Chapter 3. Statistical Analysis of the Monte Carlo Ray-Trace Method in Radiation Heat Transfer

All measurements have a degree of uncertainty associated with them, and it is imperative to understand and deal with such uncertainties. There is a sense in which the Monte Carlo ray-trace method may be thought of as a numerical experiment involving "measurements." This encourages the idea that the uncertainties of the results obtained using the MCRT method may be expected to be predictable using standard statistical methods. The sources of uncertainties in a computer model are commonly due to (1) limitations in the resolution or discretization in the model, (2) lack of repeatability, (3) under-sampling, and (4) compromises in the mathematics involved in the model. A protocol is needed to identify and quantify the contributions of these factors to the ultimate uncertainty in the results obtained and the related confidence interval associated with this uncertainty.

3.1 Uncertainty and confidence intervals

A very general definition of uncertainty is that it is the measure of the indeterminacy, or error, in the results of any experiment or set of calibrated measurements [Dietrich, 1991]. The confidence interval associated with this uncertainty is the numerical interval that, with a specified probability, includes the true value of the variable that is being measured. Different approaches exist to calculate a confidence interval in a Monte Carlo algorithm [Fishman, 1996]. Some methods use the properties of binomial distributions, others use probabilistic limit theorems, and still others use

probabilistic inequalities. In the present study the confidence intervals are calculated treating the distribution factors as population proportions.

3.2 Distribution factors as population proportions

The basic idea of treating the exchange fraction of energy (modeled as discrete energy bundles) between two surfaces as a series of independent Bernoulli trials was first seen by the current author in a publication by Maltby and Burns [1991]. However the current author considered the statistical inferences to be incomplete and decided to pursue a development of her own.

The following ideas are based on a paper by Sanchez *et al*. [2001] where the authors describe the approach used to calculate the confidence intervals of the distribution factors in a Monte Carlo method by defining such distribution factors as proportions. This is also the approach taken in Mahan [2002]

Koopmans [1987] defines the population proportion as the probability of "success" in a binomial experiment (e.g., the number of heads in repeated coin tosses). In other words, if Y is a binomial variable, the population proportion π is the probability of obtaining one of the two possible outcomes of the experiment. Sometimes it is important to know not only the variable Y but also this variable divided by the sample size n. This new value is an estimator of the population proportion π. In the MCRT environment, the tracing of N_i energy bundles emitted by surface i can be considered a binomial experiment where "success" is achieved when one of the energy bundles is absorbed by the surface j. Accordingly, the diffuse-specular radiation distribution factor, which can be estimated as

$$D_{ij} \cong \frac{N_{ij}}{N_i} \tag{2.7}$$

where N_{ij} is the number of energy bundles emitted by surface i and absorbed by surface j, can be thought of as a sample proportion that estimates the value of the population proportion. As is the case in any proportion, the two possible outcomes in the ray trace

are D_{ij} and $1 - D_{ij}$. It is important to note that the "true" value of the distribution factor is based on an infinite population of uniformly distributed random numbers while the estimate is based on a large but finite sample of this population.

3.3 Confidence interval for distribution factors

Using the approach suggested above, the confidence interval for the binomial parameter D'_{ij}, the diffuse-specular, total distribution factor, can be estimated using the expression

$$p - W_c s \le D'_{ij} \le p + W_c s \tag{3.1}$$

where p is an estimator of π, W_c is the critical value of the W statistic, and s is the estimated sample standard deviation given by

$$s = \sqrt{\frac{p(1-p)}{N}} \tag{3.2}$$

with N being the number of energy bundles traced. The W statistic for binomial distributions is tabulated in standard statistics texts as a function of the confidence interval and number of degrees of freedom.

Equation 3.1 can be written in terms of the estimated distribution factor

$$D'^{e}_{ij} - W_c \sqrt{\frac{D'^{e}_{ij}(1-D'^{e}_{ij})}{N}} \le D'^{t}_{ij} \le D'^{e}_{ij} + W_c \sqrt{\frac{D'^{e}_{ij}(1-D'^{e}_{ij})}{N}} \tag{3.3}$$

where now D'^{t}_{ij} is the (unknown) true value of the distribution factor and D'^{e}_{ij} is the MCRT estimate based on tracing N energy bundles.

