SOME ASYMPTOTIC STABILITY RESULTS FOR THE BOUSSINESQ EQUATION

by

FANG-LAN LIU

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Approved:

D. L. RUSSELL, Chairman

J. KIM

T. LIN

Y. RENARDY

S.-M. SUN

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Committee Chairman: D. L. RUSSELL

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(ABSTRACT)

We prove that the solution of the Boussinesq equation with relatively small initial data exists globally and decays exponentially under some boundary conditions. In fact, it is shown that an operator A defined by

$$A(g,h) = (h', g' - g'''), \qquad \text{for } (g,h) \in H^3(0,1) \times H^1(0,1),$$

whose domain $D(A)$ consists all $(g,h) \in H^3(0,1) \times H^2(0,1)$ with boundary conditions

$$g'(1) = a^{-1}g'(0), \qquad g''(1) = ag''(0) + Kh(0),$$

$$g(1) = ag(0), \qquad h(1) = a^{-1}h(0), \qquad h'(1) = ah'(0),$$

generates a $C_0 - semigroup$ $T(t)$ on $H = \{(g,h) \in H^1(0,1) \times L^2(0,1) \mid g(1) = ag(0)\ \}$. Moreover, there is a constant $\gamma > 0$, such that

$$\|T(t)\| \leq e^{-\gamma t}, \qquad \text{for all } t > 0.$$

Also using the operator A, we prove that the solution of the Boussinesq equation

$$u_t = v_x$$

$$v_t = u_x - u_{xxx} + (u^2)_x$$

with small initial data in H decays exponentially.

Using a pair of operators $P(t)$ and $Q(t)$ produced by Fourier series, we prove that the initial value problem (IVP) of a general Boussinesq equation

$$u_{tt} - u_{xx} + u_{xxxx} + (u^2)_{xx} + K_1 u_t + K_2 u = 0, \qquad t > 0,$$

$$u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), \qquad x \in [0, 2\pi],$$

for constants $K_1 \geq 0$ and $K_2 \geq 0$ with periodic boundary conditions is well-posed and that the solution u decays exponentially for $K_1 > 0$ and $K_2 > 0$. These are local results (in the sense that the initial data are relatively small). Later , we consider the global properties which are somewhat different.

It is known that the KdV equation is globally well-posed on periodic domains [5] [39] but it is hard to obtain the similar results for the Boussinesq equation unless we require some global conditions. In Chapter 4, we consider the solutions of a generalization of the Boussinesq equation

$$u_{tt} = u_{xx} - (u^2 + u_{xx})_{xx}$$

on periodic domains. It is shown that the solution of the initial-value problem on a periodic domain exists globally if a weak priori condition is satisfied.

Finally in Chapter 5, using the results obtained in Chapter 4, we consider a feedback controlled Boussinesq equation on a periodic domain. It is proved by conservation laws that the velocity component of the solution decays asymptotically while certain weak priori condition is imposed.

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Contents

Chapter 1. Introduction

The commonly called "Boussinesq equation" is

$$u_{tt} - u_{xx} + \frac{3}{2}(u^2)_{xx} + bu_{xxxx} = 0, \tag{1.1}$$

which was proposed by Boussinesq [8] in 1872 to describe a surface water wave whose horizontal scale is much larger than the depth of the water. In [11], it was showed that (1.1) describes large-horizontal-scale disturbances for certain marginally stable, inviscid, shear flows. Then in 1895, Korteweg and de Vries [18] studied the same physical problem and derived the so-called KdV equation

$$u_t - u_x + \frac{3}{2}(u^2)_x + bu_{xxx} = 0. \tag{1.2}$$

This equation can be derived from the equation of inviscid fluid motion by systematic asymptotic methods and also has been found to describe large-horizontal-scale waves in many neutrally stable, inviscid flows [11].

There are two versions of the Boussinesq equation (1.1) [31]. The first one is generally called the "bad Boussinesq equation". It corresponds to $b < 0$ in (1.1) and the initial value problem (IVP) for this version is not locally well-posed since the dispersion relation $w^2 = k^2(1 - k^2)$ is justified only for small wave numbers. The second one is called the "good Boussinesq equation", corresponding to $b > 0$ in (1.1). It has a useful dispersion formula [13]; the IVP for this version is always locally well-posed [6]. For this reason, we only discuss the "good Boussinesq equation" in the remaining chapters and just refer to it as the Boussinesq equation.

Although the Boussinesq equation and the KdV equation are similar in scope, the literature on the former is much less than that on the latter. For detailed results on the Boussinesq equation, see [6] [11] [35]. Also in [28] and [1], the existence, regularity and uniqueness of the solutions to an Boussinesq system of equations were studied.

In this chapter, we give a formal derivation of the Boussiness equation which is very interesting and much simpler than many others. This derivation is taken from [32]. We then give some notations which will be used later.

§ 1.1. Formal Derivation of the Boussinesq equation.

Considering the surface water wave in a channel as described above, we denote $y = s(t,x)$ as a free surface, $p(t,x,y)$ as the pressure on the surface, ρ as the constant density, h as the mean height of the water wave (at the bottom of the channel, $y = -h$), g as the gravity, $u(t,x,y)$ and $v(t,x,y)$ as the horizontal and vertical velocities of the water wave respectively and p_0 as the pressure on the free surface. Without loss of generality, we assume that $p_0 = 0$.

The basic (Euler) equations for such fluid motion are

$$u_x + v_y = 0 \tag{1.3}$$

$$u_t + uu_x + vu_y = -p_x \tag{1.4}$$

$$v_t + uv_x + vv_y = -p_y - \rho g. \tag{1.5}$$

Applying ∂_y to (1.4) and ∂_x to (1.5), we have

$$(u_y - v_x)_t + (u_y - v_x)u_x + (u_y - v_x)v_y + u(u_y - v_x)_x + v(u_y - v_x)_y = 0,$$

which becomes

$$(u_y - v_x)_t + u(u_y - v_x)_x + v(u_y - v_x)_y = 0$$

by (1.3). Adjusting the initial conditions, we then obtain

$$u_y - v_x = 0. \tag{1.6}$$

On the surface $y = s(t, x)$, by the assumption $p_0 = 0$,

$$p = p - p_0 = -S\frac{s_{xx}}{(1+s_x^2)^{\frac{3}{2}}}, \tag{1.7}$$

$$s_t + us_x = v. \tag{1.8}$$

where S is the surface tension, (1.7) is the normal-stress continuity condition which requires the jump in pressure across the free surface by the surface-tension forces [11], and (1.8) is a kinematic condition.

On $y = -h$, the bottom of the channel, the vertical fluid velocity is zero:

$$v = 0. \tag{1.9}$$

Our long wave assumption corresponds to the approximations:

$$x = lX, \qquad t = \frac{lT}{\sqrt{\rho g h}}, \qquad y = hY, \qquad s = \epsilon h N,$$

$$p = \rho(-gy + \epsilon g h P), \qquad u = \epsilon\sqrt{\rho g h}U, \qquad v = \epsilon^{\frac{3}{2}}\sqrt{\rho g h}V,$$

where $l = h\epsilon^{-\frac{1}{2}} >> 1$ is the horizontal scaling parameter. Using these approximations, we change (1.3)~(1.9) to the following:

$$U_X + V_Y = 0 \tag{1.10}$$

$$U_T + \epsilon U U_X + \epsilon V U_Y = -P_X \tag{1.11}$$

$$\epsilon(V_T + \epsilon U V_X + \epsilon V V_Y) = -P_Y \tag{1.12}$$

$$U_Y - \epsilon V_X = 0. \tag{1.13}$$

On $Y = \epsilon N$,

$$P = N - \tau \frac{\epsilon N_{XX}}{(1 + \epsilon^{\frac{3}{2}} N_X)^{\frac{3}{2}}}, \tag{1.14}$$

where $\tau = \dfrac{S}{\rho g h^2}$ is the Bond number, and

$$N_T + \epsilon U N_X = V. \tag{1.15}$$

On $Y = -1$,

$$V = 0. \tag{1.16}$$

Integrating (1.10) and using (1.16), we have

$$V = -\int_{-1}^{Y} U_X dY. \tag{1.17}$$

By (1.13)

$$U_Y = \epsilon V_X = -\epsilon \int_{-1}^{Y} U_{XX} dY.$$

Then integrating by parts:

$$U = f(T, X) - \epsilon \int_{-1}^{Y} \int_{-1}^{Z} U_{XX}(T, X, W) dW dZ,$$

and, iterating, we have

$$U = f(T, X) - \epsilon \int_{-1}^{Y} \int_{-1}^{Z} f_{XX}(T, X) dW dZ + O(\epsilon^2)$$

$$= f(T, X) - \epsilon f_{XX} \frac{(Y+1)^2}{2} + O(\epsilon^2). \tag{1.18}$$

Substituting (1.18) into (1.17), we have

$$V = -f_X(Y+1) + \epsilon f_{XXX} \frac{(Y+1)^3}{6} + O(\epsilon^2). \tag{1.19}$$

Integrating (1.12) with (1.14), (1.18) and (1.19), we have

$$\begin{aligned}
P &= N - \tau\epsilon N_{XX} + O(\epsilon^2) - \epsilon\int_{\epsilon N}^{Y}(V_T + \epsilon U V_X + \epsilon V V_Y)dY \\
&= N - \tau\epsilon N_{XX} + \epsilon f_{XT}\frac{(Y+1)^2}{2}\Big|_{\epsilon N}^{Y} + O(\epsilon^2) \\
&= N - \tau\epsilon N_{XX} + \epsilon f_{XT}\left(\frac{(Y+1)^2}{2} - \frac{1}{2}\right) + O(\epsilon^2) \qquad (1.20)
\end{aligned}$$

since (1.14) shows that on $Y = \epsilon N$,

$$P = N - \tau\epsilon N_{XX} + O(\epsilon^2).$$

Substituting (1.18)~(1.20) into (1.11), we then have

$$f_T + N_X + \epsilon f f_X - \frac{\epsilon}{2} f_{XXT} - \tau\epsilon N_{XXX} = O(\epsilon^2). \qquad (1.21)$$

Combining (1.15), (1.18) and (1.19) on the surface $Y = \epsilon N$, we obtain another equation for N and f

$$N_T + f_X + \epsilon(fN)_X - \frac{\epsilon}{6} f_{XXX} = O(\epsilon^2). \qquad (1.22)$$

Let $f = h + \frac{1}{6}\epsilon h_{XX}$; then (1.21) and (1.22) become

$$h_T + N_X + \epsilon h h_X - \frac{\epsilon}{3} h_{XXT} - \tau\epsilon N_{XXX} = O(\epsilon^2), \qquad (1.23)$$

$$N_T + h_X + \epsilon(hN)_X = O(\epsilon^2), \qquad (1.24)$$

from which we have the Boussinesq system of equations

$$N_T + h_X + \epsilon(hN)_X = 0,$$

$$h_T + N_X + \epsilon h h_X - \frac{\epsilon}{3} h_{XXT} - \tau\epsilon N_{XXX} = 0.$$

By iterating (1.23), we have the equation

$$h_T + N_X + \epsilon h h_X + \frac{\epsilon}{3} N_{XXX} - \tau \epsilon N_{XXX} = O(\epsilon^2). \tag{1.25}$$

By (1.24) and (1.25), we have the following equations

$$(N+h)_T + (N+h)_X = O(\epsilon)$$

$$N_{TT} - N_{XX} + \epsilon(hN)_{TX} - \frac{\epsilon}{2}(h^2)_{XX} + \epsilon(\tau - \frac{1}{3})N_{XXXX} = O(\epsilon^2) \tag{1.26}$$

$$N_{TT} - N_{XX} = O(\epsilon)$$

$$h_{TT} - h_{XX} = O(\epsilon)$$

which imply that $h = -N + O(\epsilon)$ by adjusting the initial conditions of h and N. Together with $h_T = -N_X + O(\epsilon)$ and $N_T = -h_X + O(\epsilon)$ from (1.25) and (1.24) respectively, (1.26) gives

$$\epsilon(hN)_{TX} = \epsilon(h_T N + h N_T) = -\epsilon(N^2)_{XX} + O(\epsilon^2)$$

and (1.26) becomes

$$N_{TT} - N_{XX} - \frac{3\epsilon}{2}(N^2)_{XX} + \epsilon(\tau - \frac{1}{3})N_{XXXX} = O(\epsilon^2),$$

which finally provides the derivation of the commonly called Boussinesq equation

$$N_{TT} - N_{XX} - \frac{3\epsilon}{2}(N^2)_{XX} + \epsilon(\tau - \frac{1}{3})N_{XXXX} = 0.$$

For $\tau > \frac{1}{3}$, it is the "good Boussinesq equation" and for $\tau < \frac{1}{3}$, it is the "bad Boussinesq equation", as described earlier.

§ 1.2. Notations.

We denote by H_L^k the Hilbert space

$$H_L^k = \{g \in H_{loc}^k | \partial^j g(x) = \partial^j g(x+L), \quad \text{for } 0 \leq j < k\}$$

with norm

$$\|g\|_k := \|g\|_{H_L^k} = \|g\|_{H^k(0,L)}$$

and by V_T^k the Banach space

$$V_T^k = C(0,T; H_L^{k+1} \times H_L^k) = C([0,T] : H_L^{k+1} \times H_L^k)$$

with norm

$$\|(u,v)\|_{V_T^k} = \sup_{0<t\leq T} (\|u(.,t)\|_{H_L^{k+1}} + \|v(.,t)\|_{H_L^k}).$$

In particular, H_L^0 is identified by $L^2(0,L)$.

For convenience, $\dfrac{\partial^k u}{\partial x^k}$ is identified by $u_{(k)}$ in this paper.

Chapter 2. Exponential Decay for Solutions of the Boussinesq Equation with a Boundary Control

In connection with the control of the Korteweg-de Vries equation, a related third order linear equation has recently been studied in [26]. Similarly, in the present chapter, we first consider the linear equation

$$u_{tt} = u_{xx} - u_{xxxx}, \qquad t > 0,\ 0 < x < L, \tag{2.1}$$

which we rewrite as the following system:

$$\begin{cases} u_t = v_x \\ v_t = u_x - u_{xxx} \qquad t > 0,\ 0 < x < 1, \end{cases} \tag{2.2}$$

with boundary conditions (for convenience, we set $L = 1$ in this chapter):

$$\begin{cases} u(1,t) = au(0,t), \qquad v(1,t) = a^{-1}v(0,t) \\ u_x(1,t) = a^{-1}u_x(0,t), \qquad u_{xx}(1,t) = au_{xx}(0,t) + Kv(0,t) \end{cases} \tag{2.3}$$

where $a \neq 1$, $a > 0$, and $K > 0$ (generally, we only need $|a| \neq 1$, and $aK > 0$). Let A be defined as

$$A(g,h) = (h', g' - g'''), \qquad for\ (g,h) \in D(A), \tag{2.4}$$

where $' = \dfrac{d}{dx}$ and the domain of A, $D(A)$, consists of all $(g,h) \in H^3(0,1) \times H^2(0,1)$ satisfying the boundary conditions

$$g'(1) = a^{-1}g'(0), \qquad g''(1) = ag''(0) + Kh(0),$$
$$g(1) = ag(0), \qquad h(1) = a^{-1}h(0), \qquad h'(1) = ah'(0). \tag{2.5}$$

We will see that A generates a $C_0 - semigroup$ on a Hilbert space $H = H_a^1(0,1) \times L^2(0,1)$, where

$$H_a^1(0,1) = \{g \in H^1(0,1) \ : \ g(1) = ag(0)\}.$$

Our goal is to prove that the solution (u,v) satisfying (2.2) and (2.3) decays exponentially. We notice that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^1 (u^2 + u_x^2 + v^2)dx &= \int_0^1 (uu_t + u_x u_{xt} + vv_t)dx \\ &= \int_0^1 (uv_x + u_x v_{xx} + v(u_x - u_{xxx}))dx \\ &(integrating\ by\ parts) \\ &= (uv + u_x v_x - u_{xx}v)|_{x=0}^{x=1} \\ &(by\ (2.3)) \\ &= -\frac{K}{a}|v(0,t)|^2 \leq 0. \end{aligned} \tag{2.6}$$

Thus it is reasonable for us to suppose that $(u(.,t), v(.,t))$ converges appropriately as $t \to \infty$. The detailed results for the linear equation/operator are contained in §2.1. Then in §2.2, we give a local exponential decay result for the nonlinear equation by using the properties of the operator A.

§ 2.1. Exponential Decay Of The Linear Semigroup.

We mainly use the following result (c.f. [12]) to achieve our goal.

Theorem 2.1 (by F. L. Huang). *Let exp(tB) be a $C_0 - semigroup$ in Hilbert space H for which there exists a positive number M such that $\|exp(tB)\| \leq M$ for all $t > 0$. Then exp(tB) is exponentially stable if and only if $\{iw \ : \ -\infty < w < +\infty\} \subset R(iw, B)$ (resolvent of B) and* $\sup_{-\infty<w<\infty} \|(iw - B)^{-1}\| < +\infty$.

We divide our proof of the exponential decay of e^{tA}, where A is given by (2.4), (2.5) into the following lemmas.

Lemma 2.1. *The operator A of (2.4), (2.5) generates a* $C_0 - semigroup$ *on H such that*

$$\|e^{tA}\| \leq 1, \qquad for\ all\ t > 0.$$

Proof. For $G = (g, h) \in D(A)$, we have, as in (2.6)

$$\begin{aligned}
&(G, AG)_H + (AG, G)_H \\
&= \int_0^1 (g\bar{h}' + g'\bar{h}'' + h(\bar{g}' - \bar{g}''') + h'\bar{g} + h''\bar{g}' + (g' - g''')\bar{h})dx \\
&= -\frac{2K}{a}|h(0)|^2 \leq 0. \qquad (2.7)
\end{aligned}$$

Then for $\lambda > 0$ and $G = (g, h) \in D(A)$,

$$\begin{aligned}
\|(\lambda I - A)G\|_H^2 &= \lambda^2 \|G\|_H^2 - \lambda[(G, AG)_H + (AG, G)_H] + \|AG\|_H^2 \\
&\geq \lambda^2 \|G\|_H^2. \qquad (2.8)
\end{aligned}$$

Thus

$$\|(\lambda I - A)^{-1}\| \leq \lambda^{-1}, \qquad for\ all\ \lambda > 0.$$

By Hille-Yosida Theorem, A generates a $C_0 - semigroup$ on H such that

$$\|e^{tA}\| \leq 1, \qquad for\ all\ t > 0,$$

if the range of $\lambda I - A$ is all of H for $\lambda > 0$.

To check the range of $\lambda I - A$, for $\lambda > 0$, we consider the equation

$$\lambda G - AG = (f_1, f_2) \in H, \qquad for\ \lambda > 0,\ G = (g, h) \in D(A), \qquad (2.9)$$

which is equivalent to the system

$$\begin{cases} \lambda g - h' = f_1 \in H_a^1(0,1) \\ \lambda h - g' + g''' = f_2 \in L^2(0,1). \end{cases} \qquad (2.10)$$

Let $g_1 = h$, $g_2 = g$, $g_3 = g'$, and $g_4 = g''$. Then with $H = (g_1, g_2, g_3, g_4)^T$ (2.10) becomes

$$H' := \begin{bmatrix} g_1' \\ g_2' \\ g_3' \\ g_4' \end{bmatrix} = \begin{bmatrix} 0 & \lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\lambda & 0 & 1 & 0 \end{bmatrix} H + \begin{bmatrix} -f_1 \\ 0 \\ 0 \\ f_2 \end{bmatrix} := F(\lambda)H + \phi, \tag{2.11}$$

that is,

$$H = H(x,\lambda) = e^{F(\lambda)x}H(0) + \int_0^x e^{F(\lambda)(x-y)}\phi(y)dy, \qquad 0 \le x \le 1. \tag{2.12}$$

By the boundary conditions,

$$H(1) = \begin{bmatrix} a^{-1} & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ K & 0 & 0 & a \end{bmatrix} H(0) := B(a,K)H(0). \tag{2.13}$$

Combining (2.12) and (2.13), we have

$$T(a,K,\lambda)H(0) := (B(a,K) - e^{F(\lambda)})H(0) = \int_0^1 e^{F(\lambda)(1-y)}\phi(y)dy. \tag{2.14}$$

If $T(a,K,\lambda)^{-1}$ exists, then by (2.12) and (2.14),

$$H(x,\lambda) = e^{F(\lambda)x}T(a,K,\lambda)^{-1}\int_0^1 e^{F(\lambda)(1-y)}\phi(y)dy + \int_0^x e^{F(\lambda)(x-y)}\phi(y)dy. \tag{2.15}$$

Suppose that for some $\lambda_0 > 0$, $\det T(a,K,\lambda_0) = 0$. Then there exists an $H_0(0) \neq 0$ such that

$$T(a,K,\lambda_0)H_0(0) = 0, \ \textit{i.e.}\ B(a,K)H_0(0) = e^{F(\lambda_0)}H_0(0),$$

and $H_0 = H_0(x,\lambda_0) := e^{F(\lambda_0)x}H_0(0)$ satisfies

$$H_0' = F(\lambda_0)H_0, \ \textit{i.e.}\ (\lambda_0 I - A)G_0 = 0,$$

where $G_0 = (g_2^0, g_1^0)$ if $H_0 = (g_1^0, g_2^0, g_3^0, g_4^0)^T$, with boundary condition

$$H_0(1) = B(a,K)H_0(0), \qquad \textit{i.e.}\ G_0 \in D(A),$$

which contradicts (2.8) for $\lambda_0 > 0$.

Therefore for $\lambda > 0$, $T(a,K,\lambda)^{-1}$ exists, and the range of $\lambda I - A$ is all of H. ■

Lemma 2.2. *The resolvent operator, $R(iw, A)$, exists for $w \in R \setminus \{0\}$.*

Proof. Suppose that there exists a real $w_0 \neq 0$, such that $R(iw_0, A)$ does not exist. By the same argument as used in Lemma 2.1, we have

$$H_0(0) \neq 0, \qquad H_0 = e^{F(iw_0)x} H_0, \qquad and \qquad AG_0 = iw_0 G_0,$$

where $G_0 = (g_2^0, g_1^0) \in D(A)$ if $H_0 = (g_1^0, g_2^0, g_3^0, g_4^0)^T$.

