypothesis (H3) defined in Chapter 4 will be replaced by

(H3').  $k(r, v, v')$  is a real measurable function, and there exist nonnegative constants

$\delta < 1$  and  $M$  such that  $|k(r, v, v')| \leq Mv^{-\delta}$ .

Remark: Obviously, (H3') is satisfied if  $k(r, v, v')$  is bounded measurable.

We recall that with the transformation

$$\begin{aligned} x &= r\mu, \\ y &= r\sqrt{1 - \mu^2}, \\ \psi(x, y, v, t) &= f(r(x, y), \mu(x, y), v, t), \end{aligned}$$

the transport equation can be written as

$$(II) \quad \left\{ \begin{array}{l} \frac{\partial \psi(x, y, v, t)}{\partial t} = -v \frac{\partial \psi(x, y, v, t)}{\partial x} - \sigma(r, v) \psi(x, y, v, t) \\ \quad + \frac{1}{2r} \int_0^{v_M} \int_{-r}^r k(r, v, v') \psi(z, \sqrt{r^2 - z^2}, v', t) dz, \\ y \in V, 0 \leq r = \sqrt{x^2 + y^2} \leq R, v \in E, t > 0, \\ \psi(-\sqrt{R^2 - y^2}, y, v, t) = \alpha \left( v, -\sqrt{1 - R^{-2}y^2} \right) \psi(\sqrt{R^2 - y^2}, y, v, t), y \in V, \\ \psi(x, y, v, 0) = \psi_0(x, y, v). \end{array} \right.$$

Set  $D = \{(x, y) : y \geq 0, 0 \leq \sqrt{x^2 + y^2} \leq R\}$ ,  $G = D \times E$ , and let  $L^1(G)$  represent the Banach space composed of all measurable complex functions defined and absolutely integrable over  $G$  with norm  $\|\cdot\|$  given by

$$\|\psi\| = \int_0^{v_M} dv \int_0^R y dy \int_{-\sqrt{R^2 - y^2}}^{\sqrt{R^2 - y^2}} |\psi(x, y, v)| dx.$$

Define operators on  $L^1(G)$  as follows:

$$\begin{aligned} B\psi &= -v \frac{\partial \psi}{\partial x} - \sigma(r, v) \psi, \\ K\psi &= \frac{1}{2r} \int_0^{v_M} dv' \int_{-r}^r k(r, v, v') \psi(z, \sqrt{r^2 - z^2}, v') dz, \\ A\psi &= B\psi + K\psi, \end{aligned}$$

with  $D(B) = D(A) = \{\psi \in L^1(G) : B\psi \in L^1(G) \text{ and } \psi(-\sqrt{R^2 - y^2}, y, v) = \alpha \left( v, -\sqrt{1 - R^{-2}y^2} \right) \psi(\sqrt{R^2 - y^2}, y, v) \text{ for every } y \in V \text{ and } v \in E\}$ ,  $D(K) = L^1(G)$ . Then Eq. (II) can be written as

$$\frac{d\psi(t)}{dt} = A\psi(t), \quad \psi(0) = \psi_0.$$

## 5.2 Spectral Properties of $A$ in $L^1$

From the hypothesis (H3'), it is easy to verify that  $K$  is a bounded operator.

Set  $\lambda^* = \text{ess inf}_{(r,v) \in V \times E} \sigma(r, v)$ . (In fact, all the conclusions given in the following part still hold if  $\lambda^*$  is replaced by  $\lambda_0^* = \text{ess inf}_{r \in V} \lim_{v \rightarrow 0^+} \sigma(r, v)$  if there exists a constant  $c \geq 0$  such that  $\sigma(r, v) \geq \lambda_0^* - cv$ , where  $\lambda_0^* = \text{ess inf}_{r \in V} \lim_{v \rightarrow 0^+} \sigma(r, v)$  (cf. [18, page 1988]).

**Lemma 5.2.1.** *In the setting of  $L^1(G)$ ,  $B$  is a densely defined operator,  $\{\lambda : \text{Re}\lambda > -\lambda^*\} \subset \rho(B)$ ,  $R(\lambda I - B) = L^1(G)$ , and  $\|(\lambda I - B)^{-1}\|_1 \leq (\text{Re}\lambda + \lambda^*)^{-1}$  for every  $\lambda$  with  $\text{Re}\lambda > -\lambda^*$ .*

**Proof:** For every  $g \in L^1(G)$ , solving  $(\lambda I - B)f = g$ , we obtain

$$\begin{aligned}
f(x, y, v) &= (\lambda I - B)^{-1}g(x, y, v) \\
&= \frac{1}{v} \int_{-\sqrt{R^2-y^2}}^x \exp[-\Sigma_1(\lambda, v, x, y, z)]g(z, y, v)dz \\
&\quad + \frac{\alpha(v, y) \exp[-\Sigma_2(\lambda, v, y)]}{v\{1 - \alpha(v, y) \exp[-\Sigma_2(\lambda, v, y)]\}} \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} \exp[-\Sigma_1(\lambda, v, x, y, z)]g(z, y, v)dz \\
&= R_\lambda^{(1)}g + R_\lambda^{(2)}g
\end{aligned} \tag{5.1}$$

where

$$\begin{aligned}
\Sigma_1(\lambda, v, x, y, z) &= \int_z^x \frac{\lambda + \sigma(\sqrt{t^2 + y^2}, v)}{v} dt, \\
\Sigma_2(\lambda, v, y) &= \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} \frac{\lambda + \sigma(\sqrt{t^2 + y^2}, v)}{v} dt.
\end{aligned}$$

$$\begin{aligned}
\|R_\lambda^{(1)}g\|_1 &= \int_0^{v_M} dv \int_0^R ydy \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} |R_\lambda^{(1)}g(x, y, v)| dx \\
&\leq \int_0^{v_M} dv \int_0^R ydy \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} dx \frac{1}{v} \int_{-\sqrt{R^2-y^2}}^x \exp\left[-\frac{(\text{Re}\lambda + \lambda^*)(x-z)}{v}\right] |g(z, y, v)| dz.
\end{aligned}$$

By exchanging the order of integrations ( $\int dx$  and  $\int dz$ ), we have

$$\begin{aligned}
&\|R_\lambda^{(1)}g\|_1 \\
&\leq \int_0^{v_M} \frac{dv}{v} \int_0^R ydy \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} dz |g(z, y, v)| \int_z^{\sqrt{R^2-y^2}} \exp\left[-\frac{(\text{Re}\lambda + \lambda^*)(x-z)}{v}\right] dx
\end{aligned}$$

$$\begin{aligned}
&= \int_0^{v_M} \frac{dv}{v} \int_0^R y dy \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} dz |g(z, y, v)| \cdot \frac{-v}{\operatorname{Re}\lambda + \lambda^*} \exp \left[ -\frac{(\operatorname{Re}\lambda + \lambda^*)(x-z)}{v} \right] \Big|_z^{\sqrt{R^2-y^2}} \\
&= \frac{1}{\operatorname{Re}\lambda + \lambda^*} \int_0^{v_M} dv \int_0^R y dy \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} dz \\
&\quad \cdot \left\{ 1 - \exp \left[ -\frac{(\operatorname{Re}\lambda + \lambda^*)(\sqrt{R^2-y^2} - z)}{v} \right] \right\} |g(z, y, v)|. \tag{5.2}
\end{aligned}$$

$$\begin{aligned}
\|R_\lambda^{(2)} g\|_1 &= \int_0^{v_M} dv \int_0^R y dy \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} |R_\lambda^{(2)} g(x, y, v)| dx \\
&\leq \int_0^{v_M} dv \int_0^R y dy \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} dx \frac{\alpha(v, y) \exp[-2\sqrt{R^2-y^2}(\operatorname{Re}\lambda + \lambda^*)/v]}{v\{1 - \alpha(v, y) \exp[-2\sqrt{R^2-y^2}(\operatorname{Re}\lambda + \lambda^*)/v]\}} \\
&\quad \cdot \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} \exp \left[ -\frac{(\operatorname{Re}\lambda + \lambda^*)(x-z)}{v} \right] |g(z, y, v)| dz \\
&\leq \int_0^{v_M} \frac{dv}{v} \int_0^R y dy \frac{\exp[-2\sqrt{R^2-y^2}(\operatorname{Re}\lambda + \lambda^*)/v]}{1 - \exp[-2\sqrt{R^2-y^2}(\operatorname{Re}\lambda + \lambda^*)/v]} \\
&\quad \cdot \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} dx \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} \exp \left[ -\frac{(\operatorname{Re}\lambda + \lambda^*)(x-z)}{v} \right] |g(z, y, v)| dz.
\end{aligned}$$

By exchanging the order of integrations ( $\int dx$  and  $\int dz$ ), we have

$$\begin{aligned}
\|R_\lambda^{(2)} g\|_1 &\leq \int_0^{v_M} \frac{dv}{v} \int_0^R y dy \frac{\exp[-2\sqrt{R^2-y^2}(\operatorname{Re}\lambda + \lambda^*)/v]}{1 - \exp[-2\sqrt{R^2-y^2}(\operatorname{Re}\lambda + \lambda^*)/v]} \\
&\quad \cdot \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} dz |g(z, y, v)| \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} dx \exp \left[ -\frac{(\operatorname{Re}\lambda + \lambda^*)(x-z)}{v} \right] \\
&= \int_0^{v_M} \frac{dv}{v} \int_0^R y dy \frac{\exp[-2\sqrt{R^2-y^2}(\operatorname{Re}\lambda + \lambda^*)/v]}{1 - \exp[-2\sqrt{R^2-y^2}(\operatorname{Re}\lambda + \lambda^*)/v]} \\
&\quad \cdot \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} dz |g(z, y, v)| \frac{-v}{\operatorname{Re}\lambda + \lambda^*} \exp \left[ -\frac{(\operatorname{Re}\lambda + \lambda^*)(x-z)}{v} \right] \Big|_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} dx \\
&\leq \frac{1}{\operatorname{Re}\lambda + \lambda^*} \int_0^{v_M} dv \int_0^R y dy \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} dz |g(z, y, v)| \cdot \frac{1}{1 - \exp[-2\sqrt{R^2-y^2}(\operatorname{Re}\lambda + \lambda^*)/v]} \\
&\quad \cdot \left\{ \exp[-(\sqrt{R^2-y^2} - z)(\operatorname{Re}\lambda + \lambda^*)/v] - \exp[-(3\sqrt{R^2-y^2} - z)(\operatorname{Re}\lambda + \lambda^*)/v] \right\}
\end{aligned}$$

$$= \frac{1}{\operatorname{Re}\lambda + \lambda^*} \int_0^{v_M} dv \int_0^R y dy \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} dz |g(z, y, v)| \exp \left[ -\frac{(\operatorname{Re}\lambda + \lambda^*)(\sqrt{R^2-y^2} - z)}{v} \right] \quad (5.3)$$

Therefore, from Eq. (5.1) – Eq. (5.3), we have

$$\begin{aligned} \|f\|_1 &= \|R_\lambda^{(1)}g + R_\lambda^{(2)}g\|_1 \leq \|R_\lambda^{(1)}g\|_1 + \|R_\lambda^{(2)}g\|_1 \\ &\leq \frac{1}{\operatorname{Re}\lambda + \lambda^*} \int_0^{v_M} dv \int_0^R y dy \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} dz |g(z, y, v)| \\ &\leq \frac{1}{\operatorname{Re}\lambda + \lambda^*} \|g\|_1. \end{aligned}$$

This indicates that  $\{\lambda : \operatorname{Re}\lambda > -\lambda^*\} \subset \rho(B)$  and  $\|(\lambda I - B)^{-1}\|_1 \leq (\operatorname{Re}\lambda + \lambda^*)^{-1}$  for every  $\lambda$  with  $\operatorname{Re}\lambda > -\lambda^*$ . Q. E. D.