Subtracting D'^{e}_{ij} from each term of Equation 3.3, yields

$$-W_c \sqrt{\frac{D'^{e}_{ij}\left(1-D'^{e}_{ij}\right)}{N}} \le D'^{t}_{ij} - D'^{e}_{ij} \le W_c \sqrt{\frac{D'^{e}_{ij}\left(1-D'^{e}_{ij}\right)}{N}} \tag{3.4}$$

The magnitude of the difference between a true value and its estimated value can be defined as the error in the estimate. Equation 3.4 therefore provides a bound on the error of the estimate, which can be written as

$$\mathrm{Err}_{D'_{ij}} \equiv \left| D'^{t}_{ij} - D'^{e}_{ij} \right| \le W_c \sqrt{\frac{D'^{e}_{ij}(1 - D'^{e}_{ij})}{N}} = \omega_{D'_{ij}} \tag{3.5}$$

The bounds described in Equation 3.5 can be thought of as the uncertainty in the estimate of the distribution factor between surface i and j, ω'_{Dij}. In other words we can be sure to a stated level of confidence that the absolute value of the error can have values between zero and the uncertainty as predicted by the right-hand side of Equation 3.

It is useful to calculate the mean value of the error of all the radiation distribution factors in an enclosure. At this point it is important to keep in mind that the radiation distribution factor matrix is obtained by modeling the emission of energy bundles from each surface in the cavity and counting the energy bundles absorbed in every one of the surfaces. In this case

$$\left\langle \left\langle \mathrm{Err}_{D'_{ij}} \right\rangle \right\rangle \equiv \mathrm{Err}_{D'} \le \left\langle \left\langle W_c \sqrt{\frac{D'^{e}_{ij}(1 - D'^{e}_{ij})}{N}} \right\rangle \right\rangle \tag{3.6}$$

Equation 3.6 can be evaluated using the Jensen's inequality, which states that for a real, continuous, convex function

$$E[f(x)] \ge f(E[x]) \quad \text{if f is concave upward} \tag{3.7}$$

and

$$E[f(x)] \le f(E[x]) \quad \text{if f is concave downward} \tag{3.8}$$

where the expected value E is the mean value of the function f(x) over x. In other words

$$\frac{1}{M}\sum_{m=1}^{M} f(x_m) \geq f\left(\frac{1}{M}\sum_{m=1}^{M} x_m\right) \quad \text{if f is concave upward} \tag{3.9}$$

or

$$\frac{1}{M}\sum_{m=1}^{M} f(x_m) \leq f\left(\frac{1}{M}\sum_{m=1}^{M} x_m\right) \quad \text{if f is concave downward} \tag{3.10}$$

Now consider the definition of $Err_{D'} = f(D'^{e}_{ij})$ in Equation 3.6. Figure 3.1 shows that this function is concave downward over the entire range of possible values for the distribution factors. Then the magnitude of the error of the distribution factors in Equation 3.6 is bounded according to

$$Err_{D'} \leq W_c \sqrt{\frac{\langle\langle D'^{e}_{ij}\rangle\rangle\left(1-\langle\langle D'^{e}_{ij}\rangle\rangle\right)}{N}} \tag{3.11}$$

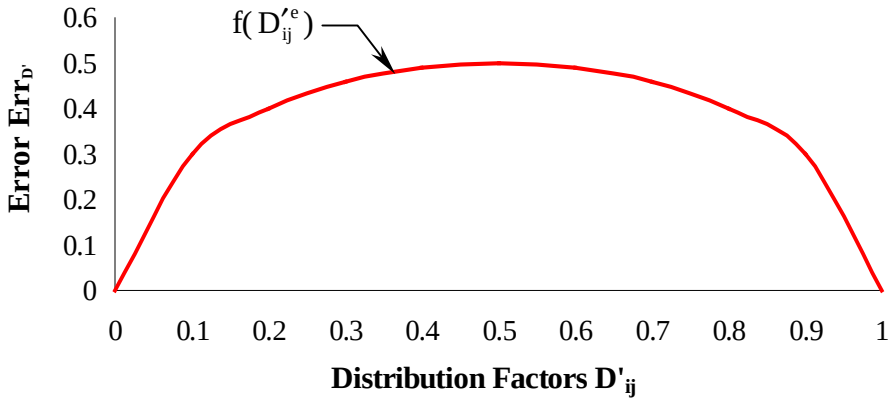

Figure 3.1 Graphical representation of $W_c\sqrt{\frac{D'^{e}_{ij}(1-D'^{e}_{ij})}{N}}$

