Cooperation and Local Interactions in the Prisoners' Dilemma Game*

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Abstract

We consider a population of players playing a Prisoners' Dilemma Game in a local interaction setting, using a formalism of automata networks. Sufficient conditions for existence of an equilibrium where cooperation coexists with non-cooperation (a mixed equilibrium) are derived and properties of an equilibrium are discussed. In a mixed equilibrium the highest payoff will be obtained by a cooperator. For a special one-dimensional case the equilibrium set is fully characterized. We further consider a model where agents can choose (to some extent) whom of their neighbors they want to play with. Results of computer simulations are reported. The most striking feature of simulations is the fact that very organized structures can be observed starting with completely random initial conditions.

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1. Introduction.

The puzzle of cooperation has attracted and continues to attract a considerable amount of research effort. Three major ways proposed to explain cooperation are: (1) Kin selection, Hamilton (1964); (2) Reciprocal altruism, Axelrod and Hamilton (1981); and (3) Group selection, Wynne-Edwards (1962).

Most of the relevant literature on the subject of cooperation makes the assumption that players are best response players. I believe that imitation should also be given serious consideration. We are living in an interconnected world and during the course of life our decisions and experiences are often affected by people whom we have never met. Even small decisions can produce unexpected externalities. This notion of interdependence is to a large extent foreign to a population of best response players, but is captured very well by a population of imitators. If we are looking at a population of players who each imitate the neighbor with the highest payoff, we are essentially looking at a system of small interconnected sub-populations, the neighborhoods where what a player will do depends not only on her neighbors, but on the neighbors of neighbors as well.

An elegant way to explain cooperation in such a context was suggested by Nowak and May (1993). They show that local interactions are dramatically different from the global ones: when agents play with only a subset of the population, a significant level of cooperation can be sustained in a single group even in the presence of defectors. It is assumed that a player imitates a player with best payoff in her neighborhood or herself. The model does not have a stochastic element in it, but the authors demonstrate that even deterministic local interactions can create beautiful, sometimes seemingly chaotic spatial patterns. A model employing local interactions could also be used in a hierarchical framework. An example of a trade network with hierarchical structure can be found in Gilles, Haller and Ruys (1994). Clearly, hierarchical and local interactions are closely related to each other: to allow meaningful interactions between the groups we need to have a special kind of interaction among the members of a given group. Conversely, local interaction could lead to the partitioning of the population into groups or hierarchies, Whittle (1982).

There is not necessarily a sharp boundary between imitation and best response. If players do not know or do not understand the nature of a game they will have to make inferences from the behavior of others, which may lead naturally to some form of imitation.

There is yet another way to look at cooperation in the context of local interaction: it may be costly per se, but there may be circumstances when it is the only reasonable choice. For example, a grandmother, who used to bring gifts for the children in her every visit. This seems to be a highly altruistic act. If she decides to skip the habit and save some money this year: will she benefit? Maybe not. From the following discussion we can also see that in a mixed equilibrium for some of the players deviation to non-cooperation is a bad strategy: by doing so they will not become better off in the next round (even though in the models discussed in the paper agents will not be able to foresee that).

When we are talking about cooperation in a Prisoners' Dilemma Game, one may argue that imitation is a silly strategy, because players often can do better by non-cooperating. On the other hand, if we look at a mixed population of imitators and best response players, in the dynamic models suggested below imitators will do better. Long-run success might be a better measure to evaluate a strategy.

We can also argue that in the local interaction setting an imitator uses more information about the population than a best response player in otherwise exactly the same game. The reason is obvious: even under very simple rules the strategy played by an imitator will generally depend on larger set of players in the population than the strategies of a best response player. This can be seen in the examples discussed in the paper. Of course, more information does not always mean a better outcome. However it can be shown (see Rhode and Stegeman (1997)) that in certain circumstances an imitator matched with a best response player will fare better.

I think the main force working in favor of cooperation in the population of imitators is the non-linear character of interaction. In this respect cooperation in a population of imitators is very similar to cooperation against a common enemy. In both cases we have a system of interacting populations, where cooperation can survive even in the long-run. As an example of cooperation induced by a common enemy can serve the game played in Dr. Hans Haller's game theory class: every player plays several tours of Cournot oligopoly with some opponent unknown to him. The goal of every player is to obtain the highest payoff in the population, which essentially implies that a player has to look and see what others are doing. As a result in the described game cooperation was observed to survive for almost the full length of the game.

In my formal model(s) a population of players is playing a Prisoners' Dilemma Game in a local interactions setting. Sufficient conditions for existence of an equilibrium where cooperation coexists with non-cooperation (a mixed equilibrium) will be derived and properties of equilibria are discussed. In a mixed equilibrium the highest payoff will be obtained by a cooperator. For a special one-dimensional case the equilibrium set is characterized. It will be shown that in a mixed equilibrium the size of the cluster of non-cooperators is directly related to the advantage of non-cooperators. I further analyze a model where agents can choose (to some extent) whom of their neighbors they want to play with. Results of computer simulations are reported.

The current paper analyzes two main models. The first model, where players imitate the best strategy around, was introduced originally in Nowak and May (1993), who reported the results of computer simulations. The results derived here for the first model (sufficient conditions for existence of an equilibrium, characterization of the equilibrium set for the one-dimensional case, highest possible payoff in a mixed equilibrium attained by a cooperator) are original contributions of the author. The second model, where agents have some freedom in choosing their partners, has its roots in the computer science literature related to learning in neural networks1, while its application to economics and the particular rule what the effective strengths of connections will be were first developed in the present paper.

We conduct surveys of the most relevant literature on group interaction and local interaction before we begin our own formal analysis with the introduction of a model of automata networks that encompasses many types of local interaction.

2. Group Interaction.

Much of the evolutionary game theory related to economics builds on the framework developed by Kandori, Mailath and Rob (1991). Several features of their model have drawn criticism. One is the rate of convergence - KMR's model takes an arbitrarily long time to reach its equilibrium. A second is the fact that in coordination games with sufficiently large populations the long run equilibrium is the risk dominant one, which need not be Pareto efficient .

Like work on local interaction, several articles on group interaction have been motivated in part by the shortcomings of the KMR model and in part by the quest for a more realistic description of human interaction. Models of group interaction assume that the population of players is partitioned into groups. They distinguish between interaction among groups and interaction within groups.

