

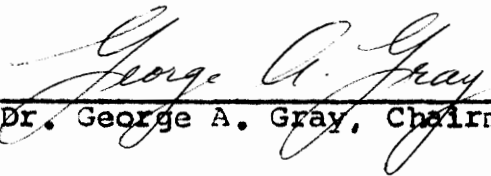
SPACE FRAME ANALYSIS

by

Fu-Tien Lin


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APPROVED:

  
Dr. George A. Gray, Chairman

  
Dr. H. M. Morris

  
Dr. R. M. Barker

  
Dr. C. B. Ling

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Blacksburg, Virginia

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## TABLE OF CONTENTS

	Page
I. INTRODUCTION . . . . .	5
II. LIST OF SYMBOLS. . . . .	9
III. ANALYSIS OF PIN-CONNECTED SPACE FRAMES . . . . .	11
A. Outline of Analysis. . . . .	11
B. Basic Concepts . . . . .	11
1. Stiffness Coefficients. . . . .	11
2. Bar Stiffness Matrices. . . . .	15
3. Bar Influence Stiffness Matrices. . . . .	16
4. Bar Matrix Relationships . . . . .	16
5. Bar Force Equations . . . . .	17
6. Joint Stiffness Matrix. . . . .	19
C. Analysis . . . . .	21
1. Stiffness Matrix for the Whole Structure . . . . .	21
2. Modified Stiffness Matrix . . . . .	25
IV. ANALYSIS OF RIGIDLY-CONNECTED SPACE FRAMES . . . . .	27
A. Basic Concepts . . . . .	27
1. Methods of Analysis . . . . .	27
2. Stiffness Coefficients. . . . .	29
3. Rotation of Axes. . . . .	31
4. Rotation Matrix . . . . .	35
5. Member Stiffness. . . . .	39
6. Rotation for Unknown Principal Axes. . . . .	41
7. Vertical Member Rotation Matrix . . . . .	42
B. Summary. . . . .	43
V. EXPERIMENTAL TEST . . . . .	46
A. Test Description . . . . .	46
B. Test Procedure . . . . .	46
C. Test Results . . . . .	49
D. Analysis of the Test Model . . . . .	50
1. Description . . . . .	50
2. Computer Program. . . . .	52
3. Analytical Results. . . . .	52
E. Comparison of Test and Analysis. . . . .	54
1. Comparison of Results . . . . .	54
2. Discussion . . . . .	54

	Page
VI. CONCLUSION . . . . .	59
VII. BIBLIOGRAPHY . . . . .	61
VIII. ACKNOWLEDGEMENT. . . . .	62
IX. VITA . . . . .	63
X. APPENDIX . . . . .	64

## LIST OF FIGURES

Figure	Page
1. A Pin-ended Prismatic Bar in Three-Dimensional Space . . . . .	12
2. A Prismatic Bar with Its Deformed Position . . . . .	18
3. The Pin-connected Space Frame with Only One Joint Moveable . . . . .	20
4. A General Pin-connected Space Frame and Its Deformed Shape . . . . .	22
5. Numbering System for Space Frame . . . . .	28
6. Fixed-end Frame Bar with Unit Displacements. . . . .	30
7. Rotation of Reference Axes . . . . .	33
8. Rotation of the x, y and z Axes. . . . .	33
9. Rotation Axes and Point P for A Vertical Member. . . . .	44
10. Experimental Test Model . . . . .	47
11. Photograph of Test Model . . . . .	48

## LIST OF TABLES

Table	Page
1. Space Frame Member Stiffness Matrix. . . . .	32
2. Test Results . . . . .	51
3. Analytical Results . . . . .	53
4. Differences Between Test and Analytic Results. . . . .	55

## I. INTRODUCTION

Structural analysis involves computing not only the external reactions and the internal forces (and stresses) of a structure, but also strains and deflections throughout.

There are many techniques to achieve these purposes. In statically determinate structures, the external reactions and the internal forces are computed first, then the strains and the deflections can be computed subsequently. In statically indeterminate structures, it is usually desirable to express the relation between the given forces and the unknown joint displacements in a simple and systematic form. Recent applications of matrix algebra to the solution of structural problems have achieved this purpose. The invention of the electronic computer and its wide usage in solving engineering problems has led the structural engineer to solve more complicated and highly redundant structures by matrix methods.

In the matrix method of statically indeterminate structural analysis, two approaches have been developed. One is called the "force method" or flexibility method" (2, 4, 6, 8); the other is called the "displacement method" or "stiffness method" (1, 4, 5, 6, 7). The force method expresses the deflection of the primary structure in terms of the unknown forces and the problem is solved by the conditions of continuity. The displacement method expresses the internal forces in terms of the displacement components which are determined

by the conditions of equilibrium. Both methods have their advantages (4, 7). In general, both methods of analysis serve useful purposes for hand calculations. The preferred method of solution will usually be the one that involves the smaller number of unknowns. For computer programming the stiffness method is normally much more suitable than the flexibility method. The advantage of the stiffness method arises from the automatic determination of the restrained structure and from the fact that all effects are localized. Of course, exceptions to the general rule occasionally will be encountered. This thesis presents the solution of indeterminate space frames using the "stiffness method."

The stiffness method is one of the most fundamental methods available to the structural analyst and is applicable to a wide variety of structures. In this method of analysis, the basic equations to be solved are those which express the equilibrium conditions at the joints of the structures. These equations can be written as relationships between loads and displacements in the structure and form the so-called stiffness matrix of the whole structure. There are two approaches to obtain the stiffness coefficients of the structure. In Section III the direct approach is used for the three-dimensional pin-connected frames. The concepts of the stiffness coefficients, the bar stiffness matrix, the bar influence stiffness matrix, and the joint stiffness matrix are introduced. In Section IV an indirect approach is used for

the three-dimensional rigidly-connected frames. The concepts of stiffness coefficients (corresponding to member axes), rotation of axes (in three-dimensions), the rotation matrix, the rotation transformation matrix, and the member stiffness matrix are introduced.

Using the principle of superposition and the above concepts, the equations of equilibrium at the joints are established. Then, by means of matrix algebra, a matrix equation for an entire structure is established. The expression of this method in matrix terms permits immediate generalization from a very simple structure to very complicated structures, and is one of the principal advantages of the matrix notation. Also, the use of matrices casts the structural problem in a form which is ideally suited for programming on a digital computer.

The objective of this study was to develop a highly systematic procedure for the analysis of articulated continuous structures and use it for the analysis of a simple three bar rigidly-connected space frame that was tested experimentally. This was accomplished and comparisons were made between analytical and experimental results with the intent to reconcile the differences by modification of the stiffness coefficients. The differences were found to be too large to be modified by the analysis, so an evaluation of the test set-up was made to suggest corrections to be incorporated in the test procedure before considering a change in the analysis.

It should be pointed out that the application of the theory is limited to structures that (a) experience small displacements and rotations and (b) have linear load-deformation relations. This infers (a) negligible effects due to slight changes in geometry, (b) elastic properties in the working range for the materials and the structure, (c) no slip at the joints and (d) no buckling of any members.

The presentation of the theory in this thesis has been prepared with reference to many writers as listed in the Bibliography in Section VII, but parts of Section IV have been developed directly from material presented in Analysis of Framed Structures by Gere and Weaver (4).

## II. LIST OF SYMBOLS

$A$	Cross-section area of member
$[D]$	Column matrix of the joint displacement components of the whole structure
$[D_A]$	Column matrix of the joint displacement components at joint A
$D_1$	Joint displacement component at joint and in direction denoted by 1
$E$	Modulus of elasticity of the member
$F_{AB}$	Axial force of member AB
$[F_A]$	Column matrix of the external loads at joint A
$[F_{AA}]_{AB}$	Column matrix of the force components induced at joint A due to the displacement at joint A of member AB
$[F]$	Column matrix of the external loads of the whole structure
$F_i$	Joint force component at joint and in direction denoted by i
$F_{ij}$	The force induced at joint and in direction denoted by i due to the displacement at joint and in direction denoted by j
$I$	Moment of inertia
$k_{ij}$	Stiffness coefficient at joint and in direction denoted by i due to unit displacement at joint and in direction denoted by j when all other joint displacement components are prevented
$L$	Length of member
$[R]$	Rotation matrix of direction cosines of member axes with respect to structure axes
$[R]^{-1}$	Inverse of rotation matrix
$[R_T']$	Transpose of rotation transformation matrix
$[R_T]$	Rotation transformation matrix

$[S]$	Stiffness matrix of the whole structure with respect to structure axes
$[S]^{-1}$	Inverse of stiffness matrix
$[S_{AA}]_{AB}$	Bar stiffness matrix of the end A of member AB
$[S_{AB}]_{AB}$	Bar influence stiffness matrix of the end A due to the displacement of the end B of member AB
$x_A, y_A, z_A$	Coordinates of joint A

### III. ANALYSIS OF PIN-CONNECTED SPACE FRAMES

#### A. Outline of Analysis

In order to introduce the concepts of matrix analysis of rigidly connected articulated three-dimensional structures, the concepts are first introduced and illustrated in this section with pin-connected frames using a direct approach and extended to rigidly-connected frames using an indirect approach in Section IV.

In pin-connected frames, there are no bending moments induced in the members at the joints so that equilibrium is established with three equations of balanced forces at each joint. The  $3 \times 3$  stiffness matrices are simple to write and manipulate and the equations are relatively simple to solve.

For the three-dimensional rigidly-connected frames explained in Section IV, six equations of equilibrium are needed at each joint (three force equations and three moment equations). The stiffness matrices are  $6 \times 6$  and are formulated in the same general way as the simpler  $3 \times 3$  matrices, but an indirect approach is used for the total solution.

#### B. Basic Concepts

1. Stiffness Coefficients. Consider a pin-ended prismatic bar AB in three-dimensional space as shown in Figure 1a. It is assumed that the system of three-dimensional Cartesian axes,  $x$ ,  $y$  and  $z$ , is a right-handed system. Assume that joint A is displaced a distance  $D_1$  in the positive  $x$

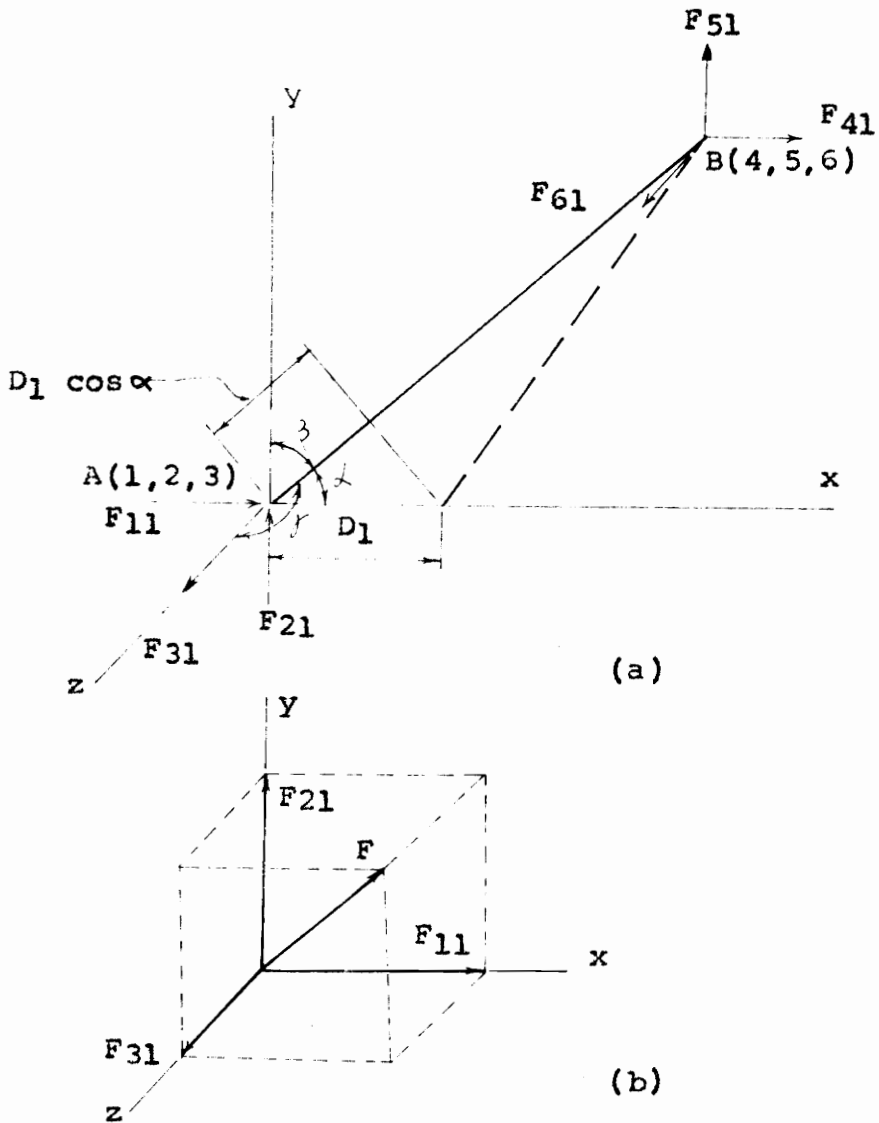


Figure 1. A Pin-Ended Prismatic Bar in Three-Dimensional Space

direction. Let the angles between bar AB and the x, y, and z axes be  $\alpha$ ,  $\beta$ , and  $\gamma$ , respectively. Then the change in length and the force required to produce this change are

$$\Delta L = D_1 \cos \alpha$$

$$F = \frac{\Delta L}{L} AE = D_1 \frac{AE}{L} \cos \alpha$$

where E, A, and L are the modulus of elasticity, the cross-sectional area, and the length of the bar, respectively. The force may be resolved into three components,  $F_{11}$ ,  $F_{21}$ , and  $F_{31}$ , parallel to the x, y and z area, as shown in Figure 1a and b. The first subscript of F refers to the axis direction of the component and the second subscript refers to the direction of the displacement  $D_1$  at A with all other end displacement components equal to zero. Thus

$$F_{11} = F \cos \alpha = D_1 \frac{AE}{L} \cos^2 \alpha$$

$$F_{21} = F \cos \beta = D_1 \frac{AE}{L} \cos \alpha \cos \beta \quad (1)$$

$$F_{31} = F \cos \gamma = D_1 \frac{AE}{L} \cos \alpha \cos \gamma$$

Similarly, if the components of force and displacement at B are denoted with subscripts 4, 5, and 6, respectively, then the reaction components at B are given by the equilibrium conditions as

$$F_{41} = -F_{11} = -D_1 \frac{AE}{L} \cos^2 \alpha$$

$$F_{51} = -F_{21} = -D_1 \frac{AE}{L} \cos \alpha \cos \beta \quad (2)$$

$$F_{61} = -F_{31} = -D_1 \frac{AE}{L} \cos \alpha \cos \gamma$$

In Equations 1 and 2,  $\cos \alpha$ ,  $\cos \beta$ , and  $\cos \gamma$  are direction cosines of the member AB. Let the coordinates of A and B be  $(x_A, y_A, z_A)$  and  $(x_B, y_B, z_B)$ , respectively. Then

$$\cos \alpha = \frac{x_B - x_A}{L}$$

$$\cos \beta = \frac{y_B - y_A}{L}$$

$$\cos \gamma = \frac{z_B - z_A}{L}$$

With  $D_1 = 1$  in Equations 1 and 2, the stiffness coefficients are found, where the stiffness coefficient  $k_{21}$  may be defined as the force at A in the y direction due to a unit displacement  $D_1$  at A in the x direction.

$$k_{11} = (x_B - x_A)^2 \frac{AE}{L^3}$$

$$k_{21} = (x_B - x_A) (y_B - y_A) \frac{AE}{L^3}$$

$$k_{31} = (x_B - x_A) (z_B - z_A) \frac{AE}{L^3}$$

$$k_{41} = -(x_B - x_A)^2 \frac{AE}{L^3}$$

$$k_{51} = -(x_B - x_A) (y_B - y_A) \frac{AE}{L^3}$$

$$k_{61} = -(x_B - x_A) (z_B - z_A) \frac{AE}{L^3}$$

The subscripts of the stiffness coefficients have the same meaning as for the forces. For example,  $k_{51}$  refers to the stiffness coefficient at B in the y direction (denoted by 5) of the bar AB due to the unit displacement  $D_1 = 1$  at A in

the x direction (denoted by 1). The stiffness coefficients for other cases can be derived in a similar way.

