NEW EXOTIC 4-MANIFOLDS VIA LUTTINGER SURGERY ON LEFSCHETZ FIBRATIONS

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ABSTRACT. In [2], the first author constructed the first known example of *exotic minimal symplectic* $\mathbb{CP}^2\#5\overline{\mathbb{CP}}^2$ and minimal symplectic 4-manifold that is homeomorphic but not diffeomorphic to $3\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$. The construction in [2] uses Yukio Matsumoto's genus two Lefschetz fibrations on $\mathbb{T}^2 \times \mathbb{S}^2\,\#4\overline{\mathbb{CP}}^2$ over $\mathbb{S}^2$ along with the fake symplectic $\mathbb{S}^2 \times \mathbb{S}^2$ construction given in [1]. The main goal in this paper is to generalize the construction in [2] using the higher genus versions of Matsumoto's fibration constructed by Mustafa Korkmaz and Yusuf Gurtas on $\Sigma_k \times \mathbb{S}^2\,\#4n\overline{\mathbb{CP}}^2$ for any $k \geq 2$ and $n = 1$, and $k \geq 1$ and $n \geq 2$, respectively. Using our symplectic building blocks, we also construct new symplectic 4-manifolds with the free group of rank $s \geq 1$, the free product of the finite cyclic groups, and various other finitely generated groups as the fundamental group.

1. Introduction

The main goal of this paper is to exhibit a new family of simply connected minimal symplectic and infinitely many non-symplectic 4-manifolds that is homeomorphic but not diffemorphic to $(2n+2k-3)\mathbb{CP}^2\#(6n+2k-3)\overline{\mathbb{CP}}^2$ for any $n \geq 1$, $k \geq 1$, and $(n, k) \neq (1, 1)$. Our construction is a generalization of the first author's work in [2] (the case $(n, k) = (1, 1)$), where he used Yukio Matsumoto's genus two Lefschetz fibrations on $\mathbb{T}^2 \times \mathbb{S}^2\,\#4\overline{\mathbb{CP}}^2$ over $\mathbb{S}^2$, arising from the factorization of the elliptic involution of the genus two surface with two fixed points in terms of Dehn twists, along with the fake symplectic $\mathbb{S}^2 \times \mathbb{S}^2$ construction in [1] obtained via a combination of the knot surgeries along the fibered knots in $\mathbb{S}^3$ and the twisted fiber sums. In the aforementioned paper [2], using the symplectic connected sum along the genus two surfaces, the first author has constructed the first known exotic minimal symplectic $\mathbb{CP}^2\#5\overline{\mathbb{CP}}^2$ and minimal symplectic 4-manifold that is homeomorphic but not diffeomorphic to $3\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$. The article [2] led to many articles such as [6, 3, 4, 7, 17], among many others. For example, the generalization of these symplectic examples to an exotic $(2l-1)\mathbb{CP}^2\#(2l+3)\overline{\mathbb{CP}}^2$ (for $l \geq 3$), using Matsumoto's genus two fibration and an iterated fiber

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sum, can be found in [4], where the geography problem for simply connected minimal symplectic 4-manifolds with signature less than or equal -2 studied in details (see also a subsequent paper [7] for signature equal to -1 case). Considering that Mustafa Kormaz's and Yusuf Gurtas' higher genus (i.e. the genus $g \geq 2$) Lefschetz fibrations on $\Sigma_k \times \mathbb{S}^2 \# 4n\overline{\mathbb{CP}}^2$, arising from the Dehn twist factorization of a certain involution of the genus $2k+n-1$ surface, which are the generalizations of Matsumoto's genus two fibration, it was a natural question whether the similar construction can be carried out using these higher genus Lefschetz fibrations. The fundamental group of these Lefschetz fibrations are not abelian if $k \geq 2$ and the relations in the fundamental group coming from the vanishing cycles are far more complicated, thus the fundamental group computations become more difficult and delicate than in [2]. In this article, by carefully analysing the fundamental group, we prove the following theorem:

Theorem. *Let M be $(2n+2k-3)\mathbb{CP}^2\#(6n+2k-3)\overline{\mathbb{CP}}^2$ for any pair of integers $n \geq 1$, $k \geq 1$, and $(n,k) \neq (1,1)$. There exists a new family of smooth closed simply-connected minimal symplectic 4-manifold and an infinite family of non-symplectic 4-manifolds that are homeomorphic but not diffeomorphic to M that can obtained by the sequence of Luttinger surgeries and a single generalized torus surgery on Lefschetz fibrations.*

One difference, though not essential, with the construction in [2] (see for example [1, 3]) is that we use the Luttinger surgery instead of the knot surgery. The Luttinger surgery will have some advantages for us in the computation of the fundamental groups, since the presentation of the fundamental groups looks much simpler via Luttinger surgery. This is due to the fact that the relations introduced by our Luttinger surgeries all involve a single commutator relation expression, while the fundamental group computations via knot surgery is more challenging to work with. Our construction can be also interpreted using the knot surgery.

The second goal of our paper is to construct new symplectic 4-manifolds with various fundamental groups (and with a small size) by applying the Luttinger surgeries to a certain family of Lefschetz fibrations. More general results along these lines are recently obtained in [9] and [10]. However, the constructions in [9, 10] use the fiber summing of $\Sigma_g \times \mathbb{T}^2$ along T^2 with the elliptic surfaces $E(n)$, whereas in this paper we consider the fiber summing of the product manifolds along the higher genus surfaces with $\Sigma_k \times \mathbb{S}^2 \# 4n\overline{\mathbb{CP}}^2$ for $k \geq 1$. For previous work on this subject, we refer the reader to [14, 7, 5, 24], and references therein.

2. Mapping class group and Lefschetz fibrations

2.1. **Mapping Class Groups.** Let Σ_g denote an orientable 2-dimensional closed and connected surface of genus $g > 0$.

Definition 2.1. Let $Diff^+(\Sigma_g)$ denote the group of all orientation-preserving diffeomorphisms $\Sigma_g \to \Sigma_g$, and $Diff_0^+(\Sigma_g)$ be the subgroup of $Diff^+(\Sigma_g)$ consisting of all orientation-preserving diffeomorphisms $\Sigma_g \to \Sigma_g$ that are isotopic to the identity. *The mapping class group* M_g of Σ_g is defined to be the group of isotopy classes of orientation-preserving diffeomorphisms of Σ_g, i.e.,

$$M_g = Diff^+(\Sigma_g)/Diff_0^+(\Sigma_g).$$

Definition 2.2. Let α be a simple closed curve on Σ_g. A *right handed Dehn twist* t_α about α is the isotopy class of a self-diffeomophism of Σ_g obtained by cutting the surface Σ_g along α and gluing the ends back after rotating one of the ends 2π to the right.

In Section 4, we will use some well known relations that hold in the mapping class group M_g.

2.2. **Lefschetz Fibration.** Now, we recall the definition and basic facts about the Lefschetz fibrations on 4-manifolds.

Definition 2.3. Let X be a compact, connected, oriented, smooth 4-manifold. A *Lefschetz fibration* on X is a smooth map $F : X \longrightarrow \Sigma_h$, where Σ_h is a compact, oriented, smooth 2-manifold of genus h, such that F is surjective and each critical point of F has an orientation preserving chart on which $F : \mathbb{C}^2 \longrightarrow \mathbb{C}$ is given by $F(z_1, z_2) = {z_1}^2 + {z_2}^2$.

By Sard's theorem, F is a smooth fiber bundle away from finitely many critical points. We denote the critical points of F by $P_1, \cdots, P_n$. The genus of the Lefschetz fibration F is defined to be the genus of the regular fiber of F. If a fiber of the Lefschetz fibration F passes through the critical point set $P_1, \cdots, P_n$, then it is called a *singular fiber*. The singular fiber is an immersed surface with a single transverse self-intersection. Moreover, a singular fiber of the genus g Lefschetz fibration can be described by its monodromy, i.e., an element of the mapping class group M_g. This element is a right-handed (or a positive) Dehn twist along a simple closed curve on Σ_g, called the *vanishing cycle*. If this curve is a nonseparating curve, then the singular fiber is called *nonseparating*, otherwise it is called *separating*. For a genus g Lefschetz fibration over S^2, the product of right handed Dehn twists t_{α_i} along the vanishing cycles α_i, for $i = 1, \cdots, n$, determines the global monodromy of the Lefschetz fibration, the relation $t_{\alpha_1} \cdot t_{\alpha_2} \cdot \ \cdots \ \cdot t_{\alpha_n} = 1$ in M_g. Conversely, any such relation in M_g determines a genus g Lefschetz fibration over S^2 with the vanishing cycles $\alpha_1, \cdots, \alpha_n$.

The following lemma and example will be needed in the sequel.