Noting that $\left\langle\left\langle D'_{ij}\right\rangle\right\rangle = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} D'_{ij} = \frac{1}{n^2}\sum_{i=1}^{n} 1 = \frac{1}{n}$, with n being the number of surfaces in the enclosure, then Equation 3.11 gives

$$Err_{D'} \le W_c\sqrt{\frac{\frac{1}{n}\left(1-\frac{1}{n}\right)}{N}} \tag{3.12}$$

or

$$Err_{D'} \le \omega_{D'} \le W_c\sqrt{\frac{n-1}{Nn^2}} \tag{3.13}$$

Equation 3.13 provides an upper bound on the mean error and mean uncertainty of the distribution factors as a function of the number of surface elements, n, and the number of energy bundles traced per surface element N. It also can be used to calculate the minimum number of energy bundles that need to be traced to ensure that the error of the distribution factors is less than a specified value. It is emphasized that this result is independent of the enclosure geometry.

Equation 3.13 is similar to Equation 15.52 in Mahan [2002]. This latter equation represents the mean of the relative uncertainty as

$$\left\langle\left\langle\frac{\omega_{D'_{ij}}}{D'_{ij}}\right\rangle\right\rangle \equiv \frac{\omega_{D'}}{D'} = \pm W_c\sqrt{\frac{n-1}{N}} \qquad \text{(15.52 [Mahan, 2002])}$$

Equation 15.52 from Mahan was developed under the assumption that $E[f(x)] = f(E[x])$, which contradicts the Jensen's Inequality. However, note that if Equation 3.13 is divided by the double mean of the distribution factors (1/n), it reduces to Equation 15.52 in Mahan except with the inequality (≤) rather than the equality (=). Unfortunately equality or the sense of the inequality cannot be formally stablished because a general relation between the double mean of a fraction and the fraction of the double mean apparently does not exist.

3.4 Uncertainty in radiation heat transfer

The following discussion closely follows the development in Mahan [2002].

Kline and McClintock [1953] present definitions for errors and uncertainties. For a single observation, the *error* is the difference between the true and observed values of the observation. On the other hand, the *uncertainty*, which is what the observer thinks the error might be, may vary depending on the circumstances of the observation. If a result R is a linear function of n independent variables, according to

$$R = R(X_1, X_2, \ldots, X_n) \tag{3.14}$$

the relation between the uncertainty of the variables X_n and that of the result is

$$\omega_R = \sqrt{\left(\frac{\partial R}{\partial X_1}\omega_{X_1}\right)^2 + \left(\frac{\partial R}{\partial X_2}\omega_{X_2}\right)^2 + \ldots + \left(\frac{\partial R}{\partial X_n}\omega_{X_n}\right)^2} \tag{3.15}$$

where ω_{X_i} are the uncertainties of the variable X_i, i = 1, 2,…, n, and all the uncertainties are random and normally distributed. Also note that for Equation 3.15 to be valid all of the component uncertainties must be stated with the same level of confidence.

Consider a case where a gray, diffuse-specular enclosure has specified surface temperatures. The unknown net heat fluxes are given by the difference between the energy emitted on the surface and the energy absorbed by the surface

$$q''_i = q''_{i,e} - q''_{i,a}, \qquad 1 \le i \le n \tag{3.16}$$

where

$$q''_{i,e} = \varepsilon_i \sigma T_i^4 \tag{3.16a}$$

$$q''_{i,a} = \varepsilon_i \sum_{j=1}^{n} \sigma T_j^4 D'_{ij} \tag{3.16b}$$

In Equations 3.16, 3.16a, and 3.16b and in the ensuing sections q_i'' is the net radiation heat flux (W/m^2) from surface i, $q_{i,e}''$ is the heat flux emitted from surface i (W/m^2), $q_{i,a}''$ is the heat flux absorbed in surface i (W/m^2), ε_i is its emissivity, T_j is the specified temperature of surface j, σ is the Stephan-Boltzmann constant (= 5.689×10^{-8} W/m^2·K^4), and D_{ij}' is the estimate of the total, diffuse-specular radiation distribution factor. In this case, Equation 3.15 becomes