The following articles have the common feature that there is an individually inferior strategy that is beneficial for the group as a whole. Making group selection a more important factor than individual selection can make the Pareto-efficient equilibrium the long-run outcome of the evolutionary process. One weakness of this approach is that the exact mechanics of group selection remains unclear. It also might be of interest to look at a model of group selection with the interaction inside of each group being local.

Canals and Vega-Redondo (1992) consider an "evolutionary process which proceeds in "parallel" at different "hierarchical" levels." They analyze a generalization of KMR's model. The population is partitioned into disjoint groups and evolution takes place at two levels. At a lower level the agents participate in pairwise contests within each group. The two basic components are imitation and experimentation. At each stage an agent has, with positive probability, an opportunity to change her strategy. The assumption is that an agent imitates the

[1] There is a huge and still growing literature on learning in automata networks. It is nearly impossible to give an exhaustive or representative list of articles on this subject. As an example can serve Rojas (1996), Nadal (1994) and

behavior that was most successful in the previous round. The experimentation dynamics induce an agent to mutate with positive probability away from the strategy selected at the imitation stage. At an upper level the groups themselves face some positive probability of adjusting the group's strategy profile, a process essentially analogous to the imitation on the intra-group level. When the set of all groups subject to the adjustment is identified, the average payoff of agents in a group is compared to the highest average payoff among all groups. If the former is strictly lower, then the group strategy profile changes to match that of a group having the highest average payoff. The motivation for the model is that in the real world interaction does not take place within the population as a whole. Economic agents communicate within different groups (families, neighborhoods, companies, . . .), and the influence of the outside population is largely channeled through inter-group dynamics. The population structure in the paper is not close to that of a human society, but the authors believe that it provides a good first order approximation.

Unlike in the KMR paper, it is shown that "Pareto dominant equilibria are the only long run states of the process if the number of groups is relatively large." The authors also point out that "...the segmented kind of interaction contemplated here speeds up the desired convergence quite significantly" (compared to the KMR paper). An intuition for the importance of the large number of groups is provided with the comments for the next article.

One criticism of the last article is that the groups are not really competing. The strategy profiles of groups are compared and the worst ones are adjusted. Considering a situation in which the well being of a group (for example, probability of survival) depends on its payoff profile should be more interesting. The next article appears to be more realistic in this respect.

Peretto et.al. (1992).

Vega-Redondo (1995) develops a ”hierarchic evolutionary model leading to long-run cooperation in the Prisoners Dilemma.” The model is similar to the one in Canals and Vega-Redondo (1992). The difference is that in the 1995 paper the underlying game is a Prisoners' Dilemma and the probability of a group dispersal (strategy profile adjustment) may differ across groups. In the 1995 article the groups are subject to **dispersal** with positive probability, but the only difference is in interpretation, rather than in the model itself. The author assumes that the probability of dispersal is strictly positive for each group, for any strategy profile. After a group disperses, the empty niche is recolonized by individuals from groups having the highest average payoffs. It may be misleading to call this process a dispersal, because a dispersed group can in fact be imitated if it had the highest average payoff. The author represents the dynamics as a Markov process. The main result is that if the number of groups is sufficiently large (for a given size of a group), then the whole population will play the cooperative strategy most of the time. The mechanism that generates convergence to the cooperative outcome here is the same as the one which ensures Pareto efficiency in the previous article. The probability that the whole population will move from a “bad” equilibrium to a “good” one decreases in the group size, while the probability of the opposite transition decreases with the number of groups. Specifically, when the whole population is in the “good” equilibrium and the number of groups is large relative to the number of individuals in each group, then a single group that switches to the “bad” equilibrium is likely to imitate a more successful cooperative group and switch back to cooperation before individual mutations in every group cause the whole population to switch to non-cooperation. Hence as the mutation rate goes to zero the competition among the groups becomes infinitely fierce. Author considers interactions in the population as having a local character, because agents interact only with other agents in their group. It seems that the large (relative to the size of a group) number of globally interacting groups undermines the local character of interaction.

Earlier work by Boyd and Richerson (1990) considers the effects of group selection on the equilibrium outcome. They show that "when there are multiple evolutionary stable strategies, selection among groups can cause the spread of the strategy that has the lowest extinction rate or highest possibility of contributing to the colonization of empty habitats."

3. Local Versus Global Interaction.

The message from Nowak and May (1993) that local interactions are drastically different from global ones is confirmed by other authors. Equally significant prove to be differences between imitation and best response dynamics.

Ellison (1993) studies the rate of convergence for a model that is essentially the same as in KMR, except for the local character of interaction. Players choose a best response to the actions of neighbors. He shows that local interaction generates dynamics that converge faster than those in KMR.

Nowak and May (1993) use a model similar to Ellison's. Interactions are local, but after each round a player is replaced by the most successful of his neighbors (including himself). This is a form of imitation dynamics. In computer simulations the authors found that the local interactions model with agents imitating the best paying strategy in their neighborhood produces interesting global behavior. Full non-cooperation does not seem to be a globally attractive point.

Cycles in which cooperation coexists with non-cooperation have appeared in simulations. The underlying model is nonlinear.

Berninghaus and Schwalbe (1996b) study the evolution of conventions in a finite population of boundedly rational players. They consider a model with agents interacting with a set of their neighbors, playing a coordination game. Players play pure strategies, and at the beginning of each period they choose the best response to the actions of their neighbors in the previous run. The authors show that this model can be analyzed in the framework of threshold automata networks. They confine their attention to analyses of evolutionary stability of conventions in one- and two-dimensional structures. In the one dimensional case they consider a model equivalent to a model with agents distributed over a circle. The authors show that the two possible strategy profiles with all agents playing the same strategy are the only fixed points of the dynamics given that all players have symmetric reference groups of the same size. It is also shown for the reference group of size two that the risk-dominant equilibrium is evolutionary stable[2] of degree n-1 (n is the population size). For the two-dimensional case the authors discuss conditions that allow peaceful coexistence of conventions when an agent has 4 neighbors (von Neumann neighborhood[3]). They relate the coexistence of different conventions to the dimensionality of the problem and to the degree of anonymity in the society. It is not clear whether the conclusion that anonymity yields possible coexistence of different conventions is justified, because they do not consider other possible two-dimensional types of neighborhoods.

[2] A convention (a strategy tuple, where all players choose the same strategy) is called evolutionary stable of degree k when, after k players simultaneously deviated from it, the convention will be played again after finitely many steps. This definition of evolutionary stability is from Berninghaus, S. K., and U. Schwalbe (1996b).