Lemma 2.4. (*cf.* [15]) *Let $F : X \to \mathbb{S}^2$ be a genus g Lefschetz fibration with global monodromy given by the relation $t_{\alpha_1} \cdot t_{\alpha_2} \cdot \ \cdots \ \cdot t_{\alpha_n} = 1$. Suppose that F has a section. Then the fundamental group of X is isomorphic to the fundamental group of Σ_g divided out by the normal closure of the simple* *closed curves* $\alpha_1, \alpha_2, \ldots, \alpha_n$*, considered as elements in* $\pi_1(\Sigma_g)$*. In particular, there is an epimorphism* $\pi_1(\Sigma_g) \to \pi_1(X)$

Example 2.5. Let α_1, α_2, ... , α_{2g}, α_{2g+1} denote the collection of simple closed curves given in Figure 1, and c_i denote the right handed Dehn twists t_{α_i} along the curve α_i. It is well-known that the following relations hold in the mapping class group M_g:

$$\Gamma_1(g) = (c_1 c_2 \cdots c_{2g-1} c_{2g} {c_{2g+1}}^2 c_{2g} c_{2g-1} \cdots c_2 c_1)^2 = 1. \tag{1}$$

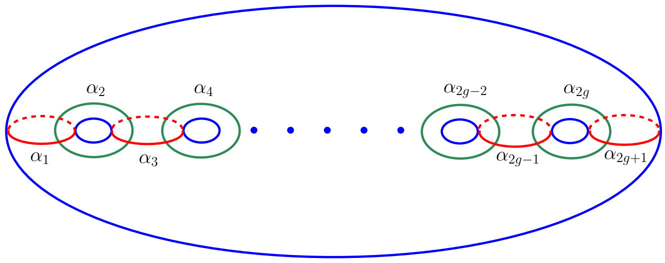

FIGURE 1. Vanishing Cycles of the Genus g Lefschetz Fibration on $X(g,1) = \mathbb{CP}^2 \# (4g+5)\overline{\mathbb{CP}}^2$,

The monodromy relation given above, corresponding to the genus g Lefschetz fibration over S^2, has total space $X(g,1) = \mathbb{CP}^2 \# (4g+5)\overline{\mathbb{CP}}^2$, the complex projective plane blown up at $4g+5$ points. Furthermore, it is well known that for $g \geq 2$, the above fibration on $X(g,1)$ admits $4g+4$ disjoint (-1)-sphere sections (see [22] for a proof of this fact using a mapping class group argument or [8] for a geometric argument).

3. Symplectic connected sum and Luttinger surgery

3.1. Symplectic connected sum.

Definition 3.1. Let X_1 and X_2 are closed, oriented, smooth 4-manifolds, and $F_i \subset X_i$ are 2-dimensional, smooth, closed, connected submanifolds in them. Suppose that $[F_1]^2 + [F_2]^2 = 0$ and the genera of F_1 and F_2 are equal. We choose an orientation-preserving diffemorphism $\psi : F_1 \longrightarrow F_2$ and lift it to an orientation-reversing diffemorphism $\Psi : \partial\nu F_1 \longrightarrow \partial\nu F_2$ between the boundaries of the tubular neighborhoods νF_i of F_i. Using Ψ, we glue $X_1 \setminus \nu F_1$ and $X_2 \setminus \nu F_2$ along the boundary. This new oriented smooth 4-manifold $X_1 \#_\Psi X_2$ is called the *connected sum* of X_1 and X_2 along F_1 and F_2, determined by Ψ.

Definition 3.2. Let $e(M)$ and $\sigma(M)$ denote the Euler characteristic and the signature of an orientable closed 4-manifold M, respectively. We define $c_1^2(M) := 2e(M) + 3\sigma(M)$ and $\chi_h(M) := (e(M) + \sigma(M))/4$.

Lemma 3.3. *Let X_1 and X_2 be smooth 4-manifolds as above. Then*

$$\begin{aligned} c_1^2(X_1 \#_\Psi X_2) &= c_1^2(X_1) + c_1^2(X_2) + 8(g-1), \\ \chi_h(X_1 \#_\Psi X_2) &= \chi_h(X_1) + \chi_h(X_2) + (g-1), \end{aligned}$$

where g is the genus of the above surfaces $F_i \subset X_i$.

Proof. The proof immediately follows from the above definition, and the formulas

$$e(X_1 \#_\Psi X_2) = e(X_1) + e(X_2) - 2e(\Sigma_g), \quad \sigma(X_1 \#_\Psi X_2) = \sigma(X_1) + \sigma(X_2)$$

□

If X_1, X_2 are symplectic manifolds and F_1, F_2 are symplectic submanifolds then according to theorem of Gompf [14] $X_1 \#_\Psi X_2$ admits a symplectic structure.

The following theorem of M. Usher [25] will be used to show that the symplectic 4-manifolds constructed in Sections 5 and 6 are minimal.

Theorem 3.4. [25] **(Minimality of Sympletic Sums)** *Let $Z = X_1 \#_{F_1=F_2} X_2$ be symplectic fiber sum of manifolds X_1 and X_2. Then:*

(i) If either $X_1 \backslash F_1$ or $X_2 \backslash F_2$ contains an embedded symplectic sphere of square -1, then Z is not minimal.

(ii) If one of the summands X_i (say X_1) admits the structure of an S^2-bundle over a surface of genus g such that F_i is a section of this fiber bundle, then Z is minimal if and only if X_2 is minimal.

(iii) In all other cases, Z is minimal.

3.2. **Luttinger surgery.** In this subsection, we briefly review a Luttinger surgery. For further details, we refer the reader to the papers [21] and [11]. Luttinger surgery has been very effective tool recently for constructing exotic smooth structures on 4-manifolds.

Definition 3.5. Let (M, ω) be a symplectic 4-manifold, and the torus Γ be a Lagrangian submanifold of M with self-intersection 0. Assume that γ is a simple closed loop on Γ, and γ' is a simple loop on $\partial(\nu\Gamma)$ that is parallel to γ under the Lagrangian framing. For any integer m, the $(\Gamma, \gamma, 1/m)$ *Luttinger surgery* on M is defined as $M_{\Gamma,\gamma}(1/m) = (M - \nu(\Gamma)) \cup_\phi (S^1 \times S^1 \times D^2)$, where, for a meridian μ_Γ of Γ, the gluing map $\phi : S^1 \times S^1 \times \partial D^2 \to \partial(M - \nu(\Gamma))$ satisfies $\phi([\partial D^2]) = m[\gamma'] + [\mu_\Gamma]$ in $H_1(\partial(M - \nu(\Gamma))$.

It is shown in [11] that $M_{\Gamma,\gamma}(1/m)$ admits a symplectic form which agrees with the original symplectic form ω on $M \setminus \nu\Gamma$. The following lemma is easy to verify, the proof will be omitted.

Lemma 3.6.

(1) $\pi_1(M_{\Gamma,\gamma}(1/m)) = \pi_1(M-\Gamma)/N(\mu_\Gamma \gamma'^m)$, *where* $N(\mu_\Gamma\gamma'^m)$ *denote the smallest normal subgroup of* $\pi_1(M-\Gamma)$ *that contains* $\mu_\Gamma\gamma'^m$
(2) $\sigma(M) = \sigma(M_{\Gamma,\gamma}(1/m))$ *and* $e(M) = e(M_{\Gamma,\gamma}(1/m))$.

4. Symplectic Building Blocks

In this section, we collect symplectic building blocks that are needed in our construction of exotic 4-manifolds. Our building blocks will be the total space of the Lefschetz fibrations over 2-sphere constructed by M. Korkmaz [18] and Y. Gurtas [16], along with the symplectic building blocks that were constructed by the first author in [1]. The symplectic 4-manifolds in [1] were obtained via knot surgery along the fibered knots, but they can be constructed via the sequence of Luttinger surgeries as well. In this paper we use Luttinger surgery, which will have some advantages for us in the computation of the fundamental groups.

4.1. **Korkmaz's fibration.** For the convenience of the reader, we provide necessary background and state the main results in [18]. Let us first consider the case when $g = 2k$. Recall that the four manifold $Y(k) = \Sigma_k \times \mathbb{S}^2 \# 4\overline{\mathbb{CP}}^2$ is the total space of the well known genus g Lefschetz fibration over $\mathbb{S}^2$ [19, 18]. This was shown by Y. Matsumoto for $k = 1$, and in the case $k \geq 2$ by M. Korkmaz, by factorizing the *vertical* involution θ of the genus $2k$ surface (See Figure 2).