From Lemma 5.2.1, similar to that of Chapter 2 §2.2 in this dissertation, it is easy to get the following results.

**Lemma 5.2.2.** *B generates a positive  $C_0$  semigroup  $S(t)$  in  $L^1(G)$ .*

**Lemma 5.2.3.** *In the setting of  $L^1(G)$ ,  $\{\lambda : \operatorname{Re}\lambda > \|K\|_1 - \lambda^*\} \subset \rho(A)$ , and  $R(\lambda I - A) = L^1(G)$ ,  $\|(\lambda I - A)^{-1}\|_1 \leq (\operatorname{Re}\lambda + \lambda^* - \|K\|_1)^{-1}$  for every  $\lambda$  with  $\operatorname{Re}\lambda > \|K\|_1 - \lambda^*$ .*

**Lemma 5.2.4.** *A generates a  $C_0$  semigroup  $T(t)$  in  $L^1(G)$ . For every  $\psi_0 \in D(A)$ , the solution  $\psi(t)$  of Eq. (I) exists and is uniquely given by  $\psi(t) = T(t)\psi_0$ .*

Let  $\beta_1$  be any constant satisfying  $\beta_1 > -\lambda^*$ . Then we have

**Theorem 5.2.5.** *Set  $\tilde{k}(r, v, v') = v^\delta k(r, v, v')$  and let  $\beta_1 > -\lambda^*$  be any constant. If the following conditions are satisfied:*

(H4').  $\alpha(v, \mu)$ ,  $\sigma(r, v)$  and  $\tilde{k}(r, v, v')$  are partially differentiable with respect to  $v, v'$ , and the corresponding partial derivatives  $\frac{\partial \alpha}{\partial v}$ ,  $\frac{\partial \sigma}{\partial v}$ ,  $\frac{\partial \tilde{k}}{\partial v}$  and  $\frac{\partial \tilde{k}}{\partial v'}$  are uniformly bounded.

then there exist positive constants  $C_1$ ,  $C_2$  and  $\tilde{\tau}$  independent of  $\beta \in [\beta_1, +\infty)$ , such that for all  $\lambda \in \{\lambda = \beta + i\tau : \beta \geq \beta_1, |\tau| \geq \tilde{\tau}\}$ ,

$$\|K(\lambda I - B)^{-1}K\|_1 \leq C_1 |\beta_1 + \lambda^* + i\tau|^{\delta-1} \log |\beta_1 + \lambda^* + i\tau|$$

when  $0 < \delta < 1$ , and

$$\|K(\lambda I - B)^{-1}K\|_1 \leq C_2|\beta_1 + \lambda^* + i\tau|^{-1} \log^2 |\beta_1 + \lambda^* + i\tau|$$

when  $\delta = 0$ .

**Proof.** Note that here and in subsequent proofs,  $C$  and  $c$  are positive constants, but may have different values at different places.

$K(\lambda I - B)^{-1}K$  is an integral operator defined on  $L^1(G)$ , i.e., for every  $\psi \in L^1(G)$ ,

$$\begin{aligned} & K(\lambda I - B)^{-1}K\psi(x, y, v) \\ &= \iiint_G k(\lambda, x, x', y, y', v, v') dx' dy' dv', \end{aligned} \quad (5.4)$$

where

$$\begin{aligned} & k(\lambda, x, x', y, y', v, v') \\ &= \frac{y'}{4r\rho} \int_0^{v_M} dv_1 \frac{k(r, v, v_1)k(\rho, v_1, v')}{v_1} \int_{|r-\rho|}^{r+\rho} \frac{1}{s} \\ & \cdot \frac{\exp[-\Sigma_3(\lambda, \sigma, \dots)] + \alpha(v_1, \sqrt{\cdot}) \exp[-\Sigma_4(\lambda, \sigma, \dots) + \Sigma_3(\lambda, \sigma, \dots)]}{1 - \alpha(v_1, \sqrt{\cdot}) \exp[-\Sigma_4(\lambda, \sigma, \dots)]} ds, \end{aligned} \quad (5.5)$$

with

$$\begin{aligned} r &= \sqrt{x^2 + y^2}, \quad \rho = \sqrt{x'^2 + y'^2}, \\ \alpha(v_1, \sqrt{\cdot}) &= \alpha\left(v_1, \sqrt{r^2 - \Delta_2^2(r, \rho, s)}\right), \\ \Sigma_3(\lambda, \sigma, \dots) &= \int_{\Delta_1(r, \rho, s)}^{\Delta_2(r, \rho, s)} \frac{\lambda + \sigma(\sqrt{t^2 + r^2 - \Delta_2^2(r, \rho, s)}, v_1)}{v_1} dt, \\ \Sigma_4(\lambda, \sigma, \dots) &= \int_{-\Delta(r, \rho, s)}^{\Delta(r, \rho, s)} \frac{\lambda + \sigma(\sqrt{t^2 + r^2 - \Delta_2^2(r, \rho, s)}, v_1)}{v_1} dt, \\ \Delta_1(r, \rho, s) &= -\frac{\rho^2 - r^2}{2s} - \frac{s}{2}, \quad \Delta_2(r, \rho, s) = -\frac{\rho^2 - r^2}{2s} + \frac{s}{2}, \\ \Delta(r, \rho, s) &= \sqrt{R^2 + \frac{1}{4s^2}(\rho^2 - r^2)^2 + \frac{s^2}{4} - \frac{1}{2}(\rho^2 + r^2)}. \end{aligned}$$

Since  $0 \leq \alpha(v_1, \sqrt{\cdot}) \leq \alpha_0 < 1$ , we have

$$\frac{1}{1 - \alpha(v_1, \sqrt{\cdot}) \exp[-\Sigma_4(\lambda, \sigma, \dots)]} = \sum_{n=0}^{\infty} \alpha^n(v_1, \sqrt{\cdot}) \exp[-n\Sigma_4(\lambda, \sigma, \dots)].$$

For every  $n = 0, 1, 2, \dots$ , let

$$\begin{aligned} & g_{n,1}(\lambda, x, x', y, y', v, v') \\ &= \frac{y'}{4r\rho} \int_0^{v_M} dv_1 \frac{k(r, v, v_1)k(\rho, v_1, v')}{v_1} \\ & \cdot \int_{|r-\rho|}^{r+\rho} \alpha^n(v_1, \sqrt{\cdot}) \exp[-n\Sigma_4(\lambda, \sigma, \dots) - \Sigma_3(\lambda, \sigma, \dots)] s^{-1} ds, \end{aligned}$$

$$\begin{aligned} & g_{n,2}(\lambda, x, x', y, y', v, v') \\ &= \frac{y'}{4r\rho} \int_0^{v_M} dv_1 \frac{k(r, v, v_1)k(\rho, v_1, v')}{v_1} \\ & \cdot \int_{|r-\rho|}^{r+\rho} \alpha^{n+1}(v_1, \sqrt{\cdot}) \exp[-(n+1)\Sigma_4(\lambda, \sigma, \dots) + \Sigma_3(\lambda, \sigma, \dots)] s^{-1} ds, \end{aligned}$$

and define operators  $G_{n,1}, G_{n,2}$  on  $L^1(G)$  by

$$G_{n,j}\psi = \iiint_G g_{n,j}(\lambda, x, x', y, y', v, v') \psi(x', y', v') dx' dy' dv', \quad n = 0, 1, \dots, \quad j = 1, 2.$$

Then we get

$$K(\lambda I - B)^{-1}K = \sum_{n=0}^{\infty} G_{n,1} + \sum_{n=0}^{\infty} G_{n,2}. \quad (5.6)$$

First, we consider  $G_{n,2}$ . For any  $\psi \in L^1(G)$ ,

$$\begin{aligned} & \|G_{n,2}\psi\|_1 = \int dx \int y dy \int dv |g_{n,2}(\lambda, x, x', y, y', v, v') \psi(x', y', v')| \\ & \leq \int dx' \int y' dy' \int dv' |\psi(x', y', v')| \int dx \int dy \int dv \frac{y}{4r\rho} \cdot \left| \int_0^{v_M} dv_1 \frac{k(r, v, v_1)k(\rho, v_1, v')}{v_1} \right. \\ & \quad \left. \cdot \int_{|r-\rho|}^{r+\rho} \alpha^{n+1}(v_1, \sqrt{\cdot}) \exp[-(n+1)\Sigma_4(\lambda, \sigma, \dots) + \Sigma_3(\lambda, \sigma, \dots)] s^{-1} ds \right|. \end{aligned}$$

By virtue of the transform

$$x = z, \quad y = \sqrt{r^2 - z^2}; \quad x' = z', \quad y' = \sqrt{\rho^2 - z'^2},$$

we get

$$\begin{aligned}
& \|G_{n,2}\psi\|_1 \\
& \leq C \int dx' \int y' dy' \int dv' |\psi(x', y', v')| \int dv \int_0^R dr \int_{-r}^r \frac{r}{y} dz \cdot \frac{y}{4r\rho} \\
& \quad \cdot \left| \int_0^{v_M} dv_1 \frac{k(r, v, v_1)k(\rho, v_1, v')}{v_1} \right. \\
& \quad \cdot \left. \int_{|r-\rho|}^{r+\rho} \alpha^{n+1}(v_1, \sqrt{\cdot}) \exp[-(n+1)\Sigma_4(\lambda, \sigma, \dots) + \Sigma_3(\lambda, \sigma, \dots)] s^{-1} ds \right| \\
& \leq C \int dx' \int y' dy' \int dv' |\psi(x', y', v')| \int dv \cdot \frac{1}{\rho} \int_0^R r dr \int_{|r-\rho|}^{r+\rho} |F(n, \lambda, r, \rho, \dots)| s^{-1} ds,
\end{aligned} \tag{5.7}$$

where

$$\begin{aligned}
& F(n, \lambda, r, \rho, \dots) \\
& = \int_0^{v_M} \frac{k(r, v, v_1)k(\rho, v_1, v')}{v_1} \\
& \quad \cdot \alpha^{n+1}(v_1, \sqrt{\cdot}) \exp[-(n+1)\Sigma_4(\lambda, \sigma, \dots) + \Sigma_3(\lambda, \sigma, \dots)] dv_1.
\end{aligned}$$

By virtue of the technique of integration by parts, we obtain the following estimation of  $F(n, \lambda, r, \rho, \dots)$ :

$$|F(n, \lambda, r, \rho, \dots)| \leq \frac{C\alpha_0^n}{|\tau| \cdot |r - \rho|v^\delta}. \tag{5.8}$$

Also, using the fact that the function  $\frac{1}{\log u} \int_1^\infty \frac{e^{-ut}}{t} dt$  is analytic in the region  $\{u \in \mathbb{C} : 0 \leq \text{Re} u < 1\}$ , we get the following estimation when  $|\lambda + \lambda^*| \cdot |r - \rho| \leq 1/2$

$$\begin{aligned}
& |F(n, \lambda, r, \rho, \dots)| \\
& \leq C\alpha_0^n (\text{Re}\lambda + \lambda^*)^{-\delta} |r - \rho|^{-\delta} v^{-\delta} \\
& \quad \cdot \{|\log(|\lambda + \lambda^*| \cdot |r - \rho|)| + |\log|\lambda + \lambda^*|| + |\log(\text{Re}\lambda + \lambda^*)| + 1\}.
\end{aligned} \tag{5.9}$$

Denote the right-hand side of the above relation by  $RHS(n, \lambda, |r - \rho|, v)$  and define the following function:

$$P(n, \lambda, |r - \rho|, v) = \begin{cases} \frac{C\alpha_0^n}{|\tau| \cdot |r - \rho|v^\delta}, & \text{if } |\lambda + \lambda^*| \cdot |r - \rho| \geq 1/2, \\ RHS(n, \lambda, |r - \rho|, v), & \text{if } |\lambda + \lambda^*| \cdot |r - \rho| \leq 1/2. \end{cases} \tag{5.10}$$

