A Mixed Methods Study of Chinese Students' Construction of Fraction Schemes: Extending the Written Test with Follow-Up Clinical Interviews

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ABSTRACT

Understanding fractions is fundamental for expanding number knowledge from the whole number system to the rational number system. According to the National Council of Teachers of Mathematics (NCTM) *Principles and Standards for School Mathematics* (2000), learning fractions is an important mathematical goal for students in grades three through five in the U.S. Moreover, the NCTM suggests that fraction instruction start in Pre-K and continue through 8^{th} grade. At the same time, the *Common Core State Standards for Mathematics* (CCSSM) suggests that fraction instruction should occur from Grade 3 to 7. In contrast to the time spent on learning fractions in the U.S., students in China spend a relatively short time learning fractions (Zhang & Siegler, 2022). According to the Chinese national curriculum standards, the *Chinese National Mathematics Curriculum Standards* (CNMCS) for five-four system, the fundamental fraction concepts are taught in grades 3 and 5 only. However, Chinese students continue to have higher performance on fraction items in international assessments when compared with American students (Fan & Zhu, 2004). Consequently, over the last several years, researchers have investigated subject content knowledge and pedagogical content knowledge of Chinese in-service teachers and pre-service teachers via fraction division (e.g., Li & Huang, 2008; Ma, 1999). There are also studies exploring Chinese written curricula of fraction division (e. g., Li, Zhang, & Ma, 2009). Recently, a quantitative study from Norton, Wilkins, and Xu (2018) investigated the process of Chinese students' construction of fraction knowledge through the lens of fraction schemes, a model established by western scholars Steffe (2002) and his colleague Olive (Steffe & Olive, 2010). However, there is a lack of qualitative research that attempts to use fraction schemes as an explanatory framework to interpret the process of Chinese students' construction of fraction knowledge. The main purpose of this study was to investigate Chinese students' understanding of the fundamental fraction knowledge in terms of their understanding of the "fraction unit," referred to as a "unit fraction" in the U.S., using Steffe and Olive's (2010) fraction schemes as the conceptual framework.

A sequential mixed methods design was used in this study. The design included two consecutive phases, namely a quantitative phase followed by a qualitative phase (Creswell & Plano Clark, 2011). During the quantitative phase, five hundred and thirty-four Chinese fourth and fifth grade students were administered an assessment. The quantitative data was first analyzed using a Cochran's Q test to determine if the Chinese participants in this study follow the same progression of fraction schemes as their American peers. Results indicate that the development of fractional schemes among Chinese 4th and 5th grade participants in this study is similar to their U.S. counterparts and the Chinese participants in Norton et al.'s (2018) study regardless of the curricula differences across countries or areas in the same country, the textbook differences, and the language differences. Next, two different analysis of variances (ANOVA), a three-way mixed ANOVA and a two-way repeated measures ANOVA were conducted. The three-way mixed ANOVA was used to inform the researcher as to the fraction schemes these students had constructed before the concept of fraction unit is formally introduced and after the concept of fraction unit is formally introduced. The results showed that the fraction knowledge

of the students' in this study developed from 4^{th} grade to 5^{th} grade. The analysis of clinical interview data confirmed this conclusion.

The two-way repeated measures ANOVA was used to determine which model (i.e., linear, circular, or rectangular) is more or less problematic for Chinese students when solving fraction tasks. The results suggest that generally students' performance on linear model tasks was better than their performance on circular model tasks, but there was no statistically significant difference between performance on circular model and its corresponding rectangular model tasks. The results from the quantitative analyses were also used to screen students to form groups based on their highest available fraction scheme for a clinical interview in the second phase, the qualitative phase.

In the qualitative phase, a clinical interview using a think-aloud method was used to gain insight into the role of students' conceptual understanding of the fraction unit in their construction of fraction knowledge. In this phase, students were asked to solve the tasks in the clinical interview protocol using the think aloud method. Two main findings were revealed analyzing the clinical interview data. First, a conceptual understanding of fraction units as well as a conceptual understanding of a unit whole play a critical role in the construction of Chinese students' fraction knowledge. Second, the lack of the understanding of a fraction unit as an iterable unit may be one of the reasons that obstructs students move from part-whole concept of fractions to the measurement concept of fractions.

This study also demonstrates that a conceptual understanding of fraction units and the unit whole are a necessary condition for constructing of a conceptual understanding of fraction knowledge. Thus, implications of this study suggest that teachers not only should help students build a conceptual understanding of fraction units, but also need to confirm that students have constructed the concept of what the unit whole is before asking students to identify the fraction units for the referent whole. On the other hand, the tasks used in the present study only include continuous but not discrete wholes. Therefore, future research may focus on investigating how students identify fraction units and in what way the iterating operation could be used when students encounter a discrete whole.

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GENERAL AUDIENCE ABSTRACT

Understanding fractions is fundamental for expanding number knowledge from the whole number system to the rational number system. According to the National Council of Teachers of Mathematics (NCTM) *Principles and Standards for School Mathematics* (2000), learning fractions is an important mathematical goal for students in grades three through five in the U.S. At the same time, the *Common Core State Standards for Mathematics* (CCSSM) suggests that fraction instruction should occur from grades 3 to 7. In contrast to the time spent on learning fractions in the U.S., students in China spend a relatively short time learning fractions. According to the Chinese national curriculum standards, the *Chinese National Mathematics Curriculum Standards* (CNMCS) for five-four system, the fundamental fraction concepts are taught in grades 3 and 5 only. However, Chinese students continue to have higher performance on fraction items in international assessments when compared with American students. Consequently, over the last several years, researchers have investigated fraction knowledge of Chinese in-service teachers and pre-service teachers via fraction division. There are also studies exploring Chinese written curricula of fraction division. Recently, Norton, Wilkins, and Xu (2018) collected and analyzed numerical data from Chinese students and investigated the process of how Chinese students learn fraction knowledge through a model established by western scholars Steffe (2002) and his colleague Olive. However, there is a lack of research study that attempts to seek an in-depth understanding of how Chinese students learn their fraction knowledge.

This study used both numerical data and data gathered from interviewing 29 4th and 5th grade Chinese students. It aimed to investigate Chinese students' understanding of the fundamental fraction knowledge in terms of their understanding of the "fraction unit," referred to as a "unit fraction" in the U.S., using Steffe and Olive's (2010) fraction schemes as the conceptual framework.

This study demonstrates that a comprehensive and practical understanding of fraction units and the whole of a given fraction are a necessary condition for building a comprehensive understanding of fraction knowledge. The implications of this study suggest that teachers not only should help students build a comprehensive understanding of fraction units, but also need to confirm that students have built the concept of what the whole of a given fraction is before asking students to identify the fraction units for the referent whole.

Dedication

This dissertation is dedicated to my parents, Benzhi Xu and Zhijun Zou. I cannot succeed without your love, motivation, and unwavering support over the years. I also dedicate this work to my sweet daughter, Melissa Hu. Thank you for your patience as I pursued and completed this degree.

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Table of Contents

List of Tables

List of Figure

List of Abbreviation

TIMSS	Trends in International Mathematics and Science Study
CCSSM	Common Core State Standards for Mathematics instruction
NCTM	National Council of Teachers of Mathematics
CNMCS	Chinese National Mathematics Curriculum Standards
PWS	Part-Whole Fraction Scheme
PUFS	Partitive Unit Fraction Scheme
PFS	Partitive Fraction Scheme
SO	Splitting Operation
RPFS	Reversible Partitive Fraction Scheme
IFS	Iterative Fraction Scheme
PEP	People’s Education Press
ANOVA	Analysis of Variance
PPCF	Part-Part Concept of Fractions

Chapter One: Introduction

Over the past three decades, mathematics education researchers have investigated and modeled children's construction of fraction knowledge from a psychological perspective. Particularly, Steffe and his colleagues' research studies (e.g., Steffe, von Glasersfeld, Richards & Cobb, 1983; Steffe & Cobb, 1988) indicate that the understanding of units not only influences the construction of children's whole number knowledge (Hackenberg, Norton & Wright, 2016; Wilkins & Norton, 2018; Ulrich & Wilkins, 2017), but also plays an important role in the construction of children's fraction knowledge (e.g., Wilkins, Woodward, & Norton, 2021; Norton & Wilkins, 2012, 2013; Wilkins & Norton, 2018, 2011; Watanabe, 1995). However, the target population for these research studies were students from western countries and especially from the U.S. Therefore, the current study aims to investigate the way Chinese students construct fraction knowledge through the lens of Steffe's fractional schemes.

Unlike whole numbers, which results in representations for counting items, fractions represent the relationship between two quantities while at the same time representing *a quantity*. In other words, students need to deal with two quantities simultaneously. The characteristic of this complexity of notion of fractions makes fractions more abstract and difficult for young children to understand when compared to whole numbers (Wynn, 1995). Moreover, even though fractions are an extension from whole numbers, many properties of whole numbers cannot be directly applied to fractions (Steffe, 2002, Kieren, 1993). For example, when comparing two positive numbers, such as 23 and 25, we use the place values of the numbers by starting at the left-most place where the value is the largest and then move to the right where the value is smaller. It can be considered as a one-dimension horizontal comparison action. However, fraction comparison is more complicated. When comparing two fractions, such as $\frac{3}{5}$ and $\frac{4}{7}$, for instance, we cannot simply compare the numerators 3 and 4, nor the denominators 5 and 7 separately. The comparison requires both the numerators and the denominators at the same time. This comparison can be considered as a set of two-dimensional, namely vertical and horizontal, actions. This is also true for adding and subtracting fractions, especially adding and subtracting fractions with unlike denominators. That is because adding and subtracting fractions with unlike denominators requires changing the different denominators to a common denominator. In other words, the fractions with unlike denominators have different unit fractions; they should be unitized to the same unit fraction before performing the addition and subtraction operation.

It is a challenge for students in the U.S. to understand why it is necessary to find a common denominator when adding and subtracting fractions with unlike denominators (e.g., Siegler & Lortie-Forgues, 2015). For example, when adding or subtracting fractions with unlike denominators, 44% of sixth grade students in Siegler, Thompson, and Schneider's (2011) study performed both operations on the numerator and/or the denominator separately. In contrast, results from the Trends in International Mathematics and Science Study (TIMSS) (2011), show fewer Asian students have this misconception. For instance, one of the fraction items on the eighth grade TIMSS (2011) test is "Which shows a correct method for finding $\frac{1}{3}-\frac{1}{4}$?" American eighth grade students answered this question with only a 29% accuracy rate whereas the accuracy rate among students from Hong Kong, Singapore, and Chinese Taipei was 83%, 82%, and 77%, respectively. Among those American students who circled the wrong answers, more than half of them, 58.6%, circled either $\frac{1-1}{3-4}$ or $\frac{1}{4-3}$. Some researchers link this kind of misconception to the interference the students make from their whole number knowledge (e.g., Ni & Zhou, 2005; Siegler & Lortie-Forgues, 2017; Van Hoof, Lijnen, Verschaffel, & Dooren, 2013), or they misapply the whole number knowledge (e.g., Newton, 2008; Siegler, Thompson,

& Schneider, 2011). Accordingly, over the past several years, some research studies have focused on developing curriculum and textbooks with an emphasis on conceptual understanding with procedural fluency by comparing and contrasting the curriculum and textbooks from other countries, especially those from Asian countries (e.g., Anthony & Ding, 2011; Li, Chen, & An, 2009; Son, 2011; Son & Senk, 2010; Sun, 2011; Yang, Reys & Wu, 2010; Zhang & Sigeler, 2022). Other researchers have focused on improving teachers' content and/or pedagogical knowledge (e.g., Ball, 1990; Lortie-Forgues, Tian, & Siegler, 2015; Olive & Vomvoridi, 2006; Tzur, 1999) by comparing and contrasting fraction teaching of different countries, particularly within Asian countries (e.g., An, Kulm, & Wu, 2004; Li, Chen, & Kulm, 2009; Ma, 1999; Watanabe, 1995, 2006). There are also other empirical studies aimed at developing different teaching methods (e.g., Mack, 1995, 2001; Torbeyns, Schneider, Xin, & Siegler, 2015) and instructional interventions (e.g., Fuchs et al., 2013) to help students overcome their misconceptions. However, results from the Lortie-Forgues, Tian, and Siegler (2015) study reveal that there is no significant improvement in the understanding of fraction operations over three decades. In their study, Lortie-Forgues, et al. (2015) repeated the same assessment that had been given to more than 20,000 U.S. 8^{th} grade students in 1978 by the National Assessment of Educational Progress. The test problem they used was the following: "Choose the closest whole number to the sum of 12/13 + 7/8" (p. 201). The possible answers for this question were 1, 2, 19, 21, and "I don't know" (p. 201). The same problem was given to 48 eight-grade students who were taking an Algebra 1 course in 2014. The results were surprisingly consistent. In 1978, among more than 20,000 eighth graders, 24% of those students chose the correct answer, versus 27% of students in 2014 (Lortie-Forgues et al., 2015). The same procedural error continues to persist. This suggests that most students in the U.S. remain deficient in understanding fraction arithmetic. It makes one wonder why fewer Asian students make operational mistakes than American students. The operational error made by U.S. students may indicate that they may lack a basic conceptual understanding about fractions.

Among all the fundamental concepts of fractions, Chinese researchers (e.g., Hua, 2011; Jiang, 2009; Lin, 2015; Yang, Xian, Hang, & Chen, 2013; Zhu, 2005) as well as researchers in the U.S. (e.g., Hackenberg & Tillema, 2009; Olive & Vomvoridi, 2006; Norton & Wilkins, 2010; Tzur & Hunt, 2015; Wilkins & Norton, 2018) consider that a conceptual understanding of fraction units, which are called unit fractions in the U.S., is crucial for learning more advanced fraction concepts. In China, the fraction unit is defined as follows: "The unit whole '1' is evenly divided into several shares. The number representing one of these shares is called the *fraction unit*" (PEP 5th grade Mathematics, p.46). This definition will be used in this study. Chinese researchers believe that the concept of fraction unit is an important part of a child's initial fraction concept (Hua, 2011). A lack of understanding of fraction units could be one of the main reasons that more sophisticated fraction concepts, such as improper fractions, equivalent fractions, and advanced fraction knowledge, such as fraction comparison and fraction arithmetic remain out of reach for some students (Zhang, 2011). This implicitly suggests that most Asian students understand that fractions have different denominators, meaning that they have different fraction units. To solve problems involving fractions, those fractions with different fraction units must be unitized. That is, they must have the same fraction unit (Zhang, 2011). Consequently, it is reasonable to infer that a conceptual understanding of fraction units may facilitate Chinese students learning and understanding of other fundamental fraction concepts and advanced fraction concepts. Therefore, the overarching purpose of this study is to investigate Chinese students' understanding of fraction units and its role in facilitating Chinese students' construction of their fraction knowledge. This in turn will support Chinese mathematics teachers to explicitly understand how their students construct their own fraction knowledge.

Unit Fractions Versus Fraction Units

In the U.S. a fraction is called a *unit fraction* when the numerator is one, such as $\frac{1}{a}$ (where a is an integer and $a \neq 0$) (Baroody & Cosilick, 1998; Lamon, 2012). However, the U.S. phrase "unit fraction" is in reverse order when compared to the Chinese phrase "fraction unit." The order of the words in these phrases reflect the different parts of the fraction name. In the phrase "unit fraction," the subject is "fraction" and "unit" is an adjective that modifies the subject "fraction." In other words, unit is the attribute describing the subject "fraction." Therefore, students may understand the "unit fraction" as a fraction, which may be a complex number for them. However, in China, the phrase is "fraction unit." The subject is "unit," and "fraction" is an adjective that modifies the subject "unit." With the phrase adjusted in this way, students may be able to reorganize the meaning of unit from their whole number knowledge for their fraction knowledge. Steffe (2002) disagrees that children's whole number knowledge interferes with children's ability to learn fractions. Instead, he suggests that learning fractions requires children to reorganize their whole number knowledge to understand fraction knowledge. It can be argued that the Chinese phrase, fraction unit, may play an important role in Chinese students' construction of more sophisticated fraction concepts.

The Time Teachers Devote to Fraction Teaching in Chinese Elementary Schools

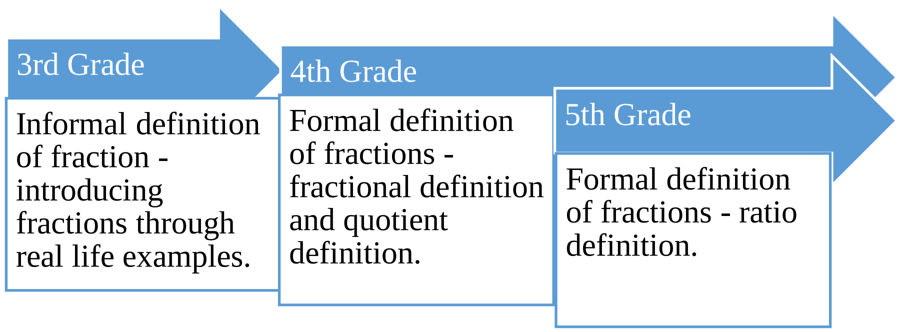
A review of different fraction teaching and learning standards in U.S., such as the Common Core State Standards for Mathematics instruction (CCSSM), the National Council of Teachers of Mathematics (NCTM), and the Virginia's Mathematics Standards of Learning, indicates that fraction learning in U.S. lasts longer than fraction learning in China. For example, according to the CCSSM, which was published in 2010 and widely adopted by most of the states in the U.S. (Porter, McMaken, Hwang, & Yang, 2011), fraction content is first introduced in grade 3 and continues throughout the middle school years (Watanabe, 2007). The National Council of Teachers of Mathematics (NCTM) (2000), which provide guidelines and frameworks for mathematics teaching and learning in the U.S., also suggests that fraction instruction should start in Pre-K and continue through 8^{th} grade. Particularly, the expectation of understanding fractions for Pre-K to 2^{nd} grade is to "understand and represent commonly used fractions in context" (NCTM, p. 32). The Commonwealth of Virginia follows the NCTM prescribed guidelines. Accordingly, the Virginia Standards of Learning (SOL) (2009) starts to introduce the concept of fractions beginning in kindergarten.

In contrast to the time spent on learning fractions in the U.S., students in China spend a relatively short time learning fractions. Through a series of curriculum reforms from the late 1990s to 2001 in China, the *National Teaching and Learning Syllabus* was replaced by the Chinese National Mathematics Curriculum Standards (CNMCS) for Grades 1 to 9 in 2001 (Zhang, 2005). Therefore, even though Chinese elementary schools do not currently use unified mathematics textbooks, the CNMCS provides the guidelines for teaching and learning. This means that all textbooks structure the mathematics content to align with the CNMCS. In addition, there are two different elementary school education systems in China: (1) 5-4 system (five years of elementary school and four years of junior secondary school) or (2) 6-3 system (six years of elementary school and 3 years of junior secondary school). Hence, textbooks for these different school systems present knowledge structure and content in slightly different ways. Because the school of the participants of the present study implements the 5-4 system, the CNMCS and Chinese textbooks described in this study are designed for the 5-4 system.

According to the CNMCS, Chinese students of both systems begin learning fractions in the grade 3. Commonly there are four ways to define fractions in China that progress from least to most mathematically rigorous (Zhang, 2008). The CNMCS requires introducing three out of the four different, but mutually connected fraction definitions in elementary school mathematics teaching. These three definitions are fractional, quotient, and ratio definition (p.78). These definitions and contexts are addressed sequentially and build on each other from grade 3 to grade 5 in the CNMC as shown in Figure 1-1.

Figure 1. 1

Sequence of teaching fraction definitions in China from 3rd grade through 5th grade.



Based on this sequence, the CNMCS (2001) arranges teaching fractions after learning the division operation with whole numbers. At the same time, the CNMC separates fraction teaching into two phases (Zhu, 2008). The first teaching phase is named the “initial phase,” which starts in grade 3. Because fractions are an extension of the whole number system, most of the textbooks use the phrase “分数的初步认识”, translated as “preliminary recognition of fractions.” The main teaching goal in this learning phase is to introduce basic fraction concepts, such as fraction units by using real life situations. In particular, the basic fraction knowledge is introduced with part-whole relations. At the end of this phase, students should have the initial understanding of the meaning of fractions embedded in real life situations, including the ability to: 1) recognize, read, and write fractions; 2) compare fractions with visual models; and 3) do simple fraction addition and subtraction with like denominators (CNMCS, 2001). In addition, the CNMCS and *Teacher's Guide Book* (PEP, 2003) suggest that the denominators used in Grade 3 should be less than 10 so that it would be easier for teachers to use models to present these simple fractions.

The second learning phase is from grade 5 to grade 6. The main goal is to extend students understanding about fractions to include decimals, percentages, as well as the relationships between fractional forms. At the end of this phase, students should be able to successfully use arithmetic operations with fractions and compare fractions with different forms (i.e., decimals, fractions, and percentages (Zhu, 2008). Specifically, in the fifth grade the focus is placed on "分数再认识", translated as "re-recognize fractions." This means the formal definition of fraction is introduced and students extend their conceptual understanding about fractions by exploring the meaning of fractions and fraction operations. In China, the meaning of fractions includes understanding fraction units, which allows students to do fraction comparisons, understand the relationship between fractions and division, understand proper fractions and improper fractions, and mixed fractions (CNMCS, 2001). At the end of 5^{th} grade, students should be able to perform the following: reducing fractions to simplest form, finding common denominators, and converting between fractions and decimals. The intensive instruction of advanced fraction properties and fraction arithmetic occurs in 6th grade. At the end of this second phase, students can successfully perform arithmetic operations on fractions, mixed with fractions, decimals, and percentages, and solve word problems related to fractions, decimals, and percentages (CNMCS, 2001).

Therefore, the short fraction-learning period in China, when compared to the U.S., raises a question: Why do Chinese students outperform U.S. students when they spend less time learning fractions? One of the main goals for the present study is to investigate whether the conceptual understanding of the fraction unit catalyzes Chinese students building of advanced fraction knowledge.

Rationale for the Study and Significance of the Study

The fundamental concepts of fraction, such as the basic meanings of fractions, the meaning of fraction units, and the meaning of proper fractions, create a solid foundation for students' fraction knowledge. Ni (1999) considered mastering the fundamental fraction concepts as a milestone in students' number development. Specifically, understanding fraction units is a critical goal addressed in both Chinese curriculum and the teacher guides.

In fact, the understanding of fraction units plays a significant role in Steffe's fraction schemes. Identifying the fraction unit is a key action in almost all fraction schemes. Only the part-whole fraction scheme does not relate to fraction units. According to Steffe and Olive (2010), to comprehend the meaning of fraction units, students not only need to understand that a fraction unit means one of several equal pieces of a given partitioned whole, but also need to understand that iterating the fraction unit a certain number of times creates any fraction (Norton & Wilkins, 2013). This is another reason that Steffe's fractional schemes are adopted as an explanatory conceptual framework for this study.

Although many researchers agree that a conceptual understanding of fraction units plays a critical role in students' construction of fraction knowledge (e.g., Hackenberg & Tillema, 2009; Steffe & Olive, 2010, Watanabe, 1995), the percent of accuracy of fraction test items among U.S. fourth and eighth grade students is lower than the percent of accuracy among their counterparts from East Asian countries in international assessments, such as TIMSS and IAEP (International Assessment of Educational Progress) (Fan & Zhu, 2004). This study, therefore, aims to investigate and explain how a conceptual understanding of the fraction unit is involved in the process of solving fraction-related tasks for Chinese students. In particular, to determine the role that a strong conceptual understanding of fraction units plays in accelerating Chinese students' construction of their knowledge of fractions within only a few years. The results of this study may inform U.S. mathematics educational researchers and teachers about how Chinese students construct fractional knowledge. In addition, findings from this study could have instructional implications for Chinese educators' enactment of curricula that could exemplify effective teaching practices, which may inform professional development models and future research studies in the U.S.

Research Questions

The current study is designed to investigate how Chinese 4th and 5th grade students construct their fraction knowledge using the fraction schemes developed by Steffe (2002) and his colleague Olive (Steffe & Olive, 2010). In particular, this study focuses on the role that a student's conceptual understanding of the fraction unit plays in Chinese students' learning transition from the initial phase of learning to the re-recognition phase of learning found in the Chinese curriculum for grades 3 and 5, respectively. To explore this, the following research questions are addressed:

1. What fraction schemes have the participating 4th and 5th grade Chinese students constructed before and after the second teaching phase, "Re-recognize Fraction" teaching phase?

2. Which model (i.e., linear, circular, and rectangular) is more or less problematic for Chinese students?
3. How does the understanding of fraction units facilitate Chinese students' ability to solve tasks involving advanced fraction schemes?

Limitation

The current study aims to investigate and explain the phenomena related to the interaction between students' fraction learning and teachers' curricula material in China. At the same time, the discussions related to the intended teaching practice in China are based on curricula analyses of Chinese textbooks, but does not include actual teaching practice. Therefore, results from this study are limited in that curricula may be enacted in very different ways by different educators in an actual classroom.

Outline

Chapter 2 discusses the conceptual framework of this study along with a review of the current literature. The radical constructivism paradigm will be discussed followed by scheme theory as introduced by von Glasersfeld (1995, 1998). Specifically, Steffe's fraction schemes that are used to ground the current study will mainly be discussed (Olive & Steffe, 2002; Steffe, 2002; 2004; Steffe & Olive, 2010). Next, one version of an elementary textbook based on the 5-4 system, 数学, translated as Mathematics for grade 3 and grade 4 published by Qingdao Press (2015), is reviewed relative to fractions. This review includes the scope and sequence of the topics. Textbooks implement the goals defined in the CNMCS curriculum. It also presents the arrangement of fraction knowledge and the structure of fraction knowledge shown in the curriculum.

Chapter 3 presents the methods used to collect and analyze data for this study. This study uses an explanatory sequential mixed methods study design (Creswell & Plano Clark, 2011). This chapter discusses why the explanatory sequential mixed methods design is an appropriate design for this study. The quantitative phase is introduced first followed by the description of the qualitative phase. The results from the quantitative phase are not only used to answer the first research question, but also used to screen students into groups based on their highest available fraction scheme for the clinical interview conducted in the second phase, the qualitative phase. In the qualitative phase, students were asked to solve fractional tasks associated with their highest available scheme as well as the two adjacent schemes above and below using a think aloud method. A process of analyzing the qualitative data involved the initial coding process using tentative categories, such as fraction unit, scheme, and gesture. Then different patterns, new categories and themes were developed through comparing clinical interview data within and across grades.

Chapter 4 includes two main sections. The first section presents the results from analyzing the quantitative data and the changes in the understanding of the fraction units from the initial phase of learning to the re-recognition phase of learning. The second phase presents the results from analyzing the clinical interview data. In addition, analyses involved comparing and contrasting the data within and across grades as well as the integration of quantitative and qualitative results.

Chapter 5 includes four main sections. The first three sections aim to answer the three research questions based on the results and the literature discussed in Chapter 2. Findings associated with each question are displayed and discussed. At the same time, this section also explains the phenomena that appeared when analyzing those quantitative data.

Chapter 6 provides implications of this study as well as addresses the future research with regard to how the iterating operation is triggered when the unit whole is a discrete whole.

Chapter Two: Review of Literature

This chapter consists of two sections. The first section describes the learning theories that grounds this study: constructivist learning theory, von Glasersfeld's (1995) scheme theory and fraction schemes (Steffe & Olive, 2010). The second section of this chapter describes the scope and sequence of fraction teaching as presented in one Chinese elementary mathematics textbook for 3rd and 4th grade students. Attention is given to comparing terminology related to fractions used in the U.S. and China.

Conceptual Framework

Fraction scheme framework (e.g., Steffe & Olive, 2010) is grounded in scheme theory and provides an explanatory framework for investigating Chinese students' understanding of fractions in terms of their understanding of fraction units. Thus, this section opens with a discussion of radical constructivism and scheme theory, followed by a discussion about fraction schemes.

Radical Constructivism and Scheme Theory

Constructivists believe that "knowledge is actively constructed by the cognizing subject, not passively received from the environment" (Lerman, 1989, p.211). In other words, knowledge cannot simply transfer from the external environment to learners. Learning builds upon learners' previous knowledge and previous personal experiences. As a result, the validity of interpretation of knowledge depends on whether the knowledge can fit into the experiential world of the learners. Thus, Cooper (1993) claimed that "personal experiences determine reality" (p.16). Accordingly, an individual learner plays an active role in the process of a person's construction of knowledge. In particular, in radical constructivism, one of the fundamental tenets considers that learning is an adaptive process that results from individuals organizing and reorganizing his or her previous knowledge and experiences.

Radical constructivism is one of the principle trends of constructivism developed by von Glasersfeld (1989). It entails two key tenets: “(1) knowledge is not passively received but actively built up by the cognizing subject; (2) the function of cognition is adaptive and serves the organization of the experiential world, not the discovery of ontological reality” (p.114). That is, radical constructivism not only considers learning as an actively constructing process, but also as an adaptive process. This adaptive process includes two key terms, assimilation and accommodation (von Glasersfeld, 1995).

von Glasersfeld (1995) described learning as associated with assimilation and accommodation through *scheme theory*. Developed from Piaget’s cognitive development theory related to reflexes and the constructivist theory of knowing, the scheme theory is about the construction of a scheme at the reflective abstraction level (von Glasersfeld, 1995, 1998). From Piaget’s action scheme, von Glasersfeld (1998) summarized and defined the pattern of the operative scheme as comprised of three parts: (1) the perceived situation refers to a triggering situation that can activate a scheme, (2) the activity refers to the mental activity associated with the situation, and (3) the expected result. According to von Glasersfeld (1995), to activate a scheme, students need to recognize the triggering situation, which means the conditions of the triggering situation satisfy the past experiences and are incorporated into the existing scheme. Once the existing scheme is activated, the associated activity is triggered. There are two possible results after an activity. If the result of the activity can be assimilated as an expected one, this generates equilibrium for a student’s conceptual structure so that the existing scheme maintains a similar structure. However, if the result of the associated activity cannot be assimilated to the existing scheme, a perturbation is generated which causes the individual to re-organize their knowledge structure by considering those different characteristics related to the existing scheme that they ignored before. At this point, the accommodation happens. The accommodation eliminates the perturbation involving the unviability of the existing scheme, so that a new equilibrium is attained (Glasersfeld, 1995, 1998). Consequently, based on Piaget's constructivist learning theory, learning happens. The pattern of schemes establishes a goal-directed environment, which "provide[s] a perfect context for the functioning of assimilation and accommodation" (Glasersfeld, 1998, p. 9). It also offers a theoretical framework for researchers trying to establish models of how students learn mathematical knowledge and how they change and reorganize their understanding and reasoning during the problem solving process. Moreover, the models of students' thinking conversely serve educators in designing instruction that supports students' conceptual learning.

More specifically, Steffe and Olive (2010) hypothesized that fraction knowledge "emerge[s] as a reorganization of whole number knowing" (p. 2). Based on this hypothesis, through a longitudinal study, Steffe and Olive built models that can be used to explain students' actions on fraction tasks and model their fraction understanding. The next section focuses on fraction schemes and the related operations established by Steffe and Olive.

Fraction Schemes and Related Operations

Building upon a longitudinal study of Steffe and his colleagues (Olive, 1999; Olive & Steffe, 2002; Steffe, 2002; 2004), researchers tried to understand students' ways of constructing their fraction knowledge through sketching students' learning trajectories (Norton & McCloskey, 2008, Norton & Wilkins, 2009). Thus, fraction schemes, models that researchers use to explain students' learning process, and related operations are identified (Hackenberg, 2007; Norton &

Wilkins, 2009a; 2010; Olive & Steffe, 2002; Steffe, 2002). Furthermore, based on Steffe's hypothesis of a hierarchy of fraction schemes, Norton and Wilkins (2009, 2010, 2012, & 2013; Wilkins & Norton, 2011, 2018) quantitatively tested the hierarchical relationship between fraction schemes and related operations that students should experience in their fraction learning progression. In addition, Steffe and Olive (2010) argued that the process of constructing fractional knowledge cannot simply be considered as an extension of children's whole number knowledge. They claimed that fraction knowledge can "emerge as a reorganization of whole number [knowledge]" (p. 2). Thus, the following sections focus on the description of the fraction schemes and operations associated with children's development of their fraction knowledge. Table 2-1 and Table 2-2 summarizes operations and all of the schemes to be discussed in the subsequent sections.

Table 2 - 1

Summary of Operations Related to Fraction Schemes

Operations	Key Action
Partitioning Operation	Divide a given figure into equal parts
Disembedding Operation	Mentally take a part or number of parts out from a partitioned whole without destroy the whole.
Iterating Operation	Repeat any of the parts in the partitioned whole to reproduce a referent whole or a fractional part of the referent whole.
Splitting Operation	Operate with partitioning and iterating operations simultaneously

Table 2 - 2

Summary of Fractional Schemes

Schemes	Key Action(s)
PWS	Disembedding a fractional part
PUFS	Sequentially partitioning and iterating a unit fraction to reproduce the whole
PFS	Partitioning a whole → Disembedding a unit fraction from the partitioned whole → Iterating unit fraction to form a fractional part of the whole (proper fraction)
RPFS	Partitioning a fractional part of a whole (proper fraction) → Disembedding unit fraction from the partitioned fractional part of the whole →Iterating the unit fraction to form the whole
IFS	Partitioning a fractional part of a whole (improper fraction) → Disembedding unit fraction from the partitioned improper fractional part → Iterating the unit fraction to find the whole

The Part-Whole Fraction Scheme (PWS)

The part-whole fraction scheme (PWS) refers to the way that students recognize a fraction as disembedding some number of distinct pieces out of a partitioned whole (Steffe, 2003). For those students who have only constructed a PWS, all pieces in the partitioned whole are not yet identical, meaning that one piece is "only equal in length to the other [pieces]" (p. 242). According to Steffe, only when the pieces are identical, does each one become a unit fractional piece that can be used to reproduce the whole by iterating the piece a certain number of times.

Disembedding Operation. The definition of the PWS suggests that the construction of this fraction scheme involves an important operation, the *disembedding operation*. The disembedding operation is defined as a mental operation in which a child can "imagine separating out some number of parts within the partitioned whole, while leaving the whole intact" (Norton, 2008, p. 406). A lack of ability to perform mental disembedding would cause students to change the referent whole and then miscount the pieces of a partitioned whole (Olive &Vomvoridi, 2006). Olive and Vomvoridi's study provided a typical example of a student who failed to construct a mental disembedding operation. A sixth grader, Tim, worked on a fractional task with a computer program. He first partitioned a whole into four unequal pieces. Then he partitioned the leftmost part of the four pieces into three equal pieces. After partitioning the unit whole into six unequal pieces, he physically disembedded a small piece (one third of one fourth) from the partitioned whole, and put it under the whole. The interviewer then asked him what fraction this piece was. Tim responded that it was one-seventh. After the researcher asked him to show how he came up with seven, he counted the totally uneven pieces in the unit whole and changed his answer to one-sixth. Tim's initial response showed his lack of a mental disembedding operation because he included the piece that he had physically disembedded from the whole. Olive and Vomvoridi argued that Tim's second response also implied that he had not constructed the concept of fraction units. According to Tim's entire explanations for his different answers, Olive and Vomvoridi also pointed out that Tim had not yet constructed an equi-partitioning scheme, which is another important operation involved in the construction of a fractional concept.

Equi-partitioning Operation. The partitioning operation is one of the essential operations throughout the entire construction of fractional schemes. Steffe (1999) described the concept of partitioning as “a psychological structure” (p.24) consisting of two basic and important operations. One is the partitioning operation that involves “breaking a continuous unit into equal sized parts” (p.24). The other is the iterating operation referring to “iterating any of the parts to reconstitute the whole” (p.24).

In the early stage of construction of fraction knowledge, students may construct a simultaneous partitioning operation while they constructed a division operation of whole numbers in their early elementary years. They could easily use this operation when dealing with discrete collections in fraction learning. For instance, in Hunting’s (1983) study, when working with a whole made up of discrete items, Easter eggs, a fourth-grade student used his division knowledge to partition all Easter eggs into three equal shares and showed (physically disembedding) the fractional part (a third of the whole) of the partitioned whole to the interviewer. According to Steffe, when working with a continuous whole, the simultaneous partitioning operation indicates that a child can take a number sequence “as material of its own operating” (2002, p.272), and, at the same time, the number sequence is used as a composite unit by projecting equal pieces into a continuous quantity (Steffe, 2002). However, with the simultaneous partitioning alone, the child is able to find a unit fractional piece of the given whole, but is not able to test if the unit fractional piece is in the right size until the child has constructed the equi-partitioning operation. That means when a student uses a simultaneous partitioning operation, the student is able to project the number of equal parts onto the given whole, but each of the parts is not iterable.

The equi-partitioning operation is the basic operation required for children to construct the fractional schemes that are more advanced than the PWS. This is necessary because children who have developed the equi-partitioning operation are able to accommodate their whole number concepts to project “separated but connected units” (Steffe, 2002, p.274) into a given whole. Moreover, they are able to examine the size of the estimated unit part through iterating this part a certain number of times and seeing if the iterations could reconstruct the given whole (Steffe, 2013). Before students can perform the equi-partitioning operation, they may use the equi-segmenting operation to divide the given whole. The main difference between equi-segmenting and equi-partitioning, according to Steffe (2002), is that the action of equi-segmenting is a sequential use of the estimated fraction unit to partition the unit whole, whereas the action of equi-partitioning involves simultaneously projecting a number of fraction units onto the unit whole.

In other words, equi-segmenting occurs when a student uses the first piece marked on the unpartitioned whole as a unit fractional piece, and as a reference for finding the other pieces sequentially through trial-and-error. After the student estimates the unit piece, s/he is unable to use iteration to identify if the unit piece is of the right size. In contrast to the equi-segmenting, equi-partitioning occurs when a student is able to project all the required number of equal parts onto the unpartitioned whole simultaneously. Furthermore, students who utilize the equi-partitioning operation can also test the size of the unit fractional piece through mentally or physically iterating the piece and reproducing the whole.

Limitations of a Part-whole Concept of Fractions. A child who has only constructed a part-whole fraction scheme understands that each piece of the partitioned whole has the same size, yet does not understand these pieces are also identical (Norton & Wilkins, 2009). To this student, each piece in the partitioned whole is individual and cannot be replaced by another. For instance, to find, say, $\frac{3}{5}$ of a partitioned whole, a student possessing the part-whole fractional scheme can disembed three unit-pieces from the whole, but they are not able to iterate one of five unit-pieces three times to produce $\frac{3}{5}$ of the whole. Thus, a student who has constructed a PWS is able to find the fraction unit of a given unit whole, but may not be able to find afractional part of a given unit. For example, the response from a student in Norton and Wilkins (2009) study showed that each shaded piece and the other unshaded two pieces were different sizes.

Another limitation is that a student who only has a part-whole concept of fractions does not recognize that the size of each unit fraction should be the same. This student may name the fractional part according to the number of visible pieces in a part to the number of visible pieces in a partitioned whole, regardless of the size of each part. In Norton's study (2008), for example, after one of two students accidentally partitioned one of the two-half segments into two parts, Norton asked them what fraction one of the smaller pieces was. Both students responded that it was one third by comparing the piece with the number of pieces in the whole, regardless of the size of each piece. After they pulled out that piece and physically measured the piece as a fourth of the whole, one of the students realized each part was not equal, but he still could not use fractional language to name the part because he physically discovered there were only three parts in the whole segment. To understand the concept of fraction units, students need to construct the next fraction scheme, the partitive unit fraction scheme.

The Partitive Unit Fraction Scheme (PUFS)

The main difference between the PWS and the PUFS is that the students who comprehend the PUFS can reproduce the whole by iterating a fraction unit. A child who has constructed a PUFS not only understands that a unit fractional piece means one out of the partitioned whole, and, but also that iterating this piece a sufficient number of times can produce a partitioned whole that is equal to the original whole (Steffe, 2003). Jason and Laura's performance in Steffe's (2002) study illustrated this difference. Both children were asked to mark a piece that was one-eighth of the whole stick. Although Laura was able to mark a piece that was very close to one-eighth of the stick, she still tried to use her knowledge of composite units to locate all eight units into the stick. However, Jason could disembed the piece and iterate it eight times to test whether it was one-eighth of the whole. What Jason did is referred to as the equi-partitioning operation.

Different from the simultaneous partitioning operation, which partitions a whole in action, the equi-partitioning operation refers to the operation that a child can partition a whole into equal pieces mentally, taking one piece out of the whole, and iterating it a sufficient number of times in a test to verify if the original whole can be reproduced (Norton, 2008; Steffe, 2002). Building upon the simultaneous partitioning operation, children are capable of using the equi-partitioning operation understands that pieces in a partitioned whole are not only equal, but are identical. In other words, iterating any one of these pieces can produce the original whole. When a child is asked to find a unit piece of the whole, using "one-eighth" of the whole as an example, the child can mark off one estimated unit fractional piece, disembed it from the whole, and test whether s/he can reproduce a whole that is equal to the original whole after iterating it eight times (Steffe, 2003). At the same time, the child is also able to name this unit piece by using the fractional language explicitly (Steffe, 2002), naming the fraction of the unit piece is one eighth. Therefore, it can be argued that understanding the concept of fraction units and using fractional language to name the fraction unit is a milestone that marks the transition from PWS to PUFS.

However, the iterating of a unit fraction can determine the size of this fraction but not the factional size of a non-unit fraction (Norton & Wilkins, 2009a). This is because iterating a non-unit fraction cannot reproduce the original whole. Also, to determine the fractional size of a non-unit fraction $\frac{n}{m}$, students not only need to understand that iterating the unit fractional part $\frac{1}{m}$ n

times to get $\frac{n}{m}$, but they also need to recognize that iterating the same unit fractional part m times can reproduce the whole. In this case, the child needs to establish the next fraction scheme, the partitive fraction scheme.

The Partitive Fraction Scheme (PFS)

According to the result of a quantitative study from Norton and Wilkins (2009a), the partitive unit fraction scheme is developed prior to the construction of a partitive fraction scheme. Steffe and Olive's (2010) study also reveals that the partitive fraction scheme is a generalization or "a functional accommodation of the partitive unit fraction scheme" (p.325). Children who have a PFS are able to produce composite fractions by iterating unit fractions. For example, when a child solves a problem such as "make a stick that is two-fifths as long as the given stick," the child can equi-partition the whole into five equal pieces, disembed one piece as a fraction unit, and iterate it two times to form the fractional part of the stick. Thus, a child who has constructed a partitive fraction scheme understands at least two factors of a fraction. First, the child understands that iterating a fraction unit sufficient times (the number is equal to the denominator) can reproduce the original whole. Secondly, the child realizes that the fractional part $\frac{n}{m}$ is n iterations of the fraction unit.

However, children who have only constructed a PFS cannot yet understand improper fractions. Steffe and Olive (2010) posited that this is because the operation of iterating the unit fraction merely transfers from the whole number knowledge of the iterable unit of one. When a unit fraction, say "one-fifth," is iterated seven times, the child who only has a PFS forms a "whole" consisting of 7 unit fractional pieces, which is not the same whole as what he/she thought, a whole consisting of 5 unit fractional pieces. Therefore, the extension of the fraction cannot be gained because of the loss of the whole (Norton & Wilkins, 2009; Steffe & Olive, 2010).

The Splitting Operation (SO)

Successfully constructing the next level of fractional scheme, the RPFS requires the operation of splitting, an operation involving the simultaneous use of partitioning and iterating (Steffe, 2002). Different perspectives on the splitting operation exist. One example is the splitting operation described by Confrey (1994) in the context of multiplication. Confrey argues that exclusively using counting or repeated addition cannot sufficiently explain multiplicative actions, such as “sharing, folding, dividing symmetrically, and magnifying” (p. 292). Thus, she proposes an alternative operation, the splitting operation, to explain multiplication. A splitting operation as defined by Confrey is “an action of creating simultaneously multiple versions of an original, an action often represented by a tree diagram” (p. 292). However, the splitting operation discussed here follows Steffe’s definition. Steffe (2002) defines the splitting operation in the context of constructions of fractions. It refers to a mental operation that involves performing partitioning and iterating simultaneously (Steffe & Olive, 2010; Wilkins & Norton, 2011).

Steffe and Olive (2010) assumed that the splitting operation is actually based on the reorganization of the equi-partitioning operation. According to their definition of the splitting operation, students may emerge with the understanding that “the one [unit whole] constitutes many,” (p.289) and the many are the ones conveying “one-to-many” relationship. When conducting the splitting operation, the child realizes that the unit item is an iterable unit of one and is gained from partitioning the continuous unitary whole. Furthermore, the child also realizes the number of parts is the number that s/he intends to partition the continuous whole.

In fact, results from Norton and Wilkins' (2009, 2011, 2013) quantitative research indicated that the splitting operation even emerges in early stages of fractional scheme constructions, like the construction of PUFS and PFS. Norton and Wilkins (2010, 2013) used a quantitative approach to confirm the role of splitting in the construction of fractional knowledge. In their quantitative research, Norton and Wilkins (2010) gave 56 seventh-grade students a written test about their fractional knowledge. They found that the construction of splitting operation had a statistically significant association with the construction of a partitive unit fraction scheme. Although students construct their partitive unit fraction scheme in early grades and the splitting operation was not required in this construction, the research results revealed that "the construction of splitting operation and partitive unit fraction largely coincided" (p.191). The other result of their study indicated that their students constructed the partitive fraction scheme after the splitting operation, which differed from the study of Steffe and Olive (2010). This result confirmed what Steffe and Olive believed: that the construction of the partitive unit fraction scheme in earlier grades "go[es] on to construct splitting as an interiorization of that scheme" (p. 192). Norton, Wilkins, and Xu (2018) conducted a similar study to examine whether Chinese students who learned fraction knowledge from different curriculum and language would perform differently. Their results aligned with the results from their previous studies of students in the U.S.

The Reversible Partitive Fraction Scheme (RPFS)

As the name indicates, the RPFS is the reverse of the PFS. Within this context, the word "reversible" means that a child can maintain his or her partitive fraction scheme "as a situation of the scheme and [reverse] the relation between the situation and the results of the original scheme" (Steffe & Olive, 2010, p. 328). Hence, given a non-unit proper fractional part, if a child has a reversible partitive fraction scheme, then s/he can produce a complete whole from the given non-unit proper fractional part by partitioning the given fractional part into units based on the numerator, and iterating one of the units a sufficient number of times to create the whole (Norton & Wilkins, 2009, Steffe & Olive, 2010). For example, if a child is given a stick which is $\frac{n}{m}$ as long as the whole and asked to draw the whole, the child can partition the given stick into n pieces to gain the fraction unit of the unknown whole. Then the whole is gained by iterating one of the pieces m times. Therefore, a child who has constructed the RPFS is able to determine the unknown whole from the given non-unit fraction.

It is important to note that students who have constructed a partitive fraction scheme and reversible fraction scheme still believe that a fraction should be smaller than a whole (Norton, 2008). Improper fractions like $\frac{3}{2}$ and $\frac{5}{3}$ are difficult for such students to understand because improper fractions are larger than a unitary whole. The one-third unit of $\frac{5}{3}$ loses its status as one-third of the original whole. In other words, for these students, the meaning of one-third in the fraction of $\frac{5}{3}$ is no longer the same as the meaning of one-third in the original whole. Thus, "the whole is lost" (Norton & Wilkins, 2009). To understand the improper fraction, students need to construct the next fraction scheme: the iterative fraction scheme.

The Iterative Fraction Scheme (IFS)

Improper fractions are fractions that exceed the whole (e.g., $\frac{7}{5}$). To understand the meaning of an improper fraction, students need to surpass part-whole conceptions of fractions (Steffe & Olive, 2010) which requires the construction of an IFS. A student who has an IFS is able to identify the unit fraction by partitioning the given fractional part into pieces so that the number of the pieces equals to the numerator, and each piece has the same size of the piece in the partitioned whole. Based on this understanding, now the unit fractional piece is free, and can be iterated any number of times to form a fractional part that may be bigger than the original whole. If a student has constructed an IFS, the student can also produce the whole by iterating the identified fraction unit to the number as the denominator (Norton & Wilkins, 2009). At this point, it is reasonable to say that a student has constructed a conceptual understanding of fraction units.

When constructing the IFS, the splitting operation plays an important role. With the splitting operation, the iterative fraction scheme allows a child to generate any fraction by iterating a unit fraction or a non-unit fraction and "maintain its relationship to the whole" (Hackeberg, 2007, p.30). At this point, the conception of fractions surpasses the part-whole conception and is "freed from relying on the whole for meaning" (p.30).

Chinese Fraction Teaching and Learning

The Chinese educational system is a centralized educational system, which is a very different system compared to the educational system of the United States. Because textbooks are a critical content resource for Chinese teachers to guide and structure their teaching activities, textbooks play an important role in everyday teaching practices (Ma, 1999; Zhang & Sigler, 2022). Understanding how Chinese textbooks structure the sequence of fraction knowledge reveals the potential teaching practice and the learning opportunities for students. In this section, I begin with a review of the Chinese curriculum that relates to the arrangement of fraction learning followed by a review of Chinese elementary mathematics textbooks.

Chinese mathematics education has its own unique history, cultural context, and national characteristics. Unlike the U.S. educational system in which each state has its own curriculum, China follows a national curriculum. In the 1950s, as part of an effort to unify teaching and learning for the whole country, the Chinese Ministry of Education founded the People's Education Press (PEP). Until 2001, PEP was the only official press that studied, compiled, and published textbooks for elementary and secondary schools (Li, Zhang, & Ma, 2009; Zhang, 2011). However, in the 1990's, due to the need for "Citizenship Education" and the requirement of the "Compulsory Educational Act," China initiated curriculum reform to develop the nation's education system (Li, Zhang, & Ma, 2009; Jiang, 2009; Zhang, 2005). The Chinese Education Administration developed nationwide unified curriculum standards and started writing the *Mathematics Curriculum Standards for Full-time Compulsory Education* in 1999 (Zhang, 2005). After undergoing many practices and revisions, the draft of *Chinese National Mathematics Curriculum Standard (CNMCS)* was announced and published in 2001 (Zhang, 2005). One of the critical changes to the CNMCS draft was to allow a variety of textbooks to enter the Chinese school system for the first time. In other words, the Chinese Education Administration provided opportunities for other education presses to compile and publish textbooks based on the CNMCS. For example, the Education Bureau of the city where the target school in this study is located chose to use the textbook series called 数学, translated as *Mathematics* (2015). This series is published by Qingdao Publishing House (青岛出版社) for the education system of the target school. Therefore, the review of textbooks in this study focuses on the Qingdao third and fourth grade textbooks. These textbooks will be called Qingdao in the following literature review.

After ten years of drafting the CNMCS, the formal CNMCS was published in 2011. Upon comparing and contrasting the descriptions regarding fraction teaching and learning, and no significant difference in the teaching arrangements in the 2001 and 2011 versions was found. For this reason, the review of fraction teaching and learning goals and teaching arrangement will be based on the draft of CNMCS.

The Fraction Teaching Arrangement in Chinese National Curriculum

There are two different kinds of primary school education systems in China: 5-year and 6-year. Therefore, textbooks for these different school systems present knowledge structure and content in slightly different ways. Since the target school of this study implements the 5-year system, the review of the curriculum focuses on the fraction teaching arrangement for the 5-year system.

The CNMCS (2001) separates fraction teaching and learning into two phases (Jiang, 2009; Zhu, 2008) for both systems. The first teaching phase is called the "initial recognition phase" in the CNMCS, and begins during the fall semester of Grade 3 for both systems. Because fractions are an extension of the whole number system, most textbooks, including the Qingdao textbook, also calls this phase the "分数的初步认识", translated as "preliminary recognition of fractions." The main teaching goal of this learning phase is to introduce basic fraction concepts by using real life situations (CNMCS 2001). The CNMCS (2001) also requires that the unit wholes used in the examples be confined to continuous wholes. Therefore, the basic fraction knowledge is introduced with part-whole relations. At the end of this phase, students should have gained the initial understanding of the meaning of fractions embedded in real life situations. These learning objectives include the ability to:1) recognize, read, and write fractions; 2) compare fractions with visual models; 3) do simple fraction addition and subtraction with like denominators (CNMCS, 2001). In addition, the CNMCS and the *Teacher's Guide Book* (PEP, 2003) suggest that the denominators used in Grade 3 should be less than 10 so that it is easier for teachers to use models to present simple fractions.

The second teaching phase happens during grade 4 to grade 5 in the 5-year system. The main teaching goal is to let students gain further understanding of fractions, decimals, and percentages, as well as their relationships to one another. At the end of this phase, students should be able to successfully use the four basic arithmetic operations with fractions and four arithmetic operations mixed with fractions, decimals, and percentages (Jiang, 2009). The second learning phase begins in the spring semester of fourth grade for 5-year system and is called "分数再认识", translated as "re-recognition of fractions." In this phase, the formal definition of fractions is introduced. Students also should establish a conceptual understanding of fractions and fraction operations.

In China, the meaning of fractions includes understanding fraction units, being able to do fraction comparison, understanding the relation between fractions and division, along with understanding proper fractions and improper fractions, and mixed fractions (CNMCS, 2001). Fraction properties include reduction of fractions, finding equivalent fractions, finding common denominators, and reciprocal exchanging of fractions and decimals. Thus, the intensive instruction of fraction properties and fraction arithmetic occurs in 4th through 5th grade, especially 5th grade. At the end of this phase, students should be able to successfully (a) compare fractions with unlike denominators, (b) use the four arithmetic operations with fractions, (c) solve the four arithmetic operations mixed with fractions, decimals, and percentages, and (d) solve word problems related to fractions, decimals, and percentages (CNMCS, 2001). Table 2-3 describes the arrangement of fraction teaching from 3rd grade to 5th grade for the 5-year system.