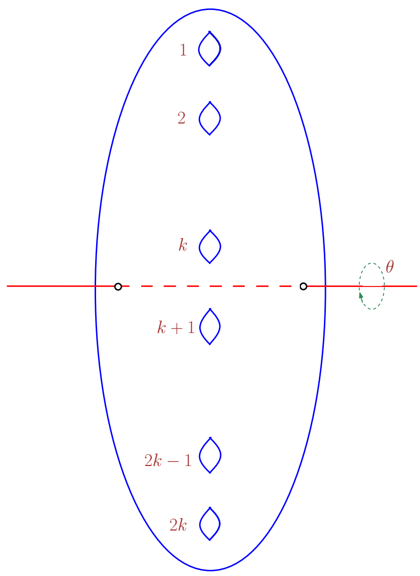

Figure 2. The vertical involution θ of the genus $2k$ surface

The branched-cover description of the above Lefschetz fibrations can be given as follows: take a double branched cover of $\Sigma_k \times \mathbb{S}^2$ along the union of two disjoint copies of $pt \times \mathbb{S}^2$ and two disjoint copies of $\Sigma_k \times pt$ (See Figure 3). The resulting branched cover has four singular points, corresponding to the number of the intersection points of the horizontal spheres and the vertical genus k surfaces in the branch set. Next, we desingularize this singular manifold to obtain $Y(k) = \Sigma_k \times S^2 \# 4\overline{\mathbb{CP}}^2$. Observe that a generic fiber of the horizontal fibration is the double cover of $\mathbb{S}^2$, branched over two points. This gives a sphere fibration on $Y(k) = \Sigma_k \times \mathbb{S}^2 \# 4\overline{\mathbb{CP}}^2$. Also, a generic fiber of the vertical fibration is the double cover of Σ_k, branched over two points. Thus, a generic fiber is a genus g surface. According to [19, 18], each of the two singular fibers of the vertical fibration can be perturbed into $g+2$ Lefschetz type singular fibers.

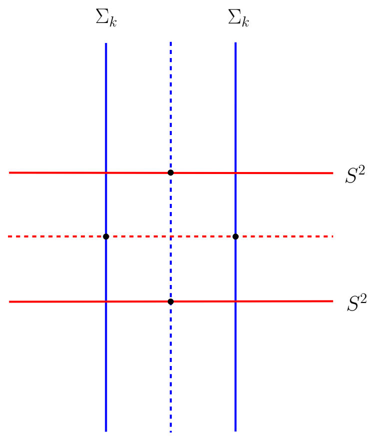

FIGURE 3. The branch locus for $\Sigma_k \times S^2 \# 4\overline{\mathbb{CP}}^2$

If $g = 2k+1$, then the total space of the corresponding Lefschetz fibration is $\Sigma_k \times \mathbb{S}^2 \# 8\overline{\mathbb{CP}}^2$. In this case, the appropriate branched-cover description of the Lefschetz fibrations is given as follows: take a double branched cover of $\Sigma_k \times \mathbb{S}^2$ along the union of 4 disjoint horizontal copies of $pt \times \mathbb{S}^2$ and two disjoint vertical copies of $\Sigma_k \times pt$. The resulting branched cover has 8 singular points. By desingularize this manifold, we obtain $\Sigma_k \times S^2 \# 8\overline{\mathbb{CP}}^2$. A generic fiber of the vertical fibration is the double cover of Σ_k, branched over 4 points. Thus, a generic fiber has genus $2k+1$. See subsection 4.2 for more details, where this particular case occurs if we set $n = 2$ and $k \geq 1$.

The following theorem was proved in [18], which computes the global monodromy of the given Lefschetz fibration for both an even and an odd g.

Theorem 4.1. *Let θ denote the vertical involution of the genus g surface with 2 fixed points. In the mapping class group M_g, the follwing relations between right Dehn twists hold:*

a) $(t_{B_0} t_{B_1} t_{B_2} \cdots t_{B_g} t_c)^2 = \theta^2 = 1$ *if g is even,*

b) $(t_{B_0} t_{B_1} t_{B_2} \cdots t_{B_g} (t_a)^2 (t_b)^2)^2 = \theta^2 = 1$ *if g is odd.*

where B_k, a, b, c are the simple closed curves defined as in Figure 4.

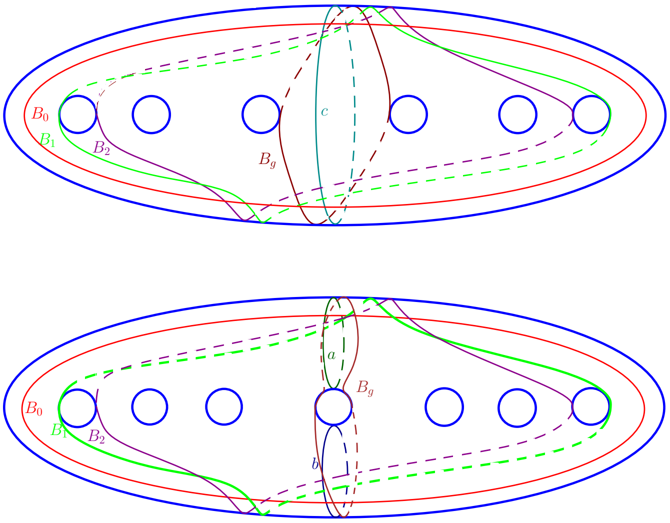

FIGURE 4. The vanishing cycles

Let us denote a regular fiber of the given Lefschetz fibration as Σ_{2k} and the standard generators of fundamental group of Σ_{2k} under the inclusion as $a_1, b_1, \cdots a_{2k}$ and b_{2k}. Using the homotopy exact sequence for a Lefschetz fibration, we have

$$\pi_1(\Sigma_{2k}) \longrightarrow \pi_1(Y(k)) \longrightarrow \pi_1(\mathbb{S}^2)$$

According to [18], we have the following identification of the fundamental group of $Y(k)$:

$$\pi_1(Y(k)) = \pi_1(\Sigma_{2k})/ < B_0, B_1, \cdots, B_{g-1}, B_g, c >$$

It follows that the fundamental group of $Y(k)$ has a presentation with the generators a_1, b_1, a_2, b_2, $\cdots$, a_g, b_g and the relations $[a_1, b_1][a_2, b_2] \cdots [a_g, b_g] = 1$, $B_0 = B_1 = B_2 = \cdots = B_g = c = 1$. It was shown in [18] that the following identities hold

$B_0 = b_1 b_2 \cdots b_g$

$B_{2i-1} = a_i b_i b_{i+1} \cdots b_{g+1-i} c_{g+1-i} a_{g+1-i}$, $1 \leq i \leq k$

$B_{2i} = a_i b_{i+1} b_{i+2} \cdots b_{g-i} c_{g-i} a_{g+1-i}$ $1 \leq i \leq k-1$

$B_g = B_{2k} = a_k c_k a_{k+1}$

$c = c_k = [a_1, b_1][a_2, b_2] \cdots [a_k, b_k]$

Now, we prove a lemma which we use in the sequel.

Lemma 4.2. *The following relations hold in the fundamental group of $Y(k)$*

$$a_1a_{2k} = 1, \quad a_2a_{2k-1} = 1, \quad \cdots, \quad a_ka_{k+1} = 1,$$
$$b_1b_2\cdots b_{2k} = 1, \quad b_2b_3\cdots b_{2k-1} = [a_{2k}, b_{2k}], \quad \cdots,$$
$$b_{i+1}b_{i+2}\cdots b_{g-i} = [a_{2k-i+1}, b_{2k-i+1}]\cdots[a_{2k-1}, b_{2k-1}][a_{2k}, b_{2k}]. \tag{2}$$

Proof. Using the relations $B_0 = b_1b_2\cdots b_g = 1$, $B_1 = a_1b_1b_2\cdots b_gc_ga_g = 1$, and $c_g = 1$ in the fundamental group of $Y(k)$, we easily see that $a_1a_g = a_1a_{2k} = 1$. Next, using the relations $B_2 = a_1b_2b_3\cdots b_{g-1}c_{g-1}a_g = 1$, $B_3 = a_2b_2b_3\cdots b_{g-1}c_{g-1}a_{g-1} = 1$, and $a_1a_g = 1$, we obtain $a_2a_{g-1} = a_2a_{2k-1} = 1$. By continuing in this fashion (i.e. using the relations $B_{2i-2} = a_{i-1}b_ib_{i+1}\cdots b_{g-i+1}c_{g-i+1}a_{g-i+2}$, $B_{2i-1} = a_ib_ib_{i+1}\cdots b_{g-i+1}c_{g-i+1}a_{g+1-i} = 1$, and $a_{i-1}a_{g-i+2} = 1$), we have $a_ia_{g-i+1} = 1$ for any i between 1 and k. Furthemore, by considering the relations $B_0 = b_1b_2\cdots b_g = 1$, $c_{g-1} = [a_g, b_g]^{-1}$, and $a_1a_g = 1$, we have $1 = B_2 = a_1b_2b_3\cdots b_{g-1}c_{g-1}a_g = b_2b_3\cdots b_{g-1}c_{g-1} = b_2b_3\cdots b_{g-1}(b_ga_g{b_g}^{-1}{a_g}^{-1}) = 1$. Consequently, using the relations $B_{2i} = a_ib_{i+1}b_{i+2}\cdots b_{g-i}c_{g-i}a_{g+1-i} = 1$, and $a_ia_{g-i+1} = 1$, we have $b_{i+1}b_{i+2}\cdots b_{g-i} = [a_{g-i+1}, b_{g-i+1}]\cdots[a_{g-1}, b_{g-1}][a_g, b_g]$ for any i between 1 and k.

