MOTIVIC CHERN CLASSES OF SCHUBERT CELLS, HECKE ALGEBRAS, AND APPLICATIONS TO CASSELMAN'S PROBLEM

PAOLO ALUFFI, LEONARDO C. MIHALCEA, JÖRG SCHÜRMANN, AND CHANGJIAN SU

ABSTRACT. Motivic Chern classes are elements in the K-theory of an algebraic variety X, depending on an extra parameter y. They are determined by functoriality and a normalization property for smooth X. In this paper we calculate the motivic Chern classes of Schubert cells in the (equivariant) K-theory of flag manifolds G/B. We show that the motivic class of a Schubert cell is determined recursively by the Demazure-Lusztig operators in the Hecke algebra of the Weyl group of G, starting from the class of a point. The resulting classes are conjectured to satisfy a positivity property. We use the recursions to give a new proof that they are equivalent to certain K-theoretic stable envelopes recently defined by Okounkov and collaborators, thus recovering results of Fehér, Rimányi and Weber. The Hecke algebra action on the K-theory of the Langlands dual flag manifold matches the Hecke action on the Iwahori invariants of the principal series representation associated to an unramified character for a group over a nonarchimedean local field. This gives a correspondence identifying the Poincaré duals of the motivic Chern classes to the standard basis in the Iwahori invariants, and the fixed point basis to Casselman's basis. We apply this correspondence to prove two conjectures of Bump, Nakasuji and Naruse concerning factorizations and holomorphy properties of the coefficients in the transition matrix between the standard and the Casselman's basis.

1. INTRODUCTION

Let X be a complex algebraic variety, and let $G_0(var/X)$ be the (relative) Grothendieck group of varieties over X. It consists of classes of morphisms $[f : Z \to X]$ modulo the scissor relations; cf. [Loo02, Bit04] and §4 below. Brasselet, Schürmann and Yokura [BSY10] defined the *motivic Chern transformation* $MC_y : G_0(var/X) \to K(X)[y]$ with values in the K-theory group of coherent sheaves in X to which one adjoins a formal variable y. The transformation MC_y is a group homomorphism, it is functorial with respect to proper push-forwards, and if X is smooth, it satisfies the normalization condition

$$MC_y[id_X : X \to X] = \sum [\wedge^j T^*(X)]y^j.$$

Here $[\wedge^j T^*(X)]$ is the K-theory class of the bundle of degree j differential forms on X. If $Z \subseteq X$ is a constructible subset, we denote by $MC_y(Z) := MC_y[Z \hookrightarrow X] \in K(X)[y]$ the motivic Chern class of Z. Because MC_y is a group homomorphism, it follows that if $X = \bigsqcup Z_i$ is a disjoint union of constructible subsets, then $MC_y(X) = \sum MC_y(Z_i)$. As

explained in [BSY10], the motivic Chern class $MC_y(Z)$ is related by a Hirzebruch-Riemann-Roch type statement to the Chern-Schwartz-MacPherson class $c_{\mathrm{SM}}(Z)$ in the homology of X, a class whose existence was conjectured by Deligne and Grothedieck and proved by MacPherson [Mac74]. Both the motivic and the CSM classes give a functorial way to attach K-theory, respectively (co)homology classes, to constructible subsets, and both satisfy the usual motivic relations. There is also an equivariant version of the motivic Chern class transformation, which uses equivariant varieties and morphisms, and has values in the appropriate equivariant K-theory group. Its definition was given in [FRW18], following closely the approach of [BSY10].

We take this opportunity to provide further details on the construction and properties of equivariant motivic Chern classes, such as functoriality and a Verdier-Riemann-Roch formula; see §4 below. However, the main goals of this paper are to build the computational foundations for the study of the (equivariant) motivic Chern classes of Schubert cells in the generalized flag manifolds, and to relate this to the representation theory of p-adic groups. Our main application consists of formulas for the transition coefficients between the standard and the Casselman bases of the module of Iwahori invariants of the principal series representation, in terms of localizations of motivic Chern classes.

