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Theory of high-temperature superconductivity and effective gravity

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We argue that an approach involving effective gravity could play a crucial role in elucidating the properties of the high-temperature superconducting materials. In particular we propose that the high critical temperature might be naturally explained in a framework constructed as a direct condensed-matter analog of the Randall-Sundrum approach to a geometrization of the hierarchy problem in high-energy physics.

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I. INTRODUCTION

The problem of high-temperature superconductivity is one of the most outstanding puzzles in contemporary physics.[1] Recently much activity has been devoted to the possible relevance of duality between gauge theory and gravity (i.e., the anti-de Sitter space or conformal field theory correspondence) to this remarkable problem.[2] The general concept of duality, on which a lot of recent progress in quantum field theory and string theory hinges, has also been used recently in various condensed-matter settings.[3,4] Furthermore, there has been a lot of work regarding the induced gravitational processes in various condensed-matter systems (see, for example, Refs. 5 and 6 for some recent samplings of the literature).

In this paper we observe that if indeed effective gravity could arise in condensed-matter systems then there are possible dramatic effects associated with its geometric physics. For example, widely separated energy scales can be related through a warping (redshift) factor. This possibility has been widely discussed in the recent high-energy literature. In particular, in the context of the Randall-Sundrum scenario,[7] a geometric warping factor is used to naturally bump up the 1 TeV scale to the Planck scale, thus evading the hierarchy problem.

The main aim of this paper is to point out a close analogy between the naturalness problem associated with the Higgs mass from the realm of particle physics and the naturalness problem associated with the high critical temperature (T_C) of high-temperature superconductors. In view of the recent geometric proposals for dealing with the particle physics naturalness problem, i.e., the Randall-Sundrum scenarios (which indeed have not been tested experimentally and are, at the moment, only theoretical suggestions), we point out that a similar geometric mechanism can be naturally used in a condensed-matter context. We propose a condensed-matter analog of the Randall-Sundrum scenario for a geometrically induced hierarchy of scales. We argue that in the context of superconductivity, such a scenario can lead to the high T_C of cuprate (Cu-O) and oxypnictide (Fe-As) superconductors, both representing families of layered anisotropic compounds with complex normal-state Fermi surfaces. Our discussion is based entirely on effective-field theory (i.e., a gravitationally dressed Landau-Ginsburg approach) and does not address deeper issues concerning the microscopic mechanism of high T_C superconductivity. Nevertheless, the effective description may point toward previously unexplored avenues for understanding the underlying microscopic physics. Because our approach concentrates on the subsection of the phase diagram containing the strange metal/superconducting phase, we ask how the usual canonical Wilsonian thinking associated with naturalness of Fermi liquids should be modified for a geometric scenario akin to Randall-Sundrum to be realized in an effective Landau-Ginsburg description of the superconducting phase. Thus our proposal can be summarized as (a) a description of high T_C superconductors based on a gravitationally dressed version of the canonical Landau-Ginsburg Lagrangian (and the gravitationally dressed Fermi liquid as its normal phase) presented in Sec. II and (b) a specific proposal for the geometry of this induced gravitational background, which is appropriate for the anisotropic geometric properties of the known high T_C materials presented in Secs. III and IV. The main result of the paper is presented in Sec. IV. The specific geometry, as in the Randall-Sundrum scenario, uses a slice of anti-de Sitter space (AdS) and that is the gravitational metric that appears in the gravitationally dressed Landau-Ginsburg Lagrangian for the high-temperature superconductors.

Note that even though a comparison can be drawn with the approach used for the currently pursued AdS superconductors[2] (which indeed build on an AdS black hole, more specifically the Abelian Higgs model in that background) we do not rely on the same gravitational description. Our background, like the original Randall-Sundrum description, represents a slice of the AdS space. A dual description of our proposal can be in principle envisioned and it is natural to conclude that this would involve a nonrelativistic AdS or conformal field theory correspondence.[4]

II. FERMI LIQUIDS VS NON-FERMI LIQUIDS

As noticed by many authors, various condensed-matter systems, such as superfluids or Bose-Einstein condensates, exhibit physical phenomena that can be interpreted by invoking effective gravity.[5] In these contexts, which can be viewed as analog models of gravity, one speaks, for example, of an effective acoustic metric, acoustic black holes (i.e., "dumb holes"), or emergent relativity.

Given the intricacies of the physics of high-temperature superconducting materials and the complexity of their phase diagram, one might ask whether the strong electron correlations responsible for these phenomena can induce effective gravitational effects, thus opening a possibility for a geometric explanation of some of the outstanding puzzles.