$$\omega_{q_i''} = \sqrt{\left(\frac{\partial q_i''}{\partial q_{i,e}''}\omega_{q_{i,e}''}\right)^2 + \left(\frac{\partial q_i''}{\partial q_{i,a}''}\omega_{q_{i,a}''}\right)^2} \quad , \qquad 1 \le i \le n \qquad (3.17)$$

with

$$\omega_{q_{i,e}'} = \sqrt{\left(\frac{\partial q_{i,e}''}{\partial \varepsilon_i}\omega_{\varepsilon_i}\right)^2 + \left(\frac{\partial q_{i,e}''}{\partial T_i}\omega_{T_i}\right)^2} \qquad (3.17a)$$

$$\omega_{q_{i,a}'} = \sqrt{\left(\frac{\partial q_{i,a}''}{\partial \varepsilon_i}\omega_{\varepsilon_i}\right)^2 + \left(\frac{\partial q_{i,a}''}{\partial T_i}\omega_{T_i}\right)^2 + \left(\frac{\partial q_{i,a}''}{\partial D_{ij}'}\omega_{D_{ij}'}\right)^2} \qquad (3.17b)$$

where the sensitivities are

$$\frac{\partial q_i''}{\partial q_{i,e}''} = 1, \qquad \frac{\partial q_i''}{\partial q_{i,e}''} = -1 \qquad (3.18)$$

$$\frac{\partial q_{i,e}''}{\partial \varepsilon_j} = \sigma T_i^4 , \qquad \frac{\partial q_{i,e}''}{\partial T_j} = 4\varepsilon_i \sigma T_i^3 \qquad (3.19)$$

$$\frac{\partial q_{i,a}''}{\partial \varepsilon_i} = \sum_{j=1}^{n} \sigma T_i^4 D_{ij}' , \quad \frac{\partial q_{i,a}''}{\partial T_j} = 4\varepsilon_i \sum_{j=1}^{n} \sigma T_i^3 D_{ij}' , \quad \frac{\partial q_{i,a}''}{\partial D_{ij}'} = \varepsilon_i \sum_{j=1}^{n} \sigma T_i^4 \qquad (3.20)$$

With the introduction of these sensitivities, Equation 3.17 becomes

$$\omega_{q_i'} = \sqrt{\left(\sigma T_i^4 \omega_{\varepsilon_i}\right)^2 + \left(4\varepsilon_i \sigma T_i^3 \omega_{T_i}\right)^2 + \left(\omega_{\varepsilon_i} \sum_{j=1}^{n} \sigma T_j^4 D_{ij}'\right)^2 + \left(4\varepsilon_i \sum_{j=1}^{n} \sigma T_j^3 \omega_{T_j} D_{ij}'\right)^2 + \left(\varepsilon_i \sum_{j=1}^{n} \sigma T_j^4 \omega_{D_{ij}'}\right)^2}$$

(3.21)

Dividing by the common factor $\varepsilon_i \sigma T_i^4$ and noting that $\left(\frac{T_j}{T_i}\right)^3 \frac{\omega_{T_j}}{T_i} = \left(\frac{T_j}{T_i}\right)^4 \frac{\omega_{T_j}}{T_j}$, Equation 3.21 becomes

$$\frac{\omega_{q_i'}}{\varepsilon_i \sigma T_i^4} = \sqrt{\left(\frac{\omega_{\varepsilon_i}}{\varepsilon_i}\right)^2 + \left(4\frac{\omega_{T_i}}{T_i}\right)^2 + \left(\frac{\omega_{\varepsilon_i}}{\varepsilon_i} \sum_{j=1}^{n} \left(\frac{T_j}{T_i}\right)^4 D_{ij}'\right)^2 + \left(4\sum_{j=1}^{n} \left(\frac{T_j}{T_i}\right)^4 D_{ij}' \frac{\omega_{T_j}}{T_j}\right)^2 + \left(\sum_{j=1}^{n} \left(\frac{T_j}{T_i}\right)^4 \omega_{D_{ij}'}\right)^2}$$