[3] For an agent on a lattice the von Neumann neighborhood is the set of players (4 for the two dimensional lattice), closest to her in metric distance.

Berninghaus and Schwalbe (1996a) discuss applications of discrete iterations analysis to games with local matching. The model is the same as in the previous article, but a different mathematical apparatus is used - analysis of discrete iterations, which allows more general types of interactions to be considered, because it requires less information about the network. Discrete iterations theory can be especially useful when a network cannot be represented as a threshold network. The authors discuss the theory of discrete iterations (mostly convergence issues) and then apply it to games with local interactions and best response players. Concepts of global and local convergence are discussed. It is shown that in a heterogeneous population (where some players play best response and others imitate the most popular strategy in their neighborhood) dynamics will not converge globally. For a local Nash equilibrium, which is also a fixed point of the dynamics, the authors describe conditions sufficient to ensure local convergence to that point.

Eshel et. al. (1996a) consider a population of players on a line who imitate their neighbors with probability proportional to the neighbor's payoff. The authors make a distinction between the set of agents a player is interacting with and the set of players whom they imitate, with the latter being called the propagation neighborhood. They consider a special type of dynamics which they call boundary preserving[4]. In the part related to the Prisoners' Dilemma Game they show that cooperation will have a frontier advantage[5] over non-cooperation if the propagation neighborhood is sufficiently larger that the interaction neighborhood. In this case "a player is likely to observe a large group of altruists who do better than groups of defectors and will

[4] "Boundary preserving" roughly means that no new clusters will be created.
[5] In a world where an infinite but countable population is divided into two groups: players on the "left" all play strategy X and players on the right all play strategy Y, authors say that strategy X has a frontier advantage over Y if the probability of a boundary player Y to switch to strategy X is higher than the analogous probability for a boundary player X.

therefore tend to imitate the altruists." The mechanism provided by this explanation seems to be quite different from the mechanism of survival of cooperation we observe in the present paper. What frontier advantage means for the future of cooperation in a dynamic process or in an equilibrium is not quite clear, as the authors point out "we do not know whether frontier advantage plays a role in a general situation, beginning from a state which has many, possibly an infinite number, of clusters."

Eshel et. al. (1996b) consider a problem very similar to the one in the current paper, i.e. how cooperation (they call it altruism) can survive in the presence of defectors. The authors find that the key for cooperation survival is imitation together with the local nature of public goods provided by a cooperator. The authors consider model where agents are placed on a circle. They derive results for two special cases: 1) everyone interacts with one immediate neighbor on the right and on the left or 2) same with two neighbors on the right and on the left. The authors consider deterministic dynamics as well as dynamics with random mutations. They characterize the absorbing states of deterministic dynamics and, in the case with one neighbor on the left and one on the right, provide conditions for which a Markov process, representing dynamics with random mutations, will have a limiting distribution that puts positive weight on the states where cooperation survives.

4. Automata Networks.

We see from the above articles that there is an essential difference between models where agents imitate and models where they play best responses. In a large population having no spatial structure, these two types of behavior can often be used interchangeably (and they are!), but if interactions have a local character then the resulting dynamics for those two types of behavior is dramatically different. The May and Nowak model provides an excellent example. With best response dynamics the only outcome after just one non-cooperator appears is full non-cooperation. In the case of imitation dynamics limit cycles appear for some spatial structures.

In this paper we investigate imitation dynamics for the Prisoners' Dilemma Game where at the beginning of each period, a player imitates the most successful of her neighbors or herself. It seems that automata networks is one of the most applicable mathematical tools to model this.

Let us first describe a general automata network that allows to formalize many types of local interaction. Consider a finite population of agents $I = \{1,..., N\}$ with each agent interacting only with her neighbors (for example, we could think of agents distributed over a lattice).

A *neighborhood system* $\Omega = \{N_i, i \in I\}$ is a collection of non-empty subsets of I such that:

1) $i \notin N_i, \forall i \in I$.

2) $i \in N_j \Leftrightarrow j \in N_i, \forall i, j \in I$.

We will call the set N_i the *neighborhood* of *i*.

Time is discrete. At the beginning of each period players are matched with their neighbors to play a symmetric two person game:

	σ_1	σ_2
σ_1	a,a	b,c
σ_2	c,b	d,d

We assume that only pure strategies are played. The set of pure strategies available for each player i is

$$\Sigma = \{\sigma_1, \sigma_2\} = \{0,1\}.$$

The state space for the whole system (which we will also call "the set of output vectors") is represented by

$$\Sigma^N = \{0,1\}^N.$$

We will denote the set of all possible input vectors as Z^N. An element of Z^N consists of N components, where each component is an input from one of N different sources.

In many cases input and output vectors belong to the same space (for example, if we look at a best response dynamics, where all the relevant information for an agent is the strategy profile of the neighbors), but they do not have to in general (for example, if an agent imitates others, she might need to know their payoffs).

An agent updates her state according to some deterministic or stochastic function, depending on the input vectors she receives.

At the beginning of each period t a player i chooses a pure strategy $s_i(t) \in \Sigma$, according to her strategy rule, specified by a function

$$f_i : Z^N \to \Sigma.$$

Due to the local character of interaction a function f_i will depend on at most k arguments with k $\leq N$. The largest k (on the set I) will be called the *connectivity* of the network.

The strategy profile for the whole population at each t, $s(t) = (s_1(t), \ldots, s_N(t))$ is updated according to the function F

$F: Z^N \rightarrow \Sigma^N$,

with F being a composition of individual f_i's.

The interaction structure of a network could be represented by a graph characterizing the connections between the automata, which motivates the following definition:

Definition: An automata network of order N is given by the pair $\Im = (G, \Phi)$, where

• $N \in \aleph$ is the number of players;

• G is a graph[6] characterizing the connections between the players and

• Φ is a set of functions from Z^n to Z^n, describing the set of possible transition functions of the network.

5. Prisoners' Dilemma Game.

Consider a Prisoners' Dilemma Game. Instead of a game matrix with four parameters, we will consider a matrix with only one parameter *b*. This will simplify notation, without sacrificing the generality of the results.

	σ_1	σ_2
σ_1	0,0	b,0
σ_2	0,b	1,1

For the game to be a Prisoners' Dilemma Game, $b > 1$ has to hold.

Additional notation.