By (2.7),

$$\frac{2K}{a} |g_1^0(0)|^2 = (G_0, AG_0)_H + (AG_0, G_0)_H = 0.$$

Thus $g_1^0(1) = ag_1^0(0) = 0$, $g_4^0(1) = ag_4^0(0)$, that is

$$H_0(1) = (0, ag_2^0(0), a^{-1}g_3^0(0), ag_4^0(0))^T = e^{F(iw_0)} H_0(0). \tag{2.16}$$

For $F(\lambda)$, there is a nonsingular matrix

$$Q(\mu) = \begin{bmatrix} \lambda & \lambda & \lambda & \lambda \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 \\ \mu_1^2 & \mu_2^2 & \mu_3^2 & \mu_4^2 \\ \mu_1^3 & \mu_2^3 & \mu_3^3 & \mu_4^3 \end{bmatrix} \tag{2.17}$$

such that

$$Q(\mu)^{-1} F(\lambda) Q(\mu) = diag[\mu_1, \mu_2, \mu_3, \mu_4], \tag{2.18}$$

where μ_i, $i = 1, 2, 3, 4$ are the roots of $\mu^4 - \mu^2 + \lambda^2 = 0$. For $\lambda = iw_0 \neq 0$, we have

$$\mu_1 = \sqrt{\frac{1 + \sqrt{1 + 4w_0^2}}{2}} = -\mu_2, \qquad \mu_3 = i\sqrt{\frac{-1 + \sqrt{1 + 4w_0^2}}{2}} = -\mu_4. \tag{2.19}$$

Combining (2.16) and (2.18),

$$Q(\mu)^{-1}(0, ag_2^0(0), a^{-1}g_3^0(0), ag_4^0(0))^T = [e^{\mu_1}, e^{\mu_2}, e^{\mu_3}, e^{\mu_4}] Q(\mu)^{-1} H_0(0). \tag{2.20}$$

Let $Q(\mu)^{-1} = (q_{ij})_{4\times 4}$, then (2.20) becomes

$$(e^{\mu_i} - a) q_{i2} g_2^0(0) + (e^{\mu_i} - a^{-1}) q_{i3} g_3^0(0) + (e^{\mu_i} - a) q_{i4} g_4^0(0) = 0, \; i = 1, 2, 3, 4.$$

But

$$rank \begin{bmatrix} (e^{\mu_1}-a)q_{12} & (e^{\mu_1}-a^{-1})q_{13} & (e^{\mu_1}-a)q_{14} \\ (e^{\mu_2}-a)q_{22} & (e^{\mu_2}-a^{-1})q_{23} & (e^{\mu_2}-a)q_{24} \\ (e^{\mu_3}-a)q_{32} & (e^{\mu_3}-a^{-1})q_{33} & (e^{\mu_3}-a)q_{34} \\ (e^{\mu_4}-a)q_{42} & (e^{\mu_4}-a^{-1})q_{43} & (e^{\mu_4}-a)q_{44} \end{bmatrix} = 3$$

.

In fact

$$q_{j1} = \frac{1}{iw_0}\prod_{k\neq j}\frac{\mu_k}{\mu_k-\mu_j}, \qquad q_{j2} = \frac{\sum_{k,l\neq j}\mu_k\mu_l}{\prod_{k\neq j}(\mu_j-\mu_k)},$$

$$q_{j3} = \frac{\sum_{k\neq j}\mu_k}{\prod_{k\neq j}(\mu_k-\mu_j)}, \qquad q_{j4} = \prod_{k\neq j}\frac{1}{\mu_j-\mu_k}, \qquad j=1,2,3,4.$$

For $e^{\mu_1}\neq a$, and $e^{\mu_2}=e^{-\mu_1}\neq a$, we have, using $\mu_1=-\mu_2$, $\mu_3=-\mu_4$, $|e^{\mu_3}|=1$, and $\mu_1^2-\mu_3^2=\rho:=\sqrt{1+4w_0^2}$,

$$I := det \begin{bmatrix} (e^{\mu_1}-a)q_{12} & (e^{\mu_1}-a^{-1})q_{13} & (e^{\mu_1}-a)q_{14} \\ (e^{\mu_2}-a)q_{22} & (e^{\mu_2}-a^{-1})q_{23} & (e^{\mu_2}-a)q_{24} \\ (e^{\mu_3}-a)q_{32} & (e^{\mu_3}-a^{-1})q_{33} & (e^{\mu_3}-a)q_{34} \end{bmatrix}$$

$$= (e^{\mu_1}-a)(e^{\mu_2}-a)(e^{\mu_3}-a)q_{14}q_{24}q_{34}q_{44}\mu_2\rho\frac{e^{\mu_1}-a}{e^{\mu_1}-a^{-1}}[(\frac{e^{\mu_1}-a^{-1}}{e^{\mu_1}-a})^2+\frac{1}{a^2}]$$

$$\neq 0;$$

for $e^{\mu_1}=a\neq a^{-1}$, i.e. $e^{\mu_2}=a^{-1}\neq a$,

$$I = det \begin{bmatrix} 0 & (e^{\mu_1}-a^{-1})q_{13} & 0 \\ (e^{\mu_2}-a)q_{22} & 0 & (e^{\mu_2}-a)q_{24} \\ (e^{\mu_3}-a)q_{32} & (e^{\mu_3}-a^{-1})q_{33} & (e^{\mu_3}-a)q_{34} \end{bmatrix}$$
$$= -(e^{\mu_1}-a^{-1})(e^{\mu_2}-a)(e^{\mu_3}-a)\rho q_{13}q_{24}q_{34}\neq 0;$$

for $e^{\mu_1}=a^{-1}\neq a$, i.e. $e^{\mu_2}=a\neq a^{-1}$,

$$I = det \begin{bmatrix} (e^{\mu_1}-a)q_{12} & 0 & (e^{\mu_1}-a)q_{14} \\ 0 & (e^{\mu_2}-a^{-1})q_{23} & 0 \\ (e^{\mu_3}-a)q_{32} & (e^{\mu_3}-a^{-1})q_{33} & (e^{\mu_3}-a)q_{34} \end{bmatrix}$$
$$= (e^{\mu_2}-a^{-1})(e^{\mu_1}-a)(e^{\mu_3}-a)\rho q_{23}q_{14}q_{34}\neq 0.$$

So $g_2^0(0)=g_3^0(0)=g_4^0(0)=0$, contradicting $H_0(0)\neq 0$ since we already have $g_1^0(0)=0$. ■

Lemma 2.3. *The resolvent operator, $R(\lambda, A)$, is uniformly bounded near $\lambda = 0$.*

Proof. By (2.13), the only thing we need to check is the existence at $\lambda = 0$ of

$$T(a,K,\lambda)^{-1} = (B(a,K) - e^{F(\lambda)})^{-1}.$$

By (2.9),

$$F(\lambda) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} := C + \lambda B.$$

We can see that

$$C^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad C^{2k} = C^2, \; C^{2k+1} = C, \; k = 1, 2, 3, \ldots$$

and

$$e^{F(\lambda)} = e^C + O(\lambda) = I + a_1 C^2 + a_2 C + O(\lambda),$$

where $a_1 = \sum_{k=1}^{\infty} \frac{1}{(2k)!}$, $a_2 = \sum_{k=1}^{\infty} \frac{1}{(2k-1)!}$, and $a_1 + a_2 = e - 1$. Then

$$\begin{aligned} T(a,K,\lambda)^{-1} &= (B(a,K) - I - a_1 C^2 - a_2 C + O(\lambda))^{-1} \\ &:= (D(a,K) + O(\lambda))^{-1}, \end{aligned}$$

and

$$\begin{aligned} &det D(a,K) \\ =&a^{-1}(a^{-1} - 1)(a-1)[a^2(1+a_1) - a(1 + (1+a_1)^2 - a_2^2) + (1+a_1)] \\ \neq&0 \end{aligned}$$

since $a_1 < a_2$, $2 + a_1 > a_2$, $a \neq 1$, $a \neq 0$, and the determinant of the quadratic equation $det D(a,K) = 0$ in a,

$$[1 + (1+a_1)^2 - a_2^2]^2 - 4(1+a_1)^2 = (a_1^2 - a_2^2)((2+a_1)^2 - a_2^2) < 0.$$

Thus $T(a,K,\lambda)^{-1} = D(a,K)^{-1} + O(\lambda)$, and $R(\lambda, A)$ is uniformly bounded near $\lambda = 0$. ■

Lemma 2.4. *The resolvent operator $R(iw, A)$, is uniformly bounded as $|w| \to \infty$, $w \in R$.*

Proof. Instead of writing the solution $G = (g, h)$ of (2.10) in the form of (2.15), we use a Green's function expression

$$G = (g, h)^T = \int_0^1 U(\lambda, x, y) f(y) dy, \qquad f = (f_1, f_2)^T \in H,$$

where $U(\lambda, x, y) = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}(\lambda, x, y)$ and $\lambda = iw$. Then

$$\begin{cases} g(\lambda, x, y) = \int_0^1 (U_{11} f_1(y) + U_{12} f_2(y)) dy \\ h(\lambda, x, y) = \int_0^1 (U_{21} f_1(y) + U_{22} f_2(y)) dy. \end{cases} \tag{2.21}$$

So that g, h will satisfy (2.10), we require

$$\begin{cases} \lambda U_{11} - U'_{21} = \delta(x - y) \\ \lambda U_{21} - U'_{11} + U'''_{11} = 0 \end{cases} \tag{2.22}$$

and

$$\begin{cases} \lambda U_{12} - U'_{22} = 0 \\ \lambda U_{22} - U'_{12} + U'''_{12} = \delta(x - y) \end{cases} \tag{2.23}$$

To satisfy the boundary conditions, we have

$$\begin{cases} U_{11}(\lambda, 1, y) = aU_{11}(\lambda, 0, y) \\ U'_{11}(\lambda, 1, y) = a^{-1} U'_{11}(\lambda, 0, y) \\ U''_{11}(\lambda, 1, y) = aU''_{11}(\lambda, 0, y) + KU_{21}(\lambda, 0, y) \\ U_{21}(\lambda, 1, y) = a^{-1} U_{21}(\lambda, 0, y) \\ U'_{21}(\lambda, 1, y) = aU'_{21}(\lambda, 0, y) \end{cases} \tag{2.24}$$

and

$$\begin{cases} U_{12}(\lambda, 1, y) = aU_{12}(\lambda, 0, y) \\ U'_{12}(\lambda, 1, y) = a^{-1} U'_{12}(\lambda, 0, y) \\ U''_{12}(\lambda, 1, y) = aU''_{12}(\lambda, 0, y) + KU_{22}(\lambda, 0, y) \\ U_{22}(\lambda, 1, y) = a^{-1} U_{22}(\lambda, 0, y) \\ U'_{22}(\lambda, 1, y) = aU'_{22}(\lambda, 0, y) \end{cases} \tag{2.25}$$

where $' = \dfrac{d}{dx}$, and $\int_R \delta(x-y)f(y)dy = f(x)$, for all $f \in L^2(R)$. From (2.13), we see that $U = (U_{ij})_{2\times 2}$ has the form

$$U_{ij}(\lambda, x, y) = \sum_{k=1}^{4} C_{ij}^{k} e^{\mu_k(x-y)} + H(x-y)\sum_{k=1}^{4} \hat{C}_{ij}^{k} e^{\mu_k(x-y)} \tag{2.26}$$

where H is the Heaviside function

$$H(x-y) = \begin{cases} 1, & x > y \\ 0, & x \leq y. \end{cases}$$

We will choose C_{ij}^k and $\hat{C}_{ij}^k$, $i, j, k = 1, 2, 3, 4$ to satisfy (2.22) $\sim$ (2.25). Substituting (2.26) in (2.22), we have

$$\begin{aligned} &\lambda C_{11}^k - \mu_k C_{21}^k = 0 \\ &\lambda \hat{C}_{11}^k - \mu_k \hat{C}_{21}^k = 0 \\ &\sum_{k=1}^{4} \hat{C}_{21}^k = 1, \qquad k = 1, 2, 3, 4 \end{aligned} \tag{2.27}$$

and

$$\sum_{k=1}^{4} \hat{C}_{11}^k = 0,\ \sum_{k=1}^{4} \mu_k \hat{C}_{11}^k = 0,\ \sum_{k=1}^{4} \mu_k^2 \hat{C}_{11}^k = 0,\ \sum_{k=1}^{4} \mu_k^3 \hat{C}_{11}^k = -\lambda, \tag{2.28}$$

by$\mu_k^4 - \mu_k^2 + \lambda^2 = 0$ and (2.27). Then we can solve for $\hat{C}_{11}^k$ and $\hat{C}_{21}^k$:

$$\hat{C}_{11}^K = \frac{\lambda}{\prod_{j\neq k}(\mu_j - \mu_k)}, \qquad \hat{C}_{21}^k = \frac{\lambda}{\mu_k}\hat{C}_{11}^k,\ for\ k = 1, 2, 3, 4. \tag{2.29}$$

Substituting (2.26) in (2.24) and using (2.27), we have

$$V(\mu)e^{-\mu y}C_{11} = a_{11}(\mu, y) \tag{2.30}$$

where $V(\mu) = (v_{jk})_{4\times 4}(\mu)$, and

$$v_{1k} = a - e^{\mu_k},\ v_{2k} = (1 - ae^{\mu_k})\mu_k,\ v_{3k} = (a - e^{\mu_k})\mu_k^2 + K\frac{\lambda}{\mu_k},$$

$$v_{4k} = \frac{1 - ae^{\mu_k}}{\mu_k}, \ \mu = diag(\mu_1, \mu_2, \mu_3, \mu_4) = [\mu_1, \mu_2, \mu_3, \mu_4] \tag{2.31}$$

$C_{11} = (C_{11}^1, C_{11}^2, C_{11}^3, C_{11}^4)^T$, and

$$a_{11}(\mu, y) = \sum_{k=1}^{4} (\hat{C}_{11}^k, a\mu_k \hat{C}_{11}^k, \mu_k^2 \hat{C}_{11}^k, \frac{a}{\mu_k} \hat{C}_{11}^k) e^{\mu_k(1-y)} \tag{2.32}$$

After solving (2.30), we have

$$e^{-\mu y} C_{11} = \Delta(\mu)^{-1} b_{11}(\mu, y) := \hat{a}_{11}(\mu, y) \tag{2.33}$$

where $\Delta(\mu) = detV(\mu)$, and $b_{11}(\mu, y)$ is a column vector whose components, $b_{11}^k(\mu, y)$, $k = 1, 2, 3, 4$ are the determinants of the matrices obtained from $V(\mu)$ by replacing the k-th column of that matrix by $a_{11}(\mu, y)$. By (2.27), we also have

$$e^{-\mu y} C_{21} = \Delta(\mu)^{-1} b_{21}(\mu, y) := \hat{a}_{21}(\mu, y) \tag{2.34}$$

where $b_{21}^k(\mu, y) = \frac{\lambda}{\mu_k} b_{11}^k(\mu, y)$, $k = 1, 2, 3, 4$.

Similarly, substituting (2.26) in (2.23) and (2.25), we have

$$\begin{gathered} \lambda C_{12}^k - \mu_k C_{22}^k = 0 \\ \lambda \hat{C}_{12}^k - \mu_k \hat{C}_{22}^k = 0 \\ \sum_{k=1}^{4} \hat{C}_{22}^k = 0, \end{gathered} \tag{2.35}$$

and

$$\sum_{k=1}^{4} \hat{C}_{12}^k = 0, \ \sum_{k=1}^{4} \mu_k \hat{C}_{11}^k = 0, \ \sum_{k=1}^{4} \mu_k^2 \hat{C}_{11}^k = 0, \ \sum_{k=1}^{4} \mu_k^3 \hat{C}_{11}^k = -\lambda,$$

by $\mu_k^4 - \mu_k^2 + \lambda^2 = 0$ and (2.35). Then we have

$$\begin{cases} \hat{C}_{12}^1 = \dfrac{\mu_2 + \mu_3 + \mu_4}{(\mu_2 - \mu_1)(\mu_3 - \mu_1)(\mu_4 - \mu_1)} = \dfrac{1}{2\rho} \\ \hat{C}_{12}^2 = \dfrac{\mu_1 + \mu_3 + \mu_4}{(\mu_1 - \mu_2)(\mu_3 - \mu_2)(\mu_4 - \mu_2)} = \dfrac{1}{2\rho} \\ \hat{C}_{12}^3 = \dfrac{\mu_1 + \mu_2 + \mu_4}{(\mu_1 - \mu_3)(\mu_2 - \mu_3)(\mu_4 - \mu_3)} = -\dfrac{1}{2\rho} \\ \hat{C}_{12}^4 = \dfrac{\mu_1 + \mu_2 + \mu_3}{(\mu_1 - \mu_4)(\mu_2 - \mu_4)(\mu_3 - \mu_4)} = -\dfrac{1}{2\rho} \end{cases} \tag{2.36}$$

where $\rho = \sqrt{1+4w^2}$ for $\lambda = iw$. and

$$V(\mu)e^{-\mu y}C_{12} = a_{12}(\mu, y) \tag{2.37}$$

where $C_{12} = (C_{12}^1, C_{12}^2, C_{12}^3, C_{12}^4)^T$, and

$$a_{12}(\mu, y) = \sum_{k=1}^{4}(1, 2\mu_k, \mu_k^2, \frac{a}{\mu_k})^T \hat{C}_{12}^k e^{\mu_k(1-y)}. \tag{2.38}$$

From (2.37), we have

$$e^{-\mu y}C_{12} = \Delta(\mu)^{-1}b_{12}(\mu, y) := \hat{a}_{12}(\mu, y) \tag{2.39}$$

where $b_{12}(\mu, y)$ is obtained in the same way as $b_{11}(\mu, y)$ and, using (2.35), we have

$$e^{-\mu y}C_{22} = \Delta(\mu)^{-1}b_{22}(\mu, y) := \hat{a}_{22}(\mu, y), \tag{2.40}$$

where $b_{22}(\mu, y) = \frac{\lambda}{\mu_k}b_{12}(\mu, y)$.

By (2.33), (2.34), (2.39), and (2.40), we can rewrite (2.26) as

$$\begin{aligned} U_{jk}(\lambda, x, y) =& \epsilon^*(e^{\mu(x-y)}C_{jk} + H(x-y)e^{\mu(x-y)}\hat{C}_{jk}) \\ =& \epsilon^*(e^{\mu x}\hat{a}_{jk}(\mu, y) + H(x-y)e^{\mu(x-y)}\hat{C}_{jk}) \end{aligned} \tag{2.41}$$

where $\epsilon^* = (1, 1, 1, 1)$.

We now estimate $\hat{a}_{jk}(\mu, y)$ as $\rho = \sqrt{1+w^2} \to \infty$. We start with $\hat{a}_{11}(\mu, y)$. Since $\mu_1 = -\mu_2 = \sqrt{\frac{\rho+1}{2}}$, and $\mu_3 = -\mu_4 = i\sqrt{\frac{\rho+1}{2}}$,

$$\Delta(\mu) = detV(\mu) \approx -e^{\mu_1}D(\mu), \qquad as\ \rho \to \infty, \tag{2.42}$$

where by (2.31), letting $d_2 = a\mu_2^2 + K\frac{\lambda}{\mu_2}$, $d_3 = (a - e^{\mu_3})\mu_3^2 + K\frac{\lambda}{\mu_3}$, and $d_4 =$

$(a-e^{\mu_4})\mu_4^2+K\frac{\lambda}{\mu_4}$, we have

$$
\begin{aligned}
&D(\mu)\\
=&det\begin{pmatrix} 1 & a & a-e^{\mu_3} & a-e^{\mu_4} \\ a\mu_1 & \mu_2 & (1-ae^{\mu_3})\mu_3 & (1-ae^{\mu_4})\mu_4 \\ \mu_1^2 & d_2 & d_3 & d_4 \\ \frac{a}{\mu_1} & \frac{1}{\mu-2} & \frac{1-ae^{\mu_3}}{\mu_3} & \frac{1-ae^{\mu_4}}{\mu_4} \end{pmatrix}\\
=&\frac{(1+a^2)\rho^2}{\mu_1\mu_3}[(1+a^2)(e^{\mu_3}+e^{-\mu_3})-4a]-\frac{K\lambda a\rho}{\mu_1^2\mu_3^2}[(1+a^2)(e^{-\mu_3}-e^{\mu_3})\mu_1\\
&+\mu_3((1+a^2)(e^{\mu_3}+e^{-\mu_3})-4a)]\\
=&-i\frac{4(1+a^2)\rho^2}{\sqrt{\rho^2-1}}[(1+a^2)cos\sqrt{\frac{\rho-1}{2}}-2a]\\
&+4Ka\rho[\frac{1+a^2}{\sqrt{\rho-1}}sin\sqrt{\frac{\rho-1}{2}}-\frac{1}{\sqrt{\rho+1}}((1+a^2)cos\sqrt{\frac{\rho-1}{2}}-2a)]\\
\approx&\rho^{1/2}\delta(K,a,\rho),
\end{aligned}
\tag{2.43}
$$

where

$$|\delta(K,a,\rho)|>min\{4Ka|2a-\sqrt{2}(1+a^2)|,4Ka|1-a|^2\}>0,\ \ as\ \rho\to\infty$$

and, by (2.32),

$$a_{11}(\mu,y)\approx(1,a\mu_1,\mu_1^2,\frac{a}{\mu_1})^T\hat{C}_{11}^1e^{\mu_1(1-y)},\qquad as\ \rho\to\infty. \tag{2.44}$$