Table 2 - 3

The Curriculum Arrangement from 3rd Grade to 5th Grade

Grades		Teaching Phase	Contents in Textbook
3rd Grade	Fall	The 1st teaching phase: "initial recognition of fraction."	1) The preliminary understanding of the fractions including: • Introduce fraction units with activities (几分之一) • Informal definition of proper composite fractions (几分之几) 2) Comparing Fractions by using different visual models 3) Simple calculations with fractions (i.e., addition and subtraction of fractions with the denominators less than 10) 4) Simple application of fractions (i.e., the unit whole is a set of objects)
	Spring	N/A	N/A
4th Grade	Fall	N/A	N/A
	Spring	The 2nd teaching phase: "re-recognition of fraction	1) The meaning of fractions, including: • Formal definition of fraction units • Formal definition of fractions 2) The relationship between fractions and division 3) Proper and improper fractions 4) The basic properties of fractions • Fraction simplification • Common fraction 5) Convert fraction and decimal 6) Addition and subtraction of fractions with like denominator
5th Grade	Fall	The 2nd teaching phase: "re-recognition of fraction	1) Compare fractions with unlike denominator 2) Addition and subtraction of fractions with unlike denominators 3) Fraction multiplication and division 4) Mixed operation of addition and subtraction of fractions. 5) Fraction application word problems

Grades		Teaching Phase		Contents in Textbook
5th Grade	Spring	The 2nd teaching phase: “re-recognition of fraction	1)	Percentage and Ratio

Textbooks Related to Fractions and Teaching Practice in the First Teaching Phase – Grade 3

In contrast to American teachers who have more freedom in the ways to use the textbooks, textbooks are the main reference for teaching in China. Chinese teachers closely follow the scope and sequence of topics and knowledge presented in the textbooks (Li, Zhang, & Ma, 2009; Ma, 1999). Because the goal of this study is to investigate the role of fraction units in the construction of Chinese students' fraction knowledge, the following sections review how the series of Chinese Qingdao textbooks conceptualizes the concepts of fraction and structures the content of fraction learning with an emphasis on the teaching and learning of the concept of the fraction unit.

Naming Fractions Linguistically and Literally

The goal of the first lesson on fractions in the Qingdao 3rd grade textbook is to introduce how to read and write fractions. Reading and writing fractions in Chinese are dramatically different to reading and writing fractions in English. A number of cross-national research studies (e.g., Lee, DeWolf, Bassok, & Holyoak, 2016; Miura, Okamoto, Vlahovic-Stetic, Kim, & Han,1999; Mix & Paik, 2008; Paik & Mix, 2003) have found that the transparency of East Asian languages (Paik & Mix, 2003, p.145), such as Korean, can facilitate children's early understanding and construction of fraction concepts, especially the part-whole fraction concepts. For example, in their cross-national fraction-identification study, Miura, et al. (1999) investigated the impact of different languages on the early fraction understandings of young children through comparing and contrasting children from three countries (i.e., the U.S., Croatia, and Korea). For this purpose, they selected only first and early second graders to ensure that these young children had not yet received formal instruction in fraction concepts. In their study, the participants needed to choose the correct pictorial representation for each given written fraction. The only instruction given to these young children was that teachers read aloud the name of the fractions 1/2, 1/3, and 1/4 in their own language without any explanation at the beginning of the test. Therefore, students' understanding of these fractions were merely based on what they heard. Then the participants needed to choose a pictorial representation out of four pictorial representations that matched the corresponding fractions, such as 2/3, 2/4, 3/4, 2/5, 3/5, and 4/5, based on their understanding of what they heard. The researchers collected data three different times, namely at the middle of the semester of the first grade, at the end of the semester of the first grade, and at the beginning of the second grade. They found that the performance of first graders from the three countries who were tested at the middle of the semester did not have significant differences. When they tested another group of first graders in the same schools at the end of the semester, the performance of the Korean children was marginally higher than the performance of the children from the U.S. and Croatia. However, by the beginning of second grade the performance of Korean children was significantly higher than the performance of U.S. and Croatian children. Most Korean second graders in this study (76%) were able to correctly identify all tested fractions compared to only one American child, and none of the Croatian. The findings of this study demonstrated that Korean mathematical language could support young children's understanding of fundamental fraction concepts. China and Korea have similar numerical systems and numerical language characteristics, including fractions. In the following section, the Chinese mathematical languages and representations related to fractions will be described in detail.

In China, fractions are named as "几分之一" or "几分之几", which is literally translated as "several shares of one," or "several shares of several," The first group of characters "几分之

一” is for fraction units. The second group of characters is for general fractions. The character “分” means dividing and the character “几” means several. The first part of the characters “几分” in both groups altogether means dividing into some shares. It represents the denominator of a fraction and literally represents the total number of shares in a partitioned whole. In contrast with using an ordinal number in the denominator in English, the character “几” in the denominator in Chinese will be substituted by a cardinal number. The characters “之一” and “之几” in both groups represent the numerator of the fraction. In Chinese, the character “一” represents the cardinal number “one.” Literally, “之一” means one of several. The characters “之几” mean some shares. The character “几” in “之几” will also be substituted by a cardinal number. Altogether “几分之一” means one of all shares. “几分之几” means some shares of all shares. For example, the linguistic representation of $\frac{1}{5}$ is 5 分之 1. The linguistic representation of $\frac{3}{7}$ is 7 分之 3. Thus, when a fraction is presented in written Chinese, the denominator is written first followed by the numerator, which is the opposite of the English fraction writing order.

When verbalizing a fraction, the order of the linguistic representation of a fraction is to call out the denominator before calling out the numerator. In addition, instead of using a cardinal number in the numerator and an ordinal number in the denominator as in the English linguistic representation of fractions, the Chinese linguistic representation of fractions uses cardinal numbers in both the denominator and numerator, which maintains the numerical consistency of denominator and numerator. When a fraction is presented in numerical form, Chinese students are also taught to write the denominator first followed by the numerator. Apparently, the transparent way in which fractions are written in Chinese explicitly expresses the part-whole relation of fractions. At the same time, the action of writing the denominator followed by the numerator implies that while writing the numerator, the whole, which is the denominator written first, remains intact. This action implicitly promotes the idea of the disembedding operation as well.

The order of verbalizing a fraction is the same as the order of writing the fraction in Chinese. This means the denominator is verbalized first, followed by the numerator. Paik and Mix's (2003) cross-cultural study (i.e., Korean and U.S.) argued that the order of numerator and denominator when verbalizing fraction names may be one of the factors that influence young children to make connections between fractions and their pictorial representations. In Korean, the order of naming fractions is the same as Chinese. For example, "one third" in English, is said literally as "of three *parts*, one" in Korean (Miura et al, 1999; Paik & Mix, 2003). They postulated that the denominator expressing first might help young children make sense of fraction names in addition to the use of the word "parts" in Korean which explicitly reflects the notion of fractional parts in the fraction names. Therefore, in the second fraction-identification experiment Miura, et al. (1999) conducted, 51 American English-speaking first graders and 48 American English-speaking second graders were randomly separated into five different word condition groups. Students in two-word condition groups with explicit "parts" groups heard the fraction names from their teacher that literally imitated Korean fraction names, but in different order, namely, either "denominator→numerator" or "numerator→denominator" (p. 150). That is, in the denominator→numerator group instead of saying "one third" to the students, the teacher said "of three parts, one." In the numerator→denominator group, the teacher said "one of three parts." Students in the other two groups were word condition without explicit parts. Students in these two groups heard fraction names in different orders without the explicit word "parts."

Thus, the fraction 1/3 was said verbalized as “three-one” or “one-three.” The last group was the control group that used the normal English way to name the fraction “one third.” The results surprisingly showed that those students who heard fraction names exactly literally translated from Korean dramatically outperformed all students in the other groups. Consequently, it is reasonable to assume that the order of verbalizing a fraction in Chinese may help Chinese young children make sense of the relationship between the numerator and denominator.

In summary, the advantage of how Chinese students read and write fractions provides a cue for the disembedding operation and understanding the part-whole relation between the numerator and denominator. Furthermore, the word “分” (“divide” in English) explicitly places emphasis on the partitioned whole and the idea of the dis-embedded relationship between numerator and denominator. Therefore, it could be assumed that writing and verbalizing fractions in Chinese may improve Chinese students’ ability to construct their fraction knowledge conceptually when it is first introduced to them. While acknowledging the advantages that culture and language play in fraction learning in China, the current study only focuses on the cognitive development of fraction knowledge among Chinese students.

Fraction Units and Composite Fractions

According to Zhu (2008), in the first learning phase, “preliminary” refers to (1) the whole as a continuous quantity, (2) proper fractions with small denominators (i.e., less than 10), and (3) the informal definition of fractions. Therefore, the first phase starts with an informal introduction of the fraction units and proper fractions with different unit wholes so that students gain an initial comprehension of fractions. As such, the Qingdao textbook starts with recognizing fraction units and proper fractions as well as the representations of a fraction. An example presented early in the textbook involves a scenario of two children sharing a mooncake. Next, the “Discover

Together" (合作探索) section provides an illustration of students figuring out a fraction problem. The girl on the top asks how to represent one half. The boy in the next line says he uses a half circle to represent one half, which is a pictorial representation of a half. The girl on the right side says she represents one half as $\frac{1}{2}$. The bottom text introduces the linguistic and literal representations of the fraction one-half. It says "一半用$\frac{1}{2}$表示，读作：二分之一". The translation of this statement is "One half can be represented as $\frac{1}{2}$. Read it as "二分之一." The textbook attempts to help students transform a real-life sharing experience into different mathematical representations.

Next, the textbook explains the meaning of one-half, one fourth, and three eighths using different area models, namely rectangular and circular models. The textbook first provides two paper-folding activities to help students understand the meaning of one-half. The first activity asks students to fold a rectangular piece of paper into half. The caption of the first activity tells students that "each share" ("每份") of the mooncake is its "half" (written as word form 二分之一 in the textbook). For the second activity, instead of explicitly instructing students "to fold the paper", students are asked to evenly divided (平均分) a circular piece of paper into two. This activity then introduces the concept that each share of the circle is its half ("它的二分之一"). To explain the meaning of one-fourth, the textbook first asks a question: "If a pancake is evenly divided into four shares, what fraction is each share?" Then the boy in the illustration says he uses a circular paper to represent a pancake. The illustration shows the circle is evenly partitioned into four equal shares with one share shaded in dark pink. The corresponding caption describes that the circle is "evenly divided" (平均分) into four shares, and "each share" (每份) of the pancake is "its one-fourth" (它的四分之一). This share can also be represented as ¼. A rectangular model partitioned into four shares with one shaded blue is also provided as an alternative model of one-fourth.

When explaining the fraction three eighths, a circular piece of paper is used to represent a birthday cake, which partitioned into eight shares. The caption of this model tells students that the birthday cake is “evenly divided” (平均分) into eight shares. “Each share” (每份) of the cake is “its one-eighth” (它的八分之一). Three-shares of the cake is “its three eighths” (它的八分之三), and can be represented as $\frac{3}{8}$. A rectangular model (i.e., a partitioned square) is also presented as an alternative model for explaining three-eighths. What is similar for all three activities is that students discover the meaning of a fraction through different representations, namely pictorial, linguistic, and mathematical representations. Each activity provides different models to represent the same fraction, enabling students to realize that a whole of the same fraction can be different.

It can be observed that the phrase “evenly divided” (平均分) appears with a high frequency in textbook descriptions. “平均分” is also the same phrase used in everyday life when students do their fair share of activities requiring mutual efforts. “平均” conveys a sense of evenness. “分” means to share or divide. Therefore, saying “平均分” implies sharing or dividing a whole evenly.

Unlike English, Chinese is a *classifier language* (Yi, 2009, p.210), meaning that Chinese nouns are not categorize as count nouns and non-count nouns, and do not have singular and plural forms. Describing quantities requires the use of classifiers for all nouns. Additionally, specific classifiers are used for specific types of nouns, depending upon how nouns are categorized. The following two examples demonstrate the use of classifiers. “三只老虎” begins with the number “three” (三), followed by the classifier 只 used for “tigers” (老虎) to translate as “three tigers.”. “四个苹果” starts with the number “four” (四) and then the classifier (个) is written before “apple” (苹果), meaning “four apples.” When describing a fraction in Chinese, the classifier “份” indicates share(s) both in Chinese everyday vocabulary and in the context of a fraction. Paik and Mix (2003) argued that children’s informal life experiences may facilitate their understanding of the meaning of the same mathematics vocabulary. Furthermore, it is observed that when the textbook presents fractions, it always describes the referent whole by saying “its one-half”, “its one-fourth”, and so on. This may indicate that Chinese fraction teaching makes an effort to construct the part-whole relation between a fractional part and the referent whole during the initial fraction teaching phase.

After activities for understanding three different fraction units and one composite fraction, the textbook tells students that “Numbers like $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{3}{8} \dots$ are fractions” (Qingdao, p. 113) as part of an informal fraction definition. It also informally introduces the numerator and the denominator by expressing the name of each part in the numerical representation of a fraction. Following the informal definition, the textbook provides different types of exercises, including four non-context exercises and four real-life word problems, that help students gain an insight of fraction units and composite proper fractions from a real-life exercise to more abstract exercises with different models. The first and the second exercises are fill-in-the-blank exercises for understanding fraction units. The first one provides four different real-life items. These items are a mooncake, an egg, a loaf of bread and a chocolate bar. Each of items are partitioned into two, four, or three shares, respectively, and one part is slightly disembedded from the whole. Students practice writing fraction names for each disembedded part. The second exercise includes four different area models, namely a circle, a triangle, a square, and a pentagon. Each of the models is evenly divided into a different number of shares. Students practice shading a fraction unit piece according to the given fraction unit.

The third, fourth, sixth and seventh exercises are for students to practice with composite proper fractions, evenly dividing, and the relationship between composite fractions and fraction units. The third exercise provides four different area models, and requires students to fill in the denominator according to the total number of shares in a whole for each given figure, and to fill in the numerator to represent the shaded part in the given figure.

The fourth exercise requires students to identify which of the given figures shows the fraction 2/5. The first figure is a square connected with an equilateral triangle with one side of the equilateral triangle equal to the side of the square. The square is divided into four equal shares and two of them are shaded. Obviously, the equilateral triangle is bigger than the triangles inside the square. The second figure is an upside-down V shape that is evenly divided into five rhombi and two of them are shaded. The third figure is a regular hexagon and is divided into six congruent triangles. Two of them are shaded. The last figure is an evenly divided five-angle star shape with two shaded shares. These exercises support students in understanding the meaning of evenly dividing through differentiating the correct models representing the given fraction. The two figures in the sixth exercise show students a pitcher of juice that is evenly poured into five glasses. Students are required to write a fraction to represent one glass, and three glasses of juice from this pitcher.

The fifth exercise provides two contexts with underlined fractions. It requires students to convert those underlined fraction units from a written form into a numerical form. This helps students become familiar with the literal and numerical representations of fractions.

The seventh and eighth exercises are more challenging tasks. The first task of the seventh exercise presents a flowerbed which is a shaded part inside a rectangular playground that is evenly divided into six smaller congruent squares. It asks students to write a fraction that represents the area shaded in green (i.e., the flowerbed) out of the whole area (i.e., the rectangular playground). However, the shaded area which is in an arrow shape contains two vertical full smaller squares and two halves of two different smaller squares on the top. To successfully solve this task, students need to coordinate with their geometry knowledge they have learned in 2nd grade. The second task of the seventh exercise is an open-ended task. After finishing the first task, students are required to design their own flowerbed within a rectangular playground and find a fraction to represent their designed flowerbed.

The eighth exercise presents a school newspaper to students and asks them to estimate a fraction representing the whole left section out of the entire newspaper. In particular, the right half of the newspaper is divided into three parts. The top part is one third of the right half section. Each of the two bottom parts is one-fourth of the right half section. Students are required to estimate a fraction representing the bottom left part of the right half of the newspaper. The researcher wonders if students are able to use the iterating operation to mentally or physically examine the size of that section with their informal concept of fractions.

In summary, all the exercises described above indicate that understanding equal sharing and the relationship between a composite fraction and its fraction unit are the teaching focal points in Grade 3, even though this is the initial fraction teaching phase.

After informally learning of fraction units and composite fractions, the next section of this textbook first introduces fractions comparison with like denominators followed by comparing different fraction units. It first posts a scenario of who ate more oranges. Two children are sharing an orange. One child gets 5/8 of the orange while the other child gets 3/8 of the orange. The question posted in the Discovery Together is “Who ate more?” Two circles with the same size are shown at the bottom of the page. One circle is divided evenly into eight equal shares and five of them are shaded in blue. The other circle is also divided evenly into eight equal shares and three shares are shaded into blue. This illustration helps students visually discover which fraction is the larger fraction.

It is worth noting that the message displayed on the right side of the illustration is the linguistic interpretation of the proper fractions 5/8 and 3/8 in terms of multiplicative relation. The first part of the sentence “ $\frac{5}{8}$里面有 5 个$\frac{1}{8}$” translated into English as “there are 5 $\frac{1}{8}$s in $\frac{5}{8}$.” The second part of the sentence “$\frac{3}{8}$里面有 3 个$\frac{1}{8}$” translated into English as “there are 3 $\frac{1}{8}$s in $\frac{3}{8}$.”

Again, it is important to emphasize the use of classifiers in Chinese language. The classifier 个 used after the number 5 in the exercise explained above “5 个” means “there are five items or groups of something.” When introducing multiplication in China, for example, 3×4 can be interpreted as “3 个 4 相加”, meaning adding 4 three times. Thus, in the Chinese language “5 个$\frac{1}{8}$” and “3 个$\frac{1}{8}$” indicate a multiplicative relationship between each proper fraction and the related fraction unit. Chinese educators (Zhang, 2011) believed that a composite fraction is constructed by accumulating (累积) certain numbers of fraction units. Instead of using the term “iterating” as in the U.S., in China, the phrase “累积” is frequently used in the teaching of fractions. The literal translation of “累积” is accumulation. It can be interpreted as adding a quantity repeatedly. Consequently, a phrase such as “5 个 1/8” is easy for Chinese students to understand as accumulating 1/8 five times. Accordingly, the phrase “累积” (accumulation) in here also conveys the multiplicative reasoning although the action involves iterating a fraction unit. This is similar to Kieren's (1980) measurement sub-construct. Hence, the introduction of proper fractions addresses the relationship between a proper fraction and its fraction unit via both additive and multiplicative relationships.

When comparing different fraction units, the textbook asks students to compare the fractions $\frac{1}{9}$ and $\frac{1}{4}$, alongside $\frac{1}{2}$ and $\frac{1}{4}$ using different pictorial representations. Both pictorial representations help students visualize that the size of the whole for each is the same, that there are more shares in one of the wholes, and that the size of each single share is smaller in the whole with more shares. In the exercise part, students practice comparing different fraction units and composite fractions with like denominators in different models, i.e., circular, triangular, and pentagonal models, respectively.

It is noticed that there are again two challenging problems. Exercise 9 says that there is a bottle of juice. One child drank 1/5 of the bottle. The other child drank less than the first student did. It gives the following fractions, 2/5, 4/5, 1/4, and 1/6, and asks students to circle the fraction that can be used to represent the amount of juice the second child drank. To successfully solve this problem, students need to understand the fractions with the same fraction unit, and the fractions with different fraction units. The last exercise, number 11, is an open-ended problem. It shows two fraction comparison expressions. One is "$\frac{1}{3} > \frac{1}{\ }$." The other one is "$\frac{\ }{9} < \frac{5}{9}$." Students need to write a correct denominator for the first expression and a correct numerator for the second expression to provide correct answer for both problems.

Simple Fraction Addition and Subtraction

According to one of the CNMCS guideline (2001), teaching simple fraction addition and subtraction is another way to strengthen students' initial understanding of fractions. Students have gained additive and multiplicative reasoning of whole numbers prior to learning fraction knowledge. They understand, for example, that 4 is composed of four 1s. In other words, accumulating four 1s can form a number of 4. Combined with pictorial expressions, simple fraction addition and subtraction during third grade helps students transition their cardinal number knowledge to fraction knowledge. For example, at the beginning of the section on preliminary recognition of fractions in the Qingdao textbook (p.119), a story of two children sharing a pitcher of juice is presented. The boy drank 3/5 of the juice, and the girl drank 1/5 of the juice. The textbook asks students to discover how much juice these two children drank in total. The textbook first gives the mathematical expression $\frac{3}{5}+\frac{1}{5}=\frac{(\quad)}{(\quad)}$. Then a rectangular model, acted as an analog to the pitcher, is divided into five equal smaller parts. Counting from the bottom, three of them are shaded into light orange and marked with a fraction 3/5 beside the model. One part on the top of those three parts is shaded into dark orange and marked with a fraction 1/5 beside the model. The rectangular model helps students visually add the fractional parts together within a partitioned whole. The portion of dialogue spoken by a girl in the middle right of the illustration explains this fraction addition by saying, "Three $\frac{1}{5}$ s plus one $\frac{1}{5}$ is four $\frac{1}{5}$s, which is $\frac{4}{5}$. Specifically, the girl pictured in the lower left changes the adding of fractions problem to adding whole numbers, namely, adding the numerators. She stats "$\frac{3}{5}$ means 3 of 5 equal parts. $\frac{1}{5}$ means 1 of 5 equal parts. 3 parts and 1 part altogether equal …" This prompts students to add the numerators 3 and 1 to find the sum of 3/5 and 1/5. At this point, the textbook provides multiple representations of simple fraction addition that allows students to make connections between perceptual representations, verbal representations, and symbolic representations. The textbook presents simple fraction subtraction in a similar way.

Two additional tasks in an exercise appearing on page 120 of the Qingdao textbook help students understand that the sum of two fractions may produce the whole. The first task displays a rectangle that is divided into six equal parts. The first two parts are painted red and the fractional name, 2/6, is given. The rest parts are painted blue. This task requires students to find what fraction represents the blue part of the whole. Next, students are required to fill in both numerator and denominator of the fraction that represents the sum of the girl's fraction (the part she had painted red) and the boy's fraction (he had painted blue), and fill in the number that is equivalent to this fraction. Because the definition of the whole had not yet been formally introduced to student by this time, it is doubtful that students can realize the number should be 1. Students lastly need to fill in the mathematical adding expression ( ) + ( ) = ( ).

Another task presented on the same page is a more challenging. This exercise shows a red rectangular sheet of paper partitioned into eight equal pieces. The task explains that one-eighth of the paper was used to make a flag and three-eighths was used to make a star. The fractional part used to make a flower is one-eighth more than the fractional part used to make a star. Students need to figure out a fraction that represents the part used to make a flower by using fraction addition. Then students need to use fraction subtraction to find how much more paper is needed to make a flower than a flag. Finally, students need to add three fractions to decide whether the paper is all used up after making these three items.

The addition and subtraction tasks in the Qingdao textbook indicate that composite fractions are produced through adding numbers of fraction units, meaning that a composite fraction $\frac{n}{m}$ is formed by adding n fraction unit $\frac{1}{m}$s. Conversely, the whole is formed by adding m of fraction unit $\frac{1}{m}$s. In particular, these written messages and visual models in the textbook explicitly demonstrate the idea of accumulation. Therefore, learning simple fraction addition and subtraction in this fashion actually conveys the multiplicative reasoning linguistically although students can find the sums and differences by counting the total shaded pieces or the rest of the shaded pieces according to the pictorial representation. The addition and subtraction activities arranged in the textbook not only help students understand the relationship between fraction units and composite fractions, but also helps them to move beyond the part-whole conception in their future learning.

In summary, analysis of the activities of implicitly introducing fraction units and composite fractions taught in the Qingdao 3^{rd} grade textbook reveals that the emphasis on fractions in third grade is based on a part-whole relationship. All of the activities and exercises underscore the idea that a fraction represents the relationship between parts and a whole. In addition, the activities and exercises in the textbook convey the multiplicative reasoning linguistically that creates the basis for student's future learning. Both construction of composite fractions and doing simple fraction addition and subtraction are introduced by using fraction units implicitly.

Textbook Related to Fraction and Teaching Practice in the Second Learning Phase – Grade 4

As discussed above, teaching fractions in China is separated into two teaching phases. Fraction teaching in the spring of grade 4 comprises the second teaching phase. In this phase, the two main teaching goals are to introduce the formal definitions of fractions and fraction units, and to re-recognize the meaning of fractions. The following sections discuss how these two goals are attained during grade 4.

The Whole as a Continuous Quantity or a Discrete Quantity

Fractions are introduced within the context of both continuous and discrete quantities in 4th grade. The CNMCS (2001) states that the purpose of using both types of quantities is, still, to strengthen the understanding of fractions in different ways. First, using both continuous and discrete quantities helps students recognize the necessity of extending the whole number system. Second, using discrete quantities strengthens the understanding of fractions as numbers. Instead of using the number line to introduce fractions as numbers (CCSM, 2010), the Qingdao textbook uses discrete quantities to impart the conception of fractions as numbers. For example, one of the activities is to evenly split six sheets of square paper into three shares. The textbook first describes that these six sheets form a unit whole. After evenly dividing this whole into three shares, each share is 1/3 of the whole. The textbook also provokes students to think about why each share contains two sheets of paper yet still is represented by a fraction 1/3. Then the textbook explains that this is because these two sheets of paper form one of three shares of the whole.

A Chinese mathematics educator, Jiang (2008) argued that using discrete quantities can enhance the understanding of the relationship between numerators and denominators. However, Jiang also pointed out that it is easier to cause students' confusion about the meaning of numerators and denominators when they encounter tasks in the context of discrete quantities. He found that, for instance, when students were asked questions such as, "What fraction can represent the relationship between the peaches each monkey gets and a plate of peaches if four monkeys share a plate of four peaches?", most students could answer $\frac{1}{4}$. However, when asked "What does 1 mean?", some students answered "1 means one peach" but not one share of a plate of peaches. Obviously, these students confused the number of objects of a share with one share because they all could be represented by “1.” Jiang pointed out another mistake his students made. Students were given a graph of 12 small squares arranged into three rows. Each row had four small squares. When students were asked to color $\frac{1}{4}$ of the 12 squares, some students only colored one square because it was one square out of four squares in the row. Jiang states that the mistake indicates that these students did not have the conceptual understanding of the meaning of a unit whole and the meaning of a denominator that represents the total shares of a partitioned whole. Thus, they confused the total of four shares with four squares in a row, which was the number of elements in one row.

Formal Fraction Definitions

The concept of fraction has different implied meanings across different contexts (Ni, 1999; Xing & Zhang, 2012; Yang & Liu, 2008). The word “meaning” here has the same explanation as Kieren’s (1980), which “applies to the process of building up or developing the elements” (p.126) in the field of construction of knowledge. Even though some Chinese researchers (Ni, 1999; Xing & Zhang, 2012) adopt the meanings of fractions from Western countries, such as Kieren’s five subconstructs (1980), in general, Chinese curricula and textbooks accept four different, but interrelated meanings of fractions and define fractions according to these four meanings (Zhang, 2008). The four definitions are fractional definition, quotient definition, ratio definition, and axiomatic definition (p.78). In the United States, Kieren (1980) identified five interrelated subconstructs of fractional knowledge that were widely accepted by other American researchers (e.g., Lamon, 2007; Norton & Wilkins, 2010). They are part-whole relations, quotients, measures, ratios, and operators (Kieren, 1980, p.134). As we can see, three of the American subconstructs, part-whole relations, quotients, and ratios, are the same as the first three Chinese definitions of fractions.

The goal of teaching fractions in Grade 4 is to re-recognize the meaning of fractions. In other words, students need to gain a conceptual understanding of fractions. Based on the CNMCS (2001) requirements, the formal definitions of fractions, including the fractional definition, the quotient definition, and the ratio definition, need to eventually be introduced to students during Grade 4. According to Ni (1999) and Zhang (2008), explicitly using part-whole relationships to define fractions is more perceptual and easier for students to understand at the beginning of the learning stage. Thus, the formal fractional definition of fractions is introduced first while reviewing fractions knowledge in Grade 4. The Qingdao textbook starts with showing $\frac{1}{4}$ of a unit whole by presenting both a continuous whole (a circle) and a discrete whole (four pieces of clay the same size). Moreover, to strengthen students' understanding of a discrete unit whole, the textbook arranges different discrete whole examples to explain why a fraction can represent a share of the discrete whole that consists of some number of items. Then the definition of a unit whole is given as "an object, a measurement unit, or a set of objects [that] can be viewed as a unit whole." Starting from this point, the whole will be more often referred to as a unit whole. The formal definitions of fractions and fraction units are given as follows: "when a unit whole '1' is equally divided into certain number of shares, the number represents one or some shares of these shares is called a fraction. A number represents one share is called fraction unit" (Qingdao 4th grade mathematics textbook, p. 64).

Teaching Fractional Units in China

As discussed in prior sections, Chinese fraction teaching emphasizes the idea of additive reasoning of fractional units. Although the formal conception of fractional units is not taught to students in Grade 3, accumulating shares or counting shaded parts implicitly promotes the idea of accumulating certain fractional units to form a composite fraction. This is formed through simple addition and subtraction of fractions. Hua (2011) contends that understanding the meaning of fractional units is crucial for students to develop a conceptual understanding of fractions through the idea of accumulating fraction units. Zhang (2011) goes further to state that understanding the meaning of fraction units facilitates students future conceptual understanding of fraction operations, especially finding common denominators and reducing fractions. For example, in Zhang's study (2011) before teaching addition and subtraction of fractions with unlike denominators, a teacher interviewed six of her students at different performing levels (high-, medium-, and low-performing), with the question, "What is $\frac{8}{9}+\frac{8}{15}$ equal to? How do you solve it?" (p. 19) Only one student correctly solved this problem, while the five other students had difficulty solving the problem. After analyzing the interview data, Zhang pointed out that the main challenge the other five students encountered was the lack of a conceptual understanding of a proper fraction as the process of accumulating fraction units. Although all six students could masterfully recite the fractional definition of fractions, most of them did not recognize that the fractional units of $\frac{8}{9}$ and $\frac{8}{15}$ are $\frac{1}{9}$ and $\frac{1}{15}$, respectively. Thus, $\frac{8}{9}$ and $\frac{8}{15}$ can be viewed as accumulating 8 $\frac{1}{9}$s and 8 $\frac{1}{15}$s, but the two fractions obviously have different fractional units. Zhang (2011) commented that if students could understand that a composite fraction is created by accumulating certain fraction units, they would realize that both fractions have different units and the units should be unified before adding or subtracting them, just like before adding 2feet and 10 inches, one should unitize these two units to either feet, or inches.

Yet, fractional units defined by Chinese textbooks are different from the units for measuring. The units that students have learned from the lessons on measurement in Grades 1 and 2, and at the beginning of Grade 3 are literal units. In other words, the units for measuring time, length, weight, and Chinese money are seconds, minutes, hours, meters, decimeters, centimeters, …, grams, kilograms, …, Yuans, Fens, and Jiaos (the last three units are units for Chinese money). However, fractional units are numbers. More accurately, they are also fractions. Furthermore, fractional units are nonstandard since they change based on the referent wholes they are associated with. Thus, students need to possess the ability to identify the fractional units based on different reference wholes. For instance, a student is asked to solve a fractional problem that presents the following question: "If a watermelon was divided into 7 equal shares and a monkey ate 3 pieces, how much watermelon, out of the whole, was eaten by the monkey?" To solve this problem, students must be able to identify the size of a fractional unit. In this case, the student should know that because the whole is partitioned into seven equal shares, the size of the fractional unit (measuring unit) should be $\frac{1}{7}$, based on the total number of shares in the whole. The student should also understand that eating three pieces means accumulating the unit $\frac{1}{7}$ three times to get the answer $\frac{3}{7}$. Furthermore, to solve more complicated problems, students need the ability to identify the whole they are working with and know how to differentiate the referent whole and the original whole. For example, given the following fractional task, "The short candy bar is $\frac{4}{5}$ of the whole candy bar. Draw the whole candy" (Norton & Wilkins, 2009b, p.153), a student might solve the problem correctly, partitioning the short candy bar into four pieces and then iterating one of the four pieces five times to get the whole candy bar. However, when the student is asked to name one of the small pieces in the partitioned short candy bar, he might answer $\frac{1}{4}$, instead of $\frac{1}{5}$. The mistake indicates that the student confused the reference whole with the original whole. This may indicate that the student only has a PWS because the original whole is changed.

To enhance the understanding of fractional units, Chinese mathematics educators (Hua, 2011; Zhang, 2011) suggest borrowing the idea of measurement in the practices of teaching to help students understand the meaning of the fractional units. Using the idea of measurement to construct the understanding of fractions is in accord with one of Kieren's (1980) five subconstructs of students' constructions of rational number knowledge, the "measure" subconstruct. The subconstruct of measure described by Kieren is that the fractional tasks could be finished "through an iteration of the process of counting the number of whole units usable in 'covering' the region" (p. 134). Although Kieren used the word "iteration" and Chinese textbooks and scholars (Hua, 2011; Zhang, 2011) use the word "accumulation," the results are the same, passing on the idea of multiplicative reasoning. This method of approaching composite fractions is beyond the part-whole relationship.

Results from a longitudinal study conducted by Lamon (2007) demonstrate that teaching fractions using the measure subconstruct develops a strong understanding of fractions, along with fraction addition and subtraction, among students. Similarly, Chinese teacher Hua's (2011) class demonstrates how students establish a conceptual understanding of fractions based on their understanding of measurement units and fractional units. The following description discusses Hua's class in detail.

As students know from their real-life experiences and prior knowledge, they need to identify a unit when making measurements. For example, when a rope is used to measure the length of the classroom, the rope is the unit of measure. If three ropes fit the length, we can say that the length of the classroom is three ropes. However, in real life, more complex measurement problems are encountered. Sometimes the measurement of the length of a classroom is equal to three ropes plus a small length of an additional rope. How is this remaining length of classroom measured by using a portion of the rope?

One of the famous Chinese elementary mathematics teachers, Yinglong Hua (2011), set up the following scenario for his students in his lesson plan and this lesson plan which serves as an exemplification of teaching fractional units in China.

Translated Scenario from one of Hua's (2011) lesson plans:

> Xiaohua's father went to a furniture store and wanted to buy a single couch. However, he forgot the length of the couch. He, then, called Xiaohua and asked him to measure the length of the couch. Xiaohua could not find a ruler. Finally, he came up with the idea of using his father's tie as a ruler because his father wore a tie that day. He put the tie on the couch. The couch was a little bit longer than the tie. Then he folded the tie into half and measured the couch. But the folded tie was a little bit shorter than the couch this time. So he folded the tie into another half and measured the couch again. Xiaohua iterated the folded tie three times and covered most [of the] length of the couch. Nevertheless, there was still a short length that the folded tie could not cover. Xiaohua then folded tie into another half and measured the couch one more time. This time Xiaohua iterated the folded tie 7 times and completely covered the length of the couch. Xiaohua is not sure how to tell his father the length of the couch by using the length of the tie. Can you help him?

In this scenario, students need to detect the fractional unit of the tie after Xiaohua folded the tie three times. After students discern that the fractional unit is $\frac{1}{8}$, the teacher introduces the definition of fraction unit. Students are then easily able to figure out the length of the couch is seven fractional units, which is $\frac{7}{8}$. As Hua (2011) states, it is imperative to set different units for different objects in order to make accurate measurements. For example, measuring the length of a classroom uses meters as a unit. However, when measuring a pencil, centimeters function as the unit of measurement. To develop an understanding of the value of different fractional units, Hua and his students had the following conversation.

Hua: 请回头来看，把一根领带对折一次就创造了一个什么分数单位？ [Let's reflect, please. What fractional unit is created after he folded the tie in half one time?]
Student: $\frac{1}{2}$.
Hua: 对折两次呢 [How about after folding the tie twice?]
Student: $\frac{1}{4}$.
Hua: 既然$\frac{1}{2}$、$\frac{1}{4}$ 也是分数单位，那么刚才量沙发长的时候为什么不用这两个单位量，用$\frac{1}{8}$ 来量呢？ [Since $\frac{1}{2}$ and $\frac{1}{4}$ are fractional units, why not use these two units to measure the length of the couch just now, but use $\frac{1}{8}$?]
Student: 因为如果用$\frac{1}{2}$来量的话，量沙发就会缺一点；如果用$\frac{1}{4}$作单位的话，还是缺一点；只有用$\frac{1}{8}$作单位量沙发才正好。 [Because if you use $\frac{1}{2}$ as the unit to measure the couch, the length of the couch falls short of being completely measured. If you use $\frac{1}{4}$ as the unit, you still don't measure the entire length. Only using $\frac{1}{8}$ as the unit to measure the couch would it be measured accurately.]
…
Hua: 看来，用$\frac{1}{2}$ 和 $\frac{1}{4}$作单位不能正好数出来，而用$\frac{1}{8}$ 作单位，刚好可以数出来。由此看来，我们可以根据需要创造合适的单位…[So it seems that using $\frac{1}{2}$ and $\frac{1}{4}$ as units cannot be counted out exactly, but using $\frac{1}{8}$ as the unit can. Thus, we can create an appropriate unit based on what we need (Hua, 2009, p.14-15).

After introducing fractional units, the textbook provides exercises that are similar to what students already learned in Grade 3. However, the exercises include both continuous and discrete quantities. For each problem, students either give the fractional unit first and then use a fraction to represent the shaded part within the unit whole, or students partition different given unit wholes, including continuous and discrete wholes, and shade the correct fractional parts according to the given fractions. By doing these exercises students connect what they learned in Grade 3 with new fractional unit knowledge.

Chinese researchers, such as Hua (2009), Zhang (2008), and Zhang (2011), believe fractional units can facilitate understanding of fraction learning, including improper fractions and fraction operations. Zhang (2011) provided an example in her study to show how one student solved a fraction division problem by reasoning about the unit. A teacher asked students to solve

$2 \div \frac{2}{3}$ according to their previous learning experiences prior to learning fraction division. One of the students solved the problem by changing 2 into $\frac{6}{3}$ and used $\frac{6}{3}$ divided by $\frac{2}{3}$ and got 3. The teacher asked how he got 3. The student responded that "6 $\frac{1}{3}$s divided by 2 $\frac{1}{3}$s is 3". Then the teacher asked the student to explain his reasoning. The student replied that because 2 and $\frac{1}{3}$ had different units, he needed to *unitize* the unit. Unitize is a word commonly used in Chinese mathematics teaching after the introduction of measurement in 4^{th} grade. When students need to convert one unit to another unit, for example, converting cm to mm, they are taught that this action is linguistically expressed as "unitizing units." Obviously, the student in Zhang's (2011) study assimilated the whole number knowledge to fraction knowledge. Moreover, the understanding of fractional units provides an alternative way to understand fraction operations.

Teaching Improper Fractions with the Concept of Fractional Units

The CNMCS (2001) explicitly states that learning improper fractions should be a part of fraction learning content and it should be taught after students have learned the meaning of fraction units. Therefore, establishing the concept of fraction units in association with simple fractional addition actually provides Chinese teachers a way to teach improper fractions.

The Qingdao textbook first provides a coloring activity to color each proper fractional part and improper fractional part that corresponds to the partitioned wholes. Students seemingly use operations of accumulating and counting the number of fraction units to identify the required fractional parts. Next, the textbook presents a learning provocation to encourage students to discover the relation between the numerator and denominator of each fraction. Based on the students findings, the definitions of proper fractions and improper fractions are provided. The definition of proper fractions is given as follows: "分子比分母小的分数叫真分数" ("when the numerator is smaller than the denominator, this fraction is called a proper fraction") (Qingdao, p. 65). The improper fractions are defined as "分子比分母大或者分子和分母相等的分数叫假分数" ("When the numerator is greater or equal to the denominator, this fraction is called an improper fraction") (Qingdao, p.65). Although both definitions are simply given by comparing the numerator and denominator, the following exercises help students to relate the improper fractions to their fraction units through the literal description of improper fractions. More examples from Qingdao textbook are fill-in-the-blank questions where students are expected to write the correct answer inside the provided parentheses. One question requires a fraction as the answer and appears in Chinese text as "9 个$\frac{1}{7}$ 是(   )", meaning that "9 $\frac{1}{7}$s is ( )." A second question, needing a number to correctly identify a quantity, is presented as "$\frac{17}{8}$是( )个$\frac{1}{8}$", translated as "$\frac{17}{8}$ is ( ) $\frac{1}{8}$s." Another exercise on the same page presents a number line containing three units. Each unit is partitioned into four subunits. Students are asked to use points they mark on the exercise to identify the positions of the fractions $\frac{1}{4}$, $\frac{3}{4}$, $\frac{4}{4}$, $\frac{7}{4}$, $1\frac{1}{4}$, and $2\frac{3}{4}$. Apparently, these exercises help students understand that improper fractions are numbers. After practicing different tasks related to proper fractions and improper fractions, the textbook also connects proper fractions and improper fractions to the operation of division, explaining to students that fractions are the result of division.

In summary, the improper fractions in Chinese textbooks are introduced with demonstrative illustrations that involve coloring fractional part of the given whole, and adding the colored fractional units or fractional parts together. This perceptual empirical approach allows students to visualize the process of forming improper fractions through hand-on activities. It is worth noting that most of the wholes presented in the textbook are partitioned wholes.

According to Steffe (2002), the splitting operation plays an important role in constructing improper fractions. However, it is difficult to find activities in the Chinese 4th grade textbook that teach students to form improper fractions that involves the splitting operation. Moreover, the teaching of improper fractions in the Chinese textbook is still related to a part-to-whole relation with the addition transferring from the whole number system. Skepticism remains regarding whether students truly understand the meaning of improper fractions with the teaching approaches presented in the textbook. Yet, the unknown information provides opportunities for future research, such as clarifying what fractional schemes Chinese students have while the students use accumulating or adding fractional units to form improper fractions. It should also be necessary to discover, without the assistance of those pictures, whether students can still notice that improper fractions are larger than their original whole.

Summary and Conclusion of Chapter Two

This chapter mainly reviews the construction of fraction knowledge in terms of the concept of fraction units in two countries, i.e., the U.S. and China. The review of developmental progression of fraction schemes developed by U.S. scholars highlights the critical role of fraction units in students' construction of fraction knowledge (e.g., Hackenberg & Tillema, 2009; Steffe & Olive, 2010, Watanabe, 1995; Wilkins & Norton, 2018). Except for the part-whole scheme, operating with the fraction unit is the main component of the fraction schemes. To successfully solve the fraction tasks involving PUFS, PFS, RPFS, and IFS, students must possess the ability to first identify the fraction unit. The student, Tim, mentioned above in Olive and Vomvoridi (2006), finally constructed the concept of fraction units after the classroom teacher realized the defect of his teaching, assuming that all students understood fraction units, and changed his teaching to include an emphasis on the fraction unit. The review of the literature on fraction schemes indicates that the U.S. scholars recognize the importance of understanding the fraction unit in the construction of fraction knowledge, and emphasize the need of conceptual understanding of fraction units.

The review of a series of Chinese textbooks demonstrates some characteristics and potential teaching practices in Chinese classrooms. The first characteristic is that each teaching topic has corresponding illustrations that provides students with visual representations, especially for those teaching topics in the 3rd grade textbook. The pictorial representation helps students make connections between abstract fraction topics and real-life experiences. For this reason, one of the goals of this study is to investigate the role of different pictorial representations in Chinese students' success with different fraction tasks.

The second characteristic revealed by a review of Chinese textbooks is that the teaching activities always relate to the fraction unit, no matter whether students are being asked to find composite fractions, to compare proper fractions, or to add and subtract fractions. In the 3rd grade textbook, the fill-in-the-blank activities reinforce the understanding that the numerator is equal to the number of parts that constitute the given whole. Filling in the fraction unit for each activity not only helps students make the connection between fraction units and composite fractions, it also helps students understand the initial meaning of the fraction unit, one of some parts of the given whole. In the 4th grade textbook, the unit whole is extended to a set of objects, and students need to be able to use fractions to represent one share of a set of objects after the set of objects is divided into a certain number of shares. Unlike using a whole number to represent the number of objects in one share, students need to abstract the concrete number of objects to a fraction. The review of one Chinese textbook reveals that the teaching practice in Chinese classrooms may put a huge emphasis on understanding the meaning of fraction units.

What this discussion seems to indicate is a commonality between the U.S. literature on fractional schemes and the focus of fraction teaching in China. That is, the importance of building students' understanding of fraction units (as referred to in China) or unit fractions (as referred to in the U.S.). The ultimate goal of this study is to investigate the role that a conceptual understanding of fraction units plays in the construction of Chinese students' fraction knowledge. Because of the emphasis on the fraction units, it is reasonable to ground this study with Steffe's fraction schemes as a theoretical and explanatory framework. With this grounding, the researcher investigated the development of Chinese students' fractions knowledge, and attempt to discover if a conceptual understanding of fraction units helps Chinese students accelerate their construction of fraction knowledge.

Chapter Three: Methodology

This study uses an exploratory sequential mixed methods design (Creswell & Plano-Clark, 2011) with the purpose of investigating the role of understanding fraction units in Chinese students' construction of fraction knowledge. This chapter describes the research design and methodology implemented. This discussion begins with a brief history of mixed methods.

Mixed Methods Methodology

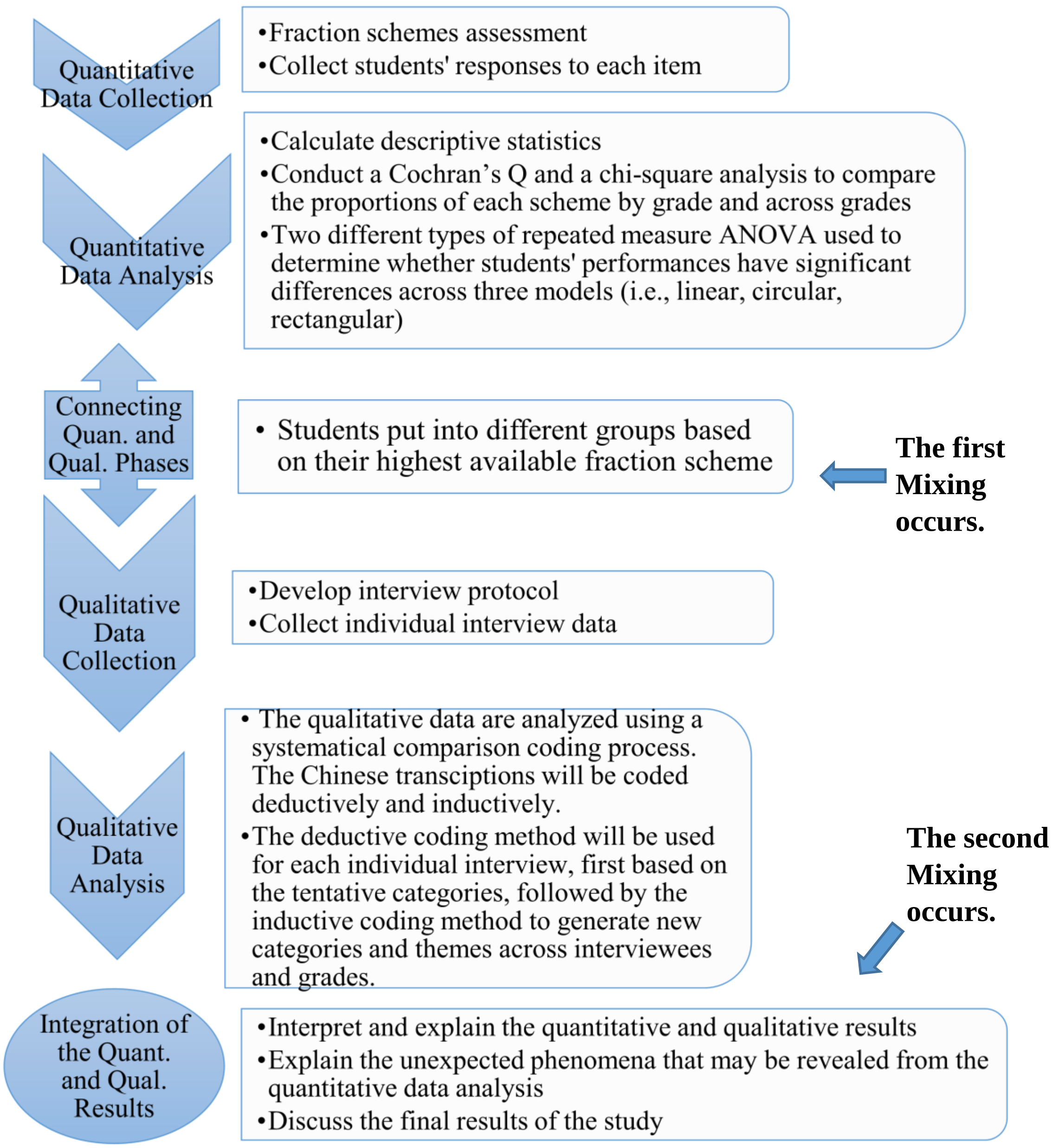
In the early portion of the twentieth century, quantitative research methodology was a dominant approach used in mathematics education research. In contrast, over the past few decades, qualitative research methods have gained prominence in mathematics education (Silver, 2004). However, there are many debates about which methodology can be used to best understand mathematical teaching and learning (Johnson & Onwuegbuzie, 2004; Silver, 2004). The major advantage for quantitative approaches is that it allows researchers to generalize their results. However, researchers cannot explore or explain some of the phenomenon revealed from analyzing those quantitative data. That is because quantitative data do not include crucial information that allows researchers to address the "why" questions linked to the phenomenon. Qualitative approaches, on the other hand, can provide researchers with detailed information about the research participant(s) so they are able to understand the participants to answer the "why" question. Yet, one of the major limitations is an inability to generalize the findings to wider populations. Therefore, mixed methods research methodology as an alternative to traditional quantitative and qualitative methods "draws from the strengths and minimizes the weaknesses of both in single research studies and across studies" (Johnson & Onwuegbuzie, 2004, pp.14-15). Thus, by adopting both forms of data collection and analyses, this researcher aims to use the strengths from both types of methods to better understand the phenomenon under investigation. For example, in this study, the collected and analyzed quantitative data will be used to inform a qualitative data collection process. The qualitative data were then used to enhance the explanation of the analyses from the quantitative data.

In particular, this study uses an explanatory sequential mixed methods design (see Figure 3-1). An explanatory sequential mixed methods design uses two distinct phases of collection and analyses by first collecting and analyzing the quantitative data followed by collecting and analyzing qualitative data (Creswell & Plano Clark, 2011). In this study, after analyzing the quantitative data, the results are used to identify students for individual interviews during the qualitative phase. In addition, quantitative analyses will also be used to identify the questions that will be used in the student interview.

Quantitative analyses may be limited without follow-up qualitative analyses because they do not explain students' conceptual knowledge and the process by which a student does mathematics. For example, in a previous pilot study conducted during spring 2015, the written solutions from students' written test informed this researcher of struggles a student was experiencing when solving one of the tasks related to the PFS. By only reading students' written solutions it was impossible to distinguish between the student's lack of an appropriate scheme and lack of an appropriate strategy to solve the task. A follow-up interview allowed the researcher to distinguish between the two. Thus, for this study, using only quantitative data would be insufficient. The collection and analysis of the qualitative data during the second phase provides opportunities to gain insights into the role of fraction units as students construct their fraction knowledge.

Figure 3. 1

Visual Model of Explanatory Sequential Design for This Study (adapted from Creswell & Plano Clark, 2011, p.69)



Research Questions

The purpose of this study was to investigate the role of Chinese students' understanding of fraction units in their construction of fractional knowledge after the second teaching phase, called the Re-recognize Fraction teaching phase in China. In this study, the researcher will examine Chinese students' conceptual understanding of fraction units and how this understanding relates to their ability to solve tasks involving more sophisticated fraction schemes. In addition, this study explores Chinese students' success rates for solving fractional tasks presented using different representational models. These purposes were addressed by answering the following research questions:

1. What fraction schemes do the participating Chinese 4^{th} and 5^{th} grade students have before and after the second teaching phase, "Re-recognize Fraction" teaching phase?
2. Which model (i.e., linear, circular, and rectangular) is more or less problematic for Chinese students?
3. How does Chinese students' understanding of fraction units facilitate their ability to solve tasks involving advanced fraction schemes?

Quantitative Phase

Participants

The participants for this study were fourth and fifth grade students from China. To understand the role that fraction units play in Chinese students' construction of fraction knowledge, the researcher compared and contrasted students' understanding of fraction units before the concept of fraction units is formally introduced and after the concept of fraction units is formally introduced. According to Chinese elementary curriculum and the arrangement of content topics in textbooks (see Table 2-1 in chapter 2), fractions are first taught in grade 3. At this time, the definition of fractions is introduced in an informal way. At the same time, simple fraction operations, namely, fraction addition and subtraction with common denominators less than or equal to 10 are introduced to help students understand the composite proper fractions. The concept of fraction units is included during this teaching phase, but it is not the focus of teaching fractions. The formal definition of fraction units is not introduced at this time. The second teaching phase of fraction knowledge happens during the spring semester, which is the second semester of the 4^{th} grade. At this time, the formal definition of fraction, the definition of fraction units, and the meaning of fraction units are the teaching foci. Therefore, in order to take advantage of the Chinese curriculum structure, data were collected in September of 2018 from fourth grade students after the informal definition of fractions had been introduced in grade 3, but prior to the introduction of the formal definition of fractions; and in fifth grade after the formal definition of fractions and fraction units had been introduced.

The participants were students enrolled in one elementary school located in Eastern China. The population of the school is predominately middle class and predominately Asian. The vice principal of this elementary school who assisted the researcher in conducting data collection was also the Head of Mathematics Teaching and Research section. Although the vice principal decided to include the assessment adopted in this study as part of the 4^{th} and 5^{th} grade teachers' lesson plans, the Research Subject Information Sheet (See Appendix D, English version, and E, Chinese version) was sent home three days prior to the day when the assessment was to be administered. The information sheet informed parents that the assessment was not only a part of the teachers' lesson plan, but would also be used in this study. Participation in this study was voluntary. The information sheet provided all necessary contact information to parents. On the day the assessment was administered, we had not received any parents' dissent.

This study involved 532 4th and 5th grade students who attempted the survey. Of these students, 26 were removed because they did not sufficiently complete the survey (criterion for removal will be discussed later in the "Missing data" section of the chapter). This resulted in a working sample of 506 students.

Procedures for the Quantitative Phase

Given that this study utilizes a sequential mixed methods design, two consecutive phases, namely a quantitative phase followed by a qualitative phase, were enacted (Creswell & Plano Clark, 2011). The following sections will outline how the research project progressed during the quantitative phase.

The goal of this phase was twofold. The first goal was to evaluate the fraction schemes that the 4th and 5th grade Chinese students had constructed. The second goal was to screen these students for participation in subsequent semi-structured clinical interviews based on their available schemes and scores on the assessment.

