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The Case eigenfunction expansion for a conservative medium

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By using the resolvent integration technique introduced by Larsen and Habetler, the one-speed, isotropic scattering, neutron transport equation is treated in the infinite and semi-infinite media. It is seen that the results previously obtained by Case's "singular eigenfunction" approach are in agreement with those obtained by resolvent integration.

I. INTRODUCTION

The linear transport equation with $c=1$ was treated by Shure and Natelson,[1] who used the Case singular eigenfunction approach.[2] Larsen and Habetler[3] later rederived Case's formulas using a contour integration technique which was not subject to some of the criticism which had been levelled at Case's approach through the years, mainly that the derivations were in fact only heuristic arguments. However, Larsen and Habetler were unable to treat the conservative case, $c=1$, but claimed (Ref. 3, p. 536) that the results for that specific case could be obtained by taking the limit $c\to 1$ in their derivation for $c\neq 1$. This contention has recently been attacked by Kaper,[4] and since Kaper's remarks seem to have merit, we present here the explicit analysis, along the lines developed in Ref. 3, for the case $c=1$. This case, incidentally, which corresponds to a critical half-space in neutron transport theory, has more physical significance in the context of radiative transport theory in stellar atmospheres, where it corresponds to a gray, conservative atmosphere in local thermodynamic equilbrium. (See Ref. 2, Sec. 10.5.)

An alternative to the Larsen–Habetler analysis was independently developed by Hangelbroek,[5] who proved that for $c<1$ the transport operator was similar to a self-adjoint operator, and so was able to apply von Neumann spectral theory. Lekkerkerker[6] has extended Hangelbroek's work to the case $c=1$ by defining a suitable subspace of the original Hilbert space, on which the transport operator is similar to a self-adjoint operator, obtaining a spectral theorem for the restriction of the transport operator to this subspace, and finally extending the results to the full space.

Our technique, following Larsen–Habetler, was inspired by Lekkerkerker. Specifically, the Larsen–Habetler technique fails for $c=1$ because the transport operator, K^{-1} in their notation, is not invertible. However, a suitable restriction of K^{-1} is invertible, and the entire Larsen–Habetler method of analysis can be carried out for this restriction. The extension of the results to the full space is then almost trivial. We feel that our analysis has some advantages over that of Ref. 6, in that it is considerably shorter and simpler, and in addition, is not restricted to a Hilbert space. Furthermore, the Larsen–Habetler technique appears to have some real advantages over both the Case and Hangelbroek methods in the analysis of the multigroup transport equation,[7,8] and it is planned to use techniques similar to those reported here to attempt to extend the multigroup results, which are so far restricted to the subcritical medium (but see Ref. 9).

II. THE RESOLVENT OPERATOR AND THE FULL RANGE EXPANSION

As in Ref. 6, we consider the one speed transport equation with isotropic scattering for a conservative medium, $c=1$, i.e.,

$$u\frac{\partial\psi}{\partial x}(x,u)+uK^{-1}\psi(x,u)=q(x,u),\quad u\neq 0 \tag{1a}$$

with

$$(K^{-1}f)(x,u)=(1/u)[f(x,u)-\tfrac{1}{2}\int_{-1}^{1}f(x,u')\,du']. \tag{1b}$$

A solution of Eq. (1) is understood to be a differentiable function $\psi:\mathbb{R}\to X_p$, $p>1$, where X_p is the Banach space of functions $f:[-1,1]\to\mathbb{C}$ satisfying

$$\|f\|_p=(\int_{-1}^{1}|uf(u)|^p\,du)^{1/p}<\infty,$$

and the vector $\psi(x)$ has been written $\psi(x,u)$. The nonhomogeneous term $q(x,u)$ is specified with $(1/u)q(x,u)\in X_p$.

Equation (1b) defines a densely defined, closed, unbounded, noninvertible operator $K^{-1}: X_p\to X_p$ with domain $D(K^{-1})=\{f\in X_p\,|\,f=ug,\ g\in X_p\}$. The choice of X_p-norm has the result that the operator $K^{-1}=u^{-1}A$ corresponds, for $p=2$, essentially to the product Au^{-1} of operators on L_2 used by Kaper[10] for a related problem in the kinetic theory of gases, rather than the product $u^{-1}A$ used by Lekkerkerker in Ref. 6. In fact, the unitary transformation $U: X_2\to L_2$ given by $(Uf)(u)=uf(u)$, transforms K^{-1} into $UK^{-1}U^{-1}=uu^{-1}Au^{-1}=Au^{-1}$. This avoids considerable difficulties encountered in Ref. 6; in particular, in our treatment, $D(u^{-1})=D(K^{-1})$.