We concentrate on one such puzzle: the actual high value of the critical temperature. In particular, given the fact that in recent years high energy physicists have invoked gravity to explain the unnatural mass scale hierarchy, we propose a condensed-matter analog of the same mechanism. In the context of high-temperature superconductivity we argue for a geometric reinterpretation of the hierarchy that exists between the ordinary critical temperatures for low-temperature superconductors and their high-temperature relatives. This geometric reinterpretation may point to a new effective gravitational physics which may be responsible, among other things, for the high critical temperature of the high-temperature superconducting materials.

A geometrically induced hierarchy of scales forces a revisiting of the Wilsonian approach to the Fermi surface to ascertain consequences for Fermi-liquid theory in the renormalization-group approach and for the emergence of non-Fermi-liquid behavior. From the effective-field theory standpoint, the question can be formulated: how does the low-energy Wilsonian action with effective gravity modify the usual scalings of the effective low-energy field theory of the Fermi surface?

A. Fermi liquid

To set the stage for our discussion let us summarize the classic effective-field theory of Landau Fermi liquids as most succinctly presented by Polchinski.[8] One starts with a *natural* (i.e., not finely tuned) Fermi surface and decomposes the momenta into the Fermi momentum and a component orthogonal to the Fermi surface

$$\vec{p} = \vec{k} + \vec{l} \tag{1}$$

and then one considers scaling of energy and momentum toward the Fermi surface. In other words

$$E \to sE, \quad \vec{k} \to \vec{k}, \quad \vec{l} \to s\vec{l}. \tag{2}$$

The lowest-order action, to quadratic order, is then given as

$$S_{\mathrm{FL}} = \int L_{\mathrm{FL}}(\psi, \partial\psi) \equiv \int dt d^3\vec{p}\{i\psi^*(\vec{p})\partial_t\psi(\vec{p}) - [E(\vec{p}) - E_F(\vec{p})]\psi^*(\vec{p})\psi(\vec{p})\}. \tag{3}$$

Close to the Fermi surface

$$E(\vec{p}) - E_F(\vec{p}) \sim l v_F, \quad v_F = \partial_{\vec{p}} E, \tag{4}$$

so that after the agreed renormalization orthogonal to the Fermi surface (note that also $t \to s^{-1}t$) one obtains, in both $2+1$ and $3+1$ space-time dimensions,

$$\psi \to s^{-1/2}\psi. \tag{5}$$

Note that this scaling leads immediately to the usual two-point function for a free quasiparticle with a single-particle pole. Now, by considering four-Fermi interactions one can see that for generic momenta the four-Fermi interaction scales as a *positive* power of s and is thus irrelevant at low energy. The measure over time contributes one negative power, the measure over the momenta orthogonal to the Fermi surface contributes four powers, and the four-Fermi interaction contributes 4/2 negative powers. The delta function over the four momenta generically does not scale. So the overall generic scaling for the four-Fermi vertex is indeed

$$s^{-1+4-4/2} = s^1. \tag{6}$$

This is valid, except if the momenta are *paired*. In that case the scaling goes as s^0 because now the delta function depends only on the sum of momenta orthogonal to the Fermi surface and due to

$$\delta(sl) \to s^{-1}\delta(l), \tag{7}$$

the four-Fermi interaction indeed scales as s^0 and is marginal. This encapsulates the usual Cooper pairing phenomenon. Note that this scaling is true both in $2+1$ and $3+1$ dimensions, with the $2+1$ dimensional case of interest for the layered anisotropic compounds. The case of $1+1$ dimensions is special because then the Fermi surface consists of two points and hence the four-Fermi interaction is automatically marginally relevant. A one-loop calculation of the beta function does ensure the asymptotic freedom provided the interaction is perturbatively attractive, as is the case for phonon-electron interaction, which leads to the strong-coupling regime (with bound states forming) in the infrared. The resulting wave function describing the superconducting state is of a BCS kind

$$\Psi_{\mathrm{BCS}} \sim \prod_k (\alpha + \beta a_k^* a_{-k}^*)\Psi_{\mathrm{FL}}, \tag{8}$$

where α and β are the usual variational parameters and a^* denote the creation operators of the electron quasiparticle.