Quantitative Data Instruments

Pilot Study

Prior to this study, the researcher conducted a pilot study of six Chinese students during spring 2015. The fraction tasks used in the pilot study were similar to the tasks that were used in Norton, Wilkins and Xu's (2018) study. These tasks were designed to elicit the PUFS, PFS, RPFS, and SO in the Norton and Wilkins' studies (2011, 2012, & 2018) and used either a linear or circular model. Some interesting results were noted from the pilot study. In the pilot study, six fifth grade Chinese students were tested using the same Chinese version of the written fractional tasks to be used in the current study, although IFS tasks were not used in the pilot study. Almost all students could partition the linear model to find the requested fraction units. However, compared to the linear models, circular models were more challenging for most of the students. Four out of six students struggled solving the circular models. The most notable example came from a child's solution to a PFS task that asked what fraction could be used to represent a piece of a given fractional pie out of a whole pie. The strategy that three students used was to make up the fractional piece of pie to create a whole pie, and then draw two lines in the middle to cut the made-up whole pie into 4 pieces (see Figure 3-2). In particular, one of the students wrote the answer $\frac{2}{4}$ on the paper after drawing two lines. The researcher asked him for an equivalent fraction for $\frac{2}{4}$. He said it was ½. He, then, realized that the given piece of pie was obviously not a half of the whole. Thus, he erased the lines and became lost in thought again. Another example also came from a task designed to elicit the PFS scheme. Students were given half of a pizza. The task asked to draw a piece of pizza so that the drawn piece of pizza was 4/5 of the given half pizza. One student drew a vertical line in the middle of the half pizza to cut the half pizza into half, and then drew two other vertical lines on both sides of the first line so that the half pizza was cut into 4 pieces (see Figure 3-3a). However, he realized that there were four, instead of five pieces. He erased all of the lines. Another student also drew a vertical radius in the middle of the half pizza first. Then he drew another two radii on both sides of the first radius so that the half pizza was cut into four sectors. He then stopped. He might have realized that there were only four sectors. It appeared that it was very difficult for these students to partition the circular models to find the unit piece. A similar struggle was observed by Ball (1993) when one of the students in her study tried to share one dozen cookies with five family members. After giving two cookies to each member, two cookies were left. To share these two cookies with five family members, the student first cut each of the cookies into four equal pieces. She realized that she had five family members. Thus, she finally drew a line into one piece of each cookie to make each cookie have five pieces. The common characteristic of the examples described above is that the representation in these fractional tasks is a circular model. From her observation, Ball argued that it is difficult to draw equal parts inside a circle, and she noted that it would be hard to differentiate between the lack of understanding that each piece must be equal or struggling to find a way to create equal size pieces.

Thus, another model, a rectangular model was added to the assessment in this study. There were two main reasons for investigating a rectangular model. First, both rectangular and circular models are considered examples of an area model. Second, Chinese students may have more life experience of partitioning a rectangular model object than a circular model object. When one of the students in the pilot study struggled with partitioning the given half circular pizza into 5 pieces, the researcher asked her whether she had eaten pizza before, she said no. According to the Chinese mentor who assisted the researcher during the pilot study, students in China seldom eat pizza. Birthday cakes in China have different shapes, not only a circular shape. In addition, when reviewing Chinese textbooks, the researcher found that the textbooks use more rectangular models as the representation in examples and exercises than circular models.

Fraction Assessment Adaptations

The fractional tasks used by Norton, Wilkins, and Xu (2018), which were written in Chinese, were adopted and modified for this study to distinguish between students' understanding and strategy used for the linear and circular models. In addition, in order to examine the possibility that other models may be more accessible to students, the researcher designed one rectangular model task for each of the schemes and operation (i.e., PUFS, PFS, RPFS, IFS and SO).

Figure 3. 2

One Student's Response for One of PFS Tasks in Pilot Study

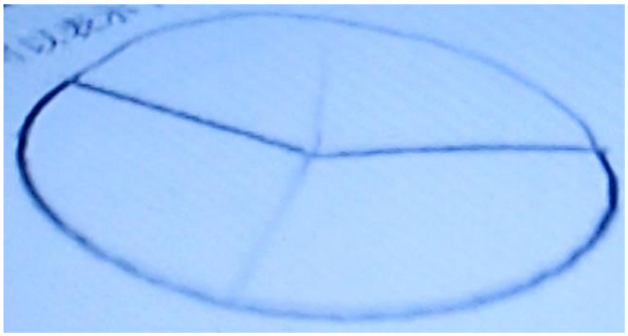
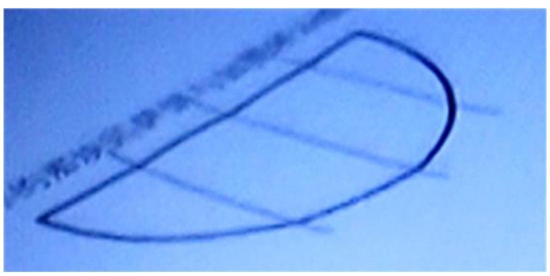
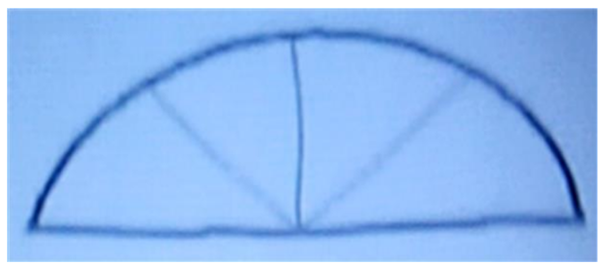


Figure 3. 3

Two Students Responses for One of PFS Tasks in Pilot Study



3-3a



3-3b

The Chinese version of the instrument used in Norton, Wilkins, and Xu (2018) includes four fractional schemes (i.e., PWS, PUFS, PFS, and RPFS) and one operation, splitting (SO), but did not have tasks to elicit an IFS. Thus, the fractional tasks for designed to elicit an IFS in Norton and Wilkins (2012) were translated and included in this study. Consequently, two instruments including two different sets of tasks were used to collect data for 4th grade and 5th grade in this study. Norton, Wilkins, and Xu (2018) conducted a study in China to investigate whether the fraction schemes identified by U.S. scholars could be used to describe Chinese students' construction of fraction knowledge. In their study, they used a set of fractional tasks that were used in Norton and Wilkins' earlier studies (2011, 2012), plus tasks designed to elicit a PWS. These tasks were administered to 5th and 6th grade Chinese students.[1] The results of their study showed that all 31 sixth graders had constructed a PWS and 81% of the sixth graders had constructed a PUFS. This suggests that there is no need to readdress the PWS for Chinese 5th grade students who had learned the formal definition of fractions and fraction units in this study. Instead of testing PWS, this study included items for IFS in the set of fractional tasks for Chinese 5th Grade. In contrast with the 6th graders in the Norton, Wilkins, and Xu (2018) study, only 80% of the 5th graders in their study had constructed a PWS and 67% had constructed a PUFS. Accordingly, the current study set up two different sets of fractional tasks. The set of fractional tasks for 4th graders includes items for PWS, PUFS, PFS, RPFS, and SO. The set of fractional tasks for 5th graders includes PUFS, PFS, RPSF, IFS, and SO. Hence, the instrument for 4th graders has 24 items (see Appendix A). There were four items for each scheme (i.e., PWS, PUFS, PFS, and RPFS) and operation (i.e., SO) in addition there is one rectangular model task for each of PUFS, PFS, RPFS, and SO. The instrument for 5th graders has 25 items (see Appendix B). This instrument includes four items for each scheme (i.e., PUFS, PFS, RPFS, IFS) and operation (i.e., SO) in addition to one rectangular model task for each of PUFS, PFS, RPFS, IFS, and SO.

To control for possible learning effects, there were two different versions for each set of the tasks. The difference between the two versions was the order of the circular and rectangular models. Half of the students in the same class used the version in which the circular models

[1]Students in Norton, Wilkins, and Xu's (2018) study were from a school in a 6-3 system, that is, six-year elementary schooling and 3-year middle schooling. Students in the fifth grade have been informally introduced to the definition of fractions in grade 3, but have not been introduced to the formal definition of fraction; students in the sixth grade were introduced to the formal definition of fractions and fraction units.

come first followed by the rectangular models. The other half of the students did the other version in which the rectangular models come first followed by the circular models. In addition, other tasks will be arranged in different orders. Table 3-1 shows the distribution of the task items and the model types across different schemes for one version[2] of the survey in each grade.

Table 3 - 1

Distribution of Items Across Different Schemes and Model Types

Fraction Scheme	Grade	Item	Model Type
PWS	4th	5, 12, 21, 24	Rectangular
PUFS	4th	14, 23	Linear
		18, 20	Circular
		10	Rectangular
	5th	10, 14	Linear
		2, 11	Circular
		25	Rectangular
PFS	4th	2, 9	Linear
		6, 16	Circular
		7	Rectangle
	5th	1, 17	Linear
		15, 16	Circular
		5	Rectangular
RPFS	4th	3, 19	Linear
		1, 22	Circular
		15	Rectangular
	5th	7, 13	Linear
		20, 24	Circular
		3	Rectangular
IFS	5th	8, 12	Linear
		6, 22	Circular
		19	Rectangular
SO	4th	11, 17	Linear
		4, 8	Circular
		13	Rectangular
	5th	4, 23	Linear
		9, 18	Circular
		21	Rectangular

[2] For the 4th grade written assessments, this table shows the distribution of the task items and the model types across different schemes in version B. For the 5th grade written assessments, this table shows the distribution of the task items and the model types across different schemes in version B.

Note: The rectangular models were compared with the underlined circular models.

Quantitative Data Collection

As described before, the vice principal of this elementary school assisted the researcher with collecting data. Hence, the researcher sent electronic versions of the surveys to the vice principal in China. He administered the surveys. As part of the directions for completing the survey, students were not allowed to use calculators or rulers on the assessment. The vice principal sent all of the completed surveys with the students' responses to the researcher via the post mail system. The researcher and a second rater with expertise in mathematics education and an understanding of fraction schemes scored all student responses independently.

The Basic Rubric

The basic rubric to grade the students' responses used in this study was the same as the grading rubric used in Norton and Wilkins (2010, 2011, & 2012) and Norton, Wilkins, and Xu (2018). Because Norton and Wilkins designed each task to "provoke responses that would indicate a particular scheme or operation" (Norton & Wilkins, 2012, p. 569), the rubric focuses on student responses irrespective of whether they provided the correct answers or not. Student responses were rated based on written responses, markings, drawings, and shading. The items were scored on the following scale, 0, 0.4, 0.6 or 1 based on the sufficiency with which student's written responses provided indication that their actions were consistent with the particular schemes or operations. The score showed the indication or counter-indication of whether the student had constructed a scheme or not. Table 3-2 describes the rubric for this study.

Table 3 - 2

Scoring Rubric for the Paper and Pencil Tasks

Score	Student Responses
0	The rater found strong counter-indication that the student operated in a manner compatible with the theorized scheme or operation. Counter-indication might include incorrect responses and marking that are incompatible with actions that would fit the scheme.
0.4	The rater found only weak counter-indication that the student operated in a manner compatible with the theorized scheme or operation.
0.6	The rater found only weak indication that the student operated in a manner compatible with the theorized scheme or operation.
1	The rater found strong indication that the student operated in a manner compatible with the theorized scheme or operation. Indication might include correct responses, partitions, and iteration.

Note: From Norton & Wilkins (2013), p. 13

Since there were four items for each scheme, the overall score for each scheme was between 0 and 4. According to Norton and Wilkins (2010, 2012, 2013, 2018), if a student's overall raw score for a scheme or operation is greater than or equal to 3, it is inferred that the student has constructed the scheme or operation. If a student's score for a scheme is less than or equal to 2, the student failed to demonstrate construction of the scheme or operation. If a student's overall score for a scheme is between 2 and 3, the individual raters revisit all four items for that scheme or operation and decide if there is enough information to infer that the student's responses indicates he or she has constructed the scheme or operation.

The Construction of the Grading Guide

From the basic scoring rubric, we used fifty randomly selected written assessments to create a grading guide. After receiving all 532 written assessments, a total of 50 written assessments were randomly selected from both grades. These 50 written assessments were used to develop the grading guide for this study. The process of developing the final grading guide was as follows. First, an initial grading guide was created according to the experience of one of the raters, the advisor of this researcher. Next, twelve written assessments, including 4th and 5th graders, were randomly selected from those 50 randomly selected written assessments. Two raters scored tasks for each set of items for a specific scheme or operation, and summed the scores for all the tasks, not including the rectangular task, in that set. Then, the two raters compared the scores of each item. If the raters were in disagreement, they re-examined the student's responses, and discussed them in order to reach a consensus. Based on the consensus, the details of how to grade each item were added into the initial grading guide. For example, one of the PFS tasks asks what fraction the given piece of pizza is out of a whole pizza. The correct answers could be $^6/_{10}$ or $^3/_5$. When solving this task, some students made a whole pizza from the given piece of pizza, partitioned the made-up whole into seven pieces, and gave the answer $^4/_7$. After discussion, we considered that the drawings showed a strong indication. We decided to give this answer a score of 1, namely, an indication of the PFS. After grading these twelve written assessments, another 11 written assessments were randomly selected again. Two raters graded them with the more detailed grading guide. Then the two raters compared, discussed, and reconciled the score for each item and scheme again. More details were added in the grading guide again because there were some drawings and responses that were not seen before. For example, when doing the same task mentioned before, one student made a whole circle based on the given piece, and partitioned the made-up whole into nine pieces so that the given part included 5 pieces and gave the answer of $^5/_9$. We considered that the response showed weak indication of the PFS. At last, the two raters independently scored the remaining 27 written assessments, and discussed and reconciled any disagreements again to further improve and finalize the grading guide. Through these step-by-step procedures, both raters believed that the grading guide covered most of the solutions revealed in the written assessments, and used it to grade the remaining 482 students.

Missing Data

To give enough space for students to draw and explain their solutions, in most cases, three tasks were arranged on one side of each page. If a student missed one page, the student would not compete 3 tasks. If a student missed two pages, the students would not complete 6 tasks, which was a half of all tasks in the assessment. Consequently, it would be difficult to determine the overall scores of some schemes. For this reason, we decided that the data from the students who skipped two or more pages in the assessment, and from the students who did not finish more than two pages were considered as missing data. There was a total of 26 students who were removed based on this criterion. After removing these 26 students from the data, there were 506 students in the working sample, 254 from 4^{th} grade and 252 from 5^{th} grade.

Quantitative Data Analysis

The purpose of the quantitative data analysis was threefold. The first purpose was to indicate the schemes Chinese students have constructed before and after being introduced to the formal definition of fractions and fraction units (i.e., respectively, theoretically in 4^{th} and 5^{th} grade for this school). This would be determined by analyzing students' responses to the tasks. The analysis of the quantitative data also served the purpose of comparing children's responses to the circular model and rectangular model tasks. The last purpose was to identify students to participate in the clinical interviews conducted in the second phase of this study. The interview was used to further investigate students' fraction knowledge based on their understanding of fraction units. Following is a discussion of the methods for analyzing the quantitative data. The discussion first begins with the results from an examination of the inter-rater reliability of the scores.

The Inter-rater Reliability

The inter-rater reliability of the scores for all items and schemes was examined before analyzing the quantitative data. Cohen's Kappa was used as a measure of inter-rater reliability. Cohen's kappa is first calculated for the scores for each item in the assessment. Cohen's kappa mainly measures the absolute agreement between two raters, but does not consider the degree of disagreement between two raters (Landis & Koch, 1977). In other words, Cohen's Kappa only measures the proportions of perfect agreements on the main diagonal in a 4×4 table (including the four ratings for each rater). In fact, some partial agreements, meaning certain types of disagreement, were acceptable in this study. Those disagreements such as 0.4 – 0 and 0.6 – 1 could be considered partial agreements. Thus, the weighted kappa was also used in this study to take into account the degree of disagreement between two raters.

As described above, the items in the instruments included original items from Norton, Wilkins, and Xu (2018) and Norton and Wilkins (2012) studies, and rectangular model items designed for this study. Therefore, there were three different kappa and weighted kappa statistics that were calculated. First, the kappa and the weighted kappa statistics for each original item, not including the rectangular model items, were first calculated to determine the necessity to reconcile the scores of items for all 482 students.[3] The kappa statistics, K, and the weighted kappa statistics, K_w, for each item in both grades indicated that the scores for most items were at

[3] At the beginning we randomly selected 50 of 532 students (appropriately 10%) from both grades to create a scoring guide, therefore the scores for these 50 students were completely reconciled. Therefore, to calculate Kappa scores the researcher created data files from the original files excluding these 50 students and the 26 students who are considered as missing data to test the inter-rater reliability.

an acceptable level (see Appendix G). In 4th grade, three kappa statistics (i.e., PWS2, PWS3, and RPFS3) were less than 0.4. In 5th grade, only one kappa statistic (PUFS3) was less than 0.4. But the weighted kappa statistics for all of these items were greater than 0.5. These suggested an acceptable level of reliability between the two raters (Landis & Koch, 1977), and thus it was felt that there was no need to reconcile the scores for all students for those original items. Next, the kappa and the weighted kappa statistics for the rectangular model items were calculated (see Table 3-3). Although the kappa statistic, K, for RPFS R in both grades (K = .55, p < .05 for RPFS R in 4th grade, K = .57, p < .05 for RPFS R in 5th grade) and the kappa statistic, K, for IFS R (K = .55, p < .05) in 5th grade indicated a "moderate" level of agreement (Landis & Koch, 1977, p.165), the weighted kappa, K_w, for these items ($K_w = .63, p < .05$ for RPFS R in 4th grade, $K_w = .66, p < .05$ for RPFS R in 5th grade, and $K_w = .71, p < .05$ for IFS R in 5th grade) were greater than .61, indicating "substantial" level of agreement (p. 165). Therefore, it also suggested that there was no need to reconcile the scores of those rectangular model items.

Table 3 - 3

Inter-rater Reliability for Rectangular Items

Items	4th Grade		5th Grade		4th & 5th Grades	
	K	K_w	K	K_w	K	K_w
PUFS R	0.88	0.94	0.83	0.89	0.86	0.92
PFS R	0.86	0.84	0.85	0.85	0.86	0.84
RPFS R	0.55	0.63	0.57	0.66	0.56	0.64
IFS R			0.55	0.71	0.55	0.71
SO R	0.73	0.84	0.72	0.85	0.72	0.84

The Kappa statistic, K, was also calculated for the five schemes and one operation (see Table 3-4). In 4th grade, the kappa statistics were calculated from 235 students. The kappa statistic for PWS (K = .80, p < .05) and PFS (K = .70, p < .05) indicated a "substantial" level of agreement (Landis & Koch, 1977, p. 165). The kappa statistic for RPFS (K = .87, $p < .05$) and SO (K = .89, $p < .05$) indicated an "almost perfect" level of agreement (p. 165). The kappa statistic for PUFS (K = .50, $p < .05$) indicated a "moderate" level of agreement (p. 165). All of the kappa statistics for 4th grade students suggested an acceptable level of reliability for the scheme scores between the two raters. In 5th grade, the kappa statistics were calculated from 221 students. The kappa statistic for PFS (K = .81, $p < .05$) and SO (K = .81, $p < .05$) indicated an "almost perfect" level of agreement (p. 165). The kappa statistic for PUFS (K = .59, $p < .05$) and RPFS (K = .59, $p < .05$) indicated a "moderate" level of agreement (p. 165). The kappa statistic for IFS (K = .71, $p < .05$) indicated a "substantial" level of agreement (p. 165). All of the kappa statistics for 5th grade students also suggested an acceptable level of reliability for the scores between the two raters. Even though the kappa statistics suggested acceptable inter-rater reliability, the two raters decided to discuss the scores for the schemes and operation that were in disagreement, and reconciled those scores. The reason is that schemes will be one of the independent variables when analyzing quantitative data.

Table 3 - 4

The Cohen's Kappa Statistic, K, of Inter-rater Reliability for Schemes From 4th, 5th, and Both Grades

Items	4th Grade (N = 235)	5th Grade (N = 221)	4th & 5th Grades (N = 456)
	K	K	K
PWS	0.80		0.80
PUFS	0.50	0.59	0.56
PFS	0.70	0.81	0.78
RPFS	0.87	0.60	0.75
IFS		0.71	0.71
SO	0.89	0.81	0.84

Creation of Variables

Initially there were eight data files created, four files for each rater. The four data files for each rater included a data file of 4th grade version A, 4th grade version B, 5th grade version A, and 5th grade version B. After reconciling the scores of the schemes and splitting operation, the researcher used SPSS, a statistical software program, to merge all eight files into a final data file. After creating the final data file, different variables were created for answering the two quantitative research questions. The creation of these variables is discussed next.

Different variables were created for different quantitative analysis methods. To answer the first research question, a Cochran's Q test was conducted to exam the development of Chinese students' fraction schemes. Hence, five variables based on student scores for schemes, PWS, PUFS, PFS, RPFS, and IFS were created as independent variables. In order to answer the second research question, three different ANOVA tests were conducted to discover if there exists a model among three models (i.e., linear, circular, and rectangular) that is more or less problematic for Chinese students. Therefore, the researcher created two independent variables. One independent variable was Model which had five levels. The researcher used the computing variable function in SPSS to generate these five levels. They were (1) PUFS_L, PUFS_C, PUFS_R, (2) PFS_L, PFS_C, PFS_R, (3) RPFS_L, RPFS _C, RPFS _R, (4) IFS_L, IFS_C, IFS_R, and (5) SO_L, SO_C, SO_R. The levels related to the linear models (i.e., PUFS_L, PFS_L, RPFS_L, IFS_L, and SO_L) were generated by computing the average scores of all linear items of each particular scheme from both raters. The levels related to the circular model were generated in the same way. Recall that there is only one rectangular model task created per scheme (except PWS) and the splitting operation, therefore, in order to compare tasks involving a rectangular model to circular model tasks (i.e., _CtoR), another five levels, PUFS_CtoR, PFS_CtoR, RPFS_CtoR, IFS_CtoR, and SO_CtoR, were generated by computing the average score of the items from the circular model tasks for each particular scheme that corresponded to the designed rectangular model item from both raters. The dependent variables were students' responses to the items for each scheme and operation. Students' responses were quantified as scores using the grading rubric described above.

Quantitative Data Analysis

Descriptive statistics were used to calculate and compare the percentages of students in 4th and 5th grades who have constructed each scheme and operation. This was further broken down by the type of models. Inferential statistics were used to determine the differences between the proportions for 4th and 5th grade students for each scheme and operation, and to compare students' performance across the three different model types: linear, circular, and rectangular model.

Descriptive Statistics. The study was based on the assumption that Chinese students should have the same fraction learning progression as their U.S. counterparts (Norton, Wilkins, & Xu, 2018). Thus, after the proportions of students who have constructed each scheme and operation were found, a Cochran's Q test was used first to compare these proportions by grade and to test if the Chinese participants in this study follow the same progression of fraction schemes identified by Steffe and Olive (2010). Thus, the independent variables were different schemes. In 4th grade, the independent variables were the four schemes: PWS, PUFS, PFS, and RPFS. In 5th grade, the independent variables were also four schemes: PUFS, PFS, RPFS, and IFS. The dependent variables were the outcome of these schemes. The score for scheme and operation entered was either 1 or 0. A score of 1 meant that the student has constructed that particular scheme or operation. Otherwise, a score of 0 meant that the student had not yet constructed that particular scheme or operation. If the Cochran's Q test results indicate that there is a statistically significant difference in the proportion of students who had constructed a particular scheme across the schemes, a pairwise comparison post hoc test, Dunnett's test, needs to be carried out.

After the development of the participants' fraction schemes in each grade was identified, next, the development of the participants' fraction schemes across grade needed to be examined. In order to test the differences between the constructions of each fraction scheme across grade, a chi-square test of independence was conducted to compare the proportions from 4^{th} and 5^{th} grades for each scheme and operation. The results of this analysis were used to answer the first research question of this study.

Inferential Statistics. To answer the second research question, two different analysis of variance (ANOVA) designs were used to determine whether there is an interaction effect associated with grade for students' performance with different models: linear, circular, and rectangular model. First, a $2 \times 3 \times 2$ three-way mixed repeated measure ANOVA (three-way mixed ANOVA) analysis with two within-subjects factors and one between-subjects factor was conducted to determine whether there is a grade effect associated with student's performance with different models as well as different schemes and the splitting operation: PUFS, PFS, RPFS, and SO. In this analysis, the within-subjects factors were Models and Scheme. The between-subjects factor was Grade. When conducting this analysis, the within-subjects factor Models was tested in two different conditions. One condition was to determine if there was a mean difference between performance on the linear model and circular model across grades. The other condition was to determine if there was a mean difference between performance on the circular model and the corresponding rectangular model across grades. The results of the three-way mixed ANOVA were used to determine whether the mean differences between different models was related to the grade of the students.

Then a two-way repeated measures ANOVA with a within-subjects factors design was used to compare mean differences between linear and circular models as well as between circular and its corresponding rectangular model. The within-subjects factors (independent variables) were model (i.e., linear and circular model, circular and corresponding rectangular model) and scheme (i.e., PUFS, PFS, RPFS, and IFS). The two-way repeated measure ANOVA was used to determine whether there was a significant difference in student performance across different models. If the results indicate a significant difference across models, a post-hoc test was conducted using pairwise *t*-tests to identify which model was significantly different from the others. The result of the two-way repeated measures ANOVA provides evidence as to the role of the different models, that is, whether a particular model is more or less problematic for the Chinese students.

Screening Process – First Mixed Phase

The first data mixing occurs in the screening process. One of the goals of the study is to determine if the conceptual understanding about the fraction unit facilitates the ability of Chinese students to solve fractional tasks related to the more advanced fraction schemes. In this case, the more advanced schemes refer to the PFS, RPFS, and IFS. Therefore, the criteria for grouping the participants for the qualitative phase is based on the availability of students who have constructed each of the five schemes and the splitting operation. Accordingly, there were 7 groups (see Table 3-5). The students were placed into one of the groups based on the available highest scheme. Although there were 532 students assessed, there were only 134 students with signed parental consent to participate in a clinical interview (38 signed consent forms from 4th

grade and 96 signed consent forms from 5th grade). Hence, after a preliminary analysis of the quantitative data, there were 10 4th graders and 19 5th graders selected to participate in a clinical interview.

Table 3 - 5

Numbers of Students from Each Grade in Different Clinical Interview Groups Based on the Highest Available Scheme

Scheme	4th Grade	5th Grade
	Numbers	Number
Pre-PWS	2	2
PWS	5	
PUFS	3	4
PFS		3
RPFS		3
IFS		4
SO		5*

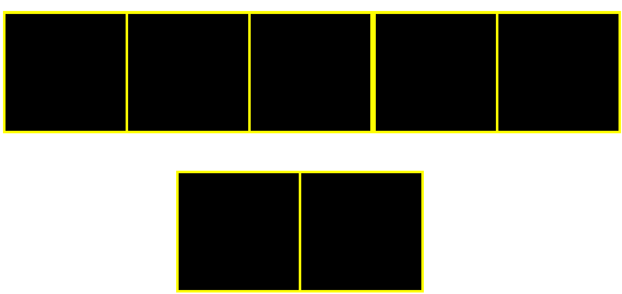
Note. *Two fifth graders in the PWS group were also in the SO group because the scores of their SO showed that they may have constructed SO.

For the Pre-PWS group, there were two fourth graders and two fifth graders. The reason for putting two fourth graders in this group was because they were not successful with any of the PWS tasks. Specifically, their answers revealed that they may not be able to identify the unit whole. For example, one of the PWS tasks requires students to write the fraction of the smaller bar out of the larger bar (see Figure 3.4). In this task, it did not explicitly tell students what the unit whole is. Students need to first identify which of the given figures is the unit whole. The answer of one selected 4th grade student considered the whole given figure to be the unit whole and gave an answer $2/7$. This suggests that the student may have a part-whole conception of fractions, but was unable to identify the unit whole. The reason for including two fifth graders in this group was to include students who were unsuccessful on all items on the assessment. Since the fifth grade assessment did not include PWS items, it is difficult to decide whether they had constructed a PWS or not. Thus, they were put in the Pre-PWS group. Two students in the PWS group also provided evidence to indicate that they had constructed the SO. That is, their scores indicated that they had only constructed PWS and SO, but no other schemes. Findings from Wilkins and Norton (2011) suggest that the construction of a PUFS occurs before the construction of the SO. For this reason, these two students were put into the SO group as well to discover how they solved the SO tasks when they had only constructed a PWS. In the end, approximately 4 to 5 students from each scheme group participated in a clinical interview.

Figure 3. 4

One of the PWS Tasks in 4th Grade Assessment

如图所示，请问小块的是大块的几分之几。
What fraction is the smaller bar out of the larger bar



Qualitative Phase

In the second phase, the goal is to gain an in-depth understanding of the role of a student’s conceptual understanding of fraction units in their ability to solve fraction tasks. For this purpose, a clinical interview (Piaget, 1952) was used. To better understand the thinking process of a student, the selected participants were also asked to think aloud when solving the fractional tasks. The data collection, analyses, and more detail regarding the think-aloud method of the interview are explained in the next section.

Procedures for the Qualitative Phase

Qualitative Data Collection

Clinical Interview

During his early research career, Piaget (1952) observed that children's mathematical cognition was "qualitatively different from adult cognition" (Ginsburg, 1981, p. 4), especially when children's responses were unanticipated. The observed evidence encouraged him to develop a new technique, the clinical interview, for his developmental psychology research. Rather than using standardized ways to rank each child, a clinical interview is considered to be a "flexible method of questioning" (Ginsburg, p. 4) used to capture the nature of children's thinking and discover thoughts through one-to-one interactions. Clinical interviews enable researchers to gain in-depth insight into children's thinking by investigating and specifying the structural patterns of children's thoughts (Cobb & Steffe, 1983; Ginsburg, 1981). During a clinical interview, an interviewee "verbalizes his thoughts, gives reasons for his actions, and generally reflects on what he has done" (Ginsburg, 1981, p. 6).

As Steffe and Thompson (2000) stated, the purpose of doing clinical interviews is not to change students' cognitive structures, but to better understand students' current reasoning patterns through discovering the fundamental activities of students' mathematical thinking. In this particular study, the main purpose of the clinical interviews is to find out how Chinese students apply fraction units when they solve fractional tasks related to different fraction schemes, and to gain insight into how their conceptual understanding of fraction units facilitates their construction of their fractional knowledge.

A semi-structured clinical interview is used so that researchers can probe and follow through a student's particular behaviors by expanding a question, or altering the problem. The semi-structured clinical interview also provides opportunities for researchers to gain deeper understanding of how students apply their knowledge about fraction units while solving fractional tasks involving different fractional schemes.

The clinical interview used in this study involved an internet based online interview similar to the clinical interviews that were conducted in the pilot study during spring 2015. The Chinese vice principal of the elementary school was responsible for setting up the necessary equipment such as a computer, and two external cameras (e.g., one was a built-in camera and the other one was connected to the computer through a USB cable) in China. All interviews were conducted in the vice principal's office. All interviews were videotaped and audiotaped across the internet through a Skype connection. Skype, as an alternative technology and data collection approach, provides an alternative way for researchers to conduct a qualitative interview in a time efficient and financially affordable manner (Seitz, 2016). However, there are some issues that need to be considered when conducting a Skype qualitative interview. These issues include "dropped calls and pauses, inaudible segments, inability to read body language and nonverbal cues, and loss of intimacy" (Seitz, 2016, p. 203). To overcome those unexpected situations such as dropped calls and pause, and/or inaudible segments, the researcher and the vice principal tested the internet connection, external camera, computer built-in camera and microphone before the interview. Before the interview started, the researcher explained to the interviewee what might happen during the interview. The vice principal asked the interviewee to sign the student assent form (see Appendix F) if they agreed to participate in the interview. To capture students' gestures, emotion, and other body language, the vice principal set up an external video camera in the interview room to ensure that words, gestures, and drawing and writing would be captured. The researcher also had a video camera to videotape through the computer. The researcher conducted the whole interview and the Chinese vice principal assisted the researcher during the whole interview. For example, when the researcher asked a question, the mathematical language used may be different from or at a higher level than the student's current language level. In this situation, the vice principal was there to restate the question more clearly. Or if a student had problems solving the circular model task, for example, the vice principal could help the researcher present the corresponding rectangular model task. All participants attended the clinical interview only one time. For 4^{th} grade students, the length of each clinical interview was between 30 to 45 minutes. For 5^{th} grade students, the length of each clinical interview was between 30 to 60 minutes.

To maintain the data fidelity, each interview was transcribed in Chinese and then coded based on the Chinese transcription. During initial coding, some aspects of fraction schemes were used deductively to determine how particular key language may relate to students' reasoning. Therefore, key words, such as fraction unit, and numbers of fraction units were used as pre-defined codes. Furthermore, descriptive coding was used inductively to best determine the novel language students may use when explaining their reasoning. As transcriptions were developed, the researcher also documented students' actions that contribute to student explanations (e.g., gestures) in a memo. These actions were also coded descriptively. In addition, the drawing and writing from the students were included for descriptive coding.

Think-aloud Method

Along with the clinical interview, the think-aloud method was also used in this study. The think-aloud method is one of the cognitive interview methods "in which participants speak aloud any words in their mind as they complete a task" (Charters, 2003). It gives researchers insight into a person's mental processes.

The written answers from students' solutions to fractional tasks in the quantitative phase of this study can be considered as the products of their thought processes (van Someren, Barnard, & Sandberg, 1994). However, the written answers do not disclose students' thought process. The goal of this study was to explore the role of the conceptual understanding of fraction units while students are solving fractional tasks. Hence, the think-aloud method allows the researcher to discover students' current cognitive experiences or thinking and reasoning processes, through students' verbal utterances while they are performing the task (van Someren, Barnard, & Sandberg, 1994). Requesting students to think aloud as they are solving the fractional tasks can make students' silent cognitive processes explicit through verbal descriptions. This technique provides a way to overcome the uncertain meaning of marks or symbols drawn by students in their written assessments.

The think-aloud technique depends on the notion that students' verbalized solutions rely upon their short-term memory, or working memory while solving a task (Charters, 2003). Consequently, their verbal explanations may be "incomplete and exclude a number of thought processes which are not held in working memory long enough to be expressed verbally" (Charters, 2003, p.73). Thus, Charters suggests that the degree of the difficulty of the chosen task needs to have moderate difficulty and be appropriate to students' cognitive abilities. If the chosen task is difficult, the cognitive load of solving the problem may be in excess of the limit of the working memory capacity, and hinder interviewees verbalizing their thought. Accordingly, participants in this study did the tasks corresponding to the fractional scheme that are below and the same as their current fractional scheme and the tasks related to the splitting operation.[4]

The third research question asks how the conceptual understanding of fraction units facilitates the construction of Chinese students' fraction knowledge. Therefore, the selected participants solved the tasks associated with their highest available scheme as well as the two adjacent schemes above and below using the think aloud method (e.g., if a student's highest fraction scheme is the PFS, they will be asked to solve tasks associated with the PUFS, PFS and RPFS).[5] Although participants might not be able to successfully solve these tasks, the verbalization of their thinking provides an opportunity to investigate the development of students' understanding of fraction units. Specifically, when a student solves the tasks associated with the schemes that are higher than his or her highest available scheme, this may provoke cognitive conflict. This may allow the researcher to discover the process of constructing the understanding of fraction units. To assist children in verbalizing their thinking, Johnstone et al. (2006) recommended that the researcher might assist a participant to voice their solution by asking questions, such as "How did you solve that?" (p. 7) when the participant's language was not clear, or "Was there anything that confused you?" (p. 7) when the participant seems to be struggling with the task.

Clinical Interview Protocol

The clinical interview of this study was conducted from late December 2018 to the second week of January 2019, which was fall semester. At that time. fourth grade participants

[4] As will be discussed in Chapter 4, this procedure was changed after interviewing several students.
[5] This plan was modified after the first day of the clinical interview. The reasons for changing the original plan and how the plan changed are described in Chapter 4.

had only learned the informal part-whole definition of fractions, but not the formal definition. However, participants in 5th grade had learned a formal part-whole definition of fractions in addition to the formal definition of fraction units during the spring semester in their 4th grade. Thus, the first two tasks in the clinical interview involved two fractions 3/5 and 5/3. Each of the participants was asked to read each fraction aloud, and explain the meaning of each fraction. The aim of these two tasks was to test whether there existed some conceptual differences between 4th grade students, who had only learned an informal part-whole definition of fractions, and 5th grade students, who had learned the formal part-whole definition of fractions and formal definition of fraction units, when explaining proper and improper fractions. For the other 10 fractional tasks used in the clinical interview, nine of them were selected from the instruments used in the quantitative phase (see Appendix C). The students in the pilot study had difficulty solving tasks involving circular models. Therefore, the selected tasks for each fraction scheme (i.e., PUFS, PFS, RPFS and IFS) included one linear and one circular model, in addition to four rectangular model tasks corresponding to the circular model tasks for each scheme (i.e., PUFS, PFS, PRFS, IFS). These rectangular model tasks were used as additional tasks for students who struggled with the circular model (see Appendix D). Furthermore, the PUFS, PFS, RPFS, and IFS tasks were purposely selected from the written assessment with the consideration of the language difference in these tasks. The tasks designed for these four schemes involve two different types of fractional language. One type of task required students to produce a proper fractional piece out of the given unpartitioned whole based on the given fractional name. The second type of task required students to determine the fractional name of the given fractional part according to the given unpartitioned whole. When examining the validity and reliability of their written assessment, Wilkins, Norton, and Boyce (2013) noticed that different fractional language may elicit students' use of schemes other than the expected schemes. For instance, some students might use their part-whole conception of fractions to solve the first type of tasks, namely finding the fractional piece of the referent unpartitioned whole from the given fractional name. Accordingly, each type of task was included in the interview protocol. For example, the linear model PUFS task asked participants to draw a fractional piece (e.g., 1/7) from the given unpartitioned whole (e.g., a stick). The circular model PUFS task asked participants to determine the fractional name based on the given fractional part (e.g., a smaller cake piece) and the given unpartitioned circle (e.g., whole cake).

For the PWS tasks, the linear model task was selected from the quantitative instrument for 4th grade. Because there were no circular model tasks in the quantitative instrument, the circular model for PWS used in the interview protocol was adopted from Olive and Vomvoridi (2006). The performance of a student who had not constructed the PWS in Olive and Vomvoridi's study indicated that this student could also have a weak understanding of the concept of fraction units. Therefore, they gave him a circle manipulative that was made up of five pieces and a missing piece. The student was asked to use a fraction to represent the missing piece. Although the student realized that the whole was partitioned into six pieces, he used one fifth to represent the missing piece. This response suggests that the student only used the number of pieces he actually saw to infer the whole, which was the denominator of the fraction. His mistake likely indicates the student's "lack of a part-to-whole relation with regard to [fraction units]" (p.27). Consequently, the same task was used in this study to discover if a student who has not constructed the mental disembedding operation or PWS may also have a weak understanding of the concept of fraction units.

Data Analysis

The main purpose of the qualitative phase is to gain insight into Chinese students' understanding of fraction units, and to investigate the role of the understanding of fraction units in the process of Chinese students' construction of fraction knowledge. The clinical interview with the think-aloud method is applied to collect the qualitative data and answer the last research question of this study.

The analysis of the Chinese transcriptions of students' language, observed gestures, and student's written responses involves a constant process of coding and comparing. The initial coding process started from coding each individual transcription deductively. It involved organizing and coding data based on tentative categories, such as scheme, operation, language, gesture, and jotting. Results from the previous pilot study indicate that not only does students' language provide information about their thinking process and reasoning behind the answers, but their gestures and jottings also provide abundant information about student' thinking and reasoning. For instance, a student tried to solve one of the PFS tasks that asked students to write a fraction representing the smaller stick out of the longer stick. She first used her pencil to measure the smaller stick. Then she put her fingers on the pencil as a mark and used it to measure the longer stick by drawing tick marks on the longer stick along her finger mark. But there was a little piece left which was shorter than the smaller stick. She then struggled with the little left out piece. Her gestures and her drawing showed that instead of finding the fraction unit of the whole, she treated the smaller stick as the fraction unit of the whole.

Another example demonstrates the inconsistency between student's language and drawing. When solving a PFS task, one student in the pilot study tried to find $\frac{4}{5}$ of a given half pizza. Theoretically, he should partition the half pizza into five equal shares through drawing four radii, and shade four of the shares. However, although this student verbally said "One share. Another one share. One share again …," his drawing indicated that rather than partitioning the half circle from the center point, he partitioned the half circle through drawing a group of parallel lines perpendicular to the diameter. It seems as if he tried to assimilate the circle model to a linear model. At the same time, he actually drew three vertical lines, but not four lines to partition the diameter. He only partitioned the half circle into four pieces, which was the same number as the numerator of the given fraction. In this situation, his drawing provides more information than his language. Consequently, the Chinese transcriptions were deductively coded with descriptive words and phrases based on three tentative categories, language, gesture, and jotting. The initial tentative codes under these categories include the name of each scheme and operation, as well as terms such as taking out, divide, evenly, iterable unit, partitioning, measure with fingers or pencil.

After the initial deductive coding process, all transcriptions were inductively coded with developing patterns, new categories, and themes. All data were analyzed and compared systematically. In order words, students' responses for the same task were compared within grade and across grades in order to develop the theoretical properties for each category or create new categories, identify patterns in the data, and reduce unrelated data. For example, after 4th grade participants explained the meaning of two fractions, one proper fraction 3/5 and the other improper fraction 5/3, a new pattern was revealed, that is, the part-part relation. Thus, the new code part-part concept of fractions (PPCF) was created under the category scheme. Through subsequent coding and comparing, other categories were developed to label particular explanations or actions that explain multiple forms of reasoning that students use at different stages of fraction understanding. For instance, it was found that the referent unit whole for some

4th grade participants was a discrete whole, rather than a continuous whole. Consequently, a new category related to the nature of the whole (i.e., whether discrete or continuous) was created. Finally, relationships between themes were determined by reflecting on the sequential aspects of the categories. These relationships drawn from fraction unit understandings informed theme development and explained the particular reasoning that students use. As new descriptive codes were developed, data were revisited. The systematical comparison coding process provides researchers iterative forms of data analysis to verify the importance of particular codes when explaining the development of the understanding of fraction units and how this understanding facilitates the construction of fraction knowledge.

Chapter Four: Results

This study applied an explanatory sequential mixed method design to discover how Chinese 4th and 5th grade students' conceptual understanding of the fraction unit relates to their construction of fraction knowledge. Accordingly, the first section focuses on the results from quantitative data analyses, followed in the second section with the results from analyzing qualitative data.

Quantitative Results

Comparing Students' Schemes and Operations within Grade and across Grades

The explanatory conceptual framework for this study suggests that children progress through the construction of fractional schemes and operations as discussed in Chapter 2 (Steffe & Olive, 2010). Furthermore, Norton, Wilkins, and Xu (2018) demonstrated that 5th and 6th grade Chinese students progressed with a similar learning progression of fraction schemes as their U.S. counterparts. Different from the students in the Norton et. al. (2018) study, the Chinese 4th and 5th grade participants in this study attend an elementary school which operates on a 5-4 system, namely, 5 years elementary school and 4 years of middle school. Therefore, it is worth examining the participants in this study to see if they too progress with a similar learning trajectory as documented previously before analyzing the quantitative data in detail to answer the research questions. Table 4-1 provides the descriptive statistics associated with students' construction of the fraction schemes and the splitting operation both within grade and across grade. Two different patterns can be observed. First, when inspecting the results in each column (by grade), it is notable that the proportion of students who had constructed the PWS, PUFS, and PFS in 4th grade and the proportion of students who had constructed the PUFS, PFS, RPFS, and IFS in fifth grade decreases by scheme. The decrease in proportion in each grade suggests that the construction of fraction schemes progresses developmentally from PWS to IFS. That is, if a student has not constructed a PWS, then this student is unlikely to construct a PUFS, and also would not be able to complete the tasks relating to the other more advanced fraction schemes. Hence, the decreasing proportions of the students who have constructed each of the schemes may reveal that the construction of fraction schemes of the participants follows a hierarchical progression, from the basic scheme PWS to PUFS, PFS, RPFS, and then IFS. Moreover, the proportion of the students constructing the splitting operation (SO) are also far greater than the frequency and proportion of students who had constructed the PFS and RPFS in both grades, and IFS in 5th grade. This may also indicate that these participants constructed the splitting operation before they constructed the more advanced fractional schemes.

To verify the schemes hierarchy in this study, a Cochran's Q test was conducted to statistically test for differences in the proportion of students constructing each scheme and the splitting operation. The Cochran's Q test is an omnibus test used to test for any overall differences between multiple measures. First, considering students in Grade 4, results from this test indicated that there were statistically significant within-grade differences between the proportions of students' construction of the different schemes (i.e., PWS, PUFS, PFS, and RPFS), $\chi^2(3, N = 254) = 403.84, p < .05$. Similarly, in fifth grade, there were statistically significant within-grade differences between the proportions of students constructing schemes (i.e., PUFS, PFS, RPFS, and IFS), $\chi^2(3, N = 252) = 101.82, p < .05$. There also exists a statistically significant difference between the proportions for the PUFS, PFS, and RPFS when considering all students, $\chi^2(3, N = 506) = 100.48, p < .05$. The results of the Cochran's Q test only indicate whether there exists a difference. In order to detect where the difference exists it is necessary to conduct a post-hoc analysis of the different pairwise comparisons. Dunn's test with a Bonferonni adjustment was used to test the different pairwise comparisons. In fourth grade, the proportions of students constructing schemes were found to be statistically different for the pairs PWS-PUFS, PWS-PFS, and PWS-RPFS (see Appendix F). For fifth grade, there are statistically significant differences between the pairs PUFS-PFS, PUFS-RPFS, and PUFS-IFS (see Appendix G). In both grades the difference between PFS and RPFS was not statistically significant. The results provide evidence that the trajectory of fraction schemes development of these Chinese participants follows the progression identified by Steffe and Olive (2010; also see Norton & Wilkins, 2012; Wilkins & Norton, 2011, 2018) in their studies of U.S. students. Results from these analyses are presented in Table 4-1 (pairwise comparisons with different subscripts represent statistically significant differences).

Table 4 - 1

The Descriptive Statistics and Comparison of Schemes and Operation within Grade and across Grades

Scheme	4th Grade (N = 254)		5th Grade (N = 252)		4th & 5th Grades (N = 506)	
	f	%	f	%	f	%
PWS	169	66.5_a	–	–	–	–
PUFS	31	12.2_b	60	23.8_a	91	18.0_a
PFS	6	2.4_c	18	7.1_b	24	4.7_b
RPFS	8	3.1_{bc}	12	4.8_{bc}	20	4.0_{bc}
IFS	–	–	8	3.2_{bc}	–	–
SO	26	10.2	46	18.3	72	14.2

Note. % with different subscripts are statistically different ($p < .05$).

Next, the frequency and proportion of each particular scheme and operation were compared across grades 4 and 5 (Table 4-1). Recall, that 4^{th} grade students in this study are not introduced to the formal notion of the fraction unit until later in fourth grade (after the administration of the study surveys), thus comparing the performance of 4th and 5th graders may shed light on the influence of the curriculum on students' construction of their fraction knowledge. In order to examine the difference in the proportions for each of the particular schemes from 4th to 5th grade, a chi-square test was conducted (see Table 4-2). The chi-square test shows that there is a statistically significant association between the development of fraction knowledge and grade: PUFS $\chi^2(1, N = 506) = 11.55$, $p < .05$; PFS, $\chi^2(1) = 6.399$, $p < .05$; and SO, $\chi^2(1) = 6.663$, $p < .05$. In other words, the results provide evidence to indicate that, on average, students' fraction knowledge changes from 4th grade to 5th grade.

Table 4 - 2

The Results of Chi-square Test

Scheme	4th Grade		5th Grade		*P*-value	χ^2	*df*
	f	%	*f*	%			
PUFS	31	.12	60	.23	.001	11.55	1
PFS	6	.02	18	.07	.011	6.99	1
RPFS	8	.03	12	.04	.352	0.866	1
SO	26	.10	46	.18	.01	6.663	1

Comparing Students' Performances for Different Models

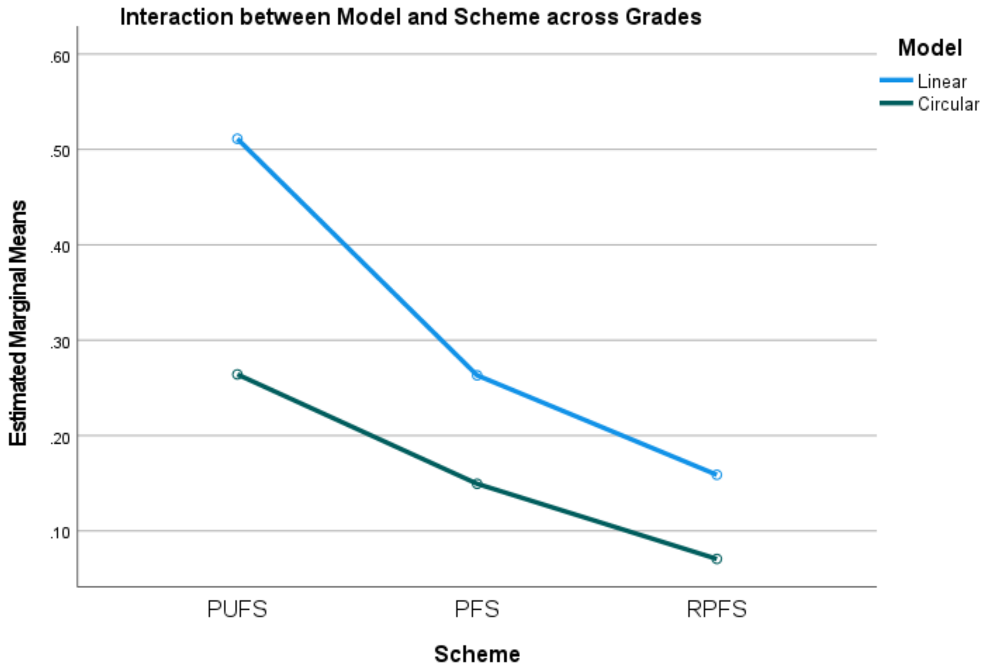
One interesting finding revealed from the prior pilot study was that the circular model was more challenging to work with than the linear model for most Chinese participants. Thus, one of the research questions of this study asks whether there are performance differences related to different models used for assessing students' construction of schemes and operations. That is, whether there are differences in performance related to linear, circular, or rectangular models, and which one of these three models is more or less problematic for Chinese students. For this purpose, the researcher designed rectangular model tasks corresponding to one of the circular model tasks for each of the fraction schemes and the splitting operation, and then compared students' performance on the three different models: linear, circular and rectangular. To examine the performance differences between different models across schemes and grade, three different ANOVAs were conducted. These analyses included a three-way mixed ANOVA, a one-way repeated measure ANOVA, and a two-way mixed ANOVA. The independent variables for these analyses include Grade, Model, and Scheme. Among them, the independent variable Grade had two levels: 4^{th} and 5^{th} grade. The independent variable Model was compared under two different conditions. The first condition is to compare students' performance on the linear and circular model tasks. The second condition is to compare students' performance on the circular and its corresponding rectangular model task. Two different conditions were used because, if you recall, the variables for the different models were created differently and are not directly comparable across all three models. The independent variable Scheme will have different levels for the different analyses. The details will be discussed for each separate analysis.

In order to investigate whether there is a grade effect associated with student performances when solving different models of fractional tasks for schemes (i.e., PUFS, PFS, and RPFS), a $3 \times 2 \times 2$ three-way mixed ANOVA with two within-student factors and one between-student factor was conducted under two different conditions. The first is to test the grade effect on students' performance when solving linear model and circular model tasks. Thus, the two within-factor independent variables were Scheme (i.e., PUFS, PFS, and RPFS), and Model (i.e., linear and circular model). The one between-subjects independent variable is Grade (i.e., 4^{th} and 5^{th} grade). The dependent variable was the scores from tasks using linear and circular models. Then the same mixed ANOVA was conducted again, but the levels of the within-subjects factor variable models changed to circular and its corresponding rectangular model tasks.

Table 4-3 presents the mean scores by model and grade. A visual examination of these means reveals apparent differences between the mean scores across model, scheme, and grade. The first $3 \times 2 \times 2$ three-way mixed ANOVA analysis comparing linear and circular models revealed a statistically significant main effect associated with Model, $F(1, 504) = 386.33$, $p < .05$, meaning that when ignoring Scheme and Grade, students' performance on linear model tasks was, overall, higher and significantly different from the performance on the circular model tasks. There was also a statistically significant main effect associated with Scheme, $F(1.89, 949.94) = 439.63$, $p < .05$, meaning that regardless of grade and the type of models, participants performed differently on tasks associated with different schemes. Finally, there was a statistically significant main effect associated with Grade $F(1, 504) = 26.76$, $p < .05$, $\eta^2 = .05$, indicating that, on average, grade 5 students performed at a higher level than grade 4 students. Before we can interpret these main effects, it is necessary to check for interaction effects. The $3 \times 2 \times 2$ three-way mixed ANOVA analysis also revealed a statistically significant three-way interaction between scheme, model, and grade (i.e., Scheme*Model*Grade), $F(1.97, 993.00) = 3.196$, $p = .042 < .05$. However, the effect size, $\eta^2 = .006$, associated with this 3-way interaction indicates that the effect is weak. This suggests that although the 3-way interaction was statistically significant, that it may not represent a practical significance. An examination of the 3-way interaction plots (Figure 4.1) provides further information for helping interpret the results. Although the plots reveal slightly different patterns suggesting the interaction, it is clear that students performed higher on tasks with linear models than those using circular models irrespective of Grade or Scheme. But what is also revealed by the plots is that for PUFS the performance on linear and circular models in both grades appeared to be different. Results from a post-hoc simple main effects test, pairwise comparison test for Scheme*Model across grades, confirmed that the difference between the linear model and circular model tasks was statistically significant for each scheme ($p < .001$).

Figure 4. 1

The Plots of Three-Way Interaction between Model and Scheme across Grades



Note. The plot for Three-way interaction between Model (linear and circular model tasks) and Scheme (PUDA, PFS, and RPFS) and Grades.