□

Using the Lemma above, we see that the fundamental group of $Y(k) = \Sigma_k \times \mathbb{S}^2 \# 4\overline{\mathbb{CP}}^2$ is isomorphic to the surface group $\Pi_k = \pi_1(\Sigma_k)$, generated by the loops a_1, b_1, $\cdots$, a_k, and b_k. Furthemore, the fundamental group of the complement $Y(k) \setminus \nu(\Sigma_{2k})$ is also Π_k. The normal circle $\mu = pt \times \partial(\mathbb{D}^2)$ to Σ_{2k} can be deformed using an exceptional sphere section, thus trivial in $\pi_1(Y(k) \setminus \nu\Sigma_{2k})$.

4.2. **Gurtas' fibration.** In [16], Yusuf Gurtas generalized the constructions in [19, 18] even further. In [16], he presented the positive Dehn twist expression for a new set of involutions in the mapping class group M_{h+v} of a compact, closed, oriented 2-dimensional surface Σ_{h+v}. These involutions were obtained by gluing the horizontal involution on a surface Σ_h and the vertical involution on a surface Σ_v, where v is a positive even number. Let θ denote the involution under consideration on the surface Σ_{h+v}, as shown in Figure 5 below. According to Gurtas [16], θ can be expressed as a product of $8h + 2v + 4$ positive Dehn twists. Observe that if we set $h = 0$ and $v \geq 2$, we recover the family of Lefschetz fibrations in [19, 18].

In the above notation, we set $n = h+1$ and $v = 2k$, for reasons which will be clear soon. Let $Y(n, k)$ denote the total space of the Lefschetz fibration defined by the word $\theta^2 = 1$ in the mapping class group M_{2k+n-1}. The manifold $Y(n, k)$ has a genus $g = 2k + n - 1$ Lefschetz fibration over $\mathbb{S}^2$ with $s = 8h + 2v + 4 = 8(n-1) + 2(2k) + 4 = 8n + 4k - 4$ singular fibers

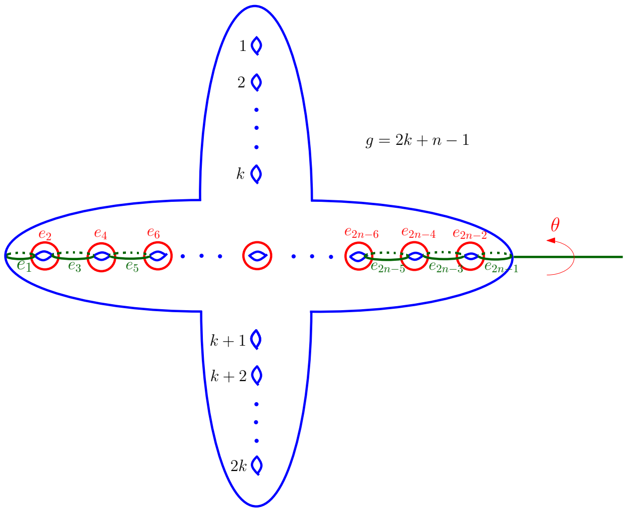

FIGURE 5. The involution θ of the surface Σ_{2k+n-1}

and the vanishing cycles all are about nonseparating curves [16]. The Euler characteristic of the symplectic 4-manifold $Y(n,k)$ can be computed using the following formula: $e(Y(n,k)) = e(\mathbb{S}^2)e(F) + s = 2(2-2(n+2k-1)) + 8n + 4k - 4 = 4n - 4k + 4$. The signature of the Lefschetz fibration described by the word $\theta^2 = 1$ was computed in [16]: $\sigma(Y(n,k)) = -4n$ (see also related work in [26]). Now, using the formulas $\chi_h = (\sigma + e)/4$, $c_1^2 = 3\sigma + 2e$, we compute: $\chi_h(Y(n,k)) = 1 - k$ and $c_1^2(Y(n,k)) = -4(n+2k-2)$.

The branched-cover description of the above Lefschetz fibrations is given as follows: take a double branched cover of $\Sigma_k \times \mathbb{S}^2$ along the union of $2n$ disjoint copies of $pt \times \mathbb{S}^2$ and two disjoint copies of $\Sigma_k \times pt$ (See Figure 6). The resulting branched cover has $4n$ singular points, corresponding to the number of the intersection points of the horizontal spheres and the vertical genus k surfaces in the branch set. We desingularize this manifold to obtain $Y(n,k) = \Sigma_k \times \mathbb{S}^2 \# 4n\overline{\mathbb{CP}}^2$. A generic horizontal fiber is the double cover of $\mathbb{S}^2$, branched over two points. Thus, we have a sphere fibration on $Y(n,k) = \Sigma_k \times \mathbb{S}^2 \# 4n\overline{\mathbb{CP}}^2$. A generic fiber of the vertical fibration is the double cover of Σ_k, branched over $2n$ points. Thus, a generic fiber of the vertical fibration has genus $n + 2k - 1$. Furthermore, two complicated singular fibers of the vertical fibration can be perturbed into $4n + 2k - 2$ Lefschetz type singular fibers (see [26] for more details and proofs).

Next, we recall the main theorem of [16], which will be needed in the proof of our Lemma 4.4. We refer the reader to [16] for any unexplained notation.

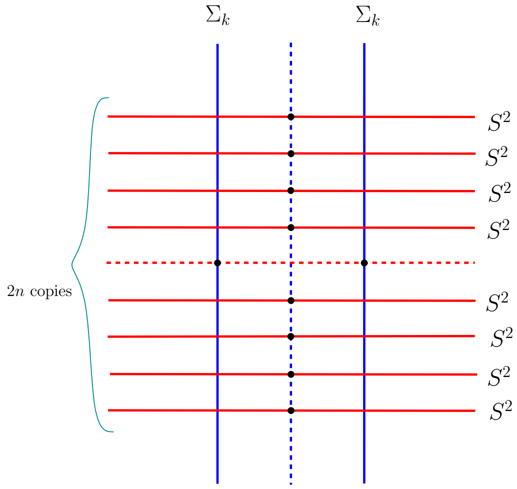

FIGURE 6. The branch locus for $\Sigma_k \times S^2 \# 4n\overline{\mathbb{CP}}^2$

Theorem 4.3. *The positive Dehn twist expression for the involution θ is given by*

$$\theta = e_{2i+2} \cdots e_{2n-2} e_{2n-1} e_{2i} \cdots e_2 e_1 B_0 e_{2n-1} e_{2n-2} \cdots e_{2i+2} e_1 e_2 \cdots e_{2i} B_1 B_2 \cdots B_{2k-1} B_{2k} e_{2i+1}.$$

Now, we prove two lemmas which we use in the sequel.

Lemma 4.4. *The following relations hold in the fundamental group of $Y(n,k)$*

$$\begin{aligned} e_1 = 1, \quad e_2 = 1, \quad \cdots, \quad e_{2n-2} = 1, \quad e_{2n-1} = 1 \\ a_1 a_{2k} = 1, \quad a_2 a_{2k-1} = 1, \quad \cdots, \quad a_k a_{k+1} = 1, \\ b_1 b_2 \cdots b_{2k} = 1, \quad b_2 b_3 \cdots b_{2k-1} = [a_{2k}, b_{2k}], \quad \cdots, \\ b_{i+1} b_{i+2} \cdots b_{g-i} = [a_{2k-i+1}, b_{2k-i+1}] \cdots [a_{2k-1}, b_{2k-1}][a_{2k}, b_{2k}]. \end{aligned} \tag{3}$$

Proof. Notice that the first set of relations simply follows from the fact that the Dehn twists along the curves $e_1, e_2, \cdots, e_{2n-1}$ appear in the factorization of θ. By using the relations $e_1 = 1$, $e_2 = 1$, $\cdots$, $e_{2n-1} = 1$, we obtian the relations $B_0 = b_1 b_2 \cdots b_{2k} = 1$, $B_1 = a_1 b_1 b_2 \cdots b_{2k} c_{2k} a_{2k} = 1$, $\cdots$, $B_{2i-1} = a_i b_i b_{i+1} \cdots b_{2k+1-i} c_{2k+1-i} a_{2k+1-i} = 1$, $B_{2i} = a_i b_{i+1} b_{i+2} \cdots b_{2k-i} c_{2k-i} a_{2k+1-i} = 1$, $\cdots$, $B_{2k} = a_k c_k a_{k+1} = 1$. To prove the remaining relations, we use the relations $B_0 = 1$, $B_1 = 1$, $\cdots$, $B_{2i-1} = 1$, $B_{2i} = 1$, $\cdots$, $B_{2k} = 1$, and $c_k = 1$ in the fundamental group of $Y(n,k)$. The proof is identical to the proof of Lemma 4.2 and therefore is omitted.