It will be found that  $k_{14} = k_{41}$  etc., and  $k_{11} = k_{44}$ ,  $k_{22} = k_{55}$  and  $k_{33} = k_{66}$ .

2. Bar Stiffness Matrices (3). When the pin-ended prismatic bar AB of Figure 1a is displaced  $[D_A]$  at A in an arbitrary direction under an arbitrary external load  $[F_A]$ , the  $[D_A]$  and  $[F_A]$  values can be resolved into x, y and z components,  $(D_1, D_2, D_3)$  and  $(F_1, F_2, F_3)$ . From the definition of stiffness coefficient and the condition of equilibrium,

$$\begin{aligned} F_1 &= k_{11} D_1 + k_{12} D_2 + k_{13} D_3 \\ F_2 &= k_{21} D_1 + k_{22} D_2 + k_{23} D_3 \\ F_3 &= k_{31} D_1 + k_{32} D_2 + k_{33} D_3 \end{aligned} \quad (3)$$

Expressed in matrix form, Equation 3 can be written as

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} \quad (4)$$

or in a more compact form

$$[F_{AA}]_{AB} = [S_{AA}] [D_A] \quad (5)$$

where  $[S_{AA}]$  is the bar stiffness matrix for space frame members,  $[F_{AA}]_{AB}$  is force column matrix of member AB at joint A

and  $[D_A]$  is displacement column matrix of joint A.

3. Bar Influence Stiffness Matrices. By the same procedure used in deriving the bar stiffness matrix, the reaction at joint B due to a displacement of  $[D_A]$  at A can be expressed as

$$\begin{aligned} F_4 &= k_{41} D_1 + k_{42} D_2 + k_{43} D_3 \\ F_5 &= k_{51} D_1 + k_{52} D_2 + k_{53} D_3 \\ F_6 &= k_{61} D_1 + k_{62} D_2 + k_{63} D_3 \end{aligned} \quad (6)$$

where  $F_4$ ,  $F_5$ , and  $F_6$  are x, y, and z reaction components at B, this can be written in matrix form as

$$\begin{bmatrix} F_4 \\ F_5 \\ F_6 \end{bmatrix} = \begin{bmatrix} k_{41} & k_{42} & k_{43} \\ k_{51} & k_{52} & k_{53} \\ k_{61} & k_{62} & k_{63} \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} \quad (7)$$

or

$$[F_{BA}] = [S_{BA}] [D_A] \quad (8)$$

where  $[S_{BA}]$  is the bar influence stiffness matrix, a term introduced to differentiate  $[S_{BA}]$  from the bar stiffness matrix  $[S_{BB}]$ .

4. Bar Matrix Relationships. Using the definitions of the bar stiffness matrix and the bar influence stiffness matrix and the relations of bar stiffness coefficients,

$$[S_{BA}] = \begin{bmatrix} k_{41} & k_{42} & k_{43} \\ k_{51} & k_{52} & k_{53} \\ k_{61} & k_{62} & k_{63} \end{bmatrix} = \begin{bmatrix} -k_{11} & -k_{12} & -k_{13} \\ -k_{21} & -k_{22} & -k_{23} \\ -k_{31} & -k_{32} & -k_{33} \end{bmatrix} = - [S_{AA}] \quad (9)$$

In a similar way

$$[S_{AB}] = - [S_{BB}], \quad [S_{AA}] = [S_{BB}] \quad \text{and} \quad [S_{AB}] = [S_{BA}] \quad (10)$$

These can be generalized as

(a) The bar stiffness matrix for one end of a given prismatic bar is equal to the bar stiffness matrix for the other end of the same bar,  $[S_{AA}] = [S_{BB}]$ .

(b) The bar influence stiffness matrix is equal to the negative of the bar stiffness matrix for a given bar,

$$[S_{BA}] = - [S_{AA}].$$

(c) The bar influence stiffness matrix at one end due to the displacement of the other end for a given bar is equal to the bar influence stiffness matrix at the other end due to the displacement of the first end,  $[S_{AB}] = [S_{BA}]$ .

5. Bar Force Equations. A prismatic bar AB from a given structure is isolated, as shown in Figure 2a. The heavy solid line AB represents its original position and the heavy dashed line represents its final position under loadings. The three components of displacement at joint A are  $D_1$ ,  $D_2$  and  $D_3$ , and for joint B are  $D_4$ ,  $D_5$  and  $D_6$ . The x, y and z components of the effective displacement are  $D_4 - D_1$ ,  $D_5 - D_2$  and  $D_6 - D_3$ , respectively, and the change in length due to these displacements are

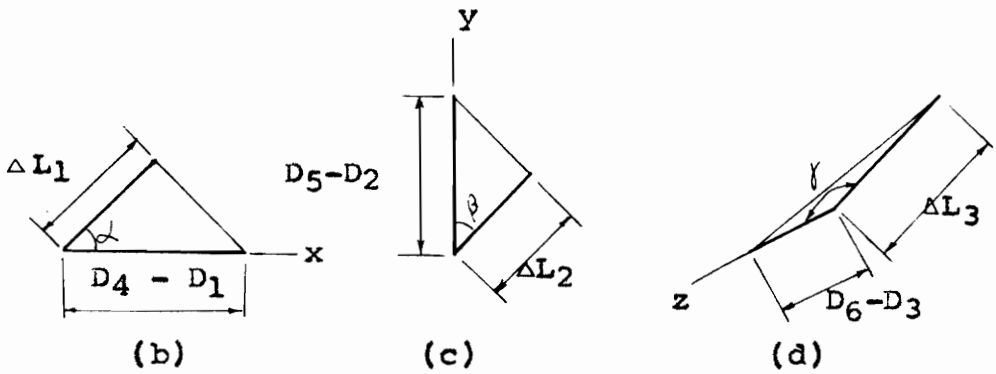
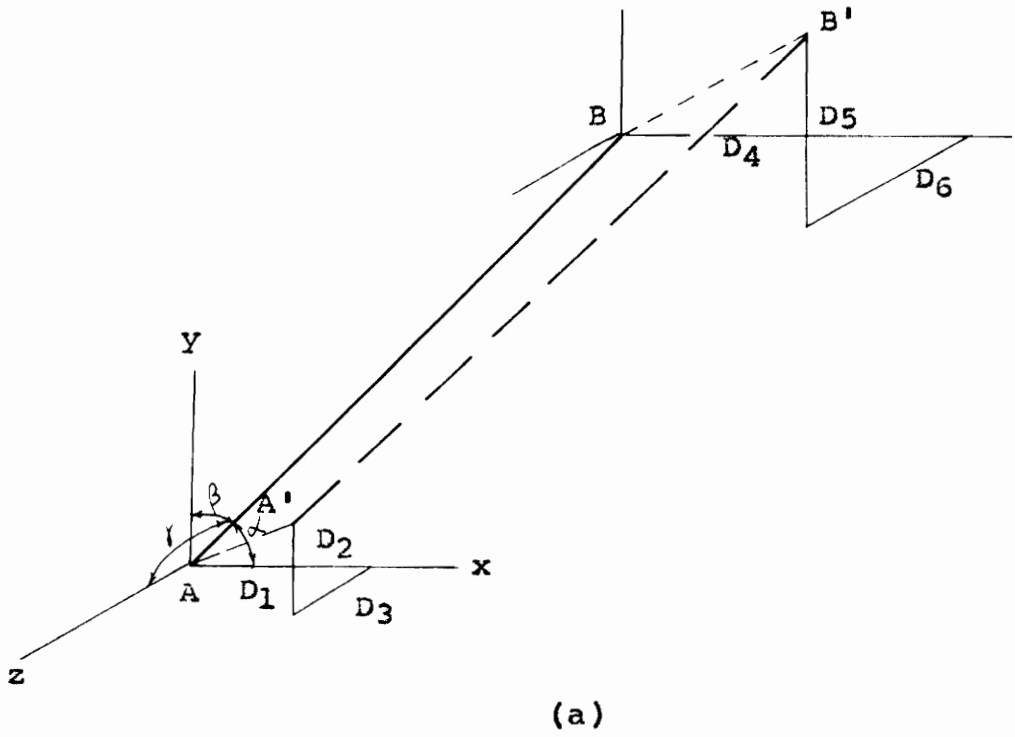


Figure 2. A Prismatic Bar With Its Deformed Position.

$$\Delta L_1 = (D_4 - D_1) \cos \alpha$$

$$\Delta L_2 = (D_5 - D_2) \cos \beta$$

$$\Delta L_3 = (D_6 - D_3) \cos \gamma$$

The total change in length is

$$\begin{aligned} \Delta L &= \Delta L_1 + \Delta L_2 + \Delta L_3 \\ &= (D_4 - D_1) \cos \alpha + (D_5 - D_2) \cos \beta + (D_6 - D_3) \cos \gamma \end{aligned}$$

By Hooke's Law, the force in the bar AB,  $F_{AB}$ , due to this change in length,  $\Delta L$ , is

$$\begin{aligned} F_{AB} &= \Delta L \frac{AE}{L} \\ &= \left[ (D_4 - D_1) \cos \alpha + (D_5 - D_2) \cos \beta + (D_6 - D_3) \cos \gamma \right] \frac{AE}{L} \end{aligned} \quad (11)$$

which can be written as

$$\begin{aligned} F_{AB} &= \left[ (D_4 - D_1)(x_B - x_A) + (D_5 - D_2)(y_B - y_A) \right. \\ &\quad \left. + (D_6 - D_3)(z_B - z_A) \right] \frac{AE}{L} \end{aligned} \quad (12)$$

Equation 12 is the fundamental bar force equation for the three-dimensional pin-connected frame.

6. Joint Stiffness Matrix (3). A pin-connected space frame with only joint A free to move and the other joints fixed against translations is shown in Figure 3 with reference axes  $x$ ,  $y$  and  $z$ . Under the external load  $[F_A] = (F_1, F_2, F_3)$

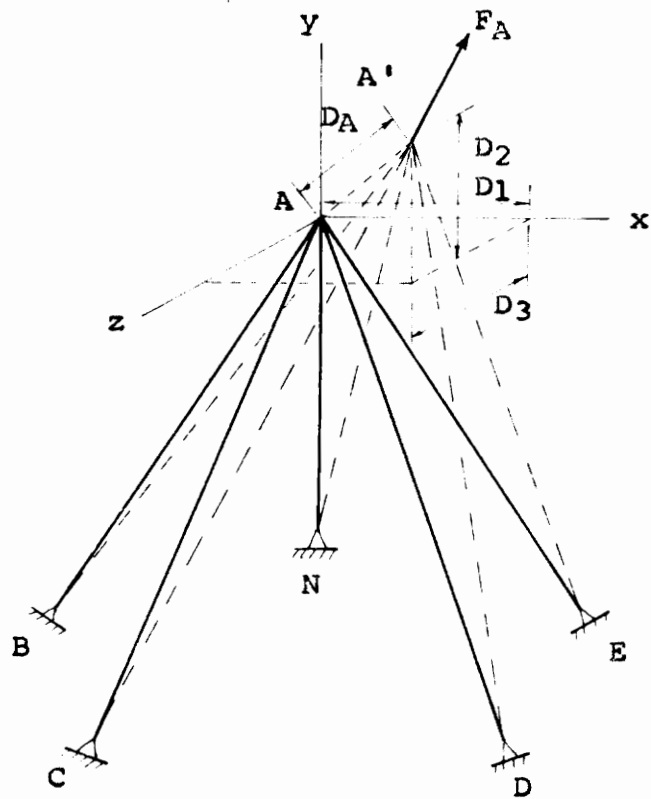


Figure 3. The Pin-Connected Space Frame with Only One Joint Movable.

at A, the joint is subjected to a displacement  $[D_A] = (D_1, D_2, D_3)$ , the subscripts of 1, 2, and 3 refer to the x, y and z directions, respectively. By Equation 5 the forces in each member due to displacement  $[D_A]$  at A are

$$\begin{aligned} [F_{AA}]_{AB} &= [S_{AA}]_{AB} [D_A] \\ [F_{AA}]_{AC} &= [S_{AA}]_{AC} [D_A] \\ &\vdots \\ [F_{AA}]_{AN} &= [S_{AA}]_{AN} [D_A] \end{aligned} \quad (13)$$

where  $[F_{AA}]_{AB}$  is the force induced in the bar AB and  $[S_{AA}]_{AB}$  is the bar stiffness matrix of the bar AB.

The structure of Figure 3 is stable and in equilibrium; the sum of the forces in each member must be equal to the external load  $[F_A]$  at A. Thus,

$$[F_{AA}] = [S_{AA}]_{AB} [D_A] + [S_{AA}]_{AC} [D_A] + [S_{AA}]_{AN} [D_A]$$

or

$$[F_{AA}] = (\sum [S_{AA}]) [D_A] \quad (14)$$

where  $\sum [S_{AA}]$  is the joint stiffness matrix which is the sum of the bar stiffness matrices of all bars connected to the joint A.