Table 4 - 3

The Mean Scores of Different Models Across Different Schemes by Grades

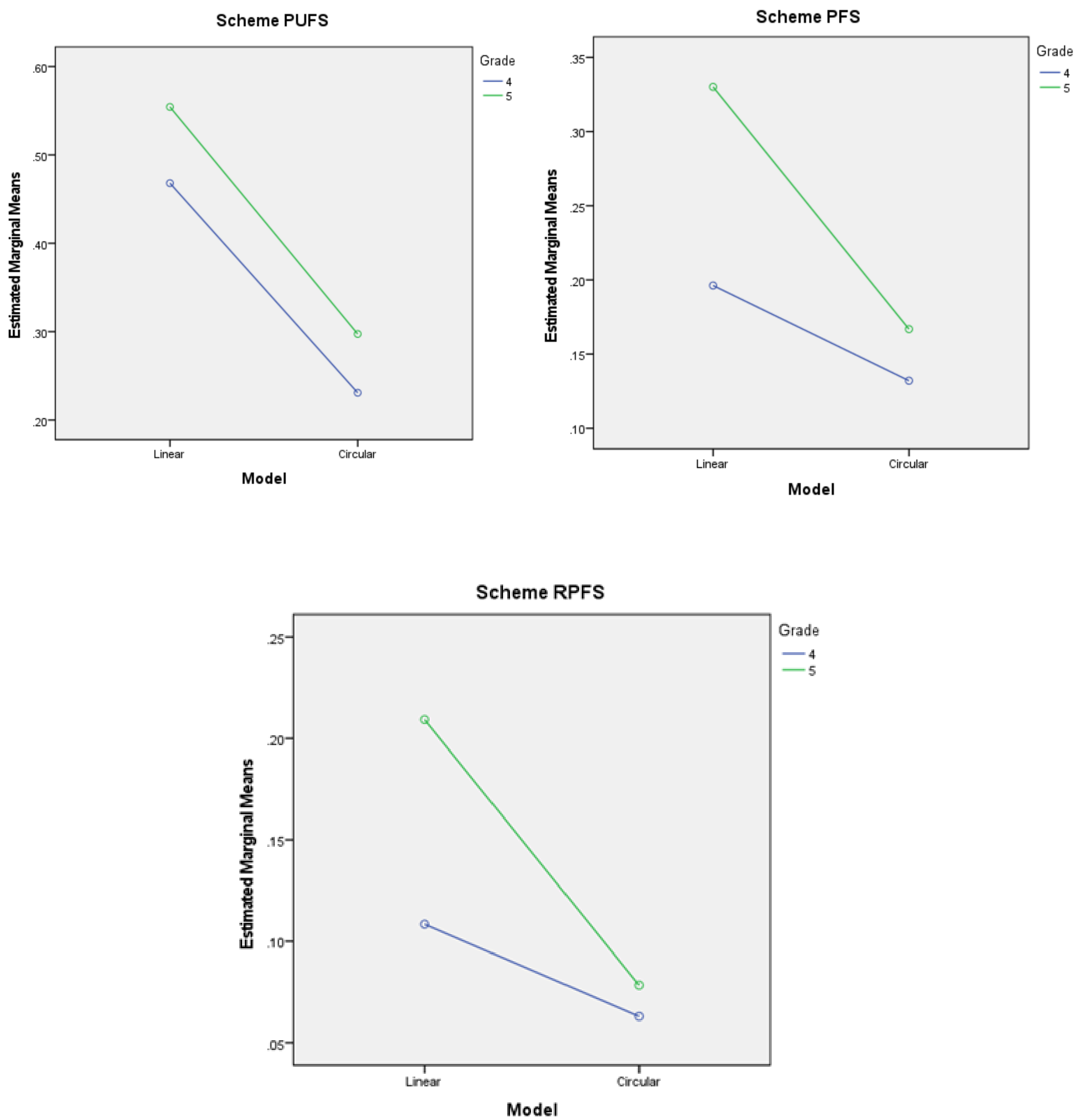
Grade	4th Grade				5th Grade			
Schemes	Linear	Circular	Rectangle	Circular Corresponding to Rectangular	Linear	Circular	Rectangle	Circular Corresponding to Rectangular
PUFS	.4681	.2309	.5701	.3374	.5544	.2974	.6159	.3714
PFS	.1963	.1321	.0933	.1610	.3302	.1669	.1393	.2131
RPFS	.1085	.0630	.0476	.0500	.2093	.0784	.0714	.0595
IFS					.1872	.0808	.1115	.0504
SO	.1581	.1565	.1516	.1327	.2853	.2131	.2135	.1984

When looking at the two-way interaction, it was found that the result of the two-way interaction between Model and Grade was statistically significant, $F(1, 504) = 20.00$, $p < .05$, indicating that there was a Grade effect associated with students' performance on different models. Figure 4.2 presents the plots for the two-way interactions between Model and Grade for each scheme. The plots illustrate that the performance on linear model tasks of all schemes for 5th grade students was consistently higher than that of 4th grade students, and students in both grades had similar lower performances on circular model tasks. Although there is a statistically significant interaction, the pattern of performance related to model is relatively consistent.

The two-way interaction between Scheme and Model for students in both grades was also statistically significant, $F(1.970, 993.0) = 51.64$, $p < .05$. Figure 4.3 shows the plots for the two-way interaction between Model and Scheme. The graph illustrates two phenomena. First, regardless of grade, on average, students had higher performance on linear models than on circular models. Second, for linear model, on average, students had better performance on PUFS than the other two schemes. Although there were statistically significant interaction effects found, the plots reveal consistent patterns of performance associated with the different models, that is, consistent with the main effect, student performance on linear model tasks appears to be higher than performance on circular model tasks. Results from the pairwise comparison test using a Bonferroni correction does show a statistically significant overall difference in students' performance between the linear model (M = .311, SD = .009) and the circular model (M = .161, SD = .007) across schemes PUFS, PFS, and RPFS.

Figure 4. 2

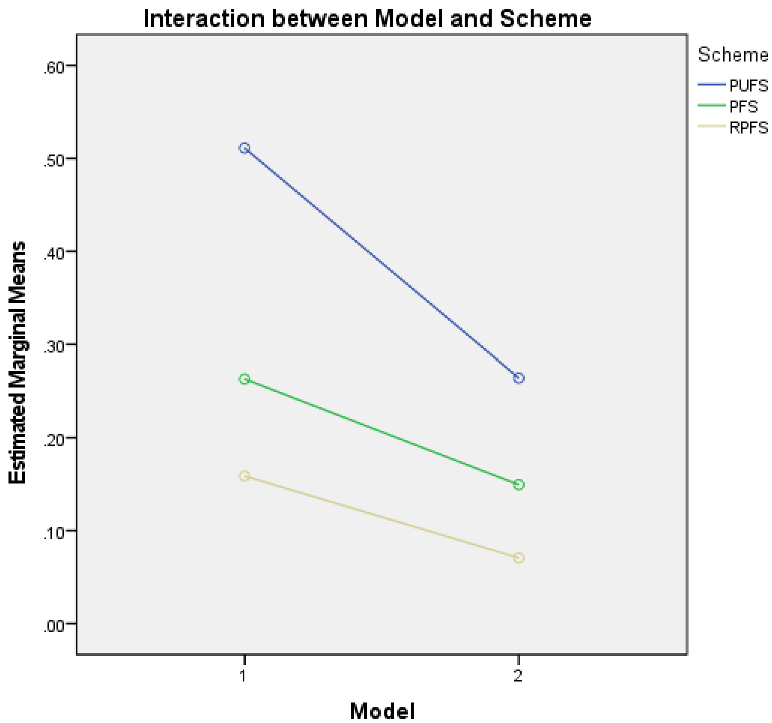
The Plots of Two-Way Interaction between Model and Grade across Schemes



Note. The plot for two-way interaction between Model (linear and circular model tasks) and Grade across schemes.

Figure 4. 3

The Plot for Two-Way Interaction between Model and Scheme across Grade



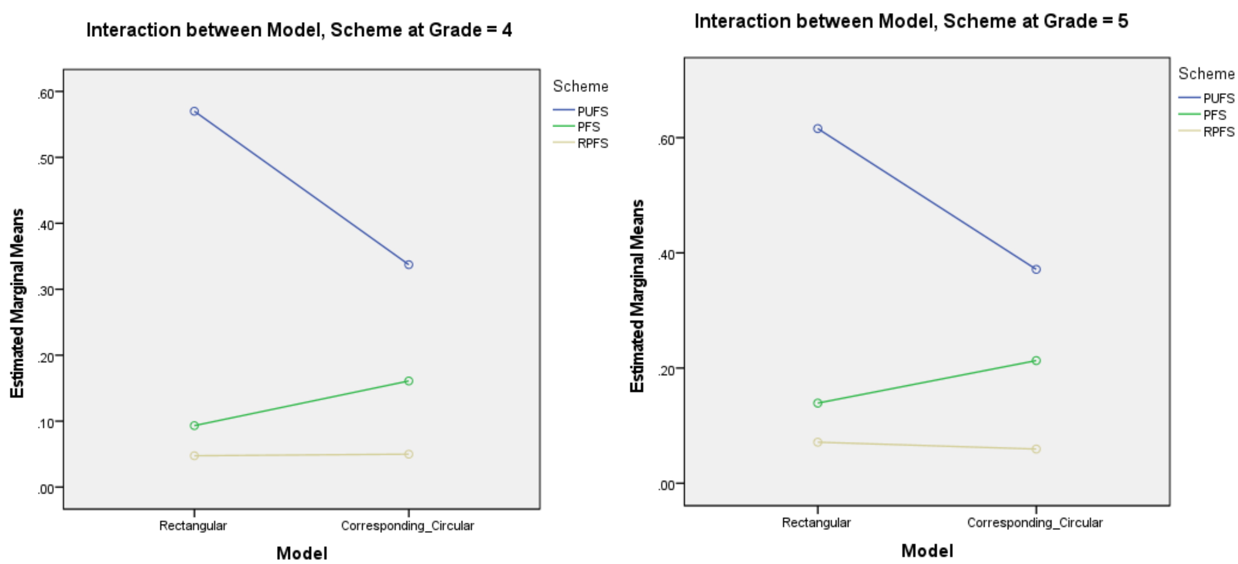
Note. The plot for two-way interaction between Model (linear and circular model tasks) and Scheme across grades.

Next, a 3 × 2 × 2 three-way mixed ANOVA was conducted to investigate the grade effect on students' performance when solving the fractional tasks relating to the circular model task and its corresponding rectangular model task. There was a statistically significant main effect associated with scheme, $F(1.59, 802.32) = 479.3$, $p < .05$, meaning that regardless of Grade and type of model, participants performed differently on different schemes. There was also a statistically significant main effect associated with Model, $F(1, 504) = 27.89$, $p < .05$, meaning that when ignoring Scheme and Grade, students' performance on tasks involving a circular model was significantly different from the performance on the corresponding rectangular model tasks. Finally, there was a statistically significant main effect associated with Grade *F*(1,

504) = 5.54, $p < .05$, $\eta^2 = .02$, indicating that, on average, Grade 5 students performed at a higher level than Grade 4 students. The three-way interaction between model, scheme, and grade was not significant, $F(2, 1008) = .09$, $p = .869$. Figure 4.4 illustrates the effect of grade on students' performance between circular and its corresponding rectangular model. Consistent with the non-significant interaction effect, the almost identical plots indicate that regardless of grade, students' performances had no significant difference when the representations of the fractional tasks relating to different schemes were area model (i.e., circular and rectangular models in this study).

Figure 4. 4

The Plot for Three-Way Interaction between Model, Scheme, and Grade



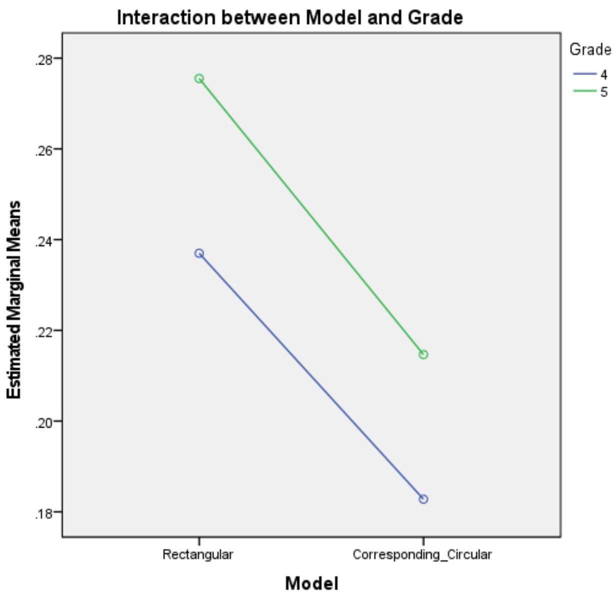
4(a) 4(b)

Note. 4(a) presents three-way interaction between model, scheme, and grade for 4th grade. 4(b) presents three-way interaction between model, scheme, and grade for 5th grade. The variables of Model were circular and its corresponding rectangular model.

The two-way interaction between Model and Grade was also not statistically significant, $F(1, 504) = .093$, $p = .76$. Figure 4.5 presents the plot of the two-way interaction between Model and Grade. The two parallel lines illustrate that the interaction between Model and Grade was not statistically significant.

Figure 4. 5

A Plot for Two-way Interaction between Model and Grade

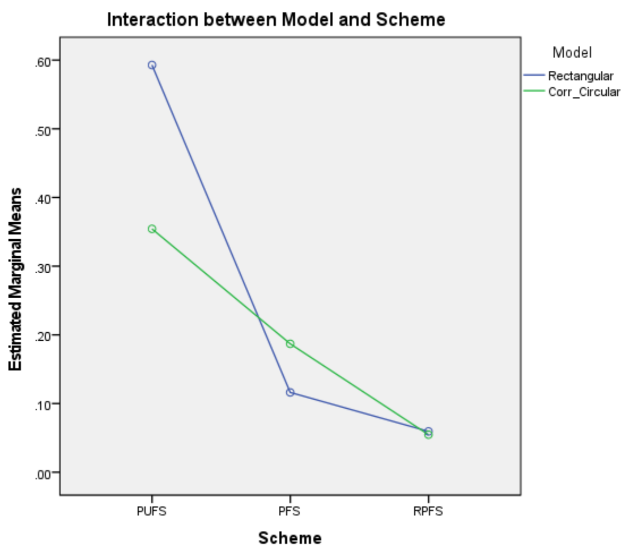


Note. The plot for two-way interaction between Model (circular model task and its corresponding rectangular model task) and Grade.

However, there was a significant interaction between Model and Scheme, $F(1.58, 797.08) = 77.67$, $p < .05$. This effect reveals that students' performance was different between circular tasks and its corresponding rectangular model task across the three schemes, regardless of grade. Figure 4.6 illustrates a clear interaction between students' performance on circular and its corresponding rectangular model tasks and the three different schemes. It shows that for PUFS, on average, students had higher performance on the rectangular model task than on its corresponding circular model tasks. However, it appears that for PFS tasks, on average, the performance of students on the circular model task was higher than the performance on the corresponding rectangular model task. At the same time, the plot also suggests that there was no difference between students' performance regardless of model for RPFS tasks. To examine these two particular features shown on Figure 4.6, three post-hoc simple effects tests were conducted. The results revealed that for PUFS there was a significant difference ($p < 0.01$) between students' performance on circular model and its corresponding rectangular model in favor of the rectangular model. For PFS tasks, there was a significant difference ($p < 0.05$) between students' performance on circular model and its corresponding rectangular model in favor of the circular model task. The results of the post-hoc test also demonstrated that for RPFS, students' performance on the circular model task and its corresponding rectangular model task was not significantly different ($p = .616$). The overlapping points on Figure 4.6 illustrate this feature.

Figure 4. 6

A Plot of Two-Way Interaction between Models and Scheme



Note. The plot of two-way interaction between Model and Scheme for circular model task and its corresponding rectangular model task.

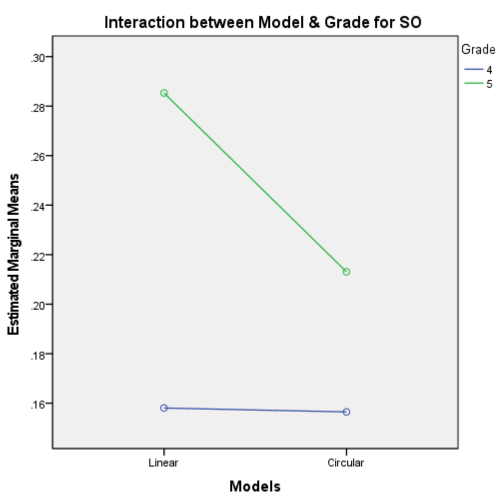
A one-way repeated measure ANOVA was conducted to analyze the mean score differences between different models for IFS (see Table 4-3). Since the fractional tasks related to IFS were only included in the 5th grade survey, the one-way repeated measures ANOVA was used to compare 5th grade students' performance between different models. The performance difference between linear and circular model tasks was analyzed first. The results of the analysis indicate that the difference in the mean scores across models was statistically significant, $F(1, 251) = 36.91$, $p < .05$. Referring to Table 4-3, this difference in student performance reflects a higher performance on the linear model task than the circular model task. Next, a repeated measures ANOVA was used again to compare the mean scores between circular and its corresponding rectangular model task. The analysis reveals that the difference in the mean scores across models was statistically significant, $F(1, 251) = 15.32$, $p < .05$; students' performance on rectangular model tasks was higher than the performance on the corresponding circular model (see Table 4-3).

To investigate students' performance on the different models for the splitting operation (SO), a 2×2 two-way mixed ANOVA with Model as a within-subjects independent variable and Grade as a between-subjects independent variable was conducted to determine the grade effect on students' performance between linear and circular models, and between circular and its corresponding rectangular model task. The analysis of this two-way mixed ANOVA for SO reveals a statistically significant main effect associated with Model, $F(1, 504) = 7.69$, $p < .05$, meaning that students' performance was significantly different between linear model and circular model regardless of grade. There was also a statistically significant main effect associated with Grade $F(1, 504) = 14.13$, $p < .05$, $\eta^2 = .03$, indicating that grade 5 students performed at a higher level than grade 4 students (see Table 4-3).

The two-way interaction between Model and Grade was statistically significant, $F(1, 504) = 7.043$, $p = .008$. The plot of the interaction is presented in Figure 4.7. From the plot we see that consistent with the significant main effect associated with Grade that students in 5th grade performed better than students in 4th grade regardless of the model. However, when considering model, it appears that in fourth grade there was no difference in performance by model, whereas in fifth grade it appears that there was a difference in performance by model with higher performance associated with the linear model. A paired samples test was conducted and the results indicated that the mean performances of 4th grade students on linear model and circular model were not significantly different ($t(253) = .092$, $p = .927$). This can be seen by the lack of any slope for the 4th grade line in Figure 4.7. The same test was conducted to compare the mean scores between linear and circular model tasks of 5th grade. The results indicated the mean performances of 5th grade students on linear and circular model tasks were significantly different ($t(252) = 3.53$, $p < .001$) in favor of the linear model.

Figure 4. 7

The Plot for Two-Way Interaction between Model and Grade for SO



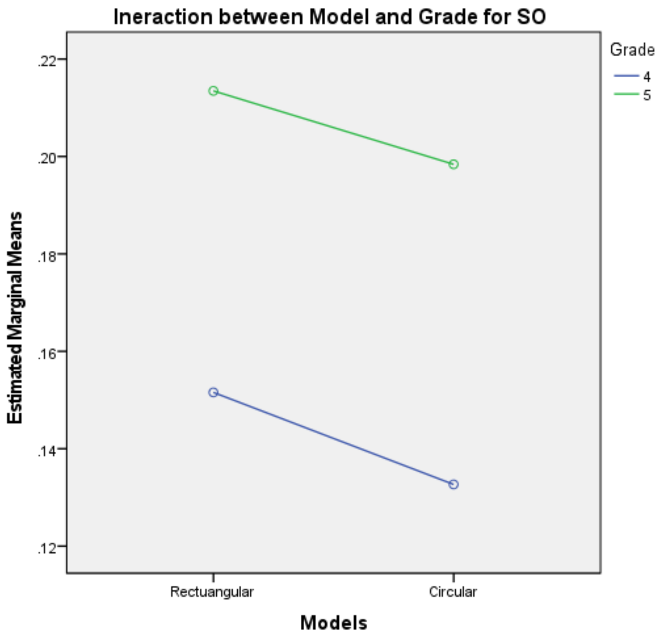
Note. The plot of two-way interaction between Model and Grade of SO for linear versus circular model tasks.

The analysis of students' performance for the SO tasks between circular and its rectangular model reveals that the difference in performance between circular model task and its corresponding rectangular model was not statistically different, $F(1, 504) = 1.241$, $p = .266$.

There was also no interaction effect by Grade, $F(1, 504) = .016$, $p = .90$. However, there was a statistically significant main effect associated with Grade, $F(1, 504) = 6.43$, $p = .012$, indicating, on average, that fifth graders outperformed fourth graders on the splitting tasks (see Figure 4.8).

Figure 4. 8

The Plot for Two-Way Interaction between Model and Grade for SO



Note. The plot of two-way interaction between Model and Grade of SO for circular model task and its corresponding rectangular model task.

Qualitative Results

Qualitative data for this study were collected through task-based interviews involving 10 fourth grade students and 19 fifth grade students. In order to investigate the differences in explaining proper and improper fractions within and across grades, all participants were first tasked with reading aloud and explaining two fractions: a proper fraction, 3/5, and an improper fraction, 5/3. After completing this task, each participant, based on the original clinical interview plan, would finish at least six fractional scheme tasks, two different models' tasks of their highest available scheme as well as tasks of two adjacent schemes, above and below. However, after comparing and contrasting the data from two 4^{th} and two 5^{th} grade participants, to gain more information about Chinese students' understandings of a unit whole and fraction units, the researcher decided that all of the rest participants would start from PWS tasks and stop at the task that the participant struggled with. After collecting all interview data from 29 participants, these qualitative data were analyzed through the process of continuously comparing and contrasting codes, categories, and themes within grade and across grades.

The goal of the qualitative data analyses is to discover the role of conceptually understanding of unit wholes and fraction units in the progression of Chinese students' construction of fraction knowledge. Thus, the development of the data codes began with the category, fraction unit. Analyzing and comparing all interview data systematically revealed that the understanding of a unit whole also played a critical role in the construction of fraction knowledge. Therefore, in this phase, findings of the relationship between students' explanation of proper and improper fractions and the unit whole and fraction units are presented first. Then students' performances on fraction schemes PWS, PUFS, PFS, RPFS and IFS, and operation SO along with the roles of unit whole and fraction units are presented next.

Verbalizing and Explaining the Given Proper and Improper Fractions

The first fraction presented was a proper fraction, 3/5, followed by an improper fraction, 5/3. The goal of explaining both proper and improper fractions was twofold. The first goal was to examine if differences in explaining proper fractions and improper fractions exists within the same grade and across grades. In particular, a student's ability to explain an improper fraction helps investigate whether 5th grade participants were able to apply the measurement concept rather than using a part-whole concept in their explanation after learning the formal definition of fractions and fraction units. Furthermore, the improper fraction was elaborately designed as the inverse of the proper fraction, with the intention of investigating whether the 4th grade participants were able to differentiate between these two fractions.

Performance of the 4th Grade Participants

Among 10 4th grade participants, two students' written assessments indicated that they have not yet established the PWS. During interviews, these two students (20%) incorrectly read the presented proper fraction, 3/5. In China, when verbalizing a fraction, the denominator is read first followed by the numerator. However, instead of saying "*五分之三*", they read the numerator first and then the denominator, *三分之五*, which literally means the unit whole is divided into three parts and take five parts out. It seems that their performances were consistent with their written assessments, not even having the PWS yet. After asking these participants to point out the denominator and the numerator of each fraction, they then correctly verbalized the proper fraction, 3/5. The same phenomenon happened again when the two of them verbalized and explained the improper fraction, 5/3. Their initial responses may suggest that they may not establish the concept of fractions yet. Thus, when they read a fraction in words, they used the common order, from top to the bottom.

The other eight 4th grade participants (80%) could easily and quickly read both fractions. However, only one participant (10%) could correctly explain the proper fraction, saying that 3/5 means "把整体分成五份*取*它的三份," meaning that a whole is divided into five shares and *takes* three shares. Specifically, her explanation included the unit whole and parts. Thus, she used a part-whole definition of fractions. When asked to explain the improper fraction 5/3, she said she did not know the meaning of 5/3, because there are not five shares, but only three shares. Her statement demonstrates that the part-whole concept of fractions hinders her ability to understand the improper fraction.

Two out of these eight students (25%) gave a partially correct explanation. They explained the proper fraction 3/5 in the same way. That is "五份*中*的三份", meaning three shares *out of* five shares. Noticeably, they did not mention the unit whole in their explanation, which is the main difference between the explanations of these two students and the previous one. Their explanation indicated that these two students also had a part-whole concept of fractions. Therefore, when they encountered an improper fraction such as 5/3, both of them did not know how to explain it because obviously the whole is less than the part. Another three out of eight students (37.5%) gave similar wrong explanations, "五个分成三份" or "五份东西切成三份." Literally, both explanations mean "five things are divided into three shares." Their explanations reveal that they did not even construct the part-whole concept of fractions although their explanations look like a part-whole relationship. This plausible part-whole concept of fraction will be referred to as a part-part concept of fractions (PPCF) in this study. PPCF shows that their concept of a unit whole was ambiguous. Therefore, it may imply that a student's understanding of a unit whole may also play an important role in the progression of constructing fraction knowledge.

One of the remaining two students (25%) did not know how to explain either fraction, whereas the other one used an interesting way to explain both fractions. For the proper fraction 3/5, he said there are five thirds. Although he surprisingly used a multiplicative structure and fraction units, he mistakenly switched the numerator and denominator of each fraction. He also used his plausible measurement concept of fractions to explain the improper fraction 5/3 as three fifths. When asked what the 5 meant and what the 3 meant in the proper fraction 3/5, he said 5 is whole number and 3 is not a whole number. His explanation may suggest that his concept of the unit whole was not clear. After asking more follow-up questions to understand his thinking, the researcher finally realized that he thought 5 meant a whole and 3 meant fractional parts of the whole. It could be assumed that he might know the unit whole contains five parts and take three shares out. Due to the limitation of his fractional language, he was not able to describe his thoughts clearly. Table 4-4 summarizes the 4^{th} grade participants' performance with verbalizing and explaining the proper fraction 3/5 and the improper fraction 5/3.

Table 4 - 4

4th Grade Participants Reading and Explaining Fractions

Reading fractions 3/5 and 5/3 aloud, respectively		*N*	%
Incorrectly		2	20
Correctly		8	80
Explanation of Fractions		*N*	%
	Part-Whole Concept of Fractions	1	10
Proper	Partial Part-Whole Concept of Fractions	2	20
Fraction 3/5	Part-Part Concept of Fractions	3	30
	Plausible Measurement Concept of Fractions	1	10
	Don't know	3	30
Improper	Part-Part Concept of Fraction	3	30
Fraction 5/3	Plausible Measurement Concept of Fraction	1	10
	Don't know	6	60

Performance of 5th Grade Participants

Nineteen 5th grade students participated in the clinical interview. Interview data for explaining proper and improper fractions from one of the participants was lost due to the unstable internet signal. Therefore, analysis of the data on verbalizing and explaining fractions is based on the data from 18 participants.

All 18 participants were able to easily and quickly verbalize each fraction correctly. When asked to explain the proper fraction 3/5, only 1 out of the 18 participants (5.6%) used the measurement concept of fractions and a multiplicative structure in her explanation. She said 3/5 is “三个五分之一,” literally meaning three one-fifths. When asked if she knew what one-fifth was, she answered that it was the fraction unit. She also knew that five one-fifths made a unit whole. When she explained the improper fraction 5/3, instead of using fraction units, she converted it to a mixed number. The following protocol presents how she explained the improper fraction 5/3. R represents the researcher and Y represents this student named Yi.

R: What does five-thirds mean?

Y: Five-fifth, um … five-thirds can be changed to a mixed number, then it is one and … (she was doing mental calculation), one and two-thirds.

R: Continue.

Y: This represents that one and two-thirds make up this fraction.

R: Then what is the one?

Y: One is the unit whole.

R: So that means in five-thirds there is a unit whole and …

Y: One two-third ….

R: So then how many one-thirds are there in two-thirds?

Y: There are two one-thirds.

R: Two one-thirds and [she tried to say something] continue.

Y: Also it can be said there is a unit whole and two one-thirds make up this fraction.

R: How many one-thirds in the unit whole?

Y: In a unit whole there is three one-thirds. Totally five one-thirds make up the fraction five-thirds.

Although this student did not use a measurement concept to explain the improper fraction 5/3, her explanation indicated that she understood 5/3 as a fraction that is more than a unit whole.

The other 17 participants (94.4%) used a part-whole concept of fraction to explain the proper fraction, 3/5, and the improper fraction, 5/3. When explaining the proper fraction 3/5, their explanations were the same or similar as “把单位一平均分成五份，*取*其中的三份,” meaning that a unit whole is divided into five shares, and take out three shares. The slight difference between these explanations was the phrase related to the unit whole. The different wordings included a thing, a cake, one, or unit whole. Yet, they all used “取” or “取出” in the second part of the sentence. “取” or “取出” literally means “take out,” which implied a part-whole relationship. When explaining the improper fraction 5/3, nine out of these 17 5th grade participants (52.9%) still used the part-whole concept in their explanation directly. They described the improper fraction 5/3 as “把单位一平均分成三份，*取其中的五份*,” literally meaning that a unit whole is evenly divided into three shares and then take out five shares. Among these nine participants, eight of them were not bothered by saying “take 5 shares out from 3 shares” although the researcher attempted to perturb their thinking. For example, the researcher tried to perturb one of the participants whose written assessment showed that he had established a RPFS. After he explained the meaning of 5/3 using a part-whole relationship, the researcher inquired how he took five shares out from three shares. He thought for a few seconds and said “divide two into three shares.” The researcher then asked him what the “two” meant. He thought for a longer time and said he did not know. Only one of these nine participants was perturbed by her own explanation. After she said “divide a unit whole into …,” she stopped and said “that is not right.” She then kept shaking her head while murmuring. Her action suggests that she may have experienced a perturbation. Yet this cognitive conflict did not drive her to adjust her part-whole concept of the fraction when the researcher encouraged her to have a try. She still used the part-whole relationship to hesitantly explain 5/3 on her second attempt.

With a part-whole concept of fractions in mind, five of the 17 participants (29%) did not know how to explain the improper fraction 5/3. After the researcher asked them to point out the fraction unit of the fraction 5/3, three students easily named the fraction unit 1/3, yet could not use the fraction unit to explain this fraction. Only one of these five students could use the fraction unit and explained 5/3 as five of one-third,

The explanations from three of the 17 participants (17.6%) could be considered as a quasi-measurement concept of fractions. One student explained 5/3 as "1 里面平均分成三份，然后再把这个*分数单位乘以* 5," literately meaning that the unit whole is divided into three shares, then *multiply this fraction unit* by five. When asked why multiply by five, he said because the numerator is greater than denominator. Although he used a part-whole relationship to explain the proper fraction, his reasoning about the improper fraction 5/3 indicates that he was able to go beyond the part-whole relationship and relate the improper fraction with the fraction units. Yet, he did not understand the true meaning of improper fractions.

Another student's explanation regarding 5/3 was "把两个物体平均分成三份，取其中的五份," literally meaning that two items are divided into three shares and take out five shares. When asked why there are two items, he said that was because this fraction is a mixed number. When asked what the fraction unit of this fraction is and how many fraction units are in this fraction, he said there were five thirds in the fraction 5/3. It is reasonable to guess that the two items in his explanation meant two unit wholes. Therefore, 5/3 means in considering two items, each of them was divided into 3 shares evenly. Then a total of five shares were taken out. Consequently, the reasoning from the above two students indicates that they might be in the transition from a part-whole concept to a measurement concept of fractions.

The explanation provided by the remaining student also suggests that this participant might be in transition from a part-whole conception to a measurement conception of fraction. He explained 5/3 using a mixed number. He said 5/3 represents “单位 1 里面还多出两份，多出三分之二.” The literal translation of this explanation is “two shares more than a whole, two thirds more.” His explanation indicates that he mentally converted the fraction 5/3 into a mixed number one and two-thirds. He knew that the one represents a unit whole. Table 4-5 summarizes 5th grade participants’ performance in explaining both proper fraction and improper fractions.

Table 4 - 5

Verbalization and Explanations of Fractions from 18[6] 5th Grade Participants

Verbalization of fractions 3/5 and 5/3, respectively		*N*	%
Incorrectly		0	0
Correctly		18	100
Explanation of Fractions		*N*	%
Proper Fraction 3/5	Part-Whole Concept of Fractions	17	94
	Measurement Concept of Fractions	1	5.6
	Don’t know	0	0
Improper Fraction 5/3	Part-Whole Concept of Fractions	9	50
	Quasi-Measurement Concept of Fractions	3	16.7
	Measurement Concept of Fractions	1	5.6
	Don’t know	5	27.8

[6] 19 5th grade students were selected to participate in the clinical interview. Due to a technical issue, one participant’s explanation of fractions was lost. Therefore, Table 4-5 shows the performance from 18 5th grade participants.

The Role of Fraction Units

After participants explained the meaning of each fraction, the researcher also asked the 4th grade participants to name one share in each given fraction and the alternative name of the one share. Only four out of 10 4th grade participants (40%) were able to give a correct fractional name of 1/5 or 1/3. They did not know 1/5 and 1/3 were also called fraction units. For the 5th grade participants, the researcher asked the participants to indicate the fraction unit of each given fraction. In contrast to the 4th grade participants, only four out of the 19 participants (21%) could not explain the fraction unit. All the other 15 participants (79%) knew what the fraction unit was for each fraction and knew how many fraction units were in each fraction. Unfortunately, it seemed that they were unable to connect the concept of fraction units to the improper fraction, and use it to explain the meaning of the improper fraction 5/3.

In summary, after only an informal introduction to the definition of a fraction, not all 4th grade participants could establish the concept of fractions through a part-whole relation at the end of the "Preliminary Recognition of Fraction" teaching phrase. Obviously, it was difficult for them to understand the meaning of improper fractions although most of them (80%) were able to verbalize the improper fractions. In contrast, after a formal introduction of fraction through a part-whole definition, all 5th grade participants established a part-whole concept of fractions. However, it seems that the part-whole concept of fractions was the dominant fraction definition for most 5th grade participants even though the formal definition of fraction units was introduced during the "Re-recognize fractions" teaching phrase. Therefore, only one participant (5.6%) could explain the improper fraction 5/3 using fraction units. All the other participants (94.4%) either still used a part-whole concept to explain the improper fraction, or did not know how to explain.

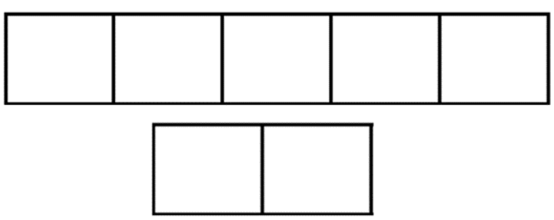
Results of Solving Part-Whole Schemes (PWS) Tasks

Figure 4.9 presents two PWS tasks presented in the clinical interview protocol. The first task (see problem #1 in Figure 4.9) was one of the tasks from the 4th grade written assessment. Because the whole in this task was a partitioned rectangular bar, it is considered to be a linear model task. The second task (see problem #2 in Figure 4.9) was a circular model, and was adopted from the Olive and Vomvoridi (2006) study. For both PWS tasks, participants needed to name the fractional relationship between the smaller part(s) out of the unit whole. The main difference between these two tasks is that the fractional part in the first task is physically disembedded from the whole, whereas the fractional part in the second task is embedded in the whole.

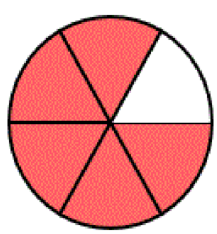
Figure 4. 9

Two PWS Tasks in the Clinical Interview Protocol

1. **如图所示，请问小块的是大块的几分之几。(PWS)**
   What fraction is the smaller bar out of the larger bar?



2. **如图所示的阴影部分是你吃剩的披萨。用什么分数可以表示你吃的那块披萨?**
   The shaded part of the pizza shown below represents the leftover pizza after you ate. What fraction of the pizza did you eat?



123

Performance of 4th grade participants

Linear Model Task

All 10 4th grade participants solved both PWS tasks during the clinical interview. For the linear model task (Figure 4.12), eight out of 10 4th grade participants (80%) gave a correct answer, 2/5. However, their explanations indicated that only one participant used part-whole reasoning. Her explanation was "大块的巧克力分成五块，*其中*的两块是五分之二." The literal translation is that "the larger chocolate bar is divided into five pieces, *two of which* are two fifths." It is noticed that the phrase *其中*, meaning "two of which" suggests that she had established the disembedding operation. In general, her explanation included the unit whole, partitioning and disembedding operations. The explanations from four other participants were the same as or similar to "大块的有五个，小块的有两个", meaning that the larger bar has five pieces, and the smaller bar has two pieces. The similarity of their explanations is that they did not explicitly speak out the unit whole. Their explanations suggested that they may have used a part-part relation. Their concept of fractions may be in transition from PPCF toward PWS. The remaining three participants' explanations indicated that their thinking was also a part-part relation. They either said "五个东西拿了两块" or "五个巧克力分成两份." The translation of the first sentence is that "there are five things and takes two things out of them." The translation of the second one is that "five chocolates are divided into two shares." The main difference of the explanations from these three participants and the above four participants is that the unit whole of these three interviewees was a discrete whole, five things, rather than a continuous whole. Thus, their concept of fractions may be considered as a PPCF.

The other two out of 10 participants' reasoning indicates that they used a part-part relation again. They explained their answer, 5/2, as five pieces on the top and two pieces on the bottom. Their explanations may suggest that their answer of 5/2 was just a ratio instead of a fraction since it seemed that these participants got their answers through direct observation.

Circular Model Task

Seven out of 10 participants (70%) provided the correct answer, 1/6. However, the reasoning behind this correct answer was different. Among these seven participants, four used part-whole reasoning, saying that the pizza is divided into six shares and one share is eaten. The three other participants reasoned that there were six slices of pizza in total and that they ate 1 slice of pizza. Their reasoning was considered as a part-part relation because the referent whole was not included in the explanation. The other three participants gave the wrong answer of 1/5 right after they read the task aloud. The explanation from two of them was that there were five slices of pizza and they ate one slice. It indicated that their reasoning was obviously a part-part relationship, as well as their referent whole was reliant upon the direct observation, the shaded fractional part. The unshaded piece was eliminated automatically. The explanation from the final participant was interesting. When asked why the fraction of the eaten slice is 1/5, he said there was a total of six slices of pizza and he ate one slice. This student obviously has not established the disembedding operation yet. For him, the unit whole is partitioned into six slices. The unit whole was changed after eating one slice.

Performance of 5th Grade Participants

According to the results of the prior written assessment and the original interview plan, nine participants were supposed to solve the PWS tasks. After interviewing two 5th grade participants, and considering the performance of the 4th grade participants, the researcher decided that the remaining 5th grade participants would also do the PWS tasks so that the researcher could compare and contrast the participants' thinking behind the PWS tasks across both grades.

Seventeen 5th grade participants completed the first task and, according to their performance on the first task, eight participants then completed the second task.

Linear Model Task

For the first task, 15 out of 17 participants (88%) gave a correct answer, 2/5. Almost all of them used a part-whole relation to explain their answer. Their explanations were either "大块的巧克力平均分成五份，取出其中的两份," meaning that the larger chocolate bar is divided into five shares, and take out two shares, or "大块的巧克力一共有五份，小块的巧克力是其中的两份", meaning that the larger chocolate bar has five shares in total and the smaller chocolate bar is two shares of the larger chocolate bar.

One of the common features of their explanation is that they all described the referent unit whole explicitly or implicitly. The second common feature is that their explanations seemed to suggest that they had constructed the disembedding operation because of the language they used, namely, taking out two shares, or two shares of it (the partitioned larger chocolate bar). To examine if these participants had truly constructed the disembedding operation, the researcher asked some participants to shade any one of the pieces in the larger chocolate bar and then give a fractional name to the shaded piece. They could name the piece as 1/5. Then the researcher asked participants to shade one piece in the smaller chocolate bar, and name that shaded piece. It might be because the question that the researcher asked at the beginning was not appropriate, but nine participants named that piece as ½. It appeared that they changed the previous unit whole to the smaller chocolate bar as a referent whole. Therefore, the researcher attempted to perturb them by asking why they used two-fifths to represent the smaller chocolate bar, but the fraction of one piece in the smaller chocolate bar was one half. Except for two of them, these participants changed their answer to one fifth. The two students who kept their answers as ½ believed there were two unit wholes and the two shaded pieces were not identical. Their thinking suggests that they might have only constructed a PWS although their written assessment indicated these two students were PUFS.

When interviewing one participant named Jiajia (after having interviewed nine other participants), the researcher realized the question asking students to name the shaded piece in the smaller chocolate bar might have caused their confusion about the referent whole. The following protocol displays this participant's thinking before and after the researcher changed the manner of posing the question. J represents Jiajia. R[7] represents the researcher.

R: Can you explain why this piece [the shaded piece in the smaller chocolate bar] is one half, but the fraction of the whole smaller chocolate bar is two fifths?

J: Because … because this small piece is one of the smaller bar, then the smaller bar has two shares in total, one share of that, it is one half. But this two-fifths is according to the larger [one], taking out this smaller [one].

R: Oh, that means when we talked about the smaller chocolate bar, the fraction you gave represents the smaller chocolate bar out of the larger chocolate bar, right?

J: uh (yes)

R: Ok, let me change my question, what fraction is the shaded piece in the smaller one out of the larger chocolate bar?

J: one fifth

This protocol suggests that after participants shaded one piece in the smaller chocolate bar, the question "what fraction is the shaded piece" may have created confusion about what the referent whole should be. Therefore, the participants considered the smaller chocolate bar as a

[7] The letter R represents the researcher in all protocols presented in this chapter.

whole. After the researcher let the participant use a fraction to represent the shaded piece out of the larger chocolate bar, she said one-fifth. The researcher then changed the question to "What fraction of the shaded piece out of the larger chocolate bar?" for the rest of the participants, and they all gave the correct answer, one-fifth.

Two out of the 17 5th grade participants gave wrong answers. One of the participants provided an incorrect answer of 5/2. Her answer indicated that the answer was her direct observation. The researcher then asked her to shade one piece in the larger chocolate bar and use a fraction to represent that piece. She wrote 1/5. Then the researcher asked her to use a fraction to represent the smaller chocolate bar again. This time she was able to give the right answer of 2/5. When asked why, she said because these two pieces was taken out from the larger chocolate bar. Accordingly, the first answer she gave might be her heuristic response, meaning that the answer was from her direct observation. With the researcher's help, she was able to finally identify the referent whole and provide a correct answer. The answer from the other participant was 2/7. It seems that she considered that the whole consisted of all the pieces in the figure. The researcher then asked her what the whole is, she said she did not know. Therefore, she could be identified as PPCF.

Circular Model Task

According to participants' performance, nine out of the 17 5th grade participants solved the second PWS task (see Figure 4.9). They all gave right answers. Their explanations were similar, that is, this pizza is divided into six shares and they ate one share. When asked to give a fraction name to one piece of the shaded part, seven participants used the correct fraction, 1/6, although the researcher did not mention "out of the whole pizza." This may be because all the fractional pieces were embedded in the whole. Two participants answered incorrectly with 1/5 to represent one of the shaded slices. When asked to explain, both of the participants said because there are five pieces in the shaded part. They both explained that the leftover pizza was another unit whole. The dynamic whole in their explanations indicated they might be in transition from PPCF to PWS.

Summary

Table 4-6 below summarized the number of participants who provided the correct answers to both tasks. It is important to point out that the way of calculating the percentage for 5th grade. Due to the changes of the original clinical interview plan after interviewing a few students, only 17 5th grade students finished the linear model task and 9 5th grade participants finished the circular model task. Thus, the percentages were calculated of by dividing the number of participants who correctly answered by the actual number of participants who completed the linear or circular model tasks respectively.

Table 4 - 6

Comparing the PWS Task Performance Across Grades

Grades / Models	4th Grade ($N = 10$)		5th Grade	
	Correct[8]		Correct	
	f	%	*f*	%
Linear Model	8	80	15	$\frac{15}{17}$ % = 88.2
Circular Model	7	70	7	$\frac{7}{9}$ % = 77.8

Note: All 10 4th grader participants completed both the linear and circular model task. For the 5th grade participants, 17 completed the linear model and 9 complete the circular model task. Due to

[8] The number of correctness shown in this table and the following summarized tables is the number of participants who solved the task correctly at their first attempt.

the changes of the original clinical interview plan after interviewing a few students, only 17 5th grade students finished the linear model task and 9 5th grade participants finished the circular model task. Thus, the percentages for 5th grade were calculated by dividing the number of participants who answered correctly by the actual number of participants who completed the linear or circular model tasks, respectively.

Lack of Disembedding Operation and Unit Whole

When solving PWS tasks, the reasoning for some 4th grade students revealed that PPCF participants have not yet constructed a disembedding operation even though their answers were correct. In particular, when students were asked to give a fractional name to one slice of the leftover pizza in the circular model task, most 4th grade students answered with the fraction 1/5 instead of 1/6. In the next paragraph two participants' explanations are discussed for the answers for the 2nd task showing two typical examples of a lack of the mental disembedding operation.

The reasoning from a participant named Yuan indicated that she did not have a conceptual understanding of the unit whole. When she performed the second task, she gave a correct answer of 1/6. She also used a quasi-part-whole relationship to explain her reasoning, saying "this pizza has six smaller slices. I ate one slice. So it is one sixth." It seemed that she had constructed a PWS and the disembedding operation although she did not specify that the pizza is divided into 6 shares. However, in her explanation of 2/5, the answer for task one, her unit whole was discrete wholes (e.g., five pieces of chocolate), but not a continuous whole (e.g., a chocolate bar is divided into five pieces).

As described, the difference between these two tasks above is that in the first task the fractional parts were physically disembedded from the whole, while in the second task the fractional part was embedded in the unit whole. Moreover, Yuan's written assessment indicated that she had not yet completely constructed a PWS. Therefore, to discover her concept of the unit whole and her actual scheme, the researcher asked her to give a fractional name to one of the slices of leftover pizza (i.e., one of the shaded slices in the pizza). She wrote 1/5 after she counted the shaded slices twice. The following protocol is the conversation between Yuan (Y) and the researcher (R).

R: Can you explain why you wrote one fifth?

Y: Because (pointing to the shaded part) five slices remain. This (pointing to the unshaded slice) one slice is eaten. Remaining … there are six slices in total, this one is eaten, leaving five slices.

R: So is it (one shaded slice) one fifth?

Y: (She looked at the figure again, and answered with hesitation] Yes

R: What do you think it should be? Do you want to change your answer?

Y: um …

R: What are you hesitating for? Can you tell me?

Y: I'm thinking … are you asking me add the eaten slice or not?

R: I just want you to use a fraction to represent one of the shaded slices. So what do you think? Do you think you should add this slice or not?

Y: uh, not add this slice.

R: Then I want to ask you another question. Is the shaded slice the same as the slice that you ate?

Y: Yes.

R: [the researcher tried to prompt her to reconsider her perspective] Then why is the slice you ate one sixth, and the shaded slice one fifth?

Y: Because I … there are six slices in total and I ate one slice, so it is one sixth.

R: So is it because there are five slices left over so one of them is one fifth?

Y: Yes.

Obviously, her unit whole was changeable, and her concept of the unit whole was ambiguous. Her lack of a disembedding operation might have been caused by the changeable unit whole.

Another example of a lack of a disembedding operation also comes from a 4th grade participant named Zhi. He gave a right answer of 2/5 for the first task, and used discrete wholes rather than a continuous whole in his explanation. He answered incorrectly with 1/5 to the second task. The following protocol presents his reasoning for his answer for the second task, and how he changed his answer after having his thought process questioned by the researcher. Z represents Zhi, and R represents the researcher.

R: Could you explain why? Explain your answer.

Z: Because (there are) five things and ate one slice, so it changed to one fifth.

R: Five things. Can you use your fingers and point out which five things?

Z: [Started to use his left middle finger and count counterclockwise from the unshaded slice] this, this, this, this, this, this [Then looked at the researcher while his middle finger was pointing to the sixth slice which was underneath the unshaded slice]

R: [Noticed that the last uttered number was not compatible with his action] Can you point out the five things again?

Z: [Used his right index figure and counted from the unshaded slice clockwise. However, when he counted to the fifth slice while pointing, he looked at the researcher again while his finger moved to the sixth slice. The mentor asked him “Is it five slices?” The researcher asked him another question at the same time.]

R: [Because his uttering and actions were still inconsistent, the researcher changed the question] Which slice did you eat?

Z: [Pointed to the unshaded piece] This

R: Okay. Can you read the problem aloud again?

Z: [He read the problem out loud again]

R: After you read the problem, do you want to change your answer?

Z: Yes

R: Okay, write your new answer next to your first answer.

Z: [He wrote ¼]

R: Can you explain?

Z: Because there are four slices not eaten

R: Which four slices?

Z: Shaded slices.

This protocol suggests that Zhi's unit whole relied more upon those perceptual items or parts. In other words, his unit whole needs to be visible. In this example, the shaded part was the visible slices compared to the unshaded slice in the circle (i.e., pizza). Therefore, he automatically eliminated the unshaded slice even though his counting action included that slice. The researcher redirected him to the first task and asked him what is the unit whole. He responded that the unit whole is seven. Because his answer was 2/5, the researcher asked him "That means none of 5 and 2 in your answer, two fifths, represents the unit whole?" He said yes. Consequently, the answer for task one, 2/5, actually showed a part-part relation, but not a part-whole relation. He should be considered as a PPCF, but not a PWS.

The Role of Fraction Units

Although solving PWS tasks did not directly involve fraction units, the conversations between the researcher and some 5th grade participants disclosed how the understanding of fraction units leads them toward developing their understanding of the concept of the unit whole.

The following protocol is one example of how to help a participant develop her understanding of the concept of the unit whole through identifying the fraction units. The participant's name is Xiaoying. Xiaoying's written assessment indicated that she had not yet constructed any of the fractional schemes (only SO). During the clinical interview, she gave a correct answer to the first task. However, when asked what fraction could represent one piece in the smaller chocolate bar, she said one-half. After being asked to identify the unit whole, she changed her answer to 1/5. To re-examine her understanding of the unit whole, the researcher had her work on the second task. Additionally, the researcher tried to discover if she had conceptually constructed the disembedding operation. Xiaoying initially gave a wrong answer of 5/6. The researcher asked her to use her finger to show which piece had been eaten. She pointed to the unshaded piece. Then the researcher asked her if the fraction of this piece is 5/6. She realized her mistake and changed her answer to 1/6. Next, the researcher asked her to use a fraction to represent one slice in the leftover pizza. She wrote down 1/5. The following conversation started from this point. Ying is Xiaoying.

R: Why?

Ying: Because to find one slice of the leftover pizza, um, the leftover pizza is a unit whole, taking out one slice is one-fifth.

R: Why is the leftover pizza a unit whole? So if it is one-fifth, Okay. Let's put this problem aside. Just now what does the five-sixths represent?

Ying: The total of the leftover pizza

R: Good. The leftover is five-sixths, right?

Ying: Yes

R: Then why do you consider one of them as one fifth?

Ying: Um …

R: Now I want to ask you, why do you use five-sixths to represent the leftover pizza?

Ying: Because it (the whole pizza) is a unit whole. It is divided into six shares, the leftover is, is five of the total shares. It is five-sixths.

R: Very good. Then what fraction is one of the slices of the pizza?

Ying: One-sixth. Just now I used the leftover pizza as a unit whole.

R: Why do you want to use the leftover pizza as a unit whole

Ying: Because I thought this is the eaten slice, this pizza [pointing to the unshaded piece] has already been eaten. To find (one slice of) the leftover, you should use the leftover pizza as the unit whole

R: Now after one slice of the pizza has been eaten, does the unit whole change? What do you think?

Ying: No change

R: Now what fraction do you think represents one slice of the leftover pizza?

Ying: One-sixth.

Her explanation indicated that her concept of unit whole was ambiguous. However, after she identified the fraction unit, she finally understood that the unit whole was unchangeable. Furthermore, she was able to correctly perform the disembedding operation.

Results of Solving PUFS Tasks

The first two tasks in Figure 4.10 are the linear and circular models for investigating participants' PUFS in the clinical interview protocol[9]. The linear model task (question #3 in Figure 4.10) requires participants to find 1/7 of the given unpartitioned line segment (i.e., the unpartitioned whole). In order to do so, participants need to mentally or physically partition the given whole, line segment, into seven equal pieces according to the denominator of the given fraction name, and draw out or show the required unit fractional part. In contrast to the linear model task described above, for the circular model task (question 4 in Figure 4.10), participants were required to determine the size of the given unit fractional part (i.e., the smaller cake piece) related to the given unpartitioned whole cake (i.e., the unpartitioned circle), and then provide the fractional name to the given smaller cake piece. To successfully solve this task, students need to apply the iterating operation to determine the size of the given smaller cake piece.

The last task in Figure 4.10 is the rectangular model task in the Backup Interview Protocol which is similar to the PUFS circular model. If the participant struggled with the circular model, the participant would complete the rectangular model to help the researcher determine if the difficulty of solving the circular model was caused by a lack of strategy to partition a circular item. On the other hand, solving the rectangular model may help the participant figure out how to solve the circular model.

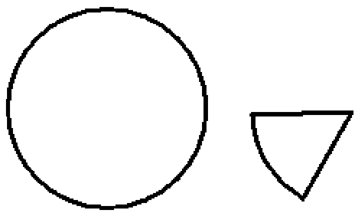
[9] The tasks' numbers in all figures are consistent with the tasks' numbers in the clinical interview protocol and the back-up interview protocol.

Figure 4. 10

PUFS Tasks in the Clinical Interview Protocol and the Back-up Protocol

3. 你的线段与下图线段的 $\frac{1}{7}$ 一样长。画出你的线段。
Your stick is 1/7 as long as the stick shown below. Draw your stick.

4. 请问图中小块蛋糕是整个蛋糕的几分之几？
What fraction is the smaller cake piece out of the whole cake?



请问图中小块蛋糕是整个蛋糕的几分之几？(PUFS)
What fraction is the smaller cake piece out of the whole cake?

Note. The first two tasks are the two PUFS tasks in the clinical interview protocol. The last task is the rectangular model task corresponding to the circular model in the Back-up interview protocol.

The Performances of 4th Grade Participants

Among 10 4th grade participants, nine of them finished both PUFS tasks. One participant did not do the PUFS tasks because it took him a long time to solve the PWS tasks. His written assessment also showed that he was not yet able to construct a PWS. Due to the time constraint, he only solved the SO tasks after he finished the PWS tasks. Below is the detailed description of the 4th grade participants' performance on the PUFS tasks.

Linear Model Task

Six participants successfully found one-seventh of the given line segment. When asked how they determined their stick (e.g., $\frac{1}{7}$ of the given line segment), most of them said "平均分成7份，取其中的一份," literally meaning "evenly divide it into seven shares and take out one share." Although their reasoning sounded like they had constructed a PUFS, only one participant was considered a PUFS student. Her methods of solving the linear and circular model tasks demonstrated that she had constructed an equi-partitioning scheme. This means she could not only use the partitioning operation to find the size of the unit fractional piece, but also applied the iterating operation to test if the size of the unit fractional piece she decided upon was correct. She first used her right thumb and index fingers to estimate the size of the unit fractional piece and then used her fingers to partition the given line segment. After that, she tested the size of this unit fractional piece by running her fingers gesturally from left to right and right to left again. Then she put tick marks to line up with her finger measurements.

The following protocol is the conversation between this student, Yuanyuan (Y) and the researcher (R) regarding how she tested the size of her unit fractional piece.