□

Lemma 4.5. *The genus $2k+n-1$ Lefschetz fibration on $Y(n,k)$ admits at least $4n$ disjoint -1 sphere sections.*

Proof. We first observe that $Y(n,k)$ is the symplectic sum of $Y(k) = \Sigma_k \times \mathbb{S}^2\#4\overline{\mathbb{CP}}^2$ and $X(n-1,1) = \mathbb{CP}^2\#(4n+1)\overline{\mathbb{CP}}^2$ along the spheres pt $\times\, \mathbb{S}^2$ and the sphere fiber of horizontal fibration on $X(n-1,1)$. Since a generic sphere fiber of $X(n-1,1)$ intersects a generic genus $n-1$ fiber at two points, after the symplectic sum we obtain a genus $2k+n-1$ fibration on $Y(n,k)$ over $\mathbb{S}^2$. Since for $n \geq 2$, the genus $n-1$ fibration on $X(n-1,1)$ admits $4n$ disjoint (-1)-sphere sections (see [22]), these -1 sphere sections extend to $Y(n,k)$.

□

4.3. **Luttinger surgeries on product manifolds $\Sigma_n \times \Sigma_2$ and $\Sigma_n \times \mathbb{T}^2$.** The following two family of symplectic building blocks will be used in our construction. They are obtained from $\Sigma_n \times \Sigma_2$ and $\Sigma_n \times \mathbb{T}^2$ by performing a sequence of Luttinger surgeries along the Lagrangian tori [12, 7]. The first family has $b_1 = 0$, and the second has $b_1 = 2$. One can relate these symplectic building blocks X_K, $V_{KK'}$ and Y_K, $W_{KK'}$ in [1, 2], where K and K' are genus 1 and genus $n-1$ fibered knots, respectively. In order to make the connection between these constructions, one should view the knot surgery manifold $M_K \times \mathbb{S}^1$ via the sequence of $2g$ Luttinger surgeries on $\Sigma_g \times \mathbb{T}^2$, where K is a fibered genus g knot in $\mathbb{S}^3$. This was carefully explained in [3] for a trefoil knot. We also refer the reader [21] (pages 225-226).

To construct the first family of examples, we proceed as follows. Let us fix integers $n \geq 2$, $p_i \geq 0$ and $q_i \geq 0$, where $1 \leq i \leq n$. We denote by $Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)$ the symplectic 4-manifold obtained by performing the following $2n+4$ Luttinger surgeries on $\Sigma_n \times \Sigma_2$, which consist of the following 8 surgeries

$$\begin{array}{ll} (a_1' \times c_1', a_1', -1), & (b_1' \times c_1'', b_1', -1), \\ (a_2' \times c_2', a_2', -1), & (b_2' \times c_2'', b_2', -1), \\ (a_2' \times c_1', c_1', +1/p_1), & (a_2'' \times d_1', d_1', +1/q_1), \\ (a_1' \times c_2', c_2', +1/p_2), & (a_1'' \times d_2', d_2', +1/q_2), \end{array}$$

followed by the following set of additional $2(n-2)$ Luttinger surgeries

$$\begin{array}{cc} (b_1' \times c_3', c_3', -1/p_3), & (b_2' \times d_3', d_3', -1/q_3), \\ \dots, & \dots, \\ (b_1' \times c_n', c_n', -1/p_n), & (b_2' \times d_n', d_n', -1/q_n). \end{array}$$

In our notation above, a_i, b_i $(i = 1, 2)$ and c_j, d_j $(j = 1, \ldots, n)$ denote the standard loops that generate $\pi_1(\Sigma_2)$ and $\pi_1(\Sigma_n)$, respectively. The Figure 7, which we borrow from [7] (with a minor modification), shows a typical Lagrangian tori along which the Luttinger surgeries performed.

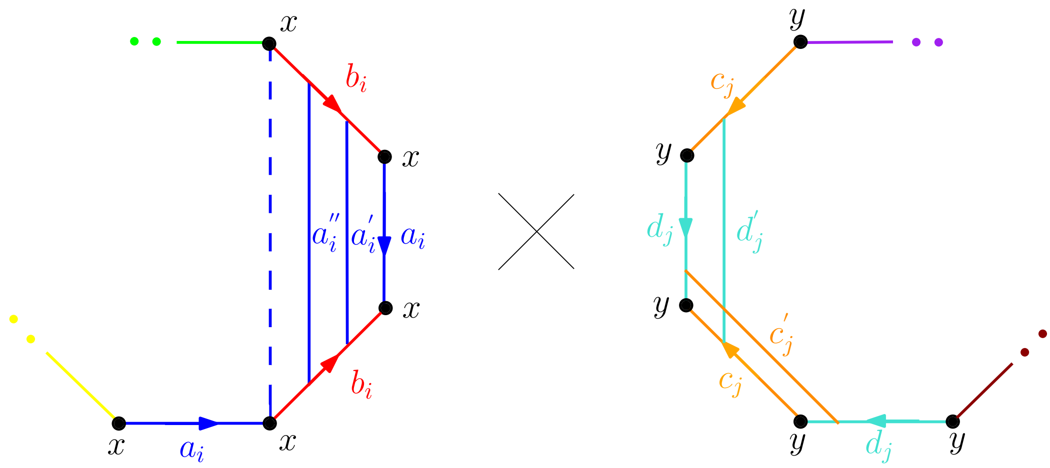

FIGURE 7. Lagrangian tori $a_i' \times c_j'$ and $a_i'' \times d_j'$

Using the Lemma 3.6, we see that the Euler characteristic of $Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)$ is $4n-4$ and the signature is 0. Furthermore, the Lemma 3.6 implies that the fundamental group $\pi_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n))$ is generated by loops a_i, b_i, c_j, d_j $(i = 1, 2$ and $j = 1, \ldots, n)$ and the following relations hold in $\pi_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n))$:

$$
\begin{gathered}
[b_1^{-1}, d_1^{-1}] = a_1, \quad [a_1^{-1}, d_1] = b_1, \quad [b_2^{-1}, d_2^{-1}] = a_2, \quad [a_2^{-1}, d_2] = b_2, \\
[d_1^{-1}, b_2^{-1}] = c_1^{p_1}, \quad [c_1^{-1}, b_2] = d_1^{q_1}, \quad [d_2^{-1}, b_1^{-1}] = c_2^{p_2}, \quad [c_2^{-1}, b_1] = d_2^{q_2}, \\
[a_1, c_1] = 1, \quad [a_1, c_2] = 1, \quad [a_1, d_2] = 1, \quad [b_1, c_1] = 1, \\
[a_2, c_1] = 1, \quad [a_2, c_2] = 1, \quad [a_2, d_1] = 1, \quad [b_2, c_2] = 1, \\
[a_1, b_1][a_2, b_2] = 1, \quad \prod_{j=1}^{n} [c_j, d_j] = 1, \\
[a_1^{-1}, d_3^{-1}] = c_3^{p_3}, \quad [a_2^{-1}, c_3^{-1}] = d_3^{q_3}, \quad \ldots, \quad [a_1^{-1}, d_n^{-1}] = c_n^{p_n}, \quad [a_2^{-1}, c_n^{-1}] = d_n^{q_n}, \\
[b_1, c_3] = 1, \quad [b_2, d_3] = 1, \quad \ldots, \quad [b_1, c_n] = 1, \quad [b_2, d_n] = 1.
\end{gathered} \tag{4}
$$

Since the surfaces $\Sigma_2 \times \{\mathrm{pt}\}$ and $\{\mathrm{pt}\} \times \Sigma_n$ in $\Sigma_2 \times \Sigma_n$ are not affected by the above Luttinger surgeries, they descend to surfaces in $Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)$. Let us denote these symplectic submanifolds by Σ_2 and Σ_n. Notice that we have $[\Sigma_2]^2 = [\Sigma_n]^2 = 0$ and $[\Sigma_2] \cdot [\Sigma_n] = 1$.