(3.22)

or

$$\frac{\omega_{q_i'}}{\varepsilon_i \sigma T_i^4} = \left\{\left(\frac{\omega_{\varepsilon_i}}{\varepsilon_i}\right)^2 \left[1 + \left(\sum_{j=1}^{n} \left(\frac{T_j}{T_i}\right)^4 D_{ij}'\right)^2\right] + (4)^2 \left[\left(\frac{\omega_{T_j}}{T_j}\right)^2 + \left(4\sum_{j=1}^{n} \left(\frac{T_j}{T_i}\right)^4 D_{ij}' \frac{\omega_{T_j}}{T_j}\right)^2\right] + \left(\sum_{j=1}^{n} \left(\frac{T_j}{T_i}\right)^4 \omega_{D_{ij}'}\right)^2\right\}^{1/2} \quad (3.23)$$

Equation 3.23 can be written for every surface i of the enclosure.

Assuming that the fractional (percentage) uncertainties of temperatures and emissivities are the same for all surface elements of the enclosure, that is, that $\omega_{Ti}/T_i = \omega_{Tj}/T_j = \omega_T/T$ and $\omega_{\varepsilon j}/\varepsilon_i = \omega_\varepsilon/\varepsilon$, where ω_T/T and $\omega_\varepsilon/\varepsilon$ are generally different constants, Equation 3.23 can be rewritten

$$\frac{\omega_{q_i''}}{\varepsilon_i \sigma T_i^4} = \left\{ \left[\left(\frac{\omega_\varepsilon}{\varepsilon} \right)^2 + \left(\frac{4\omega_T}{T} \right)^2 \right] \left[1 + \left(\sum_{j=1}^{n} \left(\frac{T_j}{T_i} \right)^4 D'_{ij} \right)^2 \right] + \left[\sum_{j=1}^{n} \left(\frac{T_j}{T_i} \right)^4 \omega_{D'_{ij}} \right]^2 \right\}^{1/2} \qquad (3.24)$$

Equation 3.24 is a simpler version of Equation 3.23 that can be used to estimate, to a stated level of confidence, the uncertainty in the net heat flux from surface element i when the surface temperatures and emissivities, the surface-to-surface distribution factors, and their uncertainties, are all known with the same level of confidence.

3.5 Design of MCRT models

It is important to clarify that in this section and the rest of this dissertation the term "design" is restricted to the statistical sense. In the present effort, this idea consists of the systematic search for the number of energy bundles N and the number of surface elements n required to bound the uncertainty in the net heat flux results.

To apply the statistical principles developed above to the design of MCRT algorithms, it is important to keep in mind that the following assumptions apply:

1. The enclosure is gray and diffuse-specular.
2. The relative uncertainties in surface temperature, surface emissivity, and the uncertainty of the heat flux, have been specified in advance, all to the same level of confidence.
3. The relative uncertainties are the same within each category.

It is desired to find an expression to calculate a bound for the uncertainty of the heat transfer as was done in the Section 3.3 for the uncertainty in the distribution factors. Equation 3.24 is rewritten as

$$\left(\frac{\omega_{q_i''}}{\varepsilon_i \sigma} \right)^2 = \left[\left(\frac{\omega_\varepsilon}{\varepsilon} \right)^2 + \left(\frac{4\omega_T}{T} \right)^2 \right] \left[\left(T_i^4 \right)^2 + \left(\sum_{j=1}^{n} T_j^4 D'_{ij} \right)^2 \right] + \left[\sum_{j=1}^{n} T_j^4 \omega_{D'_{ij}} \right]^2 \qquad (3.25)$$