Let G be a complex, simple, linear algebraic group, and B a Borel subgroup. By functoriality, the (equivariant) motivic Chern classes of Schubert cells in G/B determine those in any flag manifold G/P, where P is a parabolic subgroup. Based on previously discovered features of the CSM classes of Schubert cells [AM16, RV18, AMSS17], it was expected that the motivic classes would be closely related to objects which appear in geometric representation theory. We prove in this note that the motivic Chern classes of Schubert cells are recursively determined by the Demazure-Lusztig operators which appear in early works of Lusztig on Hecke algebras [Lus85]. Further, the motivic classes of Schubert cells are equivalent (in a precise sense) to the K-theoretic stable envelopes defined by Okounkov and collaborators in [Oko17, AO, OS16]. This coincidence was proved recently by Fehér, Rimányi and Weber [FRW18]; our approach, based on comparing the Demazure-Lusztig recursions to the recursions for the stable envelopes found by Su, Zhao and Zhong in [SZZ17] gives another proof of this result. Via this equivalence, the motivic Chern classes can be considered as natural analogues of the Schubert classes in the K-theory of the cotangent bundle of G/B.

As in the authors' previous work on CSM classes [AMSS17], the connections to Hecke algebras and K-theoretic stable envelopes yield remarkable identities among (Poincaré duals of) motivic Chern classes. We use these identities to prove two conjectures of Bump, Nakasuji and Naruse [BN11, BN19, NN16] about the coefficients in the transition matrix between the Casselman's basis and the standard basis in the Iwahori-invariant space of the principal series representation for an unramified character for a group over a non archimedean local field.

We present next a more extensive description of our results.

1.1. **Statement of results.** Let G be a complex, simple, linear algebraic group, and fix B, B^- a pair of opposite Borel subgroups of G. Denote by $T := B \cap B^-$ the maximal torus, by $W := N_G(T)/T$ the Weyl group, and by $X := G/B$ the (generalized) flag variety. For each Weyl group element $w \in W$ consider the Schubert cell $X(w)^\circ := BwB/B$, a subvariety of (complex) dimension $\ell(w)$. The opposite Schubert cell $Y(w)^\circ := B^-wB/B$ has complex codimension $\ell(w)$. The closures $X(w)$ and $Y(w)$ of these cells are the Schubert varieties. Let $\mathcal{O}_w$, respectively $\mathcal{O}^w$ be the K-theoretic Schubert classes associated to the structure sheaves of $X(w)$, respectively $Y(w)$. The equivariant K-theory ring of X, denoted by $K_T(X)$, is an

algebra over $K_T(pt) = R(T)$—the representation ring of T—and it has a $R(T)$-basis given by the Schubert classes $\mathcal{O}_w$ (or $\mathcal{O}^w$), where w varies in the Weyl group W.

If E is an equivariant vector bundle over X, we denote by $[E]$ its class in $K_T(X)$, and by $\lambda_y(E)$ the class

$$\lambda_y(E) = \sum [\wedge^i E] y^i \in K_T(X)[y].$$

For a T-stable subvariety $\Omega \subseteq X$ recall the notation

$$MC_y(\Omega) := MC_y[\Omega \hookrightarrow X] \in K_T(X)[y].$$

Our first main result is a recursive formula to calculate $MC_y(X(w)^\circ)$, the (equivariant) motivic Chern class of the Schubert cell. For each simple positive root α_i, consider the Demazure operator $\partial_i : K_T(X) \to K_T(X)$ ([Dem74], [KK90]); this is a $K_T(pt)$-linear endomorphism. Extend ∂_i linearly with respect to y, and define the Demazure-Lusztig (DL) operators $\mathcal{T}_i, \mathcal{T}_i^\vee : K_T(X)[y] \to K_T(X)[y]$ by