R: How did you know your stick is 1/7 of the stick shown in the problem?

Y: I tried.

R: I saw that you used your fingers. How did you use your fingers to figure out the answer?

Y: I tried seven times to see if it was equal.

R: What do you mean "if it was equal"? And how did you know if it was not equal?

Y: If it was not equal, it would be too long or too short.

Her explanation indicated that she understood that if the unit fractional piece is the right size, iterating it seven times should re-produce the correct length of the given stick. Her approach for solving the linear model task directly impacted her way of solving the circular model. The details of her solving the circular model task will be discussed in the next section.

The other five participants used a segmenting operation to determine the unit fractional piece. When they tried to partition the given line segment, they struggled with dividing the given stick into seven equal pieces. Their first piece was either too long (exceeding the length of the given stick after making only four or five tick marks on the stick presented in the problem), or too short (so that after making six tick marks on the given stick, part of that stick was not factored into the full length of the stick). After they adjusted their unit fractional pieces several times, four participants ultimately got seven uneven pieces even though they put in much effort to make them the same size. A typical example from one participant named Xinxin suggests that if a student has not yet formed a conceptual construct of the iterating operation, the unit fractional pieces they form measurements of will not be identical. Xinxin simply divided the given stick into seven pieces, but did not draw out 1/7 of the given line segment. When the researcher asked her "Which one is yours?" She pointed to the second piece from the left and said "the longest one."

Among the other three participants, two of them had not yet developed a conceptual construct of a PWS. To find one-seventh of the given stick, they made copies of the stick presented in the problem instead of dividing the given line segment into seven equal pieces. One of two participants made seven copies under the given stick. The other participant initially divided the given stick into seven pieces, but did not draw a stick representing one-seventh of the given stick. When the researcher asked her which one is one-seventh of the given stick, she erased all the tick marks and tried to make copies of the given stick. The last participant made a copy of the given stick and randomly put a tick mark close to the left end. When asked which one was the one-seventh of the given stick, she said the shorter one. When asked how she knew the shorter stick was one-seventh of the given stick, she said because it was short. Obviously, she randomly put a tick mark to make a shorter piece. Their performance suggests that their fraction concept was based on a part-part relation.

Circular Model Task

Based on their performance on the linear model task, eight 4th grade participants finished the circular model. Among these eight participants, only the participant who used an equi-partitioning operation on the linear model task successfully solved the circular task. She again used her fingers to measure the arc of the given small cake piece and used this measurement to partition the whole cake into six equal parts, which is an equi-segmenting operation. She then gave a correct answer of 1/6. The performance of the other seven participants indicated that the heuristic idea of finding the fractional relationship between the whole cake and the smaller cake piece was to use two perpendicular diameters to cut the circle, without using the given smaller piece to measure out the given whole. Two participants used two diameters to cut the circle into four pieces and gave their answer as ¼. When asked to explain their thinking, they said the size of the pieces was almost the same as the size of the given smaller piece. One participant even said there was a slight difference, but one-fourth was the fraction.

The other participants first cut the circle into four pieces, and compared the size of the pieces. After they realized that the size of the four pieces was bigger than the given smaller cake piece, they continued to draw another two diameters diagonally so the given whole cake was further cut into eight pieces. It seems that as long as the size was similar, the slight size differences did not bother them. When asked whether the size of the pieces they calculated was the same as the size of the given piece of cake, some of them said “almost same.”

Two participants said that the one-eighth was smaller, but did not know how to divide the circle so that the size of each piece within the given whole was the same as the given smaller piece. Then the researcher had these two participants complete the back-up rectangular model task. The purpose of using the rectangular model task was to investigate if the lack of strategy for partitioning the circular model task was the main difficulty for solving the circular model task. These two participants successfully solved the rectangular model, iterating the given smaller rectangular piece with their fingers along the given larger rectangle. However, it seems that successfully solving the rectangular model did not prompt them to use the iterating operation. They still tried to use diameters to cut the given circle.

Summary

Based on their performance in solving the linear and circular model tasks, only the participant who successfully solved both tasks using the partitioning and iterating operation are considered as a PUFS. The other five participants who solved the linear model task are considered as PWS. Their performance indicates that they employed the partitioning operation but not the iterating operation. In particular, the way they solved the circular model PUFS task indicates that they did not apply the iterating operation to determine the size of the fraction of the given smaller cake piece related to the given whole cake. Obviously, the participants relied upon the size of the unit fractional part for visual comparison. Therefore, the unit fractional pieces for these participants were not identical and iterable.

The Performances of 5th Grade Participants

The original plan of the clinical interview was that all participants would solve tasks associated with their highest available scheme as well as the two adjacent schemes above and below. After interviewing three 4th grade and two 5th grade participants, the researcher decided that the rest of the 5th grade participants (e.g., 18 out of 19 5th grade participants) would solve the circular model of the PUFS task regardless of their highest available scheme. The purpose was to investigate how the 5th grade participants use both partitioning and iterating operations to solve the PUFS tasks.

Linear Model Task

Fourteen 5th grade participants finished the linear model task. Only one participant did it incorrectly. She made a copy of the given stick under the stick presented in the problem. When asked to explain her thinking, she responded that because the phrase "as long as" means they have the same length. The researcher drew her attention to the whole phrase "1/7 as long as." She said she got confused. When she solved the circular model task, her performance suggested that she was a PPCF.

The other 13 participants successfully solved the linear model PUFS task. Their explanations were similar. That was "平均分成七份，取其中一份," literally translated as "evenly divided [it] into seven shares and take out one share." However, their actual actions of partitioning the given line segment indicated that five participants used the segmenting operation. The common characteristic of their actions was to estimate the length of the first stick, make a tick mark, and use this length as the reference to find the places to make the other tick marks. After figuring out whether six tick marks would exhaust the given line segment or the line segment would be exhausted before making six tick marks, participants were able to determine the correct unit fractional piece. If they made multiple attempts and still could not decipher the correct positions of the six tick marks, they ignored the sizes of the partitioned pieces and drew the most fitting unit fractional piece they could justify.

The following protocol presents an example from one 5th grade participant named Mengyuan. M represents Mengyuan and R represents the researcher. After making two attempts to solve the problem, she drew a line segment that was a little longer than the first piece in the partitioned whole above the given line segment.

R: According to what measurement, did you draw your stick?

M: Your stick is one-seventh as long as the one shown below.

R: Oh, I meant how did you determine the length to draw your stick.

M: … [No verbal response]

R: I'm just curious why you drew your stick this long.

M: Actually, I'm not sure. I just feel the last piece (in the partitioned whole) is a little short.

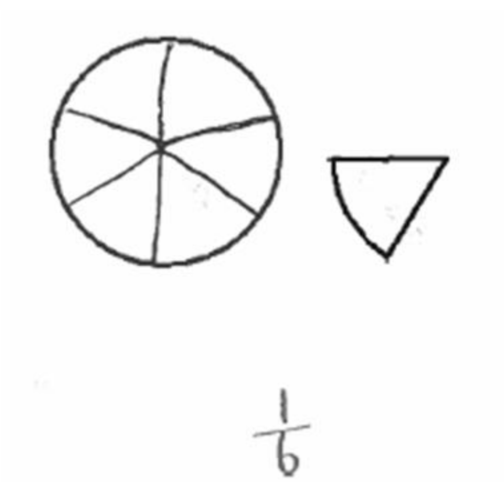
Although she understood that 1/7 means dividing the whole into seven equal pieces and taking one out, her problem-solving methods indicate that only the segmenting operation was involved, and that the size of each piece was not very important. It is difficult to determine if the other eight participants used an equi-partitioning operation to partition the given stick. Three of them mentally partitioned the given stick and drew a shorter piece beneath the given stick. The other five participants looked at the given stick for a few seconds and put six tick marks without hesitation. Then they drew out the unit fractional piece having the same length as the first piece of the partitioned whole.

Circular Model Task

Of the 19 5th grade participants interviewed, 18 of them solved the circular model PUFS task. Seven of them provided a correct answer of 1/6. Although their methods for partitioning the circle differed, they were able to equi-partition the given circular cake mentally or physically, and determine that the fraction was equal to 1/6. Two participants were able to partition the circle mentally. When asked how they knew that the fraction was 1/6, they said they divided the circle in their minds. For the other five participants, their approaches of partitioning the whole cake indicate that they used both partitioning and iterating operations. They either used pencils to trace the arc of the given smaller cake piece, or measured the arc of the given smaller cake piece with their fingers. They apparently were able to partition the given whole cake with confidence after they had the size of the given smaller piece in mind. Figure 4.11 presents the response from one of the participants.

Figure 4. 11

The Response of One 5th Grade Participant.



Note. Response to the circular model PUFS task from one of the 5th grade participants.

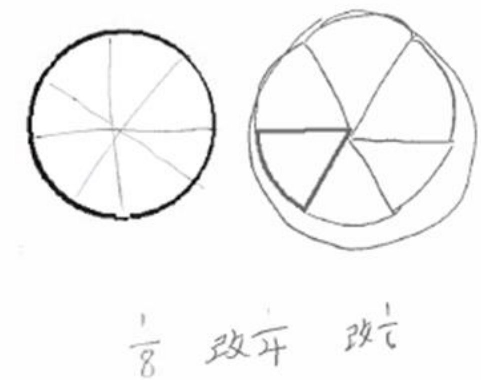
The answers from nine participants were either ¼ or 1/8. Their heuristic idea of partitioning the given whole circular cake was the same as most 4th grade participants, using two or four diameters to cut the given whole cake. Their actions of dividing the circle indicate their lack of comprehending the iterating operation. The researcher tried to provoke their perturbation by asking them if the size of each piece in the partitioned whole was the same as the size of the given smaller piece of cake. The researcher hoped that the question would elicit a conceptual understanding of the iterating operation.

One of the two participants who used two diameters started to visually compare the size, and consequently added two more diameters to cut the circle into eight pieces. After she changed her answer to 1/8, the researcher asked the same question again. She responded that "they look almost the same." The other participant erased all the diameters after visual comparison, and then used three diameters to divide the circle into six pieces. She changed her answer to 1/6 and responded to the researcher's question by saying, "They look similar now." Among the other seven participants, six were satisfied with the size of pieces in the partitioned whole and kept their answers of 1/8.

Only one participant said that it seemed the size was too small. She erased all the diameters she had drawn and drew two new diameters to cut the circle into four pieces. However, she did not provide an answer. The researcher asked her to explain her reasoning. She said that it seemed that all the pieces were a little bigger than the given smaller cake piece. Thus, the researcher asked her to complete the back-up rectangular model task. When she solved the rectangular model, she drew a horizontal line segment to cut the rectangular whole cake into half first after she measured the width of the smaller rectangular cake piece. Next, she measured the length of the smaller rectangular cake piece using her fingers. Then she kept that length, moved her fingers to the whole rectangular cake, and partitioned the top part linearly into three smaller pieces. She used the same way to partition the bottom part of the larger rectangular whole cake. When she re-did the circular model task, she surprisingly used the equi-partitioning operation to partition the circular model and provided the correct answer. Figure 4.12 presents her response.

Figure 4. 12

The Response of One 5th Grade Participant.

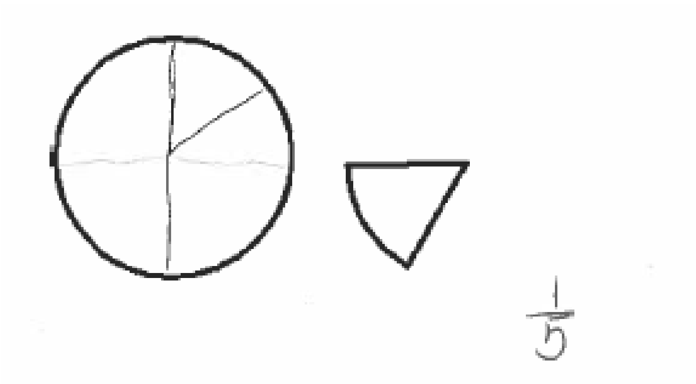


Note. Response of the circular model PUFS task from one of the 5th grade participants. The Chinese charter “改” in her answer means “change to.”

The response from the last participant was 1/5. When she solved the previous linear model task, she did not understand the PUFS language “1/7 as long as.” When she solved the circular model, she had looked at the figure for a few seconds and provided the answer of 1/5. When asked to explain her reasoning, she said that she “divided the cake into five shares and take out one share.” The researcher asked her to show on the paper how she divided the whole cake. Figure 4.13 presents her drawing. Her thinking once again demonstrated that she has not yet construct a PWS. Therefore, the different sizes of the partitioned pieces did not bother her. As long as the size of one of the pieces was close to the given one, she named the fraction based on the number of pieces in the partitioned whole.

Figure 4. 13

Response of the Circular Model PUFS Task from One of the 5th Grade Participants.



The Role of Fraction Units

The performance and explanations from both 4th and 5th grade participants suggest that their understandings of the fractional language "1/7 as long as" were in line with the levels of their fractional knowledge. For those participants who have not yet constructed a PUFS, it was a challenge for them to understand the language "1/7 as long as." The three 4th grade participants who had not constructed a PUFS could not solve the PUFS tasks because they did not understand the language "1/7 as long as." In particular, when two of the participants who did not have a PWS encountered an unpartitioned whole, they considered the given stick as the 1/7 of the whole. Thus, rather than partitioning the given stick, they made six copies of the given stick to form a whole. The following protocol displays the conversation between the researcher (R) and one of the participants name Zhi (Z) who did not have a PWS.

R: Can you explain why you drew seven line segments?

Z: [Looked at his drawing and crossed out the last one he had drawn. Then looked at me.]

R: Can you explain your answer?

Z: Because it has seven, it needs to because one-seventh.

R: What does the one-seventh mean?

Z: Take one piece from seven pieces.

R: Oh, which seven pieces?

Z: These seven lines.

Zhi's explanation suggests that first his understanding of the whole is a discrete whole. Secondly, to find 1/7 of a whole, he needed to visualize seven items. There were not seven sticks in the task. Thus, he needed to make up the whole, drawing the other six sticks that were as long as the given one so that the given stick became 1/7 of the whole.

The performance of the other 4th grade participant who had constructed a PWS, but not a PUFS, was different. Her performance on both PUFS tasks showed that she understood that the given line segment or the whole circle should be the referent whole, and that she might comprehend that the fraction unit means one out of the whole. Consequently, when she solved the linear model of the PUFS tasks, she drew a stick that was almost the same size as the given stick, and made a tick mark close to the left end to cut a smaller piece out. When asked which one was her answer, the longer stick or the shorter stick, and why, she said that the shorter stick was the answer because it was one-seventh so it should be one share. When asked to explain how she determined the size of the shorter stick, she said she did not know.

For those participants whose performance indicated that they had constructed a PUFS, they understood that "1/7 as long as the stick shown" implies that the given stick was the whole. To find the 1/7 of the referent whole, their common response was "把这条线段平均分成 7 份，取出其中的一份", meaning "divide this line segment into seven equal shares and take out one share." After the 5th grade participants drew a 1/7 of the whole, the researcher selected one of the smaller pieces in the partitioned whole and asked what fraction could represent that smaller piece. All participants responded that was 1/7 as well. Their response indicates that each piece in the partitioned whole is a fraction unit with an invariant size.

Partitioning Operation and Iterating Operation

It is difficult to determine whether the iterating operation was involved in partitioning a whole when solving the linear model task, especially when the participants used a segmenting operation. However, the circular model task provided an opportunity for the researcher to investigate whether participants applied the iterating operation when they partitioned a whole. It was reasonable to assume that those participants who used diameters to cut the circular cake were not applying the iterating operation. What they did indicates that they decided the size of the fraction unit through visual comparison, but not through the iterating operation. On the other hand, by comparing and contrasting the performance of the 4th and 5th grade participants, those participants who solved the circular model task with both partitioning and iterating operations demonstrated less of a struggle in solving the linear model task.

Summary

Table 4.7 summarizes the performance on PUFS tasks across grades. Totally, nine 4th grade participants did both linear and circular model PUFS tasks. Six of them solved the linear model task correctly and one provided the correct answer to the circular model task on the first attempt. Among the 5th grade participants, 15 participants did the linear model PUFS task and 18 participants did the circular model PUFS task. Fourteen participants correctly solved the linear model task and 8 out of 18 participants solved the circular model task correctly on their first attempt.

Table 4 - 7

Comparing the PUFS Task Performance Across Grades

Grades / Models	4th Grade ($N = 10$)				5th Grade ($N = 19$)			
	Correct		Iterating		Correct		Iterating	
	f	%	*f*	%	*f*	%	*f*	%
Linear Model	6	60			14	73.6		
Circular Model	1	10	1	10	8	42.1	5	23.6

Note: For 4th grade participants, six participants solved the linear model task correctly. One participant solved the circular model task correctly. For the 5th grade participants, 14 participants did the linear model task correctly. Eight participants solved the circular model task correctly. The percentage was calculated by using the number (*f*) shown in the table divided by the total number (*N*) of participants in each grade who participated the clinical interview, namely 10 participants for 4th grade and 19 participants for 5th grade.[10]

Results of Solving PFS Tasks

Figure 4.14 presents the PFS tasks used in the clinical interview protocol and back-up interview protocol. For the linear model task, participants needed to give the fractional name to the given shorter stick, which was a non-unit unpartitioned fractional part, with respect to the given referent whole, an unpartitioned longer stick. In contrast, the circular model task required participants to find the fractional part from the unpartitioned whole (i.e., half circle) according to the given fraction 4/5. Another main difference between the linear and circular model tasks was that it would be easier for students to identify the whole in the linear model than in the circular model. The last task in Figure 4.14 is the rectangular model task in the Backup Interview

[10] All the percentages in this and the future summarized tables are calculated in this manner.

Protocol which is similar to the above circular model. It was only used when a participant struggled with the circular model. Using this task also helped the researcher determine if the difficulty in solving the circular model was caused by a lack of strategy to partition a circular item.

Figure 4. 14

PFS Tasks in the Clinical Interview Protocol and the Back-up Protocol

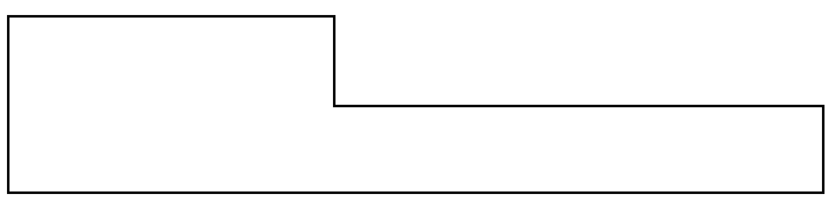
5. 如图所示，请问短线段是长线段的几分之几？

What fraction is the smaller stick out of the longer stick?

6. 你的那块匹萨饼与如图所示的这块匹萨饼的 $\frac{4}{5}$ 一样大，请画出你的那块匹萨饼。

Your piece of pizza is 4/5 as big as the piece shown below. Draw your piece of pizza.

2. 什么分数可以表示下图这块巧克力和整块巧克力的关系？
What fraction is the piece of chocolate shown below out of a whole chocolate?



Note. The first two tasks are the two PUFS tasks in the clinical interview protocol and the rectangular model task corresponding to the circular model in the Back-up interview protocol.

The Performances of 4th Grade Participants

According to their performance on the PUFS tasks, 6 out of 10 4th grade participants solved the linear model of the PFS tasks. One participant struggled with solving the linear model task and given the time restrictions, only five participants solved the circular model PFS task. The participants' performance and thinking for the linear model and circular model tasks are presented in the following sections.

Linear Model Task

Although the heuristic thinking of all six 4th grade participants solving the linear model PFS task was to treat the given short stick as a unit fractional part and iterate it into the given whole, only one participant could do it mentally. After she mentally iterated the given short stick into the given long stick, she realized that iterating the short stick would not exhaust the given whole stick. She extended the given short stick so that it had the same length as the longer stick. Next, she partitioned the given short stick into four equal pieces and continued to partition the extended part into six equal pieces. She got her answer of 4/10, which was also the correct answer. Obviously, the iterating operation helped her to determine that the given short stick was not the unit fractional part relative to the given whole. In fact, the iterating operation was activated while she solved the circular model PUFS task.

While solving the circular model PUFS task, this participant drew two diameters to cut the circle into four equal pieces. After she wrote down her answer of ¼, the researcher asked her how she knew that the given smaller cake piece was one of the partitioned pieces. She said she was not sure because it seemed a little bigger than the given smaller piece. The researcher queried how the participant adjusted the partitioned pieces in the whole so that these pieces would have the same size as the given one. She started to use her pencil to measure the arc of the given sector, and used the result of that measurement to draw the first piece. She then used that piece as the reference to draw radii clockwise and partitioned the whole circle into seven pieces. Although her answer, 1/7, was technically incorrect, the iterating operation apparently was involved in her activity suggesting the presence of a PUFS.

After the other five participants iterated the short stick on the long stick twice, one participant noticed that the last piece was shorter than the given short stick, and wrote the answer as $1/2.5$. When asked to explain why the denominator was 2.5, he said the shorter line segment at the end of the long stick was shorter than the given short stick. He said he visually estimated and felt that the smaller line segment was about half of the given smaller stick. Although his answer was correct, he did not apply the iterating operation to test his estimation.

Two other participants wrote their answers as ½. When asked if they noticed that there was a shorter piece on the long stick after they made two tick marks on the long stick, they responded yes, but it was too short. Their answer suggests that their concept about the whole was vague. Their performance also revealed that they did not construct the iterating operation yet. A similar phenomenon happened again when two other participants solved this task. The answers of these two participants were 2/4 and 3/6. They both noticed that the last piece was shorter than the given short stick after they iterated the short stick into the long stick two times. They then divided the short stick and the two short sticks within the long stick into two or three equal pieces. However, when they defined the denominator, they only counted each of the pieces in the two partitioned short sticks within the partitioned whole, but ignored the last piece to gain their denominators four or six. Their performance indicates that they may be in transition from PUFS toward the construction of a PFS as well as the progress of constructing the iterating operation.

However, their manner of defining the denominator indicates that their concept of the whole was also vague.

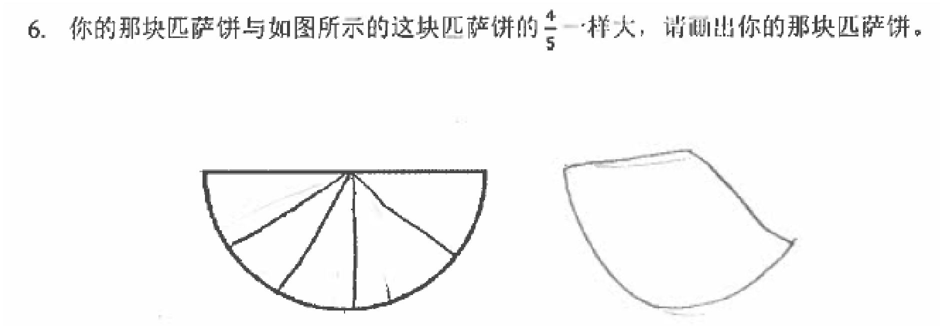
Circular Model Task

The participant who gave the answer of 1/2.5 for the linear model task did not solve the circular model. Therefore, only five 4th grade participants solved the circular model PFS task. Only one participant drew a reduced size half circle, beside the given half circle. Then she used her fingers to measure the arc of the half circle she drew, and compared the measure with the given half circle. She may have thought that was too small. She crossed out her first drawing and drew another reduced size half circle which was a little bigger than her first drawing. Her explanation indicated she had a PWS.

Two participants gave correct answers on their second attempts. Their first method of partitioning the half circle was the same, cutting the half circle into half, then cut each half into halves again to form four equal pieces. Their method may indicate that they have not yet constructed an iterating operation, but only the partitioning operation. After realizing that using this method only divided the half circle into four pieces, instead of five pieces, one of these two participants adjusted her strategy to an equi-segmenting method and started to use a radius to cut out the first fraction unit from the left side of the half circle. However, she struggled to partition the half circle into five pieces with the same size. After the researcher told her not to worry about the size too much, she was able to partition the half circle into five pieces with different sizes and drew 4/5 of the given half circle (see Figure 4.15.). Although the sizes of five different pieces were different, her method and struggling indicated that her iterating operation had been activated.

Figure 4. 15

Response of the Circular Model PFS Task From a 4th Participant



After observing the other participant struggle to divide the given half circle into five shares, the researcher became involved in her solving activity. Before the researcher provided her suggestion, she first wanted to identify whether the participant understood the language of this PFS task. The following protocol displays the conversation between the participant name Xingxing (Xing) and the researcher (R).

R: Xingxing, what are you doing? Can you explain to me? So then, I can decide how I can help you.

Xing: Divide it into five shares.

R: Why?

Xing: My piece is five … it is four fifths of my piece. So [I] need to draw five shares.

R: (Thinking that she may have said it wrong) Your piece is 4/5 of this piece, right?

Xing: Uh [yes]

R: So then what do you want to do?

Xing: Divide this [the given half circle] into five shares, and then … then shade four shares.

R: Oh, that’s great. Now you have difficulty to divide it into 5 shares, right?

Xing: Um (she nodded her head).

R: You don't need to draw preciously, almost the same will be fine. (She was still struggling to divide the half circle) I give you a hint. Don't divide it from the middle, but from one side [of the given half circle].

Xing: (She started to draw radii from the right side of the half circle, and partitioned it into five pieces with slightly different sizes. The first piece on the right was apparently the smallest one)

R: That's great. You're so smart. And then?

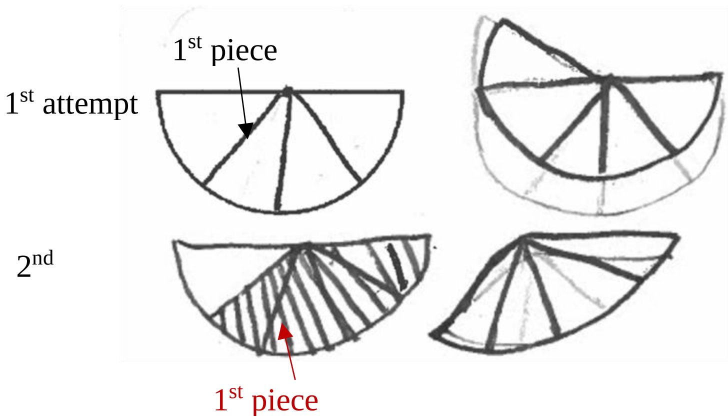
Xing: Then shade four of them (she shaded four shares from the right).

This conversation revealed that the participant understood the PFS task language, and was able to identify that the given half circle was the unit whole. However, due to the lack of the iterating operation, it was difficult for her to partition the half circle into an odd number of shares.

The fourth participant solved this task as a RPFS task on her first attempt (see Figure 4.16). However, her method of partitioning the half circle could be considered as a quasi-iterating operation. In her first attempt, because she solved the task as a RPFS task, she divided the half circle into four shares. When she partitioned the half circle, she used two radii to draw a piece slanted left (see Figure 4.16). Then she drew the other radius on the right side to form four pieces. After she realized her mistake while she was explaining her reasoning, she re-drew a half circle and drew a fraction unit piece in the middle of the half circle (see Figure 4.16). Then she drew two more radii to divide the remaining part into four pieces with different sizes, and shaded four pieces.

Figure 4. 16

Response of the Circular Model PFS Task From a 4th Grade Participant

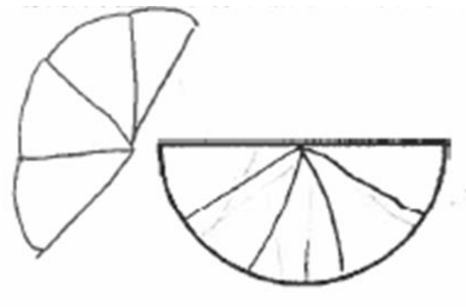


Note. The top row presents the participant's first attempt. She solved it as an RPFS task. The second row presents her second attempt.

According to her performance, the last participant was classified as a quasi-PFS. The way she solved the task indicated that she had constructed both partitioning and iterating operations. She partitioned the half circle into five shares from the left side of it, and all the pieces were almost the same size. What she did indicated that her partitioning is iterative in nature. However, after she partitioned the half circle, she drew out one piece on the left side of the given half circle. While she was explaining her reasoning, she realized she should draw four fraction unit pieces, rather than erase one piece like the other participants did when they made a mistake; she added three fraction unit pieces at the top of the first one. She once again used iteration to make the fractional part, 4/5, of the given half circle (see Figure 4.17).

Figure 4. 17

Response to circular model PFS task from one of the 4th grade participants.



The Performance of 5th Grade Participants

The number of participants who solved the linear model and circular model tasks varied. According to the original interview plan, participants would not solve the PFS tasks if their highest available scheme level were RPFS or IFS (based on the written assessment). However, after interviewing a few 4th grade students, the researcher decided that all 5th grade participants would solve all of the circular model tasks regardless of their highest available scheme. In this way, the researcher took advantage of the opportunity to investigate how those participants with advanced fraction schemes solved the circular model tasks of basic fractional schemes. Thus, 17 participants completed the linear model task, and 19 participants finished the circular model.

Linear Model Task

The first response of all 17 5th grade participants solving the linear model task was the same as the 4th grade participants, making a long tick mark from the right end point of the given short stick to the given long stick so that the short stick was allocated on the long stick. This action indicated that they used the given non-unit fractional part as the fraction unit. Next most of them either physically or mentally used the given short stick to measure the rest of the given long stick and made a second tick mark on it so that the second piece on the long stick had the same, or close, length as the given short stick. After they found that the remaining piece was shorter than the given short stick, three different responses were revealed. According to these three responses, participants were classified as having constructed a PFS, non-PFS, or quasi-PFS.

Ten participants were classified as having constructed a PFS. When they realized that the remaining piece was shorter than the given short stick, they were able to understand that the given short stick was not the fraction unit of the given whole. They started to partition the given short stick into two (seven participants) or three (three participants) equal pieces, and explained that the remaining piece was “跟它的一半一样长” or “跟它的一半差不多”, literally translated as “the same as half of it [the given short stick]” or “almost the same as its half.” Then they made tick marks in the first and the second partitioned pieces on the given longer stick, and named the given short stick as 2/5 or 3/7. Their explanations indicated that they used the remaining piece as a new fraction unit and tested it through iterating it into the given short stick and the longer stick. These actions indicated their construction of a PFS as well as the iterating operation.

Two participants were classified as quasi-PFS. After they noticed that the remaining section of the long stick was shorter than the given short stick, compared to the 4^{th} grade participants who ignored the remaining smaller piece and provided an answer 1/2, these two participants adjusted the position of the second tick mark made on the given long stick. They tried to find a place for the second tick mark so that the second piece and the remaining piece had the same or almost the same length. Then they wrote their answers of 1/3 with hesitation. Noticing their hesitation, the researcher tried to perturb their thinking. The following protocol presents one conversation between the researcher (R) and a participant name Jiajia (Jia).

R: What is your answer?

Jia: One-third

R: Just now I saw you, you first drew a short line [on the longer stick] where the short line segment is aligned, right?

Jia: Yes.

R: Then you drew the second short line, right?

Jia: Yes.

R: Then I want to ask you. You drew the second short line [on the long stick] according to what?

Jia: According to the length of this short line segment [the given short stick]

R: So you said this short line segment is one third of the long line segment. Can this explain that the long line segment contains three short line segments?

Jia: Hum, yes [with hesitation].

R: Can you carefully look at your long stick and check if it really contains three short line segments?

Jia: No [prompt reply]

It seems that the perturbation from the researcher helped this participant ensure her previous sense that the remaining piece was shorter than the given short stick. When the researcher asked if she wanted to change her answer, she said yes. The participant used the remaining piece as the new fraction unit, tested it, and provided a correct answer of 2/5. Therefore, these two participants were classified as quasi-PFS. Their hesitation suggests that their construction of an understanding of the invariant size for fraction units may be in transition.

The remaining five participants provided the answer ½, 1/3, or 2 ½. They were considered as non-PFS. They did the same as those quasi-PFS participants. The difference between the non-PFS and quasi-PFS was that the non-PFS's belief could not be perturbed by the researcher's question and they insisted that their answers were correct.

Circular Model Task

Among the 19 participants who finished the circular model task, 16 could quickly divide the half circle into five shares. Their responses suggest that they were able to identify the unit whole. Furthermore, their responses also suggest that most of the participants might have learned how to partition the circular model from solving the circular model of the PUFS task. On the other hand, their explanations were the same as or similar to "我的那块是它的五分之四，也就是说这块平均分成五份." The literal translation is "my piece is four-fifths of it [the given pizza], that means dividing this piece [the given pizza] into five shares." Their explanations can be considered PWS language. Thus, what they did next became the indicator for classifying these participants. Thirteen of them drew the fractional part (i.e., 4/5 of the given half circle) under their partitioned whole, and also divided this part into four shares. However, the four remaining participants were considered as a quasi-PFS because they only drew one piece. They realized their mistake after the researcher asked them to explain their thinking.

The final two participants were considered as non-PFS. They also could not solve the linear model PFS task. They had difficulty understanding the fractional language "4/5 as big as." Specifically, one participant explained that he was not sure if he should divide the given half circle into five shares or 4 shares. His explanation showed that he was unable to distinguish the unit whole based on the fraction language. A detailed discussion about his performance will be presented in the next section.

Comparing the numbers of students' correct responses to the linear and circular model tasks, having more participants complete the circular model correctly may indicate that the task with the given fractional name is easier than the task that requires students to find the fractional name for the non-unit fractional part. For example, one 5th grade participant struggled with solving the linear model task. She first made a tick mark on the given long stick, under the right end point of the given short stick. Next, she made another tick mark on the longer stick so that the second piece on the long stick had the same size as the given short stick. After she noticed the remaining piece was too short, rather than adjust the position of the second tick mark, she remained silent and looked at the figure. The researcher asked her to complete the circular task first. It seems the circular model task was easier for her. She quickly divided the half circle into five pieces and drew out a fractional part containing four of those pieces. After she successfully solved the circular model, she went back to the linear model. Solving the circular model task apparently helped her realize that the given short stick was not the unit fractional piece of the given whole. She made a tick mark in the middle of the given short stick and asked the researcher if she could do that. The researcher told her to test her thinking. She then continuously made two other tick marks to divide the previous two pieces into four smaller pieces, and gave the correct answer of 2/5. The circular model task seemed relatively easier than the linear model task because the fractional name was provided. The given fractional name helped her realize that the given short stick may not be the unit fractional part of the long stick because iterating it two times could not exhaust the given longer stick. That indicates that the unit fractional part should be shorter.

The Role of Fraction Units and the Unit Whole

The main difference between the linear and circular model PFS tasks was that students could easily identify the unit whole for the linear model task but not for the circular mode task. For the linear model task, it required participants to find a fraction representing the given smaller stick out of the given longer stick. Although the task does not directly tell the participant that the given long stick is the unit whole, both 4^{th} and 5^{th} grade participants considered the given long stick as the unit whole. For the circular model task, first the given figure was a half circle but not a full circle. Secondly, the circular model depicted the unit whole implicitly. Thus, it was difficult for some 4th and 5th grade participants to define the unit whole. For example, one of the participants first divided the half circle into four pieces and drew one smaller piece under the given half circle. His solution was similar to the solution shown in Figure 4.21. When asked to explain his thinking, he said that he was wrong. Then he erased that one piece he drew and tried to erase the other drawings. The researcher stopped him and asked him to re-draw the half circle under the given one. After he drew a half circle, he tried to divide it, but it seems that he did not know how to divide it. The following conversation between this participant name Kai (K) and the researcher revealed his struggle.

R: What are you thinking?

K: I am thinking why I need to divide. This is … this is the …

R: You divided the given pizza into four shares, didn't you?

K: Yes.

R: Why divide it into four shares?

K: Because of four fifths … four fifths … should divide into five shares, right?

R: Read the task again.

K: (he re-read the problem again)

R: So that means the shown pizza is four fifths, correct?

K: Wait. Let me think.

R: (after a few seconds) Which part bothers you.

K: Your piece of pizza is four fifths as big as the piece shown below … oh, divide it into four shares, and draw out one fifth of it.

R: Why?

K: Uh …

R: Kai, I want to know which part of the task bothers you most.

K: Four fifths as big as the piece shown below.

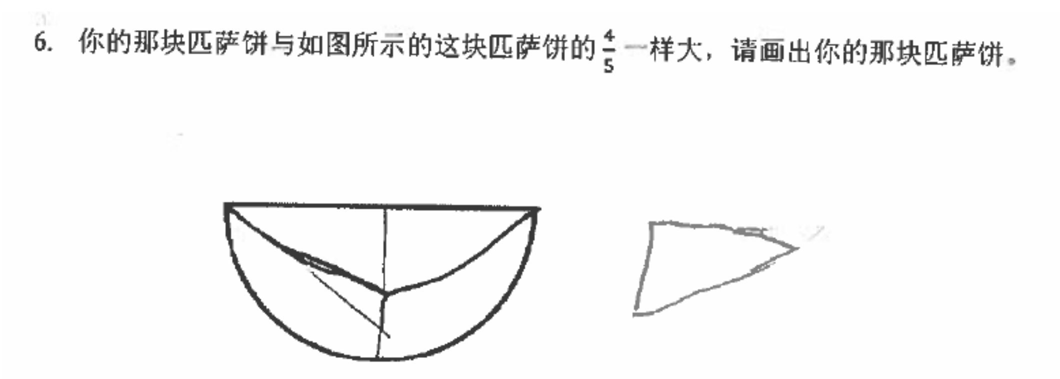
R: "as big as" is very annoying, isn't it?

K: Yes.

The conversation showed that he was unable to distinguish the whole based on the fractional language. The other participant had the same issue, but performed in a different way. He divided the half circle into four pieces from the middle of the half circle (see Figure 4.18). Then he drew one piece that was similar to one of the top two pieces. He had difficulty explaining his thinking. It seems that he understood that the unit whole should be divided into five pieces, but he was unable to identify the unit whole.

Figure 4. 18

Response of circular model PFS task from one of the 5th grade participants



Note. The student divided the given half circle into four pieces and drew out one piece. Therefore, there were a total of five pieces and the given one is 4/5 of the unit whole.

It seems that for participants in transition from a PUFS to a PFS, the size of the unit fractional piece is changeable. The performance of one of the 4th grade participants named

Tongtong provides a typical example to illustrate this phenomenon. When she solved the PFS linear model task, she first treated the given short stick as the fraction unit piece. She used her left thumb and index finger to measure the given short stick. She then moved her finger to measure the long stick so that her thumb was on the long stick, under the right end of the given short stick. Her index finger was on the longer stick close to the right end of the longer stick. She made a mark at the position of her index finger. Then she realized that the left part was too short. She erased the mark and used her left thumb and index finger to measure the short stick again. Next, she did the same as before. However, instead of making a tick mark on the position of her index finger, she placed the tick mark before her index finger so that the remaining part on the long stick was longer than her first attempt. Then she made another tick mark on the long stick, under the right end of the short stick. Now there were three short sticks in the long stick. One was the same length as the given short stick. The other two sticks had almost the same size, but were shorter than the given short stick. At this point, the researcher asked her why she made the tick mark before her index finger instead of where her index finger was. She replied "Then the left part is not long enough." Her performance suggests that she did not realize the given short stick was not the unit piece. Instead of finding the unit piece from the given short stick, she adjusted the sizes of the unit fractional piece so that iterating these unit fractional pieces would exhaust the given whole.

Another example of how one 4th grade participant, named Jingjing, solved the circular model also illustrates this assumption. After her third attempt to cut the given half circle (i.e., the given whole), she finally carefully divided the half circle into five pieces, but still with different sizes, from the right side of the half circle to the left side. Then she drew a smaller sector that was similar in size as the first left piece in the partitioned half circle. The researcher asked her to re-read the task. After she read the task aloud again, she realized her mistake. However, she drew another three connected smaller pieces of different sizes. Although the sizes of each piece was different, what she did indicates that she understood that the piece she drew was the unit fractional piece, and iterating this piece another three times gave her 4/5 of the given whole.

One Whole or Two Wholes?

An interesting phenomenon was revealed while participants solved the PFS tasks during their clinical interviews. One of the goals of this study is to identify the role that the understanding of the fraction unit plays in Chinese students' fraction knowledge construction. Thus, when participants successfully solved either the linear or circular model tasks, the researcher asked participants to find a fraction that represented one piece in the partitioned whole and the given or drawn fractional part. Interestingly, most of the participants provided different fraction units.

The following protocol presents a conversation between the researcher and a 4th grader. After she successfully drew the pizza that was 4/5 of the given pizza, the researcher asked her to find a fraction that could represent one piece within the fractional part. J represents her name Jingjing. R represents the researcher.

R: Now I want to ask you. I want you to use a fraction to represent … You divided the given pizza into five shares, didn't you?

J: Uh.

R: Can you use a fraction to represent one share within that [pizza]?

J: [She wrote down 1/5] Uh, it's one-fifth.

R: Can you use a fraction to represent one share within your pizza [the fractional pizza she drew]?

J:[She wrote down ¼] It's one-fourth.

R: So I want to ask you. Is one share of your pizza the same as one share of the given pizza?

J: Not the same

R: Oh, so I think your pizza [I started counting] one-fourth, two-fourths, three-fourths, four-fourths. So your pizza is a unit whole?

J: Yes.

R: Is the given pizza a unit whole?

J: Yes.

R: So there are two unit wholes. Is that right?

J: Yes.

Because the concepts of unit whole and fraction unit were not yet introduced to 4th grade students, the researcher's question could not perturb her thinking. Based on the participant's current level of knowledge, it seems that if a fractional part was disembedded from the referent whole physically, it became another independent whole.

The following protocol presents another typical example from a 5th grader. After she successfully solved the linear model PFS task, the researcher first had her find a fraction to represent one piece in the given partitioned non-unit fractional part. The conversation began at this point. The name of the participant was Wen.

R: Okay. I would like you to find a fraction to represent one share of the short stick.

Wen: [She wrote ½] A half.

R: So which line segment do you think is a unit whole?

Wen: Shorter. The short stick.

R: If the short stick is the unit whole, then why do you say it is two-fifths.

Wen: Because I, … what fraction is the short stick out of the long stick? I divided it into two shares and put it in the long stick. So, it represent two-fifths.

R: Let's do this. First find a fraction that represents one share of the long stick.

Wen: [She wrote1/5] One-fifth.

R: Okay. Do you think one share within the longer stick is the same as one share within the smaller stick?

Wen: Same.

R: Same. Then why is one two-fifths and the other is a half?

Wen: Because they [the shorter stick and the longer stick] are different unit wholes.

R: So the shorter stick is a unit whole, the longer stick is another unit whole. Is that right?

Wen: Yes.

In contrast to the 4th grade participant who believed one share from the short stick is different from the share from the long stick, the 5th grade participant believed they were the same, but derived from two different wholes. With this curiosity, the researcher discussed the phenomena with the mentor. He mentioned that the 5th grade students at that time were learning about discrete wholes. Those teachers did use two different wholes in the following example. "You have 12 apples. If you gave 1/3 of your apples away and ate 1/4 of the rest of the apples, how many apples did you eat?" The mentor said they taught students that the whole of 1/3 was the 12 apples. Giving away 1/3 of 12 apples meant that you gave away four apples, leaving you with eight apples. Now you ate ¼ of your apples. To find the number of apples you ate, the whole for ¼ was 8 apples, which was the new whole. This background information provided a possible explanation for the students' confusion about the whole.[11] The explanation from the 5th grade participant revealed that her fraction units were invariant and identical. However, her concept of the unit whole was vague.

The performance of both 4th and 5th grade participants suggests that the ability to identify the whole may be another critical part of fostering students' conceptual construction of fractions and fraction units. This understanding may also facilitate students to integrate their understanding of fraction units to solve fraction related problems. The examples presented in this section display different misconceptions of a whole when solving different fractional scheme tasks across grades.

Summary

The following table (Table 4.8) summarizes the performance on the PFS tasks across grades. Six 4th grade participants did the linear model task and five 4th grade participants did the circular model PFS tasks. One participant provided the correct answer when solving the linear model task on the first attempt. Two participants gave the correct answer when solving the linear model task on the first attempt. For 5th grade participants, 17 participants did the linear model PFS task and 19 participants did the circular model PUFS task. Ten out of 17 participants provided the correct answer when solving the linear model task on the first attempt. Fourteen out of 19 participants did the circular model task correct on their first attempt.

[11] The whole would remain the same: 12 apples. After giving 1/3 of 12 apples away, you ate 1/4 of the rest of the apples. Therefore, you actually ate ¼ of 1/3 of all the apples, which was 1/3 of ¾.

Table 4 - 8

Comparing the PFS Task Performance Across Grades

Grades / Models	4th grade (*N* = 10)				5th Grade (*N* = 19)			
	Correct		Iterating		Correct		Iterating	
	f	%	*f*	%	*f*	%	*f*	%
Linear Model	1	10	1	10	10	52.6	12	52.6
Circular Model	2	20	2	20	14	73.7	16	84.6

Note. For the 4th graders, one participant provided the correct answer when solving the linear model task. Two participants gave the correct answer when solving the linear model task. For the 5th graders, 10 participants provided the correct answer when solving the linear model task. Fourteen participants did the circular model task correct.

Results of Solving the RPFS Tasks

Figure 4.19 presents the linear and circular model RPFS tasks in the clinical interview protocol, and the rectangular model task in the back-up interview protocol. In general, to successfully solve RPFS tasks, students first need to be able to identify the unit fractional piece by partitioning the given non-unit proper fractional part into a number of pieces that equals the given numerator. After that, they need to recreate the unit whole according to the denominator based on the given proper fraction. Thus, constructing the Splitting Operation (SO) is a prerequisite for successfully solving RPFS tasks. The performances regarding SO will be discussed later.

Both RPFS tasks required participants to draw the implicit whole based on the given non-unit proper fractional part and the given fractional name of the given fractional part. The main difference between the linear and circular model task was that the linear model task (task #7 in

Figure 4.19) explicitly required participants to draw the whole after describing the fractional relationship between the given line segment and the implied whole. However, the circular model task (task #8 in Figure 4.19) only states the fractional relationship between the given piece of cake and your cake, the cake that participants need to draw. Furthermore, the word "whole" did not appear in the task. If a participant has constructed a RPFS, then this participant could determine that the given cake was not a referent unit whole, but the cake which they were required to draw. Furthermore, to successfully draw the required unit whole, the participant must also realize that the given part needs to be partitioned into five equal pieces because this cake piece was 5/6 of "your cake," the implicit unit whole. After identifying the unit fractional piece, the participant should iterate it six times, the denominator of the given fraction, to find the implied whole cake. The performance of the 4th and 5th grade participants are displayed below.

Figure 4. 19

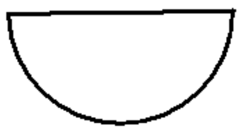
RPFS Tasks in Clinical Interview Protocol and the Back-up Protocol

7. 如图所示的线段和整条线段的 $\frac{4}{5}$ 一样长。请画出整条线段。

   The stick shown below is 4/5 as long as a whole line segment. Draw the whole line segment.

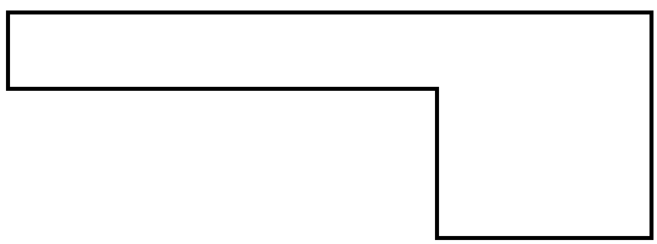
8. 如图所示的这块蛋糕和你的那块蛋糕的$\frac{5}{6}$ 一样大，请你画出你的那块蛋糕。

The piece of cake shown below is 5/6 as big as your piece of cake. Draw your piece of cake.



3. 如图所示的这块巧克力与你的那块巧克力的 $\frac{5}{6}$ 一样大，请画出你的那块巧克力。(RPFS)

The piece of chocolate shown below is 5/6 as big as your piece of chocolate. Draw your piece of chocolate.



Note. The first two tasks are the two RPFS tasks in the clinical interview protocol and the rectangular model task corresponding to the circular model in the Back-up interview protocol.

The Performances of 4th Grade Participants

Four 4th grade participants solved both linear and circular model tasks. Three of them successfully solved the linear model task, and two participants successfully solved the circular model task. Their performance is described below.

Linear Model Task

Among the four participants, three of them successfully solved the linear model task. It seems that the linear model RPFS task was not a difficult task for them. All three participants quickly partitioned the given line segment into four pieces after they read the problem. Then two of them drew a line segment that was a little longer than the given one. The other one added a smaller piece at the right end of the given line segment. Their actions and explanations demonstrated that they easily determined that the line segment they were supposed to draw was the unit whole. Their explanations further indicated that they understood that the unit whole should contain five shares. The given line segment contained four shares. To draw the whole, they simply needed to add one more share. When asked to use a fraction to represent the added piece, two of them correctly identified one-fifth. The other one said she did not know.

In particular, the explanation from one of the participants revealed a multiplicative relationship between the fraction unit and the whole. The following is the protocol of the conversation between one participant named Xiaoyu (Yu) and the researcher (R).

Yu: This line segment is four fifths, so the whole line segment is five-fifths. Adding one more piece is five-fifths.

R: How did you find the piece you added?

Yu: [Pointed to the given line segment] first evenly divide it into four shares, then find that equal, that equal one share and add.

R: Okay. Then I want you to use a fraction to represent the one share you added.

Yu: [She wrote 1/5]

R: Is it one-fifth?

Yu: Yes.

R: What fraction is the given line segment?

Yu: Four-fifths.

R: Then how many one-fifths are in the given line segment?

Yu: It has four one-fifths.

R: You divided the given line segment into four pieces, right? What fraction can represent one of them?

Yu: [She wrote 1/5]

Her explanation reveals her understanding of the relationship between a non-unit fraction and the fraction units, and the relationship between the unit whole and the fraction units.

The solution of the participant who provided a wrong answer indicates that she considered the given line segment as the unit whole. She partitioned this line segment into five shares and drew out one piece having the same length as the first piece on the left of the given segment. She seemed to interpret the RPFS task as a PUFS task. The researcher asked her if this piece was the whole. She erased all her tick marks and her drawing, and drew a very long line segment under the given one. She explained the whole should be five times as long as the given one. Her explanation suggested that she had not yet constructed a RPFS so she could not understand the language of the RPFS task.

Circular Model Task

As previously discussed, the circular model task did not clearly describe the unit whole. To successfully solve this task, a student must be able to determine that the given cake piece was a fractional part of the unit whole. Two participants provided correct answers. Surprisingly, the participant who gave an incorrect answer on the linear model task had a correct solution. She made progress while she investigated the current solution of the linear model task with the researcher's help. Her explanation indicates that she may have progressed from a PFS to a RPFS.

The other participant who provided a correct solution was the one who did the circular PFS model task as a RPFS task in her first attempt. Her performance demonstrated that she understood that the given cake was not the unit whole, but is "five-sixths of my cake." When asked to use a fraction to represent one piece in the partitioned fractional part, she wrote 1/6 without any hesitation. The following protocol displays the conversation between the researcher and the participant named Yu about the fractional name of one piece. Y represents the participant.

R: Oh, why do you say it is one-sixth, but not one-fifth?

Y: Because the total has six shares. So, it is one-sixth, not a total of five shares.

R: Oh, because it is, it is total six shares, it takes five shares, right?

Y: Yes.

R: Explain the piece of cake you drew. Why did you add one more piece?

Y: Because then it is the whole six sixths.

Her explanation is a strong indicator of a RPFS and indicates an understanding of a non-unit fraction and the relationship between a non-unit fraction and its related unit whole.

The Performance of 5th Grade Participants

According to their performance on the PFS tasks, 16 5th grade participants solved the linear model of the RPFS task and 12 participants solved the circular model of the RPFS tasks.

Linear Model Task

Among 16 participants, 14 participants solved this task correctly. Their explanations indicated that they understood that the given line segment was a non-unit fractional part. To find the whole, they needed to partition this part into the number of equal pieces as the numerator indicated. To draw the whole, they simply copied the given line segment and added one unit fractional piece.

Two participants incorrectly solved the task in a similar manner. Both partitioned the given line segment into five shares, instead of four shares. Next, they drew a line segment that was one piece longer than the given line segment. Interestingly, their explanations were similar. The following protocol presents the conversation between one of the participants about her thinking. W represents the participant's name Wenwen.

R: Can you explain your thinking?

W: It said it's four fifth as long as the whole line segment. I first divided this smaller line segment into five shares. Then it said it is four fifth as long as mine [the line segment she was asked to draw]. I look at it [the given line segment] as four-fifths, then move it down and add one small piece. It is the unit whole.

R: How many shares did you divide the line segment into?

W: Five shares.

R: Why did you divide it into five shares?

W: It said it is four-fifth as long as the whole line segment. So, I divided it into five shares.

R: Oh, so you added one share to the line segment. Can you use a fraction to represent that share?

W: [She wrote 1/5]

Her explanation indicates that she understood that the given line segment was a fractional part of the whole. To make a whole line segment, she should have added one more share to the given line segment. However, she divided the given non-unit fractional part into five shares. It seems that her thinking was still at the PUFS level. For the PUFS thinking, dividing the given figure into the number of shares as the denominator indicates could produce the unit fractional part of the referent whole. Although these two participants knew that the given line segment was a fractional part, their PUFS or PFS level of thinking prevented them from moving beyond to understand that the numerator of the given fraction indicated the number of fraction units contained in the given non-unit fractional part, but not the denominator. It seems that understanding the concept of unit whole and the ability to identify the unit whole plays a crucial role in the construction of a RPFS. The performance from those participants who failed to solve the linear RPFS task may suggest that if the referent whole was not clearly presented, then any figure in their visual field would be treated as a unit whole. Another possible reason that prevented them from moving forward may be that they lack the concept of fraction units. In other words, they have not yet constructed the relationship between a proper fraction and fraction units. Consequently, they could not reverse their way of working with PFS related tasks. This will be discussed in the next section.

Circular Model Task

Among 12 participants, 11 of them solved the task correctly. All of them understood that to find the whole, they should find the unit fractional piece. They needed to divide the given cake into five equal pieces. When asked why they divided the given part into five pieces instead of six pieces, all of them explained that was because the part contained five of one-sixth, which was a multiplicative relationship between a proper fraction and fraction units. Only one participant solved this task as a PFS task, the same way she handled the linear model RPFS task.

Fraction Unit and the Implied Whole

When solving the RPFS tasks, it seems that the main difficulty encountered by most of the participants was identifying the whole. Once participants realized that the given figure was not the whole, but a proper fractional part, they quickly applied their multiplicative reasoning about a proper fraction and fraction units to find the unit fractional piece within the given proper fractional part. This means that they understood that the given 4/5 contains four of one-fifth, and the given proper fractional part should be partitioned into four pieces to get one-fifth of the implied whole.