- $n_i = \#N_i$.
- n_i^c # of cooperators in N_i ;
- $n_i^n = n_i - n_i^c$ # of non-cooperators in N_i ;
- $z_i(t)$ payoff to i at time t, $i \in$ I;

Let $v_{ij} = \begin{pmatrix} s_i \\ s_j \end{pmatrix}$. **(1)**

The total payoff of a player i will be the sum of her payoffs obtained from matches with all her neighbors:

$$z_i = \sum_{j \in N_i} v_{ij}^{'} P v_{ij}$$

where for the above Prisoners' Dilemma Game the matrix P can be represented as:

$$P = \begin{pmatrix} 0 & -b \\ 1 & b \end{pmatrix}.$$

Equivalently, the payoff of a player i can be represented as:

$$z_i = (s_i + (1 - s_i)b)n_i^c . \quad (2)$$

6. Imitation of the Locally Most Successful Strategy.

In the beginning of each period, each player imitates the most successful of her neighbors or herself. Therefore next period she will play:

$$s_i \in \arg\max_{s_j, j \in N_i \cup i} (z_j) . \quad (3)$$

[6] Under *G* we will mean the undirected graph, where a link (*i*,*j*) is in *G* if *i* and *j* are neighbors.

Definition: An equilibrium is a strategy profile $s^* = (s_i^*, \ldots, s_N^*)$ such that:

$$s_i^* = \arg\max_{s_j, j \in N_i \cup i} (z_j), \quad i \in I.$$

A “mixed” equilibrium will mean an equilibrium strategy profile where there are cooperators as well as non-cooperators in the population.

So now we have the f_i’s and F specified, but due to the non-linear nature of the model it is not an easy object to analyze, even when everything is deterministic.

Let us consider a few examples. First, we do not have global convergence, because full cooperation and full non-cooperation are stable states of the dynamics: if the initial point is one of them, the deterministic system will stay there. But this is not the only reason for non-convergence. Even if the population starts in a mixed state (some players cooperate and others do not), the dynamics need not converge to full non-cooperation. As computer simulations show (May and Nowak (1993)), random initial configurations (the number of cooperators should be positive, of course) often lead to a limit cycle, or apparent chaos, depending on the parameter *b*. Whether there are any other stable states depends on the parameters of the problem. For a one-dimensional network with players distributed over a circle, interacting with the two closest neighbors, we can easily see that there are many equilibria, provided that $b < 2$. For example, consider an allocation where all players cooperate. Assume now that one of them decides to switch. Consider the segment that includes this non-cooperator. We have:

configuration: ...1 1 1 1 1 0 1 1 1 1... (0 means non-cooperation)

payoffs: ...2 2 2 2 1 2b 1 2 2 2...

Therefore in the next period we will have:

configuration: ...1 1 1 1 0 0 0 1 1 1

payoffs: ...2 2 2 1 b 0 b 1 2 2,

which is an equilibrium under the specified imitation dynamics.

Any allocation with clusters of three non-cooperators, separated by a sufficient number of cooperative individuals clearly will be an equilibrium point of the dynamics.

Two observations about the above example:

1) In an equilibrium a non-cooperator with the highest payoff gets less than a cooperator with the highest payoff. Therefore in any equilibrium, non-cooperation seems to be a disadvantaged strategy.

2) An equilibrium has groups of non-cooperators, surrounded by larger groups of cooperators.

In the following two statements we extend the above observations to more general interaction structures.

Theorem 1: Consider a network of players, engaged in the Prisoners' Dilemma Game on an arbitrary graph *G*, representing a neighborhood system. In a mixed equilibrium (if it exists)[7] the highest payoff obtained by a non-cooperator does not exceed the highest payoff obtained by a cooperator.

Proof. Consider a mixed equilibrium. We can find among all the non-cooperators the ones who obtain the highest payoff. Let us call that payoff n^*. We will call the highest payoff among all the cooperators c^*. We have $n^* > 0$. (If, $n^* = 0$, then there is no non-cooperator who has a cooperator as a neighbor, therefore it is not a mixed equilibrium). Therefore the highest paid non-cooperator has at least one cooperator as a neighbor, who either has a payoff no smaller than n^* or has a cooperative neighbor with a payoff of at least n^*.

For the following discussion we will consider only situations where payoffs of a cooperator and a non-cooperator cannot be equal. For this the condition $b \notin B = \{\frac{k}{n}, k, n \in I\}$ is sufficient. Consider a neighboring cooperator and non-cooperator. The payoff of the cooperator is an integer, not exceeding *N*. The payoff of the non-cooperator is also an integer no more than *N*, multiplied by *b*. So they can be equal only if $b \in B$ (actually this condition could be confined to a set smaller than *B*). This is not a significant restriction, because the set *B* is of measure zero.

Corollary 1: If $b \notin B$ then in a mixed equilibrium the highest paid cooperator has no non-cooperators in her neighborhood (they would switch in this case). And therefore her payoff equals the size of her neighborhood.

Definition: We will call a population of agents a symmetric one-dimensional interaction structure, when agents are distributed over a circle, each having n/2 neighbors on the right and n/2 neighbors on the left.

Corollary 2: In a mixed equilibrium in the symmetric one-dimensional interaction structure, the payoff of a non-cooperator does not exceed n (the size of the neighborhood) and c* = n.

Theorem 2: Consider a network of players, engaged in the Prisoner's Dilemma Game on an arbitrary graph *G*, representing a neighborhood system. When *b* is small enough (but still more than 1) at least one of the following will hold:

1) There is a mixed equilibrium or a cycle.

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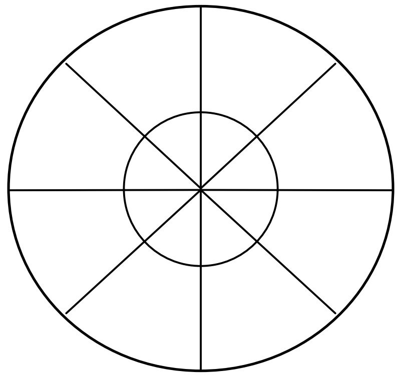
[7] Conditions on the existence of the equilibria will be discussed later.

2) There are states with mixed profile, from where imitation dynamics will lead to a fully cooperative state.