$$\mathcal{T}_i := \lambda_y(\mathcal{L}_{\alpha_i})\partial_i - id; \quad \mathcal{T}_i^\vee := \partial_i \lambda_y(\mathcal{L}_{\alpha_i}) - id,$$

where $\mathcal{L}_{\alpha_i} = G \times^B \mathbb{C}_{\alpha_i}$ is the equivariant line bundle whose fiber over $1.B$ has weight α_i. The operator $\mathcal{T}_i^\vee$ appeared classically in Lusztig's study of Hecke algebras [Lus85], and $\mathcal{T}_i$ appeared recently in related works [LLL17, BBL15]. The two operators are adjoint to each other via the K-theoretic Poincaré pairing; see §3.2 below. Our first main result is the following (cf. Theorem 5.1).

Theorem 1.1. *Let $w \in W$ and let s_i be a simple reflection such that $\ell(ws_i) > \ell(w)$. Then*

$$MC_y(X(ws_i)^\circ) = \mathcal{T}_i(MC_y(X(w)^\circ)).$$

Using the (equivariant) K-theoretic Chevalley formula [FL94, PR99, LP07] to multiply by classes of line bundles, the DL operators give a recursive formula to calculate the motivic Chern classes, starting from the class of a point.

Theorem 1.1 generalizes the analogous result from [AM16] where it was proved that the CSM classes of Schubert cells are recursively determined by operators in the degenerate Hecke algebra. The proof of Theorem 1.1 relies on the calculations of push-forwards of classes from the Bott-Samelson desingularizations of Schubert varieties.

To illustrate the result, we list below the non-equivariant motivic Chern classes of Schubert cells in $\mathrm{Fl}(3) := \mathrm{SL}_3(\mathbb{C})/B$, the manifold parametrizing flags in $\mathbb{C}^3$. In this case, the Weyl group is the symmetric group S_3, generated by simple reflections s_1 and s_2, and $w_0 = s_1 s_2 s_1$ is the longest element.

$$MC_y(X(\mathrm{id})) = \mathcal{O}_{\mathrm{id}};$$
$$MC_y(X(s_1)^\circ) = (1+y)\mathcal{O}_{s_1} - (1+2y)\mathcal{O}_{\mathrm{id}};$$
$$MC_y(X(s_2)^\circ) = (1+y)\mathcal{O}_{s_2} - (1+2y)\mathcal{O}_{\mathrm{id}};$$
$$MC_y(X(s_1 s_2)^\circ) = (1+y)^2\mathcal{O}_{s_1 s_2} - (1+y)(1+2y)\mathcal{O}_{s_1} - (1+y)(1+3y)\mathcal{O}_{s_2} + (5y^2 + 5y + 1)\mathcal{O}_{\mathrm{id}};$$
$$MC_y(X(s_2 s_1)^\circ) = (1+y)^2\mathcal{O}_{s_2 s_1} - (1+y)(1+2y)\mathcal{O}_{s_2} - (1+y)(1+3y)\mathcal{O}_{s_1} + (5y^2 + 5y + 1)\mathcal{O}_{\mathrm{id}};$$
$$MC_y(X(w_0)^\circ) = (1+y)^3\mathcal{O}_{w_0} - (1+y)^2(1+2y)(\mathcal{O}_{s_1 s_2} + \mathcal{O}_{s_2 s_1}) +$$
$$(1+y)(5y^2 + 4y + 1)(\mathcal{O}_{s_1} + \mathcal{O}_{s_2}) - (8y^3 + 11y^2 + 5y + 1)\mathcal{O}_{\mathrm{id}}.$$

One observes in these examples, and one can also prove it in general, that the specialization $y \mapsto 0$ in $MC_y(X(w)^\circ)$ yields the ideal sheaf of the boundary of the Schubert variety $X(w)$. The Schubert class $\mathcal{O}_w$ is obtained if one takes $y \mapsto 0$ in a recursion given by a renormalization of the dual operator $\mathcal{T}_i^\vee$. In fact, Theorem 1.1 generalizes the well-known fact from Schubert Calculus that the Schubert classes $\mathcal{O}_w$ are obtained recursively by the Demazure

operators ∂_i. These and other combinatorial properties of the motivic Chern classes will be studied in a continuation to this paper.