### C. Analysis

1. Stiffness Matrix for the Whole Structure. A generalized pin-connected space frame is shown in Figure 4. The solid lines represent the original structure and dashed lines

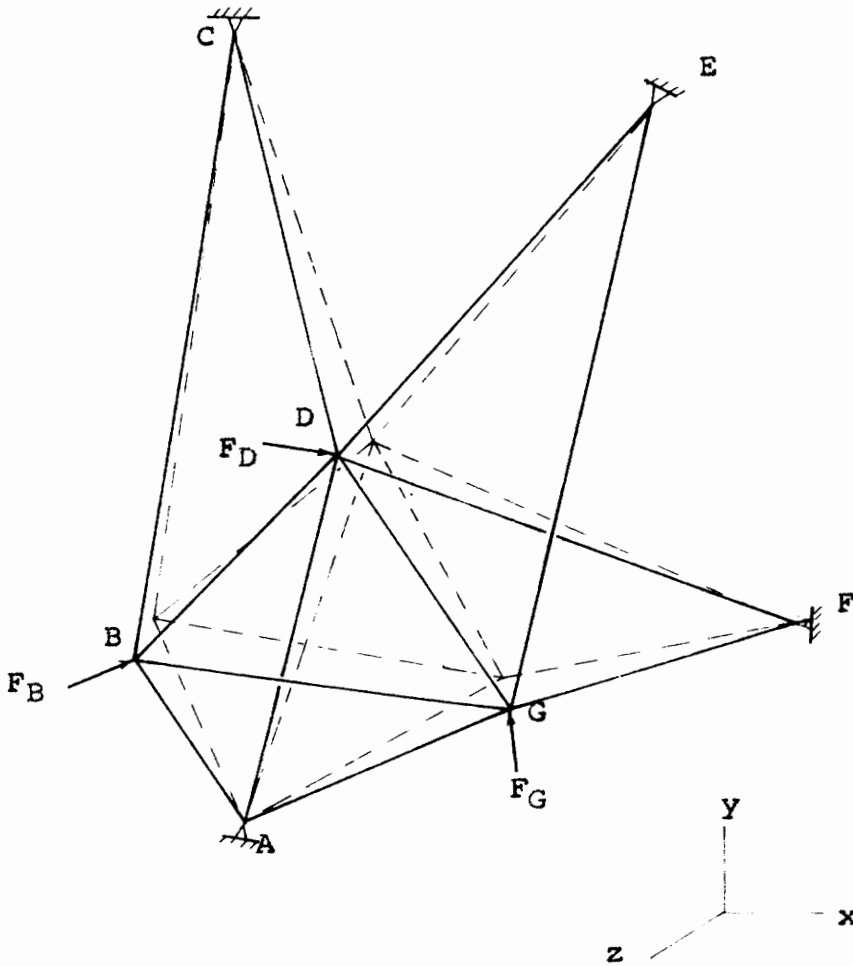


Figure 4. A General Pin-Connected Space Frame and Its Deformed Shape.

represent the deformed structure under loads. Each joint has an unknown displacement matrix which must be found to solve the problem. To derive the general equations relating the external forces and joint displacements, consider a joint G. If this joint has a final displacement  $[D_G]$  and all other joints are fixed in position, then the force required to produce this displacement  $[D_G]$  is given by Equation 14.

$$[F_{GG}] = (\sum [S_{GG}]) [D_G]$$

If joint A is displaced an amount  $[D_A]$  with all other joints, including joint G, fixed in their original position, then the reaction at joint G due to  $[D_A]$  is given by Equation 8.

$$[F_{GA}] = [S_{GA}] [D_A]$$

where  $S_{GA}$  is the bar influence stiffness matrix of bar AG. Similarly the other joint displacements will induce reactions at G. Since joint G is in equilibrium, the sum of the effects of all joint displacements must be equal to the external load acting at G. Thus

$$\begin{aligned} [F_G] &= [F_{GA}] + [F_{GB}] + \dots + [F_{GG}] \\ &= [S_{GA}][D_A] + [S_{GB}][D_B] + \dots + (\sum [S_{GG}]) [D_G] \end{aligned}$$

where  $[F_G]$  is the column matrix representing the force acting at G.

All other joints can be considered in the same way; they are summed up as follows:

$$\begin{aligned}
[F_A] &= (\sum[S_{AA}]) [D_A] + [S_{AB}][D_B] + \dots + [S_{AG}][D_G] \\
[F_B] &= [S_{BA}][D_A] + (\sum[S_{BB}]) [D_B] + \dots + [S_{BG}][D_G] \\
&\vdots \\
[F_G] &= [S_{GA}][D_A] + [S_{GB}][D_B] + \dots + (\sum[S_{GG}]) [D_G]
\end{aligned} \tag{15}$$

which can be written in matrix form as:

$$\begin{bmatrix} [F_A] \\ [F_B] \\ \vdots \\ [F_G] \end{bmatrix} = \begin{bmatrix} \sum[S_{AA}] & [S_{AB}] & \dots & [S_{AG}] \\ [S_{BA}] & \sum[S_{BB}] & \dots & [S_{BG}] \\ \vdots & \vdots & \ddots & \vdots \\ [S_{GA}] & [S_{GB}] & \dots & \sum[S_{GG}] \end{bmatrix} \begin{bmatrix} [D_A] \\ [D_B] \\ \vdots \\ [D_G] \end{bmatrix} \tag{16}$$

or in a more compact form

$$[F] = [S] [D] \tag{17}$$

where  $[F]$  is the external force matrix including reactions;  $[S]$  is the stiffness matrix of the whole structure;  $[D]$  is the joint displacement matrix. From Equation 16, it can be seen that one row of the stiffness matrix represents the effects of the displacements of all joints on the joint corresponding to that row, while one column of the stiffness matrix represents the effect of the displacement of a joint corresponding to that column on all joints.

Since  $[S_{AB}] = [S_{BA}]$ ;  $[S_{AC}] = [S_{CA}]$ , etc., it can be seen that the stiffness matrix is symmetrical. Since the elements in the stiffness matrix  $[S]$  are the joint stiffness matrices

and the bar influence stiffness matrices, and the elements in the joint stiffness matrices and in the bar influence stiffness matrices are the stiffness coefficients which depend on the properties and dimensions of the bar concerned, it can be seen that the stiffness matrix in Equation 16 is not affected by the loading conditions but depends on the properties and the dimensions of the structure only. Hence, once the stiffness matrix  $[S]$  for a given structure has been set up, it can be used for any kind of loading.

2. Modified Stiffness Matrix. In the frame shown in Figure 4 joints A, C, E and F are hinged supports. Therefore, at these joints, the force matrices  $[F_A]$ ,  $[F_C]$ ,  $[F_E]$  and  $[F_F]$  are the unknown joint reactions and the displacement matrices  $[D_A]$ ,  $[D_C]$ ,  $[D_E]$ , and  $[D_F]$  are known to be zero. Equation 16 may be modified by dropping out the corresponding rows and columns for joints A, C, E and F as follows:

$$\begin{bmatrix} [F_B] \\ [F_D] \\ [F_G] \end{bmatrix} = \begin{bmatrix} \sum[S_{BB}] & [S_{BD}] & [S_{BG}] \\ [S_{DB}] & \sum[S_{DD}] & [S_{DG}] \\ [S_{GB}] & [S_{GD}] & \sum[S_{GG}] \end{bmatrix} \begin{bmatrix} [D_B] \\ [D_D] \\ [D_G] \end{bmatrix} \quad (18)$$

or simply

$$[F'] = [S'] [D'] \quad (19)$$

where  $[S']$  is the modified stiffness matrix for the whole space frame;  $[F']$  is the modified force matrix which does

not include the unknown joint reactions; and  $[D']$  is the modified displacement matrix which does not include the known joint displacements.

By inverting the modified stiffness matrix  $[S']$  of Equation 19, the unknown modified displacement matrix may be found

$$[D'] = [S']^{-1} [F'] \quad (20)$$

Once the unknown joint displacements are found, the bar forces of each member can be found by using Equation 12 and the solution for the reactions at A, C, E and F follow.

#### IV. ANALYSIS OF RIGIDLY-CONNECTED SPACE FRAMES

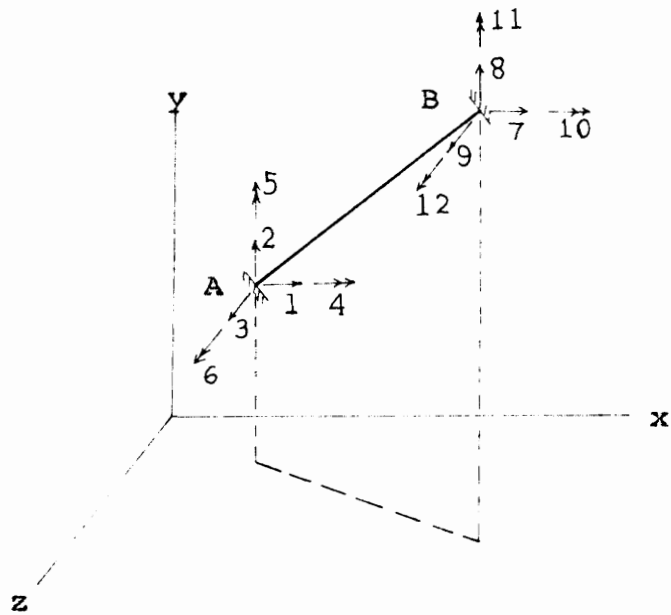
##### A. Basic Concepts

1. Methods of Analyses. The bar stiffness coefficients referred to a single set of axes for the total structure may be found by either of two methods.

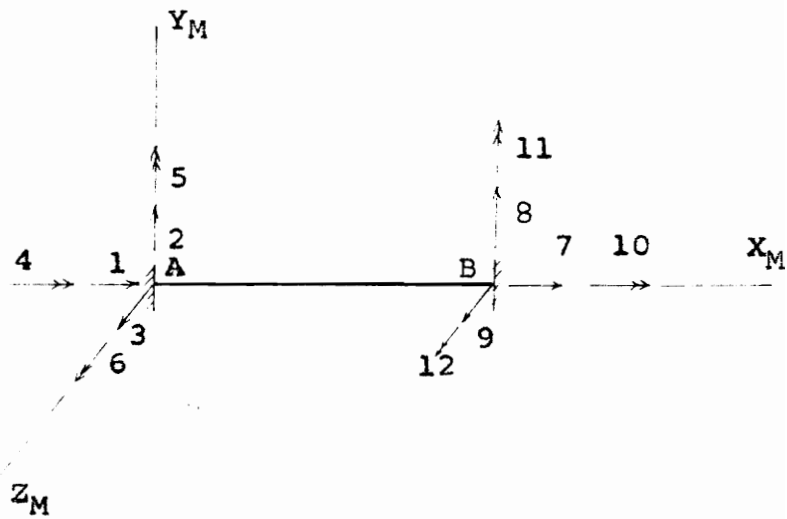
The first method consists of a direct formulation of the stiffness coefficients. In this approach, unit displacements in the directions of the reference axes, as shown in Figure 5a, are induced at the ends of the bar, and the corresponding restraint forces in the same directions are calculated. These forces become the elements of the bar stiffness matrix along each axis. These concepts were used in Section III.

The second method, on the other hand, consists of first obtaining the stiffness matrices along member-oriented axes, as shown in Figure 5b, and then transforming this matrix to the structure-oriented axes by a process of rotation of axes. Using an appropriate transformation matrix, the rotation of axes may be executed by matrix multiplications, as described in the following pages.

The direct approach to obtain the stiffness matrix is demonstrated in Section III for pin-connected frames, but this technique becomes quite involved for rigidly-connected frames with six degrees of freedom. On the other hand, the rotation of axes method is a formalized approach which is no



(a)



(b)

Figure 5. Numbering System for Space Frame.

more difficult in theory for a complicated structure than for a simple one. In order to derive the stiffness matrix for a rigid member in space with reference to the structure-oriented axes, the stiffness matrix with reference to the member-oriented axes is derived first. Then an appropriate transformation matrix is obtained by the rotation of axes method. Finally, the stiffness matrix for the structure-oriented axes is obtained by matrix multiplications.

2. Stiffness Coefficients. Consider a prismatic bar AB with fixed ends as shown in Figure 5b. The  $x_M$  axis coincides with the centroidal axis of the bar. The  $y_M$  and  $z_M$  axes are principal axes for the bar. At joint A the translations are numbered 1, 2 and 3, and the rotations are numbered 4, 5 and 6. Similarly, at joint B, 7, 8 and 9 are translations, and 10, 11 and 12 are rotations. The unit displacements are considered to be induced one at a time while all other end displacements are zero, and they are assumed to be positive in the  $x_M$ ,  $y_M$  and  $z_M$  directions.

The bar stiffnesses with respect to these axes for the end displacements at A are summarized in Figure 6. Bar stiffnesses for displacements at B can be similarly found. An arrow with a single head represents a force vector and an arrow with a double head represents a moment vector. All of the bar stiffnesses shown in the figure are derived by determining the values of the restraining forces required to hold the distorted member in equilibrium. The space frame member

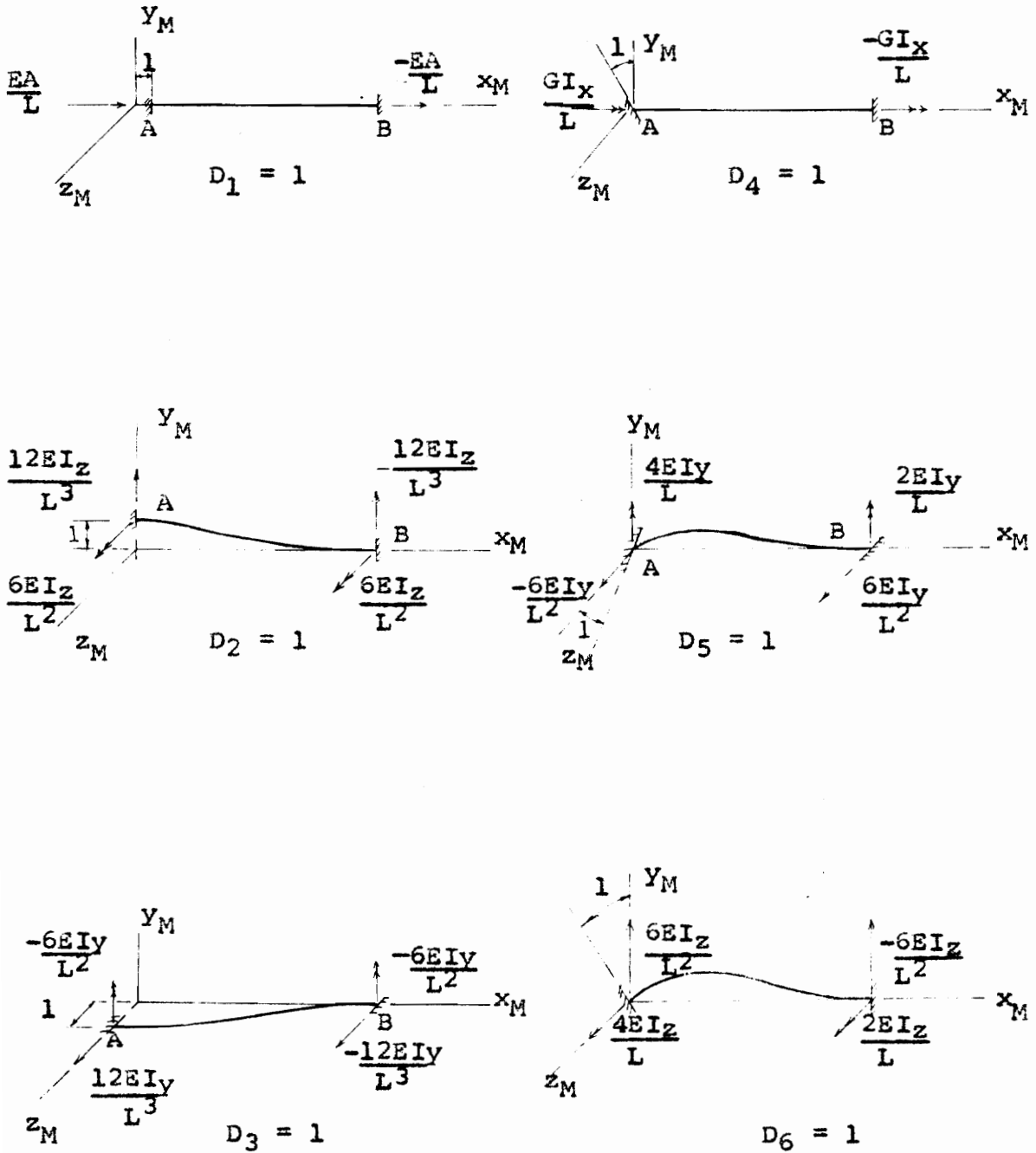


Figure 6. Fixed-End Frame Bar With Unit Displacements.

stiffness matrix  $[S_M]$  is shown in Table 1; it is of order 12 x 12 and symmetrical. Each column in the matrix represents the forces caused by one of the unit displacements. This matrix can be written as

$$[S_M] = \begin{bmatrix} [S_{AA}] & [S_{AB}] \\ [S_{BA}] & [S_{BB}] \end{bmatrix} \quad (21)$$

where the elements are described in Section III-A-2 and 3.