By (2.31) (2.44) and the definition following (2.33) for $b_{11}(\mu,y)$,

$$b_{11}^1(\mu,y)\approx\hat{C}_{11}^1e^{\mu_1(1-y)}D(\mu), \tag{2.45}$$

$$
\begin{aligned}
&b_{11}^2(\mu,y)\\
\approx&\hat{C}_{11}^1e^{\mu_1(1-y)}det\begin{pmatrix} a & 1 & a-e^{\mu_3} & a-e^{\mu_4} \\ \mu_1 & a\mu_1 & (1-ae^{\mu_3})\mu_3 & (1-ae^{\mu_4})\mu_4 \\ a\mu_1^2+K\frac{\lambda}{\mu_1} & \mu_1^2 & (a-e^{\mu_3})\mu_3^2+K\frac{\lambda}{\mu_3} & (a-e^{\mu_4})\mu_4^2+K\frac{\lambda}{\mu_4} \\ \frac{1}{\mu_1} & \frac{a}{\mu_1} & \frac{1-ae^{\mu_3}}{\mu_3} & \frac{1-ae^{\mu_4}}{\mu_4} \end{pmatrix}\\
=&\hat{C}_{11}^1e^{\mu_1(1-y)}H_2(\mu),
\end{aligned}
\tag{2.46}
$$

where

$$H_2(\mu) = \frac{(1-a^2)\rho^2}{\mu_1\mu_3}[(1+a^2)(e^{\mu_3}+e^{-\mu_3})-4a]$$
$$-\frac{K\lambda a\rho}{\mu_1^2\mu_3^2}[(1+a^2)(e^{-\mu_3}-e^{\mu_3})\mu_1+\mu_3((1+a^2)(e^{\mu_3}+e^{-\mu_3})-4a)],$$

comparing with (2.43), $h_2(\mu) := H_2(\mu)/D(\mu)$ is bounded as $\rho \to \infty$, and

$$\begin{aligned} & b_{11}^3(\mu,y) \\ \approx & \hat{C}_{11}^1 e^{\mu_1(1-y)} det \begin{pmatrix} a & a & 1 & a-e^{\mu_4} \\ \mu_1 & \mu_2 & a\mu_1 & (1-ae^{\mu_4})\mu_4 \\ a\mu_1^2+K\frac{\lambda}{\mu_1} & \mu_2^2+K\frac{\lambda}{\mu_2} & \mu_1^2 & (a-e^{\mu_4})\mu_4^2+K\frac{\lambda}{\mu_4} \\ \frac{1}{\mu_1} & \frac{1}{\mu_2} & \frac{a}{\mu_1} & \frac{1-ae^{\mu_4}}{\mu_4} \end{pmatrix} \\ = & \hat{C}_{11}^1 e^{\mu_1(1-y)} H_3(\mu) \end{aligned} \tag{2.47}$$

where $H_3(\mu) = \frac{2a^2K\lambda\rho}{\mu_1^2\mu_4}(1-ae^{\mu_4})$, and

$$h_3(\mu) := H_3(\mu)/D(\mu) \to 0,$$

as $\rho \to \infty$. Similarly,

$$b_{11}^4(\mu,y) \approx \hat{C}_{11}^1 e^{\mu_1(1-y)} H_4(\mu), \tag{2.48}$$

where

$$H_4(\mu) = \frac{2a^2K\lambda\rho}{\mu_1^2\mu_4}(1-ae^{\mu_3}),\ h_4(\mu) := H_4(\mu)/D(\mu) \to 0,\ as\ \rho \to \infty.$$

By (2.33) and (2.42) $\sim$ (2.48), as $\rho \to \infty$ we have

$$\begin{aligned} \hat{a}_{11}(\mu,y) \approx & -e^{-\mu_1}D^{-1}(\mu)\hat{C}_{11}^1 e^{\mu_1(1-y)}(D(\mu),H_2(\mu),H_3(\mu),H_4(\mu))^T \\ = & -\hat{C}_{11}^1 e^{-\mu_1 y}(1,h_2,h_3,h_4)^T \end{aligned} \tag{2.49}$$

By (2.41), $U_{11}(\lambda,x,y) = \epsilon^*(e^{\mu x}\hat{a}_{11}(\mu,y)+H(x-y)e^{\mu(x-y)}\hat{C}_{11})$. For $x \le y$, $e^{\mu x}e^{-\mu_1 y}$ is uniformly bounded as $\rho \to \infty$, h_2, h_3, and h_4 are bounded as $\rho \to \infty$ and from

(2.29), $|\hat{C}_{11}| \approx \rho^{-1/2}$ as $\rho \to \infty$ so, given any $r > 0$, there is a constant $\hat{M}_r > 0$, independent of ρ such that for $x \le y$, we have

$$|\epsilon^* e^{\mu x} \hat{a}_{11}(\mu, y)| \le \hat{M}_r \rho^{-1/2}, \qquad for\ \rho > r. \tag{2.50}$$

For $x > y$, as $\rho \to \infty$,

$$\begin{aligned} U_{11}(\lambda, x, y) \approx & - \hat{C}^1_{11} e^{-\mu_1 y}(e^{\mu_1 x} + h_2 e^{\mu_2 x} + h_3 e^{\mu_3 x} + h_4 e^{\mu_4 x}) \\ & + e^{\mu_1 (x-y)} \hat{C}^1_{11} + e^{\mu_2 (x-y)} \hat{C}^2_{11} + e^{\mu_3 (x-y)} \hat{C}^3_{11} + e^{\mu_4 (x-y)} \hat{C}^4_{11} \\ = & - \hat{C}^1_{11} e^{-\mu_1 y}(h_2 e^{\mu_2 x} + h_3 e^{\mu_3 x} + h_4 e^{\mu_4 x}) \\ & + e^{\mu_2 (x-y)} \hat{C}^2_{11} + e^{\mu_3 (x-y)} \hat{C}^3_{11} + e^{\mu_4 (x-y)} \hat{C}^4_{11} \end{aligned}$$

is uniformly bounded as $\rho \to \infty$, and we can extend (2.50) to

$$|U_{11}(\lambda, x, y)| \le M_r \rho^{-1/2}, \qquad for\ \rho > r. \tag{2.51}$$

By (2.33) and (2.34),

$$\begin{aligned} \hat{a}_{21}(\mu, y) = & \Delta(\mu)^{-1} b_{21}(\mu, y), \qquad and \\ b^k_{21}(\mu, y) = & \frac{\lambda}{\mu_k} b^k_{11}(\mu, y). \end{aligned}$$

Then

$$\begin{aligned} \hat{a}_{21}(\mu, y) &= (\frac{\lambda}{\mu_1} \hat{a}^1_{11}, \frac{\lambda}{\mu_2} \hat{a}^2_{11}, \frac{\lambda}{\mu_3} \hat{a}^3_{11}, \frac{\lambda}{\mu_4} \hat{a}^4_{11})^T \\ by\ (42) \quad &= -\hat{C}^1_{11} e^{-\mu_1 y}(\frac{\lambda}{\mu_1}, \frac{\lambda}{\mu_2} h_2, \frac{\lambda}{\mu_3} h_3, \frac{\lambda}{\mu_4} h_4)^T \\ &\approx -e^{-\mu_1 y}(\hat{C}^1_{21}, \frac{\lambda}{\mu_2} \hat{C}^1_{11} h_2, \frac{\lambda}{\mu_3} \hat{C}^1_{11} h_3, \frac{\lambda}{\mu_4} \hat{C}^1_{11} h_4)^T. \end{aligned} \tag{2.52}$$

Also by (2.41), $U_{21}(\lambda, x, y) = \epsilon^*(e^{\mu x} \hat{a}_{21}(\mu, y) + H(x - y) e^{\mu(x-y)} \hat{C}_{21})$. For $x \le y$, $e^{\mu x} e^{-\mu_1 y}$ is uniformly bounded as $\rho \to \infty$, h_2, h_3, h_4, $\frac{\lambda}{\mu_1} \hat{C}^1_{11}$, $\frac{\lambda}{\mu_2} \hat{C}^1_{11} h_2$, $\frac{\lambda}{\mu_3} \hat{C}^1_{11} h_3$, $\frac{\lambda}{\mu_4} \hat{C}^1_{11} h_4$, are bounded as $\rho \to \infty$.

Therefore for $x \leq y$, given $r > 0$, there is $M_r > 0$, such that

$$|\epsilon^* e^{\mu x} \hat{a}_{21}(\mu, y)| \leq \hat{M}_r, \qquad for\ \rho > r.$$

For $x > y$, as $\rho \to \infty$,

$$\begin{aligned}
&U_{21}(\lambda, x, y) \\
\approx &- e^{-\mu_1 y}(e^{\mu_1 x}\hat{C}_{21}^1 + e^{\mu_2 x}\frac{\lambda}{\mu_2}\hat{C}_{11}^1 h_2 + e^{\mu_3 x}\frac{\lambda}{\mu_3}\hat{C}_{11}^1 h_3 + e^{\mu_4 x}\frac{\lambda}{\mu_4}\hat{C}_{11}^1 h_4) \\
&+ e^{\mu_1(x-y)}\hat{C}_{21}^1 + e^{\mu_2(x-y)}\hat{C}_{21}^2 + e^{\mu_3(x-y)}\hat{C}_{21}^3 + e^{\mu_4(x-y)}\hat{C}_{21}^4 \\
= &- e^{-\mu_1 y}(e^{\mu_2 x}\frac{\lambda}{\mu_2}\hat{C}_{11}^1 h_2 + e^{\mu_3 x}\frac{\lambda}{\mu_3}\hat{C}_{11}^1 h_3 + e^{\mu_4 x}\frac{\lambda}{\mu_4}\hat{C}_{11}^1 h_4) \\
&+ e^{\mu_2(x-y)}\hat{C}_{21}^2 + e^{\mu_3(x-y)}\hat{C}_{21}^3 + e^{\mu_4(x-y)}\hat{C}_{21}^4
\end{aligned}$$

is uniformly bounded and

$$|U_{21}(\lambda, x, y)| \leq M_r, \qquad for\ \rho > r. \tag{2.53}$$

We continue to estimate U_{12} and U_{22}. By (2.38) $\sim$ (2.40), as $\rho \to \infty$,

$$\begin{aligned}
a_{12}(\mu, y) &\approx \hat{C}_{12}^1 e^{\mu_1(1-y)}(1, a\mu_1, \mu_1^2, \frac{a}{\mu_1})^T, \\
b_{12}(\mu, y) &\approx \hat{C}_{12}^1 e^{\mu_1(1-y)}(D(\mu), H_2(\mu), H_3(\mu), H_4(\mu))^T \\
\hat{a}_{12}(\mu, y) &= \Delta^{-1}(\mu) b_{12}(\mu, y) \\
&\approx -\hat{C}_{12}^1 e^{-\mu_1 y}(1, h_2, h_3, h_4)^T.
\end{aligned} \tag{2.54}$$

By (2.41), $U_{12}(\lambda, x, y) = \epsilon^*(e^{\mu x}\hat{a}_{12}(\mu, y) + H(x-y)e^{\mu(x-y)}\hat{C}_{12})$. From our earlier discussion and the fact that $|\hat{C}_{12}| = \frac{1}{2\rho}$ we have

$$|U_{12}(\lambda, x, y)| \leq M_r \rho^{-1}, \qquad for\ \rho > r. \tag{2.55}$$

By (2.40), (2.41) and (2.54),

$$\begin{aligned}
\hat{a}_{22}(\mu, y) &\approx -\hat{C}_{12}^1 e^{-\mu_1 y}(\frac{\lambda}{\mu_1}, \frac{\lambda}{\mu_2}h_2, \frac{\lambda}{\mu_3}h_3, \frac{\lambda}{\mu_4}h_4)^T, \ as\ \rho \to \infty, \\
U_{22}(\lambda, x, y) &= \epsilon^*(e^{\mu x}\hat{a}_{22}(\mu, y) + H(x-y)e^{\mu(x-y)}\hat{C}_{22}).
\end{aligned}$$

Similarly, we have

$$|U_{22}(\lambda, x, y)| \leq M_r \rho^{-1/2}, \qquad for\ \rho > r. \tag{2.56}$$

In the same way, we also have

$$|U'_{11}(\lambda, x, y)| \leq M_r, \qquad for\ \rho > r. \tag{2.57}$$

and

$$|U'_{12}(\lambda, x, y)| \leq M_r \rho^{-1/2}, \qquad for\ \rho > r. \tag{2.58}$$

Thus from (2.51),(2.53), and (2.55) $\sim$ (2.58), we have

$$\|(g, h)\|_H \leq M_r, \qquad \rho > r.$$

and $R(iw, A)$, $w \in R$, is uniformly bounded as $|w| \to \infty$. ■

We summarize our results in

Theorem 2.2. *The operator A generates a $C_0 -$ semigroup on H, and for $t > 0$, $exp(tA)$ decays exponentially. That is, there exists a constant $\gamma > 0$, such that*

$$\|exp(tA)\|_H \leq e^{-t\gamma}, \qquad for\ all\ t > 0.$$

Proof. Use Lemma 2.1 $\sim$ Lemma 2.4 and Theorem 2.1. ■

§ 2.2. Local Exponential Decay of the Boussinesq Equation.

By Theorem 2.2 and the result in [9] (Theorem 8, pp.10), we have the following local result:

Theorem 2.3. *For each $U_0 \in H$, there exists $\delta > 0$, such that the integral equation*

$$U(t) = T(t)U_0 + \int_0^t T(t-s)F(U(s))ds, \qquad T(t) = e^{At} \tag{2.59}$$

has a unique solution U on $[0,\delta]$, and $U \in C(0,\delta;H)$ Here $U = (u,v)$, $F(U) = (0,(u^2)_x)$, and

$$\begin{cases} u_t = v_x \\ v_t = u_x - u_{xxx} + (u^2)_x, \end{cases}$$

that is, $U_t = AU + F(U)$. ■

Let us first estimate $U(t)$ appearing in Theorem 2.3 in H. We notice that

$$\begin{aligned} &\|F(U(t))\|_H = 2\|uu_x(.,t)\|_{L^2(0,1)} \\ \le&2 \sup_{0<x<1} |u(x,t)|\|u_x(.,t)\|_0 \\ \le&2\alpha\|u(.,t)\|_1\|u_x(.,t)\|_0 \\ \le&2\alpha\|U(t)\|_H^2, \end{aligned} \tag{2.60}$$

where α is a constant. Assume that U_0 is so small that $\|U_0\|_H \le C_0 < C_1$, where $C_0 > 0$ is a constant, and

$$C_0 < \frac{\gamma}{2\alpha}, \qquad C_1 = \sqrt{\frac{\gamma}{2\alpha}C_0}. \tag{2.61}$$

Then $\|U(t)\|_H \le C_1$, on $[0,t_1]$, for some $\delta \ge t_1 > 0$. Either $t_1 = \delta$ or $\|U(t_1)\|_H = C_1$. Suppose that $C_1 = \|U(t_1)\|_H$, then by (2.59)~ (2.61) and Theorem 2.2,

$$\begin{aligned} C_1 = \|U(t_1)\|_H \le& e^{-\gamma t_1}\|U_0\|_H + \int_0^{t_1} e^{-\gamma(t_1-s)}2\alpha\|U(s)\|_H^2 ds \\ \le& e^{-\gamma t_1}(\|U_0\|_H + 2\alpha C_1^2 \int_0^{t_1} e^{\gamma s}ds) \\ =& e^{-\gamma t_1}(C_0 + \frac{2\alpha}{\gamma}C_1^2(e^{\gamma t_1}-1)) \\ =& \frac{2\alpha}{\gamma}C_1^2 = C_0 \end{aligned}$$

which is a contradiction to $C_1 > C_0$. Thus

$$\|U(t)\|_H \le C_1, \qquad for\ 0 < t \le \delta.$$

Then by (2.59), we have

$$\|U(t)\|_H \le e^{-\gamma t}\|U_0\|_H + 2\alpha C_1 \int_0^t e^{-\gamma(t-s)}\|U(s)\|_H ds.$$

Let $v(t) = \int_0^t e^{\gamma s}\|U(s)\|_H ds$. By Gronwall's inequality,

$$v'(t) \le \|U_0\|_H + 2\alpha C_1 v(t) \le \|U_0\|_H e^{2\alpha C_1 t};$$

that is,

$$\|U(t)\|_H \le \|U_0\|_H e^{-(\gamma - 2\alpha C_1)t}, \qquad 0 < t \le \delta,$$

where $\gamma - 2\alpha C_1 > 0$. Since $\|U(\delta)\|_H \le \|U_0\|_H$, the solution $U(t)$ exists globally for all $t > 0$ by extending local solutions interval by interval. Using the same steps as above, we have

$$\|U(t)\|_H \le \|U_0\|_H e^{-(\gamma - 2\alpha C_1)t}, \qquad t > 0.$$

We summarize the above arguments as the following result:

Theorem 2.4. *The solution $U(t)$ described in Theorem 2.3 exists globally and decays exponentially when the initial data U_0 are small in H.*

Chapter 3. Well-posedness and Exponential Decay for Solutions of the Boussinesq Equation with Periodic Boundary Conditions

In this chapter, we prove that the initial value problem (IVP) for a general Boussinesq equation

$$u_{tt} - u_{xx} + u_{xxxx} + (u^2)_{xx} + K_1 u_t + K_2 u = 0, \qquad t > 0,$$
$$u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), \qquad x \in [0, 2\pi] \tag{3.1}$$

for constants $K_1 \geq 0$ and $K_2 \geq 0$, with periodic boundary conditions,

$$\frac{\partial^p u(0,t)}{\partial x^p} = \frac{\partial^p u(2\pi,t)}{\partial x^p}, \qquad p = 0, 1, 2, ..., l+2,$$

$$\frac{\partial^q u_t(0,t)}{\partial x^q} = \frac{\partial^q u_t(2\pi,t)}{\partial x^q}, \qquad q = 0, 1, 2, ..., l,$$

for some integer l, is well-posed and that the solution u decays exponentially for $K_1 > 0$ and $K_2 > 0$.

We use a pair of operators $P(t)$ and $Q(t)$ obtained by Fourier series methods to identify the solution of (3.1) as a solution of the integral equation

$$u(t,x) = P(t)u_0(x) + Q(t)u_1(x) - \int_0^t Q(t-s)(u^2)_{xx}(s,x)ds. \tag{3.2}$$

Our main results in this chapter are as follows:

Theorem 3.1. *Let $k \geq 0$ be an integer. Then for any $u_0 \in H^{k+2}_{2\pi}$ and $u_1 \in H^k_{2\pi}$, there exist a positive $T = T(\|u_0\|_{k+2} + \|u_1\|_k)$ (with $T(r) \to \infty$ as $r \to 0$) and a unique strong solution $u(t)$ of the IVP (3.1) satisfying*

$$u \in C([0,T] : H^{k+2}_{2\pi}) \cap C^1([0,T] : H^k_{2\pi}).$$

for constants $K_1 \geq 0$ and $K_2 \geq 0$ where $H^k_{2\pi}$ is defined in § 1.2. □

Theorem 3.2. *For $K_1 > 0$ and $K_2 > 0$, the solution in Theorem 1 exists globally for small initial data and decays exponentially.* □

In the rest of the chapter, we will indicate how the operators $P(t)$ and $Q(t)$ are produced by Fourier series followed by the proofs of our main results.

§ 3.1. Derivation of the Operators.

First we notice that for $f \in H^k_{2\pi}$, f has a Fourier expansion

$$f(x) = \sum_{l=-\infty}^{\infty} a_l e^{ilx}$$

whose norm in that space can be written as

$$\|f\|_k := \|f\|_{H^k_{2\pi}} = c \sum_{l=-\infty}^{\infty} (1 + l^2)^k |a_l|^2. \tag{3.3}$$

Consider the linear partial differential equation obtained by suppressing the $(u^2)_{xx}$ term in (3.1):

$$u_{tt} - u_{xx} + u_{xxxx} + K_1 u_t + K_2 u = 0, \tag{3.4}$$

which is equivalent to

$$\begin{aligned} u_t &= v \\ v_t &= u_{xx} - u_{xxxx} - K_1 v - K_2 u. \end{aligned} \tag{3.5}$$

Let

$$u(t,x) = \sum_{l=-\infty}^{\infty} a_l(t)e^{ilx}, \qquad \text{and}$$

$$v(t,x) = \sum_{l=-\infty}^{\infty} b_l(t)e^{ilx}. \tag{3.6}$$

Then we have, for $-\infty < l < \infty$,

$$\begin{aligned} a_l'(t) &= b_l(t) \\ b_l'(t) &= -(l^2 + l^4 + K_2)a_l(t) - K_1 b_l(t) \end{aligned} \tag{3.7}$$

which implies

$$a_l'' + K_1 a_l' + (l^2 + l^4 + K_2)a_l = 0.$$

Thus with constants c_1, c_2,

$$\begin{aligned} a_l &= e^{\mu_1^l t}c_1 + e^{\mu_2^l t}c_2, \\ b_l &= \mu_1^l e^{\mu_1^l t}c_1 + \mu_2^l e^{\mu_2^l t}c_2, \end{aligned}$$

and the fundamental matrix of (3.7) is

$$\Psi(t) = \begin{bmatrix} e^{\mu_1^l t} & e^{\mu_2^l t} \\ \mu_1^l e^{\mu_1^l t} & \mu_2^l e^{\mu_2^l t} \end{bmatrix}, \tag{3.8}$$

where

$$\begin{aligned} \mu_1^l &= \frac{-K_1 - r_l}{2}, \qquad \mu_2^l = \frac{-K_1 + r_l}{2}, \\ \text{and} \quad r_l &= \sqrt{K_1^2 - 4K_2 - 4(l^2 + l^4)}. \end{aligned} \tag{3.9}$$

Therefore we can see that the solution of the linear inhomogeneous IVP

$$\begin{aligned} a_l'(t) &= b_l(t) \\ b_l'(t) &= -(l^2 + l^4 + K_2)a_l(t) - K_1 b_l(t) - c_l(t) \end{aligned} \tag{3.10}$$

with initial data

$$a_l(0) = a_l^0, \qquad b_l(0) = b_l^0$$

has the integral form

$$\begin{pmatrix} a_l(t) \\ b_l(t) \end{pmatrix} = \Psi(t)\Psi^{-1}(0)\begin{pmatrix} a_l^0 \\ b_l^0 \end{pmatrix} - \int_0^t \Psi(t)\Psi^{-1}(s)\begin{pmatrix} 0 \\ c_l(s) \end{pmatrix} ds. \tag{3.11}$$

Next we consider the solution of the linear inhomogeneous IVP with periodic boundary conditions:

$$\begin{aligned} &u_{tt} - u_{xx} + u_{xxxx} + (w^2)_{xx} + K_1 u_t + K_2 u = 0, \\ &u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), \end{aligned} \tag{3.12}$$

which can be rewritten as

$$\begin{aligned} &u_t = v \\ &v_t = u_{xx} - u_{xxxx} - K_1 v - K_2 u - (w^2)_{xx}. \end{aligned}$$

Expressing (u, v) in the form of (3.6) and using (3.7) $\sim$ (3.11), we obtain that the solution to the problem (3.12) has the following formal integral expression

$$u(t,x) = P(t)u_0(x) + Q(t)u_1(x) - \int_0^t Q(t-s)(w^2)_{xx}(s,x)ds \tag{3.13}$$

where

$$a_l^0 = \frac{1}{2\pi}\int_0^{2\pi} u_0(x)e^{-ilx}dx, \qquad b_l^0 = \frac{1}{2\pi}\int_0^{2\pi} u_1(x)e^{-ilx}dx,$$

$$P(t)u_0(x) = C_t * u_0(x), \qquad Q(t)u_1(x) = D_t * u_1(x),$$

$$\begin{aligned} C_t(x) &= \sum_{l=-\infty}^{\infty} r_l^{-1}(\mu_2^l e^{\mu_1^l t} - \mu_1^l e^{\mu_2^l t})e^{ilx}, \\ D_t(x) &= \sum_{l=-\infty}^{\infty} r_l^{-1}(e^{\mu_2^l t} - e^{\mu_1^l t})e^{ilx}, \end{aligned} \tag{3.14}$$

and $$f * g(x) = \frac{1}{2\pi} \int_0^{2\pi} f(x-y)g(y)dy.$$

Next we shall get some inequalities for $P(t)$ and $Q(t)$. There are two cases that we need to pay attention to.