A remarkable feature in the examples listed above is a positivity property. Based on substantial computer evidence we make the following positivity conjecture:

Conjecture 1 (Positivity Conjecture). *Consider the Schubert expansion*

$$MC_y(X(w)^\circ) = \sum c(w;u)\mathcal{O}_u \in K_T(X)[y].$$

Then the coefficients $c(w;u) \in K_T(pt)[y]$ *satisfy* $(-1)^{\ell(w)-\ell(u)}c(w,u) \in \mathbb{Z}_{\geq 0}[y][e^{-\alpha_1}, \ldots, e^{-\alpha_r}]$, *where* α_i *are the positive simple roots. In particular, in the non-equivariant case,*

$$(-1)^{\ell(w)-\ell(u)}c(w;u) \in \mathbb{Z}_{\geq 0}[y].$$

In type A, a similar positivity property was also conjectured in [FRW, §6], along with a log concavity property. In the non-equivariant case, it is conjectured in [FRW] that the polynomials

$$(-1)^{\ell(w)-\ell(u)}\frac{c(w;u)}{(1+y)^{\ell(u)}}$$

are log-concave. In cohomology, Aluffi and Mihalcea conjectured that CSM classes of Schubert cells are positive [AM09, AM16]. For Grassmannians, this was established by J. Huh [Huh16]; a few special cases were settled earlier in [AM09, Mih15, Jon10, Str11]. In full G/P generality, and in the non-equivariant case, the conjecture was recently proved in [AMSS17], using the theory of characteristic cycles in the cotangent bundle of G/B. There is also a stronger version of this conjecture, which claims that in addition $c(w;u) \neq 0$ whenever $u \leq w$. Huh's proof shows this, and also establishes (implicitly) the equivariant version for Grassmannians. The statement of the conjecture is reminiscent of the positivity in (equivariant) K-theory proved by Buch [Buc02], Brion [Bri02], and Anderson, Griffeth and Miller [AGM11].

Let $MC_y^\vee(Y(v)^\circ)$ be the classes obtained by applying the (inverse) dual operators $(\mathcal{T}_i^\vee)^{-1}$ to $\mathcal{O}^{w_0}$ (instead of $\mathcal{T}_i$ to $\mathcal{O}_{\mathrm{id}}$); see Definition 6.1. We prove (Theorem 6.2) that for every $u, v \in W$,

$$(1.1) \qquad \langle MC_y(X(u)^\circ), MC_y^\vee(Y(v)^\circ) \rangle = \delta_{u,v}(-y)^{\ell(u)-\dim G/B}\prod_{\alpha>0}(1+ye^{-\alpha}),$$

where $\langle \cdot, \cdot \rangle$ is the K-theoretic Poincaré pairing. This orthogonality property, which we call 'Hecke duality', mirrors the similar orthogonality of Chern-Schwartz-MacPherson (CSM) classes proved by the authors in [AMSS17].

1.2. **Applications to Casselman's problem.** The main application in this note is to use the Hecke algebra action on motivic Chern classes of Schubert cells to prove two conjectures of Bump, Nakasuji and Naruse [BN11, BN19, NN16] about properties of certain coefficients of the transition matrix between two natural bases of the Iwahori invariant part of the principal series representation. We briefly recall below the relevant history, definitions and the results; the details and complete proofs are given in §10 below.