3. Rotation of Axes. Consider a force vector  $F$  as shown in Figure 7 with two sets of orthogonal axes at the origin  $O$ . The  $x$ ,  $y$  and  $z$  axes are taken parallel to a set of reference axes for the structure of which this member is a part and the  $x_M$ ,  $y_M$  and  $z_M$  axes are member-oriented axes. Let the direction cosines of the  $(x_M)$ ,  $(y_M)$  and  $(z_M)$  axes with respect to the  $(x, y$  and  $z)$  axes be  $(\lambda_{11}, \lambda_{12}, \lambda_{13})$ ,  $(\lambda_{21}, \lambda_{22}, \lambda_{23})$ , and  $(\lambda_{31}, \lambda_{32}, \lambda_{33})$ , respectively. The vector  $F$  may be represented by either of two sets of orthogonal components,  $(F_x, F_y, F_z)$  or  $(F_{x_M}, F_{y_M}, F_{z_M})$ . The one set of components may be related to the other as follows.

It may be shown that  $F_{x_M}$  is equal to the sum of the projections of  $F_x$ ,  $F_y$ , and  $F_z$  on the  $x_M$  axis. The components  $F_{y_M}$  and  $F_{z_M}$  may be expressed in a similar manner,

$$\begin{aligned} F_{x_M} &= \lambda_{11}F_x + \lambda_{12}F_y + \lambda_{13}F_z \\ F_{y_M} &= \lambda_{21}F_x + \lambda_{22}F_y + \lambda_{23}F_z \\ F_{z_M} &= \lambda_{31}F_x + \lambda_{32}F_y + \lambda_{33}F_z \end{aligned} \quad (22)$$



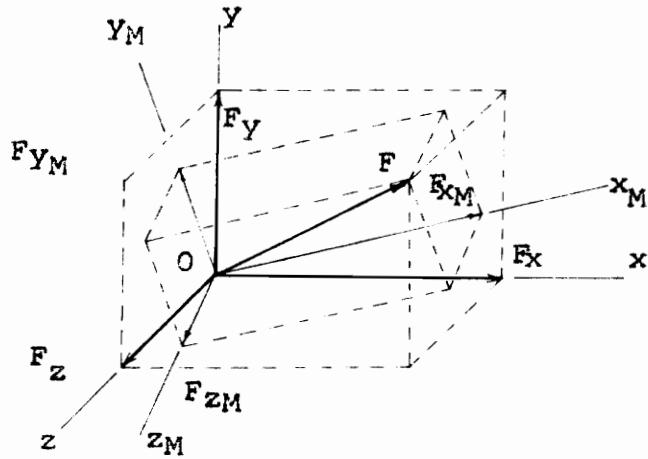


Figure 7. Rotation of Reference Axes.

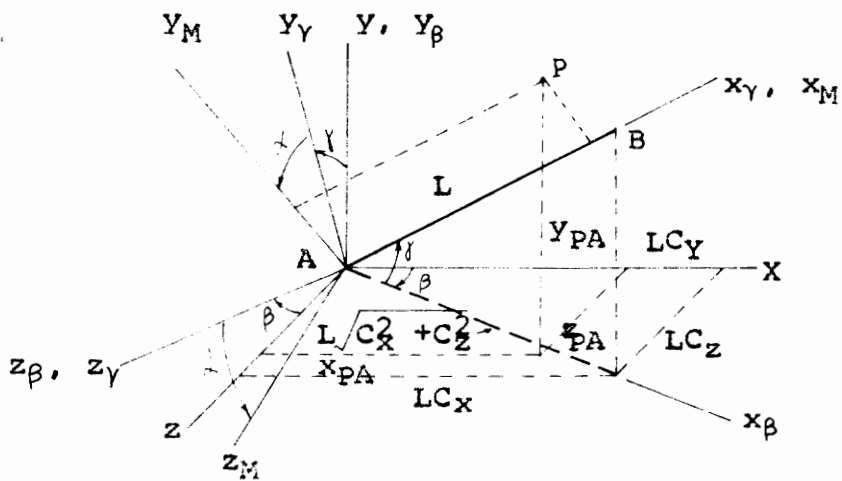


Figure 8. Rotation of the  $x$ ,  $y$  and  $z$  Axes.

in matrix form, these expressions become

$$\begin{bmatrix} F_{xM} \\ F_{yM} \\ F_{zM} \end{bmatrix} = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix} \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} \quad (23)$$

or

$$[F_M] = [R][F] \quad (24)$$

where  $[F_M]$  and  $[F]$  are vector components with respect to member axes and structure axes, respectively,  $[R]$  is the three-dimensional rotation matrix of direction cosines. Similarly, the  $x, y, z$  set of components of the vector  $F$  may be expressed in terms of the  $x_M, y_M$  and  $z_M$  set of components,

$$[F] = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} \lambda_{11} & \lambda_{21} & \lambda_{31} \\ \lambda_{12} & \lambda_{22} & \lambda_{32} \\ \lambda_{13} & \lambda_{23} & \lambda_{33} \end{bmatrix} \begin{bmatrix} F_{xM} \\ F_{yM} \\ F_{zM} \end{bmatrix} \quad (25)$$

or 
$$[F] = [R'] [F_M] \quad (26)$$

where  $[R']$  is the transpose of the rotation matrix  $[R]$ . From Equations 24 and 26 it is apparent that the transpose of  $[R]$  is equal to its inverse and the rotation matrix  $[R]$  is an orthogonal matrix.

Since displacements as well as forces may be treated as vectors, the relationships formulated above for the force  $F$  may be applied equally well to displacements,

$$\{ D_M \} = \{ R \} \{ D \} \quad (27)$$

and 
$$\{ D \} = \{ R' \} \{ D_M \} \quad (28)$$

where  $\{ D_M \}$  and  $\{ D \}$  are matrices of components of the displacement  $D$  parallel to the member axes and structure axes, respectively.

4. Rotation Matrix. The member stiffness matrix  $\{ S \}$  for structure axes is developed from  $\{ S_M \}$  for member axes using the method of rotation of axes described above. The rotation matrix required for this transformation may take one of several forms, of varying complexity, depending upon the orientation of the member in space.

Displacements of the ends of a bar AB are shown in Figure 6 for member-oriented axes, and the stiffness matrix  $\{ S_M \}$  is shown in Table 1. This matrix is transformed by means of a rotation transformation matrix  $\{ R^T \}$  into the matrix  $\{ S \}$ .

The form of the rotation matrix  $\{ R \}$  depends upon the particular orientation of the member axes. In general, the space frame member has its principal axes  $y_M$  and  $z_M$  in skew directions. The orientation of the principal axes can be specified by means of an angle of rotation about the  $x_M$  axis. Three successive rotations from the structure axes to the member axes are shown in Figure 8. The first is a rotation

of x-axis through an angle  $\beta$  about the y axis. This rotation places the axes in the positions denoted by the subscript  $\beta$ . The rotation matrix  $[R_\beta]$  for this transformation consists of the direction cosines of the  $\beta$ -axes with respect to the structure axes. That is

$$[R_\beta] = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \quad (29)$$

In the general form of the rotation matrix  $(R)$ , as in Equation 23, the three elements  $\lambda_{11}$ ,  $\lambda_{12}$ , and  $\lambda_{13}$  in the first row are the direction cosines for the member itself, thus

$$\lambda_{11} = \frac{x_B - x_A}{L} = C_x, \quad \lambda_{12} = \frac{y_B - y_A}{L} = C_y \quad \text{and} \quad \lambda_{13} = \frac{z_B - z_A}{L} = C_z$$

$$\text{where } L = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$$

The functions  $\cos\beta$  and  $\sin\beta$  in Equation 29 may be expressed in terms of  $C_x$ ,  $C_y$ , and  $C_z$  as

$$\cos\beta = \frac{C_x}{\sqrt{C_x^2 + C_z^2}} \quad \text{and} \quad \sin\beta = \frac{C_z}{\sqrt{C_x^2 + C_z^2}}$$

Therefore

$$[R_\beta] = \begin{bmatrix} \frac{C_x}{\sqrt{C_x^2 + C_z^2}} & 0 & \frac{C_z}{\sqrt{C_x^2 + C_z^2}} \\ 0 & 1 & 0 \\ \frac{-C_z}{\sqrt{C_x^2 + C_z^2}} & 0 & \frac{C_x}{\sqrt{C_x^2 + C_z^2}} \end{bmatrix} \quad (30)$$

The matrix  $[R_\beta]$  may also be used to transform the vector  $[F]$ , referred to the structure axes, into the vector  $[F_\beta]$ , referred to the  $\beta$ -axis

$$[F_\beta] = [R_\beta] [F] \quad (31)$$

The second rotation is a rotation through an angle  $\gamma$  about the  $z_\beta$  axis. This rotation places the axes in the positions denoted by the subscript  $\gamma$ . This rotation also places the  $x$  axis in its final position,  $x_M$ . The rotation matrix  $R_\gamma$  for this transformation consists of the direction cosines of the  $\gamma$ -axes with respect to the  $\beta$ -axes. That is

$$[R_\gamma] = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (32)$$

The functions  $\cos\gamma$  and  $\sin\gamma$  may be expressed in terms of  $C_x$ ,  $C_y$ , and  $C_z$

$$[R_\gamma] = \begin{bmatrix} \sqrt{C_x^2 + C_z^2} & C_y & 0 \\ -C_y & \sqrt{C_x^2 + C_z^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (33)$$

This rotation also transforms the vector  $[F_\beta]$  into the vector  $[F_\gamma]$  as

$$[F_\gamma] = [R_\gamma] [F_\beta] \quad (34)$$

Finally, the third rotation is a rotation through an angle  $\alpha$  about the  $x_\gamma$  axis, causing the  $y_\gamma$  and  $z_\gamma$  axes to

coincide with the principal axes of the member,  $y_M$  and  $z_M$ .

The rotation matrix  $[R_\alpha]$  for this final transformation consists of the direction cosines of the member axes (also  $\alpha$ -axes) with respect to the  $Y$ -axes,

$$[R_\alpha] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{pmatrix} \quad (35)$$

This final rotation also transforms  $[F_Y]$  into  $[F_M]$ , thus

$$[F_M] = [R_\alpha] [F_Y] \quad (36)$$

In summary, it will be noted that the  $x$ ,  $y$  and  $z$  axes are rotated to coincide with the  $x_M$ ,  $y_M$  and  $z_M$  axes, respectively, by multiplying them successively by  $[R_\beta]$ ,  $[R_Y]$  and  $[R_\alpha]$ .

Correspondingly the force vector  $[F]$  can be transformed into  $[F_M]$  by the same multiplication and expressed as

$$[F_M] = [R_\alpha] [R_Y] [R_\beta] [F] \quad (37)$$

$$\text{or } [F_M] = [R] [F] \quad (38)$$

$$\text{where } [R] = [R_\alpha] [R_Y] [R_\beta] \quad (39)$$

$[R]$  is the rotation matrix for the three successive rotations and is equal to

$$[R] = \begin{pmatrix} C_x & C_y & C_z \\ \frac{-C_x C_y \cos \alpha - C_z \sin \alpha}{\sqrt{C_x^2 + C_z^2}} & \sqrt{C_x^2 + C_z^2} \cos \alpha & \frac{-C_y C_z \cos \alpha + C_x \sin \alpha}{\sqrt{C_x^2 + C_z^2}} \\ \frac{C_x C_y \sin \alpha - C_z \cos \alpha}{\sqrt{C_x^2 + C_z^2}} & -\sqrt{C_x^2 + C_z^2} \sin \alpha & \frac{C_y C_z \sin \alpha + C_x \cos \alpha}{\sqrt{C_x^2 + C_z^2}} \end{pmatrix} \quad (40)$$

This rotation matrix is expressed in terms of the direction cosines of the member (or the coordinates of its ends) and the angle  $\alpha$ , which is given in the structure description.

5. Member Stiffness. The force-displacement relationships at the ends of a member for the member axes is

$$\begin{bmatrix} F_{M1} \\ F_{M2} \\ - \\ - \\ F_{M12} \end{bmatrix} = \begin{bmatrix} k_{M11} & k_{M12} & \dots & k_{M1,12} \\ k_{M21} & k_{M22} & \dots & k_{M2,12} \\ - & - & - & - \\ - & - & - & - \\ k_{M12,1} & k_{M12,2} & \dots & k_{M12,12} \end{bmatrix} \begin{bmatrix} D_{M1} \\ D_{M2} \\ - \\ - \\ D_{M12} \end{bmatrix} \quad (41)$$

From Equations 21, 24 and 27, Equation 41 becomes

$$\begin{bmatrix} [R] [F_{A1}] \\ [R] [F_{A2}] \\ [R] [F_{B1}] \\ [R] [F_{B2}] \end{bmatrix} = [S_M] \begin{bmatrix} [R] [D_{A1}] \\ [R] [D_{A2}] \\ [R] [D_{B1}] \\ [R] [D_{B2}] \end{bmatrix} \quad (42)$$

where the subscripts A and B indicate the end which is displaced, the subscript 1 denotes translation and subscript 2 denotes rotation.