In most of the remainder, explicit x-dependence will not appear, as the transport operator K^{-1} is studied in X_p. This notation agrees, except for minor variations, with that of Refs. 3 and 2, Sec. 6.9. Note that the extension of the analysis of Ref. 3 to X_p for $c\neq 1$ has been given in Ref. 11 for $p>1$. While it appears that the forthcoming analysis could be carried out in X_1, that would require substantial alteration of the technique.[12]

The essence of the Larsen–Habetler technique is to invert K^{-1} to obtain K, calculate the resolvent $(zI-K)^{-1}$, and then integrate the resolvent along a contour surrounding the spectrum of K. Application of the Cauchy theorem yields the so-called Case completeness theorem. This technique fails in the present case because K^{-1} is not invertible on its range. In fact, $\lambda=0$ is an eigenvalue of K^{-1} with eigenvector e_0 defined by

$$e_0(u)=1, \quad -1<u<1. \tag{2}$$

Furthermore,

$$K^{-1}e_1=e_0, \tag{3a}$$

where

$$e_1(u)=u, \quad -1<u<1. \tag{3b}$$

We shall see that e_0 and e_1 span the $\lambda=0$ root linear manifold of K^{-1}.

As explained in the Introduction, we now define a subspace $Y_p \subset X_p$ such that $K^{-1}| Y_p$ is invertible. To this end, define

$$Y_p=\{f\in X_p \mid \int_{-1}^{1} u^i f(u)\,du=0,\ i=1,2\}$$

and

$$Y_{p0}=\mathrm{Sp}\{e_0,e_1\}$$

Theorem 1: The direct sum decomposition $X_p=Y_p \dotplus Y_{p0}$ reduces K^{-1}.

Proof: The linear functionals

$$\rho_0: f\to \tfrac{3}{2}\int_{-1}^{1} u^2 f(u)\,du, \tag{4a}$$

$$\rho_1: f\to \tfrac{3}{2}\int_{-1}^{1} u f(u)\,du, \tag{4b}$$

have the property

$$\rho_i(e_j)=\delta_{ij}, \quad i,j=0,1.$$

Hence,

$$P: f\to \rho_0(f)e_0+\rho_1(f)e_1$$

is a continuous projection onto Y_{p0}, and Y_p is its topological supplement. The computation $\rho_i(K^{-1}f)=0$ for $f\in Y_p$ follows immediately from Eq. (1b), and since $PD(K^{-1})=Y_{p0}\subset D(K^{-1})$, the subspaces are reducing.

Theorem 2: $K^{-1}| Y_p$ is invertible, and its bounded inverse K is given by

$$Kg=ug-\tfrac{3}{2}\int_{-1}^{1} s^3 g(s)\,ds.$$

Proof: Consider the equation

$$K^{-1}f=g \text{ with } g\in Y_p.$$

This may be written

$$f-\tfrac{1}{2}\int_{-1}^{1} f(s)\,ds=ug.$$

If the equation is multiplied by u^2 and integrated over u from -1 to 1, one obtains

$$\int_{-1}^{1} f(s)\,ds=-3\int_{-1}^{1} u^3 g(u)\,du,$$

and the result follows.

Theorem 3: For $z\in \mathbb{C}/[-1,1]$ and $g\in Y_p$,

$$(zI-K)^{-1}g=\frac{1}{z-u}\left\{g-[1/2\Lambda(z)]\int_{-1}^{1}[sg(s)/s-z]\,ds\right\},$$

where

$$\Lambda(z)=\left[1-\tfrac{1}{2}\int_{-1}^{1}(z/z-s)\,ds\right]$$

is the usual dispersion function[2] for $c=1$.

Proof: The analysis of Ref. 3 can be followed to arrive at the result

$$(zI-K)^{-1}g=(1/z-u)\left\{g+\tfrac{3}{2}\left[\int_{-1}^{1}\frac{s^3 g(s)}{s-z}\,ds\right]\times\left[1+\tfrac{3}{2}\int_{-1}^{1}\frac{t^3}{z-t}\,dt\right]^{-1}\right\}.$$

Then the identities

$$u^3/(z-u)=-u^2-uz+uz^2/(z-u)$$
$$=-u^2-uz-z^2+z^3/(z-u),$$

can be used to simplify the two integrals in the expression, yielding the stated result.