B. Fermi liquid coupled to effective gravity

In what follows we outline an intuitive argument for the crucial relevance of an effective gravitational description of the normal state of the high T_C materials. How could effective gravity arise from the dynamics of the Fermi surface? First, we note that experiments indicate a highly irregular Fermi surface in the normal state, resulting from the microscopic physics.[1] This irregularity in turn could lead to an effective gravitational description. As a concrete model, let us view the Fermi surface as an incompressible fluid in momentum space. Then in analogy with the discussion of induced gravity in fluid dynamics[5,9] we can envision generating an effective metric, which in the case of real irrotational fluids is precisely the acoustic metric.[9] This effective metric is generated from the fluctuations of the fluid density ρ and the velocity potential ϕ (where the velocity $\vec{v} = \nabla\phi$). The underlying space-time action of the moving fluid is

$$S = \int dx^4 \left[\rho\dot{\phi} + \frac{1}{2}\rho(\nabla\phi)^2 + U(\rho) \right], \qquad (9)$$

with $U(\rho)$ denoting the effective potential. Note that the signs in S are consistently defined so that upon the variation of this action one obtains the equations of motion for ρ and ϕ (the Euler continuity equation and the Bernoulli energy balance equation).[5] Note also that the pressure is the negative of the action density in the expression for S. When these equations of motion are perturbed around the equilibrium values ρ_0 and ϕ_0 one is led to the equations for the fluctuations of the velocity potential φ. In particular, the equation for the fluctuations of the velocity potential can be written in a geometric form[9]

$$\frac{1}{\sqrt{-g}}\partial_a(\sqrt{-g}g^{ab}\partial_b\varphi) = 0. \qquad (10)$$

The effective space-time metric has the canonical Arnowitt-Deser-Misner form of Refs. 5 and 9 (apart from a conformal factor) and has the Lorentzian signature

$$ds^2 = \frac{\rho_0}{c}[c^2dt^2 - \delta_{ij}(dx^i - v^i dt)(dx^j - v^j dt)], \qquad (11)$$

where c is the sound velocity (or a plasmon velocity for a system of charged fermions) and v^i are the components of the fluid's velocity vector.

Note that the collective dynamics of the Fermi surface can be considered as a "bosonization" of the Fermi liquid[10] and in that approach the quasiparticle excitations can be represented by a collective mode of the "bosonized" Fermi liquid. That approach runs into trouble with the essential difference between fermions and bosons in spatial dimensions 1 (where the bosonization can be used because of the very special kinematics) and spatial dimensions above 1, where such efforts are largely prohibitive.[10] In contrast, in our proposal the non-Fermi-liquid behavior originates from the gravitational dressing, itself caused by the nontrivial geometry and topology of the Fermi surface: an experimental fact. Thus we argue that these quasiparticles are the usual fermions, but propagating in a nontrivial gravitational background and hence dressed by the gravitational fluctuations. In other words, the collective motion of the Fermi surface is of an effective gravitational kind (i.e., not spin 0 but spin 2) and the usual fermionic quasiparticles are now coupled to this collective spin 2 bosonic mode. Thus we propose that the effective theory of the strange normal state of high T_C superconductors is a gravitationally dressed Fermi liquid, i.e., the usual Fermi liquid albeit coupled to gravity

$$S_{\mathrm{NFL}} = \int d^D x\sqrt{-g}L_{\mathrm{FL}}(\psi,\nabla\psi). \qquad (12)$$

This general discussion of the gravitationally dressed Fermi liquid applies to both dimensionalities $D=2+1$ and $D=3+1$. For the normal state of the Cu-O or Fe-As planes, $D=2+1$ is relevant.

Applying this approach to the Fermi surface in momentum space, by relying on the induced effective diffeomorphism invariance, we are led to conclude that the effective action for the fermionic quasiparticle around the Fermi surface should be a gravitationally dressed action, in which the naive scaling dimensions discussed above can be changed by adding gravitational dressing. In general this would mean that instead of $\psi \rightarrow s^{-1/2}\psi$ as discussed above, we should have

$$\psi \rightarrow s^{-1/2+\alpha}\psi, \qquad (13)$$

where α denotes gravitational dressing. Such dressing can be explicitly computed in simple cases such as the coupling of two-dimensional gravity to two-dimensional matter.[11] Thus the fermionic two-point function can be changed to scale as $s^{-1+2\alpha}$, indicative of a non-Fermi-liquid behavior.

Naturally, the anomalous dimensions of Fermi propagators have been considered previously, in the marginal Fermi-liquid theory, through couplings with an effective induced gauge field, in the context of quantum critical fixed points, and other proposals.[1,12,13] Our proposal is not unique in placing emphasis on the anomalous fermionic propagators. Also, the phenomenology implied by anomalous fermionic propagators, including non-Fermi-liquid behavior such as a resistivity linear in temperature T,[12] is inherent to our proposal as well.