Let τ be an unramified character for a split reductive Chevalley group $\mathscr{G}(F)$ over a nonarchimedean local field F with finite residue field $\mathbb{F}_{q'}$. The principal series representation is the induced representation $I(\tau) := \mathrm{Ind}_{\mathscr{B}(F)}^{\mathscr{G}(F)}(\tau)$. We consider its submodule $I(\tau)^I$ of Iwahori invariants of $I(\tau)$; this is a Hecke module, with an additive basis indexed by the Weyl group W. There are two important bases: the *standard basis,* given by the characteristic

functions φ_w, and the *Casselman basis* $\{f_w\}$, defined using certain intertwiners; see (10.2). *Casselman's problem* [Cas80] is to find the transition matrix between the two bases:

$$(1.2) \qquad\qquad \varphi_u = \sum a_{u,w} f_w.$$

As observed by Bump and Nakasuji [BN11], it is better to consider the basis $\psi_u := \sum_{w \geq u} \varphi_w$ and the expansion

$$\psi_u = \sum \tilde{m}_{u,w} f_w.$$

By Möbius inversion, the problems of finding either of the transition matrices are equivalent. Recent solutions to the Casselman's problem were obtained by Naruse and Nakasuji [NN16], using the Yang-Baxter basis in the Hecke algebra introduced by Lascoux, Leclerc and Thibon [LLT97], and by Su, Zhao and Zhong [SZZ17], by means of the theory of stable envelopes developed in [Oko17, OS16, AO]. The K-theoretic stable envelopes are certain classes in the equivariant K-theory of the cotangent bundle $T^*(G/B)$, indexed by the Weyl group elements; see §8 below. Su, Zhao and Zhong proved that the Hecke algebra action on the basis of stable envelopes coincides with the Hecke algebra action on the standard basis φ_w. Under their correspondence, the Hecke action on the Casselman's basis fits with the Hecke action on the fixed point basis in equivariant K-theory.

Fehér, Rimányi and Weber [FRW18, FRW] observed that motivic Chern classes and K-theoretic stable envelopes are closely related; see also §8 below. Therefore it is not a surprise that one can recover the Hecke correspondence from [SZZ17] using motivic Chern classes. The advantage of this point of view is that motivic Chern classes satisfy strong functoriality properties, and this will allow us to obtain additional properties of the coefficients $\tilde{m}_{u,v}$.

Let G be the Langlands dual of $\mathscr{G}$. It turns out that the Hecke module $I(\tau)^I$ is more naturally related to the equivariant K-theory $K_T(G/B)$ for the Langlands dual flag manifold. Let ι_w be the fixed point basis in $K_T(G/B)$, and let b_w be the multiple of ι_w determined by the localization

$$b_w|_w = MC_y^\vee(Y(w)^\circ)|_w$$

at the fixed point e_w. The formula for $MC_y^\vee(Y(w)^\circ)|_w \in K_T(pt)[y^{-1}]$ is explicit; see Proposition 7.3. From this formula it follows that the elements b_w are in the localized ring

$$K_T(G/B)_{loc}[y^{-1}] := K_T(G/B)[y^{-1}] \otimes_{K_T(pt)} \mathrm{Frac}(K_T(pt)),$$

where Frac denotes the fraction field. We show (Theorem 10.2) that for a sufficiently general τ there is an isomorphism of Hecke modules $\Psi : K_T(G/B)_{loc}[y^{-1}] \otimes_{K_T(pt)} \mathbb{C}_\tau \to I(\tau)^I$ such that

$$\Psi(MC_y^\vee(Y(w)^\circ) \otimes 1) = \varphi_w; \quad \Psi(b_w \otimes 1) = f_w; \quad \Psi(y) = -q',$$

with $q' = |\mathbb{F}_{q'}|$ the number of elements in the finite residue field $\mathbb{F}_{q'}$, and $\mathbb{C}_\tau$ the one-dimensional $K_T(pt)$-module obtained by evaluation at τ. Using this result, we prove that $\tilde{m}_{u,w} = m_{u,w}(\tau)$, where $m_{u,w}$ are the coefficients in the expansion

$$MC_y^\vee(Y(u)) := \sum_{w \geq u} MC_y^\vee(Y(w)^\circ) = \sum m_{u,w} b_w \quad \in K_T(G/B)_{loc}[y^{-1}].$$

Implicit in this is that the coefficients $m_{u,w}$ may be regarded as complex valued functions defined on a certain Zariski open subset of the dual torus T.