Case 1. $K_1 \geq 0$ and $K_2 \geq 0$ satisfying

$$K_1^2 - 4K_2 \neq 4(l^2 + l^4), \qquad \text{for all integers } l.$$

In this case, We notice that

$$|e^{\mu_j^l t}| \leq e^{-\alpha t}, \qquad \text{for all integers } l,\ t > 0,\ j = 1, 2, \tag{3.15}$$

where $3\alpha = Re(K_1 - \sqrt{K_1^2 - 4K_2}) \geq 0$. Combining (3.3), (3.13) $\sim$ (3.15) and using the fact that $r_l^{-1} = O(l^{-2})$, $|l| \to \infty$, it is readily seen that for any integer $j \geq 0$,

$$\begin{aligned} \|P(t)u_0\|_j &\leq c\|u_0\|_j e^{-\alpha t}, & \|Q(t)u_1\|_{j+2} &\leq c\|u_1\|_j e^{-\alpha t}, \\ \|\frac{\partial}{\partial t}P(t)u_0\|_j &\leq c\|u_0\|_{j+2} e^{-\alpha t}, & \|\frac{\partial}{\partial t}Q(t)u_1\|_j &\leq c\|u_1\|_j e^{-\alpha t}. \end{aligned} \tag{3.16}$$

Case 2. There exists some integer l_0 such that

$$K_1^2 - 4K_2 = 4(l_0^2 + l_0^4), \qquad i.e. \qquad r_{l_0} = 0.$$

In this case C_t and D_t in (3.14) become

$$\begin{aligned} C_t(x) &= (1 + \frac{K_1}{2}t)e^{-\frac{K_1}{2}t + ilx} + \sum_{l \neq l_0} r_l^{-1}(\mu_2^l e^{\mu_1^l t} - \mu_1^l e^{\mu_2^l t})e^{ilx}, \\ D_t(x) &= te^{-\frac{K_1}{2}t + ilx} + \sum_{l \neq l_0} r_l^{-1}(e^{\mu_2^l t} - e^{\mu_1^l t})e^{ilx}. \end{aligned}$$

Hence one of the inequalities in (3.16) changes to

$$\|Q(t)u_1\|_{j+2} \leq c \max(t, 1)\|u_1\|_j e^{-\alpha t},$$

and the rest remains as shown.

§ 3.2. Proofs of the Main Results.

We use the contraction principle to prove Theorem 3.1 first followed by Theorem 3.2 which will be proved by a technique we have used in §2.2 of Chapter 2.

Proof of Theorem 3.1. We give the proof only in *Case 1* since *Case 2* can be treated in the same manner. Let $X_T = C([0,T] : H^{k+2}_{2\pi}) \cap C^1([0,T] : H^k_{2\pi})$. For $u_0 \in H^{k+2}_{2\pi}$ and $u_1 \in H^k_{2\pi}$, we denote $u = \Phi(w) = \Phi_{u_0,u_1}(w)$ as the solution of the linear inhomogeneous IVP (3.12) where

$$w \in X_T^a = \{v \in X_T \mid \max_{0\le t\le T}(\|v(t,.)\|_{k+2} + \|v_t(t,.)\|_k) \le a\}.$$

We shall prove that there exist $T > 0$ (depending only on $\|u_0\|_{k+2} + \|u_1\|_k$ in the appropriate manner) and $a = a(\|u_0\|_{k+2} + \|u_1\|_k) > 0$ such that if $w \in X_T^a$ then $u = \Phi(w) \in X_T^a$ and

$$\Phi \ : \ X_T^a \ \to \ X_T^a$$

is a contraction. To achieve this goal, we rely on the integral equation form of the IVP (3.12) (3.13) and (3.16). By (3.13) (3.16) and the inequality

$$\|(w^2)_{xx}(t,.)\|_k \le \|w(t,.)\|^2_{k+2},$$

we have (using Minkowsky's inequality)

$$\|u(t,.)\|_{k+2} \le c(\|P(t)u_0\|_{k+2} + \|Q(t)u_1\|_{k+2} + \int_0^t \|Q(t-s)(w^2)_{xx}\|_{k+2}ds)$$

$$\le c(\|u_0\|_{k+2} + \|u_1\|_k + \int_0^t \|w(s,.)\|^2_{k+2}ds)$$

$$\|u_t(t,.)\|_k \le c(\|\frac{\partial}{\partial t}P(t)u_0\|_k + \|\frac{\partial}{\partial t}Q(t)u_1\|_k + \int_0^t \|\frac{\partial}{\partial t}Q(t-s)(w^2)_{xx}\|_k ds)$$

$$\le c(\|u_0\|_{k+2} + \|u_1\|_k + \int_0^t \|w(s,.)\|^2_{k+2}ds).$$

Therefore

$$\|u\|_{X_T} = \max_{0\le t\le T}(\|u(t,.)\|_{k+2} + \|u_t(t,.)\|_k)$$
$$\le c(\|u_0\|_{k+2} + \|u_1\|_k + T\|w\|^2_{X_T}), \tag{3.17}$$

where we choose $c \ge 1$.

We first take

$$2c(\|u_0\|_{k+2} + \|u_1\|_k) = a \tag{3.18}$$

then we choose T such that

$$4cTa < 1 \tag{3.19}$$

and so $\Phi(w) \in X_T^a$ for $w \in X_T^a$.

Similar arguments show that

$$\|\Phi(w) - \Phi(\tilde{w})\|_{X_T} \le cT(\|w\|_{X_T} + \|\tilde{w}\|_{X_T})\|w - \tilde{w}\|_{X_T} \tag{3.20}$$

and for $T_1 \in (0,T)$,

$$\|\Phi_{u_0,u_1}(w) - \Phi_{\tilde{u}_0,\tilde{u}_1}(\tilde{w})\|_{X_{T_1}} \le c(\|u_0 - \tilde{u}_0\|_{k+2} + \|u_1 - \tilde{u}_1\|_k$$
$$+ T_1(\|w\|_{X_{T_1}} + \|\tilde{w}\|_{X_{T_1}})\|w - \tilde{w}\|_{X_{T_1}}). \tag{3.21}$$

Consequently, by (3.17) $\sim$ (3.20), there exists a unique $u \in X_T^a$ with $\Phi_{u_0,u_1} \equiv u$ so that u satisfies the integral equation (3.2).

Moreover, (3.21) along with (3.18) and (3.19) shows that for $T_1 \in (0,T)$ the map $(\tilde{u}_0, \tilde{u}_1) \to (\tilde{u}, \tilde{u}_t)$ on V (neighborhood of $(\tilde{u}_0, \tilde{u}_1)$ depending on T_1) to $C([0,T_1] : H^{k+2}_{2\pi} \times H^k_{2\pi})$ is Lipschitz. Hence our solution $u \in X_T^a$ of the integral equation (3.2) is a strong solution of the IVP (3.1) (c.f. [9]). In particular, u satisfies the equation in (3.1) at least in the distribution sense.

Next we extend our uniqueness result to the class X_T. Suppose $v \in X_{T_1}$ for some $T_1 \in (0,T)$ is a strong solution of the IVP (3.1). The argument used in (3.17) shows that for some $T_2 \in (0, T_1)$, $v \in X^a_{T_2}$. Hence (3.18) and (3.19) imply that $u \equiv v$ in $[0, 2\pi] \times [0, T_2]$. Repeating the same argument, this result can be extended to the whole interval $[0, T]$. This yields the uniqueness result in X_T. $\Box$

Now we show the exponential decay property of the solution u in (3.1).

Proof of Theorem 3.2. First in (3.15) we see that $\alpha > 0$ for $K_1 > 0$, $K_2 > 0$. By (3.2) (3.16),

$$\|u(t,.)\|_{k+2} \leq c(\|P(t)u_0\|_{k+2} + \|Q(t)u_1\|_{k+2} + \int_0^t \|Q(t-s)(u^2)_{xx}\|_{k+2} ds)$$

$$\leq c(e^{-\alpha t}(\|u_0\|_{k+2} + \|u_1\|_k) + \int_0^t \|u(s,.)\|^2_{k+2} e^{-\alpha(t-s)} ds)$$

$$\|u_t(t,.)\|_k \leq c(\|\frac{\partial}{\partial t}P(t)u_0\|_k + \|\frac{\partial}{\partial t}Q(t)u_1\|_k + \int_0^t \|\frac{\partial}{\partial t}Q(t-s)(u^2)_{xx}\|_{k+2} ds)$$

$$\leq c(e^{-\alpha t}(\|u_0\|_{k+2} + \|u_1\|_k) + \int_0^t \|u(s,.)\|^2_{k+2} e^{-\alpha(t-s)} ds).$$

Using the same arguments as in the proof of Theorem 2.4 in § 2.2, this solution u is global for small $\|u_0\|_{k+2} + \|u_1\|_k$ and

$$\|u(t)\|_{k+2} + \|u_t(t)\|_k \leq Ce^{-\beta t}, \qquad \text{for some } \beta > 0. \ \Box$$

Chapter 4. The Boussinesq Equation on Periodic Domains

Boussinesq-type equations were the first model for nonlinear dispersive wave propagation, derived by Boussinesq in the 1870's (c.f. [8]). A recent result on a generalization of one of the Boussinesq-type equations arising in the modelling of nonlinear strings, namely

$$u_{tt} - u_{xx} + (u_{xx} + f(u))_{xx} = 0, \qquad x \in \mathbb{R},\ t > 0,$$

was obtained by J. L. Bona and R. L. Sachs (c.f. [6]). They showed that the initial-value problem for this equation is always locally well posed and a global solution exists if the initial data lie relatively close to a stable solitary wave. Also there are some other results available in [35].

It is well-known that Korteweg-de Vries-type equations are similar in scope to the Boussinesq equations, but they are first order in the temporal variable instead of second order. See J. L. Bona and R. Smith [5], and B.-Y. Zhang [39] for same results on periodic domains for such equations.

One of our initial considerations is to stabilize the Boussinesq equation on periodic domains by inserting some feedback control terms. This leads us to explore the global existence of a generalization of the Boussinesq equation, which is

$$u_{tt} = u_{xx} - (u^2 + u_{xx})_{xx} - K_1 u_t - K_2(u - [u])$$

on a periodic domain $0 \leq x \leq L$. We rewrite it in the following form:

$$\begin{aligned} &u_t = v_x, \qquad 0 < x < L,\ t > 0, \\ &v_t = u_x - u_{xxx} + (u^2)_x + K_1 v + K_2 \int_0^x (u(y,t) - [u])dy, \end{aligned} \tag{4.1}$$

with initial and boundary conditions

$$\begin{cases} u(x,0) = u_0(x), \\ v(x,0) = v_0(x), \\ \partial_x^j u(x,t) = \partial_x^j u(x+L,t), \ j = 0,1,2,..., \ t \geq 0 \\ \partial_x^j v(x,t) = \partial_x^j v(x+L,t), \ j = 0,1,2,..., \ t \geq 0 \end{cases} \tag{4.2}$$

where K_1 and K_2 are constants. We denote $K_1 v + K_2 \int_0^x (u(y,t) - [u])dy$ by f(u,v).

Since the smoothness may not guarantee the global existence (c.f. [20]), we stipulate at the outset that we are concerned with solutions $v(x,t)$ of (4.1) $\sim$ (4.2) satisfying a bound

$$\sup_{0<t<T} \|v(.,t)\|_0 \leq M, \tag{4.3}$$

for some constant M, where T is finite or infinite as indicated below. Thus we are considering only solutions with bounded velocity.

The plan for this chapter is as follows. We discuss the existence theory for regularized equations in § 4.1 and a priori estimates for solutions of those equations in § 4.2. Section 4.3 contains the convergence results and § 4.4 describes the continuous dependence. Finally there is a remark on the forced equation in § 4.5.

We use the following fact very often in our proofs.

Lemma 4.0. *The imbedding: $H^s(\Omega) \to C^0(\Omega)$, for $s > \frac{1}{2}$, is continuous where Ω is bounded. In particular,*

$$\sup_{0<x\leq L} |u(x)| \leq c\|u\|_1, \qquad \text{for } u \in H^1(0,L),$$

where c is some constant.

Proof. c.f. [21] or [34].■

We will use this lemma without further indication in the following proofs.

§ 4.1. Existence of Solutions of the Regularized Initial- value Problem on the Periodic Domain.

In order to solve problem (4.1)~(4.3), we regularize (4.1) as follows for some small positive constant ϵ :

$$\begin{cases} (1-\epsilon\partial_x^2)u_t = v_x \\ (1-\epsilon\partial_x^2)v_t = u_x - u_{xxx} + (u^2)_x + f(u,v) \end{cases} \tag{4.4}$$

with initial conditions (4.2) and (4.3).

Proposition 4.1. *Let $u_0 \in H_L^{k+1}$ and $v_0 \in H_L^k$ for an integer $k \geq 1$. Then there exists some $t_0 > 0$ such that problem (4.4) with (4.2) has a unique solution in $V_{t_0}^k$ (see § 1.3).*

Before we prove this proposition, we introduce two lemmas.

Lemma 4.1. *If $f \in H_L^k$, and*

$$g(x) = \int_{-\infty}^{\infty} K_\epsilon(x-y)f(y)dy, \qquad h(x) = \int_{-\infty}^{\infty} L_\epsilon(x-y)f(y)dy$$

, where

$$K_\epsilon(z) = -\frac{1}{2\epsilon}e^{-\frac{|z|}{\sqrt{\epsilon}}}sgn(z)$$

and

$$L_\epsilon(z) = \frac{1}{2\sqrt{\epsilon}}e^{-\frac{|z|}{\sqrt{\epsilon}}},$$

then g and h are in H_L^{k+1}. Moreover, we have

$$\|g\|_{H_L^{k+1}} \leq c_1(\epsilon)\|f\|_{H_L^k}$$

and

$$\|h\|_{H_L^{k+1}} \leq c_2(\epsilon)\|f\|_{H_L^k},$$

for some positive constants $c_1(\epsilon)$ and $c_2(\epsilon)$.

Proof. An easy computation shows that for even k, $j = k/2+1$,

$$g^{(2j-1)}(x) = -\sum_{l=1}^{j} \epsilon^{-l} f^{2(j-l)}(x) + \epsilon^{-j} h(x),$$

and for odd k, $j = (k+1)/2$,

$$g^{(2j)}(x) = -\sum_{l=1}^{j} \epsilon^{-l} f^{2(j-l)+1}(x) + \epsilon^{-j} g(x)$$

together with $h^{'}(x) = g(x)$. ■

By Schwarz's inequality and Fubini's Theorem, we have

Lemma 4.2. *If* $v(x,t) = \int_0^t u(x,s)ds$, *and* $u \in C(0,T;H_L^k)$, *then*

$$\|v(.,t)\|_{H_L^k}^2 \le t \int_0^t \|u(.,s)\|_{H_L^k}^2 ds, \text{ for } 0 \le t \le T. ■$$

Proof of Proposition 4.1. By inverting $1-\epsilon\partial_x^2$ and integrating by parts, we change (4.4) to an integral system

$$u(x,t) = u_1(x,t) := u_0(x) + \int_0^t \int_{-\infty}^{\infty} K_\epsilon(x-y)v(y,s)dyds \tag{4.5}$$

$$\begin{aligned} v(x,t) = v_1(x,t) :=& v_0(x) + \epsilon^{-1}\int_0^t u_x(x,s)ds \\ &+ \int_0^t \int_{-\infty}^{\infty} K_\epsilon(x-y)[(1-\epsilon^{-1})u(y,s) + u^2(y,s)]dyds \\ &+ \int_0^t \int_{-\infty}^{\infty} L_\epsilon(x-y) f(u(y,s),v(y,s))dyds. \end{aligned} \tag{4.6}$$

Let $A(u,v) = (u_1,v_1)$ and $\|(u,v)\|_{V_T^k} \le R < \infty$, where T is to be chosen. Assume $\|u_0\|_{H_L^{k+1}} + \|v_0\|_{H_L^k} \le b < \infty$. By (4.5), (4.6), Lemma 4.1 and Lemma 4.2, we have

$$\begin{aligned} \|u_1(.,t)\|_{H_L^{k+1}} \le& \|u_0\|_{H_L^{k+1}} + t^{\frac{1}{2}}\Big(\int_0^t \|\int_{-\infty}^{\infty} K_\epsilon(.-y)v(y,s)dy\|_{H_L^k}^2 ds\Big)^{\frac{1}{2}} \\ \le& \|u_0\|_{H_L^{k+1}} + c_1(\epsilon)t^{\frac{1}{2}}\Big(\int_0^t \|v(.,s)\|_{H_L^k}^2 ds\Big)^{\frac{1}{2}} \end{aligned} \tag{4.7}$$

and

$$\|v_1(.,t)\|_{H_L^k} \le \|v_0\|_{H_L^k} + \epsilon^{-1} t^{\frac{1}{2}} (\int_0^t \|u_x(.,s)\|_{H_L^k}^2 ds)^{\frac{1}{2}}$$
$$+ t^{\frac{1}{2}} (\int_0^t \| \int_{-\infty}^{\infty} K_\epsilon(.-y)[(1-\epsilon^{-1})u(y,s) + u^2(y,s)]dy\|_{H_L^k}^2 ds)^{\frac{1}{2}}$$
$$+ t^{\frac{1}{2}} (\int_0^t \| \int_{-\infty}^{\infty} L_\epsilon(.-y) f(u(y,s), v(y,s))dy\|_{H_L^k}^2 ds)^{\frac{1}{2}}$$
$$\le \|v_0\|_{H_L^K} + \epsilon^{-1} t^{\frac{1}{2}} (\int_0^t \|u(.,s)\|_{H_L^{k+1}}^2 ds)^{\frac{1}{2}}$$
$$+ t^{\frac{1}{2}} c_1(\epsilon) (\int_0^t \|(1-\epsilon^{-1})u(.,s) + u^2(.,s)\|_{H_L^{k-1}}^2 ds)^{\frac{1}{2}}$$
$$+ t^{\frac{1}{2}} c_2(\epsilon) (\int_0^t \|f(u(.,s), v(.,s)\|_{H_L^{k-1}}^2 \qquad (4.8)$$

Then we note that

$$\|(1-\epsilon^{-1})u(.,s) + u^2(.,s)\|_{H_L^{k-1}}$$
$$\le (\epsilon^{-1} - 1)\|u(.,s)\|_{H_L^{k-1}} + \|u^2(.,s)\|_{H_L^{k-1}}$$
$$\le (\epsilon^{-1} - 1)\|u(.,s)\|_{H_L^{k-1}} + \sup_{\substack{0\le j\le k-1 \\ 0\le x\le L}} |u_{(j)}(x,s)| \|u(.,s)\|_{H_L^{k-1}}$$
$$\le (\epsilon^{-1} - 1)\|u(.,s)\|_{H_L^{K-1}} + c_0 \|u(.,s)\|_{H_L^k} \|u(.,s)\|_{H_L^{k-1}} \qquad (4.9)$$

and, by the conditions on $f(u,v)$, we have

$$\|f(u(.,s), v(.,s))\|_{H_L^{k-1}} \le c_3(\|v(.,s)\|_{H_L^{k-1}} + \|u(.,s)\|_{H_L^{k-1}}) \qquad (4.10)$$

where c_0 and c_3 are some constants. Combining (4.7) $\sim$ (4.10), we then have

$$\|u_1(.,t)\|_{H_L^{k+1}} + \|v_1(.,t)\|_{H_L^k}$$
$$\le \|u_0\|_{H_L^{k+1}} + \|v_0\|_{H_L^k} + c_1(\epsilon) \sup_{0\le s\le t} \|v(.,s)\|_{H_L^k}$$
$$+ \epsilon^{-1} t \sup_{0\le s\le t} \|u(.,s)\|_{H_L^{k+1}} + c_1(\epsilon) t (\epsilon^{-1} - 1) \sup_{0\le s\le t} \|u(.,s)\|_{H_L^{k-1}}$$
$$+ c_1(\epsilon) c_0 t \sup_{0\le s\le t} \|u(.,s)\|_{H_L^k} \|u(.,s)\|_{H_L^{k-1}}$$
$$+ c_2(\epsilon) c_3 \sup_{0\le s\le t} (\|v(.,s)\|_{H_L^{k-1}} + \|u(.,s)\|_{H_L^{k-1}})$$
$$\le b + t(c_1(\epsilon) + \epsilon^{-1} + c_1(\epsilon)(\epsilon^{-1} - 1) + c_1(\epsilon) c_0 R + c_2(\epsilon) c_3) R, \qquad (4.11)$$

where we has used the definitions of b, R appeared following (4.6). If $(U,V) \in V_T^k$, and $\|(U,V)\|_{V_T^k} \le R < \infty$, then

$$\begin{aligned} \|A(u,v) - A(U,V)\|_{V_T^k} &= \|(u-U, v-V)\|_{V_T^k} \\ &\le t(c_1(\epsilon) + \epsilon^{-1} + c_1(\epsilon)(\epsilon^{-1}-1) \\ &+ 2c_1(\epsilon)c_0 R + c_2(\epsilon)c_3)\|(u-U, v-V)\|_{V_T^k} \end{aligned} \tag{4.12}$$

as in (4.7)~ (4.10).

Let $t_0(c_1(\epsilon)+\epsilon^{-1}+c_1(\epsilon)(\epsilon^{-1}-1)+2c_1(\epsilon)c_0R+c_2(\epsilon)c_3) \le \theta < 1$, also $b+\theta R \le R$, i.e. $b \le (1-\theta)R$. Then by (4.11) and (4.12) A maps $\{(u,v) \in V_T^k | \|(u,v)\|_{V_{t_0}^k} \le R\}$ to itself and A is a contraction from $V_{t_0}^k$ to $V_{t_0}^k$. Thus the unique local solution is obtained as the unique fixed point of A in $V_{t_0}^k$. ■

Proposition 4.2. *Let $u_0 \in H_L^{k+1}$, $v_0 \in H_L^k$, for an integer $k \ge 1$ and $0 < \epsilon \le 1$ be fixed. If (u,v) is a solution pair as described in Proposition 4.1 such that*

$$\|v(.,t)\|_0 \le M, \qquad \text{for all } 0 < t < T^*,$$

where $(0,T^*)$ is the maximal existence interval of the solution, and M is a constant, then this solution exists globally, i.e., for all $t \in (0,\infty)$.

To prove this, we need a couple of lemmas.

Lemma 4.3. *Let u and v be in H_L^k, for $k \ge 2$ and $r \ge 1$, then*

$$\|\partial^k(uv) - u\partial^k v\|_0 \le c(k,r)(\|u\|_k\|v\|_r + \|u\|_{r+1}\|v\|_k).$$

Proof. See [34]. ■

Lemma 4.4. *Let $u \in H^k_L$. Then $\|u\|_{H^k_L}$ and $(\|u\|_0^2 + \|\partial_x^k u\|_0^2)^{\frac{1}{2}}$ are equivalent.*

Proof. See [21] or [34].