In this scenario, the normal state is described by a gravitationally dressed Fermi-liquid theory. Similarly, the BCS wave function would be gravitationally dressed. Note that our discussion is couched in terms of an effective-field theory and therefore does not carry information about the underlying microscopic mechanism. In particular, the specific nature of the pairing mechanism and its relationship with the effective gravitational description is outside the scope of the present approach. The approach generally does conclude that the usual effective Landau-Ginsburg theory describing the physics of the condensate is gravitationally dressed as follows ($D=2+1$ or $D=3+1$):

$$S_{\mathrm{LG}_g} = \int d^D x\sqrt{-g}[g^{\mu\nu}\nabla_\mu H^* \nabla_\nu H - V(|H|^2)], \qquad (14)$$

where the complex order parameter is denoted by H and $V(|H|^2)$ is an effective potential (for example, of a quartic type). In what follows, most important is that such a gravitationally dressed Landau-Ginsburg theory leads to a simple geometric mechanism for the high T_C of high-temperature superconductors.

III. EFFECTIVE GRAVITY AND ENERGY SCALE HIERARCHY

To implement a condensed-matter analog of the gravitationally induced energy scale hierarchy, we first present a brief summary of the Randall-Sundrum proposal[7] which precisely accomplishes this goal. The exponential hierarchy is induced by a warped extra dimensional geometry. The actual example envisions two three-branes (i.e., two 3+1 dimensional worlds), separated by a distance of r_c in one extra (fifth) spatial dimension. This *bulk* 4+1 dimensional space represents a portion of the 4+1 dimensional AdS space, whose cosmological constant is determined by another parameter of the model k. One of the branes carries the degrees of freedom of the standard model. The other carries the Planck scale degrees of freedom.

This setup can be carried over from dimensionality $D=3+1$ to $D=2+1$. In the layered anisotropic compounds, the Cu-O or Fe-As planes can be identified with $2+1$ dimensional branes and $D=2+1$ hence is relevant. The extra anisotropic dimension is to be identified with the actual anisotropy of these materials. The crucial assumption in this setup is that effective gravity is induced, perhaps by the internal dynamics of geometrically very complicated Fermi surfaces. The warped geometry of the extra dimension could originate from the interlayer coupling. We address the question of how the effective gravitational description could be induced from the dynamics of the Fermi surface in Sec. IV.

Concerning the Randall-Sundrum scenario in the anisotropic compounds, the $2+1$ dimensional branes separated in an extra anisotropic spatial dimension lead to the following *bulk* $3+1$ dimensional metric:

$$ds^2 = r_c^2 du^2 + e^{-2kr_c u}\eta_{\mu\nu}dx^\mu dx^\nu. \qquad (15)$$

Here u denotes the extra anisotropic spatial direction of size l_c and $\mu,\nu=0,1,2$ are the $2+1$ dimensional (planar) spacetime indices. Also, $\eta_{\mu\nu}$ is the planar flat (Minkowski) spacetime metric. The gravitational fluctuations $h_{\mu\nu}$ around the flat Minkowski metric define the background metric $g^0_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$.[7] For $D=2+1$, the low-energy effective action for a Higgs field (order parameter), i.e., the Landau-Ginsburg action coupled to effective gravity, is

$$S_{\mathrm{LG}_g} = \int d^{2+1}x\sqrt{-g}[g^{\mu\nu}\nabla_\mu H^*\nabla_\nu H - \lambda(|H|^2 - v_0^2)^2], \qquad (16)$$

where $g_{\mu\nu}=e^{-2kr_c l_c}g^0_{\mu\nu}$ (Ref. 7) and g is the determinant of $g_{\mu\nu}$. After rescaling the Higgs field (the Landau-Ginsburg order parameter) $H\rightarrow e^{kr_c l_c}H$ the effective action for $D=2+1$ becomes

$$S_{\mathrm{LG}_g} = \int d^{2+1}x\sqrt{-g^0}[g^{0\mu\nu}\nabla_\mu H^*\nabla_\nu H - \lambda(|H|^2 - e^{-2kr_c l_c}v_0^2)^2]. \qquad (17)$$

Note that we can take $l_c=\pi$ as in Ref. 7. Thus the physical mass scale is given by a blueshifted symmetry-breaking scale (where v_0 is the minimum of the potential)

$$v = v_0 e^{-kr_c\pi}, \qquad (18)$$

or alternatively, the relevant mass scales are related as

$$m = m_0 e^{-kr_c\pi}. \qquad (19)$$

This is the key result. Thus in order to obtain $m_0 \sim 10^{19}$ GeV and $m\sim 1$ TeV one only needs to require $kr_c \sim 50$. Note that even though this result has been derived in an effective-field theory, there exists an ultraviolet (UV) completion of this scheme from the point of view of string theory.[14] In particular, the complex issue of stability of the Randall-Sundrum scenario requires a UV complete description.