Note that this expression for the resolvent is identical to that obtained in Ref. 3, and so a great deal of the analysis given there can be taken as verbatim.

The spectrum of K can be obtained immediately from the expression for the resolvent in Theorem 3: $\sigma(K)=[-1,1]$. Although $\Lambda(z)\sim -1/3z^2$ for large z, $\int_{-1}^{1}(sg(s)/s-z)\,ds\sim 1/z^3$ in the same limit, so the resolvent $(zI-K)^{-1}$ converges to zero at infinity, reflecting of course the boundedness of K. Thus,

$$I=(1/2\pi i)\int_\Gamma (zI-K)^{-1}\,dz,$$

where Γ is any closed contour surrounding the cut $[-1,1]$.

Since the Hölder continuous functions are dense in Y_p by an easy application of the Weierstrass theorem, if

$$H_p=\{f\in Y_p \mid f \text{ is of class } H^*\},$$

then H_p+Y_{p0} is dense in X_p. It is also easy to see that $H_p\cap D(K^{-1})$ is dense in Y_p. Here by "of class H^*" is meant[2,13] that f is Holder continuous on the interior of $[-1,1]$, i.e.,

$$|f(u)-f(u')|\le \text{const}\times|u-u'|^\alpha, \quad \alpha>0,$$

and also that f near the endpoints $b=\pm 1$ of the interval is a product of a function Hölder continuous on $[-1,1]$ and the function $(u-b)^\beta$, $\beta>-1$. The Larsen–Habetler analysis utilizes the pointwise evaluation of the boundary values of certain analytic functions of z in the domain of the resolvent $(zI-K)^{-1}$. For that reason it is necessary to stay on the manifold H_p, and extend the final results as in Ref. 11.

Alternatively, we may have chosen to "compute" on functions Hölder continuous on the entire interval $[-1,1]$, whence the Case transforms $A(v)$, as well as $\lambda(v)A(v)$, would have vanished at the endpoints b [by virtue of the fact that $\lambda(v)/N(v)\to 0$ at the boundaries; see Eq. (6)]. However, this would lead to no simplification of the arguments.

In this manner, the analysis of Ref. 3 yields results analogous to the case of $c<1$; i.e., for each $f\in H_p$ there exists $A\in X_p$ of class H^* satisfying:

$$f(u)=\int_{-1}^{1} A(v)\phi_v(u)\,dv, \tag{6a}$$

$$A(v)=\frac{1}{N(v)}\int_{-1}^{1} uf(u)\phi_v(u)\,du, \tag{6b}$$

where ϕ_v is the usual Case "singular eigenfunction" corresponding to $c=1$; namely,

$$\phi_v(u)=(v/2)P(1/v-u)+\tfrac{1}{2}[\Lambda^+(v)+\Lambda^-(v)]\delta(v-u) \tag{7}$$

and

$$N(v)=v\Lambda^+(v)\Lambda^-(v)$$

converges to infinity at the endpoints ± 1. The notation is the same as that of Refs. 2,3 and 11. In the language of Ref. 2, we would say that every $f\in H_p$ can be expanded in terms of the Case continuum eigenfunctions alone.

To deal with $f\in Y_{p0}$, write

$$f=\tfrac{1}{2}a_0-\tfrac{1}{2}a_1 \tag{8}$$

where the factors $\pm\frac{1}{2}$ have boen introduced to conform with standard notation. Multiplying Eq. (8) by u or u^2 and integrating, one finds

$$a_i=3\int_{-1}^{1}(-u)^{2-i}f(u)\,du. \tag{9}$$

Let $\lambda(v)$ denote

$$\lambda(v)=\tfrac{1}{2}[\Lambda^+(v)+\Lambda^-(v)]. \tag{10}$$

We wish to show that the linear transformation $F\colon f\to\lambda A$ defined by Eq. (6b) for f of class H^*,

$$(Ff)(v)=[\lambda(v)/N(v)]\int_{-1}^{1}uf(u)\phi_v(u)\,du,$$

extends to an isomorphism $F\colon Y_p\to X_p$. Define $F'\colon\psi\to f$, the natural candidate for F^{-1}, by

$$(F'\psi)(u)=\int_{-1}^{1}[\psi(v)/\lambda(v)]\phi_v(u)\,dv$$

for any ψ of class H^*. Equations (6a) and (6b) establish the relationship $F'F=I$ on H_p, which is dense in Y_p. We must ascertain, however, that F' is a bounded transformation into Y_p, or else the extension of F to all of Y_p might no longer be invertible. Moreover, it is necessary to prove that the range of F is dense in X_p in order to insure that the solution of a transport problem solved in terms of the transformed function $A(v)$ will be the image under F of a vector in X_p.[14]