This equation can be written as

$$\begin{bmatrix} [R] & 0 & 0 & 0 \\ 0 & [R] & 0 & 0 \\ 0 & 0 & [R] & 0 \\ 0 & 0 & 0 & [R] \end{bmatrix} \begin{bmatrix} [F_{A1}] \\ [F_{A2}] \\ [F_{B1}] \\ [F_{B2}] \end{bmatrix} = [S_M] \begin{bmatrix} [R] & 0 & 0 & 0 \\ 0 & [R] & 0 & 0 \\ 0 & 0 & [R] & 0 \\ 0 & 0 & 0 & [R] \end{bmatrix} \begin{bmatrix} [D_{A1}] \\ [D_{A2}] \\ [D_{B1}] \\ [D_{B2}] \end{bmatrix} \quad (43)$$

$$\text{or } [R_T] [F] = [S_M] [R_T] [D] \quad (44)$$

where  $[R_T]$  is defined as the rotation transformation matrix

$$[R_T] = \begin{pmatrix} [R] & 0 & 0 & 0 \\ 0 & [R] & 0 & 0 \\ 0 & 0 & [R] & 0 \\ 0 & 0 & 0 & [R] \end{pmatrix} \quad (45)$$

From Equation 44,  $[F]$  becomes

$$[F] = [R_T]^{-1} [S_M][R_T][D] \quad (46)$$

Since  $[R]$  is orthogonal,  $[R_T]$  is also orthogonal, hence the transpose of  $[R_T]$  is also the inverse of  $[R_T]$ . Therefore,

$$[F] = [R_T^t][S_M][R_T][D] = [S][D] \quad (47)$$

$$\text{where } [S] = [R_T^t][S_M][R_T] \quad (48)$$

Equation 48 is quite complicated when expressed in literal form, and for this reason it will not be expanded here. The fact that such a lengthy set of relationships can be represented so concisely by Equation 48 is one of the principal advantages of matrix methods in the analysis of structures. (4)

In many instances a space frame member will be oriented so that the principal axes of the cross-section lie in horizontal and vertical planes (as an I beam with its web in a vertical plane), or has two axes of symmetry in the cross-section and the same moment of inertia about each axis (as a

circular section). For such cases the  $y_M$  and  $z_M$  axes can be selected so that  $\alpha$  is equal to zero, and  $[R_\alpha]$  becomes a unit matrix. Then Equation 39 is reduced to the form

$$[R] = [R_Y][R_\beta] \quad (49)$$

or

$$[R] = \begin{bmatrix} C_x & C_y & C_z \\ \frac{-C_x C_y}{\sqrt{C_x^2 + C_z^2}} & \sqrt{C_x^2 + C_z^2} & \frac{-C_y C_z}{\sqrt{C_x^2 + C_z^2}} \\ \frac{-C_z}{\sqrt{C_x^2 + C_z^2}} & 0 & \frac{C_z}{\sqrt{C_x^2 + C_z^2}} \end{bmatrix} \quad (50)$$

This is the rotation matrix used in the experimental structure problem in Section V.

6. Rotation for Unknown Principal Axes. In some structures the direction of the principal transverse axes for some members may not be readily available; i.e., the angle  $\alpha$  may not be known or conveniently found. In that event, a point P (Figure 8) which lies in one of the principal planes of the member but not on the longitudinal axis of the member must be specified. The angle of rotation  $\alpha$  can then be expressed in terms of the coordinates of P and the coordinates of one end of that member. If the coordinates of points P and A are  $x_P, y_P, z_P$  and  $x_A, y_A, z_A$  respectively, the coordinates of P with respect to point A are

$$x_{PA} = x_P - x_A, \quad y_{PA} = y_P - y_A, \quad z_{PA} = z_P - z_A \quad (51)$$

The coordinates of P with reference to the Y-axes are obtained from the coordinates with reference to the structure axes by a rotation of the structure axes through the angles  $\beta$  and  $\gamma$ ,

$$\begin{bmatrix} x_{PY} \\ y_{PY} \\ z_{PY} \end{bmatrix} = [R_Y][R_\beta] \begin{bmatrix} x_{PA} \\ y_{PA} \\ z_{PA} \end{bmatrix} = \begin{bmatrix} C_x x_{PA} + C_y y_{PA} + C_z z_{PA} \\ \frac{-C_x C_y}{\sqrt{C_x^2 + C_z^2}} x_{PA} + \sqrt{C_x^2 + C_z^2} y_{PA} + \frac{C_y C_z}{\sqrt{C_x^2 + C_z^2}} z_{PA} \\ \frac{C_z}{\sqrt{C_x^2 + C_z^2}} x_{PA} + \frac{C_x}{\sqrt{C_x^2 + C_z^2}} z_{PA} \end{bmatrix} \quad (52)$$

The matrix elements for rotation through angle  $\alpha$  can then be found,

$$\sin \alpha = \frac{z_{PY}}{\sqrt{y_{PY}^2 + z_{PY}^2}} \quad \cos \alpha = \frac{y_{PY}}{\sqrt{y_{PY}^2 + z_{PY}^2}} \quad (53)$$

The rotation matrix [R] (Equation 40) can now be completed.

7. Vertical Member Rotation Matrix. The rotation matrix for a vertical member AB (Figure 9) takes a different form. There is no rotation needed through angle  $\beta$ . The first rotation of the x-axis is through angle  $\gamma$ , which may be either  $90^\circ$  or  $270^\circ$ . The second rotation is through angle  $\alpha$ . The rotation matrix is

$$[R]_V = [R_\alpha][R_\gamma] = \begin{bmatrix} 0 & C_y & 0 \\ -C_y \cos \alpha & 0 & \sin \alpha \\ C_y \sin \alpha & 0 & \cos \alpha \end{bmatrix} \quad (54)$$

Figure 9c shows the case where a point P must be used and the angle  $\alpha$  is between  $90^\circ$  and  $180^\circ$ ; and Figure 9d shows the case where a point P must be used and  $\alpha$  is between  $0^\circ$  and  $90^\circ$ .

$$\sin \alpha = \frac{z_{PA}}{\sqrt{x_{PA}^2 + z_{PA}^2}} \quad \cos \alpha = \frac{-x_{PA}}{\sqrt{x_{PA}^2 + z_{PA}^2}} Cy \quad (55)$$

If  $\alpha$  is equal to zero, the rotation matrix becomes

$$[R]_V = \begin{pmatrix} 0 & Cy & 0 \\ -Cy & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (56)$$

#### B. Summary

In summary, the member stiffness matrix  $[S_M]$  is first obtained for each member (in reference to member axes). Next, a rotation matrix  $[R]$  is constructed for each member in a form which depends upon the direction of the member under consideration, using either the angle  $\alpha$  or the coordinates of a point P to identify a principal plane. Finally, the rotation transformation matrix  $[R_T]$  is formed and the member stiffness matrix in reference to the structure axes is computed.

After the stiffness matrix for each member is obtained, each joint stiffness matrix and each influence stiffness matrix is constructed. Then the stiffness matrix for the

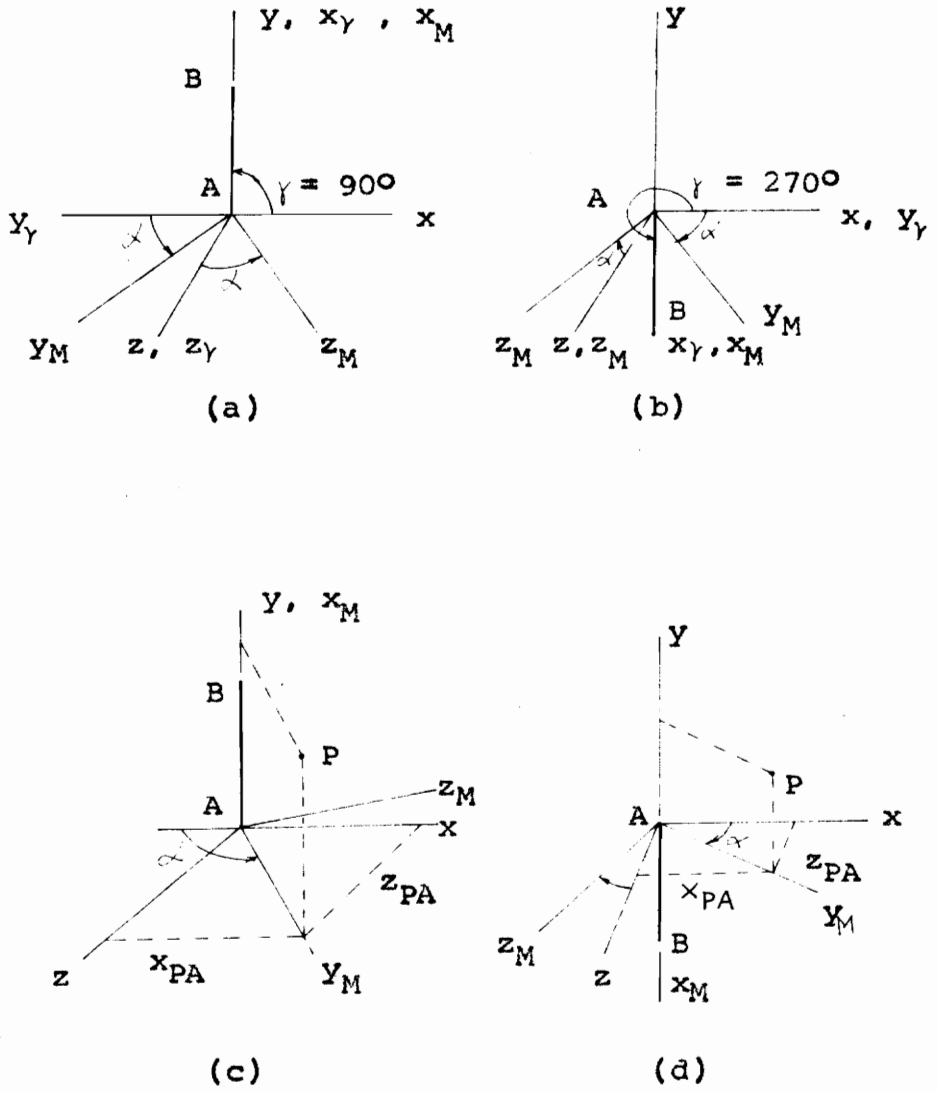


Figure 9. Rotation of Axes and Point P for a Vertical Member.

whole structure is formed. The stiffness matrix is inverted and multiplied by the load matrix to give the matrix solution of deflections of the joints of the structure.

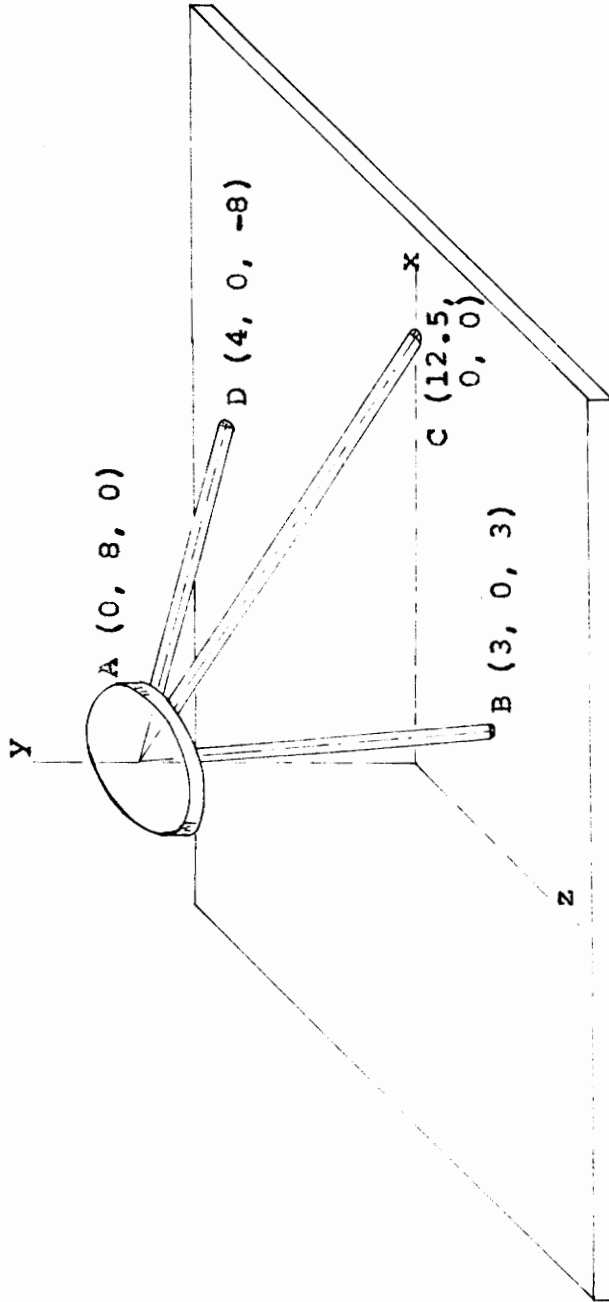
## V. EXPERIMENTAL TEST

### A. Test Description

A simple space frame was set up as shown in Figure 10. One end of each of three one-half inch diameter circular steel bars were welded together to form a typical space frame joint at A. The other ends of the bars were welded to a heavy three-quarters inch thick steel plate, 16 inches wide by 25 inches long, stiffened by four-inch channel beams. A small horizontal steel plate, four and one-half inches in diameter by one-half inch thick, was welded to joint A to support applied loads. Small holes were drilled on the surface to provide spherical seats for the applied loads, one and three-quarters inches from the axis of the joint. Four horizontal extension arms, five inches long, were attached to the plate 90 degrees apart to be used for determining joint displacement. Displacement components were measured with dial gages to the nearest .0001 inch. Four gages were set vertically under extension arms at the ends. Four gages were set horizontally against the sides of the extension arms at the ends. The complete test set-up is shown in the photograph of Figure 11.

### B. Test Procedure

The frame was loaded in a Baldwin mechanical screw type testing machine with a 24,000 pound capacity. The 6,000 pound low range was used. Load was applied gradually up to



Bar	Length	Slenderness Ratio $L/r$
AB	9.055"	72.44
AC	14.841"	118.73
AD	12.000"	96.00

Figure 10. Experimental Test Model



Figure 11. Photograph of Test Model

500 pounds at the center of joint A and at each of three points around the edge of the load plate, thus producing pure vertical axial load and axial load with each of three eccentric moments. The load was applied through a short strut to assure that the machine head would not restrict horizontal movement of the joint. The dial gages were read during loading and unloading sequences and the loads were applied and released a sufficient number of times until there were no differences in successive cycles.

Initially there were apparent errors and some inconsistencies in the measurements that were made. The test apparatus and the procedure were examined and modified until the measurements appeared to be consistently good.