Then

$$|F(n, \lambda, r, \rho, \dots)| \leq P(n, \lambda, |r - \rho|, v).$$

From Eq. (5.7), we have

$$\begin{aligned} \|G_{n,2}\psi\|_1 &\leq C \int dx' \int y' dy' \int dv' |\psi(x', y', v')| \\ &\quad \cdot \int dv \cdot \frac{1}{\rho} \int_0^R r dr \int_{|r-\rho|}^{r+\rho} P(n, \lambda, |r - \rho|, v) s^{-1} ds. \end{aligned} \quad (5.11)$$

Investigate the following integration (which is part of the right hand side of Eq. (5.11))

$$\begin{aligned} &\frac{1}{\rho} \int_0^R r dr \int_{|r-\rho|}^{r+\rho} P(n, \lambda, |r - \rho|, v) s^{-1} ds = \frac{1}{\rho} \int_0^R r dr P(n, \lambda, |r - \rho|, v) \int_{|r-\rho|}^{r+\rho} \frac{1}{s} ds \\ &= \frac{1}{\rho} \int_0^\rho r dr P(n, \lambda, |r - \rho|, v) \int_{|r-\rho|}^{r+\rho} \frac{1}{s} ds + \frac{1}{\rho} \int_\rho^R r dr P(n, \lambda, |r - \rho|, v) \int_{|r-\rho|}^{r+\rho} \frac{1}{s} ds \\ &= T_1 + T_2. \end{aligned} \quad (5.12)$$

For term  $T_1$ , since  $\frac{r}{\rho} \leq 1$ , using transform  $w(r) = \rho - r$ , we obtain

$$\begin{aligned} T_1 &\leq \int_0^\rho dr P(n, \lambda, |r - \rho|, v) \int_{|r-\rho|}^{2R} \frac{1}{s} ds \\ &\leq \int_0^\rho dw P(n, \lambda, w, v) \int_w^{2R} \frac{1}{s} ds \leq C \int_0^R |\log w| P(n, \lambda, w, v) dw. \end{aligned} \quad (5.13)$$

As for term  $T_2$ , using transform  $w = r - \rho$ , we obtain

$$\begin{aligned} T_2 &= \frac{1}{\rho} \int_0^{R-\rho} (w + \rho) dw P(n, \lambda, w, v) \int_w^{2\rho+w} \frac{1}{s} ds \\ &= \frac{1}{\rho} \int_0^{R-\rho} w dw P(n, \lambda, w, v) \int_w^{2\rho+w} \frac{1}{s} ds + \frac{1}{\rho} \int_0^{R-\rho} \rho dw P(n, \lambda, w, v) \int_w^{2\rho+w} \frac{1}{s} ds \\ &\leq \frac{1}{\rho} \int_0^{R-\rho} w dw P(n, \lambda, w, v) \int_w^{2\rho+w} \frac{1}{w} ds + \int_0^{R-\rho} dw P(n, \lambda, w, v) \int_w^{2\rho+w} \frac{1}{s} ds \\ &\leq 2 \int_0^{R-\rho} P(n, \lambda, w, v) dw + C \int_0^{R-\rho} |\log w| P(n, \lambda, w, v) dw \\ &\leq C \int_0^R |\log w| P(n, \lambda, w, v) dw. \end{aligned} \quad (5.14)$$

Set  $W_1 = \{w \in [0, R] : |\lambda + \lambda^*|w \leq 1/2\}$ ,  $W_2 = \{w \in [0, R] : |\lambda + \lambda^*|w > 1/2\}$ . Then from Eq. (5.10), we have

$$\begin{aligned}
& \int_0^R |\log w| P(n, \lambda, w, v) dw \\
& \leq \int_{W_2} |\log w| \frac{C\alpha_0^n}{|\tau| \cdot wv^\delta} dw \\
& \quad + \int_{W_1} |\log w| \frac{C\alpha_0^n \{|\log(|\lambda + \lambda^*|w)| + |\log|\lambda + \lambda^*|| + C\}}{(\operatorname{Re}\lambda + \lambda^*)^\delta w^\delta v^\delta} dw \tag{5.15}
\end{aligned}$$

From Eqs. (5.11) — (5.15), we have

$$\begin{aligned}
& \|G_{n,2}\psi\|_1 \\
& \leq C \int dx' \int y' dy' \int dv' |\psi(x', y', v')| \int dv \int_0^R |\log w| P(n, \lambda, w, v) dw \\
& \leq C\alpha_0^n \|\psi\|_1 \left\{ \frac{1}{|\tau|} \int_{W_2} \frac{|\log w|}{w} dw \right. \\
& \quad \left. + \int_{W_1} \frac{|\log w| \{|\log(|\lambda + \lambda^*|w)| + |\log|\lambda + \lambda^*|| + C\}}{w^\delta} dw \right\} \tag{5.16}
\end{aligned}$$

Consider  $\frac{1}{|\tau|} \int_{W_2} \frac{|\log w|}{w} dw$ . If  $\delta = 0$ , then by virtue of the transform  $z(w) = |\lambda + \lambda^*|w$ , we have

$$\begin{aligned}
& \frac{1}{|\tau|} \int_{W_2} \frac{|\log w|}{w} dw \\
& \leq \frac{1}{|\tau|} \int_{1/2}^{|\lambda + \lambda^*|R} \frac{|\log|\lambda + \lambda^*|| + |\log z|}{z} dz \\
& = \frac{1}{|\tau|} [|\log|\lambda + \lambda^*|| \cdot [|\log|\lambda + \lambda^*|| + C] + \log^2|\lambda + \lambda^*| + C] \\
& \leq C \frac{\log^2|\lambda + \lambda^*|}{|\tau|} \leq C \frac{\log^2|\lambda + \lambda^*|}{|\beta_1 + \lambda^* + i\tau|}.
\end{aligned}$$

If  $0 < \delta < 1$ , then there exists a constant  $\delta_0$  such that  $\max\{\delta, 1 - \delta\} < \delta_0 < 1$ . By the Hölder inequality (let  $p^* = \delta_0^{-1}$ ,  $q^* = (1 - \delta_0)^{-1}$ ) and the transform  $z(w) = |\lambda + \lambda^*|w$ , we have

$$\frac{1}{|\tau|} \int_{W_2} \frac{|\log w|}{w} \cdot 1 dw \leq \frac{1}{|\tau|} \left[ \int_{W_2} \frac{|\log w|^{1/\delta_0}}{w^{1/\delta_0}} dw \right]^{\delta_0} \cdot \left[ \int_{W_2} dw \right]^{1-\delta_0}$$

$$\begin{aligned}
&\leq \frac{C}{|\tau|} \left[ \int_{1/2}^{\infty} \frac{(|\log |\lambda + \lambda^*|| + |\log z|)^{1/\delta_0} \cdot |\lambda + \lambda^*|^{1/\delta_0 - 1}}{z^{1/\delta_0}} dz \right]^{\delta_0} \\
&\leq \frac{C \log |\beta_1 + \lambda^* + i\tau|}{|\beta_1 + \lambda^* + i\tau|^{\delta_0}} \\
&\leq \frac{C \log |\beta_1 + \lambda^* + i\tau|}{|\beta_1 + \lambda^* + i\tau|^{1-\delta}}. \tag{5.17}
\end{aligned}$$

In the same manner, using the transform  $z(w) = |\lambda + \lambda^*|w$ , we have

$$\begin{aligned}
&\int_{w_1} \frac{|\log w| \{ |\log(|\lambda + \lambda^*|w)| + |\log |\lambda + \lambda^*|| + C \}}{w^\delta} dw \\
&\leq C \int_0^{1/2} \frac{(|\log |\lambda + \lambda^*|| + |\log z|) \{ |\log z| + |\log |\lambda + \lambda^*|| + C \}}{z^\delta |\lambda + \lambda^*|^{1-\delta}} dz \\
&\leq \frac{C \log |\beta_1 + \lambda^* + i\tau|}{|\beta_1 + \lambda^* + i\tau|^{1-\delta}}. \tag{5.18}
\end{aligned}$$

Therefore, we have

$$\|G_{n,2}\|_1 \leq C\alpha_0^n \frac{\log |\beta_1 + \lambda^* + i\tau|}{|\beta_1 + \lambda^* + i\tau|^{1-\delta}}$$

when  $0 < \delta < 1$ , and

$$\|G_{n,2}\|_1 \leq C\alpha_0^n |\beta_1 + \lambda^* + i\tau|^{-1} \log^2 |\beta_1 + \lambda^* + i\tau|$$

when  $\delta = 0$ .

By a similar procedure, we can get a similar estimation about  $\|G_{n,1}\|$ . Since  $\alpha_0 < 1$ , it follows from Eq. (5.5) that

$$\begin{aligned}
\|K(\lambda I - B)^{-1}K\|_1 &= \left\| \sum_{n=0}^{\infty} G_{n,1} + \sum_{n=0}^{\infty} G_{n,2} \right\|_1 \leq \sum_{n=0}^{\infty} \|G_{n,1}\|_1 + \sum_{n=0}^{\infty} \|G_{n,2}\|_1 \\
&\leq C \frac{\log |\beta_1 + \lambda^* + i\tau|}{|\beta_1 + \lambda^* + i\tau|^{1-\delta}}
\end{aligned}$$

when  $0 < \delta < 1$ , and

$$\|K(\lambda I - B)^{-1}K\|_1 \leq C |\beta_1 + \lambda^* + i\tau|^{-1} \log^2 |\beta_1 + \lambda^* + i\tau|$$

when  $\delta = 0$ . This completes the proof. Q. E. D.

The conclusion obtained in Theorem 5.2.5 heavily depends on the differentiability conditions. What happens if these differentiability conditions are not satisfied? We are going to investigate the asymptotic behavior for  $[K(\lambda I - B)^{-1}]^2 K$  as  $|\tau| \rightarrow \infty$  using the perturbation method.

Since  $\alpha(v, \mu)$ ,  $\sigma(r, v)$  and  $\tilde{k}(r, v, v') = v^\delta k(r, v, v')$  are bounded measurable, it can be shown that there exist three sequences  $\{\alpha_n(v, \mu)\}$ ,  $\{\sigma_n(r, v)\}$  and  $\{\tilde{k}_n(r, v, v')\}$  such that

- (a). for every  $n$ ,  $\alpha_n(v, \mu)$ ,  $\sigma_n(r, v)$  and  $\tilde{k}_n(r, v, v')$  are infinite order smooth functions;
- (b).  $|\alpha_n(v, \mu)| \leq \alpha_0 + (1 - \alpha_0)/2 < 1$ , and  $\alpha_n(v, \mu)$  converges to  $\alpha(v, \mu)$  almost everywhere in  $E \times \Omega$ ;
- (c).  $(\lambda^* - \beta_1)/2 \leq \sigma_n(r, v) \leq \text{ess sup}_{(r, v)} \sigma(r, v) + (\beta_1 + \lambda^*)/2$ , and  $\sigma_n(r, v)$  converges to  $\sigma(r, v)$  almost everywhere in  $V \times E$ ;
- (d).  $|\tilde{k}_n(r, v, v')| \leq \text{ess sup}_{(r, v, v')} \tilde{k}(r, v, v') + 1$ , and  $\tilde{k}_n(r, v, v')$  converges to  $\tilde{k}(r, v, v')$  almost everywhere in  $V \times E \times E$ .