In Ref. 2 it is shown that if f is of class H^*, then A will be of class H^*, and hence so will $\lambda(v)A(v)$. Furthermore, any A of class H^* will yield a function f of class H^* via Eq. (6a), since

$$f(u)=\lambda(u)A(u)+\tfrac{1}{2}P\int_{-1}^{1}[vA(v)/v-u]\,dv, \tag{11}$$

and the boundary values of the Cauchy integral of a class H^* function are also of class H^*.

In Ref. 11, the inequality

$$\int_{-1}^{1}|v\lambda(v)A(v)|^p\,dv\le M_p\int_{-1}^{1}|uf(u)|^p\,du, \tag{12}$$

where M_p is a constant depending upon p, proves that $\lambda A\in X_p$ if $f\in H_p$, and that F is a bounded transformation. Let

$$H_p^1=\{A\in X_p\,|\,\lambda A\in X_p \text{ of class } H^*\}.$$

Then the same argument used to derive Eq. (12) also yields

$$\int_{-1}^{1}|uf(u)|^p\,du\le M_p'\int_{-1}^{1}|v\psi(v)|^p\,dv,$$

for $\psi\in H_p^1$, which implies that F is one-one on H_p. Combining these remarks, we obtain bounded transformations F and F' on Y_p and X_p, respectively, with $F'F=I$ on Y_p, and $FF'=I$ on $\mathrm{Ran}(F)$.

A direct computation shows $F'\psi\in Y_p$ for $\psi\in H_p^1$. For example,

$$\rho_0(F'\psi)=\int_{-1}^{1}[v\psi(v)/\lambda(v)]P\int_{-1}^{1}(u^2/v-u)\,du\,dv$$
$$+\int_{-1}^{1}[\psi(v)/\lambda(v)]v^2\lambda(v)\,dv$$
$$=0,$$

since

$$P\int_{-1}^{1}(u^2/v-u)\,du=-2v\lambda(v).$$

Thus, to prove $\mathrm{Ran}(F)$ is dense in X_p, suppose $A\in H_p^1$ and

$$0=\int_{-1}^{1}A(v)\phi_v(u)\,dv. \tag{13}$$

Defining

$$n(z)=\int_{-1}^{1}A(v)v/(v-z)\,dv, \tag{14}$$

expanding Eq. (13) as in Eq. (11), and using the Plemelj formulas with Eq. (14), yields

$$(1/2\pi i)[n^+(u)-n^-(u)]\lambda(u)+\tfrac{1}{2}[n^+(u)+n^-(u)](u/2)=0.$$

With the substitution

$$u=(1/\pi i)[\Lambda^+(u)-\Lambda^-(u)],$$

and Eq. (10), this becomes

$$(1/2\pi i)(n^+\Lambda^+-n^-\Lambda^-)=0,\quad -1<u<1. \tag{15}$$

If $J(z)$ is defined by

$$J(z)=n(z)\Lambda(z),$$

then Eq. (15) proves that J is an entire function. But $\Lambda(\infty)=n(\infty)=0$, so by Liouville's Theorem, $J\equiv 0$, which proves $A(v)\equiv 0$. Hence $FF'=I$ and $F'=F^{-1}$. Using the density of H_p and H_p^1 in Y_p and X_p, the transformations in Eqs. (6a) and (6b) may be extended by continuity to all of X_p.

The above results can be summarized in Theorem 4.

Theorem 4: Let $f\in X_p$. Then f has an eigenfunction expansion of the form

$$f=\tfrac{1}{2}a_0-\tfrac{1}{2}a_1u+\int_{-1}^{1}A(v)\phi_v(u)\,dv, \tag{16}$$

where a_i are given by Eq. (9), $A(v)$ is given by Eq. (6b), and ϕ_v is the Case singular eigenfunction defined in Eq. (7). The linear transformation $F\colon f\to\lambda A$ is an isomorphism $F\colon Y_p\to X_p$.