The question arises why the Randall-Sundrum warp factor should occur in the interlayer coupling. We notice that the physics of interlayer tunneling considered in the past[13] naturally involves exponential wave functions and thus can be directly compared to the geometric exponential factors that appear in the Randall-Sundrum case. Thus the proposed geometrization captures the relevant interlayer physics, at least on the level of an effective, and not microscopic, description. Here we emphasize the weak nature of the interlayer coupling as opposed to the strong intralayer coupling; the latter leading to non-Fermi-liquid behavior and induced gravitational dressing as argued in Sec. II. Finally we remark that in principle the geometric argument applies to both cuprates and oxypnictides and indeed stresses the very features they bear in common.

IV. WHY HIGH T_c?

Here we present a simple computation of T_C based on the gravitationally dressed Landau-Ginsburg effective description and the geometric argument presented above. This constitutes the main result of this paper.

Remembering the usual BCS dispersion relation given a gap Δ,

$$E = \sqrt{\Delta^2 + k^2}, \qquad (20)$$

in which the gap plays the role of the Higgs mass (given the well-known analogy with the relativistic dispersion relation for a particle of mass m, i.e., $E=\sqrt{m^2+k^2}$), we see that there exists a distinct possibility of similarly warping the gap, as in Eq. (5), and thus warping the T_C.

Normally T_C is related to the gap evaluated at low momenta[15]

$$T_C = \mathrm{const}\,\Delta(0). \qquad (21)$$

In the superconductors considered, one obtains a geometrically shifted temperature so that the following relation results:

$$T_{\mathrm{HT}_c} = T_C e^{kr_c\pi}. \qquad (22)$$

Here T_C denotes the low critical temperature of conventional superconductors—typically a few K. T_{HT_c} denotes the critical temperature of high T_C superconductors—equal to around 50–150 K.

Therefore, a high T_C naturally arises from the above exponential factor that is responsible for a hierarchy of energy scales. An exponential factor multiplying the usual 1–10 K critical temperature (determined by the value of the BCS-type gap at zero momenta) produces a new critical temperature in the 50–150 K range or higher. The representative samples of, from top to bottom, cuprate, oxypnictide (Fe-As),[16] BCS type II, and BCS type I superconducting critical temperatures are collected in Table I. For example, transitioning from the lowest (Al) to the highest (Tl compound) requires only an argument of 5 in the exponent.

V. CONCLUSIONS

In this paper we have proposed a condensed-matter analog of the Randall-Sundrum scenario for a geometrically in-

TABLE I. Representative samples of cuprate, oxypnictide (Fe-As), BCS type II, and BCS type I superconducting critical temperatures.

Compound	T_C (K)
$Tl_2Ba_2Ca_2Cu_3O_{10}$	125
$Bi_2Sr_2Ca_2Cu_3O_{10}$	110
$YBa_2Cu_3O_7$	92
$La_{2-x}Sr_xCuO_4$	38
$SmFeAsO_{0.85}F_{0.15}$	42
$CeFeAsO_{0.84}F_{0.16}$	41
$LaFeAsO_{0.89}F_{0.11}$	26
Nb_3Sn	18
NbTi	10
Pb	7.2
Al	1.1

duced hierarchy of scales. We have argued that in the context of the physics of high-temperature superconductivity, such a scenario can be responsible for the high T_C of cuprate and oxypnictide superconductors. The conjecture spontaneously leads to emphasis on the unusual features shared by cuprates and oxypnictides. Needless to say we have only scratched the surface of the manifold of issues relevant for high T_C superconductivity. Of primary importance remains a specific microscopic mechanism as well as a more detailed phenomenology derived from the gravitationally dressed Landau-Ginsburg approach. Apart from advancing understanding of the unusual normal as well as the superconducting phase, this approach can illuminate other parts of the phase diagrams of high-temperature superconductors. For instance, of future interest are the relevance of spin-spin interaction[17] for the effective gravitational description proposed in this paper, a study whether the d-wave nature of the order parameter[1] naturally follows from a gravitationally dressed Landau-Ginsburg description, an understanding of the pseudogap region of the phase diagram, and finally connecting the recent discussion of Mottness[18] to our context. These and other questions are left for more detailed future work.

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