Care had to be exercised in the placement of the apparatus in the machine to assure vertical loading. Gages were carefully set to read horizontal and vertical displacements without being affected by the joint rotations.

Errors due to bending or warping of the base plate in early loadings were essentially eliminated by added channel stiffeners to the plate and isolating the dial gages from the base on a plate referenced to the fixed ends of the three bars.

### C. Test Results

During the loading sequences for each of four load conditions, the eight dial gages on the test set-up were read (See Figures 10 and 11 and Sections VA and VB). These

deflection readings were then used to compute the six components of motion of the movable joint. The results are shown in Table 2.

#### D. Analysis of the Test Model

1. Description. A matrix method for the analysis of rigidly-connected space frames was discussed and developed in Section IV. This method was used for the analysis of the test model described in Section V-A, and involves the following steps:

- a. Establishing an origin and directions for three orthogonal axes for the structure.
- b. Determining the coordinates of all joints with reference to the structure-oriented axes.
- c. Determine the directions of the member-oriented axes and the orientation angles of these directions with reference to the structure-oriented axes.
- d. Determine the lengths of each member and the rotation matrices for each member.
- e. Determine the stiffness coefficients for each member, the stiffness matrix for each joint and the combined stiffness matrix for the whole structure.
- f. Solve the problem of matrix inversion and multiplication to determine the joint displacements and support reactions.

Table 2. Test Results

Load Cond.	F <sub>y</sub> lbs.	M <sub>x</sub> in-lbs.	M <sub>z</sub> in-lbs.	x in.	y in.	z in.	x̄ rad.	ȳ rad.	z̄ rad.
1	-500	0	0	-.00200	-.00222	-.00015	-.00055	.00022	.00068
2	-500	0	875	-.00400	-.00290	-.00080	-.00150	.00117	.00995
3	-500	875	0	-.00260	-.00718	-.01655	.00932	-.00386	.00008
4	-500	619	619	-.00415	-.00265	.00040	.00721	-.00369	.00662

F<sub>x</sub> = F<sub>z</sub> = M<sub>y</sub> = 0      See Figure 10 for axes orientation.

- g. Determine the axial forces and bending moments at each end of each member.

2. Computer Program. The steps for the analysis of a rigidly-connected space frame were programmed for solution on the IBM 1401-7040 Computer at Virginia Polytechnic Institute. The program was written in Fortran IV language and is reproduced in the Appendix. The total program consists of a main program and four subroutine programs:

- a. A subroutine for generating the stiffness matrix for the whole structure.
- b. A subroutine for matrix inversion.
- c. A subroutine for the calculation of joint displacements and reactions.
- d. A subroutine for the calculation of forces and bending moments on each member.

It should be noted that this general solution of the stiffness matrix for the whole structure is independent of load and may easily be used for any number or type of load condition after it has been developed for a given structure.

3. Analytical Results. Structure-oriented axes were established for the test frame as shown on Figure 10 and the basic data on geometry, material properties and loads were prepared in a proper form to be used as data for the computer program described above.

The results of the analysis are shown in Table 3 in the same manner as the test results were shown in Table 2.

Table 3. Analytical Results

Load Cond.	F <sub>y</sub> lbs.	M <sub>x</sub> in-lbs.	M <sub>z</sub> in-lbs.	x in.	y in.	z in.	x rad.	y rad.	z rad.
1	-500	0	0	-.00212	-.00180	-.00010	-.00001	-.00001	.00038
2	-500	0	875	-.00318	-.00247	-.00018	-.00043	.00044	.01128
3	-500	875	0	-.00206	-.00178	-.00068	.01348	-.00596	-.00004
4	-500	619	619	-.00282	-.00226	.00009	.00923	-.00390	.00779
F <sub>x</sub> = F <sub>z</sub> = M <sub>y</sub> = 0									
See Figure 10 for axes orientation.									

## E. Comparison of Test and Analysis

1. Comparison of Results. The analysis was developed to predict test results. The degree of success is presented in Table 4 in the same manner as in Tables 2 and 3. However, for each condition, two values are given:

- a. The numerical value of the test result minus the analytical result.
- b. The difference as a percentage of the test result. A positive percentage indicates that the test result is numerically larger than the analytical result.

2. Discussion. A cursory examination of Table 4 would indicate that the analysis did not provide a satisfactory solution for displacements and rotations of the loaded joint of the test frame. Differences range from -57% to +150%.

A closer examination of the differences would reveal that they are of the order of magnitude of a few ten-thousandths of an inch in deflection or a few ten-thousandths of a radian in rotation. Considering the sensitivity of the test frame measuring apparatus, the fact that dial gages measured only to the nearest 0.0001 inch and the fact that almost 90 per cent of the deflection components were less than 0.0042 inch, the results can be of value.

Furthermore, a study of the test frame construction, the support base, the loading plate and the deflection

Table 4. Differences Between Test and Analytic Results

Load Cond.	F <sub>y</sub> lbs.	M <sub>x</sub> in-lbs.	M <sub>z</sub> in-lbs.	x in.	y in.	z in.	x rad.	y rad.	z rad.
1	-500	0	0	.00012 -6%	-.00042 19%	-.00005 33%	-.00054 98%	.00023 105%	.00030 44%
2	-500	0	875	-.00082 20%	-.00043 15%	-.00062 78%	-.00107 71%	.00073 62%	-.00133 -13%
3	-500	875	0	-.00054 21%	-.00540 75%	-.01587 96%	-.00416 -45%	.00210 -55%	.00012 150%
4	-500	619	619	-.00133 32%	-.00039 15%	.00031 78%	-.00202 -28%	.00021 -57%	-.00117 -18%
$F_x = F_z = M_y = 0$									
See Figure 10 for axes orientation.									

measuring appurtenances would indicate that the differences cited above were due more to discrepancies in the test than in the analysis.

Although the scope of this thesis was primarily intended to include only the development of the analytical technique and an illustration of its use, the construction of the test frame and the collection of reliable data proved to be a major problem. Rebuilding the test frame with changes and improvements, now recognized as essential, is beyond the scope of this work, but some recommendations are made for further studies and are listed in Section VI.

Some improvements were incorporated in the test set-up during the building process and these have been described in Section V-A.

Originally it was anticipated that the three-quarters inch thick base plate would provide fixed-end conditions for the bars welded to it, but either because of variations from a plane surface or warping due to the load application, it was suspected that errors would arise. It was felt that these errors were essentially eliminated when the base plate was stiffened by welding channels to it.

It would have been ideal to measure deflections with reference to a plane through the fixed ends of the three bar members. A plate mounted on supports at these three points would have provided a good reference surface, but it would have been difficult to add to the completed frame. Stiffening

the base was assumed adequate to make it a satisfactory reference plane and the gage mountings were clamped to it.

The application of the load to the circular load plate circumference caused bending of that plate and thus affected the movement of the extension arms to give erroneous readings. A simplified computation was made of the bending deflections and subtracted from the recorded deflections. Although this reduced the differences, it did not bring the test results in line with the analytical results; while it decreased some differences, it increased others. Evidently the load caused some warping of the plate and this could not be easily calculated.

The extension arms were welded on the circular plate after the frame was constructed and it was found to be impossible to place the large dial gages where they were needed. Smaller dial gages were not available. The contact surfaces for the dial gages were not perfectly plane and they were not all normal to the principal axes so that readings might be non-linear and displacements normal to a dial gage axis might also indicate a small amount of movement of the dial. An accurate evaluation of these effects could not be made; when approximate adjustments were made the results still gave large differences that could not be accounted for. The results presented in Table 2, therefore, are not adjusted for this effect.

It is likely that many of the errors arising from the use of dial gages and the deflection measuring apparatus could have been off-set if bar forces and bending moments could have been determined from electric strain gage measurements and compared with those analytical values. Such test data would require extensive instrumentation and was not considered for this initial study.

The results of this study will certainly provide a sound base on which to make further studies.

## VI. CONCLUSION

The principal objective of this thesis has been accomplished. The technique for the analysis of a space frame has been developed and described. A computer program has been written and presented. A test frame was constructed, analyzed and tested and the results for deflections and rotations compared.

Although the differences between test results and analytical results were too large to verify the analytical solution, there are several valuable conclusions to be drawn from the work:

1. There is no indication that the analysis is not a good one. The differences are more likely to have been due to the test results.
2. Improvement of the test apparatus and frame design is necessary before modifications are made in the analysis.
3. If the same basic test set-up is used, the dial gage holders must be isolated from any effect of bending of the base plate. A "floating plane" supported only at the bases of the three bars would provide a satisfactory support.
4. The extension arms used to activate the dial gages should be attached directly to the joint of the members and remain free of the loading plate.

5. The extension arms should be circular in cross-section and the dial gages should have flat contact surfaces. This would eliminate errors due to displacements and rotations in directions other than those being measured.
6. The base plate must be stiff enough to assure rigid fixity of the attached bars as intended.
7. With a satisfactory test set-up some of the bars should be instrumented with electric strain gages to provide means of calculating axial forces, torque and bending moments that may be compared with analytical results.
8. The study should be continued and extended to evaluate the effect of the relative size of the joint connections on the action of a loaded space frame.

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To his wife, Chii-mei Kuo Lin, the author wishes to express his deep gratitude for her constant encouragement to continue his graduate studies.

## IX. VITA

The author was born on September 1, 1934, in Taichung, Taiwan, China. He was graduated from the Civil Engineering Department of the National Taiwan University in June, 1958. After 21 months of reserve officer's training in the Chinese Army, he worked for the Taiwan Provincial Water Conservancy Bureau as a junior engineer for four and one-half years. In December, 1964 he came to the United States of America. He has been a graduate student in the Civil Engineering Department at Virginia Polytechnic Institute since March, 1965.

*Ju-Tien Lin*

## X. APPENDIX

The Computer Program used for the analysis of the space frame described in Section V is listed on the following pages. It consists of a main program and four subroutine programs:

- a. Subroutine 1 - for generating the structure stiffness matrix.
- b. Subroutine 2 - for matrix inversion.
- c. Subroutine 3 - for determining joint displacements and reactions.
- d. Subroutine 4 - for determining bar forces and bending moments.

BJOB 10 826140369 LIN, F. T.

SIBJOB

SIBFTC D1

C PROGRAM FOR SPACE RIGID FRAME ANALYSIS

DIMENSION X(4),Y(4),Z(4),JJ(3),JK(3),AX(3),AIX(3),AIY(3),AIZ(3),  
 1EL(3),R(3,9),RL(24),JKL(24),SM(12,12),SMR(12,12),SMD(12,12),IK(6)  
 2,A(24),AML(3,12),AE(24),AC(24),AR(24),D(24),AF(12),IJ(6),S(24,6)

DIMENSION F(6,7)

COMMON X,Y,Z,JJ,JK,AX,AIX,AIY,AIZ,EL,R,RL,JKL,SM,SMR,SMD,S,A,  
 1AML,AE,AC,AR,D,AF,IJ,IK,M,N,NJ,NR,NRJ,NN,E,G

C INPUT AND OUTPUT STRUCTURE DATA

C M=NO OF MEMBERS

C N=NO OF DEGREES OF FREEDOM

C NJ=NO OF JOINTS

C NR=NO OF SUPPORT RESTRAINTS

C NRJ=NO OF RESTRAINED JOINTS

C E=ELASTIC MODULUS FOR TENSION OR COMPRESSION

C G=ELASTIC MODULDS FOR SHEAR

READ(5,2) M,NJ,NR,NRJ,E,G

2 FORMAT(4I4,2F10.1)

NN=6\*NJ

N=NN-NR

WRITE(6,101)

101 FORMAT(1H1///50X,29HANALYSIS OF SPACE RIGID FRAME///)

WRITE(6,102)

102 FORMAT(/58X,14HSTRUCTURE DATA)

WRITE(6,103) M,N,NJ,NR,NRJ,E,G

103 FORMAT(/47X,1HM,4H N,4H NJ,4H NR,5H NRJ,5X,1HE,9X,1HG/  
 144X,5I4,2F10.1////)

WRITE(6,104)

104 FORMAT(55X,21HCOORDINATES OF JOINTS)

WRITE(6,105)

105 FORMAT(/55X,5HJOINT,4X,1HX,5X,1HY,5X,1HZ)

DO 4 I=1,NJ

C J,K=JOINT INDEX

C X(J),Y(J),Z(J)=X,Y,Z COORDINATS OF JOINT J

READ(5,3) J,X(J),Y(J),Z(J)

3 FORMAT(I3,3F6.1)

4 WRITE(6,106) J,X(J),Y(J),Z(J)

106 FORMAT(55X,I3,2X,3F6.1)

WRITE(6,107)

107 FORMAT(/41X,49HMEMBER DESIGNATIONS, PROPERTIES, AND ORIENTATIONS)

WRITE(6,108)

108 FORMAT(/11X,6HMEMBER,3H JJ,3H JK,7X,2HAX,12X,3HAIX,12X,3HAIY,12X,  
 13HAIZ,7X,2HAA,5H L,6X,2HCX,6X,2HCY,6X,2HCZ,7X,1HQ)

5 DO 15 J=1,M

C I=MEMBER INDEX

C JJ(I),JK(I)=DESIGNATION FOR END J AND K OF MEMBER I

```

C   AX(I)=CROSS-SECTIONAL AREA OF MEMBER I
C   Y   AIX(I)=TORSION CONSTANT OF MEMBER I
C   Y   AIY(I),AIZ(I)=MOMENTS OF INERTIA ABOUT THE Y AND Z AXES
C   Y   AA=ORIENTATION ANGLE OF THE PRINCIPAL AXES OF THE CROSS-SECTION
C   Y   READ(5,6) I,JJ(I),JK(I),AX(I),AIX(I),AIY(I),AIZ(I),AA
6   FORMAT(3I3,4E15.8,F3.0)
      J1=JJ(I)
      K1=JK(I)
C   XCL,YCL,ZCL=X,Y,Z COMPONENTS OF LENGTH OF MEMBER
      XCL=X(K1)-X(J1)
      YCL=Y(K1)-Y(J1)
      ZCL=Z(K1)-Z(J1)
C   EL(I)=LENGTH OF MEMBER I
      EL(I)=(XCL**2+YCL**2+ZCL**2)**.5
C   CX(I),CY(I),CZ(I)=X,Y,Z DIRECTION COSINES OF MEMBER I
      CX=XCL/EL(I)
      CY=YCL/EL(I)
      CZ=ZCL/EL(I)
      Q=(CX**2+CZ**2)**.5
      WRITE(6,109) I,JJ(I),JK(I),AX(I),AIX(I),AIY(I),AIZ(I),AA,EL(I),CX,
1   CY,CZ,Q
109  FORMAT(12X,I3,2X,2I3,4E15.8,F3.0,4F8.5,F7.5)
C   R(I,K)=ROTATION MATRIX OF MEMBER I
      DO 7 K=1,9
6   R(I,K)=0.
      IF(AA-1.) 10,8,10
C   XP,YP,ZP=COORDINATS OF POINT P ON PRINCIPAL PLANE
8   READ(5,9) I,XP,YP,ZP
9   FORMAT(I3,3F6.1)
      XPS=XP-X(J1)
      YPS=YP-Y(J1)
      ZPS=ZP-Z(J1)
10  IF(Q-0.001) 11,13,13
11  R(I,2)=CY
      R(I,4)=-CY
      R(I,9)=1.0
      IF(AA-1.) 15,12,15
12  SQ=(XPS**2+ZPS**2)**.5
      COSA=-XPS*CY/SQ
      SINA=ZPS/SQ
      R(I,4)=-CY*COSA
      R(I,6)=SINA
      R(I,7)=CY*SINA
      R(I,9)=COSA
      GO TO 15
13  R(I,1)=CX
      R(I,2)=CY
      R(I,3)=CZ