Let

$$\begin{aligned} k_n(r, v, v') &= v^{-\delta} \tilde{k}_n(r, v, v'), \\ \delta k_n(r, v, v') &= k(r, v, v') - k_n(r, v, v'), \\ \delta \alpha_n(v, \mu) &= \alpha(v, \mu) - \alpha_n(v, \mu), \\ \delta \sigma_n(r, v) &= \sigma_n(r, v) - \sigma(r, v) \end{aligned}$$

and define operators on  $L^1(G)$  as follows:

$$\begin{aligned} \delta F \psi &= \delta \sigma_n(r, v) \psi; \\ K_n \psi &= \frac{1}{2r} \int_E dv' \int_{-r}^r k_n(r, v, v') \psi(z, \sqrt{r^2 - z^2}, v') dz, \\ \delta K_n \psi &= \frac{1}{2r} \int_E dv' \int_{-r}^r \delta k_n(r, v, v') \psi(z, \sqrt{r^2 - z^2}, v') dz, \end{aligned}$$

with  $D(\delta F_n) = D(K_n) = D(\delta K_n) = L^1(G)$ , and

$$B_{\alpha_n} \psi = -v \frac{\partial \psi}{\partial x} - \sigma(r, v) \psi,$$

with domain  $D(B_{\alpha_n}) = \{\psi \in L^1(G) : B_{\alpha_n}\psi \in L^1(G) \text{ and } \psi(-\sqrt{R^2 - y^2}, y, v) = \alpha_n(v, -\sqrt{1 - R^{-2}y^2})\psi(\sqrt{R^2 - y^2}, y, v) \text{ for every } y \in V \text{ and } v \in E\}$ ,

$$B_{\alpha_n, \sigma_n}\psi = -v \frac{\partial \psi}{\partial x} - \sigma_n(r, v)\psi,$$

with  $D(B_{\alpha_n, \sigma_n}) = \{\psi \in L^1(G) : B_{\alpha_n, \sigma_n}\psi \in L^1(G) \text{ and } \psi(-\sqrt{R^2 - y^2}, y, v) = \alpha_n(v, -\sqrt{1 - R^{-2}y^2})\psi(\sqrt{R^2 - y^2}, y, v) \text{ for every } y \in V \text{ and } v \in E\}$ .

Obviously,  $K = K_n + \delta K_n$ .

Similar to that of Lemma 5.2.1, it can be shown that  $\{\lambda : \operatorname{Re}\lambda \geq \beta_1\} \subset \rho(B_{\alpha_n})$ ,  $\{\lambda : \operatorname{Re}\lambda \geq \beta_1\} \subset \rho(B_{\alpha_n, \sigma_n})$ , and  $\|(\lambda I - B_{\alpha_n})^{-1}\| \leq (\operatorname{Re}\lambda + \lambda^*)^{-1}$  for every  $\lambda$  with  $\operatorname{Re}\lambda > -\lambda^*$ ,  $\|(\lambda I - B_{\alpha_n, \sigma_n})^{-1}\| \leq 2(\beta_1 + \lambda^*)^{-1}$  for every  $\lambda$  with  $\operatorname{Re}\lambda \geq \beta_1$ .

**Lemma 5.2.6.** *Let  $q = \frac{4}{3+\delta}$ ,  $c$  be a positive constant; then*

$$\frac{1}{\rho^q} \int_0^R r dr \int_0^{v_M} \frac{\exp[\frac{-c|r-\rho|}{v_1}]}{v_1^{q(1+\delta)}} dv_1 \left( \int_{|r-\rho|}^{r+\rho} \frac{1}{s} ds \right)^q < +\infty. \quad (5.19)$$

**Proof.** Write

$$\begin{aligned} & \frac{1}{\rho^q} \int_0^R r dr \int_0^{v_M} \frac{\exp[\frac{-c|r-\rho|}{v_1}]}{v_1^{q(1+\delta)}} dv_1 \left( \int_{|r-\rho|}^{r+\rho} \frac{1}{s} ds \right)^q \\ &= \frac{1}{\rho^q} \int_0^\rho r dr \int_0^{v_M} \frac{\exp[\frac{-c|r-\rho|}{v_1}]}{v_1^{q(1+\delta)}} dv_1 \left( \int_{|r-\rho|}^{r+\rho} \frac{1}{s} ds \right)^q \\ & \quad + \frac{1}{\rho^q} \int_\rho^R r dr \int_0^{v_M} \frac{\exp[\frac{-c|r-\rho|}{v_1}]}{v_1^{q(1+\delta)}} dv_1 \left( \int_{|r-\rho|}^{r+\rho} \frac{1}{s} ds \right)^q \\ &= I_1 + I_2. \end{aligned} \quad (5.20)$$

First consider the term  $I_1$ . Let  $t(r) = \rho - r$ ,  $p$  be the constant that satisfies  $\frac{1}{p} + \frac{1}{q} = 1$ , i.e.,  $p = \frac{4}{1-\delta}$ . Noting that  $\rho - t \leq \rho$  for all  $t \in (0, \rho)$ , and  $\frac{q}{p} + 1 = q$ , we have

$$\begin{aligned} I_1 &\leq \frac{1}{\rho^q} \int_0^\rho (\rho - t) dt \int_0^{v_M} \frac{\exp[\frac{-ct}{v_1}]}{v_1^{q(1+\delta)}} dv_1 \left( \int_t^{2\rho-t} \frac{1}{s} ds \right)^q \\ &\leq \rho^{1-q} \int_0^{v_M} \frac{dv_1}{v_1^{q(1+\delta)}} \int_0^\rho dt e^{-\frac{ct}{v_1}} \left( \int_t^{2\rho-t} \frac{1}{s} ds \right)^q \end{aligned}$$

$$\begin{aligned}
&= \rho^{1-q} \int_0^{v_M} \frac{dv_1}{v_1^{q(1+\delta)}} \int_0^\rho dt e^{-\frac{ct}{v_1}} \left( \int_t^{2\rho-t} \frac{1}{s} ds \right)^{q/p} \cdot \left( \int_t^{2\rho-t} \frac{1}{s} ds \right) \\
&\leq \rho^{1-q} \int_0^{v_M} \frac{dv_1}{v_1^{q(1+\delta)}} \int_0^\rho dt e^{-\frac{ct}{v_1}} \left( \int_t^{2\rho-t} \frac{1}{t} ds \right)^{q/p} \cdot (\log(2\rho - t) - \log t) \\
&\leq \rho^{1-q} \int_0^{v_M} \frac{dv_1}{v_1^{q(1+\delta)}} \int_0^\rho dt e^{-\frac{ct}{v_1}} \left( \frac{2\rho - 2t}{t} ds \right)^{q/p} \cdot C |\log t|
\end{aligned}$$

Since  $2\rho - 2t = 2r \leq 2\rho$ , we get

$$I_1 \leq C \rho^{1-q+\frac{q}{p}} \int_0^{v_M} \frac{dv_1}{v_1^{q(1+\delta)}} \int_0^\rho dt e^{-\frac{ct}{v_1}} \cdot \frac{|\log t|}{t^{\frac{q}{p}}}.$$

Let  $p_1 = \frac{3+\delta}{3\delta+1+(1-\delta)/2}$ ,  $q_1 = \frac{3+\delta}{1-\delta+(1-\delta)/2}$ . Since  $\delta < 1$ , it is easy to check  $p_1 > 1$ ,  $q_1 > 1$ ,  $\frac{1}{p_1} + \frac{1}{q_1} = 1$ . Moreover, from  $p_1 < \frac{3+\delta}{3\delta+1}$ ,  $q_1 < \frac{3+\delta}{1-\delta}$ , noting that  $p = \frac{4}{1-\delta}$ ,  $q = \frac{4}{3+\delta}$ , it is easy to see  $q(1+\delta) - \frac{1}{p_1} < 1$ ,  $\frac{qq_1}{p} < 1$ . From the Hölder inequality, noting that  $1 - q + \frac{q}{p} = 0$ , we have

$$\begin{aligned}
I_1 &\leq C \int_0^{v_M} \frac{dv_1}{v_1^{q(1+\delta)}} \left( \int_0^\rho e^{-\frac{ct p_1}{v_1}} dt \right)^{\frac{1}{p_1}} \cdot \left( \int_0^\rho \left( \frac{\log t}{t^{\frac{q}{p}}} \right)^{q_1} dt \right)^{\frac{1}{q_1}} \\
&\leq C \int_0^{v_M} \frac{dv_1}{v_1^{q(1+\delta) - \frac{1}{p_1}}} \cdot \left( \int_0^R \frac{\log^{q_1} t}{t^{\frac{qq_1}{p}}} dt \right)^{\frac{1}{q_1}} \\
&\leq \infty.
\end{aligned} \tag{5.21}$$

Next consider the term  $I_2$ . Let  $t(r) = r - \rho$ , and we have

$$\begin{aligned}
I_2 &\leq \frac{1}{\rho^q} \int_0^{R-\rho} (\rho + t) dt \int_0^{v_M} \frac{\exp\left[-\frac{ct}{v_1}\right]}{v_1^{q(1+\delta)}} dv_1 \left( \int_t^{2\rho+t} \frac{1}{s} ds \right)^q \\
&= \rho^{1-q} \int_0^{v_M} \frac{dv_1}{v_1^{q(1+\delta)}} \int_0^{R-\rho} dt e^{-\frac{ct}{v_1}} \left( \int_t^{2\rho+t} \frac{1}{s} ds \right)^q \\
&\quad + \frac{1}{\rho^q} \int_0^{v_M} \frac{dv_1}{v_1^{q(1+\delta)}} \int_0^{R-\rho} e^{-\frac{ct}{v_1}} t \left( \int_t^{2\rho+t} \frac{1}{s} ds \right)^q dt.
\end{aligned}$$

The first item is bounded by a procedure similar to that for proving  $I_1 < \infty$ , so we get

$$\begin{aligned}
I_2 &\leq C + \frac{C}{\rho^q} \int_0^{v_M} \frac{dv_1}{v_1^{q(1+\delta)}} \int_0^{R-\rho} e^{-\frac{ct}{v_1}} t \frac{\rho^q}{t^q} dt \\
&\leq C + C \int_0^{v_M} \frac{dv_1}{v_1^{q(1+\delta)}} \int_0^{R-\rho} e^{-\frac{ct}{v_1}} \cdot \frac{1}{t^{q-1}} dt.
\end{aligned}$$

Using the Hölder inequality with the same  $p_1$  and  $q_1$  as above, we have

$$\begin{aligned} I_2 &\leq C + C \int_0^{v_M} \frac{dv_1}{v_1^{q(1+\delta)}} \left( \int_0^{R-\rho} e^{-\frac{ctp_1}{v_1}} dt \right)^{\frac{1}{p_1}} \cdot \left( \int_0^{R-\rho} \frac{1}{t^{(q-1)q_1}} dt \right)^{\frac{1}{q_1}} \\ &\leq C + C \int_0^{v_M} \frac{dv_1}{v_1^{q(1+\delta) - \frac{1}{p_1}}} \cdot \left( \int_0^R \frac{1}{t^{(q-1)q_1}} dt \right)^{\frac{1}{q_1}}. \end{aligned}$$

Since  $q(1+\delta) - \frac{1}{p_1} < 1$  and  $(q-1)q_1 = \left(\frac{4}{3+\delta} - 1\right) \cdot \frac{3+\delta}{1-\delta+(1-\delta)/2} = \frac{2}{3} < 1$ , we get  $I_2 < \infty$ . This together with Eqs. (5.20) and (5.21) completes the proof of this lemma. Q.E.D.