The following protocol illustrates how a 4^{th} grader understood the proper fraction. As discussed above, one of the 4^{th} grade participants seemed to move from a PFS into a RPFS. Her reasoning in solving the circular model RPFS task displays how she understood the relationship between a non-unit fractional part, fraction units, and the related unit whole. The following protocol displayed her understanding with respect to RPFS. T represents the participant named Tontong.

R: Could you explain why you divide this cake [the given half circle] into five shares?

T: The reason why this cake is divided into five shares is, it says, my cake is five-sixths. This means my cake has six shares in total. I divided this [the given half circle] into five shares, then added one share. It [the share she added] is just this big [pointing to one piece in the partitioned given figure], six shares.

R: So what is your purpose of dividing it into five shares?

T: It is to see how big one share is.

R: Great! Do you know what fraction can be used to represent this one share?

T: One share is one-fifth.

R: Really?

T: One-sixth.

This conversation indicates a clear relationship between a non-unit fraction and the unit fraction. Although she initially incorrectly identified the fraction unit, she corrected it as the researcher asked questions to help her reconsider her misconceptions. This participant made amazing progress during the time of just one hour-long interview.

A lack of understanding of the relationship between a proper fractional part, fraction units and a unit whole may also hinder a student to solve the RPFS task. For example, the 5^{th} grade participant discussed above understood that the given fractional part was not the unit whole at the beginning. Thus, to draw the unit whole, she needed to find the fraction unit of the unit whole, which was one-fifth of the unit whole. However, she treated the given fractional part as the unit whole, and divided it into five pieces. Then she drew a line segment that was one piece longer than the given line segment. The conversation below presented her confusion.

R: You divided the above line segment [the given line segment] into five shares, right?

W: Yes.

R: So, what fraction is one share?

W: One fifth.

R: One fifth. So, you divided the above line segment into five shares, one share is one fifth. That means that one [the given line segment] is a unit whole. Is that right?

W: Yes.

R: A unit whole plus a one-fifth, that is five fifths plus one fifth. Then your line segment [the one she drew] is six fifths.

W: The one below [the line segment she drew] is a unit whole.

R: The bottom one is the unit whole. Let's look at it and see how many fraction units are in this [the one she drew] unit whole.

W: (she started to partition the part on her drawing that was exactly underneath the given line segment into four pieces and realized that she made a mistake) it seems here has an extra piece.

R: I think so. Can you read the problem again?

W: (she read the problem again)

R: Do you want to change your answer?

W: Yes.

R: How?

W: I feel that the above line segment should not be divided into five shares.

R: Then how many shares should [you] divide it into?

W: … (she thought for a few seconds, but could not figure it out)

R: Okay, let me ask you in this way. Now you have two line segments, right?

W: Yes.

R: Then I want to ask you how many unit wholes are here.

W: One.

R: (Trying to perturb her) One or two?

W: Two.

Her solution indicated that she understood that the fraction unit for this task was one fifth, and the given line segment was a fractional part. However, when she needed to find the fraction unit, she was unable to reverse the operations of the PFS and find the fraction unit from the proper fractional part.

Summary

The following table (Table 4.9) summarizes the performance on the RPFS tasks across both grades. Four 4th grade participants completed the linear and circular model RPFS tasks. Among the 5th grade participants, 16 participants completed the linear model RPFS task and 12 participants completed the circular model RPFS task.

Table 4 - 9

Comparing the Success Rate of the RPFS Task Across Grades

Grades / Models	4th grade		5th Grade	
	Correct		Correct	
	N	*%*	*N*	*%*
Linear Model	3	30	14	73.7
Circular Model	2	20	11	57.8

Note: For 4th grade participants, three of them solved the linear model task correctly. Two of them provided a correct answer to the circular model task. Among the 5th grade participants, 14 participants solved the linear model task correctly and 11 participants solved the circular model task correctly.

Results of Solving the IFS Tasks

Figure 4.20 presents the linear and circular model IFS tasks in the clinical interview protocol, and the rectangular model task in the back-up interview protocol. The IFS is the most advanced scheme of Steffe's fractional schemes. To successfully solve the IFS tasks, students need to move beyond the part-whole concept of fractions. They need to construct the understanding of the concept of fraction units, the measurement concept of fractions, which is the understanding of the relationship between fraction units and a unit whole, as well as three levels of units coordination. For the current study, the discussion would focus on the understanding of the concept of fraction units and the relationship between the fraction unit and a unit whole.

For both tasks, participants were required to draw the whole where the given fractional part was an improper fractional part relative to the unit whole. This means the given fractional part is bigger than the whole that they were asked to draw. They were similar to the RPFS tasks. The linear task explicitly required participants to draw the unit whole, but the circular model task did not.

Figure 4. 20

IFS Tasks in Clinical Interview Protocol and the Back-up Protocol

9. 下图所示的线段与整条线段的$\frac{7}{3}$一样长。请画出整条线段。

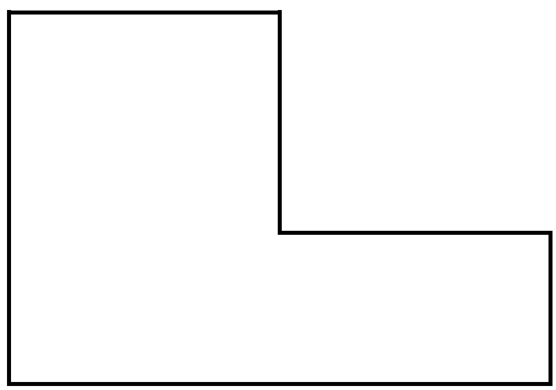
The bar shown below is 7/3 as long as a whole line segment. Draw the whole line segment.

10. 下图所示的披萨饼与你的那块披萨饼的$\frac{7}{5}$一样大。请画出你的那块披萨饼。(IFS)

The piece of pizza shown below is 7/5 as big as your piece of pizza. Draw your piece of pizza.

4. 下图所示的巧克力与你的那块巧克力的$\frac{7}{5}$一样大。请画出你的那块巧克力。(IFS)

The piece of chocolate shown below is 7/5 as big as your piece of chocolate. Draw your piece of chocolate.



Note. The first two tasks are the two IFS tasks in the clinical interview protocol and the rectangular model task corresponding to the circular model in the Back-up interview protocol.

Performance from 4th Grade Participants

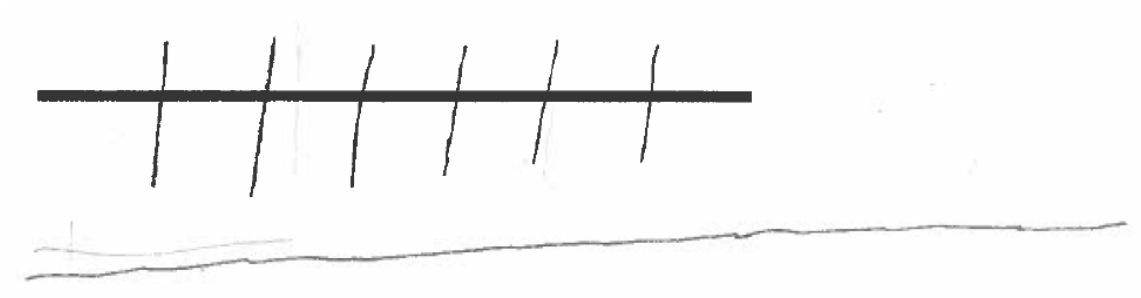
Linear Model Task

According to the participants' performance solving the previous tasks and their performance of solving the linear IFS task, only three 4th grade participants solved the linear model task. Because it seems difficult for them to understand the language of the IFS task when they solved the linear model IFS task, none of them were required to solve the circular model task. All three participants solved the linear model IFS using their PWS or RPFS knowledge.

One participant first partitioned the given line segment into three pieces. That means she understood that the whole should include three shares. Then she drew one piece below the given line segment. Then she paused and erased all of her drawing. Next, she re-partitioned the given line segment into seven equal pieces. At last, she made a copy of the given segment and extended the line segment by adding three smaller pieces that had the same size as the smaller pieces within the partitioned line segment (see Figure 4.21). When asked why she divided the line segment into seven shares, she started to say, "because it said this [the given line segment] is seven out of …", stopping her spoken thought as she hesitated. Her explanation demonstrated that her first attempt tried to take seven pieces from three shares which is impossible. When the researcher asked her to give a fraction name to one of the pieces in the partitioned line segment, she said one-seventh, again, with hesitated tone. Obviously, she was confused as she did not know whether the whole should contain three shares or seven shares.

Figure 4. 21

Response of Linear Model IFS Task from One 4th Grade Participants



Note. The response from a 4th grade participant solving the IFS linear task with her part-whole concept of fractions.

Another participant directly said she did not know how to divide the line segment into three shares and take out seven shares. The last participant did exactly the same thing the first participant did on her second attempt to solve the problem. The researcher asked why she divided the line segment into seven shares, and she responded because it was not the whole. The whole should have three shares. Their explanations indicate that they were unable to identify the unit whole when they encountered an improper fractional part with their part-whole concept of fractions.

Circular Model

It was difficult for the three participants who did the linear model task to understand the IFS linear model task. Thus, it took them a longer time to finish the linear task. To ensure these participants had enough time to finish the SO tasks, they were not required to do the circular model task.

The Performances of 5th Grade Participants

According to the participants' performances of solving the previous tasks, 12 5th grade participants solved the linear model task and nine participants solved the circular model task.

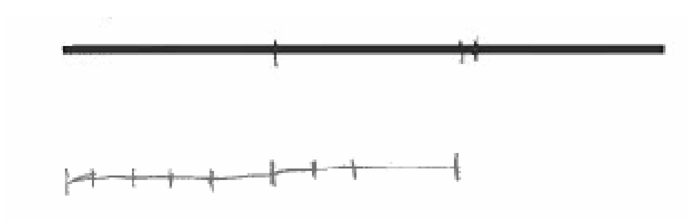
Linear Model

Among the 12 participants who solved the linear model, 11 provided correct solutions and correct explanations. In their explanations, they were able to indicate that the fraction unit of 7/3 was 1/3. The multiplicative relationship was revealed when interpreting the given fraction 7/3 as 7 of 1/3s. They explained that they divided the given line segment into seven shares in order to determine the size of the fraction unit of 1/3 of the whole. At last, they understood that the whole includes 3 of 1/3s. When asked what fraction can be used to represent one share of the partitioned given line segment, all of them could identify it as 1/3.

Figure 4.22 presents the problem-solving process provided by the participant who answered the task incorrectly. Her work illustrates a lack of ability to identify what the unit whole should be. After she partitioned the given line segment into three pieces, she drew two connected pieces below the given line segment. Next, she divided her drawing into pieces, and paused for a while. The researcher asked her to describe her thinking. The following protocol represents the conversation between the researcher and the participant named Qiqi represented by the letter Q.

Figure 4. 22

Response for the Linear Model IFS Task from One of the 5th Grade Participants



Note. The response from a 5th grade participant who failed to solve the IFS task because she failed to identify that the given stick was a fractional part of the implied unit whole.

R: Okay, Qiqi, can you tell me about your thinking? Why did you partition the given line segment into three shares?

Q: Because … (She was thinking)

R: Why?

Q: Because seven thirds of the whole line segment, so I need to divide into three shares.

R: Because the denominator is three, so you divided into three shares, right?

Q: Yes.

R: When you drew the bottom line segment, what were you thinking?

Q: I am not sure [whether] I should divide it into seven shares or three shares.

Obviously, she was confused when the numerator was greater than the denominator. She knew that the whole line segment only contains three shares, but the numerator indicated the line segment should have seven shares. It seems that she could not accept that the part was longer than the whole. She tried to fit seven shares within three shares, but she could not. In other words, her whole was lost. Her explanation also illustrates that she could not go beyond her part-whole concept of fraction to understand an improper fraction.

Circular Model Task

All nine participants were able to correctly solve the circular model task. Although some of them still struggled to partition the given half circle into seven equal shares, most of the participants' reasoning was similar to the reasoning they provided for the linear model.

The Role of Fraction Units

The explanations given by those 5th grade participants who correctly solved the IFS tasks indicated that these participants conceptually understand fraction units and the relationship between any fraction and its fraction units. For example, the following protocol presents one of the conversations between the researcher and a 5th grade participant. Jia represents the name of this participant.

R: Can you explain your thinking?

Jia: This seven-thirds, this numerator is, is this straight line [used her pencil to trace the given line segment]. So divide this line into seven shares. My line and this line, this, this is seven-thirds …

R: Okay. Let me help you a little bit. I want to ask you, what is the fraction unit of that fraction [pointing to the given fraction]?

Jia: One-third.

R: So, you divided the above line segment into seven shares, right?

Jia: Yes.

R: Why didn't you divide it into three shares instead of seven shares?

Jia: Because there are seven of one-third in seven-thirds.

R: Very good. Now I want you to use a fraction to represent one of the shares, one share on the long line segment (the given line segment).

Jia: [She wrote 1/3.]

Jia's explanation demonstrates her understanding of the improper fraction in addition to her understanding of fraction units.

Below is another example that illustrates how a conceptual understanding of the relationship between a fraction and fraction units helped the participant correct her misconception and move beyond a part-whole concept of fractions. After partitioning the given pizza (i.e., the half circle) into seven pieces, this 5th grade participant paused. She seemed to be struggling with the next step. The conversation began at this point. XY represents her name Xiaoyin.

R: You divided this pizza into seven shares, right?

XY: Yes.

R: Why?

XY: Because this pizza [the given pizza] is seven-fifths of my part. That means it has, has seven one-fifths.

R: Oh, great. Now you're supposed to draw your pizza. You didn't draw it. Why?

XY: Because I am not sure, this … uh …

R. Not sure what?

XY: uh … this …

R: [Thinking she must be confused about the whole] How do you think your pizza should be drawn?

XY: I feel my part should be this … uh … this [pointing to the partitioned figure]. One share of this pizza (the given pizza).

R: One share?

XY: [No verbal response]

R: Okay. You divided this pizza into seven shares, didn't you?

XY: Yes.

R: Can you use a fraction to represent one of the shares?

XY: [She wrote 1/5]

R: Now what do you think about your part? How can you draw your part?

XY: [She copied one of the pieces beside the given half circle and iterated it four times]

R: Okay, I want to know how many small pieces are in your pizza.

XY: Five.

Xiaoyin's last answer indicates that she understood that the denominator tells her the total number of shares in the whole. She might try to shade five shares within the partitioned whole at first, and then noticed that the unit whole would be embedded in the given fractional part. She seemed confused. After the researcher helped her to reorganize her fraction knowledge, she apparently overcame her misconceptions regarding the part-whole concept of fractions and used the fraction units to reproduce the referent whole.

Summary

The following table (Table 10) summarizes the participants' performance on the IFS tasks across both grades. Only three 4th grade participants solved the linear model IFS task. Among the 5th grade participants, 12 participants solved the linear model IFS task and nine participants solved the circular model IFS task.

Table 4 - 10

Comparing the Correction Rate of IFS Task Across Grades

Grades / Models	4th grade		5th Grade	
	Correct		Correct	
	N	*%*	*N*	*%*
Linear Model	0	0	11	57.9
Circular Model	0	0	9	47.4

Note. For 4th grade participants, three of them completed the linear model IFS task, but none of them provided a correct answer. Among the 5th grade participants, 11 out of 12 participants completed the linear model IFS task. All nine participants who completed the circular model IFS task provided a correct answer.

Results of Solving the SO Tasks

Two selected SO tasks used in the clinical interview are shown in Figure 4.23. For this task, a line segment was given and participants were asked to draw a stick if the given line segment was five times as long as the stick that they would draw. For the circular task, participants needed to draw a slice of pizza if the given pizza (a half circle) was three times as big as the pizza slice they would draw.

Figure 4. 23

SO Tasks in Clinical Interview Protocol and the Back-up Protocol

1. 如图所示的线段长度和另一条线段的 5 倍一样长，请画出另一条线段。

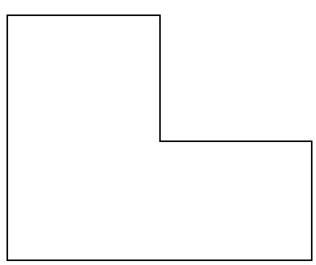
The stick shown below is 5 times as long as another stick. Draw the other stick.

2. 如图所示的这块匹萨饼跟你的那块匹萨饼的 3 倍一样大，请画出你的那块匹萨饼。

The amount of pizza shown below is 3 times as big as your slice. Draw your slice.

5. 下图所示的这块巧克力和你的那块巧克力的6倍一样大，请画出你的那块巧克力。

The amount of chocolate shown below is 6 times as big as your piece of chocolate. Draw your piece.



Note. The first two tasks are the two IFS tasks in the clinical interview protocol and the rectangular model task corresponding to the circular model in the back-up interview protocol.

The Performances of 4th Grade Participants

Due to time constraint, one participant did not have time to solve the SO tasks. The other nine 4th grade participants finished both linear and circular SO tasks.

Linear Model Task

Only two participants were able to solve the linear model task on their first attempt. Although they divided the given line segment into five pieces with slightly different sizes, they understood that "the given stick is five times as long as another" means the one they were required to draw should be one fifth of the given stick. Thus, the given line segment should be divided into five equal shares and then draw out one share. When asked to represent the one share using a fraction, they provided the correct answer of 1/5.

After reading the problem, three participants made five connected copies of the given line segment. Explanations from two participants indicates that they swapped the subject of the sentence. One participant said, "The stick shown [pointing to the given stick] is five times as long as the other stick. The stick I drew was about five times the length of this stick (the given stick)." The researcher asked him to read the problem again. After he re-read the problem, he stated that his answer was correct. His response was clear evidence that he had misinterpreted the problem. Another two participants realized their mistake while explaining their thinking. One of these two participants divided the given line segment into five pieces and drew out one piece. When the researcher asked her to explain why she made the mistake. She said that she realized the given line segment should be the whole. Then she used a multiplicative relation to explain the relationship between the given one and the one required to draw. She said that five times of the line segment required her to draw equal the given line segment. It seems that the area model helped them gain more understanding of the SO task.

The other participant solved the task correctly. Then she erased all her marks and started to continually draw five connected copies of the given line segments. After she was done, she erased her drawing again. The following conversation happened at this point. Q represents the participant named Qingqing.

R: Okay, Qingqing. I saw you first divided the given line segment into five shares, right?

Q: Yes.

R: Then you erased all of your drawing. The second time you drew a longer stick, including three or four shares, right?

Q: Uh [yes]

R: Why do you change your solution?

Q: Um … I feel I am wrong.

R: Okay. Can you read the problem again?

Q: (she read the problem out loud. Then, it seemed she still did not know how to solve it)

R: (Feeling that the pizza task, the circular SO task, may be easier because it was closer to the real life) Qingqing, how about you do the next task [the circular SO task] first. She solved the circular SO task easily, and then she quickly solved the linear model task in the correct way. It seems that the circular model helped her gain more understanding of the linear model task.

Two other participants made one copy of the given line segment. They explained that the phrase "as long as," meant they should make a copy of the given line segment. Their explanations indicate that they did not comprehend the language of the SO tasks. The last participant solved this task as a PWS task. He continually drew four connected line segments under the given line segment. When asked to explain, he responded that four line segments plus one line segment [the given line segment] made five line segments. His PPCF explanation was similar to the PPCF explanation that he gave when he solved the linear PWS task.

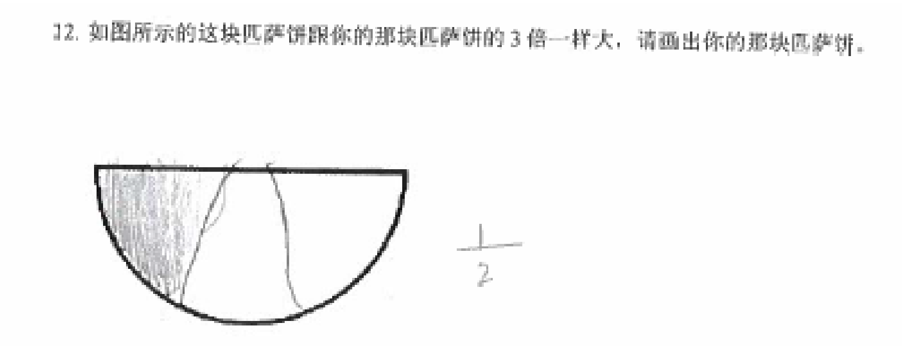
Circular Model

Five of the participants successfully solved the circular model task. It seems that the circular model was easier for these participants than the linear model task although two out of five participants still struggled with the phrase “as big as.” The other two participants’ solutions were similar to the one shown in Figure 4.22. However, they explained that they should evenly divide the given pizza into three pieces and take one out even though the size of the pieces were obviously different. When asked to give a fraction name to the piece they drew, they all provided the correct answer of 1/3.

Figure 4.24 displays the solution from one of the participants. His solution alone may provide a weak indicator of the SO. However, when asked to name the shaded part, he wrote his answer as ½. His answer indicates that he was a PPCF. The solution provided by the last participant seemed to be a weak indicator of the SO. He drew a smaller piece that was close to 1/3 of the half circle. However, when asked how he knew that piece was the correct size, he drew a radius in the middle of the half circle and explained: “separate in the middle and add this piece [pointed to the one he drew]. There are three pieces.” His explanation suggests a PPCF even though he had not constructed a partitioning operation yet.

Figure 4. 24

Response for the Circular Model SO Task from One of the 4th Grade Participants



Note. A 4th grade participant's response to the SO circular task with the shaded piece representing the fraction.

The Performance of 5th Grade Participants

All 19 5th grade participants finished both linear and circular model tasks. Only two participants were unable to solve both linear and circular model tasks. One of these two participants solved the linear model task correctly. When asked to explain her thinking, this participant changed her solution to draw five copies of the given line segment. The researcher asked her to explain the reason for changing her answer. The participant responded, "Five times does not mean 'divide into five shares,' but rather 'make bigger.'" Thus, she also produced three copies of the half circle when she solved the circular model task. After reviewing her performance on other tasks, the other participant who provided an incorrect solution was classified as a PPCF. It is reasonable to assume that she did not comprehend the language of the SO task.

Seventeen participants successfully solved both linear and circular model tasks. When asked to explain their reasoning, 14 out of 17 participants presented similar explanations. In their minds, the given stick is five times as long as their stick, meaning that their stick is one-fifth of the given stick. They converted the phrase "5 times," the multiplicative relationship, to a fractional relationship. In particular, there were two participants who used division while one participant used multiplication to explain how they found the required piece. The explanation for using division was that "5 times" means the given stick is divided by five. The multiplicative explanation was that multiplying their stick by five produced the given stick. By comparing the performance of 5th grade participants on the SO tasks with their performance on the circular model of PUFS, noticeably the number of participants capable of using the iterating operation to solve the circular model of PUFS (i.e., 5 participants) was less than the number of participants who successfully solved both SO tasks (i.e., 17 participants). A possible explanation for this might be that some participants may not have truly constructed the splitting operation, but merely applied their existing concepts about division to solve the SO tasks. Table 4.11 summarizes the number of participants in each grade solving the tasks correctly on their first attempt.

Table 4 - 11

Comparing the Correction Rate of SO Task Across Grades

Grades / Models	4th grade		5th Grade	
	Correct		Correct	
	N	%	*N*	%
Linear Model	2	20	16	84.2
Circular Model	5	50	17	89.5

Note. For the 4th grade participants, two of them solved the linear model task correctly. Five participants provided a correct answer to the circular model task. For the 5th grade participants, 16 out of 19 participants successfully solved the linear model task, 17 out of 19 participants successfully solved the circular model task at their first attempt.

Chapter Five: Discussion

The goal of this study was to explore how a conceptual understanding of fraction units facilitates Chinese students' construction of their fraction knowledge. Using an explanatory sequential mixed methods research design, two consecutive phases were included in this study. Therefore, the discussion presented in this chapter integrated the results from analyzing both quantitative and qualitative data. In particular, the discussion focused on investigating the role of Chinese students' conceptual understanding of fraction units in their construction of fraction knowledge.

Fraction Schemes Constructed Before and After the Second Teaching Phase

In 2018, Norton, Wilkins and Xu extended Norton and Wilkins quantitative studies (e.g., Norton & Wilkins, 2009, 2010, 2012; Wilkins & Norton 2011) to Chinese students. They used the same written assessment given to the American students to test the fraction schemes of 76 5th and 6th grade Chinese students. The results indicate that the fraction learning trajectory of these Chinese students was similar to the developmental progression of their U.S. counterparts. The present study used a sequential mixed methods design. It not only repeated the Norton, et al (2018) study to assess 534 Chinese 4th and 5th grade students quantitatively, but focused on exploring how Chinese students construct their fraction knowledge with respect to the conceptual understanding of fraction units as well as the understanding of the relationship between fraction units and the referent unit whole qualitatively.

Norton, Wilkins and Xu (2018) examined 76 Chinese 5th and 6th grade students' fraction knowledge using fraction schemes, a developmental progression identified by Western scholar Steffe and his colleague (Steffe & Olive, 2010). Their results indicate that the process of Chinese students constructing their fraction knowledge is similar to their U.S. counterparts. However, the written assessment they used did not include the most advanced fraction scheme, the iterative fraction scheme (IFS), when they assessed the 6th grade students. To gain a better understanding of the developmental progression of Chinese students, the written assessment for 5th grade students in this study included the IFS tasks. In addition, the results from the Norton et al. study revealed that all 6th grade students successfully solved the PWS tasks, meaning that they all had constructed the PWS. Thus, the written assessment used for 5th grade students in this study excluded the PWS tasks. This was the first difference between the current study and Norton et al.'s study although the quantitative phase of this study replicated the assessment from Norton et al.'s study.

The second difference between these two studies is the number of participants and the grades of the participants. The present study involved 534 4th and 5th grade Chinese students enrolled in one elementary school located in Southwest China. As described in Chapter 2, China has adopted two different elementary and middle school systems. One system is the 5-4 system, consisting of five years of elementary and four years of junior secondary school study. The other system, the 6-3 system, involves six years of elementary and three years of middle school study. The elementary school that participated in the Norton et al. study was enrolled in a 6-3 system. Therefore, the 5th grade students in their study were informally introduced to the concept of fractions in 3rd grade, and would learn the formal definition and notion of the fraction units after the assessment of their study. The 6th grade students in their study had learned the formal definition and notion of the fraction units when the written assessment was administrated. The elementary school that participated in the current study implements a 5-4 system. Under the same national curriculum guideline, the arrangement of fraction topics in the textbooks that these participants used differs slightly from the arrangement in the textbook those participants used in Norton et al.'s study.

For both systems, the initial fraction teaching phase starts during 3rd grade. However, the second teaching phase, "Re-recognize Fraction," in the 5-4 system begins during the spring semester of 4th grade, while this same teaching phase started in late fall semester of 5th grade in the 6-3 school system. Therefore, when the quantitative data was collected in September 2018, the 4th grade students had finished their initial learning phase prior to their re-cognizing fractions learning phase. The 5th grade students in this study had learned the formal definition and notion of the fraction units.

The Progression of Fraction Schemes in 4th and 5th Grade

The results from analyzing both quantitative and qualitative data suggested that both 4th and 5th grade Chinese participants processed through a similar developmental progression of fraction schemes as their U.S. counterparts. Table 4-1 presents the frequency and proportion of all 4th and 5th grade students in this study who had constructed each scheme and operation. Table 5-1 summarizes the frequency and proportions of correct answers provided by 4th and 5th grade participants on their first attempt during the clinical interview.

Table 5 - 1

Comparison of Schemes and Operation within Grade and Across Grade

Schemes &Models \ Grades		4th Grade (*N* = 10)				5th Grade (*N* = 19)			
		Correct		Iterating		Correct		Iterating	
		f	%	*f*	%	*f*	%	*f*	%
PWS	Linear	8	80			15	$\frac{15}{17}$ % = 88.2		
	Circular	7	70			7	$\frac{7}{9}$ % = 77.8		
PUFS	Linear	6	60			14	73.6		
	Circular	1	10	1	10	8	23.6	5	23.6
PFS	Linear	1	10	1	10	10	52.6	12	63.2
	Circular	2	20	2	20	14	73.7	16	84.6
RPFS	Linear	3	30			14	73.7		
	Circular	2	20			11	57.9		
IFS	Linear	0	0			11	57.9		
	Circular	0	0			9	47.4		
SO	Linear	2	20			16	84.2		
	Circular	5	50			17	89.5		

Note. The data collected in this table were based on participants' performance during the clinical interview. All the percentages were calculated based on the clinical interview data. The frequencies of correctness in the table were counted when a participant correctly solved the task on his or her first attempt. The total number *N* represents the numbers of participants in each grade:10 4th grade participants and 19 5th grade participants.

Both tables show a declining trend in the proportion of schemes for each grade. The lower percentages may indicate that those tasks are more challenging. Students with available fraction knowledge were not able to solve the tasks related to higher level fractional schemes. Moreover, the Cochran's Q test presented in Chapter 4 confirmed that the difference between the proportions of participants constructing schemes in each grade was statistically significant ($\chi^2(3, N = 254) = 403.84$, $p < .05$ for 4th grade and $\chi^2(3, N = 252) = 101.82$, $p < .05$ for 5th grade, retrieved from Chapter 4). The post-hoc Dunn's test from Chapter 4 confirmed that, for 4th grade participants, the differences between PWS and PUFS, PWS and PFS, and PWS and RPFS were statistically significant.

Particularly worth noting is the comparison of the number of student constructions for PWS with the number for PUFS among 4th grade students (Table 4-1). The number of correct responses drops rapidly from 169 for PWS, a scheme at the lowest level of the set of fractional schemes, to only 31 for PUFS, the scheme right after PWS at the level of fraction schemes. For 5th graders, comparing the number of student construction for PUFS with the number for PFS also drops significantly from 60 to 18.

The performance of participants in the clinical interview supports the findings from the quantitative analysis (see Table 5-1). Among 4th grade participants, 80% provided correct answers for the linear model PWS task and 70% correctly answered the circular model PWS task. Starting with PUFS the numbers rapidly dropped. In addition, none of the 4th grade participants were able to solve the IFS tasks, the scheme located at the highest level in the set of fractional schemes. For 5th grade participants, the number of correct responses for each scheme eventually decreased as well.

Analyzing the 4th grade participants' performance during the clinical interview indicates that some students may not completely construct PWS. For example, one of the 4th grade participants provided correct answers to the linear model task. When he explained his thinking for the linear model task, he used the phrase "taking out two things from five things." His explanation seems to indicate that he used part-whole reasoning. His written assessment also revealed that he had constructed a PWS. When the researcher asked him what the unit whole was, he responded that the unit whole consisted of all seven things. His response suggests that he had failed to construct a part-whole concept of fractions by using a part-part concept of fraction (PPCF). His explanation for the circular model task confirms that he used a PPCF.

For the circular model PWS task, a circle (i.e., pizza) was partitioned into six slices and five of them were shaded. The one unshaded slice represents the eaten piece. The task required students to give a fraction name for this eaten slice of pizza. This 4th grade participant used the fraction 1/5 to represent the eaten slice. When asked to explain his thinking, he stated, "There are five things and one was eaten. It changed to one fifth." When asked how many pizza slices were in the whole pizza, he said five, even though he counted the number of slices starting from the unshaded eaten one. His counting action indicates that he may be aware of the eaten slice. However, the unit whole changed after one slice was eaten. His explanation shows that he used a PPCF. He had not constructed the concept of unit whole as well.

The participants having a PPCF in mind were unable to solve the PUFS tasks because the given unit wholes in both PUFS tasks were unpartitioned. When the same participant solved the PUFS linear model task, he made seven copies of the given line segment. When asked why he drew seven line segments, he erased one and said that he needed to draw one-seventh of the given line segment. In other words, he needed to visualize seven items to form one-seventh of the given item. This example illustrates one of the reasons why the numbers dropped quickly from PWS to PUFS. It also provides a caution to classroom teachers that students could use PPCF to solve some PWS tasks. After students solve the tasks, teachers may need to ask some follow-up questions to ensure that students have accurately constructed a PWS.

Fraction Knowledge Before and After "Re-recognize Fraction" Teaching phase

Comparing the corresponding frequency and proportion for each scheme across grades in Table 4-1 and Table 5-1 suggests that the number of participants who were able to successfully solve each scheme increased from 4th to 5th grade, especially for the more advanced schemes. An across-grades chi-square test was conducted to examine if these changes were statistically significant. Since the written assessment for 5th grade participants in this study did not include the PWS tasks and the written assessment for 4th grade participants excluded the IFS tasks, the chi-square test examined the difference in the proportions for the following three schemes: PUFS, PFS, and RPFS. Recall from Chapter 4 that a statistically significant association between the development of fraction knowledge across grades was found (PUFS $\chi^2(1, N = 506) = 11.55$, $p < .05$; PFS, $\chi^2(1) = 6.399$, $p < .05$; and SO, $\chi^2(1) = 6.663$, $p < .05$, retrieved from Chapter 4). The results from a three-way mixed ANOVA analysis also revealed a statistically significant main effect associated with grade levels ($F(1, 504) = 26.76$, $p < .05$, $\eta^2 = .05$, retrieved from Chapter 4) when comparing 4th grade participants' performance with 5th grade participants' performance on linear model and circular model task for each scheme. That means, on average, 5th grade participants performed better than 4th grade participant regardless of schemes and models.

Iterating Operation and Splitting Operation

As described above, the descriptive statistics from analyzing quantitative data shows a big change in the number of successes related to the PWS to the numbers related to the PUFS. The clinical interview reveals that another main reason for this change may be the lack of the iterating operation.

Several research studies (e.g., Hackenberg, 2007, Norton & Wilkins, 2009, 2010, 2013; Steffe, 2002, 2004; Wilkins & Norton, 2013) indicated that the development of advanced schemes relies upon the splitting operation. The splitting operation requires a coordination of partitioning and iterating operations (Hackenberg, 2007, Norton & Wilkins, 2012, 2013; Wilkins & Norton 2011, 2018). Studies (e.g., Norton & Wilkins, 2013; Olive & Vomvoridi, 2006; Wilkins & Norton, 2018) have also demonstrated that the iterating operation plays a critical role in the construction of a measurement concept of fractions and the splitting operation. Specifically, when exploring the relationship between the partitioning and iterating operations, the operations of partitioning and iterating and PUFS, PUFS and SO, and PFS and SO, Wilkins and Norton (2011) found that the construction of the partitioning and iterating operations were not parallel. Students constructed PUFS before they had constructed the splitting operation, but after they had constructed the partitioning and iterating operations. Furthermore, students could not construct PFS until they had constructed SO. Findings from the clinical interview in this study aligned with their findings.

The descriptive statistics shown in Table 4-1 reveal that there were more students in both grades capable of solving the PUFS tasks compared to the number of students who successfully solved the SO tasks. To successfully solve the SO tasks, students need to be able to coordinate both the partitioning and iterating operations. However, the data in Table 5-1 presents a different case. Table 5-1 shows that there were more 4th grade participants who could successfully solve the linear model PUFS tasks than participants who solved the linear model SO task successfully during the clinical interview, which aligns with the descriptive statistics table (Table 4-1). However, there were less 4th grade participants who solved the circular model PUFS task compared to the number of 4th grade participants who solved the circular model SO task. At the same time, it was noticeable that Table 5-1 also shows that the number of participants who successfully solved the circular model PUFS task in both grades was far less than the numbers of participants who successfully solved the linear model PUFS tasks. One of the main reasons revealed from the clinical interview indicates that the only available operation for almost all 4th grade, and some 5th grade, participants was the partitioning operation. It seems that it was feasible for participants to use the partitioning operation alone to solve the PUFS linear model task. However, when they encountered the circular model PUFS task which required the use of the iterating operation, they failed to solve the task. The linear model PUFS task required participants to find one-seventh of the given line segment. Although this task called for the partitioning operation, the performance from most 4th grade and some 5th grade participants reveals that their partitioning operation may just aim to divide the line segment and make seven pieces with their available PWS. Thus, their partitioning operation may not be an iteration in nature, but rather a reflection of their part-whole concept of fractions. The circular model PUFS provided an opportunity for the researcher to gain a deeper understanding about whether participants had constructed the iterating operation or not.

With a given unpartitioned circular cake, the circular model PUFS task required participants to estimate the size of the given unit fractional cake piece relative to the given referent whole. The method used by 5th grade participants who successfully solved this task was to measure the arc of the given unit fractional piece, and iterate this measurement into the given circle (i.e., the unit whole). They meant to identify the size of the unit fractional piece through finding the number of iterations that allowed them to reproduce the given referent unit whole. This approach indicates that they used the iterating operation to solve the task alone, but their iterating operation used in service of solving the circular model PUFS task by nature simultaneously involved the partitioning operation. It is reasonable to argue that the participants who used the iterating operation to solve this circular task had constructed the splitting operation. Their performance of successfully solving the tasks related to the more sophisticated schemes later demonstrated this assumption. Furthermore, having17 out of 19 5th grade participants successfully solve both SO tasks affirms this argument (Table 5-1).

In fact, not all 5th grade participants were able to use the iterating operation to determine the size of the given unit fraction on their first attempt. The heuristic thinking for solving the circular PUFS model for some 5th grade participants and most 4th grade participants was to cut the circle into four or eight equal pieces using diameters. Their thought process indicated that they likely only used the partitioning operation but not the iterating operation. However, having been introduced to the formal definition of fraction units, it seems that these 5th graders could accommodate the situation and adjust their previous methods after the perturbation from the researcher.

Ten out of 18 5th grade participants, in the clinical interview, initially used diameters to cut the given circle on their first attempt. After the researcher asked them if the size of the pieces in the partitioned circle was the same as the given smaller unit fractional piece, three participants promptly modified their approach by using their fingers to measure the arc of the given fraction unit piece, iterating their finger measurement into the given circle, and finding the correct answer. Their performance indicates that the iterating operation was triggered by perturbation. The other two participants realized that the sizes between the pieces in the partitioned circle differed from the size of the given smaller unit fractional piece. However, they struggled to find a way to partition the given circular cake so that the size of the pieces had the same size as the given unit fractional piece. Then the researcher let them solve the rectangular model back-up task. After they read the rectangular model task, they first drew a horizontal line to cut the larger given rectangle into half. Then they used their fingers to measure the length of the given smaller rectangle and iterated this measurement onto the length of the larger rectangle. Obviously, their iterating operation was triggered at this moment. Figuring out how to solve the rectangular model helped them solve the circular model PUFS task in the same manner, using their fingers to measure the arc of the given smaller piece, iterating their finger measurement. It is reasonable to assume that having a relatively deep understanding of fractions makes the iterating operation easier to be triggered. Moreover, after these 5th grade participants successfully solved the PUFS circular model task, they could also use the iterating operation to solve the circular model tasks for the other more sophisticated schemes tasks.

These findings may provide qualitative evidence that affirms the findings from the Norton and Wilkins (2011) study. Their quantitative data suggests that the construction of the partitioning and iterating operations is not in parallel. Participants in this study demonstrated that students construct the partitioning operation first. At this point, it was reasonable to consider that those 5th grade participants who were able to solve the PUFS circular model task with the researcher's involvement were quasi-SO, meaning they had not constructed the SO yet, but were in the transition toward the SO at the time they solved the PUFS task. They may need more support from teachers who provide instructional activities to scaffold the transition to the construction of the iterating operation.

In comparison to the 5th grade participants, the iterating operation of the 4th grade participants was not easily triggered. The scheme theory presented by Glasersfeld (1995) states that the associated activity is triggered only if the existing scheme is activated. It is reasonable to conjecture that students' informal knowledge about dividing a circle may come from their everyday life experience. Their available limited fraction knowledge combined with their life experience may obstruct the activation of the iterating operation. When a 4th grade participant needed to determine the size of the unit fractional part, with the partitioning operation alone, students who had not constructed the iterating operation determined the size based on their visual estimation. After they cut the circle into four or eight pieces, the researcher tried to perturb their thinking using the same question posed to 5th grade participants. Instead of comparing the sizes with the measurement, some 4th grade participants responded that the sizes were almost the same. A review of their solution for the PUFS linear model task found that these participants divided the given line segment into seven pieces with different sizes. When they drew out the one-seventh of the given line segment, they either drew the one similar to the longest piece among the seven pieces, or the shortest one among the seven pieces.

The remaining 4th grade participants recognized the size difference, but did not know how to partition the circular cake. Only one participant's iterating operation was triggered after the researcher's question. She started to use her fingers to measure the arc of the given fraction unit piece and iterated it into the given circle. She was also the only 4th grade participant who successfully solved the circular model of the PFS and RPFS tasks. The rest of the participants experienced difficulty solving the circular model PFS task with their available partitioning operation. When they tried to partition the half circle into five shares for the PFS circular model task, they used a similar method of drawing a radius in the middle of the given half circle to cut it into half, then using two radii to cut the two halves into halves again. Obviously, they still used the partitioning operation, but not the iterating operation. The performance of these 4th grade participants was not surprising. According to Chinese curriculum guidelines, they had just finished the initial learning phase about fractions. They had not yet been introduced to the formal definition of fractions and the formal definition of fraction units. With their limited fraction knowledge, their iterating operation was difficult to trigger.

In summary, after analyzing the performance of participants in both grades, it may be assumed that the partitioning operation used by the students who only had a PWS may not contain the iterating meaning. To find the required fractional part of the whole, they simply used the partitioning operation to create the number of pieces that were the same as the denominator. It can be concluded that the iterating operation may play an important role in the transition from PWS to PFS.

Solving the Fractional Tasks without a Given Fraction Name

Another interesting phenomenon appeared during the pilot study. During the pilot study, four out of six participants could easily solve the RPFS tasks, but experienced difficulty solving the PFS tasks. A similar phenomenon also revealed in Table 4-1 and Table 5-1. Table 4-1 shows that the number of successes with RPFS tasks is greater than the number for PFS among 4th grade participants. Specifically, this phenomenon can also be seen in Table 5-1 when comparing data for the frequencies of correctness between linear model PFS and RPFS tasks for both grades. Comparing the linear model of PFS and RPFS tasks reveals that the difference between these two tasks is that the linear model RPFS task provided the fraction name of the given non-unit fractional part. However, the linear model PFS task required participants to determine the fraction name for the given non-unit fractional part in terms of the given referent whole.

In other words, participants needed to use the iterating operation to determine that the given shorter line segment was not a unit fractional part of the given referent whole. Then they should use both partitioning and iterating operations sequentially to find the correct unit fractional part of the referent whole. It seems that coordinating both operations proved challenging for most 4th grade participants. Thus, the linear PFS task without a given fraction name was more problematic than the linear RPFS task for most 4th grade participants, as well as for some 5th grade participants whose iterating operation was not available. Having the fraction in view, participants (especially a few 4th grade participants) were able to identify the number of pieces consisting in the given stick once they determined that the given stick was not a unit whole even though they had not yet learned the formal notion and definition of fraction units.

The linear model PFS task presented a non-unit fractional part and the referent unit whole. It required participants to identify the fractional name of the given fractional part (i.e., the given shorter stick) in term of the given referent whole (i.e., the given longer stick). Apparently, this task created more difficulties to those participants who had not yet constructed the iterating operation. The difficulties participants experienced in solving the PFS liner model task once again affirms the findings from Norton and Wilkins (2011). That is, it is impossible for students to solve the PFS tasks without the splitting operation because the construction of SO is prior to the construction of PFS.

The performance of some 4th grade and 5th participants also indicates a lack of conceptual understanding of fraction units. For example, when “iterating” the given non-unit fractional part into the given referent whole two times and not exhausting the given whole, two 4th grade and two 5th grade participants ignored the remaining smaller piece, and gave ½ as the fraction name to the given shorter stick. It could be argued that their plausible iterating operation was not applied to test the size of the fraction unit. Their "iteration" may serve to identify how many times the length of the given shorter stick went into the given longer stick. The performance from four 5th grade participants suggests a lack of construction of iterating operation in another manner. After these participants "iterated" the given shorter stick into the longer stick two times and a smaller piece remained, they ignored the size differences and used a fraction 1/3 to represent the given non-unit fractional part. Their action indicated that they treated the given factional part as a fraction unit of the referent whole. Their plausible iterating operation was actually a partitioning operation that aimed to divide the given unit whole.

Which Model is More or Less Problematic: Linear, Circular, or Rectangular

The pilot study revealed that the tasks involving a circular model may be more challenging for Chinese students. The performance of the participants in the pilot study suggests that these students may not have developed sufficient strategies to partition the circular models. The common approach Chinese participants used in the pilot study to partition a circle was to cut the circle using diameters. To identify whether the circular model was more or less problematic than the linear model in general, the researcher designed five different rectangular model tasks corresponding to one of the circular model tasks for each scheme.

Three different ANOVA analyses were conducted with different foci. First, a three-way mixed ANOVA was conducted to compare the mean scores between two model pairs: 1) a linear and circular model pair, and 2) a circular and corresponding rectangular model pair. The results indicate that a statistically significant main effect is associated with the variable Model ($F(1, 504) = 386.33$, $p < .05$, retrieved from Chapter 4). That means the performance of students in both grades on linear model and on circular model tasks was significantly different. Figure 4.1 in Chapter 4 presents the analysis across grades. A visual investigation of Figure 4.1 indicates that generally students' performance on linear model tasks was better than their performance on circular model tasks.

Next the same three-way mixed ANOVA analysis was conducted again to analyze if a significant difference exists between the performances on circular and rectangular models. The results indicate that there are statistically significant main effects associated with variables Scheme ($F(1.59, 802.32) = 479.3$, $p < .05$, retrieved from Chapter 4) and Model ($F(1, 504) = 27.89$, $p < .05$, retrieved from Chapter 4). This means the different performances on different models is associated with different schemes. A two-way interaction analysis between Model and Grade reveals that the difference is not statistically significant, meaning that students' different performance on both circular and its corresponding rectangular models is not associated with grade levels. However, another two-way interaction analysis between variables Model and Scheme indicates that the performance difference on circular and its corresponding rectangular model tasks is significant and dependent on schemes. The statistics of this two-way ANOVA revealed that students performed differently ($p < .05$, retrieved from Chapter 4) between circular PUFS and PFS model and its corresponding rectangular PUFS and PFS model task, but not on RPFS tasks ($p = .616$, retrieved from Chapter 4). Particularly, for PFS tasks, student performance on the corresponding rectangular model was better than their performance on the circular model task. Figure 4.6 in Chapter 4 illustrated this feature.

A one-way ANOVA was conducted to test the performance of 5^{th} grade students on different models for IFS. The results show that students' performance on the linear model task was higher than their performance on the circular model task, and the difference is statistically significant ($F(1, 251) = 36.91$, $p < .05$, retrieved from Chapter 4). However, students' performance was higher on the rectangular model task than their performance on the circular model task. For SO there was not a statistically significant difference in performance by models for 4th grade students. For 5th grade students, the performance on linear model tasks was significantly different from their performance on circular model tasks ($t(252) = 3.53$, $p < .001$, retrieved from Chapter 4), but there was no significant difference between performance on circular model and its corresponding rectangular model tasks.

Both the circular model and the rectangular model can be considered area models. The analyses of quantitative data indicate that students' performance on linear model tasks, in general, was higher than their performance on area model tasks. The performance of all participants during the clinical interview affirms that overall the linear model tasks for all schemes (except the PFS linear model task in general) seem easier than the circular model tasks. However, when solving the PUFS circular model task, one 4th grade participant and two 5th grade participants were asked to solve the corresponding rectangular model back-up task. Solving the rectangular model PUFS task did help 5th grade participants modify their methods used to successfully solve the circular model PUFS. However, due to the lack of an iterating operation, the 4th grade participants still were unable to solve the rectangular model PUFS task.

The Role of Understanding of Fraction Units

According to the Chinese National Mathematics Curriculum Standard (CNMCS) guidelines, fraction teaching and learning is divided into two phases. The first teaching phase, Initial Recognition Phase, happens during the fall semester of 3rd Grade. The second teaching phase, Re-recognition Phase, varies based on the school system the school district has adopted. Recall that two different school systems exist in China: 1) the 6-3 system with six-years of elementary school and three years of middle school, and 2) the 5-4 school system comprised of five years of elementary schooling and four years of middle school.

The school district that the participants in this study were enrolled implements a 5-4 school system. Therefore, the second teaching phase happens during the spring semester of Grade 4. The written assessment was administrated in September 2018. The clinical interview data were collected in December 2018. By this time, the 4th grade participants had learned the informal concept of fractions, such as "像$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{3}{8}$...这样的数， 都是分数", translated as "the numbers such as $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{3}{8}$ ... are all fractions." The concepts of unit wholes and fraction units were not introduced until the second teaching phase. During the first teaching phase, the basic terminologies and the comparison between two fractions were also introduced. Fraction comparison was limited to fractions with like denominators and the denominators being less than 10. During the second teaching phase, the formal definition of unit wholes, the formal notation and definition of proper and improper fractions were formally introduced. Thus, it was expected that the 5th grade participants had been introduced to these concepts and had the opportunity to construct a conceptual understanding of fractions and fraction units. Moreover, a review of textbooks used by the school these participants were enrolled in reveals that learning and applying fraction units were a teaching focus point for the second teaching phase. Therefore, this study aims to explore whether the conceptual understanding of fraction units facilitated Chinese students to construct their fraction knowledge.

The results from quantitative analyses suggest that, after the Re-recognized fractions teaching phase, 5th grade participants performed higher on fraction related tasks than 4th grade participants, especially those tasks related to more advanced fractional schemes. Comparing the explanations between 4th and 5th grade participants suggests that not only did the understanding of fraction units play an important role in Chinese students' construction of fraction knowledge, but also the understanding of a unit whole. The following sections present a detailed discussion regarding the role of an understanding of the unit whole and fraction units.

The Role of an Understanding a Unit Whole: Perceptual Whole vs. Conceptual Whole

When describing a unit whole in terms of fractions, a unit whole may consist of a set of items or just one continuous item. If a set of items forms a unit whole, then the unit whole is called a discrete whole. If a unit whole is one item, the whole is called a continuous whole. In this study, the unit whole refers to a continuous whole because the referent or implicit wholes presented in all tasks were continuous wholes. Hence, it is an abstract concept for most beginning learners to understand that a fraction can represent a number or a relationship. In Chinese, the whole is called "单位一" or "单位 1". "一" is the Chinese character for the number 1. Hence, the English translation of "单位一" or "单位 1" is "unit one." This is the reason why "a unit whole" is used in this study.

After the formal definition of fractions is introduced during 4th grade in China, the concept "unit one" is endowed with a new meaning. It not only represents a collection of several items, but also could represent one item. Being able to identify what the referent or implicit unit whole is becomes critically important when solving fraction related word problems. Consider the following two fraction word problems, and note the difference between the italicized text in each problem.

1. Melissa's mom baked a cheesecake for her yesterday. She ate ¼ of the cake yesterday. *She ate ½ of the cake today*. Use a fraction to represent the amount of cake remaining out of the whole cheesecake.

2. Melissa's mom baked a cheesecake for her yesterday. She ate ¼ of the cake yesterday. *She ate ½ of the remaining cake today*. Use a fraction to represent the amount of cake remaining out of the whole cheesecake.

If a student has not constructed a conceptual understanding of a unit whole, this student may consider these two problems to be the same. In fact, the referent unit whole for the fraction 1/2 in these two problems is completely different. In the first question the whole of 1/2 is the original cake whereas the referent unit whole of 1/2 in the second question is the remaining cake after Melissa ate some cheesecake yesterday. Thus, understanding the concept of a unit whole is considered to be fundamental basic knowledge.

To explore the differences in understanding fractions between 4^{th} and 5^{th} participants, the first task for all participants was to verbalize the proper fraction 3/5 and the improper fraction 5/3, and explain the meanings of these two fractions. One of the differences was the description of the unit whole. According to their performance and explanations, most 4^{th} grade participants had not yet constructed a part-whole concept of fractions, they had developed only a part-part concept of fractions (PPCF). When they explained the proper fraction 3/5, two of them did not include a unit whole in their explanations. The unit whole included in the explanation of three participant was five things. When they solved the PWS and PUFS tasks, their performance indicates that their unit whole comprehension was that of a discrete whole. Thus, their concept of a unit whole was defined as a perceptual concept of a unit whole.

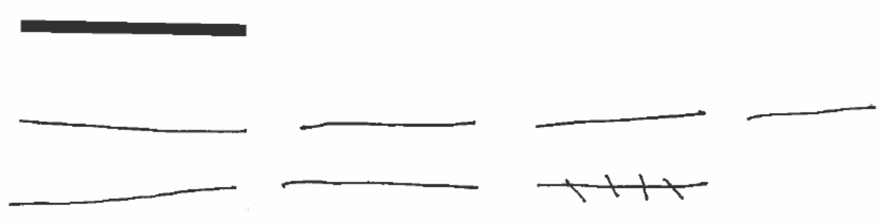
At this point, the perceptual concept of a unit whole is defined as the whole that consists of a set of items presented in students view. In order words, they need to have concrete countable items in front of them. If they saw a pizza that was partitioned into six slices and all shaded, then the unit whole consisted of 6 pieces. If a pizza that was partitioned into six slices had only five pieces shaded, then the unit whole consisted of 5 pieces. In contrast, the conceptual concept of a unit whole is defined as an abstract concept of a unit whole, meaning that the unit whole is unchangeable even when a disembedding operation is applied physically or mentally. The performance of participants in this study indicates that if a student has only constructed a PPCF, their whole consists of all the items based on their perception. In other words, they assimilate their whole number knowledge and their unit whole as the total number of items based on their counting. Therefore, when a fractional part is disembedded from the unit whole, the whole changes. However, if a student has accurately constructed a conceptual concept of a unit whole, their whole is invariant after the disembedding operation is applied. Consequently, it can be concluded that establishing a conceptual concept of a unit whole is a prerequisite for the disembedding operation. The next section will display the performances from those participants who had a perceptual concept of the unit whole and will discuss their performance in detail.

Perceptual Concept of Unit Whole and Disembedding Operation

Steffe & Olive (2010) state that the disembedding operation is one of the essential operations involved in the construction of fraction schemes, especially regarding the construction of the PWS. Moreover, Olive and Vomvoridi (2006) point out that the lack of a disembedding operation causes students to change the referent whole and then miscount the pieces of a partitioned whole. However, after analyzing the 4th grade participants' performance and explanations, it may be argued that a perceptual concept of a unit whole might be the reason that students are unable to conceptually disembed a part or parts from a whole. For example, one of the 4th grade participants considered to have a PPCF held a perceptual concept of unit wholes. When he solved the linear model PUFS tasks, he was required to draw one-seventh of the given line segment. He knew that 1/7 meant one out of seven pieces. However, rather than partition the given line segment into seven equal pieces, he first made seven copies of the given line segment under the given line segment. When asked to explain his reasoning, he counted the line segments again and crossed out the last copy of the line segment (see Figure 5.1). Now including the given line segment, he had seven line segments in his view. He explained, “There are seven lines and need one-seventh.” Apparently, based on his perception, the given line segment was not the unit whole but one-seventh of the unit whole. He understood that he needed seven things because the denominator was seven. However, there was only one thing in his sight. To create seven things, he made six more copies so that the given shorter line segment now became one-seventh of the unit whole.