```

```

R(I,4)=-CX*CY/Q
R(I,5)=Q
R(I,6)=-CY*CZ/Q
R(I,7)=-CZ/Q
R(I,9)=CX/Q
IF(AA-1.) 15,14,15
14 YPG=R(I,4)*XPS-R(I,5)*YPS+R(I,6)*ZPS
ZPG=R(I,7)*XPS+R(I,8)*YPS+R(I,9)*ZPS
SQ=(YPZ**2+ZPG**2)**.5
COSA=YPG/SQ
SINA=ZPG/SQ
R(I,4)=(-CX*CY*COSA-CZ*SINA)/Q
R(I,5)=Q*COSA
R(I,6)=(-CY*CZ*COSA+CX*SINA)/Q
R(I,7)=(CX*CY*SINA-CZ*COSA)/Q
R(I,8)=-Q*SINA
R(I,9)=(CY*CZ*SINA+CX*COSA)/Q
15 CONTINUE
DO 20 I=1,M
20 WRITE(6,113) I,R(I,1),R(I,2),R(I,3),R(I,4),R(I,5),R(I,6),
1R(I,7),R(I,8),R(I,9)
113 FORMAT(//6CX,2HR(,11,8H) MATRIX/12X,9E12.4)
WRITE(6,110)
110 FORMAT(//58X,16HJOINT RESTRAINTS)
WRITE(6,111)
111 FORMAT(/41X,50HJOINT XTRANS. YTRANS. ZTRANS. XROTA. YROTA. ZROTA.)
C RL(J)=JOINT RESTRAINT LIST OF JOINT J
DO 16 K=1,NN
16 RL(K)=0.
DO 18 J=1,NRJ
READ(5,17) K,RL(6*K-5),RL(6*K-4),RL(6*K-3),RL(6*K-2),RL(6*K-1),
1RL(6*K)
17 FORMAT(I3,6F3.0)
18 WRITE(6,112) K,RL(6*K-5),RL(6*K-4),RL(6*K-3),RL(6*K-2),RL(6*K-1),
1RL(6*K)
112 FORMAT(41X,I3,4X,F3.0,2(5X,F3.0),5X,F3.0,2(4X,F3.0))
C JKL(J)=CUMULATIVE RESTRAINT LIST OF JOINT J
JKL(1)=RL(1)
DO 99 K=2,NN
II=RL(K)
99 JKL(K)=JKL(K-1)+II
C GENERATION OF STIFFNESS MATRIX
C STMA=STIFFNESS MATRIX
CALL STMA
C INVERSION OF STIFFNESS MATRIX
KK=N
LL=KK+1
DO 201 I=1,N

```

```
DO 201 J=1,N
201 F(I,J)=S(I,J)
CALL INVERT(F, KK, LL)
DO 202 I=1,N
DO 202 J=1,N
202 S(I,J)=F(I,J)
WRITE(6,115)
115 FORMAT(1H1//////53X,26H INVERTED STIFFNESS MATRIX//)
WRITE(6,114) ((S(I,J),J=1,N),I=1,N)
114 FORMAT(15X,6E17.2)
C DISPL=DISPLACEMENTS AND SUPPORTS REACTIONS
CALL DISPL
C STRESS=MEMBER END-ACTIONS AF(I) AND AM(I)
C AF(I)=AXIAL FORCE IN I DIRECTION
C AM(I)=MOMENT ABOUT I AXIS
CALL STRESS
STOP
END
```

\*IRFTC SUP1

C GENERATION OF STIFFNESS MATRIX FOR 9 WHOLE STRUCTURE  
SUBROUTINE STMA

DIMENSION X(4),Y(4),Z(4),JJ(3),JK(3),AX(3),AIX(3),AIY(3),AIZ(3),  
1EL(3),R(3,9),RL(24),JKL(24),SM(12,12),SMR(12,12),SMD(12,12),IK(6)  
2,A(24),AML(3,12),AE(24),AC(24),AR(24),D(24),AF(12),IJ(6),S(24,6)  
COMMON X,Y,Z,JJ,JK,AX,AIX,AIY,AIZ,EL,R,RL,JKL,SM,SMR,SMD,S,A,  
1AML,AE,AC,AR,D,AF,IJ,IK,M,N,NJ,NK,NRJ,NN,E,G  
EQUIVALENCE (IJ(6),J1),(IJ(5),J2),(IJ(4),J3),(IJ(3),J4),(IJ(2),J5)  
1,(IJ(1),J6),(IK(6),K1),(IK(5),K2),(IK(4),K3),(IK(3),K4),(IK(2),K5)  
2,(IK(1),K6)

C S(I,J)=STIFFNESS MATRIX FOR WHOLE STRUCTURE

DO 59 J=1,N

DO 59 K=1,N

59 S(J,K)=0.

DO 80 I=1,M

DO 18 J=1,6

IJ(J)=6\*JJ(I)+1-J

18 IK(J)=6\*JK(I)+1-J

C SCM=STIFFNESS CONSTANT FOR MEMBER

ELL=EL(I)

SCM1A=E\*AX(I)/ELL

SCM1B=G\*AIX(I)/ELL

SCM2Y=4.\*E\*AIY(I)/ELL

SCM3Y=1.5\*SCM2Y/ELL

SCM4Y=2.\*SCM3Y/ELL

SCM2Z=4.\*E\*AIZ(I)/ELL

SCM3Z=1.5\*SCM2Z/ELL

SCM4Z=2.\*SCM3Z/ELL

IF(RL(J1)) 20,19,20

19 J1=J1-JKL(J1)

GO TO 21

20 J1=N+JKL(J1)

21 IF(RL(J2)) 23,22,23

22 J2=J2-JKL(J2)

GO TO 24

23 J2=N+JKL(J2)

24 IF(RL(J3)) 26,25,26

25 J3=J3-JKL(J3)

GO TO 27

26 J3=N+JKL(J3)

27 IF(RL(J4)) 29,28,29

28 J4=J4-JKL(J4)

GO TO 30

29 J4=N+JKL(J4)

30 IF(RL(J5)) 32,31,32

31 J5=J5-JKL(J5)

GO TO 33

```

32 J5=N+JKL(J5)
33 IF(RL(J6)) 35,34,35
34 J6=J6-JKL(J6)
   GO TO 36
35 J6=N+JKL(J6)
36 IF(RL(K1)) 38,37,38
37 K1=K1-JKL(K1)
   GO TO 39
38 K1=N+JKL(K1)
39 IF(RL(K2)) 41,40,41
40 K2=K2-JKL(K2)
   GO TO 42
41 K2=N+JKL(K2)
42 IF(RL(K3)) 44,43,44
43 K3=K3-JKL(K3)
   GO TO 45
44 K3=N+JKL(K3)
45 IF(RL(K4)) 47,46,47
46 K4=K4-JKL(K4)
   GO TO 48
47 K4=N+JKL(K4)
48 IF(RL(K5)) 50,49,50
49 K5=K5-JKL(K5)
   GO TO 51
50 K5=N+JKL(K5)
51 IF(RL(K6)) 53,52,53
52 K6=K6-JKL(K6)
   GO TO 54
53 K6=N+JKL(K6)
C  SM(I,J)=MEMBER STIFFNESS MATRIX FOR MEMBER-ORIENTED AXES
54 DO 55 J=1,12
   DO 55 K=1,12
55 SM(J,K)=0.
   SM(1,1)=SCM1A
   SM(1,7)=-SCM1A
   SM(2,2)=SCM4Z
   SM(2,6)=SCM3Z
   SM(2,8)=-SCM4Z
   SM(2,12)=SCM3Z
   SM(3,3)=SCM4Y
   SM(3,5)=-SCM3Y
   SM(3,9)=-SCM4Y
   SM(3,11)=-SCM3Y
   SM(4,4)=SCM1B
   SM(4,10)=-SCM1B
   SM(5,5)=SCM2Y
   SM(5,9)=SCM3Y
   SM(5,11)=SCM2Y/2.

```

```

SM(6,6)=SCM2Z
SM(6,8)=-SCM3Z
SM(6,12)=SCM2Z/2.
SM(7,7)=SCM1A
SM(8,8)=SCM4Z
SM(8,12)=-SCM3Z
SM(9,9)=SCM4Y
SM(9,11)=SCM3Y
SM(10,10)=SCM1B
SM(11,11)=SCM2Y
SM(12,12)=SCM2Z
DO 56 J=1,12
DO 56 K=J,12
56 SM(K,J)=SM(J,K)
WRITE(6,110) I
110 FORMAT(1H1//60X,3HSM(,I1,8H) MATRIX//)
WRITE(6,114) ((SM(K,J),J=1,12),K=1,12)
114 FORMAT(6X,12E10.3)
C SMR(I,J)=PRODUCTED MATRIX OF SM AND RT MATRICES
C RT=ROTATION TRANSFORMATION MATRIX
DO 57 K=1,4
DO 57 J=1,12
SMR(J,3*K-2)=SM(J,3*K-2)*R(I,1)+SM(J,3*K-1)*R(I,4)+
1SM(J,3*K)*R(I,7)
SMR(J,3*K-1)=SM(J,3*K-2)*R(I,2)+SM(J,3*K-1)*R(I,5)+
1SM(J,3*K)*R(I,8)
57 SMR(J,3*K)=SM(J,3*K-2)*R(I,3)+SM(J,3*K-1)*R(I,6)+SM(J,3*K)*R(I,9)
WRITE(6,111) I
111 FORMAT(//59X,4HSMR(,I1,8H) MATRIX//)
WRITE(6,114) ((SMR(K,J),J=1,12),K=1,12)
C SMD(I,J)=MEMBER STIFFNESS MATRX FOR STRUCTURE-ORIENTED AXES
DO 58 J=1,4
DO 58 K=1,12
SMD(3*J-2,K)=R(I,1)*SMR(3*J-2,K)+R(I,4)*SMR(3*J-1,K)+
1R(I,7)*SMR(3*J,K)
SMD(3*J-1,K)=R(I,2)*SMR(3*J-2,K)+R(I,5)*SMR(3*J-1,K)+
1R(I,8)*SMR(3*J,K)
58 SMD(3*J,K)=R(I,3)*SMR(3*J-2,K)+R(I,6)*SMR(3*J-1,K)+
1R(I,9)*SMR(3*J,K)
WRITE(6,112) I
112 FORMAT(//59X,4HSMD(,I1,8H) MATRIX//)
WRITE(6,114) ((SMD(K,J),J=1,12),K=1,12)
JJ1=6*JJ(I)-5
IF(RL(JJ1)) 61,60,61
60 DO 17 J=1,6
MM=7-J
K=IJ(MM)
S(K,J1)=S(K,J1)+SMD(J,1)

```

```
K=IK(MM)
17 S(K,J1)=SVD(J+6,1)
61 JJ2=6*JJ(I)-4
  IF(PL(JJ2)) 63,62,63
62 DO 16 J=1,6
  MM=7-J
  K=IJ(MM)
  S(K,J2)=S(K,J2)+SMD(J,2)
  K=IK(MM)
16 S(K,J2)=SMD(J+6,2)
63 JJ3=6*JJ(I)-3
  IF(RL(JJ3)) 65,64,65
64 DO 15 J=1,6
  MM=7-J
  K=IJ(MM)
  S(K,J3)=S(K,J3)+SMD(J,3)
  K=IK(MM)
15 S(K,J3)=SMD(J+6,3)
65 JJ4=6*JJ(I)-2
  IF(RL(JJ4)) 67,66,67
66 DO 14 J=1,6
  MM=7-J
  K=IJ(MM)
  S(K,J4)=S(K,J4)+SMD(J,4)
  K=IK(MM)
14 S(K,J4)=SMD(J+6,4)
67 JJ5=6*JJ(I)-1
  IF(RL(JJ5)) 69,68,69
68 DO 13 J=1,6
  MM=7-J
  K=IJ(MM)
  S(K,J5)=S(K,J5)+SMD(J,5)
  K=IK(MM)
13 S(K,J5)=SMD(J+6,5)
69 JJ6=6*JJ(I)
  IF(RL(JJ6)) 71,70,71
70 DO 12 J=1,6
  MM=7-J
  K=IJ(MM)
  S(K,J6)=S(K,J6)+SMD(J,6)
  K=IK(MM)
12 S(K,J6)=SMD(J+6,6)
71 KK1=6*JK(I)-5
  IF(RL(KK1)) 73,72,73
72 DO 11 J=1,6
  MM=7-J
  K=IJ(MM)
  S(K,K1)=SMD(J,7)
```

```

      K=IK(MM)
11  S(K,K1)=S(K,K1)+SMD(J+6,7)
73  KK2=6*JK(I)-4
      IF(RL(KK2)) 75,74,75
74  DO 10 J=1,6
      MM=7-J
      K=IJ(MM)
      S(K,K2)=SMD(J,8)
      K=IK(MM)
10  S(K,K2)=S(K,K2)+SMD(J+6,8)
75  KK3=6*JK(I)-3
      IF(RL(KK3)) 77,76,77
76  DO 9 J=1,6
      MM=7-J
      K=IJ(MM)
      S(K,K3)=SMD(J,9)
      K=IK(MM)
77  KK4=6*JK(I)-2
      IF(RL(KK4)) 79,78,79
  9  S(K,K3)=S(K,K3)+SMD(J+6,9)
78  DO 8 J=1,6
      MM=7-J
      K=IJ(MM)
      S(K,K4)=SMD(J,10)
      K=IK(MM)
  8  S(K,K4)=S(K,K4)+SMD(J+6,10)
79  KK5=6*JK(I)-1
      IF(RL(KK5)) 82,81,82
81  DO 7 J=1,6
      MM=7-J
      K=IJ(MM)
      S(K,K5)=SMD(J,11)
      K=IK(MM)
  7  S(K,K5)=S(K,K5)+SMD(J+6,11)
82  KK6=6*JK(I)
      IF(RL(KK6)) 80,83,80
83  DO 6 J=1,6
      MM=7-J
      K=IJ(MM)
      S(K,K6)=SMD(J,12)
      K=IK(MM)
  6  S(K,K6)=S(K,K6)+SMD(J+6,12)
80  CONTINUE
      WRITE(6,113)
113  FORMAT(1H1/////52X,26HSTRUCTURE STIFFNESS MATRIX//)
      WRITE(6,115) ((S(I,J),J=1,N),I=1,N)
115  FORMAT(15X,6E17.8)
      RETURN
      END