**Lemma 5.2.7.** *For any  $\lambda$  with  $\operatorname{Re}\lambda > -\lambda^*$ ,  $\lim_{n \rightarrow \infty} \|\delta K_n(\lambda I - B)^{-1}K\|_1 = 0$ . The convergence is uniform with respect to  $\lambda \in \{\lambda : \operatorname{Re}\lambda \geq \beta_1\}$ , where  $\beta_1$  is any constant satisfying  $\beta_1 > -\lambda^*$ .*

**Proof:**  $\delta K_n(\lambda I - B)^{-1}K$  is an integral operator defined on  $L^1(G)$  with integral kernel given by (cf. Eq. (5.4))

$$\begin{aligned} &\frac{y'}{4r\rho} \int_0^{v_M} dv_1 \frac{\delta k_n(r, v, v_1)k(\rho, v_1, v')}{v_1} \int_{|r-\rho|}^{r+\rho} \frac{1}{s} \\ &\cdot \frac{\exp[-\Sigma_3(\lambda, \sigma, \dots)] + \alpha(v_1, \sqrt{\cdot}) \exp[-\Sigma_4(\lambda, \sigma, \dots) + \Sigma_3(\lambda, \sigma, \dots)]}{1 - \alpha(v_1, \sqrt{\cdot}) \exp[-\Sigma_4(\lambda, \sigma, \dots)]} ds, \end{aligned}$$

where

$$r = \sqrt{x^2 + y^2}, \quad \rho = \sqrt{x'^2 + y'^2},$$

and  $\alpha(v_1, \sqrt{\cdot})$ ,  $\Sigma_3(\lambda, \sigma, \dots)$  and  $\Sigma_4(\lambda, \sigma, \dots)$  have been previously defined.

So, for every  $\psi \in L^1(G)$ ,

$$\begin{aligned} &\|\delta K_n(\lambda I - B)^{-1}K\psi\|_1 \\ &\leq C \int dx \int y dy \int dv \int dx' \int dy' \int dv' |\psi(x', y', v')| \frac{y'}{4r\rho} \int_0^{v_M} dv_1 \frac{|\widehat{\delta k_n}(r, v, v_1)|}{v_1^{1+\delta} v^\delta} \\ &\cdot \int_{|r-\rho|}^{r+\rho} \frac{1}{s} \exp \left[ -\frac{(\operatorname{Re}\lambda + \lambda^*)|r-\rho|}{v_1} \right] ds. \end{aligned}$$

By virtue of the transform

$$x = z, \quad y = \sqrt{r^2 - z^2},$$

we get

$$\begin{aligned}
& \|\delta K_n(\lambda I - B)^{-1}K\psi\|_1 \\
& \leq C \int dx' \int y'dy' \int dv' |\psi(x', y', v')| \cdot \frac{1}{\rho} \int_0^R dr \int_{-r}^r \frac{r}{y} dz \frac{y}{r} \\
& \quad \cdot \int_0^{v_M} \frac{dv}{v^\delta} \int_0^{v_M} \frac{dv_1}{v_1^{1+\delta}} |\widetilde{\delta k_n}(r, v, v_1)| \exp \left[ -\frac{(\operatorname{Re}\lambda + \lambda^*)|r - \rho|}{v_1} \right] \int_{|r-\rho|}^{r+\rho} \frac{1}{s} ds \\
& \leq C \int \int \int dx' y' dy' dv' |\psi(x', y', v')| \cdot \frac{1}{\rho} \\
& \quad \cdot \int_0^R dr \int_0^{v_M} dv \int_0^{v_M} dv_1 |\widetilde{\delta k_n}(r, v, v_1)| \cdot \frac{r \exp[-(\operatorname{Re}\lambda + \lambda^*)|r - \rho|/v_1]}{v^\delta v_1^{1+\delta}} \int_{|r-\rho|}^{r+\rho} \frac{1}{s} ds \\
& \leq C \|\widetilde{\delta k_n}\|_p \int \int \int dx' y' dy' dv' |\psi(x', y', v')| \cdot \frac{1}{\rho} \\
& \quad \cdot \left\{ \int_0^R r^q dr \int_0^{v_M} dv \int_0^{v_M} dv_1 \frac{\exp[-q(\operatorname{Re}\lambda + \lambda^*)|r - \rho|/v_1]}{v^{\delta q} v_1^{(1+\delta)q}} \left( \int_{|r-\rho|}^{r+\rho} \frac{1}{s} ds \right)^q \right\}^{\frac{1}{q}}
\end{aligned}$$

where  $q = \frac{4}{3+\delta}, \frac{1}{p} + \frac{1}{q} = 1$ .  $q > 1$ , so  $r^q < Cr$ . Also,  $\delta q < 1$  (which can be easily verified), so

$$\begin{aligned}
& \|\delta K_n(\lambda I - B)^{-1}K\psi\|_1 \\
& \leq C \|\widetilde{\delta k_n}\|_p \int \int \int dx' y' dy' dv' |\psi(x', y', v')| \cdot \\
& \quad \cdot \left\{ \frac{1}{\rho^q} \int_0^R r dr \int_0^{v_M} dv_1 \frac{\exp[-q(\operatorname{Re}\lambda + \lambda^*)|r - \rho|/v_1]}{v_1^{(1+\delta)q}} \left( \int_{|r-\rho|}^{r+\rho} \frac{1}{s} ds \right)^q \right\}^{\frac{1}{q}}.
\end{aligned}$$

From Lemma 5.2.6, we obtain

$$\|\delta K_n(\lambda I - B)^{-1}K\psi\|_1 \leq C \|\psi\|_1 \cdot \|\widetilde{\delta k_n}\|_p,$$

So  $\|\delta K_n(\lambda I - B)^{-1}K\|_1 \leq C \|\widetilde{\delta k_n}\|_p$ , which completes the proof. Q. E. D.

Similarly, we can show the following.

**Lemma 5.2.8.** *For any  $\lambda$  with  $\operatorname{Re}\lambda > -\lambda^*$ ,  $\lim_{n \rightarrow \infty} \|\delta K_n(\lambda I - B_{\alpha_n})^{-1}K\|_1 = 0$ . The convergence is uniform with respect to  $\lambda \in \{\lambda : \operatorname{Re}\lambda \geq \beta_1\}$ , where  $\beta_1$  is any constant satisfying  $\beta_1 > -\lambda^*$ .*

**Lemma 5.2.9.** For any  $\lambda$  with  $\operatorname{Re}\lambda > -\lambda^*$ ,

$$\lim_{n \rightarrow \infty} \| [(\lambda I - B_{\alpha_n})^{-1} - (\lambda I - B)^{-1}] K(\lambda I - B)^{-1} K \|_1 = 0.$$

The convergence is uniform with respect to  $\lambda \in \{\lambda : \operatorname{Re}\lambda \geq \beta_1\}$ , where  $\beta_1$  is any constant satisfying  $\beta_1 > -\lambda^*$ .

**Proof.** From the expressions for  $(\lambda I - B)^{-1}$  (cf. Eq. (5.1)) and  $K(\lambda I - B)^{-1}K$  (cf. Eq. (5.4)), it is not difficult to see that for any  $\psi \in L^1(G)$ ,

$$\begin{aligned} & [(\lambda I - B_{\alpha_n})^{-1} - (\lambda I - B)^{-1}] K(\lambda I - B)^{-1} K \psi(x, y, v) \\ = & \left\{ \frac{\alpha_n(v, y) \exp[-\Sigma_2(\lambda, v, y)]}{v\{1 - \alpha_n(v, y) \exp[-\Sigma_2(\lambda, v, y)]\}} - \frac{\alpha(v, y) \exp[-\Sigma_2(\lambda, v, y)]}{v\{1 - \alpha(v, y) \exp[-\Sigma_2(\lambda, v, y)]\}} \right\} \\ & \cdot \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} du \exp[-\Sigma_1(\lambda, v, x, y, u)] \int dx' \int dy' \int dv' \psi(x', y', v') \\ & \cdot \frac{y'}{4\tilde{r}\rho} \int_0^{v_M} dv_1 \frac{k(\tilde{r}, v, v_1)k(\rho, v_1, v')}{v_1} \\ & \cdot \int_{|\tilde{r}-\rho|}^{\tilde{r}+\rho} \frac{1}{s} \cdot \frac{\exp[-\Sigma_3(\lambda, \sigma, \dots)] + \alpha(v_1, \sqrt{\cdot}) \exp[-\Sigma_4(\lambda, \sigma, \dots) + \Sigma_3(\lambda, \sigma, \dots)]}{1 - \alpha(v_1, \sqrt{\cdot}) \exp[-\Sigma_4(\lambda, \sigma, \dots)]} ds, \end{aligned}$$

where

$$\tilde{r} = \sqrt{u^2 + y^2},$$

and the expressions for  $\alpha$ ,  $\Sigma_3$  and  $\Sigma_4$  are as before, but with  $r$  replaced by  $\tilde{r}$ .

So,

$$\begin{aligned} & | [(\lambda I - B_{\alpha_n})^{-1} - (\lambda I - B)^{-1}] K(\lambda I - B)^{-1} K \psi(x, y, v) | \\ \leq & C \frac{|\delta\alpha_n(v, y)|}{v^{1+\delta}} \exp \left[ -\frac{2(\operatorname{Re}\lambda + \lambda^*)\sqrt{R^2 - y^2}}{v} \right] \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} du \\ & \cdot \exp \left[ -\frac{(\operatorname{Re}\lambda + \lambda^*)(x - u)}{v} \right] \int dx' \int dy' \int dv' |\psi(x', y', v')| \frac{y'}{4\tilde{r}\rho} \int_0^{v_M} \frac{dv_1}{v_1^{1+\delta}} \\ & \cdot \int_{|\tilde{r}-\rho|}^{\tilde{r}+\rho} \frac{\exp[-\Sigma_3(\lambda, \sigma, \dots)] + \exp[-\Sigma_4(\lambda, \sigma, \dots) + \Sigma_3(\lambda, \sigma, \dots)]}{s} ds. \end{aligned}$$

Noting that

$$|\exp[-\Sigma_3(\lambda, \sigma, \dots)]| \leq \exp \left[ -\frac{c|\tilde{r} - \rho|}{v_1} \right], \quad (5.22)$$

$$|\exp[-\Sigma_4(\lambda, \sigma, \dots)]| \leq \exp\left[-\frac{c|\tilde{r} - \rho|}{v_1}\right], \quad (5.23)$$

we obtain

$$\begin{aligned} & \| [(\lambda I - B_{\alpha_n})^{-1} - (\lambda I - B)^{-1}] K(\lambda I - B)^{-1} K\psi \|_1 \\ & \leq C \int dx' \int y' dy' \int dv' |\psi(x', y', v')| \frac{1}{\rho} \int y dy \int dv \frac{|\delta\alpha_n(v, y)|}{v^{1+\delta}} \exp\left[-\frac{2(\operatorname{Re}\lambda + \lambda^*)\sqrt{R^2 - y^2}}{v}\right] \\ & \quad \cdot \int_{-\sqrt{R^2 - y^2}}^{\sqrt{R^2 - y^2}} dx \int_{-\sqrt{R^2 - y^2}}^{\sqrt{R^2 - y^2}} du \exp\left[-\frac{(\operatorname{Re}\lambda + \lambda^*)(x - u)}{v}\right] \frac{1}{4\tilde{r}} \int_0^{v_M} \frac{dv_1}{v_1^{1+\delta}} \\ & \quad \cdot \int_{|\tilde{r} - \rho|}^{\tilde{r} + \rho} \frac{\exp[-c|\tilde{r} - \rho|/v_1]}{s} ds. \end{aligned}$$