Figure 5. 1

The Solution of Linear Model PUFS Task from a 4th Grade Participant

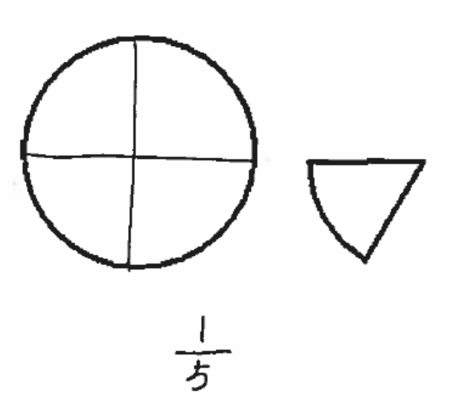


His perceptual concept of unit wholes was demonstrated again when he solved the circular model PUFS task. For the PUFS circular model, he needed to give a fraction name to the given unit fractional piece according to the given referent whole. After a few seconds of thinking, he wrote a fraction 1/5, which was the correct answer. When asked where the one fifth came from, he responded in Chinese, “五个小蛋糕拼起来等于一个大蛋糕,” (translated into English as “putting five smaller cakes together equals a large cake”). To understand his reasoning, the researcher required him to show the small cakes on the given figure by drawing.

Instead of making copies of the given smaller piece, he drew two perpendicular diameters to *cut* the given circle into four pieces (see Figure 5.2). Using the word "cut" describes his action of drawing two perpendicular diameters, which cannot be considered partitioning or even segmenting. In addition, using diameters might be the easiest and most common way to divide a circle. After he cut the circle into four pieces, the researcher asked him if the pieces in the circle had the same size as the given unit fractional piece. He said that the sizes were similar. With the given unit fractional piece, he had five pieces now and labeled the smaller piece as 1/5. His explanation clearly indicates that he is a PPCF and possesses a perceptual concept of a unit whole. Even though the problem explicitly explained that the circle was a unit whole, he did not consider that it was a whole because the size of the circle was too big, compared to the given smaller piece. The same phenomena happened again when he solved the linear and circular model SO tasks.

Figure 5. 2

The Solution of Circular Model PUFS Task from a 4th Grade Participant



When this participant solved the linear model PWS tasks in the clinical interview protocol (see problem #1 in Figure 5.3), it seems that he used the disembedding operation to find a correct answer, 2/5. After he solved the linear and circular model PUFS tasks, the researcher tried to gain a deeper understanding of his concept of unit wholes. Thus, the researcher asked him to review the PWS linear task. To examine his disembedding operation again, the researcher asked him to identify the unit whole. He responded that the unit whole was the total. The following conversation presents how the researcher tried to clarify the meaning of the total in his explanation. R represents the researcher and Z represents the participant name Zhi.

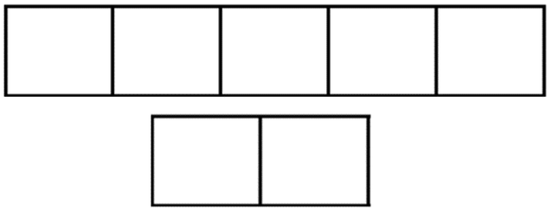
> R: Just now, on question one, you said the answer is two fifths. Does any part in two fifths represent a unit whole (the researcher meant to let him identify whether the numerator 2 is the whole or the denominator 5 is the whole)?
>
> Z: All the pieces of chocolate.
>
> R: How many pieces of chocolate? It is five, or two, or another number? How many pieces of chocolate in all?
>
> Z: Seven.
>
> R: Oh, so, you mean none of the five and two in your fraction two fifths represents the unit whole, right?
>
> Z: Yes.

His response indicates that all pieces of chocolate, according to his perception, makes up the unit whole. He actually did not use the disembedding operation to come up with his answer. His answer of 2/5 simply shows the relationship between the larger chocolate bar consisting of five pieces and the smaller chocolate bar made up of two pieces. This example may illustrate that to successfully use a disembedding operation, students need to construct a conceptual concept of a unit whole.

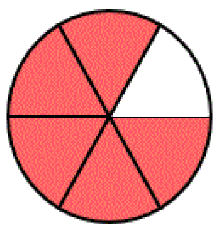
Figure 5. 3

The Linear and Circular Model Tasks in the Clinical Interview Protocol

1. **如图所示，请问小块的是大块的几分之几。(PWS)**
   What fraction is the smaller bar out of the larger bar?



2. **如图所示的阴影部分是你吃剩的披萨。用什么分数可以表示你吃的那块披萨?**
The shaded part of the pizza shown below represents the leftover pizza after you ate. What fraction of the pizza did you eat?



An example from a 5th grade participant confirms that a lack of a conceptual concept of the unit whole obstructs students from using the disembedding operation correctly. When this participant solved the linear model PWS task, her answer was two sevenths. Obviously, her unit whole consisted of all the pieces in her view. Her unit whole was considered a perceptual concept of unit wholes. However, she provided a correct answer to the circular model PWS task (see problem #2 in Figure 5.3), 1/6. To examine her disembedding operation, she was required to find a fraction to represent the shaded part. She used the fraction 5/6, which was correct again. Furthermore, the researcher asked her to give a fraction name to one shaded piece, she wrote 1/6, which was also correct. Her performance indicates that she could disembed a fractional part from the referent whole. However, her performance on the linear model PUFS task indicates that her perceptual concept of a unit restrained her ability to move forward and construct a PUFS. When she solved the PUFS linear task, she made a copy of the given line segment. When asked if her drawing is “1/7 as long as” the given line segment, she responded, “Yes, because they have the same length.” When she tried to determine the size of the given smaller piece in terms of the given circle, she drew two perpendicular diameters. Then in the top left quarter of area, she drew a radius and provided the answer 1/5. Although the answer was correct, her answer was based on her perceptual concept of a unit whole, five pieces regardless of the size difference.

According to their performance, it could be argued that students’ inability to employ the disembemdding mental action is due to a lack of a conceptual understanding of a unit whole. The 5th grade participant’s performance indicates that she could disembed a fractional part from the referent whole. For the PWS linear task, her answer could have been correct, if she had been able to determine the unit whole correctly. Unfortunately, because of her perceptual concept of unit wholes, she determined the unit whole based on the total number of pieces in her view.

One Whole or Two Wholes

Another interesting phenomenon emerged during the clinical interviews: when the fractional part was physically disembedded from the unit whole some 5th grade participants provided two different fraction names for the unit pieces in each part. For example, after they had solved the PFS linear model task, the researcher required participants to give a fraction name for one piece in each part. Ten out of 12 participants gave ½ to one piece of the top non-unit fraction part and 1/5 to one piece of the bottom referent unit whole (see Figure 5.4). The following conversation displays one participant’s thinking related to the number of unit wholes in the figure. R represents the researcher. Y represents the participant name Yueyue.

R: Can you use a fraction to represent one share in the longer line segment?

Y: (promptly wrote) one fifth.

R: Good. I also want you to use a fraction to represent one share in the shorter line segment.

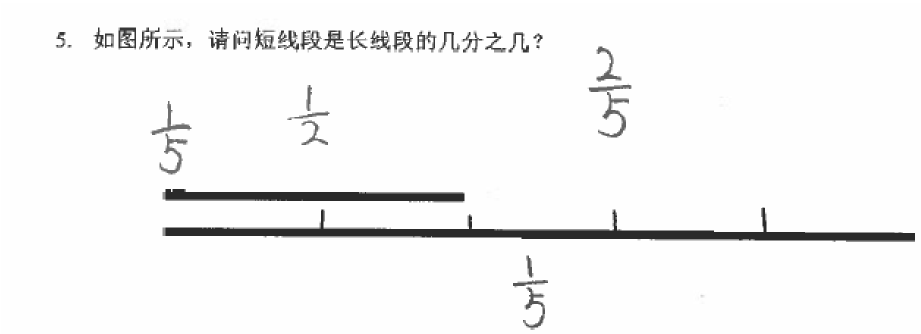
Y: (She wrote) ½

R: Oh, okay. Do you think that the shorter line segment is actually part of the longer line segment? Or do you think the shorter line segment is a unit whole and the longer line segment is also a unit whole.

Y: The short line segment is a unit whole. The longer line segment is also a unit whole.

Figure 5. 4

The Response of Fraction Name from a 5th Grade Participant



Note. A 5th grade participant gave a fraction name to one piece in each part.

After the researcher discussed this phenomenon with the mentor (the chair of the mathematics department of the school the participants are enrolled in), he mentioned that 5th grade students were learning fractions with a discrete whole at that time. He shared how he and his colleagues were confused about the fractional word problems related to a discrete whole. He provided the following example: "You have 12 apples. If you gave 1/3 of your apples away and ate 1/4 of the rest of the apples, how many apples did you eat?" The Chinese mentor explained that the students were told that to find how many apples were given away, they should consider 12 apples as the unit whole for the first fraction 1/3. Then four apples were given away. Taking away four apples from the total 12 apples would give eight apples left. The eight apples would be a new unit whole for the second fraction ¼. Although this approach is correct, it could be argued that participants' misconception about the unit whole was caused by this teaching method. However, with participants' understanding of the fraction units, it was not difficult to correct the participants' misconception. This issue will be discussed in the next section.

The Role of Understanding Fraction Units

The measurement subconstruct is one of the five basic subconstructs for students' construction of rational numbers identified by Kieren (1980). This subconstruct states that a fraction is the result of measuring the fraction by its fraction unit. The partitive unit fraction scheme (PUFS) is the second level of fraction schemes established by Steffe and Olive (Steffe & Olive, 2010), which uses the multiplicative relationship of a fraction and its fraction unit, and describes a fraction as multiples of its fraction units. The common key for these two subconstructs is the fraction unit. Results from Norton and Wilkins' (2010) quantitative analysis indicates that Steffe's PUFS closely aligns with Kieren's measurement subconstruct (also see Wilkins & Norton, 2018). A literature review of Chinese national curriculum and fraction learning reveals that teaching fractions with an emphasis on understanding the concept of fraction units and applying fraction units is the focus of the second fractions teaching phase, the re-recognizing fraction phase.

A review of the textbooks used by the participants in this study shows that although the concept of fraction units was not formally introduced to students during the first teaching phase, the phrases displayed with the highest frequency in 3rd grade textbooks include evenly divide and shares. This may be one reason why even 4th grade participants struggled with where the tick marks should be placed so that each piece was evenly divided. On the other hand, when 3rd grade textbooks described proper fractions, they not only use a part-whole concept of fractions, but also describe a multiplicative relationship, which is similar to the PUFS structure.

For example, a 3rd grade textbook introduces a fraction 5/8 as “$\frac{5}{8}$ 表示 8 等份中的 5 份,” and “$\frac{5}{8}$里面有 5 个$\frac{1}{8}$.” In the description, 份 means share(s). 等份 means equal shares. Thus, the English translation of this sentence is “5/8 represents 5 shares out of 8 equal shares” and “there are 5 1/8s in 5/8.” It can be believed that students start to apply the fraction units in the initial informal fraction learning phase. In a 4th grade textbook, after the formal definition of unit wholes and the formal notation and definition of fraction units are introduced, there are plenty of exercises for students to practice. These exercises include a partitioned whole and shaded fractional part that require students to identify the fraction unit of each figure. Students are then asked to write the fraction for each figure. One type of exercise, a fill in the blank, is impressive. The given sentences are the descriptions of multiples of a fraction unit in different ways, such as “3 个$\frac{1}{13}$是( )”, translated as “ 3 1/13s is ( )”; “$\frac{5}{17}$是 5 个( )”, translated as “5/17 is 5 ( )”; “( )个$\frac{1}{6}$是$\frac{5}{6}$”, translated as “( ) $\frac{1}{6}$s is $\frac{5}{6}$”; and “4 个( )是$\frac{4}{9}$”, translated as “4 ( ) is $\frac{4}{9}$.” It can be argued that students would be familiar with different word representations of fractions using a multiplicative relationship. While completing the first task during the clinical interview, all the 5th grade participants, except one, were able to fluently call out the fraction unit for each presented fraction, and at the same time use a multiplicative relationship to describe each of the presented fractions.

According to the written assessment, only two out of 19 5th grade participants had constructed a SO; three had constructed RPFS; and four had constructed IFS. However, at the end of the clinical interview, more than half of the participants had successfully solved all of the scheme tasks. In fact, not all the 5th grade participants could solve the PUFS tasks on their first attempt. However, with those given fraction name tasks, nearly all participants were able to identify the fraction unit for that task. They also understood that they should partition the given fractional part or unit whole to find the size of the fraction unit. The more problematic task was the RPFS task. The issue for those struggling participants was identifying the unit whole because the language reversed the language of the PFS tasks. However, most participants could solve the linear model task after the researcher asked them to re-read the problem and decide whether the given stick was a fractional part or a unit whole. These participants could solve the PFS tasks and experienced no problems solving the circular model task.

One of the major difficulties the 5th grade participants encountered was the number of unit wholes in a figure. After the researcher discovered that it was difficult for 4th grade participants to identify the unit whole of the tasks, the researcher started to ask participants to give a fraction name to one piece of each part starting from the linear model PFS task. Surprisingly, 10 out of 13 5th grade participants provided wrong fractions. Figure 5.5 presents one of the examples. When the researcher asked the participant named Yueyue to use a fraction to represent one piece of the shorter stick and one piece of the longer stick, she used 1/5 to represent one piece of the longer stick, which was correct. However, she incorrectly used ½ to represent one piece of the shorter stick. The following conversation continues from a previous one regarding how she resolved the problem with the researcher's prompt (see the conversation under the section "One Whole or Two Whole"). R represents the researcher and Y represents the participant name Yueyue.

R: (In the prior conversation she said the shorter stick was a unit whole and the longer stick was another unit whole) So, if the shorter line segment is a unit whole, why did you say that the shorter line segment is 2/5 [of the longer line segment]? If the shorter line segment is a unit whole, the longer line segment is another unit whole, then [the fraction] is one out of one, right?

Y: The shorter line segment is part of the longer line segment.

R: Is it?

Y: Uh (A common Chinese verbal response to express "yes" or "okay".)

R: It is a partial. How come?

Y: The short line segment just counts two fifths of the longer line segment.

R: That means the shorter line segment contains two of one-fifths. Can I understand in this way?

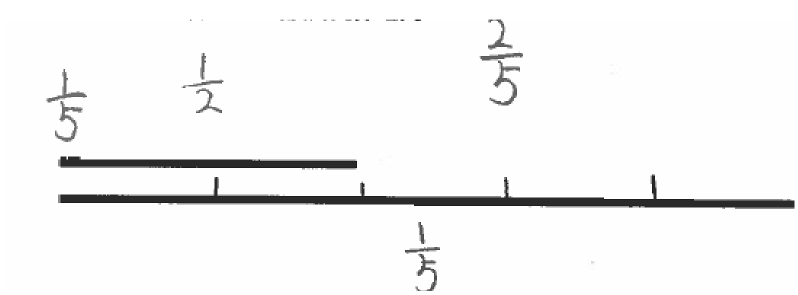
Y: Uh.

R: Then what fraction can you use to represent one piece of the shorter line segment?

Y: One fifth.

Figure 5. 5

The Response of Giving a Fraction Name to One Piece of Each Part of the PFS Task



Note. When a 5th grade participant was asked to give a fraction name to one piece of the shorter stick and one piece of the longer stick, she used 1/5 to represent one piece of the longer stick, but ½ to represent one piece of the shorter stick.

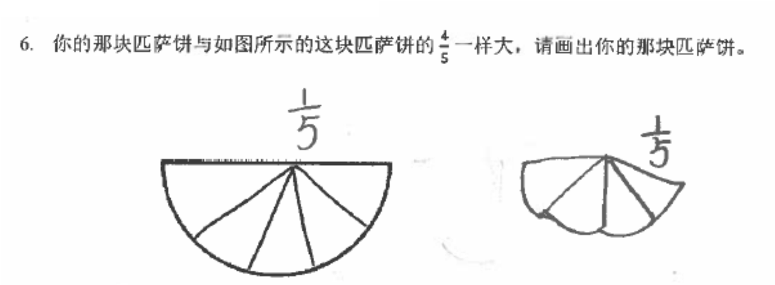
It seems that Yueyue had begun to gain a deeper understanding of the concept of fraction units and a unit whole. She gave correct fraction names to all the following tasks. Figure 5.6

displays her answers for the tasks relative to different advanced schemes. Performances from other 5th grade participants were similar to that of this participant. Performances from these 5th grade participants confirm the findings from other studies (e.g., Hackenberg, Norton, & Wright, 2016; Norton & Wilkins, 2010; Wilkins & Norton, 2018). That is, a conceptual understanding of fraction units and the unit whole are a necessary condition for constructing a conceptual understanding of fraction knowledge. Moreover, findings from this study also suggest that it would be difficult to trigger the iterating operation when a student has not yet constructed a conceptual understanding of fraction units.

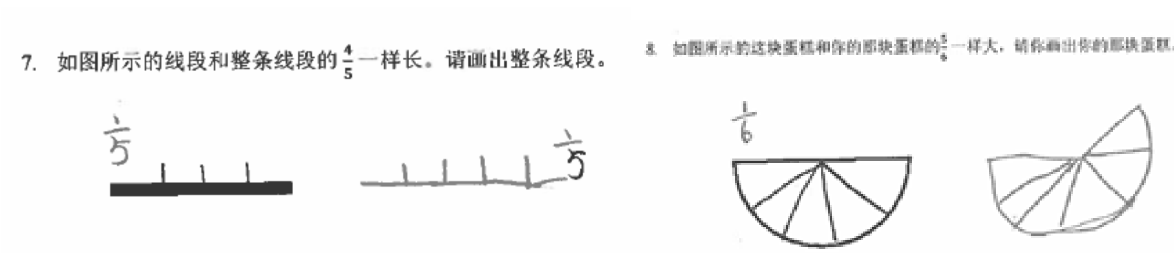
Figure 5. 6

The Answers for the Other Tasks Relative to Different Advance Schemes from the Same Participant as Figure 5.5

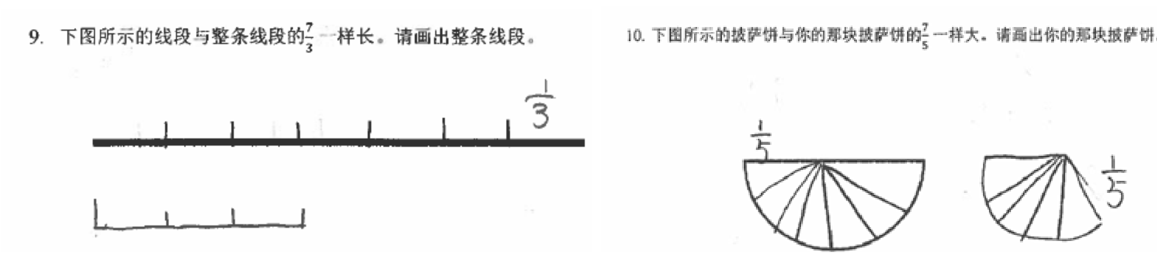
PFS circular model task



RPFS circular model task



IFS circular model task



Chapter Six: Conclusion

The concept of fractions is complicated and difficult for many students. Understanding fractions is much more than recognizing that a fraction is a number that represents a concrete quantity. For example, consider the fraction in the sentence "I walked 3/4 of a mile today." The fraction ¾ in this context refers to an actual distance. A fraction also represents a quantitative relationship between two quantities. The following sentence provides another example: Today I walked ¾ of the distance I walked yesterday. The fraction ¾ in this example does not represent an actual quantity, but instead describes the relationship between the walking distances of today and yesterday. Being able to identify the meaning of a given fraction and flexibly apply fraction knowledge to solve fraction related tasks requires students to understand different aspects of fraction concepts. Furthermore, conceptual understanding of fraction knowledge will facilitate students' future learning of more advanced concepts such as rational numbers, proportional reasoning, and algebra.

From their longitudinal study, Steffe and his colleague (e.g., Steffe, 2002; Steffe & Olive, 2010) identified a learning trajectory and a set of associated operations that are necessary in the construction of fraction knowledge through scheme theory. This fraction learning trajectory involves a progression from the part-whole scheme (PWS), to partitive unit fraction scheme (PUFS), to partitive fraction scheme (PFS), to reversible partitive fraction scheme (RPFS), and then to iterative fraction scheme (IFS). In several quantitative studies, Norton and Wilkins (e.g., Norton & Wilkins, 2009, 2010, 2012; Wilkins & Norton 2011, 2018) demonstrated that American students' construction of fraction knowledge aligned with this progression. Their studies also affirmed the progression of these fractional schemes occurs in a hierarchical order, meaning that if students could not conceptually construct the previous fraction scheme, it would be difficult for them to solve the fractional tasks related to the more advanced fractional schemes. Norton, Wilkins and Xu (2018) also examined the fraction schemes of 76 5th and 6th grade Chinese students. Their results indicated that these Chinese students' fraction knowledge also aligned with the progression identified by Steffe and Olive (2010).

The purpose of present study not only extended Norton, Wilkins and Xu's study by including more Chinese students, but also places an emphasis on exploring the role of a conceptual understanding of fraction units in Chinese students' construction of their fraction knowledge. Thus, using an explanatory sequential mixed methods research design, two consecutive phases were included in this study. This chapter centers on conclusions based on the results and discussion in the previous chapters, and is organized around each research question. In addition, limitations of the study and possible future research directions are discussed.

Research Question One

What fraction schemes did the 4th and 5th grade Chinese students construct before and after the second teaching phase ("Re-recognize Fraction" teaching phase)?

The discussion in Chapter 5 confirms that the development of fractional schemes among Chinese 4th and 5th grade participants in this study is similar to their U.S. counterparts and the Chinese participants in Norton et al.'s (2018) study regardless of differences in the curricula between the countries or various areas within the same country, differences in textbooks, and differences in language. Furthermore, the discussion in Chapter 5 also indicated that most 4th grade participants have only constructed a PWS prior to the second teaching phase, namely the "re-recognize fractions" teaching phase. Comparing and contrasting the performance of 4th and 5th grade participants indicated that fraction knowledge rapidly developed from 4th grade to 5th grade.

After learning fractions with informal definitions, most 4th grade participants only possessed a part-whole concept of fractions. Although they were able to give the fractional name for a unit fractional piece of the partitioned whole, they did not understand conceptually that fraction units are identical and iterable. They also lacked the understanding that iterating a fraction unit a certain number of times could re-produce the referent whole. Consequently, the iterating operation was not available for them to apply when they needed to determine the size of a unit fractional piece when they encountered an unpartitioned whole. The discussions in Chapter 5 revealed that during the clinical interview those 4th grade participants who were able to solve the linear model PUFS task used the partitioning operation alone. They could not use the iterating operation to examine the sizes of those smaller parts within the partitioned whole. Moreover, it was challenging for them to solve the circular model PUFS task in the clinical interview, because this task requires them to determine the size of a given unit fractional piece relative to the given unpartitioned whole via the iterating operation. Rather than iterating the given smaller sector within the circular model to determine the size of the given smaller piece, most 4th grade participants drew two diameters to cut the circle into four pieces. They accepted the size differences because of their lack of understanding of the meaning of fraction units.

However, with more understanding of fractions, 5th grade participants performed differently. During the second teaching phase, the formal definitions of fractions and fraction units are introduced. Recall from Chapter 2 that fraction teaching foci for the second teaching phase are the definition of fraction units and the meaning of fraction units. When solving the circular model PUFS task during the clinical interview, some 5th grade participants could not apply the iterating operation to determine the size of the given unit fractional piece at the beginning. However, with a deeper understanding of fraction units, it was relatively easier to trigger their iterating operation. Moreover, when solving the linear model PUFS task during the clinical interview, the performance of some 5th grade participants suggested that they not only used the partitioning operation to find the unit fractional part of the referent whole, they could also use the iterating operation to test whether the unit fractional pieces were the right size. Thus, findings suggest that gaining an in-depth understanding of fraction units during the second teaching phase facilitates students' transition from the part-whole concept of fractions to the measurement concept of fractions.

Research Question Two

Which model (i.e., linear, circular, and rectangular) is more or less problematic for Chinese students?

A comparison of participants performance on tasks involving different models revealed that the performance of students in both grades on linear model and on circular model tasks was significantly different. Chinese students' performance on linear model tasks was better than their performance on circular model tasks. However, there was no significant difference between performance on circular model and its corresponding rectangular model tasks.

The results from analyzing participants' performance during the clinical interview in both grades revealed that participants would have performed better if the tasks could have been solved using the partitioning operation alone. Tasks that involve an iterating operation became problematic for most 4th grade and some 5th grade participants. For example, the performance of most 4th grade participants and some 5th grade participants who finished the PUFS indicated that they could successfully use a partitioning operation alone to solve the linear model PUFS task, but encountered difficulties when solving the circular model PUFS tasks.

In addition, the performance from 4th grade participants indicated that the partitioning operation of some 4th grade participants was just a *plausible partitioning operation*. That means the goal of their partitioning operation was to fragment a given figure into a certain number of parts without regard to the needed size of the pieces. For example, when solving the circular model PFS task that required participants to find 4/5 of the given half circle, those participants who had a plausible partitioning operation struggled to partition the given unit whole five pieces. That was because they cut the half circle into halves, then cut the halves into halves again. Their fragmenting operation could not create five pieces within the given unit whole. Their actions appeared to be merely aimed at dividing the half circle into five pieces. For this reason, it could not be considered true partitioning operation, but the beginning stages of fragmenting (Hackenberg, Norton, & Wright, 2016).

The discussion in Chapter 5 revealed that it was easier to trigger an iterating operation of 5th grade participants. It is possible that this is because 5th grade participants had learned the formal definition of fraction units. They may have constructed partial meanings of fraction units. In contrast, 4th grade participants had only learned an informal fraction definition. Their limited fraction knowledge hindered their construction of an iterating operation. Therefore, it is reasonable to conclude that the iterating operation and the splitting operation are key to solving the fraction related tasks regardless of the model differences. Once students develop an understanding that fraction units are identical and that iterating a fraction unit a certain number of times can re-produce the referent whole, they are able to complete fractional tasks regardless of model differences.

Research Question Three

How does the understanding of fraction units facilitate Chinese students' ability to solve tasks involving advanced fraction schemes?

A number of various research studies (e.g., Norton & Wilkins, 2013; Olive & Vomvoridi, 2006; Wilkins & Norton, 2018) have confirmed the critical role of the iterating operation in the construction of a measurement concept of fractions and the splitting operation. The findings from this study suggest that the iterating operation can only be triggered when a student has constructed a conceptual understanding of fraction units. In other words, students need an understanding that each of the unit fractional pieces in a partitioned whole is identical. Moreover, any one of the unit fractional pieces is iterable, meaning that iterating any one of the unit fractional pieces a certain number of times could re-produce the referent unit whole.

Quantitative findings illustrated that 5th grade participants performed better on more advanced fractional schemes tasks than 4th grade participants. Qualitative findings demonstrated that without an understanding of fraction units, most 4th grade participants only applied a partitioning operation to fractional tasks. For example, when solving the linear model PUFS task during the clinical interview, 4th grade participants ignored the size differences between the smaller parts after they partitioned a line segment into seven parts. These participants performed in a similar manner when they solved the circular model PUFS task. They used the diameters to cut the circle into four parts, and accepted the size difference between these parts and the given smaller sector.

In contrast to the 4th grade participants, some 5th grade participants with an understanding of fraction units were able to use the iterating operation to test the size of unit fractional pieces when solving PUFS tasks during the clinical interview. Some 5th grade participants struggled with the circular model PUFS task. However, after they were perturbed by the researcher, or after they solved the rectangular model back-up PUFS task, they were able to use the iterating operation and they could successfully solve the circular model PUFS task and the other tasks related to more sophisticated fractional schemes.

Research studies in U.S. (e.g., Norton & Wilkins, 2010, Watanabe, 2007; Wilkins & Norton, 2018) indicated that the fraction teaching and learning in U.S. focuses on the part-whole concept of fractions. Although the part-whole concept of fractions is introduced to Chinese students during the second teaching phase, a review of Chinese textbooks indicates that the fraction learning activities and exercises in 3rd and 4th grade textbooks have expanded to the measurement concept of fractions. For instance, after the definition of fraction units was introduced during the second teaching phase, students practice to describe fractions using fraction units in different ways. Students are required to describe how many fraction units are in a fraction such as 5/6. Students also need to tell what the fraction unit of a fraction, say 5/17, is. Results from Wilkins and Norton (2018) demonstrated that a construction of a measurement concept of fractions could facilitate the construction of a splitting operation, which is the fundamental operation for the construction of the other more advanced fractional schemes. Thus, it can be concluded that Chinese fraction teaching with an emphasis on fraction units plays a critical role in Chinese students constructing a conceptual understanding of fraction units. The performance of 4th and 5th grade students during the clinical interview demonstrated that the iterating operation could be triggered only when students had constructed a conceptual understanding of fraction units. Furthermore, students' conceptual understanding of fraction units facilitates the transition from the part-whole concept of fractions toward a measurement concept of fractions, and the construction of more advanced fraction schemes. The conceptual understanding of fraction units also catalyzes Chinese students building of advanced fraction knowledge in a relatively short period of time compared to their U.S. counterparts. It could be argued that this may be one of the reasons why Chinese students have higher performance on fraction items in international assessments compared to the performance of U.S. students.

Implications

Implications for Teaching Practice

Findings from this study may provide instructional implications for both American and Chinese educators. Performances from 4th and 5th grade participants suggest that a conceptual understanding of fraction units and unit wholes play an important role in the construction of students' fraction knowledge. The 4th grade participants' performance suggests that students are unable to construct a part-whole concept of fractions without possessing an understanding of a unit whole. Although they seem to use the disembedding operation to find the answer and the answer seems correct, their answer was merely based on their understanding of a part-part concept of fractions. Thus, after students provide their answers for PWS tasks, teachers may still need to ask students to point out the unit whole of their fraction and confirm that the referent unit whole of their answer is correct. On the other hand, the performance of 5th grade participants suggest that although students may not have constructed the advanced fraction schemes, a conceptual understanding of fraction units and unit whole facilitates their constructions of more advanced schemes.

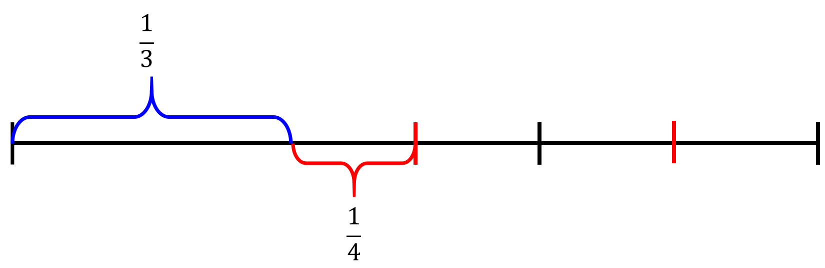
The Chinese language in particular may have advantages for representing fractions. Chinese textbooks suggest that teaching students different representations of fraction units might help students gain a deeper understanding of fraction units. The comparison between 4th and 5th grade participants performance on PUFS linear model tasks reveals that 5th grade participants were able to find a solution to the task based on their understanding of the fraction units. The 4th grade participants, due to their informal fraction knowledge, lacked the understanding that fraction units are identical and invariant. Therefore, it was difficult to trigger their iterating operation.

A review of Chinese textbooks reveals that most of the illustrations in both 3rd and 4th grade textbooks provide partitioned unit wholes. Students lack opportunities to practice partitioning a given whole into numbers of equal parts. The process of partitioning a unit whole may help trigger the iterating operation for students. Thus, teachers may provide different model tasks and let students partition according to the given fraction names.

One of the common mistakes made by participants from both grades was to cut the circular model into half and halves. Chinese teachers may need to provide fractions with odd number denominators to facilitate triggering the iterating operation. Another common mistake displayed by most 5th grade participants during the clinical interview was the misconception about identifying a unit whole. This misconception may be conveyed from teaching methods. To avoid this misconception, teachers can draw diagrams to help students identify the unit fraction and the related fractions. Recall the example task provided by the chair of the mathematics department: "You have 12 apples. If you gave 1/3 of your apples away and ate 1/4 of the rest of the apples, how many apples did you eat?" The diagram shown below (see Figure 6.1) may help students visualize the relationship between the fractional part and the unit whole.

Figure 6. 1

Diagram Used to Help Students Understand a Fraction Related Word Problems



Implication for Written Assessment

The written assessment adopted by this study included only linear model PWS tasks. To test whether students have constructed the disembedding operation, the researcher added a circular model task in the clinical interview protocol. The circular model task displays a circle representing a pizza that was divided into six equal parts, with five of them being shaded. The one unshaded part represents the eaten piece of pizza. Participants were required to use a fraction to represent the eaten slice of the pizza. Some 4th grade participants who provided the correct answer for the linear model task could not solve this task correctly. They used 1/5 to represent the eaten slice of pizza. One of the reasons found from their explanations was that there are five slices of pizza and one slice was eaten. This reasoning indicates that the participants defined a wrong unit whole. Therefore, adding an area model task may help teachers identify if students have an understanding of 1) a perceptual unit whole, 2) a conceptual unit whole, and 3) the disembedding operation.

Implication for Future Research

Explanations of the thinking processes used by some 5th grade participants when solving the SO task suggested that they used division knowledge to explain the SO task. For example, the researcher asked one 5th grade participant who successfully solved the SO linear model task why he drew the line segment shorter than the given one, he responded that the given line segment was five times as long as the line he had drawn. Therefore, he divided the given line segment by five to determine the length of the line he was supposed to draw. In contrast to other participants who used fraction knowledge to explain their solutions, this participant used his knowledge of division to explain his solution. Thus, it would be interesting to discover whether students would use their fraction knowledge or division knowledge to solve fraction problems. In addition, it would be particularly interesting to examine if the discrete unit whole could trigger students' division knowledge or an iterating operation. If the iterating operation is triggered, then in what ways could their iterating operation be used.

Limitations

One of the main limitations of the current study is that all the discussions of the study assumed the intended teaching practices in China. It was assumed that the ways students were taught aligned with the guideline from the CNMCS (2001). However, the researcher does not know for sure that this was the case, nor whether each student was given the same opportunity to learn the concepts outlined in each phase. At the same time, it may also be the case that the differences among 4th and 5th grade participants are attributed to the curriculum, and could be a result of natural growth over the course of one year.

Another limitation of this study is that the unit wholes of each task are continuous wholes. This study did not test how students apply their fraction knowledge to situations where the unit wholes are discrete or a portion of discrete wholes. Future research could expand this study and investigate how students identify fraction units when they work with a discrete whole.

It is believed that the results of this future expanded research could provide more implications for teaching discrete unit whole and fraction units relative to a discrete unit whole.

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Appendices

Appendix A: 4th Grade Fraction Scheme Assessment with English Translation Included

四年级分数水平测试

4th Grade Fraction Scheme Assessment

维吉尼亚理工大学

姓名 ______________________________ 年级 _____________

学号 ______________________________ 年龄 __________

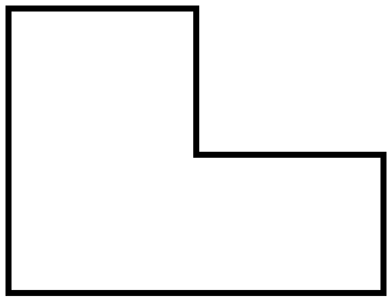
教师 ____________________________ 班级_________________

性别 __________________

此测试不允许使用尺子或计算器。

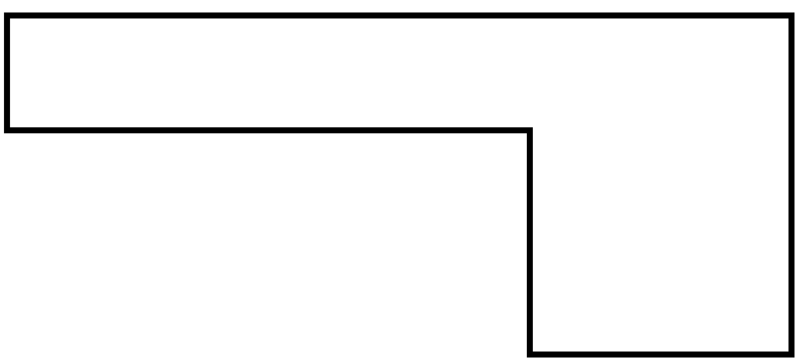
四年级分数测试

1. 下图所示的这块巧克力与你的那块巧克力的6倍一样大，请画出你的那块巧克力。(SO)
The amount of chocolate shown below is 6 times as big as your piece of chocolate. Draw your
piece.

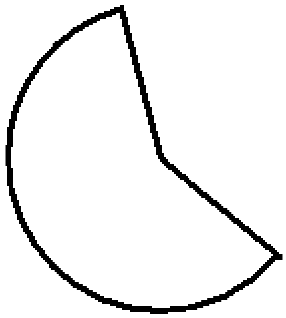


2. 你的线段与下图线段的 $\frac{1}{7}$ 一样长。画出你的线段。(PUFS)
Your stick is 1/7 as long as the stick shown below. Draw your stick.

3. 如图所示的这块巧克力与你的那块巧克力的 $\frac{5}{6}$ 一样大，请画出你的那块巧克力。(RPFS)
The piece of chocolate shown below is 5/6 as big as your piece of chocolate. Draw your piece of chocolate.

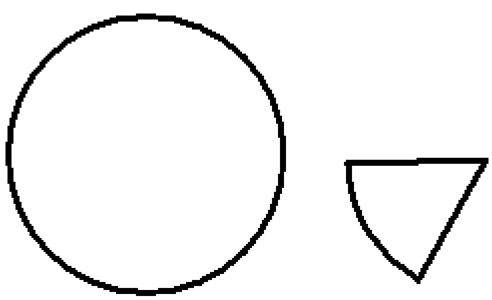


4. 什么分数可以表示下图这块匹萨饼和整个匹萨饼的关系？(PFS)
What fraction is the piece of pizza shown below out of a whole pizza?



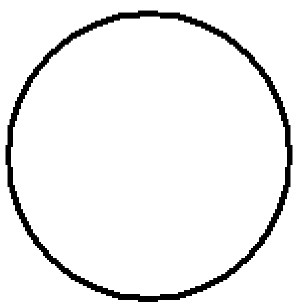
5. 如图所示的线段长度和另一条线段的 5 倍一样长，请画出另一条线段。(SO)
The stick shown below is 5 times as long as another stick. Draw the other stick.

6. 请问图中小块蛋糕是整个蛋糕的几分之几？（PUFS）
What fraction is the smaller cake piece out of the whole cake?



7. 如图所示的线段与整条线段的 $\frac{3}{7}$ 一样长，请你画出整条线段。(RPFS)
   The stick shown below is 3/7 as long as a whole line segment. Draw the whole line segment.

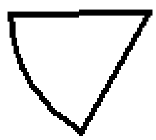
8. 如图是一块蛋糕。你的那块和这块的 $\frac{1}{5}$ 一样大，请画出你的那块。(PUFS)
   Your piece of cake is 1/5 as big as the piece shown below. Draw your piece of cake.



9. 如图所示，请问小块的是大块的几分之几。(PWS)
   What fraction is the smaller bar out of the larger bar?

10. 如图所示的这块蛋糕和你那块蛋糕的 $\frac{2}{5}$ 一样大，请画出你的那块蛋糕。(RPFS)

The piece of cake shown below is 2/5 as big as your piece of cake. Draw your piece of cake.



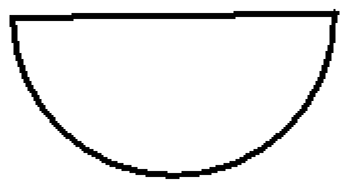
11. 请问图中短线段是长线段的几分之几？（PUFS）

What fraction is the smaller stick out of the longer stick?

12. 请指出下图巧克力糖的 $\frac{1}{9}$ 。(PWS)

Make $\frac{1}{9}$ of the chocolate shown below.

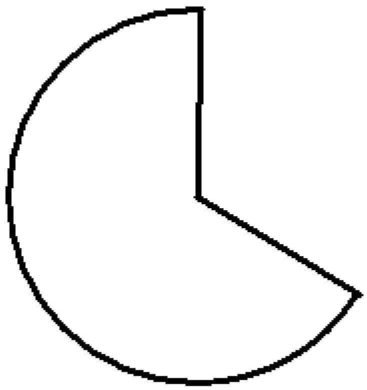
13. 如图所示的这块蛋糕与你的那块蛋糕的 $\frac{5}{6}$ 一样大，请你画出你的那块蛋糕。(RPFS)
The piece of cake shown below is 5/6 as big as your piece of cake. Draw your piece of cake.



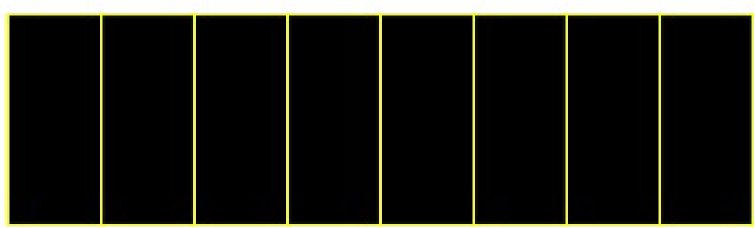
14. 如图所示，请问短线段是长线段的几分之几？(PFS)
What fraction is the smaller stick out of the longer stick?

15. 如图所示的线段与整条线段的 $\frac{4}{5}$ 一样长。请画出整条线段。(RPFS)
The stick shown below is 4/5 as long as a whole line segment. Draw the whole line segment.

16. 下图所示的这块匹萨饼和你那块匹萨饼的 6 倍一样大，请画出你的那块匹萨饼。(SO)
The amount of pizza shown below is 6 times as big as your slice. Draw your slice.

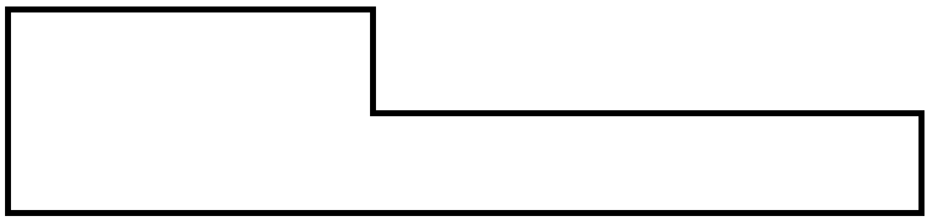


17. 请指出下图巧克力糖的 $\frac{5}{8}$ 。(PWS)
Make $\frac{5}{8}$ of the chocolate shown below.



18. 你的那块匹萨饼与如图所示的这块匹萨饼的 $\frac{4}{5}$ 一样大，请画出你的那块匹萨饼。(PFS)
Your piece of pizza is 4/5 as big as the piece shown below. Draw your piece of pizza.

19. 什么分数可以表示下图这块的巧克力与整块巧克力的关系？PFS)
What fraction is the chocolate shown below out of a whole chocolate?



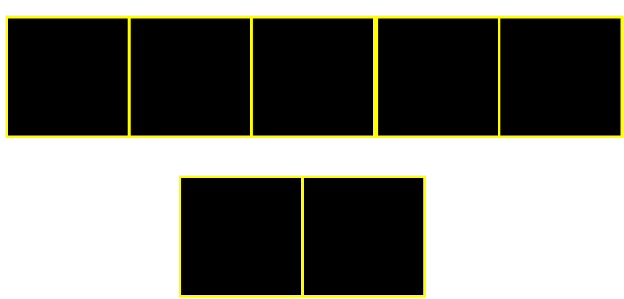
20. 如图所示的这块匹萨饼跟你的那块匹萨饼的 3 倍一样大，请画出你的那块匹萨饼。(SO)
The amount of pizza shown below is 3 times as big as your slice. Draw your slice.

21. 如果你的线段跟下图线段的 $\frac{3}{5}$ 一样长，请画出你的线段。(PFS)
Your stick is 3/5 as long as the stick shown below. Draw your stick.

22. 请问图中小块蛋糕是整个蛋糕的几分之几?（PUFS）
What fraction is the smaller cake piece out of the whole cake?

23. 下图所示的线段长度和另一条线的 3 倍一样长，请画出另一条线段。(SO)
The stick shown below is 3 times as long as another stick. Draw the other stick.

24. 如图所示，请问小块的是大块的几分之几。(PWS)
What fraction is the smaller bar out of the larger bar?



Appendix B: 5th Grade Fraction Scheme Assessment with English Translation Included

五年级分数水平测试

5th Grade Written Assessment of Levels of Fraction Schemes

维吉尼亚理工大学

姓名 ______________________________ 年级 _____________

学号 ______________________________ 年龄 __________

教师 ____________________________ 班级_________________

性别 __________________

此测试不允许使用尺子或计算器。

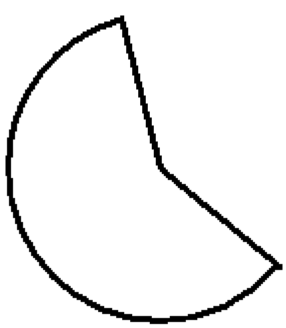
五年级分数测试 （5th Grade Fraction Test）

1. 如图所示的线段和整条线段的 $\frac{3}{7}$ 一样长，请你画出整条线段。(RPFS)
   The stick shown below is 3/7 as long as a whole line segment. Draw the whole line segment.

2. 请问图中短线段是长线段的几分之几？(PUFS)

   What fraction is the smaller stick out of the longer stick?

3. 什么分数可以表示下图这块匹萨饼和整个匹萨饼的关系？(PFS)
   What fraction is the piece of pizza shown below out of a whole pizza?



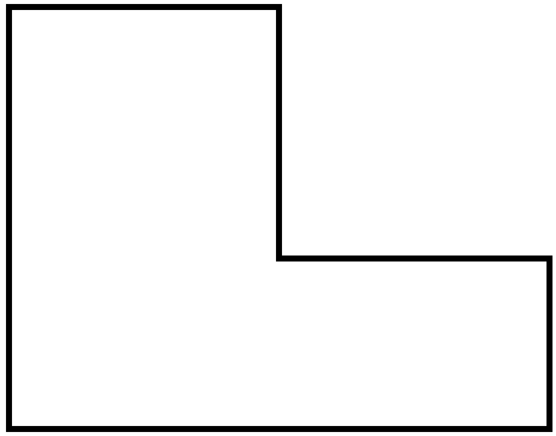
4. 你的那块匹萨饼与如图所示的这块匹萨饼的 $\frac{4}{5}$ 一样大，请画出你的那块匹萨饼。(PFS)
Your piece of pizza is 4/5 as big as the piece shown below. Draw your piece of pizza.

5. 如果你的线段跟下图线段的 $\frac{3}{5}$ 一样长，请画出你的线段。(PFS)
Your stick is 3/5 as long as the stick shown below. Draw your stick.

6. 如图所示的这块匹萨饼跟你的那块匹萨饼的 3 倍一样大，请画出你的那块匹萨饼。(SO)
The amount of pizza shown below is 3 times as big as your slice. Draw your slice.

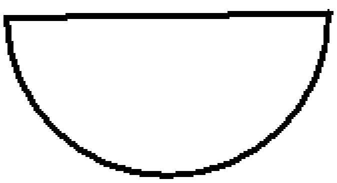
7. 下图所示的巧克力与你的那块巧克力的 $\frac{7}{5}$ 一样大。请画出你的那块巧克力。(IFS)

The piece of chocolate shown below is 7/5 as big as your piece of chocolate. Draw your piece of chocolate.



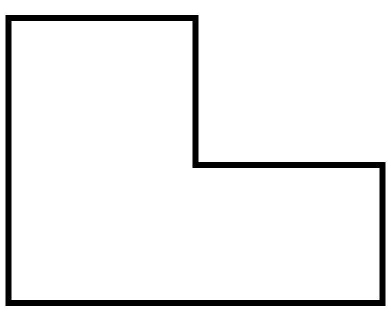
8. 如图所示的这块蛋糕和你的那块蛋糕的$\frac{5}{6}$一样大，请你画出你的那块蛋糕。(RPFS)

The piece of cake shown below is 5/6 as big as your piece of cake. Draw your piece of cake.



9. 下图所示的这块巧克力和你的那块巧克力的6倍一样大，请画出你的那块巧克力。(SO)

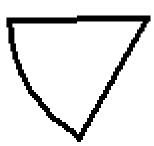
The amount of chocolate shown below is 6 times as big as your piece of chocolate. Draw your piece.



10. 下图所示的蛋糕与你的那块蛋糕的 $\frac{5}{4}$ 一样大。请画出你的那块蛋糕。(IFS)
The piece of cake shown below is 5/4 as big as your piece of cake. Draw your piece of pie.

11. 下图所示的线段长度和另一条线的 3 倍一样长，请画出另一条线段。(SO)
The stick shown below is 3 times as long as another stick. Draw the other stick.

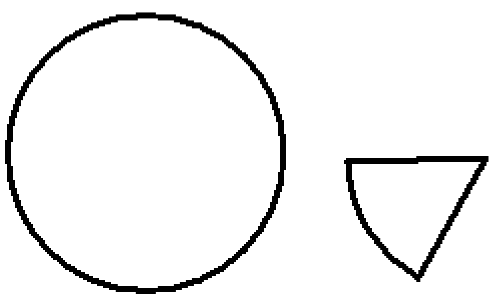
12. 如图所示的这块蛋糕和你那块蛋糕的 $\frac{2}{5}$ 一样大，请画出你的那块蛋糕。(RPFS)
The piece of cake shown below is 2/5 as big as your piece of cake. Draw your piece of cake.



13. 请问图中小块蛋糕是整个蛋糕的几分之几?（PUFS）
What fraction is the smaller cake piece out of the whole cake?

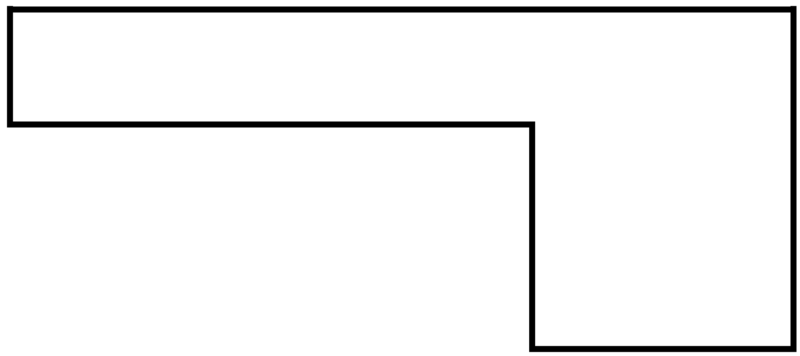
14. 如图所示，请问短线段是长线段的几分之几？(PFS)
What fraction is the smaller stick out of the longer stick?

15. 请问图中小块蛋糕是整个蛋糕的几分之几?（PUFS）
What fraction is the smaller pie cake out of the whole cake?



16. 如图所示的这块巧克力与你的那块巧克力的 $\frac{5}{6}$ 一样大，请画出你的那块巧克力。(RPFS)

The piece of chocolate shown below is 5/6 as big as your piece of chocolate. Draw your piece of chocolate.

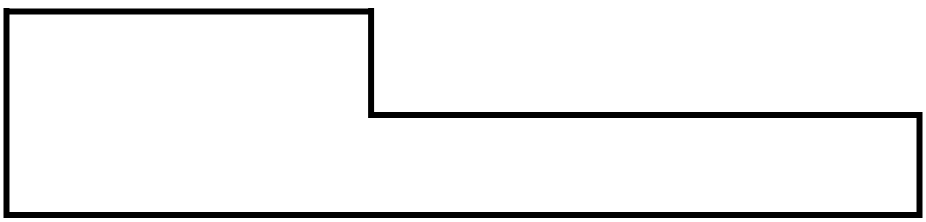


17. 如图所示的线段长度和另一条线段的 5 倍一样长，请画出另一条线段。(SO)

The stick shown below is 5 times as long as another stick. Draw the other stick.

18. 什么分数可以表示下图这块巧克力和整块巧克力的关系？(PFS)

What fraction is the piece of chocolate shown below out of a whole chocolate?



19. 下图所示的披萨饼与你的那块披萨饼的 $\frac{7}{5}$ 一样大。请画出你的那块披萨饼。(IFS)

The piece of pizza shown below is 7/5 as big as your piece of pizza. Draw your piece of pizza.

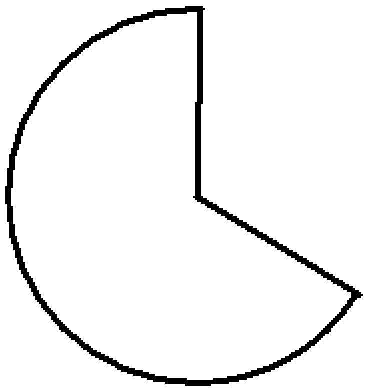
20. 如图所示的线段和整条线段的 $\frac{4}{5}$ 一样长。请画出整条线段。(RPFS)

The stick shown below is 4/5 as long as a whole line segment. Draw the whole line segment.

21. 下图所示的线段于整条线段的 $\frac{5}{4}$ 一样长。请画出整条线段。(IFS)

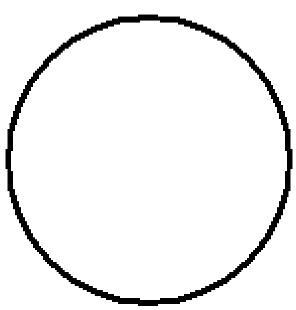
The line segment shown below is 5/4 as long as a whole line segment. Draw the whole line segment.

22. 下图所示的这块匹萨饼和你那块匹萨饼的 6 倍一样大，请画出你的那块匹萨饼。(SO)
The amount of pizza shown below is 6 times as big as your slice. Draw your slice.



23. 你的线段与下图线段的 $\frac{1}{7}$ 一样长。画出你的线段。(PUFS)
Your stick is 1/7 as long as the stick shown below. Draw your stick.

24. 如图是一块蛋糕。你的那块和这块的 $\frac{1}{5}$ 一样大，请画出你的那块。(PUFS)
Your piece of cake is 1/5 as big as the piece shown below. Draw your piece of cake.



25. 下图所示的线段与整条线段的 $\frac{7}{3}$ 一样长。请画出整条线段。(IFS)
The bar shown below is 7/3 as long as a whole line segment. Draw the whole line segment.