III. HALF RANGE EXPANSION

Let X_p' be the Banach space of functions $f\colon[0,1]\to C$ with

$$\|f\|_p'=\left[\int_0^1|uf(u)|^p\,du\right]^{1/p}<\infty.$$

The object for the half range theory is to find an operator $E\colon X_p'\to X_p$ with certain analyticity properties given below. Then the full range expansion of Ef will correspond to the "half range expansion" of f (cf. Ref. 3, Sec. 4). It will in fact be necessary to restrict E to a subspace $Y_p'\subset X_p'$ such that $E|\,Y_p'$ will have its range in Y_p. Then the expansion of $(E|\,Y_p')f$ will give the half range "continuum modes," while the discrete modes can be separately treated.

We require the operator E to have the properties:

(i) $(zI-K)^{-1}Ef$ analytic in z for all $\mathrm{Re}\, z<0$, $f\in Y_p'$.

(ii) $\rho_0(Ef)=0$ for all $f\in Y_p'$,

(iii) $\rho_1(Ef)=0$ for all $f\in X_p'$.

The first proper guarantees that the expansion of Ef contains only eigenfunctions ϕ_v with $v>0$; the second and third guarantee that $Ef\in Y_p$; while the third also insures that the discrete coefficient a_1 of Ef vanishes.

Before the subspace Y_p' may be specified explicitly, let us recall some properties of the dispersion function Λ. The Wiener–Hopf factorization of $\Lambda(z)$ provides a function $X(z)$, analytic for $\mathrm{Re}\, z<0$, such that

$$X(z)X(-z)=3\Lambda(z). \tag{17}$$

Moreover,

$$X(z)=\int_0^1[\gamma(u)/u-z]\,du, \tag{18a}$$

where

$$\gamma(u)=\frac{u}{2}\frac{X^-(u)}{\Lambda^-(u)}. \tag{18b}$$

Now we may define $Y_p'\subset X_p'$ by

$$Y_p'=\{f\in X_p'\!\int_0^1 Y(\mu)f(\mu)\,d\mu=0\}.$$

By analogy with transport in absorbing media, we are led to study the transformation $E\colon X_p'\to X_p$, defined on $f\in X_p'$ of class H^* by

$$(Ef)(u)=\begin{cases}\dfrac{1}{X(u)}\dfrac{3}{2}\displaystyle\int_0^1\frac{sf(s)\,ds}{X(-s)(s-u)}, & u<0,\\ f(u), & u>0.\end{cases} \tag{19}$$

Since $X(u)$ is analytic and bounded away from zero for $u<0$, we see from the Hölder inequality that E extends to a bounded operator from X_p' to X_p.

Property (iii) is verified by Theorem 5.

Theorem 5: For all $f\in X_p'$, $\rho_1(Ef)=0$.

Proof: From Eq. (19),

$$\int_{-1}^1 u(Ef)(u)\,du=\int_0^1 uf(u)\,du+\frac{3}{2}\int_0^1 ds\,\frac{sf(s)}{X(-s)}\int_{-1}^0\frac{u\,du}{X(u)(s-u)}$$

for f of class H^*. Changing variable from u to $-u$ in the second term above and utilizing equations (18c) and (18a), the identity

$$\int_0^1\gamma(u)\,du=1, \tag{20}$$

and the continuity of E, the result follows.

Next we shall see that property (ii) is satisfied.

Theorem 6: For all $f\in X_p'$,

$$\rho_1(Ef)=\tfrac{2}{3}\int_0^1\gamma(u)f(u)\,du.$$

Hence, if $f\in Y_p'$, then $Ef\in Y_p$.

Proof: As in the proof of Theorem 5, we compute

$$\int_{-1}^1 u^2(Ef)(u)\,du=\int_0^1 u^2f(u)\,du+\frac{3}{2}\int_0^1\frac{sf(s)}{X(-s)}\,ds\int_{-1}^1\frac{u^2\,du}{X(u)(s-u)}.$$

The change of variable $u\to -u$ along with Eq. (18b) reduces this to

$$\int_{-1}^1 u^2(Ef)(u)\,du=\int_0^1 u^2f(u)\,du+\int_0^1\frac{sf(s)}{X(-s)}\,ds\int_0^1\frac{u\gamma(u)}{s+u}\,du.$$

Finally, writing $u/(s+u)=1-s/(s+u)$ and using Eq. (18a), we obtain the desired expression for $\rho_0(Ef)$.

This result, along with Theorem 5, proves that $Ef\in H_p$ if $f\in Y_p'$ and is of class H^*, since the Cauchy integral in Eq. (19) preserves Hölder continuity.