Proof of proposition 4.2. We actually prove that if for any given finite $T > 0$ we have

$$\|v(.,t)\|_0 \leq M, \qquad \text{for } 0 < t \leq T,$$

then

$$\|u(.,t)\|_{k+1} + \|v(.,t)\|_k \leq M_0, \qquad \text{for } 0 < t \leq T.$$

where M_0 is some constant.

We multiply the first and second equations in (4.4) by u and v respectively , apply ∂_x to the first one in (4.4) and multiply it by u_x, then by adding these three and integrating by parts, we have

$$\begin{aligned}
\frac{1}{2}\dot{E}_{2\epsilon}(t) :=& \frac{1}{2}\frac{d}{dt}\int_0^L [u^2 + u_x^2 + v^2 + \epsilon(u_x^2 + u_{xx}^2 + v_x^2)]dx \\
=& \int_0^L (v(u^2)_x + vf(u,v))dx \\
\leq& 2\sup_{0\leq x\leq L}|u(x,t)|\int_0^L |v||u_x|dx + \int_0^L |v||f(u,v)|dx \\
\leq& c_4\|u(.,t)\|_1^2\|v(.,t)\|_0 + \|v(.,t)\|_0\|f(u,v)(.,t)\|_0 \\
\leq& c_4 M\|u(.,t)\|_1^2 + c_3\|v(.,t)\|_0(\|v(.,t)\|_0 + \|u(.,t)\|_0) \\
\leq& c_5(\|u(.,t)\|_1^2 + \|v(.,t)\|_0^2),
\end{aligned}$$

which clearly implies that

$$E_{2\epsilon}(t) \leq E_{2\epsilon}(0)e^{2c_5 T}, \qquad \text{for } 0 < t \leq T. \tag{4.13}$$

where c_4 and c_5 are independent of t.

We then apply ∂_x^k and ∂_x^{k-1} to the first and second equations and multiply by $\partial_x^k u$ and $\partial_x^{k-1} v$ respectively, and then apply ∂_x^{k-1} to the first one in (4.4) and multiply by $\partial_x^{k-1} u$. by adding these three and integrating by parts, we have

$$
\begin{aligned}
&\frac{1}{2}\dot{E}_{(k+1)\epsilon}(t) \\
:=&\frac{1}{2}\frac{d}{dt}\int_0^L [u_{(k-1)}^2 + u_{(k)}^2 + v_{(k-1)}^2 + \epsilon(u_{(k)}^2 + u_{(k+1)}^2 + v_{(k)}^2)]dx \\
=&\int_0^L [v_{(k-1)}(u^2)_{(k)} + v_{(k-1)}(f(u,v))_{(k-1)}]dx \\
=&\|v_{(k-1)}(.,t)\|_0\|(u^2)_{(k)}(.,t)\|_0 + \|v_{(k-1)}(.,t)\|_0\|f(u,v)_{(k-1)}(.,t)\|_0
\end{aligned} \tag{4.14}
$$

From Lemma 4.3, we obtain

$$
\begin{aligned}
&\|(u^2)_{(k)}(.,t)\|_0 \\
\leq&\|(u^2)_{(k)}(.,t) - uu_{(k)}(.,t)\|_0 + \|uu_{(k)}(.,t)\|_0 \\
\leq&c(k)(\|u(.,t)\|_k\|u(.,t)\|_1 + \|u(.,t)\|_2\|u(.,t)\|_k) + \|uu_{(k)}(.,t)\|_0 \\
\leq&c_6\|u(.,t)\|_2\|u(.,t)\|_k \qquad (4.15) \\
\leq&c_7\|u(.,t)\|_k, \text{ for } 0 < t \leq T, \text{ by } (4.13)
\end{aligned}
$$

where c_6 and c_7 are constants and independent of t. Using (4.10) and (4.14), we finally have

$$
\begin{aligned}
&\frac{1}{2}\dot{E}_{(k+1)\epsilon}(t) \\
\leq&c_7\|v_{(k-1)}(.,t)\|_0\|u(.,t)\|_k + c_3\|v_{(k-1)}(.,t)\|_0(\|v_{(k-1)}(.,t)\|_0 + \|u_{(k-1)}(.,t)\|_0) \\
\leq&c_8(\|u(.,t)\|_k^2 + \|v_{(k-1)}(.,t)\|_0^2).
\end{aligned}
$$

Integrating and using Lemma 4.4, we have

$$
\begin{aligned}
&\|u(.,t)\|_{k+1}^2 + \|v(.,t)\|_k^2 \\
\leq&c_9(E_{(k+1)\epsilon}(t) + \|u(.,t)\|_0^2 + \|v(.,t)\|_0^2) \\
\leq&c_9(E_{(k+1)\epsilon}(0) + 2c_8\int_0^t (\|u(.,s)\|_k^2 + \|v_{(k-1)}(.,s)\|_0^2)ds + \|u(.,t)\|_0^2 + \|v(.,t)\|_0^2)
\end{aligned}
$$

$$\leq c_{10} + c_{11} \int_0^t (\|u(.,t)\|_{k+1}^2 + \|v(.,t)\|_k^2) ds$$
$$\leq c_{10} e^{c_{11} T}, \qquad \text{for } 0 \leq t \leq T,$$

where $c_8 \sim c_{11}$ are constants and independent of t. This completes the proof. ■

§ 4.2. A Priori Estimates For Solutions Of The Regularized Initial-value Problem On the Periodic Domain.

Proposition 4.3. *Let $T > 0$ and $u_0,\ v_0\ \in H_L^\infty$ be given. Assume that*

$$\sup_{t>0} \|v(.,t)\|_0 \leq M,$$

for some $M > 0$. Then the solution (u,v) to the regularized initial-value problem (4.4) and (4.2) is bounded in V_T^k, for $k \geq 1$ with a bound dependent only on M, T, $\|(u_0, v_0)\|_{H_L^{k+1} \times H_L^k}$ and $\epsilon^{\frac{1}{2}} \|(u_0, v_0)\|_{H_L^{k+2} \times H_L^{k+1}}$.

Proof. As we see from (4.14), we have

$$\begin{aligned} \frac{1}{2} \dot{E}_{(l+2)\epsilon}(t) =& \frac{1}{2} \frac{d}{dt} \int_0^L [u_{(l)}^2 + u_{(l+1)}^2 + v_{(l)}^2 + \epsilon(u_{(l+1)}^2 + u_{(l+2)}^2 + v_{(l+1)}^2)] dx \\ =& \int_0^L [v_{(l)}(u^2)_{(l+1)} + v_{(l)}(f(u,v))_{(l)}] dx, \qquad \text{for } 0 \leq l \leq k. \end{aligned} \tag{4.16}$$

From (4.13), we have already seen that

$$\sup_{0<t\leq T} E_{2\epsilon}(t) \leq E_{2\epsilon}(0) e^{2c_5 T}$$

where c_5 depends on M, which means

$$\sup_{0<t\leq T} \|u(.,t)\|_1 + \|v(.,t)\|_0 \leq c_{12} \tag{4.17}$$

where c_{12} depends only on M, T, $\|(u_0, v_0)\|_{H_L^1 \times H_L^0}$ and $\epsilon^{\frac{1}{2}} \|(u_0, v_0)\|_{H_L^2 \times H_L^1}$.

By (4.16) (4.17), we have

$$
\begin{aligned}
&\frac{1}{2}\dot{E}_{3\epsilon}(t)\\
=&\int_0^L [v_x(u^2)_{xx} + v_x(f(u,v))_x]dx\\
=&\int_0^L [2v_x(u_x^2 + uu_{xx}) + v_x(f(u,v))_x]dx\\
\leq&2\sup_{0\leq x\leq L}|u_x|\int_0^L |v_x||u_x|dx\\
&+2\sup_{0\leq x\leq L}|u|\int_0^L |v_x||u_{xx}|dx + \|v_x(.,t)\|_0\|f(u,v)_x(.,t)\|_0\\
\leq&c_4\|u(.,t)\|_2\|v_x(.,t)\|_0\|u_x(.,t)\|_0 + c_4\|u(.,t)\|_1\|v_x(.,t)\|_0\|u_{xx}(.,t)\|_0\\
&\quad + c_3\|v_x(.,t)\|_0(\|v_x(.,t)\|_0 + \|u_x(.,t)\|_0)\\
\leq&c_4c_{12}\|u(.,t)\|_2\|v_x(.,t)\|_0 + c_4c_{12}\|v_x(.,t)\|_0\|u_{xx}(.,t)\|_0\\
&\quad + c_3\|v_x(.,t)\|_0(\|v_x(.,t)\|_0 + \|u_x(.,t)\|_0)\\
\leq&c_{13}(\|u(.,t)\|_2^2 + \|v_x(.,t)\|_0^2)
\end{aligned}
\tag{4.18}
$$

where c_{13} depends only on M, T, $\|(u_0,v_0)\|_{H_L^1\times H_L^0}$ and $\epsilon^{\frac{1}{2}}\|(u_0,v_0)\|_{H_L^2\times H_L^1}$. Integrating by parts and using Lemma 4.4 and (4.17), one obtains

$$
\begin{aligned}
&\|u(.,t)\|_2^2 + \|v_x(.,t)\|_0^2\\
\leq&c_{14}(\|u(.,t)\|_0^2 + \|u_{xx}(.,t)\|_0^2 + \|v_x(.,t)\|_0^2)\\
\leq&c_{14}(c_{12}^2 + E_{3\epsilon}(t)),\\
\leq&c_{14}(c_{12}^2 + E_{3\epsilon}(0) + 2c_{13}\int_0^t(\|u(.,s)\|_2^2 + \|v_x(.,s)\|_0^2)ds)\\
\leq&c_{14}(c_{12}^2 + E_{3\epsilon}(0))e^{2c_{14}c_{13}T},
\end{aligned}
$$

where c_{14} is constant, that is:

$$
\sup_{0\leq t\leq T}\|u(.,t)\|_2^2 + \|v_x(.,t)\|_0^2 \leq c_{15}^2, \tag{4.19}
$$

where c_{15} depends only on M, T, $\|(u_0, v_0)\|_{H_L^2 \times H_L^1}$ and $\epsilon^{\frac{1}{2}}\|(u_0, v_0)\|_{H_L^3 \times H_L^2}$.

Returning to (4.16) and taking $l = k$, we have

$$
\begin{aligned}
&\frac{1}{2}\dot{E}_{(k+2)\epsilon}(t) \\
&= \int_0^L [v_{(k)}(u^2)_{(k+1)} + v_{(k)}(f(u,v))_{(k)}]dx, \\
&\leq \|v_{(k)}(.,t)\|_0(\|(u^2)_{(k+1)} - uu_{(k+1)}\|_0 + \|uu_{(k+1)}\|_0) \\
&\quad + \|v_{(k)}\|_0\|(f(u,v))_{(k)}\|_0 \\
&\quad (\textit{by Lemma } 4.3) \\
&\leq c(k)\|v_{(k)}(.,t)\|_0\|u(.,t)\|_{k+1}\|u(.,t)\|_2 \\
&\quad + c_3\|v_{(k)}(.,t)\|_0(\|v_{(k)}(.,t)\|_0 + \|u_{(k)}(.,t)\|_0) \\
&\quad (\textit{by } (4.19)) \\
&\leq c(k)c_{15}\|v_{(k)}(.,t)\|_0\|u(.,t)\|_{k+1} \\
&\quad + c_3\|v_{(k)}(.,t)\|_0(\|v_{(k)}(.,t)\|_0 + \|u_{(k)}(.,t)\|_0) \\
&\leq c_{16}(\|u(.,t)\|_{k+1}^2 + \|v_{(k)}(.,t)\|_0^2) \qquad (4.20)
\end{aligned}
$$

Using (4.20) and an argument like (4.18) $\sim$ (4.19), one can also obtain

$$\sup_{0 \leq t \leq T} \|u(.,t)\|_{k+1}^2 + \|v_{(k)}(.,t)\|_0^2 \leq c_{17}^2 \qquad (4.21)$$

where c_{17} depends only on M, T, $\|(u_0, v_0)\|_{H_L^{k+1} \times H_L^k}$ and $\epsilon^{\frac{1}{2}}\|(u_0, v_0)\|_{H_L^{k+2} \times H_L^{k+1}}$.

By Lemma 4.4, we have

$$
\begin{aligned}
&\sup_{0 < t \leq T} \|u(.,t)\|_{k+1}^2 + \|v(.,t)\|_k^2 \\
&\leq \sup_{0 < t \leq T} c_{18}(\|u(.,t)\|_{k+1}^2 + \|v_{(k)}(.,t)\|_0^2 + \|v(.,t)\|_0^2) \\
&\quad (\text{by (4.21) and the assumption on v}) \\
&\leq c_{18}(c_{17}^2 + M^2) = c_{19}
\end{aligned}
$$

with constants c_{18} and c_{19} depending only on

$$M, T, \|(u_0, v_0)\|_{H_L^{k+1}\times H_L^k} \text{ and } \epsilon^{\frac{1}{2}}\|(u_0, v_0)\|_{H_L^{k+2}\times H_L^{k+1}}. \blacksquare$$

§ 4.3. Convergence of the Approximation.

Lemma 4.5. *Let $g \in H_L^k$ with $k \geq 3$, then there exists $g_\epsilon \in H_L^\infty$, such that*

$$\|g_\epsilon\|_{k+j} = O(\epsilon^{-\frac{j}{6}}),\ i.e.\ \epsilon^{\frac{j}{6}}\|g_\epsilon\|_{k+j} \leq c\|g\|_k, \qquad \text{for j=1, 2,...}$$

$$\|g - g_\epsilon\|_{k-j} = o(\epsilon^{\frac{j}{6}}), \qquad \text{for j=1, 2,...}$$

$$\|g - g_\epsilon\|_k = o(1),$$

as $\epsilon \downarrow 0$. Furthermore, the first bound holds uniformly on bounded subsets of H_L^k, and the last two bounds hold uniformly on compact subsets of H_L^k. The second bound holds uniformly on bounded subsets of H_L^k if $o(\epsilon^{\frac{j}{6}})$ is replaced by $O(\epsilon^{\frac{j}{6}})$.

Proof. See [5]. ■

Corollary 4.1. *Let $u_\epsilon(x,t) = u(x,t,\epsilon)$ and $v_\epsilon(x,t) = v(x,t,\epsilon)$ be a pair of solutions to the problem (4.4) with initial conditions, related to u_0, v_0 as g_ϵ is related to g in Lemma 4.5,*

$$\begin{cases} u(x,0) &= u_{0\epsilon}(x) \\ v(x,0) &= v_{0\epsilon}(x) \end{cases} \tag{4.22}$$

where $u_0 \in H_L^{k+1}$ and $v_0 \in H_L^k$. By Lemma 4.5, one may suppose that

$$\begin{cases} \|u_{0\epsilon}\|_{k+1+j} = O(\epsilon^{-\frac{j}{6}}),\ i.e.\ \epsilon^{\frac{j}{6}}\|u_{0\epsilon}\|_{k+1+j} \leq c\|u_0\|_{k+1}, & \text{for j=1, 2,...} \\ \|u_0 - u_{0\epsilon}\|_{k+1-j} = o(\epsilon^{\frac{j}{6}}), & \text{for j=1, 2,...} \\ \|u_0 - u_{0\epsilon}\|_{k+1} = o(1), \end{cases} \tag{4.23}$$

and

$$\begin{cases} \|v_{0\epsilon}\|_{k+j} = O(\epsilon^{-\frac{j}{6}}),\ i.e.\ \epsilon^{\frac{j}{6}}\|v_{0\epsilon}\|_{k+j} \leq c\|u_0\|_k, & \text{for j=1, 2,...} \\ \|v_0 - v_{0\epsilon}\|_{k-j} = o(\epsilon^{\frac{j}{6}}), & \text{for j=1, 2,...} \\ \|u_0 - u_{0\epsilon}\|_k = o(1), \end{cases} \tag{4.24}$$

as $\epsilon \downarrow 0$. Suppose that for any given $T > 0$,

$$\sup_{0<t\leq T} \|v_\epsilon(.,t)\|_0 \leq M, \tag{4.25}$$

with a constant M independent of small ϵ. Then (u_ϵ, v_ϵ) is bounded in V_T^k with a bound independent of small ϵ for each finite $T > 0$. Further, $\epsilon^{\frac{m}{6}}(u_\epsilon, v_\epsilon)$ is bounded in V_T^{k+m} with a bound independent of small ϵ for each finite $T > 0$ and $m \geq 1$.

Proof. This follows from Lemma 4.5 and Proposition 4.3.

By Proposition 4.3, $\sup_{0<t\leq T} \|(u_\epsilon, v_\epsilon)\|_{H_L^{k+1}\times H_L^k}$ has a upper bound depending only on M, T, $\|(u_{0\epsilon}, v_{0\epsilon})\|_{H_L^{k+1}\times H_L^k}$ and $\epsilon^{\frac{1}{2}}\|(u_{0\epsilon}, v_{0\epsilon})\|_{H_L^{k+2}\times H_L^{k+1}}$. From Lemma 4.5,

$$\|(u_{0\epsilon}, v_{0\epsilon})\|_{H_L^{k+1}\times H_L^k} \leq \|(u_0, v_0)\|_{H_L^{k+1}\times H_L^k}$$

and

$$\begin{aligned}\epsilon^{\frac{1}{2}}\|(u_{0\epsilon}, v_{0\epsilon})\|_{H_L^{k+2}\times H_L^{k+1}} &\leq c\epsilon^{\frac{1}{2}}\epsilon^{-\frac{1}{6}}\|(u_0, v_0)\|_{H_L^{k+1}\times H_L^k} \\ &\leq c\|(u_0, v_0)\|_{H_L^{k+1}\times H_L^k}.\end{aligned}$$

Therefore $\sup_{0<t\leq T} \|(u_\epsilon, v_\epsilon)\|_{H_L^{k+1}\times H_L^k}$ has a upper bound depending only on M, T and $\|(u_0, v_0)\|_{H_L^{k+1}\times H_L^k}$.

Also by Proposition 4.3 and Lemma 4.5, $\epsilon^{\frac{m}{6}}(u_\epsilon, v_\epsilon)_{V_T^{k+m}}$ has a upper bound depending only on

$$M, T, \epsilon^{\frac{m}{6}}\|(u_{0\epsilon}, v_{0\epsilon})\|_{H_L^{k+m+1}\times H_L^{k+m}}$$

$$\text{and} \qquad \epsilon^{\frac{1}{2}+\frac{m}{6}}\|(u_{0\epsilon}, v_{0\epsilon})\|_{H_L^{k+m+2}\times H_L^{k+m+1}}.$$

But by (4.23) and (4.24),

$$\epsilon^{\frac{m}{6}}\|(u_{0\epsilon}, v_{0\epsilon})\|_{H_L^{k+m+1}\times H_L^{k+m}} \leq c\|(u_0, v_0)\|_{H_L^{k+1}\times H_L^k}$$

and

$$\epsilon^{\frac{1}{2}+\frac{m}{6}}\|(u_{0\epsilon},v_{0\epsilon})\|_{H_L^{k+m+2}\times H_L^{k+m+1}} = \epsilon^{\frac{1}{2}-\frac{1}{6}}\epsilon^{\frac{m+1}{6}}\|(u_{0\epsilon},v_{0\epsilon})\|_{H_L^{k+m+2}\times H_L^{k+m+1}}$$
$$\leq c\|(u_0,v_0)\|_{H_L^{k+1}\times H_L^{k}}.$$

So $\epsilon^{\frac{m}{6}}(u_\epsilon,v_\epsilon)_{V_T^{k+m}}$ has a upper bound independent of ϵ. ■

Corollary 4.2. *$(\partial_t u_\epsilon, \partial_t v_\epsilon)$ is bounded in V_T^{k-3}, $k \geq 3$ and $\epsilon^{\frac{m}{6}}\partial_x^{k+m-3}(\partial_t u_\epsilon, \partial_t v_\epsilon)$ is bounded in V_T^0 with a bound independent of small $\epsilon > 0$, for all finite $T > 0$ and $m = 1, 2, ..., 5$.*

Proof. Apply ∂_x^{j+1}, ∂_x^j to the first and the second equations in (4.22), respectively. Inverting $1-\epsilon\partial_x^2$, we have

$$\partial_t u_\epsilon = \int_{-\infty}^{\infty} L_\epsilon(x-y)\partial_y v_\epsilon(y,t)dy,$$
$$\partial_x^{j+1}\partial_t u_\epsilon = \int_{-\infty}^{\infty} L_\epsilon(x-y)\partial_y^{j+2} v_\epsilon(y,t)dy$$

and

$$\partial_x^j\partial_t v_\epsilon = \int_{-\infty}^{\infty} L_\epsilon(x-y)[\partial_y^{j+1}u_\epsilon - \partial_y^{j+3}u_\epsilon + \partial_y^{j+1}u_\epsilon^2 + \partial_y^j f(u_\epsilon,v_\epsilon)](y,t)dy.$$

Since $\int_{-\infty}^{\infty} L_\epsilon(x-y)dy = 1$, we then have

$$|\partial_t u_\epsilon| \leq \sup_{0\leq y\leq L} |\partial_y v_\epsilon(y,t)| \tag{4.26}$$
$$|\partial_x^{j+1}\partial_t u_\epsilon| \leq \sup_{0\leq y\leq L} |\partial_y^{j+2} v_\epsilon(y,t)| \tag{4.27}$$

and

$$|\partial_x^j\partial_t v_\epsilon| \leq \sup_{0\leq x\leq L}(|\partial_y^{j+1}u_\epsilon| + |\partial_y^{j+3}u_\epsilon| + |\partial_y^{j+1}u_\epsilon^2| + |\partial_y^j f(u_\epsilon,v_\epsilon)|)(y,t) \tag{4.28}$$
$$|\partial_t v_\epsilon| \leq \sup_{0\leq x\leq L}(|\partial_y u_\epsilon| + |\partial_y^3 u_\epsilon| + |\partial_y u_\epsilon^2| + |f(u_\epsilon,v_\epsilon)|)(y,t) \tag{4.29}$$