By exchanging the order of integrations ( $\int dx$  and  $\int du$ ), noting that

$$\int_{-\sqrt{R^2 - y^2}}^{\sqrt{R^2 - y^2}} \exp\left[-\frac{(\operatorname{Re}\lambda + \lambda^*)(x - u)}{v}\right] dx = \frac{-v}{\operatorname{Re}\lambda + \lambda^*} \exp\left[-\frac{(\operatorname{Re}\lambda + \lambda^*)(x - z)}{v}\right] \Big|_{-\sqrt{R^2 - y^2}}^{\sqrt{R^2 - y^2}},$$

we obtain

$$\begin{aligned} & \| [(\lambda I - B_{\alpha_n})^{-1} - (\lambda I - B)^{-1}] K(\lambda I - B)^{-1} K\psi \|_1 \\ & \leq C \int dx' \int y' dy' \int dv' |\psi(x', y', v')| \cdot \frac{1}{\rho} \int y dy \int_{-\sqrt{R^2 - y^2}}^{\sqrt{R^2 - y^2}} du \int \frac{dv}{v^\delta} |\delta\alpha_n(v, y)| \\ & \quad \cdot \frac{1}{\tilde{r}} \int_0^{v_M} \frac{dv_1}{v_1^{1+\delta}} \int_{|\tilde{r} - \rho|}^{\tilde{r} + \rho} \frac{\exp[-c|\tilde{r} - \rho|/v_1]}{s} ds. \end{aligned}$$

By virtue of the transform

$$u = z, \quad y = \sqrt{\tilde{r}^2 - z^2},$$

we have

$$\begin{aligned} & \| [(\lambda I - B_{\alpha_n})^{-1} - (\lambda I - B)^{-1}] K(\lambda I - B)^{-1} K\psi \|_1 \\ & \leq C \int dx' \int y' dy' \int dv' |\psi(x', y', v')| \cdot \frac{1}{\rho} \int_0^R d\tilde{r} \int_{-\tilde{r}}^{\tilde{r}} \frac{\tilde{r}}{y} dz \cdot y \\ & \quad \cdot \int_0^{v_M} \frac{dv}{v^\delta} |\delta\alpha_n(v, \sqrt{\tilde{r}^2 - z^2})| \frac{1}{\tilde{r}} \int_0^{v_M} \frac{dv_1}{v_1^{1+\delta}} \int_{|\tilde{r} - \rho|}^{\tilde{r} + \rho} \frac{\exp[-c|\tilde{r} - \rho|/v_1]}{s} ds \end{aligned}$$

$$\begin{aligned}
&= C \int dx' \int y' dy' \int dv' |\psi(x', y', v')| \cdot \frac{1}{\rho} \\
&\quad \cdot \int_0^R d\tilde{r} \int_0^{v_M} dv \int_{-\tilde{r}}^{\tilde{r}} dz |\delta\alpha_n(v, \sqrt{\tilde{r}^2 - z^2})| \cdot \frac{1}{v^\delta} \int_0^{v_M} \frac{dv_1}{v_1^{1+\delta}} \int_{|\tilde{r}-\rho|}^{\tilde{r}+\rho} \frac{\exp[-c|\tilde{r} - \rho|/v_1]}{s} ds \\
&\leq C \int dx' \int y' dy' \int dv' |\psi(x', y', v')| \left\{ \int_0^R d\tilde{r} \int_0^{v_M} dv \int_{-\tilde{r}}^{\tilde{r}} dz |\delta\alpha_n(v, \sqrt{\tilde{r}^2 - z^2})|^p \right\}^{\frac{1}{p}} \\
&\quad \cdot \frac{1}{\rho} \left\{ \int_0^R d\tilde{r} \int_0^{v_M} dv \int_{-\tilde{r}}^{\tilde{r}} dz \frac{1}{v^{\delta q}} \left( \int_0^{v_M} \frac{\exp[-c|\tilde{r} - \rho|/v_1]}{v_1^{1+\delta}} dv_1 \int_{|\tilde{r}-\rho|}^{\tilde{r}+\rho} \frac{1}{s} ds \right)^q \right\}^{\frac{1}{q}}
\end{aligned}$$

Since  $q = \frac{4}{3+\delta}$ , it is easy to see  $\delta q < 1$ , and hence  $\int_0^{v_M} \frac{dv}{v^{\delta q}} < +\infty$ . Using the Hölder inequality, we see that

$$\begin{aligned}
&\int_0^{v_M} 1 \cdot \frac{\exp[-c|\tilde{r} - \rho|/v_1]}{v_1^{1+\delta}} \int_{|\tilde{r}-\rho|}^{\tilde{r}+\rho} \frac{1}{s} ds dv_1 \\
&\leq \left\{ \int_0^{v_M} 1 dv_1 \right\}^{\frac{1}{p}} \cdot \left\{ \int_0^{v_M} \frac{\exp[-cq|\tilde{r} - \rho|/v_1]}{v_1^{(1+\delta)q}} dv_1 \left( \int_{|\tilde{r}-\rho|}^{\tilde{r}+\rho} \frac{1}{s} ds \right)^q \right\}^{\frac{1}{q}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\| [(\lambda I - B_{\alpha_n})^{-1} - (\lambda I - B)^{-1}] K(\lambda I - B)^{-1} K\psi \|_1 \\
&\leq C \int dx' \int y' dy' \int dv' |\psi(x', y', v')| \left\{ \int_0^R d\tilde{r} \int_0^{v_M} dv \int_{-\tilde{r}}^{\tilde{r}} dz |\delta\alpha_n(v, \sqrt{\tilde{r}^2 - z^2})|^p \right\}^{\frac{1}{p}} \\
&\quad \cdot \left\{ \frac{1}{\rho^q} \int_0^R \tilde{r} d\tilde{r} \int_0^{v_M} \frac{\exp[-cq|\tilde{r} - \rho|/v_1]}{v_1^{(1+\delta)q}} dv_1 \left( \int_{|\tilde{r}-\rho|}^{\tilde{r}+\rho} \frac{1}{s} ds \right)^q \right\}^{\frac{1}{q}}.
\end{aligned}$$

From Lemma 5.2.6 and Lebesgue's dominated convergence theorem (noting that  $\delta\alpha_n \rightarrow 0$  almost everywhere), we have completed the proof. Q. E. D.

**Lemma 5.2.10.** *For any  $\lambda$  with  $\operatorname{Re}\lambda > -\lambda^* + (\beta_1 + \lambda^*)/2$ ,*

$$\lim_{n \rightarrow \infty} \|\delta F_n (\lambda I - B_{\alpha_n, \sigma_n})^{-1} K(\lambda I - B)^{-1} K\|_1 = 0.$$

*The convergence is uniform with respect to  $\lambda \in \{\lambda : \operatorname{Re}\lambda \geq \beta_1\}$ , where  $\beta_1$  is any constant satisfying  $\beta_1 > -\lambda^*$ .*

**Proof:** From the expressions for  $(\lambda I - B_{\alpha_n, \sigma_n})^{-1}$  (cf. Eq. (5.1)) and  $K(\lambda I - B)^{-1}K$  (cf. Eq. (5.4)), it is not difficult to see that for any  $\psi \in L^1(G)$ ,

$$\begin{aligned}
& \|\delta F_n (\lambda I - B_{\alpha_n, \sigma_n})^{-1} K(\lambda I - B)^{-1} K \psi\|_1 \\
& \leq C \int dx \int y dy \int dv |\delta \sigma_n(r, v)| \frac{1}{v^{1+\delta}} \int_{-\sqrt{R^2-y^2}}^x \exp\left[-\frac{c(x-u)}{v}\right] du \\
& \quad \cdot \int dx' \int dy' \int dv' |\psi(x', y', v')| \frac{y'}{4\tilde{r}\rho} \int_0^{v_M} dv_1 \frac{\exp[-c|\tilde{r}-\rho|/v_1]}{v_1^{1+\delta}} \int_{|\tilde{r}-\rho|}^{\tilde{r}+\rho} \frac{1}{s} ds \\
& \quad + C \int dx \int y dy \int dv |\delta \sigma_n(r, v)| \frac{1}{v^{1+\delta}} \exp\left[-\frac{2c\sqrt{R^2-y^2}}{v}\right] \\
& \quad \cdot \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} \exp\left[-\frac{c(x-u)}{v}\right] du \\
& \quad \cdot \int dx' \int dy' \int dv' |\psi(x', y', v')| \frac{y'}{4\tilde{r}\rho} \int_0^{v_M} dv_1 \frac{\exp[-c|\tilde{r}-\rho|/v_1]}{v_1^{1+\delta}} \int_{|\tilde{r}-\rho|}^{\tilde{r}+\rho} \frac{1}{s} ds \\
& = \|S_1\psi\|_1 + \|S_2\psi\|_1. \tag{5.24}
\end{aligned}$$

For the term  $S_1\psi$ , by exchanging the order of integrations ( $\int dx$  and  $\int du$ ), we obtain

$$\begin{aligned}
\|S_1\psi\|_1 & \leq C \int dx' \int y' dy' \int dv' |\psi(x', y', v')| \cdot \frac{1}{\rho} \int_0^R y dy \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} du \\
& \quad \cdot \int_0^{v_M} \frac{dv}{v^{1+\delta}} \int_{-\sqrt{R^2-y^2}}^u dx \exp\left[-\frac{c(x-u)}{v}\right] |\delta \sigma_n(\sqrt{x^2+y^2}, v)| \\
& \quad \cdot \frac{1}{4\tilde{r}} \int_0^{v_M} dv_1 \frac{\exp[-c|\tilde{r}-\rho|/v_1]}{v_1^{1+\delta}} \int_{|\tilde{r}-\rho|}^{\tilde{r}+\rho} \frac{1}{s} ds.
\end{aligned}$$

Noting that

$$\begin{aligned}
& \int_0^{v_M} \frac{dv}{v^{1+\delta}} \int_{-\sqrt{R^2-y^2}}^u dx \exp\left[-\frac{c(x-u)}{v}\right] |\delta \sigma_n(\sqrt{x^2+y^2}, v)| \\
& = \int_0^{v_M} dv \int_{-\sqrt{R^2-y^2}}^u dx \frac{\exp[-c(x-u)/v]}{v^{1+\delta}} \cdot |\delta \sigma_n(\sqrt{x^2+y^2}, v)| \\
& \leq \left\{ \int_0^{v_M} dv \int_{-\sqrt{R^2-y^2}}^u dx \frac{\exp[-cq(x-u)/v]}{v^{(1+\delta)q}} \right\}^{\frac{1}{q}} \\
& \quad \cdot \left\{ \int_0^{v_M} dv \int_{-\sqrt{R^2-y^2}}^u dx |\delta \sigma_n(\sqrt{x^2+y^2}, v)|^p \right\}^{\frac{1}{p}}
\end{aligned}$$

$$\begin{aligned}
&\leq C \left\{ \int_0^{v_M} \frac{1}{v^{(1+\delta)q-1}} dv \right\}^{\frac{1}{q}} \cdot \left\{ \int_0^{v_M} dv \int_{-\sqrt{R^2-y^2}}^u dx |\delta\sigma_n(\sqrt{x^2+y^2}, v)|^p \right\}^{\frac{1}{p}} \\
&\leq C \left\{ \int_0^{v_M} dv \int_{-\sqrt{R^2-y^2}}^u dx |\delta\sigma_n(\sqrt{x^2+y^2}, v)|^p \right\}^{\frac{1}{p}},
\end{aligned}$$

(it is easy to verify that  $(1+\delta)q-1 < 1$ ) we have

$$\begin{aligned}
\|S_1\psi\|_1 &\leq C \int dx' \int y' dy' \int dv' |\psi(x', y', v')| \cdot \frac{1}{\rho} \\
&\quad \cdot \left\{ \iint \frac{y}{\tilde{r}} dy du \left\{ \int_0^{v_M} dv \int_{-\sqrt{R^2-y^2}}^u dx |\delta\sigma_n(\sqrt{x^2+y^2}, v)|^p \right\}^{\frac{1}{p}} \right. \\
&\quad \cdot \int_0^{v_M} dv_1 \frac{\exp[-c|\tilde{r}-\rho|/v_1]}{v_1^{1+\delta}} \int_{|\tilde{r}-\rho|}^{\tilde{r}+\rho} \frac{1}{s} ds \\
&\leq C \int dx' \int y' dy' \int dv' |\psi(x', y', v')| \cdot \frac{1}{\rho} \\
&\quad \cdot \left\{ \iint \frac{y}{\tilde{r}} dy du \int_0^{v_M} dv \int_{-\sqrt{R^2-y^2}}^u dx |\delta\sigma_n(\sqrt{x^2+y^2}, v)|^p \right\}^{\frac{1}{p}} \\
&\quad \cdot \left\{ \iint \frac{y}{\tilde{r}} dy du \left( \int_0^{v_M} \frac{\exp[-c|\tilde{r}-\rho|/v_1]}{v_1^{1+\delta}} dv_1 \int_{|\tilde{r}-\rho|}^{\tilde{r}+\rho} \frac{1}{s} ds \right)^q \right\}^{\frac{1}{q}}.
\end{aligned}$$