Now, consider the term in Equation 3.25 involving the distribution factor. Since $D'_{ij} \le 1$ then $\sum_{j=1}^{n} T_j^4 D'_{ij} \le \sum_{j=1}^{n} T_j^4$. Equation 3.25 then becomes

$$\left(\frac{\omega_{q''_i}}{\varepsilon_i \sigma}\right)^2 \le \left[\left(\frac{\omega_\varepsilon}{\varepsilon}\right)^2 + \left(\frac{4\omega_T}{T}\right)^2\right]\left[\left(T_i^4\right)^2 + \left(\sum_{j=1}^{n} T_j^4\right)^2\right] + \left[\sum_{j=1}^{n} T_j^4 \omega_{D'_{ij}}\right]^2 \qquad (3.26)$$

The Cauchy inequality [Beckenbach, 1971] states that for any x_k and y_k real,

$$\left(\sum_{k=1}^{m} x_k y_k\right)^2 \le \left(\sum_{k=1}^{m} {x_k}^2\right)\left(\sum_{k=1}^{m} {y_k}^2\right) \qquad (3.27)$$

Applying Equation 3.27 to the third term on the right-hand side of Equation 3.26 yields

$$\left(\frac{\omega_{q''_i}}{\varepsilon_i \sigma}\right)^2 \le \left[\left(\frac{\omega_\varepsilon}{\varepsilon}\right)^2 + \left(\frac{4\omega_T}{T}\right)^2\right]\left[\left(T_i^4\right)^2 + \left(\sum_{j=1}^{n} T_j^4\right)^2\right] + \left[\sum_{j=1}^{n} \left(T_j^4\right)^2 \sum_{j=1}^{n} \omega_{D'_{ij}}^2\right]. \qquad (3.28)$$

Once the uncertainty of the distribution factors is estimated, Equation 3.28 can be used to establish an upper bound on the uncertainty of the heat transfer for each one of the surfaces in an enclosure. Unfortunately the calculation of the uncertainty of the distribution factors using Equation 3.5 involves the estimation of the distribution factors themselves, which requires that a ray-trace program be executed. This inconvenience can be avoided by establishing a bound for the mean of the uncertainties of the net heat flux from <u>all</u> the surfaces in the enclosure, that is

$$\left\langle\left(\frac{\omega_{q''_i}}{\varepsilon_i \sigma}\right)^2\right\rangle_i \le \left[\left(\frac{\omega_\varepsilon}{\varepsilon}\right)^2 + \left(\frac{4\omega_T}{T}\right)^2\right]\left\langle\left[\left(T_i^4\right)^2 + \left(\sum_{j=1}^{n} T_j^4\right)^2\right]\right\rangle_i + \left\langle\left[\sum_{j=1}^{n} \left(T_j^4\right)^2 \sum_{j=1}^{n} \omega_{D'_{ij}}^2\right]\right\rangle_i \qquad (3.29)$$

Noting that the summation of the temperatures does not depend on the subscript i, Equation 3.29 becomes

$$\left\langle \left(\frac{\omega_{q_i''}}{\varepsilon_i \sigma} \right)^2 \right\rangle_i \le \left[\left(\frac{\omega_\varepsilon}{\varepsilon} \right)^2 + \left(\frac{4\omega_T}{T} \right)^2 \right] \left[\left\langle \left(T_i^4 \right)^2 \right\rangle_i + \left(\sum_{j=1}^{n} T_j^4 \right)^2 \right] + \left[\sum_{j=1}^{n} \left(T_j^4 \right)^2 \left\langle \sum_{j=1}^{n} \omega_{D'_{ij}}^2 \right\rangle_i \right] \quad (3.30)$$

Now consider the expression involving the uncertainty of the distribution factors,

$$\left\langle \sum_{j=1}^{n} \omega_{D'_{ij}}^2 \right\rangle_i = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \omega_{D'_{ij}}^2 = n \left\langle \left\langle \omega_{D'_{ij}}^2 \right\rangle \right\rangle \quad (3.31)$$

Using the square of Equation 3.5, the double mean of the uncertainty of the distribution factors in Equation 3.31 can be estimated as

$$\omega_{D'_{ij}}^2 \le W_c^{\,2} \left[\frac{D'_{ij} \left(1 - D'_{ij} \right)}{N} \right] \quad (3.32)$$

where N is the number of energy bundles traced and W_c is the critical value of the W statistic. Note that the function represented by Equation 3.32 has the same form—that is, it is concave downward—as the function defined by Equation 3.6 and represented in Figure 3.1. Then the Jensen inequality can be used again yielding

$$\left\langle \left\langle \omega_{D'_{ij}}^2 \right\rangle \right\rangle \le \frac{W_c^{\,2}}{N} \left[\left\langle \left\langle D'_{ij} \right\rangle \right\rangle \left(1 - \left\langle \left\langle D'_{ij} \right\rangle \right\rangle \right) \right] \quad (3.33)$$