Therefore by Lemma 4.4, we have

$$\begin{aligned}
\|\partial_t u_\epsilon\|_{j+1}^2 &\leq c(\|\partial_x^{j+1}\partial_t u_\epsilon\|_0^2 + \|\partial_t u_\epsilon\|_0^2) \\
&\quad (by\ (4.26)\ and\ (4.27)) \\
&\leq cL(\sup_{0\leq y\leq L} |\partial_y^{j+2} v_\epsilon(y,t)|^2 + \sup_{0\leq y\leq L} |\partial_y v_\epsilon(y,t)|^2) \\
&\leq c_{20}\|v_\epsilon(.,t)\|_{j+3}^2, \qquad (4.30)
\end{aligned}$$

where c and c_{20} are constant. Now, by the same argument as used to obtain (4.15),

$$\|\partial_y^{j+2} u_\epsilon^2\|_0 \leq c(j)\|u_\epsilon\|_{j+2}\|u_\epsilon\|_2$$

and also, by Lemma 4.3,

$$\begin{aligned}
\|\partial_t v_\epsilon\|_j^2 &\leq c(\|\partial_x^j \partial_t v_\epsilon\|_0^2 + \|\partial_t v_\epsilon\|_0^2) \\
&\quad (by\ (4.28)\ and\ (4.29)) \\
&\leq 3cL[\sup_{0\leq x\leq L} (|\partial_y^{j+1} u_\epsilon|^2 + |\partial_y^{j+3} u_\epsilon|^2 + |\partial_y^{j+1} u_\epsilon^2|^2 + |\partial_y^j f(u_\epsilon, v_\epsilon)|^2)(y,t) \\
&\quad + \sup_{0\leq x\leq L} (|\partial_y u_\epsilon| + |\partial_y^3 u_\epsilon| + |\partial_y u_\epsilon^2| + |f(u_\epsilon, v_\epsilon)|)(y,t)] \\
&\leq c_{21}(\|u_\epsilon\|_{j+4}^2 + \|v_\epsilon\|_{j+1}^2 + \|\partial_x^{j+2} u_\epsilon^2\|_0^2) \\
&\leq c_{21}(\|u_\epsilon\|_{j+4}^2 + \|v_\epsilon\|_{j+1}^2 + c^2(j)\|u_\epsilon\|_{j+2}^4), \qquad (4.31)
\end{aligned}$$

Combining (4.30) and (4.31) and using Corollary 4.1, we finally have

$$\|(\partial_t u_\epsilon, \partial_t v_\epsilon)\|_{V_T^{k-3}} \leq c_{22},$$

and

$$\epsilon^{\frac{m}{6}}\|\partial_x^{k+m-3}(\partial_t u_\epsilon, \partial_t v_\epsilon)\|_{V_T^0} \leq c_{23},$$

where c_{22} and c_{23} are constants independent of ϵ. ■

Proposition 4.4. *Let (u_ϵ, v_ϵ) be the solution of the regularized problems (4.4), (4.2), (4.22) and (4.25). Then $\{(u_\epsilon, v_\epsilon)\}$ is Cauchy in V_T^k as $\epsilon \downarrow 0$, for $k \geq 3$.*

Proof. Let $u = u_\epsilon,\ v = v_\epsilon,\ \tilde{u} = u_\delta,\ \tilde{v} = v_\delta,\ \phi = u - \tilde{u}$ and $\psi = v - \tilde{v}$, for $\delta \leq \epsilon$. Then

$$\begin{cases} \phi_t - \delta\phi_{txx} & = \psi_x + (\epsilon - \delta)u_{txx} \\ \psi_t - \delta\psi_{txx} & = \phi_x - \phi_{xxx} + 2u\phi_x + 2\phi\tilde{u}_x + f(\phi, \psi) + (\epsilon - \delta)v_{txx} \end{cases} \tag{4.32}$$

with initial conditions

$$\begin{cases} \phi(x, 0) & = u_{0\epsilon}(x) - u_{0\delta}(x) := \phi_0(x) \\ \psi(x, 0) & = v_{0\epsilon}(x) - v_{0\delta}(x) := \psi_0(x) \end{cases} \tag{4.33}$$

As in (4.14), we have the similar result

$$\begin{aligned} &\frac{1}{2}\dot{E}_{(l+2)\delta}(\phi, \psi)(t) \\ =&\frac{1}{2}\frac{d}{dt}\int_0^L [\phi_{(l)}^2 + \phi_{(l+1)}^2 + \psi_{(l)}^2 + \delta(\phi_{(l+1)}^2 + \phi_{(l+2)}^2 + \psi_{(l+1)}^2)]dx \\ =&\int_0^L [(\epsilon - \delta)\phi_{(l)}\partial_x^{l+2}u_t + (\epsilon - \delta)\phi_{(l+1)}\partial_x^{l+3}u_t + (\epsilon - \delta)\psi_{(l)}\partial_x^{l+2}v_t \\ &+ 2\psi_{(l)}\partial_x^l(u\phi_x + \phi\tilde{u}_x) + \psi_{(l)}\partial_x^l f(\phi, \psi)]dx. \end{aligned} \tag{4.34}$$

First we note by Lemma 4.0, Corollary 4.1 and (4.10) that

$$\begin{aligned} &\frac{1}{2}\dot{E}_{2\delta}(\phi, \psi)(t) \\ =&\int_0^L [(\epsilon - \delta)\phi\partial_x^2 u_t + (\epsilon - \delta)\phi_x\partial_x^3 u_t + (\epsilon - \delta)\psi\partial_x^2 v_t \\ &+ 2\psi(u\phi_x + \phi\tilde{u}_x) + \psi f(\phi, \psi)]dx \\ \leq&(\epsilon - \delta)\|\phi(., t)\|_0\|\partial_x^2 u_t(., t)\|_0 + (\epsilon - \delta)\|\phi_x(., t)\|_0\|\partial_x^3 u_t(., t)\|_0 \\ &+ (\epsilon - \delta)\|\psi(., t)\|_0\|\partial_x^2 v_t(., t)\|_0 2 \sup_{0\leq x\leq L}|u(x, t)|\|\psi(., t)\|_0\|\phi_x(., t)\|_0 \\ &+ \sup_{0\leq x\leq L}|\tilde{u}_x(x, t)|\|\phi(., t)\|_0\|\psi(., t)\|_0\|f(\phi, \psi)(., t)\|_0 \end{aligned}$$

$$\leq c_{24}[(\epsilon-\delta)^2\|\partial_x^2 u_t(.,t)\|_0^2+(\epsilon-\delta)^2\|\partial_x^3 u_t(.,t)\|_0^2+(\epsilon-\delta)^2\|\partial_x^2 v_t(.,t)\|_0^2$$

$$+\|\phi(.,t)\|_1^2+\|\psi(.,t)\|_0^2]$$

(by corollary 4.2)

$$\leq c_{24.5}\epsilon^{\frac{4}{3}}+c_{24.5}(\|\phi(.,t)\|_1^2+\|\psi(.,t)\|_0^2)$$

Integrating by parts,

$$\begin{aligned}\|\phi(.,t)\|_1^2+\|\psi(.,t)\|_0^2 \leq E_{2\delta}(\phi,\psi)(t) &\leq E_{2\delta}(\phi,\psi)(0)+2c_{24.5}\epsilon^{\frac{4}{3}}T\\ &+2c_{24.5}\int_0^t(\|\phi(.,s)\|_1^2+\|\psi(.,s)\|_0^2)ds\\ &\leq (E_{2\delta}(\phi,\psi)(0)+2c_{24.5}\epsilon^{\frac{4}{3}}T)e^{2c_{24.5}T} \qquad (4.35)\end{aligned}$$

But, by the definition in (4.34),

$$\begin{aligned}&\frac{1}{2}E_{2\delta}(\phi,\psi)(0)\\ =&\int_0^L[\phi_0^2+\phi_{0x}^2+\psi_0^2+\delta(\phi_{0x}^2+\phi_{0xx}^2+\psi_{0x}^2)]dx\\ \leq&\|\phi_0\|_1^2+\|\psi_0\|_0^2+\delta(\|\phi_0\|_2^2+\|\psi_0\|_1^2)\\ =&\|u_{0\epsilon}-u_{0\delta}\|_1^2+\|v_{0\epsilon}-v_{0\delta}\|_0^2+\delta(\|u_{0\epsilon}-u_{0\delta}\|_2^2+\|v_{0\epsilon}-v_{0\delta}\|_1^2)\\ \leq&(\|u_{0\epsilon}-u_0\|_1+\|u_0-u_{0\delta}\|_1)^2+(\|v_{0\epsilon}-v_0\|_0+\|v_0-v_{0\delta}\|_0)^2\\ &+\delta[(\|u_{0\epsilon}-u_0\|_2+\|u_0-u_{0\delta}\|_2)^2+(\|v_{0\epsilon}-v_0\|_1+\|v_0-v_{0\delta}\|_1)^2]\\ &(by\ (4.23)\ and\ (4.24))\\ \leq&c_{25}\epsilon^{\frac{2}{3}} \qquad (4.36)\end{aligned}$$

Substituting this into (4.35),

$$\|\phi(.,t)\|_1^2+\|\psi(.,t)\|_0^2\leq c_{26}\epsilon^{\frac{2}{3}}, \qquad (4.37)$$

where $c_{24} \sim c_{26}$ are constants independent of ϵ. Similarly, we have for $k \geq 3$,

$$
\begin{aligned}
&\frac{1}{2}\dot{E}_{(k+2)\delta}(\phi, \psi)(t) \\
&(by\ (4.34)) \\
=&\int_0^L [(\epsilon-\delta)\phi_{(k)}\partial_x^{k+2}u_t + (\epsilon-\delta)\phi_{(k+1)}\partial_x^{k+3}u_t \\
&+ (\epsilon-\delta)\psi_{(k)}\partial_x^{k+2}v_t + 2\psi_{(k)}\partial_x^k(u\phi_x + \phi\tilde{u}_x) + \psi_{(k)}\partial_x^k f(\phi,\psi)]dx \\
\leq&(\epsilon-\delta)\|\phi_{(k)}(.,t)\|_0\|\partial_x^{k+2}u_t(.,t)\|_0 + (\epsilon-\delta)\|\phi_{(k+1)}(.,t)\|_0\|\partial_x^{k+3}u_t(.,t)\|_0 \\
&+ (\epsilon-\delta)\|\psi_{(k)}(.,t)\|_0\|\partial_x^{k+2}v_t(.,t)\|_0 \\
&+ 2\|\psi_{(k)}(.,t)\|_0\|\partial_x^k(u\phi_x + \phi\tilde{u}_x)(.,t)\|_0 + \|\psi_{(k)}(.,t)\|_0\|\partial_x^k f(\phi,\psi)(.,t)\|_0.
\end{aligned}
$$

An application of Lemma 4.3 yields

$$
\begin{aligned}
&\|\partial_x^k(u\phi_x + \phi\tilde{u}_x)(.,t)\|_0 \\
\leq&\|\partial_x^k(u\phi_x)(.,t) - u\partial_x^k\phi_x(.,t)\|_0 + \|u\partial_x^k\phi_x(.,t)\|_0 \\
&+ \|\partial_x^k(\phi\tilde{u}_x)(.,t) - \phi\partial_x^k\tilde{u}_x(.,t)\|_0 + \|\phi\partial_x^k\tilde{u}_x(.,t)\|_0 \\
\leq&c(k)(\|u(.,t)\|_k\|\phi_x(.,t)\|_1 + \|u(.,t)\|_2\|\phi_x(.,t)\|_k \\
&+ \|u(.,t)\|_0\|\phi_{(k+1)}\|_0 + \|\phi(.,t)\|_k\|\tilde{u}_x(.,t)\|_1 \\
&+ \|\phi(.,t)\|_2\|\tilde{u}_x(.,t)\|_k + \|\phi(.,t)\|_0\|\tilde{u}_{(k+1)}(.,t)\|_0) \\
\leq&c_{27}\|\phi(.,t)\|_{k+1},
\end{aligned}
\tag{4.38}
$$

and Corollary 4.1 shows that $\sup_{0<t\leq T}\|u(.,t)\|_{k+1}$ is bounded by a constant independent of ϵ , where c_{27} is a constant independent of ϵ. Further by Corollary 4.2, the norms $\epsilon^{\frac{5}{6}}\|\partial_x^{k+2}(u_t,v_t)\|_{V_T^0}$ are bounded with a bound independent of ϵ. Thus

we have

$$
\begin{aligned}
&\dot{E}_{(k+2)\delta}(\phi,\psi)(t) \\
\leq &(\epsilon-\delta)^2\|\partial_x^{k+2}(u_t,v_t)\|_{V_T^0} + \|\phi_k(.,t)\|_1^2 + \|\psi_k(.,t)\|_0^2 \\
&+ 2\|\psi_{(k)}(.,t)\|_0\|\partial_x^k(u\phi_x+\phi\tilde{u}_x)(.,t)\|_0 \\
&+ c_3\|\psi_{(k)}(.,t)\|_0(\|\phi_{(k)}(.,t)\|_0 + \|\psi_{(k)}(.,t)\|_0) \\
&(by\ (4.38)) \\
\leq &c_{28}(\epsilon^{\frac{1}{3}} + \|\phi(.,t)\|_{k+1}^2 + \|\psi_{(k)}(.,t)\|_0^2).
\end{aligned}
$$

Integrating by parts and using Lemma 4.4 and (4.37), we have

$$
\begin{aligned}
&\|\phi(.,t)\|_{k+1}^2 + \|\psi(.,t)\|_k^2 \\
\leq &c_{29}(\|\phi(.,t)\|_0^2 + \|\psi(.,t)\|_0^2 + E_{(k+2)\delta}(\phi,\psi)(t)) \\
\leq &c_{29}[\|\phi(.,t)\|_0^2 + \|\psi(.,t)\|_0^2 + E_{(k+2)\delta}(\phi,\psi)(0) \\
&+ c_{28}(\epsilon^{\frac{1}{3}}T + \int_0^t(\|\phi(.,s)\|_{k+1}^2 + \|\psi_{(k)}(.,s)\|_0^2)ds)]
\end{aligned}
$$

$$
\begin{aligned}
\leq &c_{29}[c_{26}\epsilon^{\frac{2}{3}} + E_{(k+2)\delta}(\phi,\psi)(0) \\
&+ c_{28}(\epsilon^{\frac{1}{3}}T + \int_0^t(\|\phi(.,s)\|_{k+1}^2 + \|\psi_{(k)}(.,s)\|_0^2)ds)] \\
\leq &c_{29}[c_{26}\epsilon^{\frac{2}{3}} + E_{(k+2)\delta}(\phi,\psi)(0) + c_{28}\epsilon^{\frac{1}{3}}]e^{c_{28}c_{29}T},
\end{aligned}
$$

where c_{28} and c_{29} are constants independent of ϵ. But by (4.23), (4.24) and an argument similar to (4.36),

$$
E_{(k+2)\delta}(\phi,\psi)(0) = o(1), \qquad \text{as } \epsilon \downarrow 0.
$$

Therefore $\|(\phi,\psi)\|_{V_T^k} \to 0$, as $\epsilon \downarrow 0$; that is , $\{(u_\epsilon, v_\epsilon)\}$ is Cauchy in V_T^k.■

Corollary 4.3. *$\{(\partial_t u_\epsilon, \partial_t v_\epsilon)\}$ is Cauchy in V_T^{k-3}, as $\epsilon \downarrow 0$, for $k \geq 3$.*

Proof. As in the proof of proposition 4.4, we set $u = u_\epsilon$, $v = v_\epsilon$, $\tilde{u} = u_\delta$, $\tilde{v} = v_\delta$, $\phi = u - \tilde{u}$ and $\psi = v - \tilde{v}$, for $\delta \leq \epsilon$. By (4.32),

$$\begin{cases} \phi_t = & \delta\phi_{txx} + \psi_x + (\epsilon - \delta)u_{txx} \\ \psi_t = & \delta\psi_{txx} + \phi_x - \phi_{xxx} + 2u\phi_x + 2\phi\tilde{u}_x + f(\phi, \psi) + (\epsilon - \delta)v_{txx}. \end{cases}$$

Then by Corollary 4.2 and the proof of proposition 4.4,

$$\|\phi_t\|_{k-2} + \|\psi_t\|_{k-3} \to 0, \qquad \text{as } \epsilon \downarrow 0, \text{ for } k \geq 3. \qquad \blacksquare$$

Theorem 1. *Let $u_0 \in H_L^{k+1}$, $v_0 \in H_L^k$, $k \geq 3$ and let $T > 0$. Then the limit (u,v) of $\{(u_\epsilon, v_\epsilon)\}$, as $\epsilon \downarrow 0$, is a solution to the problem (4.1) $\sim$ (4.3) with initial data (u_0, v_0) and (u,v) $\in V_T^k$. Moreover if there is a constant $M_1 > 0$ such that $\sup_{0<t\leq T} \|u(.,t)\|_1 \leq M_1$, then that solution (u,v) is unique.*

Proof. From Proposition 4.4 and Corollary 4.3,

$$\begin{aligned} &(u_\epsilon, v_\epsilon) \to (u, v) \text{ in } V_T^k, && (4.39) \\ &f(u_\epsilon, v_\epsilon) \to f(u, v) \text{ in } V_T^k, \\ &(\partial_t u_\epsilon, \partial_t v_\epsilon) \to (\tilde{u}, \tilde{v}) \text{ in } V_T^k, \text{ for } k \geq 3, \\ &(u_\epsilon^2)_x \to (u^2)_x \text{ in } C(0, T;\ H_L^k), \\ &(u_\epsilon)_{xxx} \to u_{xxx} \text{ in } C(0, T;\ H_L^{k-2}). \end{aligned}$$

Since $\epsilon^{\frac{1}{3}}\partial_x^2(\partial_t u_\epsilon, \partial_t v_\epsilon)$ is bounded in V_T^{k-3}, it follows that at least in the sense of distributions (c.f. [5]),

$$\epsilon\partial_x^2(\partial_t u_\epsilon, \partial_t v_\epsilon) \to 0, \qquad \text{in } D'.$$

(4.39) implies $(u_\epsilon, v_\epsilon) \to (u, v)$ in the distribution sense. So $(\partial_t u_\epsilon, \partial_t v_\epsilon) \to (\partial_t u, \partial_t v)$ in the distribution sense and hence $\tilde{u} = u_t$, $\tilde{v} = v_t$.

Combining the previous facts, we have, at least in the sense of distributions,

$$\begin{cases} u_t = v_x \\ v_t = u_x - u_{xxx} + (u^2)_x + f(u,v) \end{cases}$$

with $u(x,0) = u_0(x)$ and $v(x,0) = v_0(x)$.

Since $(u,v) \in V_T^k$, $(u_t, v_t) \in V_T^{k-3}$, (u,v) is an L_2-solution if $k = 3$ and a classical solution if $k > 3$.

For the uniqueness result, suppose that (u,v) and $(\bar{u}, \bar{v})$ are two pairs of solutions of the problem (4.1) $\sim$ (4.3) with

$$\begin{cases} \sup_{0<t\leq T} \|u(.,t)\|_1 \leq M_1, & \sup_{0<t\leq T} \|v(.,t)\|_0 \leq M, \\ \sup_{0<t\leq T} \|\bar{u}(.,t)\|_1 \leq M_1, & \sup_{0<t\leq T} \|\bar{v}(.,t)\|_0 \leq M. \end{cases} \tag{4.40}$$

Let $\phi = u - \bar{u},\ \psi = v - \bar{v}$. Then

$$\begin{cases} \phi_t = \psi_x \\ \psi_t = \phi_x - \phi_{xxx} + 2(u\phi_x + \phi\bar{u}_x) + f(\phi,\psi) \end{cases}$$

Then proceeding as we did to get (4.14),

$$\dot{E}(t) = \frac{d}{dt}\int_0^L (\phi^2 + \phi_x^2 + \psi^2)dx$$

$$=2\int_0^L [2\psi(u\phi_x + \phi\bar{u}_x) + \psi f(\phi,\psi)]dx$$

$$\leq 4 \sup_{0\leq x\leq L} |u(x,t)| \|\psi(.,t)\|_0 \|\phi_x(.,t)\|_0 + 4 \sup_{0\leq x\leq L} |\phi(x,t)| \|\psi(.,t)\|_0 \|\bar{u}_x\|_0$$

$$+ \|\psi(.,t)\|_0 \|f(\phi,\psi)(.,t)\|_0$$

$$\leq c_{30}(\|u(.,t)\|_1 \|\psi(.,t)\|_0 \|\phi_x(.,t)\|_0 + \|\phi(.,t)\|_1 \|\psi(.,t)\|_0 \|\bar{u}_x\|_0)$$

$$+ c_3 \|\psi(.,t)\|_0 (\|\phi(.,t)\|_0 + \|\psi(.,t)\|_0)$$

$$(by\ (4.40))$$

$$\leq c_{31} E(t),$$

where c_{30} and c_{31} are constants. Therefore

$$E(t) \leq E(0)e^{c_{31}T} = 0, \qquad \text{for all } 0 < t \leq T,$$

and the uniqueness follows.■

Remark 4.1. *If the solution* (u_ϵ, v_ϵ) *of the regularized problem described in Corollary 1 satisfies*

$$\sup_{t>0}(\|u_\epsilon(.,t)\|_1 + \|v_\epsilon(.,t)\|_0) \le M, \tag{41}$$

then the unique solution (u,v) of problem (4.1) $\sim$ (4.3) *exists globally in* V_∞^k.