By virtue of the transform

$$u = z, \quad y = \sqrt{\tilde{r}^2 - z^2},$$

we get

$$\begin{aligned}
\|S_1\psi\|_1 &\leq C \int dx' \int y' dy' \int dv' |\psi(x', y', v')| \\
&\quad \cdot \left\{ \int_0^R d\tilde{r} \int_{-\tilde{r}}^{\tilde{r}} dz \frac{\tilde{r}}{y} \cdot \frac{y}{\tilde{r}} \int_0^{v_M} dv \int_{-\sqrt{R^2-y^2}}^z dx |\delta\sigma_n(\sqrt{x^2+y^2}, v)|^p \right\}^{\frac{1}{p}} \\
&\quad \cdot \left\{ \frac{1}{\rho^q} \int_0^R d\tilde{r} \int_{-\tilde{r}}^{\tilde{r}} dz \frac{\tilde{r}}{y} \cdot \frac{y}{\tilde{r}} \int_0^{v_M} \frac{\exp[-cq|\tilde{r}-\rho|/v_1]}{v_1^{(1+\delta)q}} dv_1 \left( \int_{|\tilde{r}-\rho|}^{\tilde{r}+\rho} \frac{1}{s} ds \right)^q \right\}^{\frac{1}{q}} \\
&= C \int dx' \int y' dy' \int dv' |\psi(x', y', v')| \\
&\quad \cdot \left\{ \int_0^R d\tilde{r} \int_{-\tilde{r}}^{\tilde{r}} dz \int_0^{v_M} dv \int_{-\sqrt{R^2-y^2}}^z dx |\delta\sigma_n(\sqrt{x^2+y^2}, v)|^p \right\}^{\frac{1}{p}}
\end{aligned}$$

$$\cdot \left\{ \frac{1}{\rho^q} \int_0^R \tilde{r} d\tilde{r} \int_0^{v_M} \frac{\exp[-cq|\tilde{r} - \rho|/v_1]}{v_1^{(1+\delta)q}} dv_1 \left( \int_{|\tilde{r}-\rho|}^{\tilde{r}+\rho} \frac{1}{s} ds \right)^q \right\}^{\frac{1}{q}}.$$

By Lebesgue's dominated convergence theorem and Lemma 5.2.6, we see that  $\|S_1\|_1 \rightarrow 0$ .

Similarly, we can show  $\|S_2\|_1 \rightarrow 0$ . This together with Eq. (5.24) completes the proof. Q. E. D.

**Theorem 5.2.11.** *For every  $\lambda$  with  $\operatorname{Re}\lambda > -\lambda^*$ ,  $[K(\lambda I - B)^{-1}]^2 K$  is a compact operator on  $L^1(G)$ .*

**Proof.** By virtue of the relation

$$(\lambda I - B_{\alpha_n})^{-1} - (\lambda I - B_{\alpha_n, \sigma_n})^{-1} = (\lambda I - B_{\alpha_n})^{-1} \delta F_n (\lambda I - B_{\alpha_n, \sigma_n})^{-1},$$

we get

$$\begin{aligned} & K(\lambda I - B)^{-1} K(\lambda I - B)^{-1} K \\ = & K[(\lambda I - B)^{-1} - (\lambda I - B_{\alpha_n})^{-1}] K(\lambda I - B)^{-1} K \\ & + K(\lambda I - B_{\alpha_n})^{-1} K(\lambda I - B)^{-1} K \\ = & K[(\lambda I - B)^{-1} - (\lambda I - B_{\alpha_n})^{-1}] K(\lambda I - B)^{-1} K \\ & + (K_n + \delta K_n) [(\lambda I - B_{\alpha_n, \sigma_n})^{-1} + (\lambda I - B_{\alpha_n})^{-1} \delta F_n (\lambda I - B_{\alpha_n, \sigma_n})^{-1}] \\ & \cdot (K_n + \delta K_n) (\lambda I - B)^{-1} K \\ = & K[(\lambda I - B)^{-1} - (\lambda I - B_{\alpha_n})^{-1}] K(\lambda I - B)^{-1} K \\ & + K_n (\lambda I - B_{\alpha_n, \sigma_n})^{-1} K_n (\lambda I - B)^{-1} K \\ & + \delta K_n (\lambda I - B_{\alpha_n})^{-1} K(\lambda I - B)^{-1} K \\ & + K_n (\lambda I - B_{\alpha_n})^{-1} \delta F_n (\lambda I - B_{\alpha_n, \sigma_n})^{-1} K(\lambda I - B)^{-1} K \\ & + K_n (\lambda I - B_{\alpha_n, \sigma_n})^{-1} \delta K_n (\lambda I - B)^{-1} K \end{aligned} \quad (5.25)$$

Since  $\alpha(v, \mu)$ ,  $\sigma(r, v)$  and  $\tilde{k}(r, v, v') = v^\delta k(r, v, v')$  are smooth functions, it can be shown that for every integer  $n$ ,  $K_n (\lambda I - B_{\alpha_n, \sigma_n})^{-1} K_n$  is a compact operator on  $L^1(G)$  (cf. [18]). Thus, we get the conclusion from Eq. (5.25) and Lemma 5.2.7, 5.2.8, 5.2.9, 5.2.10.

**Theorem 5.2.12.** *Suppose (H1), (H2) and (H3') are satisfied, and  $\beta_1 > -\lambda^*$  is an arbitrarily given constant. Then for every  $\varepsilon > 0$ , there exists a positive constant  $\tilde{\tau}$  independent of  $\beta \in [\beta_1, +\infty)$ , such that*

$$\| [K(\lambda I - B)^{-1}]^2 K \|_1 < \varepsilon$$

uniformly in  $\{\lambda = \beta + i\tau : \beta \geq \beta_1, |\tau| \geq \tilde{\tau}\}$ .

**Proof.** For any  $\tilde{\varepsilon} > 0$ , it follows from Lemma 5.2.7 – Lemma 5.2.10 that there exists an integer  $n_0$  independent of  $\lambda \in \{\lambda : \operatorname{Re}\lambda \geq \beta_1\}$  such that

$$\|\delta K_{n_0}(\lambda I - B)^{-1}K\|_1 < \tilde{\varepsilon}, \quad (5.26)$$

$$\|\delta K_{n_0}(\lambda I - B_{\alpha_{n_0}})^{-1}K\|_1 < \tilde{\varepsilon}, \quad (5.27)$$

$$\| [(\lambda I - B_{\alpha_{n_0}})^{-1} - (\lambda I - B)^{-1}] K(\lambda I - B)^{-1}K \|_1 < \tilde{\varepsilon}, \quad (5.28)$$

$$\|\delta F_n (\lambda I - B_{\alpha_{n_0}, \sigma_{n_0}})^{-1} K(\lambda I - B)^{-1}K \|_1 < \tilde{\varepsilon}. \quad (5.29)$$

Noting that  $\alpha_{n_0}(v, \mu)$ ,  $\sigma_{n_0}(r, v)$  and  $\tilde{k}_{n_0}(r, v, v')$  are smooth functions, we see from Theorem 5.2.5 that there exists a positive constant  $\tilde{\tau}$  independent of  $\beta \in [\beta_1, +\infty)$ , such that

$$\|K_{n_0}(\lambda I - B_{n_0, n_0})^{-1}K_{n_0}\|_1 < \tilde{\varepsilon} \quad (5.30)$$

uniformly in  $\{\lambda = \beta + i\tau : \beta \geq \beta_1, |\tau| \geq \tilde{\tau}\}$ .

From Eqs. (5.26) – (5.30) and Eq. (5.25) (with  $n = n_0$ ), we get the conclusion. Q. E. D.

From Theorem 5.2.11, 5.2.12 and the well known Gohberg's theorem (cf. [15, page 259, Cor. 11.6]), we have the following results about  $\sigma(A)$  (cf. [22, Lemma 1.1]).

**Theorem 5.2.13.** *Under the conditions (H1), (H2) and (H3'), the following assertions hold for any constant  $\beta_1 > -\lambda^*$  in the setting of  $L^1(G)$ :*

(i).  $\operatorname{Pas}(A) := \sigma(A) \cap \{\lambda : \operatorname{Re}\lambda > -\lambda^*\}$  contains at most countable isolated elements, each of which is an eigenvalue of  $A$  with finite algebraic multiplicity.

(ii). The set  $\sigma(A) \cap \{\lambda : \operatorname{Re}\lambda \geq \beta_1\}$  contains at most finite elements.

(iii). There exists a positive constant  $\tilde{\tau}$  independent of  $\beta \in [\beta_1, +\infty)$ , such that  $\|(\lambda I - A)^{-1}\|_1$  is uniformly bounded in  $\{\lambda = \beta + i\tau : \beta \geq \beta_1, |\tau| \geq \tilde{\tau}\}$ .

### 5.3 Spectral Properties of $T(t)$ in $L^1$

We will conclude by discussing some aspects of the spectral properties of  $T(t)$ , and these spectral properties are closely related to the asymptotic behavior of  $f(t)$ .

From Theorem 5.2.13, the eigenvalues of  $A$  lying in the half-plane  $\operatorname{Re}\lambda > -\lambda^*$  can be ordered in such a way that the real part decreases. Suppose  $\lambda_1, \lambda_2, \dots, \lambda_m, \lambda_{m+1}, \dots$  are eigenvalues of  $A$ ,  $\operatorname{Re}\lambda_1 \geq \operatorname{Re}\lambda_2 \geq \dots \geq \operatorname{Re}\lambda_m > \operatorname{Re}\lambda_{m+1} \geq \dots > -\lambda^*$ , and  $\{\lambda : \operatorname{Re}\lambda > -\lambda^*\} \setminus \{\lambda_n : n = 1, 2, \dots\} \subset \rho(A)$ . For convenience' sake, the eigenvalues of  $A$  are ordered repeatedly according to their algebraic multiplicities (cf. [16, page 108]), i.e.,  $\lambda_1, \lambda_2, \dots, \lambda_m, \lambda_{m+1}, \dots$  are repeated eigenvalues of  $A$  (see [16, page 42]).

Similar to the discussion for slab geometry, we can obtain the following conclusions.