Let Y_{p0}' denote the subspace of X_p' spanned by $e_0'(u)\equiv 1$, $u\in[0,1]$. As a corollary of Theorem 6, we obtain

Corollary 1:

$X_p'=Y_p'+Y_{p0}'$. reduces E.

Proof: From equation (18) we obtain

$$X(u)=\tfrac{3}{2}\int_0^1 s\,ds/X(-s)(s-u), \tag{21}$$

and thus compute

$$(Ee_0')(u)=e_0(u).$$

Defining the bounded linear functional

$$\rho_0'\colon f\to\int_0^1\gamma(u)f(u)\,du,$$

and the projection

$$P'\colon f\to\rho_0'(f)e_0',$$

the identity equation (20) and Theorem 6 prove the reduction.

The remaining property of E to be confirmed is given by the following theorem.

Theorem 7: $(zI-K)^{-1}(Ef)(u)$ is analytic in z for $\mathrm{Re}\, z<0$, $f\in Y_p'$.

Proof: Analyticity is assured except for a possible branch cut $[-1,0]$. However, using Theorem 3 and Eq. (19), and applying Eqs. (17) and (18), yields for $u<0$, after some rearranging,

$$(zI-K)^{-1}(Ef)(u)=(1/z-u)\left\{\int_0^1 dt\,\gamma(t)\times f(t)\left[\frac{1}{X(u)(t-u)}-\frac{1}{X(z)(t-z)}\right]\right\}.$$

From this, the analyticity along $[-1,0]$ can be concluded.

The expansion of a function $f\in X_p'$ is accomplished by applying the full range expansion of Sec. II to Ef. In particular, let P'' represent the "projection" onto Y_p along Y_{p0}', $P''=(I-P)E$. Then

$$(Ef)(u)=\tfrac{1}{2}a_0+P''f(u),$$

since $a_1=0$ by Theorem 5. The expansion of $P''f$ is made as in Eq. (16), while a_0 can be calculated from Theorem 6. Thus,

$$P''f(u) = \int_0^1 A(v)\phi_v(u)\,dv, \tag{22a}$$

with

$$A(v) = [v/\gamma(v)N(v)] \int_0^1 f(u)\gamma(u)\phi_v(u)\,du \tag{22b}$$

and

$$a_0 = 2\int_0^1 \gamma(u)f(u)\,du \Big/ \int_0^1 \gamma(u)\,du. \tag{22c}$$

The solution of the half range neutron transport equation at $c=1$ may now be carried out as described in Refs. 3 and 11. The eigenfunction expansions developed here are used to choose a_0 and $A(v)$ to satisfy the boundary conditions at $x=0$ and $x\to\infty$, and the full solution is expressed in the form

$$\psi(u) = \tfrac{1}{2}a_0 + \int_0^1 A(v)\phi_v(u)\exp(-x/v)\,dv.$$

For details, see the references cited.

IV. SOLUTION OF THE MILNE PROBLEM

We seek the solution $\psi_M(x,u)$, of the homogeneous transport equation in a half space subject to the conditions

$$\psi_M(0,u) = 0, \quad u > 0, \tag{23a}$$

and

$$\psi_M(x,u) \sim \tfrac{1}{2}x, \tag{23b}$$

as $x\to\infty$. The Milne problem is solved by

$$\psi_M(x,u) = \tfrac{1}{2}a_0 + \tfrac{1}{2}(x-u) + \int_0^1 A(v)\phi_v(u)\exp(-x/v)\,dv, \tag{24}$$

where

$$z_o = -a_0 = -\int_0^1 u\gamma(u)\,du \Big/ \int_0^1 \gamma(u)\,du \tag{25}$$

is the so-called "extrapolated endpoint," and

$$A(v) = [v/\gamma(v)N(v)]\,\tfrac{1}{2}\int_0^1 u\phi_v(u)\gamma(u)\,du. \tag{26}$$

It is trivial to verify that the first two terms of equation (24) do indeed satisfy Eq. (1), and since[11]

$$K^{-1}f = \int_0^1 (1/v)A(v)\phi_v(u)\,dv,$$

for all $f\in Y_p'$, that the third term does also. The coefficient a_0 has been determined by setting $x=0$, multiplying both sides of equation (24) by $\gamma(u)$, integrating over u, and using the boundary condition (23a), as well as Theorem 6 to conclude that the integral does not contribute. Similarly, to solve for $A(v)$, imposing the boundary condition (23a) and using the fact that $a_0 \in Y_{p0}'$, one obtains expression (26).

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