Remark 4.2. *For* $K_2 = 0$, $K_1 \ge 0$, *(41) can be replaced by* $\|u(.,t)\|_0 \le M_0$, *for all* $t > 0$ *when the initial data is small. In fact*

$$\dot{H}(t) := \frac{d}{dt}(\|u\|_1^2 + \|v\|_0^2 + \frac{2}{3}\int_0^L u^3 dx) = -2K_1\|v(.,t)\|_0^2,$$

so that

$$H(t) = \|u\|_1^2 + \|v\|_0^2 + \frac{2}{3}\int_0^L u^3 dx \le H(0). \tag{42}$$

Using Lemma 4.0 and the same method to get (4.18), we have

$$\begin{aligned} 2\int_0^L u^3 dx \le\ & 2c\|u(.,t)\|_1\|u(.,t)\|_0^2 \\ \le & c\theta\|u(.,t)\|_1^2 + c/\theta\|u(.,t)\|_0^4. \end{aligned}$$

Combining this with (42), we have

$$(1 - \frac{c\theta}{3} - \frac{c}{3\theta}\|u\|_0^2)\|u\|_0^2 + (1 - \frac{c\theta}{3})\|u_x\|_0^2 + \|v\|_0^2 \le H(0),$$

where θ *is a small constant. Therefore, when*

$$\frac{1 - \frac{c\theta}{3}}{2} \ge \frac{c}{3\theta}\|u\|_0^2 \text{ and } H(0) \le \frac{3\theta(1 - \frac{c\theta}{3})}{4c},$$

we will have

$$\frac{1 - \frac{c\theta}{3}}{2}\|u\|_0^2 + (1 - \frac{c\theta}{3})\|u_x\|_0^2 + \|v\|_0^2 \le H(0),$$

which implies (41). ■

4.4. Continuous Dependence of the Solution on Initial Data.

Theorem 4.2. *Let $T > 0$ be given and $P: H_L^{k+1} \times H_L^k \to V_T^k$ be the map assigning to given initial data $(u_0, v_0) \in H_L^{k+1} \times H_L^k$ the unique solution $(u, v) \in V_T^k$ described in Theorem 4.1. Then P is continuous.*

Proof. Let $(u_0^n, v_0^n) \to (u_0, v_0)$ in $H_L^{k+1} \times H_L^k$ and $(u^n, v^n) = P(u_0^n, v_0^n)$. We want $(u^n, v^n) \to (u, v)$ in V_T^k . By the triangle inequality,

$$\|u^n - u\|_{k+1} \leq \|u^n - u_\epsilon^n\|_{k+1} + \|u_\epsilon^n - u_\epsilon\|_{k+1} + \|u_\epsilon - u\|_{k+1},$$

and

$$\|v^n - v\|_k \leq \|v^n - v_\epsilon^n\|_k + \|v_\epsilon^n - v_\epsilon\|_k + \|v_\epsilon - v\|_k.$$

By the proof of Proposition 4.4 and (4.23) and (4.24),

$$\|u_\delta - u_\epsilon\|_{k+1} + \|v_\delta - v_\epsilon\|_k \leq c_{32}(\epsilon^{\frac{1}{3}} + \|u_{0\delta} - u_{0\epsilon}\|_{k+1} + \|v_{0\delta} - v_{0\epsilon}\|_k)$$

where c_{32} only depends on M, T and $\|u_0\|_{k+1} + \|v_0\|_k$. Similarly,

$$\|u_\delta^n - u_\epsilon^n\|_{k+1} + \|v_\delta^n - v_\epsilon^n\|_k \leq c_{33}(\epsilon^{\frac{1}{3}} + \|u_{0\delta}^n - u_{0\epsilon}^n\|_{k+1} + \|v_{0\delta}^n - v_{0\epsilon}^n\|_k)$$

where c_{33} only depends on M, T and $\|u_0^n\|_{k+1} + \|v_0^n\|_k$.

Since $(u_\delta, v_\delta) \to (u, v)$ in V_T^k, as $\delta \downarrow 0$, we have, letting $\delta \downarrow 0$,

$$\|u - u_\epsilon\|_{k+1} + \|v - v_\epsilon\|_k \leq c_{32}(\epsilon^{\frac{1}{3}} + \|u_0 - u_{0\epsilon}\|_{k+1} + \|v_0 - v_{0\epsilon}\|_k), \tag{4.43}$$

$$\|u^n - u_\epsilon^n\|_{k+1} + \|v^n - v_\epsilon^n\|_k \leq c_{33}(\epsilon^{\frac{1}{3}} + \|u_0^n - u_{0\epsilon}^n\|_{k+1} + \|v_0^n - v_{0\epsilon}^n\|_k). \tag{4.44}$$

Since $(u_0^n, v_0^n) \to (u_0, v_0)$ in $H_L^{k+1} \times H_L^k$,

$$\sup_{n>N}(\|u_0^n\|_{k+1} + \|v_0^n\|_k) \leq c_{34}$$

for some $c_{34} > 0$ and some large N. So, in fact, we can assert that c_{33} is independent of n.

Next we use the fact proved in Lemma 4.5 (c.f. [5]) that if $(u_0^n, v_0^n) \to (u_0, v_0)$ in $H_L^{k+1} \times H_L^k$, then $\|u_0 - u_{0\epsilon}\|_{k+1} + \|v_0 - v_{0\epsilon}\|_k$ and $\|u_0^n - u_{0\epsilon}^n\|_{k+1} + \|v_0^n - v_{0\epsilon}^n\|_k$, $n = 1, 2, ...$, all converge uniformly to zero as $\epsilon \downarrow 0$. Therefore given $\gamma > 0$ for any $t \in (0, T]$, $n > N$, by (4.43) and (4.44) we can choose small ϵ such that

$$\|u(.,t) - u_\epsilon(.,t)\|_{k+1} + \|v(.,t) - v_\epsilon(.,t)\|_k \le \frac{1}{3}\gamma,$$
$$\|u^n(.,t) - u_\epsilon^n(.,t)\|_{k+1} + \|v^n(.,t) - v_\epsilon^n(.,t)\|_k \le \frac{1}{3}\gamma.$$

Finally we show that

$$\|u_\epsilon(.,t) - u_\epsilon^n(.,t)\|_{k+1} + \|v_\epsilon(.,t) - v_\epsilon^n(.,t)\|_k \le \frac{1}{3}\gamma, \qquad \text{for } n > N.$$

Let $\phi = u_\epsilon - u_\epsilon^n$, $\psi = v_\epsilon - v_\epsilon^n$. Then (ϕ, ψ) satisfies

$$\begin{cases} (1 - \epsilon\partial_x^2)\phi_t = \psi_x \\ (1 - \epsilon\partial_x^2)\psi_t = \phi_x - \phi_{xxx} + 2(u_\epsilon^n \phi_x + \phi u_{\epsilon x}) + f(\phi, \psi) \end{cases}$$

with $\phi(x, 0) = u_{0\epsilon}(x) - u_{0\epsilon}^n(x)$, $\psi(x, 0) = v_{0\epsilon}(x) - v_{0\epsilon}^n(x)$.

As in the proof of Proposition 4.4, we have

$$\begin{aligned} &\frac{1}{2}\dot{E}_{(k+2)\epsilon}(\phi, \psi)(t) = \int_0^L [2\psi_{(k)}\partial_x^k(u_\epsilon^n \phi_x + \phi u_{\epsilon x}) + \psi_{(k)}\partial_x^k f(\phi, \psi)]dx \\ \le & 2\|\psi_{(k)}(.,t)\|_0 \|\partial_x^k(u_\epsilon^n \phi_x + \phi u_{\epsilon x})(.,t)\|_0 + \|\psi_{(k)}(.,t)\|_0 \|\partial_x^k f(\phi, \psi)(.,t)\|_0. \end{aligned} \tag{4.45}$$

For $k = 0$, using Corollary 4.1,

$$\begin{aligned} &\frac{1}{2}\dot{E}_{(k+2)\epsilon}(\phi, \psi)(t) \\ \le & 2\|\psi(.,t)\|_0 \|(u_\epsilon^n \phi_x + \phi u_{\epsilon x})(.,t)\|_0 + \|\psi(.,t)\|_0 \|f(\phi, \psi)(.,t)\|_0 \\ \le & 2\|\psi(.,t)\|_0 (\sup_{0 \le x \le L} |u_\epsilon^n(x,t)| \|\phi_x(.,t)\|_0 + \sup_{0 \le x \le L} |\phi(x,t)| \|u_{\epsilon x})(.,t)\|_0) \\ & + c_3 \|\psi(.,t)\|_0 (\|\phi(.,t)\|_0 + \|\psi(.,t)\|_0) \\ \le & c_{34}(\|\phi(.,t)\|_1^2 + \|\psi(.,t)\|_0^2), \end{aligned}$$

where c_{34} is independent of n and ϵ. So

$$\begin{aligned}
&\|\phi(.,t)\|_1^2 + \|\psi(.,t)\|_0^2 \le E_{2\epsilon}(\phi,\psi)(t) \\
\le& E_{2\epsilon}(\phi,\psi)(0) + 2c_{34}\int_0^t (\|\phi(.,s)\|_1^2 + \|\psi(.,s)\|_0^2)ds \\
\le& E_{2\epsilon}(\phi,\psi)(0)e^{2c_{34}T}
\end{aligned} \tag{4.46}$$

By the same argument as used in (4.38), we have

$$\|\partial_x^k(u_\epsilon^n\phi_x + \phi u_{\epsilon x})(.,t)\|_0 \le c_{35}\|\phi(.,t)\|_{k+1},$$

where c_{35} is independent of n and ϵ, since u_ϵ^n and u_ϵ are bounded in $C(0,T;H_L^{k+1})$ with bounds independent of n and ϵ. Then, going back to (4.45),

$$\begin{aligned}
&\dot{E}_{(k+2)\epsilon}(\phi,\psi)(t) \\
\le& 4c_{35}\|\psi_{(k)}(.,t)\|_0\|\phi(.,t)\|_{k+1} + 2c_3\|\psi_{(k)}(.,t)\|_0(\|\phi_{(k)}(.,t)\|_0 + \|\psi_{(k)}(.,t)\|_0) \\
\le& c_{36}(\|\phi(.,t)\|_{k+1}^2 + \|\psi_{(k)}(.,t)\|_0^2)
\end{aligned}$$

where c_{36} is independent of n and ϵ. Therefore

$$\begin{aligned}
&\|\phi(.,t)\|_{k+1}^2 + \|\psi(.,t)\|_k^2 \\
&(by\ Lemma\ 4.4) \\
\le& c_{37}(\|\phi(.,t)\|_0^2 + \|\psi(.,t)\|_0^2 + E_{(k+2)\epsilon}(\phi,\psi)(t)) \\
&(by\ (4.46)) \\
\le& c_{37}[E_{2\epsilon}(\phi,\psi)(0)e^{2c_{34}T} + E_{(k+2)\epsilon}(\phi,\psi)(0) \\
&+ c_{36}\int_0^t (\|\phi(.,s)\|_{k+1}^2 + \|\psi(.,s)\|_k^2)ds] \\
\le& c_{37}(E_{2\epsilon}(\phi,\psi)(0)e^{2c_{34}T} + E_{(k+2)\epsilon}(\phi,\psi)(0))e^{c_{36}T},
\end{aligned}$$

where c_{37} is independent of n and ϵ. By the construction of the regularized function and the proof of Lemma 4.5, one can obtain

$$\|u_{0\epsilon}^n - u_{0\epsilon}\|_{k+1} + \|v_{0\epsilon}^n - v_{0\epsilon}\|_k \le c_{38}(\|u_0^n - u_0\|_{k+1} + \|v_0^n - v_0\|_k),$$

and

$$\epsilon^{\frac{1}{6}}(\|u_{0\epsilon}^n - u_{0\epsilon}\|_{k+2} + \|v_{0\epsilon}^n - v_{0\epsilon}\|_{k+1}) \le c_{39}(\|u_0^n - u_0\|_{k+1} + \|v_0^n - v_0\|_k),$$

where c_{38} and c_{39} are independent of n and ϵ. Combining the above estimates, we have

$$\|\phi(.,t)\|_{k+1}^2 + \|\psi(.,t)\|_k \le \frac{1}{3}\gamma,$$

for all $0 < t \le T$, when $n > N$, for some integer N. ■

§ 4.5. The Forced Boussinesq Equation.

We now consider the forced version of system (4.1) ~ (4.2) , that is

$$\begin{cases} u_t = v_x \\ v_t = u_x - u_{xxx} + 2uu_x + h(x,t) \end{cases} \tag{4.47}$$

with initial and periodic boundary conditions (4.2).

There is no essential change in our previous proofs except the estimates on $f(u,v)$, which is now replaced by $h(x,t)$. When we estimate $h(x,t)$, we need another lemma, which is

Lemma 4.6. *Let $f \in L^2(0,T;H_L^k)$, for $k \ge 1$ and let f_ϵ be the smoothed version of f, i.e. if $f \sim \sum_k a_k(t)e^{2\pi ikx/L}$, then $f_\epsilon \sim \sum_k \phi(\epsilon^{\frac{1}{6}}k)a_k(t)e^{2\pi ikx/L} \in L^2(0,T;H_L^\infty)$, where ϕ is defined as in Lemma 4.5 (c.f. [5]). Then as $\epsilon \downarrow 0$,*

$$\int_0^T \|f_\epsilon\|_{k+j}^2(.,t)dt = O(\epsilon^{-j/3}),$$

$$i.e.\ \epsilon^{j/3}\int_0^T \|f_\epsilon\|_{k+j}^2(t)dt \le c\int_0^T \|f(.,t)\|_k^2 dt,\ j = 1,2,...,$$

$$\int_0^T \|f - f_\epsilon\|_{k-j}^2(t)dt = o(\epsilon^{j/3}),\ j = 1,2,...,$$

$$\int_0^T \|f - f_\epsilon\|_k^2(t)dt = o(1).$$

Furthermore, the first bound holds uniformly on bounded subsets of H_L^k, and the last two bounds hold uniformly on compact subsets of H_L^k. The second bound holds uniformly on bounded subsets of H_L^k if $o(\epsilon^{\frac{j}{3}})$ is replaced by $O(\epsilon^{\frac{j}{3}})$.

Proof.. See Lemma 5 in [5].

Then we have

Proposition 4.5. *Let $u_0 \in H_L^{k+1}$, $v_0 \in H_L^k$, for $k \geq 3$ and*

$$h \in C(0,T; H_L^{k-1}) \cap L^2(0,T; H_L^k), \qquad h_t \in L^2(0,T; H_L^{k-2}),$$

for a given finite T. Assume that the regularized system of (4.45):

$$\begin{cases} (1-\epsilon\partial_x^2)u_t^\epsilon = v_x^\epsilon \\ (1-\epsilon\partial_x^2)v_t^\epsilon = u_x^\epsilon - u_{xxx}^\epsilon + 2u^\epsilon u_x^\epsilon + h_\epsilon(x,t) \end{cases} \tag{4.48}$$

with $u^\epsilon(x,0) = u_{0\epsilon}(x)$, $v^\epsilon(x,0) = v_{0\epsilon}(x)$ has a solution (u^ϵ, v^ϵ) satisfying

$$\sup_{0<t\leq T} (\|u^\epsilon(.,t)\|_1 + \|v^\epsilon(.,t)\|_0) \leq M, \qquad \text{for all small } \epsilon.$$

Then there is a unique solution $(u,v) \in V_T^k(0,L)$ to (4.45) with initial value (u_0, v_0). Moreover the solution (u,v) depends continuously on (u_0, v_0), h and h_t.

Proof. We replace $f(u,v)$ by $h(x,t)$ everywhere through §4.1 ~ §4.3 and use Lemma 4.6. ■

Chapter 5. Stabilization of the Boussinesq Equation with Periodic Boundary Conditions

A nonlinear exponential stabilization of Boussinesq equation has been considered in [37]. In this chapter, we discuss the similar asymptotic behavior of the velocity in one of the Boussinesq-type equations on a periodic domain which arises in the modelling of nonlinear, dispersive wave propagation (c.f. [8]), namely

$$u_{tt} - u_{xx} + (u_{xx} - u^2)_{xx} = 0.$$

We do this by putting a linear feed back control term into the equation. That is, we consider the equation

$$u_{tt} - u_{xx} + u_{xxxx} + (u^2)_{xx} + Ku_t = 0, \qquad 0 < x < L,\ t > 0, \tag{5.1}$$

for $K > 0$. As usual, we rewrite it as

$$\begin{cases} u_t = v_x \\ v_t = u_x - u_{xxx} - (u^2)_x - Kv \end{cases} \qquad 0 < x < L,\ t > 0,\ K > 0. \tag{5.2}$$

We already have the following result from Chapter 4:

Theorem 5.1. *Given $u_0 \in H_L^4$, and $v_0 \in H_L^3$, there exists a unique solution*

$$(u, v) \in C(0, \infty; H_L^4 \times H_L^3)$$

of system (5.2) with initial conditions

$$u(x, 0) = u_0(x), \qquad v(x, 0) = v_0(x), \qquad 0 < x \leq L,$$

and periodic boundary conditions

$$\begin{cases} \partial_x^k u(0,t) = \partial_x^k u(L,t), \ \ for\ k = 0,1,2,3, \ \ t > 0 \\ \partial_x^k v(0,t) = \partial_x^k v(L,t), \ \ for\ k = 0,1,2, \ \ t > 0 \end{cases} \tag{5.3}$$

provided that the solution of the regularized version of (5.2)

$$\begin{cases} (1-\epsilon\partial_x^2)u_t = v_{\epsilon x} \\ (1-\epsilon\partial_x^2)v_t = u_{\epsilon x} - u_{\epsilon xxx} - (u_\epsilon^2)_x - Kv_\epsilon \end{cases} \tag{5.4}$$

with initial conditions

$$u_\epsilon(x,0) = u_{0\epsilon}, v_\epsilon(x,0) = v_{0\epsilon}, \qquad 0 < x \leq L$$

and periodic boundary conditions (5.3) for u_ϵ and v_ϵ satisfies

$$\|u_\epsilon(.,t)\|_1 + \|v_\epsilon(.,t)\|_0 \leq M, \ \ uniformly\ for\ t > 0\ ,\ \epsilon > 0. \tag{5.5}$$

That is, provided that, as $\epsilon \downarrow 0$,

$$\|u(.,t)\|_1 + \|v(.,t)\|_0 \leq M, \qquad t > 0, \tag{5.6}$$

where M is a constant.

By Remark 4.2 in Chapter 4, (5.5) and (5.6) can be reduced to

$$\|u(.,t)\|_0 \leq M, \qquad t > 0,$$

for small initial data.

We will use some conservation laws for the system

$$\begin{cases} u_t = v_x \\ v_t = u_x - u_{xxx} - (u^2)_x \end{cases} \tag{5.7}$$

to obtain our main result:

Theorem 5.2. *The solution (u,v) of the system (5.2) in Theorem 1 satisfies*

$$\lim_{t\to\infty} \|v(.,t)\|_2 = 0,$$

that is,

$$\lim_{t\to\infty} \|u_t(.,t)\|_1 = 0,$$

for the solution u of Boussinesq equation (5.1) with initial conditions

$$u(x,0) = u_0(x), \qquad u_t(x,0) = v_0'(x),$$

and periodic boundary conditions.

§ 5.1. Conservation Laws

In V. E. Zakharov's paper(c.f. [38]), it is proved that there exist infinitely many conservation laws for system (5.7) and also two nontrivial conservation laws are given, that is,

$$H(u,v) := \int_0^L (u^2 + v^2 + u_x^2 - \frac{2}{3}u^3)dx \tag{5.8}$$

$$I(u,v) := \int_0^L (u_x^2 + u_{xx}^2 + v_x^2 - uv^2 - \frac{1}{3}u^3 - 3uu_x^2 + \frac{1}{3}u^4)dx \tag{5.9}$$

are conserved. But the method used in that paper would require extensive calculations to obtain further conservation laws. We here seek a more convenient way to get them by developing some of the ideas of M. Kruskal et al (c.f. [19]).

Definition 5.1. *For a term in a polynomial having the form*

$$cu_0^{a_0}u_1^{a_1}...u_l^{a_l}v_0^{b_0}v_1^{b_1}...v_k^{b_k}$$

where we denote the i-th-order x-derivative of u by u_i, we define the **rank** *of that term by*

$$r = \sum_{i=0}^{l}(1+\frac{i}{2})a_i + \sum_{j=0}^{k}(1+\frac{j+1}{2})b_j$$

By using the Method of Undetermined Coefficients (c.f. [19]), we have

Proposition 5.1. *Terms with rank 1,2,...,6 form a nontrivial conservation law. That is,*

$$\begin{aligned} J(u,v) = \int_0^L (&40u_3^2 + 24v_2^2 + 16u_4^2 + 16v_3^2 - \frac{280}{3}u^3u_1^2 \\ &- \frac{16}{3}u^5 + 140u^2u_1^2 - \frac{62}{3}u_1^4 + 136u^2u_2^2 - 5u^4 \\ &+ 35uu_1^2 + 20u^2v^2 - 136uu_2^2 + \frac{20}{3}u^3 + 160uu_1vv_1 \\ &+ 20u_1^2v^2 + 40u^2v_1^2 + \frac{296}{3}u_2^3 - 80uu_3^2 - \frac{45}{2}u_1^2 \\ &+ 25uv^2 - u_2^2 + \frac{5}{2}u^2 - 40uv_1^2 + 40u_2v^2 \\ &- 8u_2v_1^2 - 64u_2vv_2 - 48uv_2^2 - 25v_1^2 \\ &+ \frac{5}{2}v^2 + \frac{10}{3}v^4 - \frac{40}{3}u^3v^2 + \frac{16}{9}u^6)dx \end{aligned} \tag{5.10}$$

is conserved.