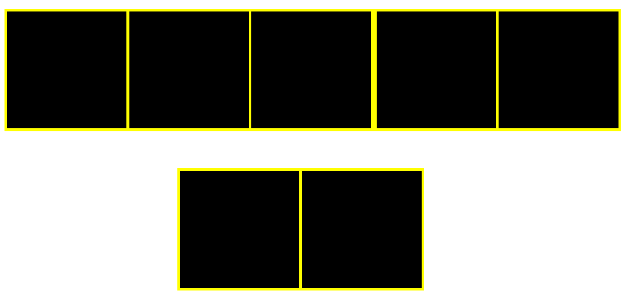
Appendix C: Interview Protocol

Interview Protocol

1. The mentor shows an index card with fraction "$\frac{3}{5}$" on the card, and ask the student what it means.
2. The mentor shows an index card with fraction "$\frac{5}{3}$" on the card, and ask the student what it means.

分数测试 （Fraction Tasks）

1. 如图所示，请问小块的是大块的几分之几。(PWS)
   What fraction is the smaller bar out of the larger bar?



Follow-up Questions:

1) If a student answers 2/5, then ask the student to explain the reasoning for the answer.

2) If a student answer 1/3, then ask the student:

   a. Can you explain the reasoning for the answer?

   b. If smaller bar is 1/3 of the whole, can you draw the whole candy bar?

   c. Ask other clarifying questions as needed.

3) If a student answer 2/7, then ask the student:

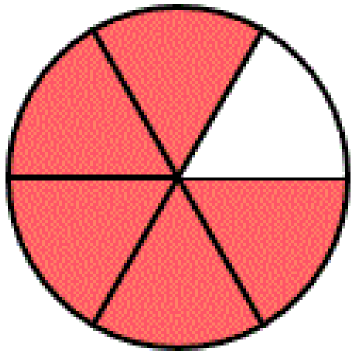
   a. Can you show me the whole?

   b. Can you tell me how many equal pieces in the whole?

   c. What is the fraction unit of this whole?

d. Ask other clarifying questions as needed.

2. 如图所示的阴影部分是你吃剩的披萨。用什么分数可以表示**你吃的**那块披萨? (PWS)
The shaded part of the pizza shown below represents the leftover pizza after you ate. What fraction of the pizza **did you eat**?



Follow-up Questions:

1) If a student answers 1/6, then ask the student to explain the reasoning for the answer.

   Then circle one of the shaded piece and ask students if the circled piece is 1/6. Why?

2) If a student answers 1/5, then ask the student:

   a. Can you explain why you think it is 1/5?

   b. Can you draw 1/6 of the whole?

   c. Ask other clarifying questions as needed.

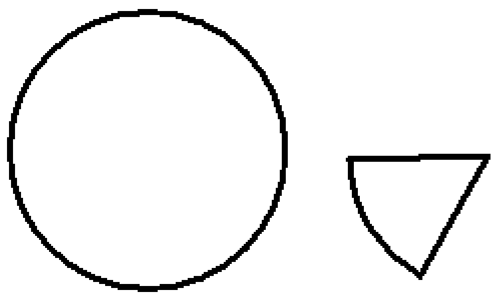
3. 你的线段与下图线段的 $\frac{1}{7}$ 一样长。画出你的线段。(PUFS)
Your stick is 1/7 as long as the stick shown below. Draw your stick.

Follow-up Question:

1) If a student gives correct answer, then ask the student to explain the reasoning for the answer?

2) If a student iterate the given stick 7 times, then ask the student to read the question again and explain what his or her think about the question. Then ask other clarifying questions as needed.

3. 请问图中小块蛋糕是整个蛋糕的几分之几？(PUFS)
What fraction is the smaller cake piece out of the whole cake?



Follow-up Question:

1) If a student answers correctly, then ask the student to explain the reasoning for the answer.
2) If a student answers wrong, then ask the student:
    a. Can you explain the way you partition the whole?
    b. Explain the reasoning for your answer.
    c. Ask other clarifying questions as needed.

4. 如图所示，请问短线段是长线段的几分之几？(PFS)
What fraction is the smaller stick out of the longer stick?

Follow-up Questions:

1) If a student answers correctly, then ask the student to explain the reasoning for the answer. Ask other clarifying questions as needed.
2) If a student gives wrong answer, or struggle to name the smaller stick, ask the student to speak out his or her struggling. Then ask other clarifying questions as needed.

5. 你的那块匹萨饼与如图所示的这块匹萨饼的 $\frac{4}{5}$ 一样大，请画出你的那块匹萨饼。(PFS)
Your piece of pizza is 4/5 as big as the piece shown below. Draw your piece of pizza.

Follow-up Questions:

1) If a student answers correctly, then ask the student to explain the reasoning for the answer. Ask other clarifying questions as needed.
2) If a student draws a line in the middle first and struggles to partition the whole into five, then the mentor will change the half circle model to a rectangular model.

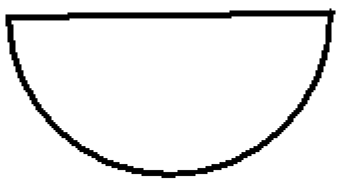
4. 如图所示的线段和整条线段的 $\frac{4}{5}$ 一样长。请画出整条线段。(RPFS)
The stick shown below is 4/5 as long as a whole line segment. Draw the whole line segment.

Follow-up Questions:

1) If a student answers correctly, then ask the student to explain the reasoning for the answer. Ask other clarifying questions as needed.

2) If a student partitions the whole into 5 pieces, then ask the student to explain what he or she think about the problem. Then ask other clarifying questions as needed.
3) If a student iterates the given piece 5 times, then ask student to explain the reasoning for the answer.
4) Ask other clarifying questions as needed.

5. 如图所示的这块蛋糕和你的那块蛋糕的$\frac{5}{6}$一样大，请你画出你的那块蛋糕。(RPFS) The piece of cake shown below is 5/6 as big as your piece of cake. Draw your piece of cake.



Follow-up Questions:

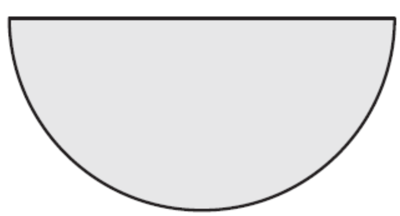
1) If a student answers correctly, then ask the student to explain the reasoning for the answer. Ask other clarifying questions as needed.
2) If a student answers wrong, then ask the student to explain the reasoning for the answer. Ask other clarifying questions as needed.
3) If a student struggle to partition the given piece, then the mentor will change to a rectangular model. Ask other clarifying questions as needed.

6. 下图所示的线段与整条线段的$\frac{7}{3}$一样长。请画出整条线段。(IFS)
The bar shown below is 7/3 as long as a whole line segment. Draw the whole line segment.

Follow-up Questions:

1) If a student answers correctly, then ask the student to explain the reasoning for the answer. Ask other clarifying questions as needed.
2) If a student partitions the whole into 3 pieces, then ask the student to read the question again and explain what his or her think about the question. Then ask other clarifying questions as needed.
3) If a student struggles to solve the problem, ask the student to explain his or her struggling.

7. 下图所示的披萨饼与你的那块披萨饼的$\frac{7}{5}$一样大。请画出你的那块披萨饼。(IFS)
The piece of pizza shown below is 7/5 as big as your piece of pizza. Draw your piece of pizza.



Follow-up Questions:

1) If a student answers correctly, then ask the student to explain the reasoning for the answer. Ask other clarifying questions as needed.

2) If a student partitions the whole into 3 pieces, then ask the student to read the question again and explain what his or her think about the question. Then ask other clarifying questions as needed.

3) If a student struggle to partition the given piece, then the mentor will change to a rectangular model. Ask other clarifying questions as needed.

Appendix D: Back-up Interview Protocol

Back-up Interview Protocol

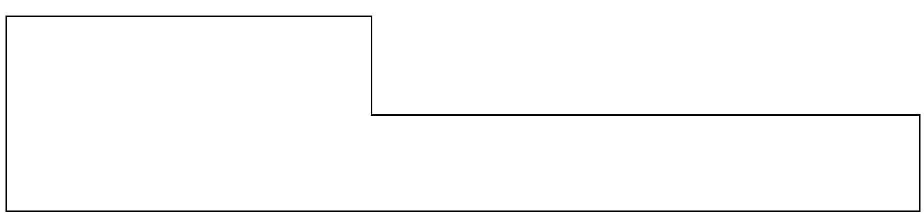
This interview protocol is used only when a student struggles with the circular model. The student will be only given the corresponding rectangular model, but not all tasks in this protocol.

1. 请问图中小块蛋糕是整个蛋糕的几分之几?（PUFS）
   What fraction is the smaller cake piece out of the whole cake?

Follow-up Questions:

1) If a student answers correctly, then ask the student to explain the reasoning for the answer. Ask other clarifying questions as needed.
2) If a student answers wrong, then ask the student to explain the reasoning for the answer. Ask other clarifying questions as needed.

2. 什么分数可以表示下图这块巧克力和整块巧克力的关系？(PFS)
   What fraction is the piece of chocolate shown below out of a whole chocolate?



Follow-up Questions:

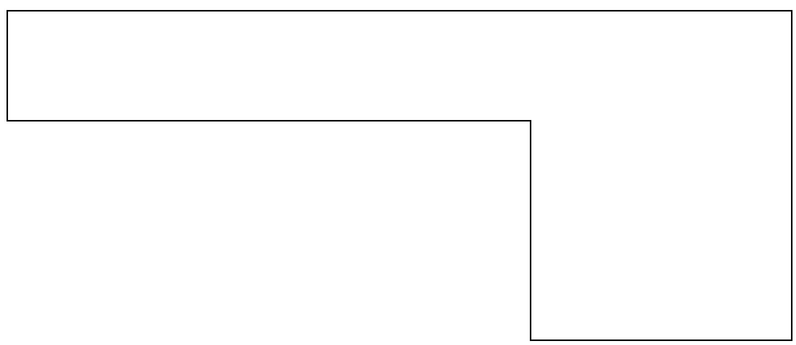
1) If a student answers correctly, then ask the student to explain the reasoning for the answer. Ask other clarifying questions as needed.

2) If a student answers wrong, then ask the student:

    a. What does the whole look like?

    b. Ask other clarifying questions as needed.

3. 如图所示的这块巧克力与你的那块巧克力的 $\frac{5}{6}$ 一样大，请画出你的那块巧克力。(RPFS)
The piece of chocolate shown below is 5/6 as big as your piece of chocolate. Draw your piece of chocolate.



Follow-up Questions:

1) If a student answers correctly, then ask the student to explain the reasoning for the answer. Ask other clarifying questions as needed.

2) If a student answers wrong, then ask the student:

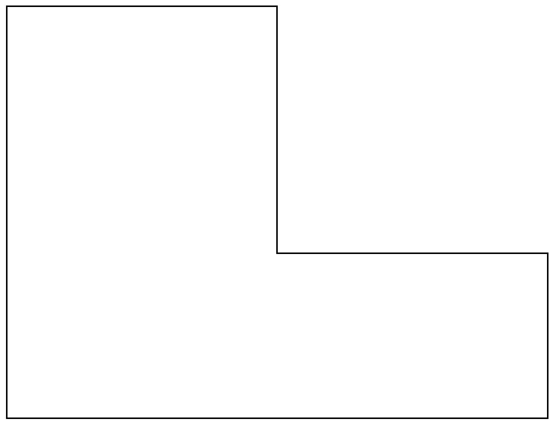
    a. What fraction can represent your piece of chocolate?

    b. To draw your piece of chocolate, what do you need to find first?

    c. Ask other clarifying questions as needed.

4. 下图所示的巧克力与你的那块巧克力的$\frac{7}{5}$一样大。请画出你的那块巧克力。(IFS)

   The piece of chocolate shown below is 7/5 as big as your piece of chocolate. Draw your piece of chocolate.



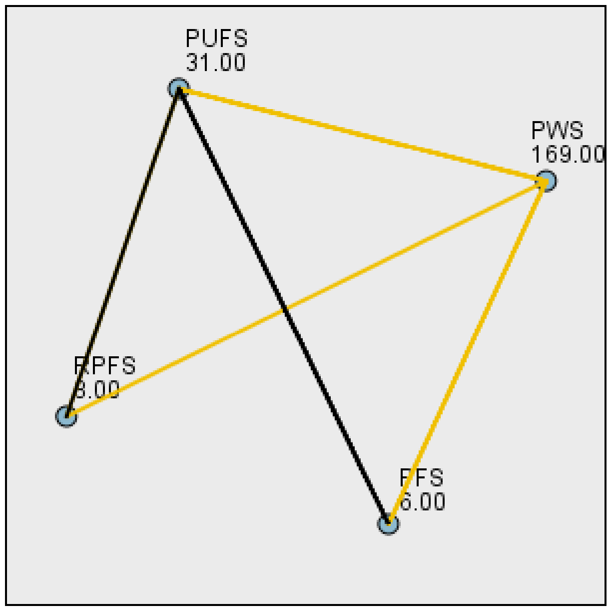
Follow-up Questions:

1) If a student answers correctly, then ask the student to explain the reasoning for the answer. Ask other clarifying questions as needed.
2) If a student answers wrong, then ask the student:
   a. What does $\frac{7}{5}$ mean?
   b. What fraction can represent your piece of chocolate?
   c. How many equal pieces should you partition the given piece of chocolate into?
   d. Ask other clarifying questions as needed.

Appendix E: Result of the Post-hoc Dunn's Test for 4th Grade

The Post-hoc Dunn's test, pairwise comparison to the PWS, PUFS, PFS, and RPFS for 4th grade.

Pairwise Comparisons



Each node shows the sample number of successes.

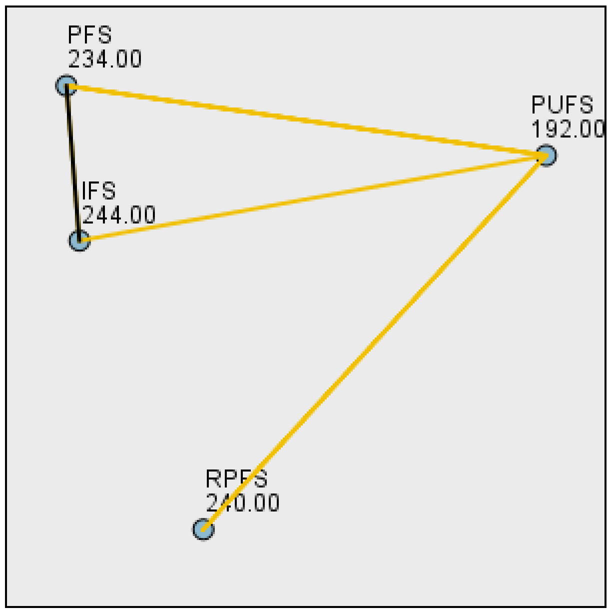
Sample1-Sample2	Test Statistic	Std. Error	Std. Test Statistic	Sig.	Adj.Sig.
PFS-RPFS	-.008	.037	-.211	.833	1.000
PFS-PUFS	.098	.037	2.635	.008	.050
PFS-PWS	.642	.037	17.182	.000	.000
RPFS-PUFS	.091	.037	2.424	.015	.092
RPFS-PWS	.634	.037	16.971	.000	.000
PUFS-PWS	.543	.037	14.546	.000	.000

Each row tests the null hypothesis that the Sample 1 and Sample 2 distributions are the same.
Asymptotic significances (2-sided tests) are displayed. The significance level is .05.
Significance values have been adjusted by the Bonferroni correction for multiple tests.

Appendix F: Result of the Post-hoc Dunn's Test for 5th Grade

The Post-hoc Dunn's test, Pairwise Comparison to the PUFS, PFS, RPFS, and IFS for 5th grade.

Pairwise Comparisons



Each node shows the sample number of successes.

Sample1-Sample2	Test Statistic	Std. Error	Std. Test Statistic	Sig.	Adj.Sig.
PUFS-PFS	-.167	.023	-7.203	.000	.000
PUFS-RPFS	-.190	.023	-8.232	.000	.000
PUFS-IFS	-.206	.023	-8.918	.000	.000
PFS-RPFS	-.024	.023	-1.029	.303	1.000
PFS-IFS	-.040	.023	-1.715	.086	.518
RPFS-IFS	-.016	.023	-.686	.493	1.000

Each row tests the null hypothesis that the Sample 1 and Sample 2 distributions are the same.
Asymptotic significances (2-sided tests) are displayed. The significance level is .05.
Significance values have been adjusted by the Bonferroni correction for multiple tests.

Appendix G: WIRB Approved Research Subject Consent Form English Version

RESEARCH SUBJECT CONSENT FORM

Title:	A Mixed Methods Study of Chinese Students' Construction of Fraction Schemes: Extending the Written Test with Follow-Up Clinical Interviews
Protocol No.:	18-465 WIRB® Protocol #20182051 18-465
Sponsor:	Virginia Tech
Investigator:	Jesse L.M. Wilkins, PhD War Memorial Hall, Rm 300-C 370 Drillfield Drive Blacksburg, VA, 24060 U.S.A. Phone Number: 001-540-231-8326
Sub-Investigator:	Congze Xu 1212 University City Blvd., Apt. I-104 Blacksburg, VA, 24060 U.S.A. Phone Number: 001-716-783-1520

You are being invited to take part in a research study. A person who takes part in a research study is called a research subject, or research participant.

In this consent form "you" generally refers to the research subject. If you are being asked as the parent or guardian to permit the subject to take part in the research, "you" in the rest of this form generally means the research subject.

What should I know about this research?

Someone will explain this research to you.
This form sums up that explanation.
Taking part in this research is voluntary. Whether you take part is up to you.
You can choose not to take part. There will be no penalty or loss of benefits to which you are otherwise entitled.
You can agree to take part and later change your mind. There will be no penalty or loss of benefits to which you are otherwise entitled.
If you don't understand, ask questions.
Ask all the questions you want before you decide.

Why is this research being done?

The purpose of this research is to investigate how Chinese students generally understand fractions. This information could help teachers and schools tailor their instruction to students' needs more effectively. Currently, we have to rely on time-intensive interviews and analysis to gain this information. As a regular part of his or her mathematics class, your child has already taken a written assessment about fractions. Selected students will then be interviewed to see how they solve the same or similar fraction problems. We are inviting fourth- and fifth-grade students to participate in the interviews.

About 30 subjects will participate in this part of the research study.

How long will I be in this research?

We expect that your taking part in this research will last about 45 minutes.

What happens to me if I agree to take part in this research?

If you agree to participate, you may be selected to take part in an approximately 45 minute interview in which researchers will ask you to solve a series of mathematical tasks involving fractions. These tasks are designed to help the students better understand basic fraction concepts as the researchers study how they construct their understanding of fraction units.

The interview will be in an individual room with a teacher and Ms. Congze Xu (you will see Ms. Xu through the internet). During the interview, you will work on math problems with a math teacher and me for about 45 minutes. You will be asked about how you are solving the problems. At the same time, you will be videotaped and your written work will be collected.

What are my responsibilities if I take part in this research?

If you take part in this research, you will be responsible to:

- Meet with the researchers for approximately 45 minutes.
- Answer mathematics questions about fractions, to the degree that you are able and comfortable.

Could being in this research hurt me?

You may experience mild frustration when solving difficult mathematics problems, although the researchers will modify problems and support your efforts through encouragement. No other harm or discomfort is expected as a result of participation in this study. You may refuse to answer questions or stop the sessions at any time if you become uncomfortable. Your grade will not be affected if you decide not to participate or decide not to finish the interview.

Will it cost me money to take part in this research?

Taking part in this research will not cost you anything.

Will I be paid if I take part in this research?

You will not be paid for being in this study.

Will being in this research benefit me?

We cannot promise any benefits to you or others from your taking part in this research. However, possible benefits to you include increased understanding of fraction knowledge. Possible benefits to others include helping teachers and researchers better understand how children develop fundamental fraction knowledge. This understanding may ultimately influence mathematics instruction for children in the future.

What other choices do I have besides taking part in this research?

This research is not designed to teach content knowledge. Your alternative is to not take part in the research.

What happens to the information collected for this research?

Your information will be shared with individuals and organizations that conduct or watch over this research, including:

The Investigator and Sub-Investigator
The Institutional Review Board (IRB) that reviewed this research

We may publish the data and results of this research. We may present the data and results of this research at professional conferences. We may also use the data and results for training mathematics educators. However, we will keep your name and other identifying information confidential.

We protect your information from disclosure to others to the extent required by law. We cannot promise complete secrecy.

Data collected in this research might be deidentified and used for future research or distributed to another investigator for future research without your consent.

Who can answer my questions about this research?

If you have questions, concerns, or complaints, or think this research has hurt you or made you sick, you can talk to your teacher, or talk to the research team at the phone numbers listed above on the first page.

This research is being overseen by an Institutional Review Board ("IRB"). An IRB is a group of people who perform independent review of research studies. You may talk to them at 001-800-562-4789, help@wirb.com if:

You have questions, concerns, or complaints that are not being answered by the research team.
You are not getting answers from the research team.
You cannot reach the research team.
You want to talk to someone else about the research.
You have questions about your rights as a research subject.

What if I am injured because of taking part in this research?

If you feel uncomfortable during the interview, tell your teacher and/or the researchers immediately. They will assist you in receiving care.

What happens if I agree to be in this research, but I change my mind later?

You are free to refuse to participate or stop taking part in this research at any time without giving any reason, and without penalty or loss of benefits to which you are otherwise entitled. You can ask to have the information related to you returned, removed from the research records, or destroyed at any time.

Statement of Consent:

All children are required to assent. If your child chooses to participate, he or she will be asked to sign an assent form prior to participating in the interview.

Your signature documents your permission for the individual named below to take part in this research.

Signature of child subject's parent	Date
Printed name of subject	Date

Please circle one option below:

I (DO / DO NOT) give permission for video footage of my child to be used in professional meetings, such as conferences. No attempt will be made to disguise my child's appearance in the video footage, although any use of my child's name in the video footage will be removed. I understand that I can revoke this permission at any time and without giving any reason.

Signature of child subject's parent	Date
Signature of person obtaining consent	Date

Appendix H: WIRB Approved Research Subject Consent Form Chinese Version

研究参与者知情书

题目：	运用混合法研究中国学生分数概念的建构及测后访谈
项目编号:	18-465
研究管理机构:	维吉尼亚理工大学
研究员:	Jesse L.M. Wilkins War Memorial Hall, Rm 300-C 370 Drillfield Drive Blacksburg, VA, 24060 U.S.A.
电话号码:	001-540-231-8326
合作研究者:	徐琮泽 1212 University City Blvd., Apt. I-104 Blacksburg, VA, 24060 U.S.A.
电话号码：	001-716-783-1520

我们邀请您参加一项教学研究。参与研究的人被称为研究对象或研究参与者.

在此知情书中，"您"泛指研究对象。如果您是允许研究对象参加研究项目的法定授权的代表，家长，或监护人，那么在此知情书中的"您"仍泛指研究对象。

我应该了解有关此研究项目什么信息?

- 有人将向您解释此项研究
- 此知情书概述相关解释
- 参加此项研究是自愿的，是否参加由您自主决定
- 您可选择不参加，不参加此项研究不会受到任何惩罚或损失
- 您同意参与，后可以改变主意，这也不会受到任何惩罚或损失
- 如果有不明白的地方，您可以提问
- 在您决定参加与否之前，请提出所有您想了解的问题

为什么要进行此项研究?

此项研究的目的是为了探究中国学生是如何理解分数概念的。了解中国学生分数概念的构建可以帮助老师更有效地结合学生认知结构设计教学课程。目前我们必须依靠时间密集的面对面访谈和分析来获得这些信息。

作为正常教学计划的一部分，五，六年级的学生包括您的孩子已经在之前的数学课进行了分数概念的书面测试。目前研究进入面谈阶段，希望通过面谈能够深入了解学生是如何解答分数问题的。我们邀请参加访谈的学生是根据他们的笔试成绩进行选取的。

将会有大概 30 个学生参加这部分研究。

我将参与多长时间?

我们预计面谈大概持续 45 分钟。

如果我决定参加此项研究，我需要做什么?

如果您被选中，作为参与者，您将参加一个大概 45 分钟的面谈，在此期间，研究者将会让您解答一系列关于分数的数学问题。这些数学问题的设计目的是帮助学生更好的理解分数基本概念，同时研究者也可以通过他们的解题过程，了解学生是如何建构对分数单位的理解的。

根据您课堂上已经做过的分数笔试成绩，如果您被选中，您将会在一间独立的房间里参加面谈，并将会与你的一位老师和徐琮泽（您将会通过网络视频看到徐老师）一起解决一些数学问题，这个过程持续大概 45 分钟。在此期间，您将回答是如何解决这些问题的。整个面谈全程将被录像，您写在纸上的解题过程也将会被收集作为研究数据。

如果我参加此项研究，我的责任是什么?

如果您参加此项研究，您就负责:

- 与研究员一起度过45分钟。
- 解答符合您认知程度的与分数有关的数学题

参与此项研究会对我造成伤害吗?

尽管当您遇到较难的题目时，研究者会适时地修改题目难度并鼓励和支持您解答这些数学问题，但是您可能还是会有一些轻微的挫折感。除此之外参与这个项目不会给您带来其它的不适与伤害。如果您在任何时候感觉到不安，都可拒绝回答问题，或停止访谈过程。如果您决定不参与或停止访谈，您的决定不会影响您在学校的成绩。

参加此项研究需要缴纳费用吗?

参加此项研究无需缴纳任何费用。

参加此项研究能获益吗?

我们不能保证您或其他人从此项研究中获益。但是，对你可能带来的好处包括提高您对分数知识的理解。对他人可能带来的好处包括能帮助老师与研究者们更好的了解学生是如何形成和发展分数的基本知识。这样的了解最终可能影响将来对其他孩子的数学教育课程。

除了参加此项研究，我还有什么其它选择?

这项研究不是为讲授学科内容知识而设计的。您的其它选择就是不参与此项研究。

此研究将如何处理收集到的信息?

有关您的信息将与参加此项研究和负责监督此项研究的个人或组织共享，包括:

- 研究员与合作研究员
- 审查此项研究的伦理审查委员会（IRB）

我们可能会发表此项研究的数据及结果。我们也可能会 在专业会议上展示这项研究的数据和结果。我们还可能会用这项研究的数据及结果培训数学教育工作者。但是您的名字以及其它可以确认您的辨识信息将会严格保密。

在法律要求的范畴内我们会保护您的信息不泄露给他人。我们不能保证完全保密。

本研究收集的数据可能会被识别并用于未来的研究，或在未经您同意的情况下给其他研究人员用于将来的研究使用。

谁能回答我的有关此项研究的问题?

如果你有任何问题，顾虑，或不满，或认为此项研究对您造成伤害或让您生病，您可以告诉您的老师，或研究团队。您可在本知情书的第一页找到他们的联系电话。

本研究项目受伦理审查委员会（IRB）监督。IRB 是一个对研究项目进行独立审核的机构。如果你有以下问题需要告知他们，可致电 001-800-562-4789 或发邮件至 help@wirb.com。

- 您的一些问题，顾虑，或不满没有从研究团队处得到满意的答复;
- 研究团队没有答复您的问题;
- 您无法联系到研究团队;
- 您想与其他人讨论此研究;
- 您有一些关于您作为参与者的权利的问题。

如果我因为参与此项研究受伤了，该怎么办?

如果在访谈期间您受伤或感到不舒服，马上告诉您的老师或研究员。他们会帮助您联系急救护理。

如果我同意参与此项目，但之后又改变主意了会发生什么事?

您可以自由地在任何时候拒绝参加此项研究，或者在研究过程中的任何时候选择退出研究，不需给出任何理由。您不会因此受到惩罚，或者丧失利益。此外，如果您决定中途退出，可以在任何时候要求返还或销毁与您有关信息。

知情同意页:

项目要求所有孩子都签订同意书。如果您的孩子选择参与，他或她在面谈开始前将被要求签署同意书。

您签名的文件将允许下列个人参与此项研究。

参与儿童家长签名 日期

参与儿童姓名（正楷） 日期

请圈选其中一个选项:

我（允许/不允许）我孩子的视频资料用于学术集会，例如学术会议。尽管我孩子的名字将会在视频资料中删除，但我孩子的外貌会出现在视频资料中。我明白我可以在 任何时候无需任何理由撤回我的许可。

参与儿童家长签名 日期

收签者签名 日期

Appendix I: IRB Approved Minor Assent Form Chinese Version

日期:_______________

未成年人知情同意书

题目： 运用混合法研究中国学生分数概念的建构及测后访谈

项目编号: 18-465
WIRB® Protocol #20182051
18-465

相关研究员电话: Jesse L.M. Wilkins 博士
001-540-231-8326

徐琮泽
001-716-783-1520

亲爱的同学，你好！

我是美国维吉尼亚理工大学教育学院数学教育专业的博士生徐琮泽。你被邀请参加一项名为“运用混合法研究中国学生分数概念的建构及测后访谈”的研究。在此研究中，我们希望通过观察学生解决和答分数问题的过程，能更好地了解中国学生对于分数概念的认知过程。

在之前的数学课上，你已经做了一份有关分数知识的笔试试卷。如果你决定继续参加这个项目的访谈部分，你将会与我，徐琮泽，（你会通过视频看到我）和一位数学老师一起解几道数学题。在此过程中，我们将会要求你解释你是如何求解这些数学问题的。整个访谈过程大概持续 45 分钟左右。如果你同意参加这个访谈，我们将会对整个过程进行录像。

你参加这个项目不会对你在学校的数学成绩有任何的影响。但是你的参与将会对我们提高关于分数的数学知识教学水平提供很大的帮助。我们承诺你的名字不会出现在今后任何有关这个项目的学术文章或报告里。

你的参与完全基于自愿，你可以在项目进行期间任何时候要求退出。如果有些问题你不想回答，你也可以选择不回答。

此致，

徐琮泽
维吉尼亚理工大学
教育学院
电子邮箱: jmf@vt.edu

Jesse Wilkins
维吉尼亚理工大学

教育学院
地址：300C War Memorial Hall, Virginia Tech, Blacksburg, VA 24061 USA
电子邮箱：wilkins@vt.edu

同意声明

我已经充分了解上述研究项目的目的与要求，我询问的问题也得到了满意的答复，我同意参加此项研究并收到了这份同意书的副本。

____ _______________

少年参与者签名 日期（月/日/年）

请圈选其中一个选项：

我（允许/不允许）我的图像出现在有非本项目团队成员出席的学术集会上，例如学术会议。我明白我可以在任何时候无需任何理由改变我的选择。

____ _______________

少年参与者签名 日期（月/日/年）

____ _______________

签收者签名 日期（月/日/年）

本知情书一式两份，参与者和研究者个保留一份。

如果你有任何问题，顾虑，或不满，可联系你的老师，或 徐琮泽，此研究项目的助理研究员。她的电话号码是 001-716-783-1520，她的电子信箱是 jmf@vt.edu。

本研究项目受伦理审查委员会（IRB）监督。IRB 是一个对研究项目进行独立审核的机构。如果你有以下问题需要告知他们，可致电 001-800-562-4789 或发邮件至 help@wirb.com。

- 你的问题，顾虑，或不满没有从研究团队处得到满意的答复；
- 研究团队没有答复你；
- 你无法联系到研究团队；
- 你想与其他人讨论此研究；
- 你有一些关于你作为参与者的权利的问题。

Appendix J: IRB Approved Minor Assent Form English Version

Date:_________________

Minor Assent Form

Dear Participant,

I am a mathematics education graduate student at Virginia Tech, U.S.A. You are invited to participate in a research project titled, "A Mixed Methods Study of Chinese Students' Construction of Fraction Schemes." Through this project, we are trying to better understand how Chinese students think about fractions when they solve fraction problems.

During your math class, you already took a written test about fractions. If you decide to participate in the interview portion of the study, you will work on math problems with a math teacher and me (you will see me through the internet) for about 30-45 minutes. You will be asked about how you are solving the problems. If you agree to participate in the interview portion of the study, we will videotape you while you are writing and talking. Your participation in this project will not affect your grades in school. We will not use your name on any papers that we write about this project. Your participation in this study will help us develop fraction instruction.

If you want to stop participating in this project, you are free to do so at any time. You can also choose not to answer questions that you do not want to answer.

Sincerely,

Congze Xu

School of Education

Virginia Polytechnic Institute and State University

Email: jmf@vt.edu

Jesse Wilkins

School of Education

Virginia Polytechnic Institute and State University

Address: 300C War Memorial Hall, Virginia Tech, Blacksburg, VA 24061 USA

Email: wilkins@vt.edu

I understand the project described above. My questions have been answered and I agree to participate in this research study. I have received a copy of this form.

Signature of child subject

Date

Circle one option below:

I (DO / DO NOT) give permission for video footage of me to be shown to others in professional meetings, such as conferences. I understand that I can change my mind at any time and without giving any reason.

Signature of child subject

Date

Signature of person obtaining assent

Date

Please sign both copies, keep one, and return one to the researcher.

If you have any questions, or concerns, or complaints talk to your teacher, or Jesse Wilkins, the investigator of this research study, at the following phone number, 001-540-231-8326, or e-mail wilkins@vt.edu.

This research is being overseen by an Institutional Review Board (IRB). An IRB is a group of people who perform an independent review of research studies. You may talk to them at 001-800-562-4789 or help@wirb.com if:

- You have questions, concerns, or complaints that are not being answered by the research team.
- You are not getting answers from the research team.
- You cannot reach the research team.
- You want to talk to someone else about the research.
- You have questions about your rights as a research subject.

Appendix K: IRB Approved Research Subject Information Sheet English Version

Research Subject Information Sheet

Sponsor:	Virginia Tech, USA
Protocol Title:	A Mixed Methods Study of Chinese Students Construction of Fraction Schemes: Extending the Written Test with Follow-up Clinical Interviews
Protocol No.:	18-465 WIRB® Protocol #20182051 18-465
Investigator:	Dr. Jesse L. M. Wilkins, PhD Phone Number: 001-540-231-8326
Sub-Investigator:	Congze Xu Phone Number: 001-716-783-1520

Dear Parents:
Your daughter/son will be participating in a class activity in their mathematics class that is part of a research study that is being conducted by Dr. Jesse Wilkins and Congze Xu, from the School of Education at Virginia Tech. The purpose of this study is to investigate how Chinese students generally understand fractions, which is an important part of the 4th and 5th grade curriculum in China. This information could help teachers and schools in the future to tailor their instruction to students' needs more effectively.

Mrs. Shaoyan Jiang, the principal of your child's elementary school, has approved this study.

As part of your child's regular mathematics class, they will be asked to solve some mathematics tasks involving fractions as part of a written assessment. It will take approximately 45 minutes to complete the tasks.

There are no physical risks associated with being in this research. There is the risk of a loss of confidentiality of your research-related information. The data and results of this research may be published in professional journals or presented at professional conferences. The data and results may also be used for training mathematics educators. However, the names of children, teachers, and schools involved in this research will not be used in publications or presentations.

There are no direct benefits associated with participating in this research. However, people in the future may benefit from the information obtained from this research.

Participation in this study is voluntary. Your alternative is to not participate in this study. Your child will not be penalized or lose benefits if they decide not to participate or decide to stop participating.

If you have questions about your child's participation, or you decide you do not want your child to participate in this activity, you can contact your child's teacher, or Mr. Haibo Liu, the Vice Principal and Director of the Department of Mathematics Teaching and Research, in your child's school, at 13562505038. You can also contact Congze Xu at 001-716-783-1520, or Dr. Jesse Wilkins at 001-540-231-8326 for questions, concerns or complaints about the research, or if you think your child has been harmed as a result of joining this research. Contact the Western Institutional Review Board (WIRB) if you have questions about your child's rights as a research subject, concerns, complaints or input: 001-800-562-4789. WIRB is a group of people who perform independent review of research.

The study staff will share the records generated from this research with the sponsor, regulatory agencies and WIRB. This information is shared so the research can be conducted and properly monitored.

Appendix L: IRB Approved Research Subject Information Sheet Chinese Version

研究项目情况书

研究管理机构: 美国维吉尼亚理工大学

题目: 运用混合法研究中国学生分数概念的建构及测后访谈

项目编号: 18-465

WIRB® Protocol #20182051

18-465

研究者: Jesse L.M. Wilkins 博士

电话号码：001-540-231-8326

助理研究者: 徐琮泽

电话号码：001-716-783-1520

尊敬的家长:

您的女儿/儿子将在数学课上参加一个课堂活动，该课堂活动属于一项由维吉尼亚理工大学教育学院的Jesse Wilkins博士及徐琮泽女士 进行的教学研究项目的一部分。此项研究的目的是为了探究中国学生是如何理解分数概念的。分数知识也是中国四五年级数学课程的重要部分，了解中国学生分数概念的构建可以帮助老师和学校在今后的工作中更有效地结合学生认知结构设计教学课程。

您孩子所在小学的校长姜少艳女士已同意我们在该校进行此项研究。

作为您孩子正常数学课程的一部分，您孩子将被要求参加一项约45分钟时间的笔试，该笔试的内容涉及解决与分数有关的一些数学问题。

除了一些与此项研究有关的、涉及您孩子的信息可能会泄露外，此项研究不会对孩子造成任何伤害。这项研究的数据和结果可能会 发表在学术杂志或在学术会议上展示。此项研究的数据和结果还可能用于培训数学教育工作者。但是您孩子的名字，老师的名字以及学校名称将不会出现在任何刊物或会议上被披露或公开。

虽然参与者难以直接从这项研究中获得利益，但是师生们将来可能会从此项研究的成果中获益。

参与此项研究是自愿的。您可以选择不参加此项研究。您的孩子不会因决定不参与，或半途终止参与此项研究而受到任何惩罚或丧失任何应得利益。

对于您孩子参与此项研究，如果您有任何问题，或您认为您决定您的孩子不参与此项研究，可以联系您孩子的老师，或者副校长及数学教研室主任刘海波校长，电话：13562505038。 如果您对此研究有任何问题，困惑，或不满，或者您认为您孩子在参与的过程中受到伤害，可以联系徐琮泽女士，电话：001-716-783-1520， 或 Jesse Wilkins 博士，电话：001-540-231-8326。 如果您对您孩子在参与此项研究中的有关权利有困惑或不满，可联系西方伦理审查委员会 （WIRB），他们的电话是： 001-800-562-4789。WIRB 是一个独立审查此项研究的审核组织。

研究团队将会与此研究的管理、监管机构，及 WIRB 分享此研究的所有记录，与这些机构分享有关信息是为了他们能更好的指导与监管此项研究。

Appendix M: IRB Approved Principal Support Letter English Version

20182051
#22979339.0

IRB Approved at the
Protocol Level
Oct 22, 2018

Document of Qixia City Daqing Road School

★

Approval

Congze Xu, Doctoral Student

School of Education

Virginia Tech

Dear Congze Xu

We have reviewed your research request and this letter serves as notification that I have approved your proposal to conduct the study, *A Mixed Methods Study of Chinese Students' Construction of Fraction Schemes*, in our school. This approval gives you permission to administer a mathematics assessment to students in 4th and 5th grade. You will also be allowed to select approximately 30 students from among the 4th and 5th grade students who completed the assessment to participate in a follow-up interview based on their performance on the written assessment. In addition, these interviews will be videotaped for later analysis of student thinking.

You are allowed to contact the Vice Principal, Mr. Haibo Liu, who is also the Director of the Department of Mathematics Teaching and Research at our school, and the classroom teachers who will assist you with carrying out your study. However, the decision whether to participate in the interviews will rest with the parents of the individual students, and the students themselves. It is our understanding that you will obtain appropriate written parental consent and student assent for students participating in the interviews, and the interviews will last approximately 45 minutes.

Daqing Road School, Qixia City, Shandong (School Stamp)

Principal: Shaoyan Jiang (signature)

Date: 2018-10-15

Appendix N: IRB Approved Principal Support Letter Chinese Version

栖霞市大庆路学校文件

同意书

维吉尼亚理工大学教育学院博士生徐琮泽女士:

我们审阅了您的研究意向与要求，我们同意您在我校进行您的研究项目"运用混合法研究中国学生分数概念的建构"。我们批准您对我校四、五年级学生进行分数知识的测试。同时我们也同意您根据笔试结果，从四年级及五年级学生中选择约 30 学生进行测后访谈，并对访谈过程进行录像以作为接下来分析学生解题思路的视频数据。

此函允许您联系我校副校长及数学教研室主任，刘海波老师，并由他及四五年级数学任课老师全程协助您的研究项目。不过学生是否参加您的测后访谈，应由学生家长及学生本人决定。据我们了解您将在获得家长签名的家长知情书，及学生本人签名的学生知情书之后进行访谈，且访谈时间不超过 45 分钟。

栖霞市大庆路学校

校长签名：姜少艳

日期 2018.10.15

Appendix O: Institutional Review Board Approval Letter

VIRGINIA TECH.

Division of Scholarly Integrity and Research Compliance
Institutional Review Board
North End Center, Suite 4120 (MC 0497)
300 Turner Street NW
Blacksburg, Virginia 24061
540/231-3732
irb@vt.edu
http://www.research.vt.edu/sirc/hrpp

MEMORANDUM

DATE: August 15, 2019

TO: Jesse L Wilkins, Cong Ze Xu

FROM: Virginia Tech Institutional Review Board (FWA00000572, expires January 29, 2021)

PROTOCOL TITLE: A mixed methods study of Chinese students' construction of fraction schemes: Extending the written test with follow-up clinical interviews

IRB NUMBER: **18-465**

Effective August 15, 2019, the Virginia Tech Institution Review Board (IRB) approved the Amendment request for the above-mentioned research protocol.

This approval provides permission to begin the human subject activities outlined in the IRB-approved protocol and supporting documents.

Plans to deviate from the approved protocol and/or supporting documents must be submitted to the IRB as an amendment request and approved by the IRB prior to the implementation of any changes, regardless of how minor, except where necessary to eliminate apparent immediate hazards to the subjects. Report within 5 business days to the IRB any injuries or other unanticipated or adverse events involving risks or harms to human research subjects or others.

All investigators (listed above) are required to comply with the researcher requirements outlined at:

https://secure.research.vt.edu/external/irb/responsibilities.htm

(Please review responsibilities before beginning your research.)

PROTOCOL INFORMATION:

Approved As:	**Expedited, under 45 CFR 46.110 category(ies) 6,7**
Protocol Approval Date:	**August 17, 2018**
Protocol Expiration Date:	**August 17, 2019**
Continuing Review Due Date*:	**August 3, 2019**

*Date a Continuing Review application is due to the IRB office if human subject activities covered under this protocol, including data analysis, are to continue beyond the Protocol Expiration Date.

ASSOCIATED FUNDING:

The table on the following page indicates whether grant proposals are related to this protocol, and which of the listed proposals, if any, have been compared to this protocol, if required.

Invent the Future

VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY
An equal opportunity, affirmative action institution

Appendix P: Western Institutional Review Board Approval Letter 01

WIRB
A WIRB-Copernicus Group Company

Certificate of Action

Investigator Name: Jesse Leroy Muller Wilkins, PhD	**Board Action Date:** 09/13/2018
Investigator Address: War Memorial Hall, Room 300-C, 370 Drillfield Drive Blacksburg, VA 24061, United States	**Approval Expires:** 08/17/2019 **Continuing Review Frequency:** Annually
Sponsor: Virginia Tech **Institution Tracking Number:** 18-465	**Sponsor Protocol Number:** 18-465 **Amended Sponsor Protocol Number:**
Study Number: 1189041	**IRB Tracking Number:** 20182051
Work Order Number: 1-1112652-1	**Panel:** 2
Protocol Title: A Mixed Methods Study of Chinese Students Construction of Fraction Schemes: Extending the Written Test with Follow-up Clinical Interviews	

THE FOLLOWING ITEMS ARE APPROVED:
Chinese 5th Grade Written Assessment of Levels of Fraction Schemes - English #18217595.0 (Client Translation)
Chinese 6th Grade Written Assessment of Levels of Fraction Schemes - English #18217597.0 (Client Translation)
Chinese Assent Form - Clinical Interview (English Approved 09-11-2018) (Client Translation)
Chinese Consent Form - Clinical Interview (English Approved 09-11-2018) (Client Translation)
Chinese Consent Information Sheet - Written Assessment Letter to Parents (English Approved 09-11-2018) (Client Translation)
Chinese Interview Protocol - English #18217599.0 (Client Translation)
Translation Certification - Assorted Chinese Documents

Please note the following information:
Please use the new and revised consent documents specified in this approval to enroll future subjects.

Custom COA Phrase: The Board found that administering the written assessment for this research meets the requirements for a waiver of documentation of consent under 45 CFR 46.117(c)(2).

THE IRB HAS APPROVED THE FOLLOWING LOCATIONS TO BE USED IN THE RESEARCH:
Virginia Tech, War Memorial Hall, 300-C, 370 Drillfield Drive, Blacksburg, Virginia 24061
Qinglong Town No1. Elementary School, 67 Qinglong Town Upper Street, Pengshan District, Meishan, 620866 China

ALL IRB APPROVED INVESTIGATORS MUST COMPLY WITH THE FOLLOWING:
As a requirement of IRB approval, the investigators conducting this research will:

- Comply with all requirements and determinations of the IRB.
- Protect the rights, safety, and welfare of subjects involved in the research.
- Personally conduct or supervise the research.
- Conduct the research in accordance with the relevant current protocol approved by the IRB.
- Ensure that there are adequate resources to carry out the research safely.
- Ensure that research staff are qualified to perform procedures and duties assigned to them during the research.
- Submit proposed modifications to the IRB prior to their implementation.
  - Not make modifications to the research without prior IRB review and approval unless necessary to eliminate apparent immediate hazards to subjects.
- Submit continuing review reports when requested by the IRB.
- Submit a closure form to close research (end the IRB's oversight) when:

This is to certify that the information contained herein is true and correct as reflected in the records of this IRB. WE CERTIFY THAT THIS IRB IS IN FULL COMPLIANCE WITH GOOD CLINICAL PRACTICES AS DEFINED UNDER THE U.S. FOOD AND DRUG ADMINISTRATION (FDA) REGULATIONS, U.S. DEPARTMENT OF HEALTH AND HUMAN SERVICES (HHS) REGULATIONS, AND THE INTERNATIONAL CONFERENCE ON HARMONISATION (ICH) GUIDELINES.

- The protocol is permanently closed to enrollment
- All subjects have completed all protocol related interventions and interactions
- For research subject to federal oversight other than FDA:
  - No additional identifiable private information about the subjects is being obtained
  - Analysis of private identifiable information is completed

- If research approval expires, stop all research activities and immediately contact the IRB.
- Promptly report to the IRB the information items listed in the IRB's "Prompt Reporting Requirements" available on the IRB's Web site.
- Not accept or provide payments to professionals in exchange for referrals of potential subjects ("finder's fees.")
- Not accept payments designed to accelerate recruitment that are tied to the rate or timing of enrollment ("bonus payments") without prior IRB approval.
- When required by the IRB ensure that consent, permission, and assent are obtained and documented in accordance with the relevant current protocol as approved by the IRB.
- Promptly notify the IRB of any change to information provided on your initial submission form.

Consistent with AAHRPP's requirements in connection with its accreditation of IRBs, the individual and/or organization shall promptly communicate or provide, the following information relevant to the protection of human subjects to the IRB in a timely manner:

- Upon request of the IRB, a copy of the written plan between sponsor or CRO and site that addresses whether expenses for medical care incurred by human subject research subjects who experience research related injury will be reimbursed, and if so, who is responsible in order to determine consistency with the language in the consent document.
- Any site monitoring report that directly and materially affects subject safety or their willingness to continue participation. Such reports will be provided to the IRB within 5 days.
- Reports from any data monitoring committee, data and safety monitoring board, or data and safety monitoring committee in accordance with the time frame specified in the research protocol.
- Any findings from a closed research when those findings materially affect the safety and medical care of past subjects. Findings will be reported for 2 years after the closure of the research.

If your research site is a HIPAA covered entity, the HIPAA Privacy Rule requires you to obtain written authorization from each research subject for any use or disclosure of protected health information for research. If your IRB-approved consent form does not include such HIPAA authorization language, the HIPAA Privacy Rule requires you to have each research subject sign a separate authorization agreement. "

Federal regulations require that the IRB conduct continuing review of approved research. You will receive Continuing Review Report forms from this IRB when the expiration date is approaching.

Thank you for using this WCG IRB to provide oversight for your research project.

DISTRIBUTION OF COPIES:
<u>Contact, Company</u>
Jennifer Farmer, Virginia Polytechnic Institute and State University (Virginia Tech)
Jesse Leroy Muller Wilkins, PhD, Virginia Tech
Congze Xu, Virginia Tech

Appendix Q: Western Institutional Review Board Approval Letter 02

Certificate of Action

Investigator Name: Jesse Leroy Muller Wilkins, PhD	**Board Action Date:** 10/22/2018
Investigator Address: War Memorial Hall, Room 300-C, 370 Drillfield Drive Blacksburg, VA 24061, United States	**Approval Expires:** 08/17/2019 **Continuing Review Frequency:** Annually
Sponsor: Virginia Tech **Institution Tracking Number:** 18-465	**Sponsor Protocol Number:** 18-465 **Amended Sponsor Protocol Number:**
Study Number: 1189041	**IRB Tracking Number:** 20182051
Work Order Number: 1-1123522-1	**Panel:** 2
Protocol Title: A Mixed Methods Study of Chinese Students Construction of Fraction Schemes: Extending the Written Test with Follow-up Clinical Interviews	

THE FOLLOWING ITEMS ARE APPROVED:
Principal Approval Letter (English) (10-15-2018) #22979339.0 - As Submitted
Consent Form - Clinical Interview [IN0]
Consent Information Sheet - Written Assessment Letter to Parents [IN0]
Revised Research Locations x1 (10-16-2018)

Please note the following information:
Please have all future subjects sign the revised Consent Form(s) specified in this approval.

THE IRB HAS APPROVED THE FOLLOWING LOCATIONS TO BE USED IN THE RESEARCH:
Virginia Tech, War Memorial Hall, 300-C, 370 Drillfield Drive, Blacksburg, Virginia 24061
Daqing Road School, 172 Culture Road, Qixia, Shandong 265300 China

ALL IRB APPROVED INVESTIGATORS MUST COMPLY WITH THE FOLLOWING:
As a requirement of IRB approval, the investigators conducting this research will:

- Comply with all requirements and determinations of the IRB.
- Protect the rights, safety, and welfare of subjects involved in the research.
- Personally conduct or supervise the research.
- Conduct the research in accordance with the relevant current protocol approved by the IRB.
- Ensure that there are adequate resources to carry out the research safely.
- Ensure that research staff are qualified to perform procedures and duties assigned to them during the research.
- Submit proposed modifications to the IRB prior to their implementation.
    - Not make modifications to the research without prior IRB review and approval unless necessary to eliminate apparent immediate hazards to subjects.
- Submit continuing review reports when requested by the IRB.
- Submit a closure form to close research (end the IRB's oversight) when:
    - The protocol is permanently closed to enrollment
    - All subjects have completed all protocol related interventions and interactions
    - For research subject to federal oversight other than FDA:
        - No additional identifiable private information about the subjects is being obtained
        - Analysis of private identifiable information is completed
- If research approval expires, stop all research activities and immediately contact the IRB.
- Promptly report to the IRB the information items listed in the IRB's "Prompt Reporting Requirements" available on the IRB's Web site.

This is to certify that the information contained herein is true and correct as reflected in the records of this IRB. WE CERTIFY THAT THIS IRB IS IN FULL COMPLIANCE WITH GOOD CLINICAL PRACTICES AS DEFINED UNDER THE U.S. FOOD AND DRUG ADMINISTRATION (FDA) REGULATIONS, U.S. DEPARTMENT OF HEALTH AND HUMAN SERVICES (HHS) REGULATIONS, AND THE INTERNATIONAL CONFERENCE ON HARMONISATION (ICH) GUIDELINES.

Board Action: 10/22/2018

- Not accept or provide payments to professionals in exchange for referrals of potential subjects ("finder's fees.")
- Not accept payments designed to accelerate recruitment that are tied to the rate or timing of enrollment ("bonus payments") without prior IRB approval.
- When required by the IRB ensure that consent, permission, and assent are obtained and documented in accordance with the relevant current protocol as approved by the IRB.
- Promptly notify the IRB of any change to information provided on your initial submission form.

Consistent with AAHRPP's requirements in connection with its accreditation of IRBs, the individual and/or organization shall promptly communicate or provide, the following information relevant to the protection of human subjects to the IRB in a timely manner:

- Upon request of the IRB, a copy of the written plan between sponsor or CRO and site that addresses whether expenses for medical care incurred by human subject research subjects who experience research related injury will be reimbursed, and if so, who is responsible in order to determine consistency with the language in the consent document.
- Any site monitoring report that directly and materially affects subject safety or their willingness to continue participation. Such reports will be provided to the IRB within 5 days.
- Reports from any data monitoring committee, data and safety monitoring board, or data and safety monitoring committee in accordance with the time frame specified in the research protocol.
- Any findings from a closed research when those findings materially affect the safety and medical care of past subjects. Findings will be reported for 2 years after the closure of the research.

If your research site is a HIPAA covered entity, the HIPAA Privacy Rule requires you to obtain written authorization from each research subject for any use or disclosure of protected health information for research. If your IRB-approved consent form does not include such HIPAA authorization language, the HIPAA Privacy Rule requires you to have each research subject sign a separate authorization agreement. "

Federal regulations require that the IRB conduct continuing review of approved research. You will receive Continuing Review Report forms from this IRB when the expiration date is approaching.

Thank you for using this WCG IRB to provide oversight for your research project.

DISTRIBUTION OF COPIES:
Contact, Company
Jennifer Farmer, Virginia Polytechnic Institute and State University (Virginia Tech)
Jesse Leroy Muller Wilkins, PhD, Virginia Tech
Congze Xu, Virginia Tech

COA Template 01-03-2018

Appendix R: IRB Approved Translation Certification

TRANSLATION CERTIFICATION

I hereby certify that I am fluent in English and Chinese and that I have, to the best of my knowledge and belief, made a true and complete translation from English to Chinese of the WIRB approved Consent Form and Research Subject Information Sheet for Virginia Tech 18-465, WIRB Protocol #20182051 this 24th day of October, 2018.

Congze Xu
(Printed name of translator)

徐聪泽
(Signature of translator)

As principal investigator of Virginia Tech 18-465, WIRB Protocol #20182051, I hereby certify, to the best of my knowledge and belief, that these translated documents correspond to the WIRB approved version of the Consent Form and Research Subject Information Sheet.

Jesse L. M. Wilkins
(Printed name of principal investigator)

Jesse L M Wilkins
(Signature of principal investigator)