Recalling that $\left\langle \left\langle D'_{ij} \right\rangle \right\rangle = \frac{1}{n}$, then

$$\left\langle \left\langle \omega_{D'_{ij}}^2 \right\rangle \right\rangle \le \frac{W_c^{\,2}}{N} \left[\frac{1}{n} \left(1 - \frac{1}{n} \right) \right] \quad (3.34)$$

Then Equation 3.31 becomes

$$\left\langle \sum_{j=1}^{n} \omega_{ij}^2 \right\rangle_i \le n\left\{ \frac{W_c^2}{N}\left[\frac{1}{n}\left(1-\frac{1}{n}\right)\right]\right\} \tag{3.35}$$

and substituting Equation 3.35 into Equation 3.30 yields

$$\left\langle \left(\frac{\omega_{q_i''}}{\varepsilon_i \sigma} \right)^2 \right\rangle_i \le \left[\left(\frac{\omega_\varepsilon}{\varepsilon} \right)^2 + \left(\frac{4\omega_T}{T} \right)^2 \right] \left[\left\langle \left(T_i^4 \right)^2 \right\rangle_i + n^2 \left\langle T_j^4 \right\rangle_j^2 \right] + \frac{W_c^2}{N}\left(1-\frac{1}{n}\right)\left\langle \left(T_j^4 \right)^2 \right\rangle_j \tag{3.36}$$

Noting that $\left\langle T_i^2 \right\rangle_i = \left\langle T_j^2 \right\rangle_j$ it is possible to divide Equation 3.36 by the mean of the square of the temperatures to yield

$$\left\langle \left(\frac{\omega_{q_i''}}{\varepsilon_i \sigma} \right)^2 \right\rangle_i \frac{1}{\left\langle \left(T_j^4 \right)^2 \right\rangle} \le \left[\left(\frac{\omega_\varepsilon}{\varepsilon} \right)^2 + \left(\frac{4\omega_T}{T} \right)^2 \right] \left[1 + n^2 \frac{\left\langle T^4 \right\rangle^2}{\left\langle \left(T^4 \right)^2 \right\rangle} \right] + \frac{W_c^2}{N}\left(1-\frac{1}{n}\right) \tag{3.37}$$

Using the Jensen inequality again, it can be stated that $\frac{1}{\left\langle \left(T^4 \right)^2 \right\rangle} \le \left\langle \frac{1}{\left(T^4 \right)^2} \right\rangle$ or

$$\left\langle \left(\frac{\omega_{q_i''}}{\varepsilon_i \sigma} \right)^2 \right\rangle_i \left\langle \frac{1}{\left(T_j^4 \right)^2} \right\rangle \ge \left\langle \left(\frac{\omega_{q_i''}}{\varepsilon_i \sigma} \right)^2 \right\rangle_i \frac{1}{\left\langle \left(T_j^4 \right)^2 \right\rangle} \le \left[\left(\frac{\omega_\varepsilon}{\varepsilon} \right)^2 + \left(\frac{4\omega_T}{T} \right)^2 \right] \left[1 + n^2 \frac{\left\langle T^4 \right\rangle^2}{\left\langle \left(T^4 \right)^2 \right\rangle} \right] + \frac{W_c^2}{N}\left(1-\frac{1}{n}\right) \tag{3.38}$$