**Theorem 5.3.1.** *Let  $\lambda_1, \lambda_2, \dots, \lambda_m, \lambda_{m+1}, \dots$  be repeated eigenvalues of  $A$ ,  $\operatorname{Re}\lambda_1 \geq \operatorname{Re}\lambda_2 \geq \dots \geq \operatorname{Re}\lambda_m > \operatorname{Re}\lambda_{m+1} \geq \dots > -\lambda^*$ ,  $\{\lambda : \operatorname{Re}\lambda > -\lambda^*\} \setminus \{\lambda_n : n = 1, 2, \dots\} \subset \rho(A)$ , and let  $P_m$  be the projection operator of  $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$  corresponding to  $A$ . If (H1), (H2), (H3') and (H4') are satisfied, then for every  $\varepsilon > 0$ , there exists a positive constant  $M_1$  such that*

$$\|T(t)(I - P_m)\|_1 \leq M_1 \exp\{(\operatorname{Re}\lambda_{m+1} + \varepsilon)t\}.$$

**Theorem 5.3.2.** *In the setting of  $L^1(G)$ , if (H1), (H2), (H3') and (H4') are satisfied, then the spectrum of  $T(t)$  outside the disk  $\{\lambda : |\lambda| \leq \exp(-\lambda^*t)\}$  consists of isolated eigenvalues of  $T(t)$  with finite algebraic multiplicity, and the accumulation points of the set  $\sigma(T(t)) \cap \{\lambda : |\lambda| > \exp(-\lambda^*t)\}$ , if they exist, could only appear on the circle  $\{\lambda : |\lambda| = \exp(-\lambda^*t)\}$ .*

If  $\operatorname{Pas}(A)$  is not empty, then the asymptotic behavior of the solution  $f(t)$  of Eq. (I) can be easily derived from Theorem 5.3.1 and Theorem 5.3.2.

# Bibliography

- [1] N. Angelescu, N. Marinescu and V. Protopopescu, Linear monoenergetic transport with reflecting boundary conditions, *Rev. Roum. Phys.*, 19 (1974), 17–26.
- [2] W. Arendt, Vector-valued Laplace transform and Cauchy problem, *Israel J. Math.*, 59:3 (1987), 327–352.
- [3] A. Belleni–Morante, The initial value problem for neutron transport in a slab with perfect reflection boundary conditions, *J. Math. Anal. Appl.*, 30 (1970), 353–374.
- [4] J. Bergh and J. Löfström, *Interpolation Spaces, An Introduction*, Springer-Verlag, 1976.
- [5] E. B. Davies, *One-parameter Semigroups*, Academic Press, London, 1980.
- [6] R. Derndinger, Über das spektrum positiver generatoren, *Math. Z.*, 172 (1980), 281–193.
- [7] G. Frosali, C. van der Mee and W. Protopopescu, Transport equation with boundary conditions of reverse reflection type, *Math. Methods Appl. Sci.*, 10 (1988), 15–35.
- [8] W. Greenberg, C. van der Mee and V. Protopopescu, *Boundary Value Problems in Abstract Kinetic Theory*, Birkhäuser Verlag, 1987.
- [9] G. Greiner and R. Nagel, On the stability of strongly continuous semigroups of positive operators on  $L^2(\mu)$ , *Ann. Scuola Norm. Sup. Pisa*, 10 (1983), 257–262.
- [10] G. Greiner, Spectral properties and asymptotic behavior of the linear transport equation, *Math. Z.*, 185 (1984), 167–177.
- [11] I. Herbst, The spectrum of Hilbert space semigroups, *J. Operator Theory*, 10 (1983), 87–94.

- [12] E. Hille and R. E. Phillips, *Functional Analysis and Semigroups*, Amer. Math. Soc. Coll. Publ., Providence, R. I., 1957.
- [13] Falun Huang, Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces, *Ann. of Diff. Eqs.*, 1:1 (1985), 43-56.
- [14] K. Jörgens, An asymptotic expansion in the theory of neutron transport, *Comm. Pure Appl. Math.*, 11 (1958), 219-242.
- [15] H. G. Kapper, C. G. Lekkerkerker and J. Hejtmanex, *Spectral Methods in Transport Theory*, Birkhäuser Verlag, 1982.
- [16] T. Kato, *Perturbation Theory for Linear Operators*, Second Corrected Printing of the Second Edition, Springer-Verlag, 1984.
- [17] H. Kellermann and M. Hilber, Integrated semigroups, *J. Func. Anal.*, 84 (1989), 160-180.
- [18] E. W. Larsen and P. F. Zweifel, On the spectrum of the linear transport operator, *J. Math. Phys.*, 15 (1974), 1987-1997.
- [19] J. Lehner and G. M. Wing, Solution of the linearized Boltzmann transport equation for the slab geometry, *Duke Math. J.*, 23 (1956), 125-142.
- [20] Peng Lei and Mingzhu Yang, On the spectrum of transport operator in a homogeneous sphere with partly reflection boundary conditions, *Kexue Tongbao*, 31:24 (1986), 1867-1871 (in Chinese).
- [21] J. Mika, R. Stankiewicz and T. Trombetti, Effective solution to the initial value problem in neutron thermalization theory, *J. Math. Anal. Appl.*, 25 (1969), 149-161.
- [22] M. Mokhtar-Kharroubi, Time asymptotic behavior and compactness in transport theory, *Eur. J. Mech., B / Fluids*, 11 (1992), 39-68.
- [23] R. Nagel ed., *One-Parameter Semigroups of Positive Operators*, Lect. Notes in Math., 1184, Springer-Verlag, 1986.
- [24] F. Neubrander, Integrated semigroups and their applications to the abstract Cauchy problem, *Pacific J. Math.*, 135:1 (1988), 111-155.
- [25] R. Van Norton, On the real spectrum of a mono-energetic neutron transport operator, *Comm. Pure Appl. Math.*, 15 (1962), 149-158.

- [26] C. V. Pao, Positive solution and subcriticality of energy dependent system in slab geometry, *J. Math. Phys.*, 18 (1977), 544–550.
- [27] C. V. Pao, Asymptotic behavior of the solution for the time dependent neutron transport problems, *J. Integral Eqs.*, 1 (1979), 131–152.
- [28] J. Paulling, Criticality estimates for neutron transport in a slab with generalized reflecting boundary conditions, *J. Integ. Equ.*, 5 (1983), 1–32.
- [29] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.
- [30] J. Pruss, On the spectrum of  $C_0$  semigroups, *Transl. Amer. Math. Soc.*, 284:2 (1984), 847–857.
- [31] Degong Song, Miansen Wang and Guangtian Zhu, Asymptotic expansion and asymptotic behavior of the solution for the time dependent neutron transport problem in a slab with generalized boundary conditions, *Sys. Sci. & Math. Scis.*, 3:2 (1990), 102–125.
- [32] Degong Song, Miansen Wang and Guangtian Zhu, A note on the spectrum of neutron transport operator in a slab with generalized boundary conditions, *Sys. Sci. & Math. Scis.*, 5:2 (1992), 97–107.
- [33] Degong Song, Miansen Wang and Guangtian Zhu, A note on the spectrum of the transport operator with continuous energy, *Transport Theory and Statistical Physics*, 22:5 (1993), 709–721.
- [34] Degong Song and Miansen Wang, On the existence of critical eigenvalue for a class of steady-state neutron transport system, *J. Xi'an Jiaotong Univ.*, 27:3 (1993), 105–112 (in Chinese).
- [35] Degong Song, An asymptotic expansion theorem in transport theory, *Transport Theory and Statistical Physics*, 24:9 (1995), 1385–1399.
- [36] Degong Song, Asymptotic behavior of the classical time-dependent solution for particle transport in a finite body with  $[0, v_M]$  energy, *Transport Theory and Statistical Physics*, 25:1 (1996), 103–120.
- [37] P. Takac, A spectral mapping theorem for the exponential function in linear transport theory, *Transport Theory and Statistical Physics*, 14 (1985), 655–667.

- [38] I. Vidav, Spectra of perturbed semigroups with applications to transport theory, *J. Math. Anal. Appl.*, 30 (1970), 264–279 .
- [39] J. Voigt, A perturbation theorem for the essential spectral radius of strongly continuous semigroups, *Mh. Math.*, 90 (1980), 153–161.
- [40] J. Voigt, Positivity in time dependent linear transport theory, *Acta Appl. Math.*, 2 (1984), 311–331.
- [41] J. Voigt, Spectral properties of the neutron transport equation, *J. Math. Anal. Appl.*, 106 (1985), 140–153.
- [42] Haiyan Wang, Mingzhu Yang and Wenwei Wang, On the spectrum of a spherical symmetric transport operator with generalized boundary conditions, *J. South-east Univ.*, 19:5 (1989), 7–17 (in Chinese).
- [43] L. W. Weis, A generalization of the Vidav–Jørgens perturbation theorem for semigroups and its application to transport theory, *J. Math. Anal. Appl.*, 129 (1988), 6–23.
- [44] V. Wrobel, Asymptotic behavior of  $C_0$ -semigroups in B-convex spaces, *Indiana Univ. Math. J.*, 38 (1989), 101–114.
- [45] Xianwen Zhang, Spectral properties of a transport operator in a slab with integral boundary conditions, *J. Math. Anal. Appl.*, 222:1 (1998), 192–207.
- [46] Xianwen Zhang and BenzhongLiang, On the spectrum of a one-velocity transport operator with Maxwell boundary condition, *J. Math. Anal. Appl.*, 202:3 (1996), 920–939.
- [47] Silvia Totaro, Study of the free streaming operator in slab geometry in dependence of the boundary conditions, *Math. Methods Appl. Sci.*, 20:8 (1997), 717–736.
- [48] Khalid Latrach and Aref Jeribi, On the essential spectrum of transport operators on  $L_1$ -spaces, *J. Math. Phys.*, 37:12 (1996), 6486–6494.
- [49] Khalid Latrach, Some remarks on the essential spectrum of transport operators with abstract boundary conditions, *J. Math. Phys.*, 35:11 (1994), 6199–6212.
- [50] Khalid Latrach, Time asymptotic behaviour for linear transport equations with abstract boundary conditions in slab geometry, *Transport Theory and Statistical Physics*, 23:5 (1994), 633–670.

- [51] Khalid Latrach and Aref Jeribi, Some results on Fredholm operators, essential spectra, and application, *J. Math. Anal. Appl.*, 225:2 (1998), 461–485.
- [52] M. Mokhtar-Kharroubi, *Mathematical topics in neutron transport theory*, Series on Advances in Mathematics for Applied Sciences, 46. World Scientific Publishing Co., Inc., River Edge, NJ, 1997.
- [53] D. C. Sahni, N. S. Garis and N. G. Sjostrand, Spectrum of one-speed neutron transport operator with reflective boundary conditions in slab geometry, *Transport Theory and Statistical Physics*, 24 (1995), 629–656.
- [54] E. B. Dahl and D. C. Sahni, Complex time eigenvalues of the one speed neutron transport equation for a homogeneous sphere, *Transport Theory and Statistical Physics*, 12:4 (1983/84), 341–368.
- [55] G. Milton Wing, *An introduction to transport theory*, John Wiley and Sons, Inc., New York-London 1962.
- [56] Zhifeng Kuang, Mingzhu Yang and Xu Wang, Complex eigenvalues of a monoenergetic neutron transport operator, *Transport Theory and Statistical Physics*, 26 (1997), 253–261.
- [57] Jun Chen and Mingzhu Yang, Linear transport equation with specular reflection boundary condition, *Transport Theory and Statistical Physics*, 20:4 (1991), 281–306.
- [58] Mingzhu Yang and Guangtian Zhu, Spectrum of neutron transport operator with anisotropic scattering and fission, *Sci. Sinica*, 24:4 (1981), 476–482.

# Vita

Degong Song was born and grew up in Tongchuan, a city in Shaanxi Province of P. R. China. In 1982, he entered Fudan University when he was 16 years old. After he graduated in 1986 with Bachelor of Science degree in Mathematics, he entered the Graduate School of Xi'an Jiaotong University, where he completed a Master's degree in Applied Mathematics in April 1989. That same year, he joined the faculty of the Mathematical Department of Xi'an Jiaotong University. During his 8-year service in Xi'an Jiaotong University, he had been in the Department of Mathematics, Center of Applied Mathematics, Institute of Information and System Sciences, Institute of Software Development. He was the leader of a project supported by the National Natural Science Foundation of China.

In 1997, he left Xi'an Jiaotong University and began his graduate work toward a Ph. D. degree in Mathematics and a M. S. degree in Computer Science at Virginia Polytechnic Institute and State University. He received his Master's degree in Computer Science in May 1999.