Now, we consider a slightly different family. Let us fix a quadruple of integers $n \geq 2$, $m \geq 1$, $p \geq 1$ and $q \geq 1$. Let $Y_n(1/p, m/q)$ denote smooth 4-manifold obtained by performing the following $2n$ torus surgeries on $\Sigma_n \times \mathbb{T}^2$:

$$(5) \qquad \begin{array}{l} (a_1' \times c', a_1', -1), \quad (b_1' \times c'', b_1', -1), \\ (a_2' \times c', a_2', -1), \quad (b_2' \times c'', b_2', -1), \\ \cdots, \quad \cdots \\ (a_{n-1}' \times c', a_{n-1}', -1), \quad (b_{n-1}' \times c'', b_{n-1}', -1), \\ (a_n' \times c', c', +1/p), \quad (a_n'' \times d', d', +m/q). \end{array}$$

Let a_i, b_i ($i = 1, 2, \cdots, n$) and c, d denote the standard generators of $\pi_1(\Sigma_n)$ and $\pi_1(\mathbb{T}^2)$, respectively. Since all the torus surgeries listed above are Luttinger surgeries when $m = 1$ and the Luttinger surgery preserves minimality, $Y_n(1/p, 1/q)$ is a minimal symplectic 4-manifold. The fundamental group of $Y_n(1/p, m/q)$ is generated by a_i, b_i ($i = 1, 2, 3 \cdots, n$) and c, d, and the Lemma 3.6 implies that the following relations hold in $\pi_1(Y_n(1/p, m/q))$:

$$(6) \qquad \begin{gathered} [b_1^{-1}, d^{-1}] = a_1, \quad [a_1^{-1}, d] = b_1, \quad [b_2^{-1}, d^{-1}] = a_2, \quad [a_2^{-1}, d] = b_2, \\ \cdots, \quad \cdots, \\ [b_{n-1}^{-1}, d^{-1}] = a_{n-1}, \quad [a_{n-1}^{-1}, d] = b_{n-1}, \quad [d^{-1}, b_n^{-1}] = c^p, \quad [c^{-1}, b_n]^{-m} = d^q, \\ [a_1, c] = 1, \quad [b_1, c] = 1, \quad [a_2, c] = 1, \quad [b_2, c] = 1, \\ [a_3, c] = 1, \quad [b_3, c] = 1, \\ \cdots, \quad \cdots, \\ [a_{n-1}, c] = 1, \quad [b_{n-1}, c] = 1, \\ [a_n, c] = 1, \quad [a_n, d] = 1, \\ [a_1, b_1][a_2, b_2] \cdots [a_n, b_n] = 1, \quad [c, d] = 1. \end{gathered}$$

Let us denote by $\Sigma_n' \subset Y_n(1/p, l/q)$ a genus n surface that desend from the surface $\Sigma_n \times \{\text{pt}\}$ in $\Sigma_n \times \mathbb{T}^2$.

5. Construction of exotic 4-manifolds

In this section we prove the following theorem.

Theorem 5.1. *Let M be $(2n+2k-3)\mathbb{CP}^2\#(6n+2k-3)\overline{\mathbb{CP}}^2$ for any $n \geq 1$, $k \geq 1$, and $(n, k) \neq (1, 1)$. There exists a new family of smooth closed simply-connected minimal symplectic 4-manifold and an infinite family of non-symplectic 4-manifolds that is homeomorphic but not diffeomorphic to M that can obtained by the sequence of Luttinger surgeries and a single generalized torus surgery on Lefschetz fibrations.*

We will accomplish the proof of this theorem in several steps. First, we provide the construction which uses a combination of the Luttinger surgery [21] and symplectic fiber sum operations [14]. Since the order of these operations do not influence the outcome, our symplectic examples can be obtained by performing Luttinger surgeries on Lefschetz fibrations over higher genus surfaces, which are obtained by symplectically summing $Y(n,k) = \Sigma_k \times \mathbb{S}^2 \# 4n\overline{\mathbb{CP}}^2$ and $\Sigma_{2k+n-1} \times \mathbb{T}^2$ along the symplectic submanifolds of genus $2k+n-1$. Next, we show that the fundamental groups are trivial, and determine the homeomorphism types of our examples. In the final step, we study the diffeomorphism types of our manifolds and distinguish them from $(2n+2k-3)\mathbb{CP}^2 \# (6n+2k-3)\overline{\mathbb{CP}}^2$ by computing their Seiberg-Witten invariants and applying Theorem [25].

Our first building block will be the symplectic 4-manifold $Y(n,k) = \Sigma_k \times \mathbb{S}^2 \# 4n\overline{\mathbb{CP}}^2$ with a genus $2k+n-1$ symplectic submanifold $\Sigma_{2k+n-1} \subset Y(n,k)$, a regular fiber of the Lefschetz fibration which we discussed Section 4. Here we endowed $Y(n,k) = \Sigma_k \times \mathbb{S}^2 \# 4n\overline{\mathbb{CP}}^2$ with the symplectic structure induced from the given Lefschetz fibration. The other building block will be the symplectic 4-manifold $Y_g(1,1)$ along the symplectic submanifold Σ'_g (see Section 4), where we set $g = 2k+n-1$ and $p = q = m = 1$. We denote by $X(n,k)$ the symplectic 4-manifold obtained by forming the symplectic fiber sum of $Y(n,k)$ and $Y_g(1,1)$ along the surfaces Σ_{2k+n-1} and Σ'_g.

Let us first remind us from Section 4 that loops a_1, b_1, $\cdots$, a_{2k}, and b_{2k} generate the inclusion-induced image of $\pi_1(\Sigma_{2k+n-1} \times S^1)$ inside $\pi_1(Y(n,k) \setminus \nu\Sigma_{2k+n-1})$. This is due to the fact that the loops e_1, e_2, $\cdots$, e_{2n-2}, e_{2n-1} and the normal circle $\mu = \{\mathrm{pt}\} \times S^1$ to Σ_{2k+n-1} are all nullhomotopic in $(Y(n,k) \setminus \nu\Sigma_{2k+n-1})$. We choose a base point x on $\partial(\nu\Sigma_{2k+n-1})$ such that $\pi_1(Y(n,k) \setminus \nu\Sigma_{2k+n-1})$ is generated by the homotopy classes of the based loops a_1, b_1, $\cdots$, a_{2k}, and b_{2k} at x.

Let a'_1, b'_1, $\cdots$ a'_g, b'_g and $\mu' = [c,d]$ generate $\pi_1(\Sigma'_g \times S^1)$ in $\pi_1(Y_g(1,1) \setminus \nu\Sigma'_g)$. We choose the base point y to lie on the boundary $\partial(\nu\Sigma'_g)$ (see [7]).

We choose an orientation-reversing gluing diffeomorphism $\psi : \Sigma_{2k+n-1} \times S^1 \to \Sigma'_g \times S^1$ that maps the generators of the fundamental groups as follows:

$$\psi_*(a_1) = a'_1,\ \psi_*(b_1) = b'_1,\ \cdots,\ \psi_*(a_k) = a'_k,\ \psi_*(b_k) = b'_k,\ \psi_*(e_1) = a'_{k+1},$$
$$\psi_*(e_2) = b'_{k+1},\ \cdots,\ \psi_*(e_{2n-3}) = a'_{k+n-1},\ \psi_*(e_{2n-2}) = b'_{k+n-1},\ \psi_*(a_{k+1}) = a'_{k+n},$$
$$\psi_*(b_{k+1}) = b'_{k+n},\ \cdots, \psi_*(a_{2k}) = a'_{2k+n-1},\ \psi_*(b_{2k}) = b'_{2k+n-1},\ \psi_*(\mu) = (\mu')^{-1}.$$

It follows from Gompf's theorem in [14] that $X(n,k)$ is symplectic.