```

\$IBFTC SUB2

```
C      MATRIX INVERSION
      SUBROUTINE INVERT(F,N,NN)
      DIMENSION E(N,NN)
      NN=N+1
  6   DO 13 K=1,N
      DO 100 J=1,N
100  E(J,N+1)=0.
      E(K,N+1)=1.
      DO 105 J=1,N
        IF (J-K) 8,105,8
  8   IF(E(K,K)) 75,14,75
 14  WRITE(6,101)
101  FORMAT(10X,47HZERO GENERATED ON DIAGONAL, INVERSE SET TO ZERO)
      DO 3 I=1,N
        DO 3 JJ=1,NN
  3   E(I,JJ)=0.0
      GO TO 5
 75  A=E(K,K)
      C=E(J,K)
      DO 9 I=1,NN
  9   E(J,I)=E(J,I)-E(K,I)*C/A
105  CONTINUE
      DO 10 J=1,NN
 10  E(K,J)=E(K,J)/A
      DO 11 J=1,N
 11  E(J,K)=E(J,N+1)
 13  CONTINUE
  5   RETURN
      END
```

```

SIBFTC SUB3
C   CALCULATION OF DISPLACEMENTS AND REACTIONS
SUBROUTINE DISPL
DIMENSION X(4),Y(4),Z(4),JJ(3),JK(3),AX(3),AIX(3),AIY(3),AIZ(3),
1EL(3),R(3,9),RL(24),JKL(24),SM(12,12),SMR(12,12),SMD(12,12),IK(6)
2,A(24),AML(3,12),AE(24),AC(24),AR(24),D(24),AF(12),IJ(6),S(24,6)
COMMON X,Y,Z,JJ,JK,AX,AIX,AIY,AIZ,EL,R,RL,JKL,SM,SMR,SMD,S,A,
1AML,AE,AC,AR,D,AF,IJ,IK,M,N,NJ,NR,NRJ,NN,E,G
C   INPUT AND PRINT LOAD DATA
WRITE(6,116)
116 FORMAT(1H1/////61X,9HLOAD DATA)
C   NLJ=NO OF LOADED JOINTS
C   NLM=NO OF LOADED MEMBERS
WRITE(6,117)
117 FORMAT(/62X,8HN LJ NLM)
READ(5,84) NLJ,NLM
84 FORMAT(2I3)
WRITE(6,118) NLJ,NLM
118 FORMAT(61X,I3,2X,I3)
IF(NLJ) 85,88,85
85 WRITE(6,119)
119 FORMAT(/53X,25HACTIONS APPLIED AT JOINTS)
WRITE(6,120)
120 FORMAT(/36X,50HJOINT XFORCE YFORCE ZFORCE XMOMENT YMOMENT ZMOMENT)
C   A(I)=ACTIONS APPLIED AT JOINTS
DO 87 J=1,NLJ
READ(5,86) K,A(6*K-5),A(6*K-4),A(6*K-3),A(6*K-2),A(6*K-1),A(6*K)
86 FORMAT(I3,6F3.0)
87 WRITE(6,121) K,A(6*K-5),A(6*K-4),A(6*K-3),A(6*K-2),A(6*K-1),A(6*K)
121 FORMAT(36X,I3,3(4X,F3.0),3(5X,F3.0))
88 IF(NLM) 89,92,89
89 WRITE(6,122)
122 FORMAT(/36X,50HACTIONS AT ENDS OF RESTRAINED MEMBERS DUE TO LOADS)
C   AML(J,K)=ACTIONS AT ENDS J AND K OF RESTRAINED MEMBER DUE TO LOADS
WRITE(6,123)
123 FORMAT(/48X,36HMEMBER AML1 AML2 AML3 AML4 AML5 AML6)
WRITE(6,124)
124 FORMAT(55X,32HAML7 AML8 AML9 AML10 AML11 AML12)
DO 91 J=1,NLM
READ(5,90) I,AML(I,1),AML(I,2),AML(I,3),AML(I,4),AML(I,5),AML(I,6)
1,AML(I,7),AML(I,8),AML(I,9),AML(I,10),AML(I,11),AML(I,12)
90 FORMAT(I3,12F4.1)
WRITE(6,125) I,AML(I,1),AML(I,2),AML(I,3),AML(I,4),
1AML(I,5),AML(I,6)
125 FORMAT(48X,I3,4X,F4.1,5(1X,F4.1))
91 WRITE(6,126) AML(I,7),AML(I,8),AML(I,9),AML(I,10),AML(I,11),
1AML(I,12)
126 FORMAT(54X,4(1X,F4.1),2(2X,F4.1))

```

```

92 IF(NLM) 93,96,93
C   CONSTRUCTION OF VECTORS ASSOCIATED WITH LOADS
C   AE(J)=EQUIVALENT JOINT LOADS
93 DO 94 I=1,NN
94 AE(I)=0.
   DO 95 I=1,M
     J1=6*JJ(I)-5
     J2=6*JJ(I)-4
     J3=6*JJ(I)-3
     J4=6*JJ(I)-2
     J5=6*JJ(I)-1
     J6=6*JJ(I)
     K1=6*JK(I)-5
     K2=6*JK(I)-4
     K3=6*JK(I)-3
     K4=6*JK(I)-2
     K5=6*JK(I)-1
     K6=6*JK(I)
     AE(J1)=AE(J1)-R(I,1)*AML(I,1)-R(I,4)*AML(I,2)-R(I,7)*AML(I,3)
     AE(J2)=AE(J2)-R(I,2)*AML(I,1)-R(I,5)*AML(I,2)-R(I,8)*AML(I,3)
     AE(J3)=AE(J3)-R(I,3)*AML(I,1)-R(I,6)*AML(I,2)-R(I,9)*AML(I,3)
     AE(J4)=AE(J4)-R(I,1)*AML(I,4)-R(I,4)*AML(I,5)-R(I,7)*AML(I,6)
     AE(J5)=AE(J5)-R(I,2)*AML(I,4)-R(I,5)*AML(I,5)-R(I,8)*AML(I,6)
     AE(J6)=AE(J6)-R(I,3)*AML(I,4)-R(I,6)*AML(I,5)-R(I,9)*AML(I,6)
     AE(K1)=AE(K1)-R(I,1)*AML(I,7)-R(I,4)*AML(I,8)-R(I,7)*AML(I,9)
     AE(K2)=AE(K2)-R(I,2)*AML(I,7)-R(I,5)*AML(I,8)-R(I,8)*AML(I,9)
     AE(K3)=AE(K3)-R(I,3)*AML(I,7)-R(I,6)*AML(I,8)-R(I,9)*AML(I,9)
     AE(K4)=AE(K4)-R(I,1)*AML(I,10)-R(I,4)*AML(I,11)-R(I,7)*AML(I,12)
     AE(K5)=AE(K5)-R(I,2)*AML(I,10)-R(I,5)*AML(I,11)-R(I,8)*AML(I,12)
     AE(K6)=AE(K6)-R(I,3)*AML(I,10)-R(I,6)*AML(I,11)-R(I,9)*AML(I,12)
95 CONTINUE
C   AC(J)=COMBINED JOINT LOADS
96 DO 100 J=1,NN
   IF(RL(J)) 98,97,98
97 K=J-JKL(J)
   GO TO 100
98 K=N+JKL(J)
100 AC(K)=A(J)+AE(J)
C   CALCULATION AND OUTPUT OF RESDLTS
C   D(J)=JOINT DISPLACEMENTS
   DO 200 J=1,N
     D(J)=0.
     DO 200 K=1,N
700 D(J)=D(J)+S(J,K)*AC(K)
C   AR(J)=SUPPORT REACTIONS
   N1=N+1
   DO 201 K=N1,NN
     AR(K)=-AC(K)

```

```
DO 201 J=1,N
201 AR(K)=AR(K)+S(K,J)*D(J)
   J=N+1
   DO 204 K=1,NN
   JF=NN+1-K
   IF(RL(JE)) 203,202,203
202 J=J-1
   D(JE)=D(J)
   GO TO 204
203 D(JE)=0.
204 CONTINUE
   K=N
   DO 207 KE=1,NN
   IF(RL(KE)-1.) 206,205,206
205 K=K+1
   AR(KE)=AR(K)
   GO TO 207
206 AR(KE)=0.
207 CONTINUE
   WRITE(6,127)
127 FORMAT(/745X,41HJOINT DISPLACEMENTS AND SUPPORT REACTIONS)
   WRITE(6,128)
128 FORMAT(/15X,5HJOINT,5X,7HXTRANS.,9X,7HYTRANS.,9X,7HZTRANS.,
19X,7HX ROTA.,9X,7HY ROTA.,9X,7HZ ROTA.)
   WRITE(6,129)
129 FORMAT(25X,7HX FORCE,9X,7HY FORCE,9X,7HZ FORCE,9X,7HXMOMENT,
19X,7HYMOMENT,9X,7HZMOMENT)
   DO 208 JE=6,NN,6
   I=JE/6
   WRITE(6,130) I,D(JE-5),D(JE-4),D(JE-3),D(JE-2),D(JE-1),D(JE)
130 FORMAT(17X,I1,2X,6(1X,E15.8))
208 WRITE(6,131) AR(JE-5),AR(JE-4),AR(JE-3),AR(JE-2),AR(JE-1),AR(JE)
131 FORMAT(20X,6(1X,E15.8))
   RETURN
   END
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SIBFTC SUB4

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C   CALCULATE MEMBER ENDS FORCES AND MOMENTS IN MEMBER-ORIENTED AXES
    SUBROUTINE STRESS
      DIMENSION X(4),Y(4),Z(4),JJ(3),JK(3),AX(3),AIX(3),AIY(3),AIZ(3),
1     EL(3),R(3,9),RL(24),JKL(24),SM(12,12),SMR(12,12),SMD(12,12),IK(6)
2     ,A(24),AML(3,12),AE(24),AC(24),AR(24),D(24),AF(12),IJ(6),S(24,6)
      COMMON X,Y,Z,JJ,JK,AX,AIX,AIY,AIZ,EL,R,RL,JKL,SM,SMR,SMD,S,A,
1     AML,AE,AC,AR,D,AF,IJ,IK,M,N,NJ,NR,NRJ,NN,E,G
      WRITE(6,51)
51  FORMAT(1H1////57X,18HMEMBER END-ACTIONS)
      WRITE(6,52)
52  FORMAT(/12X,6HMEMBER,7X,4HAF 1,13X,4HAF 2,13X,4HAF 3,13X,4HAM 4,
1     13X,4HAM 5,13X,4HAM 6)
      WRITE(6,53)
53  FORMAT(25X,4HAF 7,13X,4HAF 8,13X,4HAF 9,13X,4HAM10,
1     13X,4HAM11,13X,4HAM12)
      DO 50 I=1,M
        J1=6*JJ(I)-5
        J2=6*JJ(I)-4
        J3=6*JJ(I)-3
        J4=6*JJ(I)-2
        J5=6*JJ(I)-1
        J6=6*JJ(I)
        K1=6*JK(I)-5
        K2=6*JK(I)-4
        K3=6*JK(I)-3
        K4=6*JK(I)-2
        K5=6*JK(I)-1
        K6=6*JK(I)
        ELL=EL(I)
        SCM1A=E*AX(I)/ELL
        SCM1B=G*AIX(I)/ELL
        SCM2Y=4.*E*AIY(I)/ELL
        SCM3Y=1.5*SCM2Y/ELL
        SCM4Y=2.*SCM3Y/ELL
        SCM2Z=4.*E*AIZ(I)/ELL
        SCM3Z=1.5*SCM2Z/ELL
        SCM4Z=2.*SCM3Z/ELL
C   SM(I,J)=MEMBER STIFFNESS MATRIX FOR MEMBER-ORIENTED AXES
      DO 55 J=1,12
      DO 55 K=1,12
55  SM(J,K)=0.
      SM(1,1)=SCM1A
      SM(1,7)=-SCM1A
      SM(2,2)=SCM4Z
      SM(2,6)=SCM3Z
      SM(2,8)=-SCM4Z
      SM(2,12)=SCM3Z

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SM(3,2)=SCM4Y
SM(3,5)=-SCM3Y
SM(3,9)=-SCM4Y
SM(3,11)=-SCM3Y
SM(4,4)=SCM1B
SM(4,10)=-SCM1B
SM(5,5)=SCM2Y
SM(5,9)=SCM3Y
SM(5,11)=SCM2Y/2.
SM(6,6)=SCM2Z
SM(6,8)=-SCM3Z
SM(6,12)=SCM2Z/2.
SM(7,7)=SCM1A
SM(8,8)=SCM4Z
SM(8,12)=-SCM3Z
SM(9,9)=SCM4Y
SM(9,11)=SCM3Y
SM(10,10)=SCM1B
SM(11,11)=SCM2Y
SM(12,12)=SCM2Z
DO 56 J=1,12
DO 56 K=1,12
56 SM(K,J)=SM(J,K)
SMR(I,J)=PRODUCTED MATRIX OF SM AND RT MATRICES
RT=ROTATION TRANSFORMATION MATRIX
DO 57 K=1,4
DO 57 J=1,12
SMR(J,3*K-2)=SM(J,3*K-2)*R(I,1)+SM(J,3*K-1)*R(I,4)+
1SM(J,3*K)*R(I,7)
SMR(J,3*K-1)=SM(J,3*K-2)*R(I,2)+SM(J,3*K-1)*R(I,5)+
1SM(J,3*K)*R(I,8)
57 SMR(J,3*K)=SM(J,3*K-2)*R(I,3)+SM(J,3*K-1)*R(I,6)+SM(J,3*K)*R(I,9)
DO 30 J=1,12
AF(J)=AML(I,J)
DO 30 K=1,12
30 AF(J)=AF(J)+SMR(J,K)*D(K)
50 WRITE(6,54) I,AF
54 FORMAT(12X,I3,3X,6E17.8/18X,6E17.8)
RETURN
END

```

# SPACE FRAME ANALYSIS

by

Fu Tien Lin

## ABSTRACT

The "displacement method" or "stiffness method" for the structural analysis of space frames is developed in this thesis and used for the analysis of an experimental test frame. A direct and an indirect approach are introduced to obtain stiffness coefficients, and then, using the principles of superposition and matrix algebra, the structural analysis is completed.

A digital computer program for rigidly-connected space frames is presented. Experimental results of deflections of a simple three member space frame loaded at one joint are compared with the analytical results. The differences are large, so an evaluation of the test set-up is made to suggest corrections to be incorporated in the test procedure before considering modifications of the analysis.