Proof: Let us find a player with maximal number of neighbors. Call her player 1 and let $n_{\max}$ be the size of her neighborhood. Assume that player 1 and all of her neighbors are playing the cooperative strategy and the rest of the population plays non-cooperation. By construction there is no player in the population, who has n cooperative neighbors, except for player 1. Therefore none of the cooperative players will want to change their strategy (provided b is small enough, but still strictly larger that 1). This construction may or may not be an equilibrium, because the non-cooperators might want to change their strategy. We can see that none of the neighbors of 1 will switch to non-cooperation, because the highest paid non-cooperator they can observe receives a payoff of (n-1)b, which is smaller than 1's payoff of n if b is sufficiently small. Therefore in a finite population the described process will reach one of the following: a mixed equilibrium, the fully cooperative equilibrium, or a cycle.

The conclusion, that full cooperation may be a locally attractive state does not seem to be very intuitive. I will illustrate it with an example. Consider the following figure:



Assume that players are located on intersections of the circle with radial lines and there is also a player in the center. Assume that players interact only with those with whom they are connected by radial lines and with their two immediate neighbors on the circle. Then if the player in the center and all his neighbors cooperate, and the rest does not, cooperation will spread to the entire population (*b* has to be strictly smaller than 8). Strangely enough, the fully cooperative equilibrium in the example is stable against mutation of any single player, except for the central one.

The following statement characterizes the set of equilibria for the one-dimensional interaction structure.

Theorem 3: Assume a symmetric one-dimensional interaction structure. If $b \notin$ B, then the condition $b \in \tilde{B} = \bigcup_{k=0}^{\frac{n}{2}-1} \left(2\frac{n-1}{n+2k}, 2\frac{n}{n+2k} \right)$ is necessary and sufficient (for *N* large enough)[8] to have a mixed equilibrium.

Proof.

Necessity. Assume there is a mixed equilibrium. Consider a cooperator, earning c* (the highest payoff among all agents). Call her C1. Consider a "straightened" fragment of a circle:

...1	1	1	1	1	1	1	0	0	0 ...
	C1		C2	C3			N1		
	n		n	n - 1					

Without sacrificing generality move to the right and find the last agent C2 such that C2 and all agents passed in this movement are cooperators earning c* (C1 and C2 may coincide). Moving to the right from C2 find the first non-cooperator (if such a player does not exist, this is not a mixed equilibrium constellation). Call her N1. This player is the first player outside the C2 neighborhood. Since c* = n, Theorem 1 and $b \notin B$ imply that N1 earns less than n. Call C3 the immediate neighbor of C2 to the right. C3 earns n - 1. The equilibrium condition and $b \notin$ B implies that one of the following two conditions has to hold:

1) N1 is getting strictly more than n-1, or

2) N1 is getting strictly less than n - 1, but has a non-cooperator in the neighborhood who is getting strictly more than n-1.

Consider 1) first. It means that N1 is a highest paid non-cooperator (as $kb > n$ - 1 (k is the number of cooperators N1 has in the neighborhood) means that $(k + 1)b > n$ for $b > 1$.)

There are two ways that N1 could be a highest paid non-cooperator.

i) N1 has no cooperators in her neighborhood on the right. Therefore b must be large enough, for N1 not to be tempted to follow the agent C3 with n-1:

$$\frac{n}{2}b > n-1 \Rightarrow b > 2\frac{n-1}{n}.$$

[8] Will be discussed later.

The b must also be small enough ($b < 2$) for payoff of N1 to be smaller than n.

ii) N1 has k > 0 cooperators in her neighborhood on the right. Then:

$\left(\frac{n}{2}+k\right)b < n \Rightarrow b < 2\frac{n}{n+2k}$. By assumption the payoff to N1 must also exceed n - 1:

$(\frac{n}{2}+k)b > n-1 \Rightarrow b > 2\frac{n-1}{n+2k}$. The number k cannot exceed $\frac{n}{2}-1$

Consider 2) next. N1 is getting less than n - 1, but has as a neighbor a non-cooperator who is getting more than n - 1. This means that there must be at least one cooperator between them, who will have payoff less than n - 1 (by construction) and who does not have a cooperator with payoff equal to n as a neighbor. Therefore this cooperator has to change her strategy. Contradiction.

We have shown that 2) can be ruled out, whereas 1) implies $b \in \tilde{B}$. Therefore $b \in \tilde{B}$ is a necessary condition for a mixed equilibrium.

Remark. Assuming that we have a mixed equilibrium we implicitly assume N to be large enough For otherwise there would no cooperator with payoff of n.

Sufficiency. Suppose $b \in \tilde{B}$, and $b \notin B$. Consider a non-cooperative player (N1) inside a fully cooperative cycle.

...1 1 1 1 1 0 1 1 1 1...

N1

Here we need to have N large enough (we will determine the minimum size of N below). Find the unique value of k as a function of b and n (from the formula for $\tilde{B}$). Add $\frac{n}{2}-k$ non-cooperative players to the right of N1. For k = 0 any number of non-cooperators will do, provided that we have a cluster of cooperators of at least size n + 1 (to have a cooperator with payoff of n). We can check that this construction is an equilibrium.

The following corollary is due to Dr. Hans Haller.

Corollary 1: The same assertion (and the proof as well) will hold for an infinite symmetric one-dimensional interaction structure.

We see that if k > 0, then there is a unique size for a cluster of non-cooperators in a mixed equilibrium, $\frac{n}{2} - k$ + 1. If k = 0, then the size of a cluster of cooperators can be arbitrarily large (but at least $\frac{n}{2} - 1$), provided that there is a cluster of at least n + 1 cooperators in the population. The equilibrium size of a cluster of non-cooperators grows together with *b*. From the above we see that the lower bound for the size of the population - if the equilibrium conditions are to be satisfied - is $n + 1 + \frac{n}{2} - k + 1 = \frac{3n - 2(k-2)}{2}$. In an equilibrium, a cluster of cooperators can be arbitrarily large (provided it fits into the population and there are some non-cooperators left).

Obviously, a mixed equilibrium is not globally stable (there is no cure if a cooperator deep inside the cluster of cooperators decides to deviate), but there are types of deviations for which it is stable. For example if a cooperator on the boundary of a cooperative cluster will deviate, in the next period her payoff will be less than n - 1 (she has one cooperative neighbor less), she will still have a cooperative neighbor with payoff of n - 1, therefore next period she will move back to cooperation.

From the example we see that in an equilibrium a kind of ranking of agents emerges: there are players (inside the cooperative area) who if they change their strategy will be followed by their neighbors, and there are players (on the boundary of the area) whose behavior does not induce anyone to change their strategy. This ability or inability to influence behavior of others can be interpreted as relative power of agents in society. Here the power of an agent is determined not by characteristics of the agent, but by the state of the world.