Lemma 5.2. *$X(n,k)$ is simply-connected.*

Proof. By applying the Seifert-Van Kampen theorem, we see that

$$\pi_1(X(n,k)) = \frac{\pi_1(Y(n,k) \setminus \nu\Sigma_{2k+n-1}) * \pi_1(Y_g(1,1) \setminus \nu\Sigma'_g)}{\langle a_1 = a'_1,\, b_1 = b'_1,\, \cdots,\, a_{2k} = a'_{2k+n-1},\, b_{2k} = b'_{2k+n-1},\, \mu = \mu' = 1\rangle}.$$

Since the loops $e_1,\ e_2,\ \cdots,\ e_{2n-3},\ e_{2n-2}$, corresponding to the vanishing cycles, and the normal circle $\mu = \{\text{pt}\} \times S^1$ are all nullhomotopic in $Y(n,k) \setminus \nu\Sigma_{2k+n-1}$, we get the following presentation for the fundamental group of $X(n,k)$.

$$\begin{aligned}
\pi_1(X(n,k)) \quad = \quad & \langle a_1, b_1, \cdots, a_{2k}, b_{2k};\ c, d;\ | \\
& [b_1^{-1}, d^{-1}] = a_1, \quad [a_1^{-1}, d] = b_1, \\
& \cdots, \quad \cdots, \\
& [b_{2k-1}^{-1}, d^{-1}] = a_{2k-1}, \quad [a_{2k-1}^{-1}, d] = b_{2k-1}, \quad [d^{-1}, b_{2k}^{-1}] = c, \quad [c^{-1}, b_{2k}] = d, \\
& [a_1, c] = 1, \quad [b_1, c] = 1, \quad [a_2, c] = 1, \quad [b_2, c] = 1, \\
& [a_3, c] = 1, \quad [b_3, c] = 1, \\
& \cdots, \quad \cdots, \\
& [a_{2k-1}, c] = 1, \quad [b_{2k-1}, c] = 1, \\
& [a_{2k}, c] = 1, \quad [a_{2k}, d] = 1, \\
& [a_1, b_1][a_2, b_2] \cdots [a_{2k}, b_{2k}] = 1, \quad [c, d] = 1, \\
& a_1 a_{2k} = 1, \quad a_2 a_{2k-1} = 1, \quad \cdots, \quad a_k a_{k+1} = 1, \\
& b_1 b_2 \cdots b_{2k} = 1, \quad b_2 b_3 \cdots b_{2k-1} = [a_{2k}, b_{2k}], \\
& \cdots, \cdots, \cdots \\
& b_{i+1} b_{i+2} \cdots b_{2k-i} = [a_{2k-i+1}, b_{2k-i+1}] \cdots [a_{2k-1}, b_{2k-1}][a_{2k}, b_{2k}], \\
& [a_1, b_1][a_2, b_2] \cdots [a_k, b_k] = 1\rangle.
\end{aligned}$$

To prove $\pi_1(X(n,k)) = 1$, it is enough to prove that $b_1 = 1$, which in turn will imply that all other generators are trivial. Using the last set of identities, we have ${a_1}^{-1} = a_{2k}, \cdots, {a_k}^{-1} = a_{k+1}$. Let us rewrite the relation $[a_1^{-1}, d] = b_1$ as $[a_{2k}, d] = b_1$. Since $[a_{2k}, d] = 1$, we obtain $b_1 = 1$. This in turn imply that $a_1 = a_{2k} = b_{2k} = c = d = 1$ using the relations $[b_1^{-1}, d^{-1}] = a_1$, $a_{2k} = {a_1}^{-1}$, $b_1 b_2 \cdots b_{2k} = 1$, $b_2 b_3 \cdots b_{2k-1} = [a_{2k}, b_{2k}]$, $[d^{-1}, b_{2k}^{-1}] = c$ and $[c^{-1}, b_{2k}] = d$. Since $[b_i^{-1}, d^{-1}] = a_i$ and $[a_i^{-1}, d] = b_i$, for any $1 \le i \le 2k-1$, we have $a_2 = \cdots = a_{2k-1} = b_2 = \cdots = b_{2k-1} = 1$. Thus, we conclude that $\pi_1(X(n,k))$ is trivial.

□

The following lemma is elementary; we include a proof for completeness.

Lemma 5.3. *$e(X(n,k)) = 8n + 4k - 4$, $\sigma(X(n,k)) = -4n$, $c_1^2(X(n,k)) = 4n + 8k - 8$, and $\chi_h(X(n,k)) = n + k - 1$.*

Proof. Using the Lemma 3.3, we have $e(X(n,k)) = e(Y(n,k)) + e(Y_g(1,1)) + 4(2k+n-2)$, $\sigma(X(n,k)) = \sigma(Y(n,k)) + \sigma(Y_g(1,1))$, $c_1^2(X(n,k)) = c_1^2(Y(n,k)) + c_1^2(Y_g(1,1)) + 8(2k+n-2)$, and $\chi_h(X(n,k)) = \chi_h(Y(n,k)) + \chi_h(Y_g(1,1)) + (2k+n-2)$. Since $e(Y_g(1,1)) = 0$, $\sigma(Y_g(1,1)) = 0$, $c_1^2(Y_g(1,1)) = 0$, $\chi_h(Y_g(1,1)) = 0$, $e(Y(n,k)) = 4-4k+4n$, $\sigma(Y(n,k)) = -4n$, $c_1^2(Y(n,k)) = 8-8k-4n$, and $\chi_h(Y(n,k)) = 1-k$, the proof of lemma follows. □

Using Freedman's classification theorem for simply-connected 4-manifolds [13], the lemma above and the fact that $X(n,k)$ contains $4n$ tori of self-intersection -1 resulting from the fiber summing, we conclude that $X(n,k)$ is homeomorphic to $(2n+2k-3)\mathbb{CP}^2\#(6n+2k-3)\overline{\mathbb{CP}}^2$. Since $X(n,k)$ is symplectic, by Taubes's theorem [23] $\mathrm{SW}_{X(n,k)}(K_{X(n,k)}) = \pm 1$, where $K_{X(n,k)}$ is the canonical class of $X(n,k)$. Next, using the connected sum theorem for the Seiberg-Witten invariant, we deduce that the Seiberg-Witten invariant of $(2n+2k-3)\mathbb{CP}^2\#(6n+2k-3)\overline{\mathbb{CP}}^2$ is trivial. Since the Seiberg-Witten invariant is a diffeomorphism invariant, $X(n,k)$ is not diffeomorphic to $(2n+2k-3)\mathbb{CP}^2\#(6n+2k-3)\overline{\mathbb{CP}}^2$.

All $4n$ exceptional spheres E_1, E_2, $\cdots$, E_{4n-1}, and E_{4n}, which are the sections of the genus $2k+n-1$ fibration on $Y(n,k) = \Sigma_k \times \mathbb{S}^2 \# 4n\overline{\mathbb{CP}}^2$, meet with the fiber $\Sigma = 2(\Sigma_k \times \{\mathrm{pt}\}) + n(\{\mathrm{pt}\} \times \mathbb{S}^2) - E_1 - E_2 - \cdots - E_{4n-1} - E_{4n}$. Moreover, it follows from the adjunction formula that any embedded symplectic -1 sphere in $Y(n,k)$ has the form $r\mathbb{S}^2 \pm E_i$, thus intersect nontrivially with the fiber $\Sigma = 2(\Sigma_k \times \{\mathrm{pt}\}) + n(\{\mathrm{pt}\} \times \mathbb{S}^2) - E_1 - E_2 - \cdots - E_{4n-1} - E_{4n}$. Using the Usher's Theorem, the symplectic sum manifold $X(n,k)$ is a minimal symplectic 4-manifold. Since symplectic minimality implies smooth minimality (cf. [20]), $X(n,k)$ is also smoothly minimal.

In order to produce an infinite family of exotic $(2n+2k-3)\mathbb{CP}^2\#(6n+2k-3)\overline{\mathbb{CP}}^2$'s, we replace the building block $Y_g(1,1)$ used in our construction above with $Y_g(1,m)$, where $|m| > 1$. Let us denote the resulting smooth 4-manifold as $X(n,k,m)$. In the presentation of the fundamental group, this amounts replacing a single relation $[c^{-1}, b_{2k}] = d$ in $\pi_1(X(n,k))$, corresponding to the Luttinger surgery $(a''_{2k+n-1} \times d', d', 1)$, with $[c^{-1}, b_{2k}]^{-m} = d$. Chainging this relation has no affect on our proof of $\pi_1(X(n,k)) = 1$; all the fundamental group calculations follow the same lines of arguments, and thus $\pi_1(X(n,k,m))$ is trivial group.

Moreover, using the argument similar as in [3] (Section 4, pages 12-18), we see that $X(n,k)$ has one basic class up to sign, the canonical class $K_{X(n,k)}$ up to sign. Let us denote by $X(n,k)_0$ the symplectic 4-manifold obtained by performing the following Luttinger surgery on: $(a''_{2k+n-1} \times d', d', 0/1)$. It is easy to see that $\pi_1(X(n,k)_0) = \mathbb{Z}$ and the canonical class of $X(n,k)_0$ is given by the formula $K_{X(n,k)_0} = 2[\Sigma_k] + \sum_{j=1}^{4n}[R_j]$, where R_j are tori of self-intersection -1. Moreover, the Seiberg-Witten invariants of the basic class $\beta_{n,k,m}$ of $X(n,k,m)$ corresponding to the canonical class $K_{X(n,k)_0}$ evaluates as $SW_{X(n,k,m)}(\beta_{n,k,m}) = SW_{X_{n,k}}(K_{X(n,k)}) + (m-1)SW_{X(n,k)_0}(K_{X(n,k)_0}) = 1 + (m-1) = m$. Thus, we conclude that $X(n,k,m)$ is nonsymplectic for any $m \geq 2$.