The Hecke isomorphism Ψ provides a 'dictionary', translating all statements about $\tilde{m}_{u,w}$ into statements about $m_{u,w}$, which have geometric meaning. The key result for the representation theoretic applications is that the coefficients $m_{u,w}$ are given by localization (cf. Proposition 9.8 below):

Theorem 1.2. *(a) For every $w \geq u \in W$, the coefficient $m_{u,w}$ equals*

$$m_{u,w} = \Big(\frac{MC_y(Y(u))|_w}{MC_y(Y(w)^\circ)|_w} \Big)^\vee,$$

where $\vee$ is the operator mapping $e^\lambda \mapsto e^{-\lambda}$ for $e^\lambda \in K_T(pt)$ and $y \mapsto y^{-1}$.

(b) Assume that $Y(u)$ is smooth at the fixed point e_w and denote by $(N_{Y(w)}Y(u))_w$ the normal space at e_w in $Y(u)$, regarded as a trivial (but not equivariantly trivial) vector bundle. Then

$$m_{u,w} = \frac{\lambda_{y^{-1}}((N_{Y(w)}Y(u))_w)}{\lambda_{-1}((N_{Y(w)}Y(u))_w)}.$$

In particular, the entries $m_{1,w}$ are obtained from the motivic Chern class of the full flag variety $MC_y(Y(\mathrm{id})) = MC_y(G/B) = \lambda_y(T^*(G/B))$, and one recovers the (geometric version of the) classical Gindikin-Karpelevich formula, proved by Langlands [Lan71]:

$$m_{1,w} = \prod \frac{1 + y^{-1}e^\alpha}{1 - e^\alpha},$$

where the product of over positive roots α such that $w^{-1}(\alpha) < 0$. Let

$$S(u,w) := \{\alpha \in R^+ | u \leq s_\alpha w < w\}.$$

Our main application is the following factorization formula for $m_{u,v}$, see Theorem 9.5 below:

Theorem 1.3 (Geometric Bump-Nakasuji-Naruse Conjecture). *For every $u \leq w \in W$,*

$$m_{u,w} = \prod_{\alpha \in S(u,w)} \frac{1 + y^{-1}e^\alpha}{1 - e^\alpha},$$

if and only if the Schubert variety $Y(u)$ is smooth at the torus fixed point e_w.

This is the geometric analogue of a conjecture of Bump and Nakasuji [BN11, BN19] for simply laced types, generalized to all types by Naruse [Nar14], and further analyzed by Nakasuji and Naruse [NN16]. While this paper was in preparation, Naruse informed us that he also obtained an (unpublished) proof of the implication assuming factorization. Both Naruse's and our proofs are based on Kumar's cohomological criterion for smoothness of Schubert varieties [Kum96]; Naruse used Hecke algebra calculations; ours relies on properties of motivic Chern classes. The original conjecture of Bump and Nakasuji from [BN11] was stated in terms of conditions under which certain Kazhdan-Lusztig polynomials $P_{w_0 w^{-1}, w_0 u^{-1}}$ equal 1; we explain the equivalence to the statement above (in simply laced types) and discuss further this conjecture in sections §9.2 and §10.2.