Proof. Use (5.7) to compute $\frac{d}{dt}J(u,v)$. ∎

Remark. *In the case of the Boussinesq equation, half rank terms can also form conservation laws, unlike the KdV case.*

§ 5.2. Proof of Theorem 5.2

We first prove

Proposition 5.2. *The solution (u,v) to the system (5.2) in Theorem 1 satisfies*

$$\sup_{t>0}(\|v(.,t)\|_3 + \|u(.,t)\|_4) < \infty, \qquad \textit{and}$$

$$\int_0^\infty \|v(.,t)\|_3^2 dt < \infty.$$

Proof. By (5.2) and (5.8), we have

$$\frac{d}{dt}H(u,v) = -2K\int_0^L v^2 dx.$$

Integrating, we have

$$H(u,v) = H(u_0,v_0) - 2K\int_0^t\int_0^L v(x,s)^2dxds. \qquad (5.11)$$

Let $G(t) = \int_0^t\int_0^L v(x,s)^2dxds$, then by (5.8)

$$\begin{aligned}
\dot{G}(t) = &\int_0^L v^2dx = H(u,v) - \int_0^L (u^2+u_1^2-\frac{2}{3}u^3)dx\\
\leq &H(u,v) + \int_0^L \frac{2}{3}u^3dx\\
\leq &H(u,v) + \frac{2}{3}\sup_{0<x\leq L}|u(x,t)|\int_0^L u^2dx\\
\leq &H(u,v) + \frac{2}{3}\alpha\|u(.,t)\|_1\|u(.,t)\|_0^2\\
&\text{by (5.6) and (5.11)}\\
\leq &H(u_0,v_0) - 2KG(t) + \frac{2}{3}\alpha M^3\\
\textit{i.e.} \qquad \dot{G}(t) \leq &-2KG(t) + H(u_0,v_0) + \frac{2}{3}\alpha M^3,
\end{aligned}$$

which implies

$$\int_0^t\int_0^L v^2(x,s)dxds = G(t) \leq \frac{H(u_0,v_0)+\frac{2}{3}\alpha M^3}{2K} := c_1, \qquad t>0. \qquad (5.12)$$

Similarly by (5.2) and (5.9), we have

$$\frac{d}{dt}I(u,v) = -2K\int_0^L(v_1^2-uv^2)dx. \qquad (5.13)$$

Let $F(t) := \int_0^t\int_0^L(v_1^2-uv^2)dxdt$, then by (5.13),

$$I(u,v) = I(u_0,v_0) - 2KF(t) \qquad (5.14)$$

and by (5.9),

$$
\begin{aligned}
\dot{F}(t) =& I(u,v) - \int_0^L (u_1^2 + u_2^2 - \frac{1}{3}u^3 - 3uu_1^2 + \frac{1}{3}u^4)dx \\
\leq& I(u,v) + \int_0^L (\frac{1}{3}u^3 + 3uu_1^2)dx \\
\leq& I(u,v) + \frac{1}{3}\sup_{0<x\leq L}|u(x,t)|\int_0^L u^2dx + 3\sup_{0<x\leq L}|u(x,t)|\int_0^L u_1^2dx \\
\leq& I(u,v) + \frac{1}{3}\alpha\|u(.,t)\|_1\|u(.,t)\|_0^2 + 3\alpha\|u(.,t)\|_1\|u_1(.,t)\|_0^2 \\
& \text{by (5.6)} \\
\leq& I(u,v) + 3\alpha M^3 \\
& \text{by (5.14)} \\
=& -2KF(t) + I(u_0,v_0) + 3\alpha M^3,
\end{aligned}
$$

which means

$$
F(t) \leq \frac{I(u_0,v_0) + 3\alpha M^3}{2K} := c_2, \qquad t > 0. \tag{5.15}
$$

Then by the definition of H(t),

$$
\begin{aligned}
\int_0^t\int_0^L v_1(x,s)^2dxds =& F(t) + \int_0^t\int_0^L uv^2dxds \\
\leq& (by\ (5.15))\ c_2 + \int_0^t \sup_{0<x\leq L}|u(x,s)|\|v(.,s)\|_0^2ds \\
\leq& c_2 + \int_0^t \alpha\|u(.,s)\|_1\|v(.,s)\|_0^2ds \\
& \text{by (5.6)} \\
\leq& c_2 + \alpha MG(t) \\
& \text{by (5.12)} \\
\leq& c_2 + \alpha Mc_1 := c_3, \qquad t > 0.
\end{aligned} \tag{5.16}
$$

By (5.9) and (5.14), we have

$$\begin{aligned}
&\int_0^L (v_1^2 + u_2^2)dx \\
=&I(u,v) - \int_0^L (-uv^2 - 3uu_1^2 + \frac{1}{3}u^4 + u_1^2 - \frac{1}{3}u^3)dx \\
\leq&I(u,v) + \int_0^L (uv^2 + 3uu_1^2 + \frac{1}{3}u^3)dx \\
=&I(u_0,v_0) - 2K\int_0^t\int_0^L (v_1^2 - uv^2)(x,s)dx + \int_0^L (uv^2 + 3uu_1^2 + \frac{1}{3}u^3)dx \\
\leq&I(u_0,v_0) + 2K\int_0^t \sup_{0<x\leq L} |u(x,s)| \|v(.,s)\|_0^2 ds \\
&+ \sup_{0<x\leq L}\int_0^L (v^2 + 3u_1^2 + \frac{1}{3}u^2)dx \\
\leq&I(u_0,v_0) + 2K\int_0^t \alpha\|u(.,s)\|_1 \|v(.,s)\|_0^2 ds \\
&+ 3\alpha\|u(.,t)\|_1 \int_0^L (v^2 + u_1^2 + u^2)dx \\
\leq&(by\ (5.6))I(u_0,v_0) + 2K\alpha M\int_0^t\int_0^L v^2(x,s)dxds + 3\alpha M^3 \\
\leq&(by\ (5.12))I(u_0,v_0) + 2K\alpha M c_1 + 3\alpha M^3 := c_4 \qquad (5.17)
\end{aligned}$$

Also by (5.2) and (5.10), we have

$$\frac{d}{dt}J(u,v) = -2K\dot{E}(t) - \frac{20}{3}Kv^4 \leq -2K\dot{E}(t),$$

so that

$$J(u,v) \leq J(u_0,v_0) - 2KE(t), \qquad (5.18)$$

where

$$\begin{aligned}
E(t) = \int_0^t\int_0^L &(24v_2^2 + 16v_3^2 + 20u^2v^2 + 160uu_1vv_1 + 20u_1^2v^2 \\
&+ 40u^2v_1^2 + 25uv^2 - 40uv_1^2 + 40u_2v^2 - 8u_2v_1^2 - 64u_2vv_2 \\
&- 48uv_2^2 - 25v_1^2 + \frac{5}{2}v^2 + \frac{10}{3}v^4 - \frac{40}{3}u^3v^2)(x,s)dxds. \qquad (5.19)
\end{aligned}$$

By comparing (5.19) with (5.10), we obtain

$$
\begin{aligned}
\dot{E}(t) =& J(u,v) - \int_0^L (40u_3^2 + 16u_4^2 + \frac{296}{3}u_2^3 - 80uu_3^2)dx \\
& - \int_0^L (-\frac{280}{3}u^3u_1^2 - \frac{16}{3}u^5 + 140u^2u_1^2 - \frac{62}{3}u_1^4 + 136u^2u_2^2 \\
& - 5u^4 + 35uu_1^2 - 136uu_2^2 + \frac{20}{3}u^3 - \frac{45}{2}u_1^2 - u_2^2 + \frac{5}{2}u^2 + \frac{16}{9}u^6)dx \\
\leq & J(u,v) - \int_0^L (40u_3^2 + 16u_4^2 + \frac{296}{3}u_2^3 - 80uu_3^2)dx + c_5, \qquad (5.20)
\end{aligned}
$$

where c_5 is a constant. The last inequality is obtained by (5.6) and (5.17). For instance, the term $\frac{280}{3}u^3u_1^2$ can be estimated in this way:

$$
\begin{aligned}
& \int_0^L \frac{280}{3}u^3u_1^2dx \leq \frac{280}{3}\sup 0 < x \leq L|u(x,t)|^3 \int_0^L u_1^2dx \\
\leq & \frac{280}{3}\alpha^3\|u(.,t)\|_1^3\|u_1(.,t)\|_0^2 \leq (by\ (5.6)\ )\frac{280}{3}\alpha^3M^5. \qquad (5.21)
\end{aligned}
$$

Since (u,v) is the solution of (5.2) $\sim$ (5.3), integrating by parts, we have

$$
\int_0^L u_2^3dx = \int_0^L u_{xx}^2u_{xx}dx = -\int_0^L 2u_{xx}u_{xxx}u_xdx.
$$

Then by (5.6) and (5.17),

$$
\begin{aligned}
|\int_0^L u_2^3dx| \leq & 2\sup_{0<x\leq L}|u_1(x,t)|\int_0^L |u_2u_3|dx \\
\leq & \alpha\|u_1(.,t)\|_1\int_0^L (\frac{u_2^2}{a^2} + a^2u_3^2)dx \\
\leq & \alpha\sqrt{M^2+c_4}\frac{c_4}{a^2} + a^2\alpha\sqrt{M^2+c_4}\int_0^L u_3^2dx. \qquad (5.22)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
& \int_0^L uu_3^2dx = \int_0^L uu_{xxx}u_{xxx}dx \\
= & -\int_0^L (u_xu_{xxx}u_{xx} + uu_{xxxx}u_{xx})dx,
\end{aligned}
$$

$$
\begin{aligned}
&|\int_0^L uu_3^2 dx| \\
&\le \sup_{0<x\le L} |u_x(x,t)| \int_0^L |u_3u_2|dx + \sup_{0<x\le L} |u(x,t)| \int_0^L |u_4u_2|dx \\
&\le \alpha\|u_1(.,t)\|_1 \int_0^L (\frac{u_2^2}{2b^2} + \frac{b^2}{2}u_3^2)dx + \alpha\|u(.,t)\|_1 \int_0^L (\frac{u_2^2}{2c^2} + \frac{c^2}{2}u_4^2)dx \\
&\le \alpha\sqrt{M^2+c_4}\frac{c_4}{2b^2} + \alpha\sqrt{M^2+c_4}\frac{b^2}{2} \\
&+ \int_0^L u_3^2dx + \alpha M\frac{c_4}{2c^2} + \alpha M\frac{c^2}{2}\int_0^L u_4^2 dx, \qquad (5.23)
\end{aligned}
$$

where a, b and c are constants to be chosen. Substituting (5.22) and (5.23) into (5.20), we have

$$
\begin{aligned}
\dot{E}(t) \le & J(u,v) - \int_0^L (40u_3^2 + 16u_4^2)dx + \frac{296}{3}|\int_0^L u_2^3 dx| \\
& + 80|\int_0^L uu_3^2 dx| + c_5 \\
\le & J(u,v) + \alpha\sqrt{M^2+c_4}c_4(\frac{296}{3a^2} + \frac{40}{b^2}) + 40\alpha M\frac{c_4}{c^2} + c_5 \\
& - \int_0^L [40 - \alpha\sqrt{M^2+c_4}(\frac{296a^2}{3} + 40b^2)]u_3^2 dx \\
& - \int_0^L (16 - 40\alpha Mc^2)u_4^2 dx \\
\le & J(u,v) + \alpha\sqrt{M^2+c_4}c_4(\frac{296}{3a^2} + \frac{40}{b^2}) + 40\alpha M\frac{c_4}{c^2} + c_5 \\
:= & J(u,v) + c_6, \qquad (5.24)
\end{aligned}
$$

where a, b and c satisfy

$$
\begin{aligned}
&\alpha\sqrt{M^2+c_4}(\frac{296a^2}{3} + 40b^2) < 39, \\
\textit{and} \quad & 40\alpha Mc^2 < 16 - 1 = 15. \qquad (5.25)
\end{aligned}
$$

Combining (5.18) and (5.24), we have

$$
\dot{E}(t) \le -2KE(t) + J(u_0, v_0) + c_6
$$

which gives

$$E(t) \leq \frac{J(u_0, v_0) + c_6}{2K} := c_{6.5}, \qquad \textit{for all } t > 0. \tag{5.26}$$

Then we go back to (5.19) and notice that

$$\begin{aligned}
&\int_0^t \int_0^L (v_2^2 + v_3^2)(x,s)dx\,ds \\
=&E(t) - \int_0^t \int_0^L (23v_2^2 + 15v_3^2 - 8u_2v_1^2 - 64u_2vv_2 - 48uv_2^2)dxds \\
&- \int_0^t \int_0^L (20u^2v^2 + 160uu_1vv_1 + 20u_1^2v^2 + 40u^2v_1^2 + 25uv^2 \\
&- 40uv_1^2 + 40u_2v^2 - 25v_1^2 + \frac{5}{2}v^2 + \frac{10}{3}v^4 - \frac{40}{3}u^3v^2)dxds \\
\leq&E(t) - \int_0^t \int_0^L (23v_2^2 + 15v_3^2 - 8u_2v_1^2 - 64u_2vv_2 - 48uv_2^2)dxds \\
&+ \int_0^t \int_0^L (-160uu_1vv_1 - 25uv^2 + 40uv_1^2 - 40u_2v^2 + 25v_1^2 + \frac{40}{3}u^3v^2)dxds \\
\leq&E(t) - \int_0^t \int_0^L (23v_2^2 + 15v_3^2 - 8u_2v_1^2 - 64u_2vv_2 - 48uv_2^2)dxds + c_7,
\end{aligned} \tag{5.27}$$

where c_7 is a constant. The last inequality is obtained by (5.6), (5.12), (5.16) and (5.17). For example,

$$\begin{aligned}
&\int_0^t \int_0^L uu_1vv_1(x,s)dxds \\
\leq& \int_0^t \sup_{0<x\leq L} |u(x,s)| \sup_{0<x\leq L} |u_1(x,s)| \int_0^L |vv_1|dxds \\
\leq& \int_0^t \alpha^2 \|u(.,s)\|_1 \|u_1(.,s)\|_1 \int_0^L |vv_1|dxds \\
&(\textit{by } (5.6) \textit{ and } (5.17)\ ) \\
\leq&\alpha^2 M\sqrt{M^2 + c_4}1/2 \int_0^t \int_0^L (v^2 + v_1^2)(x,s)dxds \\
&(\textit{by } (5.12) \textit{ and } (5.16)\ ) \\
\leq&\alpha^2 M\sqrt{M^2 + c_4}1/2(c_1 + c_3),
\end{aligned}$$

Similarly, by (5.6) and (5.12),

$$\begin{aligned}
&\int_0^t \int_0^L uv^2(x,s)dxds \\
\leq &\int_0^t \sup_{0<x\leq L} |u(x,s)| \int_0^L v^2(x,s)dxds \\
\leq &\int_0^t \alpha \|u(.,s)\|_1 \int_0^L v^2(x,s)dxds \\
\leq &\alpha M \int_0^t \int_0^L v^2(x,s)dxds \leq \alpha M c_1.
\end{aligned}$$

Integrating by parts and using (5.6) (5.17) (5.12) (5.16), we have

$$\begin{aligned}
&\int_0^t \int_0^L u_2 v^2(x,s)dxds = -\int_0^t \int_0^L u_1 v v_1(x,s)dxds \\
\leq &\int_0^t \sup_{0<x\leq L} |u_1(x,s)| \int_0^L |2vv_1(x,s)|dxds \\
\leq &\int_0^t \alpha \|u_1(.,s)\|_1 \int_0^L |2vv_1(x,s)|dxds \\
\leq &\alpha\sqrt{M^2+c_4} \int_0^t \int_0^L (v^2+v_1^2)(x,s)dxds \\
\leq &\alpha\sqrt{M^2+c_4}(c_1+c_3),
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t \int_0^L v^4(x,s)dxds \\
\leq &\int_0^t \sup_{0<x\leq L} v^2(x,s) \int_0^L v^2(x,s)dxds \\
\leq &\int_0^t \alpha^2 \|v(.,s)\|_1^2 \int_0^L v^2(x,s)dxds \\
&\text{(by (5.6) and (5.17))} \\
\leq &\alpha^2(M^2+c_4) \int_0^t \int_0^L v^2(x,s)dxds \\
&\text{(by (5.12))} \\
\leq &\alpha^2(M^2+c_4)c_1.
\end{aligned}$$

Now we begin our treatment of (5.27). Integrating by parts and using (5.6) (5.17) (5.16),

$$\int_0^t \int_0^L u_2 v_1^2(x,s)dxds = -\int_0^t \int_0^L 2u_1 v_1 v_2(x,s)dxds$$
$$\leq \int_0^t \sup_{0<x\leq L} |u_1(x,s)| \int_0^L 2|v_1 v_2(x,s)|dxds$$
$$\leq \int_0^t \alpha \|u_1(.,s)\|_1 \int_0^L 2|v_1 v_2(x,s)|dxds$$
$$\leq \alpha\sqrt{M^2+c_4} \int_0^t \int_0^L (\frac{v_1^2}{e^2} + e^2 v_2^2)(x,s)dx\,ds$$
$$\leq \alpha\sqrt{M^2+c_4}\frac{c_3}{e^2} + e^2\alpha\sqrt{M^2+c_4} \int_0^t \int_0^L v_2^2(x,s)dxds, \tag{5.28}$$

and similarly,

$$\int_0^t \int_0^L u_2 v v_2(x,s)dxds$$
$$= -\int_0^t \int_0^L (u_1 v_1 v_2 + u_1 v v_3)(x,s)dxds$$
$$\leq \int_0^t \sup_{0<x\leq L} |u_1(x,s)| \int_0^L (|v_1 v_2| + |v v_3|)(x,s)dxds$$
$$\leq \int_0^t \alpha \|u_1(.,s)\|_1 \int_0^L [\frac{v^2+v_1^2}{2e^2} + \frac{e^2}{2}(v_2^2+v_3^2)](x,s)dxds$$
$$(\textit{by } (5.6),(5.12),(5.16) \textit{ and } (5.17))$$
$$\leq \alpha\sqrt{M^2+c_4}\frac{c_1+c_3}{2e^2} + \alpha\sqrt{M^2+c_4}\frac{e^2}{2} \int_0^t \int_0^L (v_2^2+v_3^2)dxds. \tag{5.29}$$

The same arguments also yield

$$\int_0^t \int_0^L u v_2^2(x,s)dx\,ds$$
$$= -\int_0^t \int_0^L (u_1 v_2 v_1 + u v_3 v_1)(x,s)dxds$$
$$\leq \int_0^t [\sup_{0<x\leq L} |u_1(x,s)| \int_0^L |v_2 v_1(x,s)|dx$$
$$+ \sup_{0<x\leq L} |u(x,s)| \int_0^L |v_3 v_1(x,s)|dx]ds$$

$$\le \int_0^t [\alpha\|u_1(.,s)\|_1 \int_0^L (\frac{v_1^2}{2\rho^2} + \frac{\rho^2}{2}v_2^2)dx$$

$$+ \alpha\|u(.,s)\|_1 \int_0^L (\frac{v_1^2}{2\rho^2} + \frac{\rho^2}{2}v_3^2)dx]ds$$

$$(\mathit{by}\ (5.6), (5.16)\ \mathit{and}\ (5.17))$$

$$\le \alpha(\sqrt{M^2+c_4}+M)\frac{c_3}{2\rho^2}$$

$$+ \int_0^t \int_0^L (\alpha\sqrt{M^2+c_4}\frac{\rho^2}{2}v_2^2 + \alpha M\frac{\rho^2}{2}v_3^2)(x,s)dx\,ds. \tag{5.30}$$

Combining (5.27) $\sim$ (5.30), we obtain that

$$\int_0^t \int_0^L (v_2^2+v_3^2)(x,s)dxds$$

$$\le E(t) + c_7 + \frac{\alpha}{\rho^2}\sqrt{M^2+c_4}(60c_3+32c_1) + 24\alpha M\frac{c_3}{\rho^2}$$

$$- \int_0^t \int_0^L [(23 - 60\alpha\sqrt{M^2+c_4}\rho^2)v_2^2$$

$$+ (15 - 32\alpha\sqrt{M^2+c_4}\rho^2 - 24\alpha M\rho^2)v_3^2](x,s)dxds \tag{5.31}$$

$$(\mathit{by}\ (5.26))$$

$$\le c_{6.5} + c_7 + \frac{\alpha}{\rho^2}\sqrt{M^2+c_4}(60c_3+32c_1) + 24\alpha M\frac{c_3}{\rho^2} := c_8, \tag{5.32}$$

for all $t > 0$, where constant ρ satisfies

$$60\alpha\sqrt{M^2+c_4}\rho^2 < 23, \qquad \mathit{and}$$

$$(32\alpha\sqrt{M^2+c_4} + 24\alpha M)\rho^2 < 15. \tag{5.33}$$

Now we have to check one more thing, that is, by (5.10), (5.6) and (5.17) and the same argument used to obtain (5.20) and (5.21),

$$\int_0^L (u_3^2 + u_4^2 + v_2^2 + v_3^2)dx$$

$$\leq J(u,v) - \int_0^L (39u_3^2 + 15u_4^2 + \frac{296}{3}u_2^3 - 80uu_3^2)dx$$

$$- \int_0^L (23v_2^2 + 15v_3^2 - 8u_2v_1^2 - 64u_2vv_2 - 48uv_2^2)dx + c_9$$

$$(by\ (5.22) \sim (5.25)\ and\ (5.27) \sim (5.33))$$

$$\leq J(u,v) + c_{10}$$

$$(by\ (5.18))$$

$$\leq J(u_0, v_0) - 2KE(t) + c_{10}.$$

But by (5.31), we have

$$-E(t) \leq c_7 + \frac{\alpha}{e^2}\sqrt{M^2 + c_4}(60c_3 + 32c_1) + 24\alpha M\frac{c_3}{e^2} := c_{11}.$$

where c_9 and c_{10} are constant. Therefore, we obtain that

$$\int_0^L (u_3^2 + u_4^2 + v_2^2 + v_3^2)dx \leq J(u_0, v_0) + 2Kc_{11} + c_{10}. \tag{5.34}$$

Combining (5.6), (5.12), (5.16), (5.17), (5.32) and (5.34), the proposition is proved.

Now we are ready to prove Theorem 5.2.

Proof of Theorem 5.2. From (5.2) we have

$$\frac{d}{dt}\int_0^L (v^2 + v_1^2 + v_2^2)dx = 2\int_0^L (vv_t + v_1v_{1t} + v_{2t})dx$$

$$=2\int_0^L [v(u_1 - u_3 - 2uu_1 - Kv) + v_1(u_2 - u_4 - 2u_1^2 - 2uu_2 - Kv_1)$$

$$+ v_2(u_3 - u_5 - 6u_1u_2 - 2uu_3 - Kv_2)]dx$$

$$(Integrating\ by\ parts)$$

$$=2\int_0^L (u_1v - u_3v - 2uu_1v + u_2v_1 - u_4v_1 - 2u_1^2v_1 - 2uu_2v_1$$

$$+ u_3v_2 + u_4v_3 - 6u_1u_2v_2 - 2uu_3v_2)dx - 2K\int_0^L (v^2 + v_1^2 + v_2^2)dx.$$

Therefore, using the inequality $2ab \leq a^2 + b^2$ and Proposition 5.2, we have, much as in the discussion used to obtain (5.20),

$$|\frac{d}{dt}\int_0^L (v^2 + v_1^2 + v_2^2)dx| \leq C,$$

where C is a constant. So $\|v(.,t)\|_2^2$ is uniformly continuous and also, by Proposition 5.2,

$$\int_0^\infty \|v(.,t)\|_2^2 dt < \infty$$

Thus we have

$$\lim_{t\to\infty} \|v(.,t)\|_2^2 = 0,$$

according to the basic results of advanced calculus. ■

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Ms. FANG-LAN LIU

2409, 1300 University City Blvd
Blacksburg, Virginia 24060
Tel: (O)703-231-7081 (H)703-552-2944
e-mail: fanglan@math.vt.edu

EDUCATION:

June 1993	Ph.D., Applied Mathematics, Virginia Tech. •Thesis advisor: Professor David L. Russell •Thesis title: Some asymptotic stability results for the Boussinesq equation •Knowledge of Software and Operating Systems: Fortran, Pascal, Matlab, UNIX system
1989	M.A. Mathematics, Fudan University, Shanghai, China
1986	B.A. Mathematics (differential geometry), Fudan University, Shanghai, China

PROFESSIONAL EXPERIENCE:

Fall 1989 -present	Graduate Research or Teaching Assistant •Taught pre-calculus for one year (100% in charge) Awarded Instructional Scholarships and Tuition Scholarships Mathematics Department, Virginia Tech, Blacksburg, Virginia

RESEARCH INTERESTS:

•Nonlinear/linear partial differential equations, control theory, and ordinary differential equations(AMS subject classification: 34 35 93 58)

PROFESSIONAL ACTIVITIES:

1992	20-minute talk at Midwest-Southeastern Atlantic 2nd Joint Regional Conference on Differential Equations, University of Kentucky, Lexington, KY •Member of American Mathematics Society and Association for Women in Mathematics

PERSONAL DATA:

•Date of Birth: June 17, 1965
•Citizenship: China
•Visa Status: F-1