The condition $b \notin B$ is not necessary for the above theorem. We could replace it with a tie-breaking rule that a player changes her strategy only if the other strategy is strictly better. Then the conclusions of the Theorem 3 still hold.

When $b \notin B \bigcup \widetilde{B}$, no mixed equilibrium is possible, but for the same type of configuration (clusters of non-cooperators, surrounded by a larger number of cooperators), we can have a cycle instead. We also see that non-cooperation can be counterproductive and in a cycle a group of non-cooperators can shrink.

For example, consider a population where a player has two neighbors on the right and two on the left (we consider a one-dimensional problem, agents distributed over a circle) and $b < 1.5$. Then a cluster of cooperators (of size more than three) will grow, unless there is an isolated group of two (or one) non-cooperators on the boundary of it. It can be easily seen, that if we have a cluster of 4 cooperators in an otherwise non-cooperative population ($b < 1.5$), the non-cooperative population will shrink. The long-run outcome will be a cycle with at most 6 non-cooperators (provided the total number of players is sufficiently large (at least 11)). This discussion demonstrates

Theorem 4: Consider a population distributed over a circle with each player interacting with two neighbors on the left and two on the right. If $b < 1.5$ and $N > 11$, then the non-cooperative convention is evolutionary stable[9] of degree 3.

So it is enough just for four players to deviate from non-cooperation. Apparently we have nothing similar to this result in the case of best response players, where the non-cooperative convention is evolutionary stable of degree $N - 1$ (it is enough just for one player to deviate from the cooperative convention for the whole population to follow). Clearly, Proposition 1 can be extended to other interaction structures as well. We could observe that the degree of stability of the non-cooperative convention will grow together with the number of neighbors an agent has. Also the minimum required size of the population will grow. This is because how fast and how far non-cooperation will spread depends on the number of neighbors of a non-cooperator; for non-cooperation to be contained, the size of the cluster of cooperators must be no less than the total number of neighbors an agent has.

It seems that one of the main differences between imitation and best response in an automata network setting is that even if the number of neighbors is the same, the strategy choice of an imitator depends not only on players in her neighborhood (as for a best response player), but on the players outside of it as well. Therefore for imitation dynamics, interactions are not as localized as for best response dynamics. Most of the results derived in the computer science literature are for the threshold automata networks. The model above does not seem to allow such a representation. A slightly different model, in which an agent imitates the strategy that earns best on average payoff in her neighborhood, can be represented in a way similar to a threshold network.

[9] See the Footnote 1.

An interesting modification of the above model can be developed by assuming that player *i* is imitating the strategy that is most successful on average in her neighborhood. This model still possesses the most striking feature of the first one, i.e. for some interaction structures non-cooperation does not necessarily spread to the entire population. Some analysis of both models could be performed using the apparatus of discrete iterations, described for example in Berninghaus and Schwalbe (1996a). The second model can also be represented in a form similar to a threshold automata network.

7. Choice of Playing Partners.

In the following modification of the original model I want to account for the fact that in real life people have to some extent the ability to choose their partners. We assume the same model as before with only one difference -- agents now can choose whether to play with a particular neighbor or not. We will assume that in the graph G every directed link between two players *i* and *j* is assigned a number $w_{ij} \in [0,1]$ -- the strength of connection. Among possible interpretations we could talk about time or effort spent on interaction with the other player. We will assume that in every period every player chooses the strength of connection she wants to have. There are many possible ways for an agent to do that. As it turns out, even very simple rules can produce rich dynamics and fascinating spatial patterns. For the following discussion I will assume that players follow these simple decision rules:

1) everyone wants to play with a cooperator;

2) no one wants to play with a non-cooperator, unless the player is a non-cooperator herself.

Therefore w_{ij} can take on only two values - 0 (do not want to play) and 1(want to play).

The present model can be considered in the framework of learning in automata networks, though I am not aware of literature considering a similar model. General references on learning in the automata networks can be found in: Rojas (1996), Nadal (1994), Peretto (1992) and elsewhere.

When two players decide on the strength of connection between them they can easily disagree. Therefore we need a rule which will assign the strength of connection between the two players. I will assume that the effective strength of connection w_{ij}^* between any two players *i* and *j* will be the average of the strengths they have chosen.

$$w_{ij}^* = \frac{1}{2}(w_{ij} + w_{ji}) .$$

This assumption captures in a stylized form the fact that in the real world we are not always able to refuse to interact with a particular person, even if we wish to. Surprisingly enough, the above rules do not always benefit a cooperator, because there is a punishment for a non-cooperator who decided to cooperate. However, in equilibrium the rules will be beneficial for a cooperator, because a non-cooperator interacting with a cooperator will be getting only half of what she would have gotten otherwise.

We can easily see that in case of agents distributed over a circle, any configuration that was a mixed equilibrium in the first model will no longer be an equilibrium. If we start in one of them, the process will converge to full cooperation. As a matter of fact for $b \in \tilde{B} = \bigcup_{k=0}^{\frac{n}{2}-1}\left(2\frac{n-1}{n+2k}, 2\frac{n}{n+2k}\right)$ there will be no mixed equlibria at all. We can also immediately see that the cooperative equilibrium becomes a locally attractive point.

For $1.75 \leq b < 2$ we can observe a new interesting equilibrium pattern -- a single non-cooperator in a cluster of cooperators will not be willing to change her strategy, neither will her neighbors..

This model might be useful for analysis of coalition formation. If we allow the strength of connection chosen by an agent to assume more values than 0 and 1, we can consider as a coalition a set of players with relatively strong connections among each other.