However, Equation 3.38 leads to an ambiguous relationship between the new left-hand side and the right-hand side of Equation 3.37

$$\left[\frac{1}{n}\sum_{i=1}^{n} \left(\frac{\omega_{q_i''}}{\varepsilon_i \sigma} \right)^2 \right] \left[\frac{1}{n}\sum_{i=1}^{n} \left(\frac{1}{T_i^4} \right)^2 \right] \overset{?}{\lessgtr} \left[\left(\frac{\omega_\varepsilon}{\varepsilon} \right)^2 + \left(\frac{4\omega_T}{T} \right)^2 \right] \left[1 + n^2 \frac{\left\langle T^4 \right\rangle^2}{\left\langle \left(T^4 \right)^2 \right\rangle} \right] + \frac{W_c^2}{N}\left(1-\frac{1}{n}\right) \tag{3.38a}$$

We were unable to find a rigorous demonstration of the correct direction of the inequality in Equation 3.38a, but in Chapter 4 numerical results will establish that the left-hand side of Equation 3.38a is smaller in magnitude than the right-hand side. Then accepting the ≤ sense of the inequality and continuing with the development of an expression that relates the number of energy bundles, N, and the number of surface elements, n, to the uncertainty of the net heat transfer, we again invoke the Cauchy inequality and transform Equation 3.38 into

$$\left\langle \left(\frac{\omega_{q_i''}}{\varepsilon_i \sigma T_i^4} \right) \right\rangle_i^2 \le \left[\left(\frac{\omega_\varepsilon}{\varepsilon} \right)^2 + \left(\frac{4\omega_T}{T} \right)^2 \right] \left[1 + n^2 \frac{\left\langle T^4 \right\rangle^2}{\left\langle \left(T^4 \right)^2 \right\rangle} \right] + \frac{W_c^2}{N} \left(1 - \frac{1}{n} \right) \qquad (3.39)$$

Equation 3.39 is hypothesized to give an upper bound for the square of the mean of the fraction of the net flux from surface i relative to the flux emitted from surface i. Subject to this hypothesis this expression can then be used to estimate the number of energy bundles, N, that must be traced corresponding to a given number of surface elements, n, to obtain a bound on the desired relative mean uncertainty of the net heat flux for the surfaces of an enclosure. Note that several nested inequalities have been invoked, which means that Equation 3.39 is expected to be a very conservative result, even taking into account the direction of the first inequality in Equation 3.38.

Once again, it is appropriate to contrast the results obtained here to those in Mahan [2002]. The development in Section 15.9 of Mahan results in

$$\begin{aligned} \frac{\omega_{q_i'}}{\varepsilon_i \sigma T_i^4} = \pm \Bigg\{ & \left(\frac{\omega_{\varepsilon_i}}{\varepsilon_i} \right)^2 \left[1 - \sum_{j=1}^{n} \left(\frac{T_j}{Ti} \right)^4 D'_{ij} \right]^2 \\ & + 16 \left[\frac{\omega_{T_i}}{T_i} - \sum_{j=1}^{n} \left(\frac{T_i}{T_j} \right)^3 \left(\frac{\omega_{T_j}}{T_i} \right) D'_{ij} \right]^2 \\ & + \left[\sum_{j=1}^{n} \left(\frac{T_j}{T_i} \right)^4 \left(\frac{\omega_{D'_{ij}}}{D'_{ij}} \right) D'_{ij} \right]^2 \Bigg\}^{1/2} \end{aligned} \qquad (15.42 \text{ [Mahan, 2002]})$$

where $\frac{\omega_{\varepsilon_i}}{\varepsilon_i}, \frac{\omega_{T_i}}{T_i}$, and $\frac{\omega_{D'_{ij}}}{D'_{ij}}$ are defined as the fractional uncertainties for the temperatures, emissivities, and distribution factors, respectively. Mahan justifies this result by assuming that the inequalities leading up to Equation 3.39 can be replaced by equalities in the limit of small relative uncertainties in D'_{ij}, ε_i, and T_i.

As shown in Chapter 4, where the author validates the theory developed here by performing numerical experiments, it is possible to obtain estimated distribution factors equal to zero, rendering Equation 15.42 in Mahan impossible to use. By avoiding Mahan's development the current author has obtained different but more practical relations for the uncertainty in the net heat flux.

In this chapter a development of equations to predict the uncertainty in the distribution factors, the uncertainty of the heat flux from one surface, and the global relative uncertainty of the heat flux of an enclosure is presented. The next chapter presents a description of numerical experiments used to validate the equations developed in this chapter and a discussion of their application to a generic enclosure.