6. Construction of symplectic 4-manifolds with various fundamental groups

In this section we modify the above construction to obtain symplectic 4-manifolds with the various finitely generated fundamental groups, such as the free groups of rank $s \geq 1$, and the finite free products of cyclic groups, using the *Luttinger surgery.* Our construction can be generalized further to obtain symplectic 4-manifolds with arbitrarily finitely presented fundamental groups and with *small size*, but we will not pursue this here.

Our first building block will be the symplectic 4-manifold $Y(n,k) = \Sigma_k \times \mathbb{S}^2 \# 4n\overline{\mathbb{CP}}^2$ along with a regular fiber of the genus $2k+n-1$ Lefschetz fibration on $Y(n,k)$ (see Section 4). We equip $Y(n,k) = \Sigma_k \times \mathbb{S}^2 \# 4n\overline{\mathbb{CP}}^2$ with the symplectic structure induced from the given Lefschetz fibration. The second building block will be the symplectic 4-manifold $Y_l(1/p_1, 1/q_1, \cdots, 1/p_l, 1/q_l)$ discussed in the Section 4.3, along with the symplectic submanifold Σ_l (see Section 4), where we set $l = 2k+n-1$. To simplify the notation, we set $\overline{p} = (p_1, \ldots, p_l)$ and $\overline{q} = (q_1, \ldots, q_l)$ throughout this section. Let $X(n,k,\overline{p},\overline{q})$ denote the symplectic 4-manifold obtained by forming the symplectic fiber sum of $Y(n,k)$ and $Y_l(1/p_1, 1/q_1, \cdots, 1/p_l, 1/q_l)$ along the surfaces Σ_{2k+n-1} and Σ_l. Let c_1, d_1, $\cdots$, c_l, d_l, and $\mu' = [a_1,b_1][a_2,b_2]$ generate $\pi_1(\Sigma_l \times S^1)$ in $\pi_1(Y_l(1/p_1, 1/q_1, \cdots, 1/p_l, 1/q_l) \setminus \nu\Sigma_l)$ (see Section 4).

We choose the gluing diffeomorphism $\psi : \Sigma_{2k+n-1} \times S^1 \to \Sigma_l \times S^1$ that maps the generators of the fundamental groups as follows:

$$\psi_*(\alpha_1) = c_1,\ \psi_*(\beta_1) = d_1,\ \psi_*(\alpha_2) = c_2,\ \psi_*(\beta_2) = d_2,\ \cdots,$$
$$\psi_*(\alpha_{2k}) = c_{2k},\ \psi_*(\beta_{2k}) = d_{2k},\ \psi_*(e_1) = c_{2k+1},\ \psi_*(e_2) = d_{2k+1},\ \cdots,$$
$$\psi_*(e_{2n-3}) = c_{2k+n-1},\ \psi_*(e_{2n-2}) = d_{2k+n-1},\ \psi_*(\mu) = \mu'$$

By Gompf's theorem in [14], $X(n,k,\overline{p},\overline{q})$ is symplectic. By Van Kampen's theorem, we have

$$\pi_1(X(n,k,\overline{p},\overline{q})) = \frac{\pi_1(Y(n,k) \setminus \nu\Sigma_{2k+n-1}) * \pi_1(Y_l(1/p_1, 1/q_1, \cdots, 1/p_l, 1/q_l) \setminus \nu\Sigma_l)}{\langle \alpha_1 = c_1,\ \beta_1 = d_1,\ \cdots,\ \alpha_{2k} = c_{2k},\ \beta_{2k} = d_{2k},\ \cdots,\ e_{2n-2} = d_{2k+n-1},\ \mu = \mu' \rangle}.$$

Since the loops $e_1, e_2, \cdots, e_{2n-3}, e_{2n-2}$ and the normal circle $\mu = \{\mathrm{pt}\} \times S^1$ are all nullhomotopic in $Y(n,k) \setminus \nu\Sigma_{2k+n-1}$, we get the following presentation for the fundamental group of $X(n,k,\overline{p},\overline{q})$.

$$
\begin{aligned}
\pi_1(X(n,k,\overline{p},\overline{q})) \;=\; & \langle c_1, d_1, \cdots, c_{2k}, d_{2k};\, a_1, b_1, a_2, b_2; \mid \\
& [b_1^{-1}, d_1^{-1}] = a_1, \quad [a_1^{-1}, d_1] = b_1, \quad [b_2^{-1}, d_2^{-1}] = a_2, \quad [a_2^{-1}, d_2] = b_2, \\
& [d_1^{-1}, b_2^{-1}] = c_1^{p_1}, \quad [c_1^{-1}, b_2] = d_1^{q_1}, \quad [d_2^{-1}, b_1^{-1}] = c_2^{p_2}, \quad [c_2^{-1}, b_1] = d_2^{q_2}, \\
& [a_1, c_1] = 1, \quad [a_1, c_2] = 1, \quad [a_1, d_2] = 1, \quad [b_1, c_1] = 1, \\
& [a_2, c_1] = 1, \quad [a_2, c_2] = 1, \quad [a_2, d_1] = 1, \quad [b_2, c_2] = 1, \\
& [a_1, b_1][a_2, b_2] = 1, \quad \prod_{j=1}^{2k} [c_j, d_j] = 1, \\
& [a_1^{-1}, d_3^{-1}] = c_3^{p_3}, \quad [a_2^{-1}, c_3^{-1}] = d_3^{q_3}, \\
& \cdots, \quad \cdots, \\
& [a_1^{-1}, d_{2k}^{-1}] = c_{2k}^{p_{2k}}, \quad [a_2^{-1}, c_{2k}^{-1}] = d_{2k}^{q_{2k}}, \\
& [b_1, c_3] = 1, \quad [b_2, d_3] = 1, \quad \cdots, [b_1, c_{2k}] = 1, \quad [b_2, d_{2k}] = 1, \\
& c_1 c_{2k} = 1, \quad c_2 c_{2k-1} = 1, \quad \cdots, \quad c_k c_{k+1} = 1, \\
& d_1 d_2 \cdots d_{2k} = 1, \quad d_2 d_3 \cdots d_{2k-1} = [c_{2k}, d_{2k}], \quad \cdots, \\
& d_{i+1} d_{i+2} \cdots d_{2k-i} = [c_{2k-i+1}, d_{2k-i+1}] \cdots [c_{2k-1}, d_{2k-1}][c_{2k}, d_{2k}] \rangle.
\end{aligned}
$$

To realize the free group of rank $s \geq 1$ as the fundamental groups, we simply set $p_3 = \cdots = p_{2k} = 0$, $p_1 = p_2 = q_1 = \cdots = q_{2k} = 1$ in the above presentation. Using the identity ${c_{2k}}^{-1} = c_1$, we rewrite the relation $[a_2^{-1}, c_{2k}^{-1}] = d_{2k}$ as $[a_2^{-1}, c_1] = d_{2k}$. Since $[a_2, c_1] = 1$, we obtain $d_{2k} = 1$. $d_{2k} = 1$ in turn implies $d_1 = 1$ using the relations $d_2 d_3 \cdots d_{2k-1} = [c_{2k}, d_{2k}]$ and $d_1 d_2 \cdots d_{2k} = 1$. Next, using $[b_1^{-1}, d_1^{-1}] = a_1$, $[a_1^{-1}, d_1] = b_1$ and $[d_1^{-1}, b_2^{-1}] = c_1$, we obtain $a_1 = b_1 = c_1 = 1$. Since $[c_2^{-1}, b_1] = d_2$ and $[d_2^{-1}, b_1^{-1}] = c_2$, we have $d_2 = c_2 = 1$, which in turn lead $a_2 = b_2 = 1$. Next, using the relations $[a_2^{-1}, c_i^{-1}] = d_i$ for any $3 \leq i \leq 2k$, we have $d_3 = d_4 = \cdots = d_{2k} = 1$. Since ${c_{2k-i+1}}^{-1} = c_i$ for any $i \leq k$ and $c_1 = c_2 = 1$, we conclude that $\pi_1(X(n,k,\overline{p},\overline{q}))$ is a free group of rank $s := k-2$ generated by $c_3, \cdots, c_k$.

To realize the finite free products of cyclic groups as the fundamental groups, we simply set $p_1 = p_2 = 1$, $p_3 = \cdots = p_l = 0$, and let $q_i \geq 1$ vary arbitrarily in the above presentation. Furthermore, by varying $p_j \geq 1$ and $q_i \geq 1$ arbitrarily, we can realize many other finitely presented groups as the fundamental groups.

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