8. Numerical Simulations.

Except for the one-dimensional case, an equilibrium allocation is not an easy prey for an analytical characterization. Dynamical behavior is even more difficult. Therefore it seems that numerical simulations may be helpful. I wrote a program that simulates the model with the choice of playing partners for the population of agents allocated on a lattice[10]. Every agent *i* has 8 neighbors, unless she is on a boundary where the number of agents is smaller. I used a game matrix of more general form than in the first model:

	σ_1	σ_2
σ_1	a,a	b,0
σ_2	0,b	1,1

All simulations discussed below have been run with the parameter a being slightly positive (usually 0.1). This eliminates two of the three Nash equilibria in the game matrix. The most important parameter is b -- the advantage of a non-cooperator. Depending on the value of b we can (only approximately) classify the dynamics. There are at least two ways to do that. One is qualitative - by inspecting the spatial and dynamical patterns generated for different values of b. Another is more quantitative - by looking at the behavior of some average value reached in series of simulations. These two methods give similar, but not identical results. Figure 1 shows average levels of cooperation which were reached after 100 simulations each for different values of b. The length of each simulation was 1000 periods. Each simulation started with a randomly selected original configuration with 50% cooperators (on average). As we can see from the above description, points on the plot will be affected not only by the configuration (or a cycle) to which a simulation converges, but also by the speed of convergence. This is why in spite of the fact that for values b < 1.5 every simulation converged to full cooperation, the average level of cooperation reached for every level of b was different. The boundaries of the following regions are extrapolated from both classifications and only approximately known.

Region 1. $b \in [1,1.5]$. In this region, beginning from almost any initial configuration (the proportion of cooperators can be quite low, even a few percent will do sometimes) deterministic dynamics converges to cooperation.

Region 2. $b \in (1.5,1.75)$. This region is interesting because cooperators still prevail over non-cooperators, but there is no longer fast convergence to full cooperation. Also a single non-cooperator in the middle of cooperators will switch to cooperation. The most fascinating feature of this region is that even if we start with a completely random initial configuration (as in Figure

[10] I have also run simulations for the same model with the population allocated on a torus. Results of it will be reported below.

5[11]), just in a few periods we will get a very organized structure (see Figure 6). The initial proportion of cooperators in Figure 6 was 95%. The overall pattern of the picture may vary greatly, but there is always one particular feature present -- moving “comets” (see Figure 6). The “comets” have a body and a tail and move as a single piece in the direction of the head. After a few periods of imitation (if a high enough proportion of cooperators was chosen originally) there are usually just a few “comets” left. At this moment it becomes clear that the fate of non-cooperators really depends on the initial random configuration. Sometimes comets will not hit each other. Then they will be destroyed by hitting the boundary. If a few comets intersect and hit each other, the result depends on the exact details of the collision. In many cases collision will destroy them, but sometimes, especially if more than two “comets” collided, a beautiful pattern may appear. Sometimes a whole starfleet can be generated by a collision (see the Figures 7 or 8). The pattern in Figures 7 and 8 was created by a slightly off line head to head collision of two “comets." After an initial period of apparent order, if non-cooperators manage to survive, there usually follow a dynamics that does not seem to follow any discernible laws (see Figure 9 and Figure 2). The same type of chaotic dynamics usually develops when one starts with a smaller initial proportion of cooperators. This region is really hostile to non-cooperators, even if we start with a large proportion of non-cooperators (50%) originally. Sometimes they are on the edge of extinction. In the simulations I have run, the historic average of the proportion of cooperators was somewhere in the neighborhood of 70%.

Region 3. $b \in [1.75, 2.3]$. This region is more favorable to non-cooperators than Region 2. Dynamics appears to possess even less order than before. Fluctuations in the proportion of

[11] In Figures 6-10 a cooperator is represented by a white square and a non-cooperator by a black square. All pictures are produces for a 100x100 lattice (10000 agents).

cooperators are very wide . If in Region 2 it appeared that non-cooperators were swimming in a sea of cooperation, here the situation seems reversed. Paradoxically, even though cooperators often appear to be on the verge of extinction, dynamics can converge to full cooperation. I observed it for the value b = 1.9 where it took at least 3000 periods. For b = 2.1 much faster convergence was observed (see Figures 3 and 4). In the region where b > 2, dynamics seem to be especially unpredictable. Fluctuations of the proportion of cooperators are especially big and dynamics can converge to either full cooperation or full non-cooperation in a relatively short time.

Both Regions 2 and 3 look like a jigsaw puzzle sometimes -- cooperators or non-cooperators can almost cover the entire area and then suddenly a few pieces in the puzzle are missing and they are starting all over again. Probably we can say that we observe deterministic chaos in Regions 2 and 3. The state space for the simulations actually is very large. The simulations were conducted on a 100 by 100 lattice. Therefore the total number of possible configurations is 2^{10000}. I think this makes my conclusion about chaos in those regions very plausible.

Region 4. *b* > 2.3. In this region cooperators have very little chance to survive. Usually one can observe very fast convergence to full non-cooperation.

One of the special features of this model is that there was no parameter region where cycling behavior (in the sense of the whole pattern or of the level of cooperation) would be unambiguously observed in the simulations. This is in sharp contrast with the Nowak and May results.

One may argue that the absence of cycles[12] in the simulations is due to the presence of the boundary. This does not seem to be the case. I have run simulations for the same model for agents distributed over a torus, where no cycles were observed for as long as 5,000 rounds of simulations.

The only model where rather sharp and unusual convergence to an apparent cycle was observed is a variation of the same model as the above, with the additional condition that an agent also interacts with herself, without fully realizing that this is a self-interaction. In this case not only abrupt convergence to a cycle was observed (see Figure 11), but also all large scale movements (like “comets”) were totally absent.

So far all the observations were made in the case of no mutations. Mutations can change the picture in some circumstances. For example in Region 3, the introduction of relatively large mutations (1% probability that a given player will adopt a strategy other than the one prescribed by imitation) will greatly reduce the chances of cooperators. However, mutations of .1% produce no visible change in patterns.

From the above discussion we can see a strong resemblance of the observed dynamics to those of the Game of Life, which was invented in 1970; see Convey et al. (1982). The following description of the Game of Life will follow the one in Robert (1987). In the Game of Life there is a population of interacting cells that ‘live’ on a two-dimensional grid. A cell can be ‘dead’ or ‘alive’. The cells update their state simultaneously according to the following rules (that depend only on 8 closest neighbors of the cell), chosen so as to ‘mimic life’:

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[12] I mean cycles of observable length. Strictly speaking we will always have cycles due to the finite nature of the system.

1) a cell can survive for the next period if and only if it has exactly 2 or 3 living cells in its neighborhood.

2) a cell will become alive in the next period if and only if it has exactly 3 living cells in its neighborhood.

There are three types of global behavior observed in the Game of Life:

- fixed points
- limit cycles and
- “traveling” configurations.

The most striking similarity is the large scale movements observed in both models. The “comets” in the present paper resemble “gliders” in the game of life, though there is a difference: “comets” travel as a whole, while “gliders” keep getting recreated (translated 1 cell diagonally) after 4 periods. In both cases the observed large scale movements are created by memories about the state of the neighborhood in the previous period.