A second conjecture refers to a holomorphy property. In relation to Kazhdan-Lusztig theory, Bump and Nakasuji [BN19] defined the coefficients

$$\tilde{r}_{u,w} := \sum_{u \leq x \leq w} (-1)^{\ell(x)-\ell(u)} \overline{\tilde{m}}_{x,w},$$

where the bar operator replaces q' by q'^{-1}. Using Möbius inversion it follows that the coefficients $a_{u,w}$ from (1.2) satisfy $\bar{a}_{u,w} = \tilde{r}_{u,w}$. Geometrically, these correspond to the coefficients $r_{u,w}$ obtained from the expansion

$$MC_y^\vee(Y(u)^\circ) = \sum \bar{r}_{u,w} b_w \quad \in K_T(G/B)_{loc}[y^{-1}],$$

where $\bar{f}(y) := f(y^{-1})$. We prove the following result (cf. Theorem 10.4), which answers affirmatively Conjecture 1 from [BN19].

Theorem 1.4. *Let $u \leq w$ be two Weyl group elements. Then the functions*

$$\prod_{\alpha \in S(u,w)} (1 - e^{\alpha}) r_{u,w} \quad , \quad \prod_{\alpha \in S(u,w)} (1 - e^{\alpha}) m_{u,w}$$

are holomorphic on the torus T.

Both Theorems 1.3 and 1.4 are consequences of Theorem 1.2. The proof of the latter requires a second orthogonality property between motivic Chern classes and their duals, proven by means of the connection with the the theory of K-theoretic stable envelopes. From this orthogonality we deduce the following key formula, proved in Theorem 8.11:

$$MC_y^{\vee}(Y(u)^{\circ}) = \prod_{\alpha > 0} (1 + ye^{-\alpha}) \frac{\mathcal{D}(MC_y(Y(u)^{\circ}))}{\lambda_y(T^*(G/B))}$$

as elements in the appropriate localized K-theory ring, where $\mathcal{D}[E] = (-1)^{\dim X}[E^{\vee}] \otimes [\wedge^{\dim X} T^*(X)]$ is the (equivariant) Grothedieck-Serre duality operator, with $X = G/B$.[1] The proof requires a precise relationship between the motivic Chern classes and stable envelopes. If $\iota : X \to T_X^*$ is the zero section, then our statement is that (roughly)

(1.3) $$\mathcal{D}(\iota^*(\text{stab}_+(w))) = N(q)MC_{-q^{-1}}(X(w)^{\circ}),$$

where $\text{stab}_+(w)$ is a stable envelope, $N(q)$ is a normalization parameter, and $q = -y^{-1}$ is determined from the dilation action of $\mathbb{C}^*$ on the fibers of the cotangent bundle. The precise statement is given in Theorem 8.5.

Formula (1.3) is part of a more general paradigm, stemming from the classical works of Sabbah [Sab85] and Ginzburg [Gin86], relating intersection theory on the cotangent bundle to that of characteristic classes of singular varieties. For instance, the (co)homological analogues of the motivic Chern classes of Schubert cells—the CSM classes—are equivalent to Maulik and Okounkov's *cohomological stable envelopes* [MO19]. This statement, observed by Rimányi and Varchenko [RV18], and by the authors in [AMSS17], is a consequence of the fact that both the stable envelopes and the CSM classes are determined by certain interpolation conditions obtained from equivariant localization; cf. Weber's article [Web12]. The relation to stable envelopes was recently extended to K-theory by Fehér, Rimányi and Weber [FRW18] (see also [FRW]). They showed that the motivic Chern classes of the Schubert cells satisfy the same localization conditions as the K-theoretic stable envelopes appearing in papers by Okounkov and Smirnov [Oko17, OS16] for a particular choice of parameters. (The result from [FRW18] is more general, involving the motivic Chern classes for orbits in a space with finitely many orbits under a group action.) We reprove this result by comparing the Demazure-Lusztig type recursions for motivic Chern classes to the recursions for the stable envelopes found by Su, Zhao and Zhong in [SZZ17]. We also discuss the relation between the motivic Chern class and various choices of parameters for the K-theoretic stable envelopes, which might be of independent interest; see §8 below.