9. Conclusions.

In the current paper, the equilibria for a population engaged in the Prisoners’ Dilemma Game have been characterized and necessary and sufficient conditions for the existence of mixed equilibria on a one-dimensional interaction structure have been derived. It has been shown that the greater the advantage of non-cooperators, the bigger is the size of an equilibrium cluster of non-cooperators. It has also been shown for the general case that in a mixed equilibrium a cooperator with the highest payoff is at least as good as any non-cooperator. Sufficient conditions for existence of a mixed equilibrium have been derived, too.

A model with agents choosing whom to play with has been considered and compared to the original model. Results of computer simulations are discussed. The most striking feature of the model is creation of very orderly pattern ("comets" for example) from a completely random initial configuration under deterministic dynamics.

It seems very promising to study coalition formation within our framework. For example, if we allow the strength of connection between agents to assume more values than 0 and 1, we can consider as a coalition a set of players with relatively strong connections among each other.

Another promising direction for future research is to incorporate physiology into game theory, to understand how people make decisions. For example; let us consider two groups of players. In one of them take a player (we will call him player 1) who gets more than anyone else and compare him with a player in the other group who gets even more than player 1, but still less than other players in his group. Who will feel happier and more content with his current situation? As Sen (1983) says: "A grumbling rich man may feel less happy than a content peasant, but he does have a higher standard of living." In a similar view, Akerlof et. al. (1991) consider worker preferences that depend on wage (esp. income), but also on the deviation from average wage (income) in the population or neighborhood. Answering this type of questions will help to understand what type of behavioral assumptions (best response, imitation or maybe something else) should be used in game theory. From an economic point of view, this may also allow us to address questions of migration and social mobility.

The deterministic dynamics explored in the paper exhibit multiplicity of equilibria and cycles. Under reasonable assumptions, a random Markov process on the same system may yield a unique stationary distribution. This suggests a further direction of future research which could follow the lead of Peretto et al. (1992), considering ensembles of networks for the statistical description of the system.

REFERENCES

Akerlof, George A., Janet L Yellen (eds.) (1986): *Efficiency Wage Models of the Labor Market*. New York: Cambridge University Press.

Axelrod, R., and Hamilton, W. D. (1981): "The Evolution of Cooperation," *Science*, 211, 1390-96.

Berninghaus, S. K., and U. Schwalbe (1996a): "Evolution, Interaction, and Nash equilibria," *Journal of Economic Behavior and Organization*, 29, 57-85.

Berninghaus, S. K., and U. Schwalbe (1996b): "Conventions, Local Interaction, and Automata Networks," *Journal of Evolutionary Economics*, 6, 297-312.

Boyd, R., and P. J. Richerson (1990): "Group Selection among Alternative Evolutionary Stable Strategies," *Journal of Theoretical Biology,* 145, 331-42.

Canals, J., and F. Vega-Redondo (1992): "Multilevel Evolution in Games," working paper.

Conway J. H., E. R. Berlekamp, and R. K. Guy (1982): *Winning Ways for your Mathematical Plays*. New York: Academic Press.

Ellison, E. (1993): "Learning, Local Interaction, and Coordination," *Econometrica*, 61, 1047-71.

Eshel, Illan, Emilia Sansone, and Avner Shaked (1996a): *Evolutionary Dynamics of Populations with a Local Interaction Structure, mimeo.*

Eshel, Illan, Larry Samuelson, and Avner Shaked (1996b): *Altruists, Egoists and Hooligans in a Local Interaction Model,* mimeo.

Gilles Robert P., Hans H. Haller, and Pieter H.M. Ruys (1994) "Modelling of Economies with Relational Constraints on Coalition Formation" in *Imperfections and Behavior in Economic Organizations*, ed. by Robert P. Gilles and Pieter H.M. Ruys. Boston: Kluwer Academic Publishers.

Hamilton, W. D. (1964): "The Generical Evolution of Social Behavior," *J. Theory Biol.* 1-52.

Kandory, M, G. Mailath and R. Rob (1991): "Learning, Mutation, and Long Run Equilibria in Games," *Econometrica*, 61, 29-56.

May, R. M. (1994): "Spatial Chaos and its Role in Ecology and Evolution" in *Frontiers in Mathematical Biology,* ed. by Simon A. Levin. New York: Springer-Verlag.

Nadal, Jean-Pierre (1994): "Formal Neural Networks: From Supervised to Unsupervised Learning" in *Cellular Automata, Dynamical Systems and Neural Networks,* ed. by Goles, Eric, and Servet Martinez. Dordrecht: Kluwer.

Nowak, Martin A., and Robert M. May (1993): "The Spatial Dilemmas of Evolution," *International Journal of Bifurcation and Chaos,* 3, 35-78.

Nowak, Martin A., Sebastian Bonhoeffer, and Robert M. May (1994): "More Spatial Games," *International Journal of Bifurcation and Chaos,* 4, 33-56.

Peretto, P, M. Gordon, and M. Rodriguez-Girones (1992): "A Brief Account of Statistical Theories of Learning and Generalization in Neural Networks" in *Statistical Physics, Automata Networks and Dynamical Systems,* ed. by Eric Goles and Servet Martinez. Dordecht: Kluwer.

Rhode, Paul, and Mark Stegeman (1997): "Darwinian Dynamics in Finite populations," manuscript.

Robert, F. (1987): "An Introduction to Discrete Iterations," in *Automata Networks in Computer Science*, ed. by Fogelman Soulie, F., Yves Robert, and Maurice Tchuente. Princeton: Princeton University Press.
Rojas R. (1996): *Neural Networks: A systematic Introduction*. Berlin: Springer-Verlag.
Sen, Amartya (1983): "Poor, Relatively Speaking," *Oxford Economic Papers*, 153-169.
Vega-Redondo, F. (1993): "Competition and Culture in the Evolution of Economic Behavior: A Simple Example," *Games and Economic Behavior,* 5, 618-31.
Vega-Redondo, F. (1995): "Long-Run Cooperation in One-Shot Prisoners Dilemma: A Hierarchical Approach," mimeo.
Whittle, P. (1982): "Criticality and the Emergence of Structure," in *Evolution of Order and Chaos*, ed. by H. Haken. New York: Springer-Verlag.
Wynne-Edwards (1962): *Animal Dispersion in Relation to Social Behavior.* Edinburgh: Oliver & Boyd.