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[1]The class $SMC_y(\Omega) := \frac{\mathcal{D}(MC_y(\Omega))}{\lambda_y(T^*(G/B))}$ may be regarded as the motivic analogue of the Segre-MacPherson class $\frac{c_*(\mathbf{1}_\Omega)}{c(T(G/B))}$.

2. Schubert varieties and their Bott-Samelson resolutions

In this section we recall the basic definitions and facts about the Bott-Samelson resolution of Schubert varieties. These will be used in the next section to perform the calculation of the motivic Chern class of a Schubert cell. Our main references are [AM16] and [BK05].

Let G be a complex semisimple linear algebraic group. Fix a Borel subgroup B and let $T := B \cap B^-$ be the maximal torus, where B^- denotes the opposite Borel subgroup. Let $W := N_G(T)/T$ be the Weyl group, and $\ell : W \to \mathbb{N}$ the associated length function. Denote by w_0 the longest element in W; then $B^- = w_0 B w_0$. Let also $\Delta := \{\alpha_1, \ldots, \alpha_r\} \subseteq R^+$ denote the set of simple roots included in the set of positive roots for (G, B). The simple reflection for the root $\alpha_i \in \Delta$ is denoted by s_i, and P_i denotes the corresponding *minimal parabolic subgroup*.

Let $X := G/B$ be the flag variety. It has stratifications by Schubert cells $X(w)^\circ := BwB/B$ and by opposite Schubert cells $Y(w)^\circ := B^- wB/B$. The closures $X(w) := \overline{X(w)^\circ}$ and $Y(w) := \overline{Y(w)^\circ}$ are the Schubert varieties. Note that $Y(w) = w_0 X(ww_0)$. With these definitions, $\dim_{\mathbb{C}} X(w) = \operatorname{codim}_{\mathbb{C}} Y(w) = \ell(w)$. The Weyl group W admits a partial ordering, called the Bruhat ordering, defined by $u \leq v$ if and only if $X(u) \subseteq X(v)$.

We recall next the definition of the Bott-Samelson resolution of a Schubert variety, following [AM16, §2.3] and [BK05, §2.2]. Fix $w \in W$ and a decomposition of w, i.e., a sequence $(i_1, \ldots, i_k)$ such that $w = s_{i_1} \cdot \ldots \cdot s_{i_k}$. If $\ell(w) = k$, then this decomposition is called *reduced*. This data determines a tower Z of $\mathbb{P}^1$-bundles and a birational map $\theta : Z \to X(w)$ as follows.

If the word is empty, then define $Z := pt$. In general assume we have constructed $Z' := Z_{i_1, \ldots, i_{k-1}}$ and the map $\theta' : Z' \to X(w')$, for $w' = s_{i_1} \cdots s_{i_{k-1}}$. Define $Z = Z_{i_1, \ldots, i_k}$ so that the left square in the diagram

(2.1)
$$
\begin{array}{ccccc}
Z & \xrightarrow{\theta_1} & G/B \times_{G/P_{i_k}} G/B & \xrightarrow{pr_1} & G/B \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle pr_2} & & \downarrow{\scriptstyle p_{i_k}} \\
Z' & \xrightarrow{\theta'} & G/B & \xrightarrow{p_{i_k}} & G/P_{i_k}
\end{array}
$$

is a fiber square; the morphisms pr_1, pr_2, p_{i_k} are the natural projections. From this construction it follows that Z is a smooth, projective variety of dimension k. The map $\theta : Z \to G/B$ is defined as the composition $pr_1 \circ \theta_1$.

The Bott-Samelson variety Z is equipped with a simple normal crossing (SNC) divisor D_Z, constructed inductively as follows. If $Z = pt$, then $D_Z = \emptyset$. In general, G/B is the projectivization $\mathbb{P}(E)$ of a homogeneous rank-2 vector bundle $E \to G/P_k$, defined up to tensoring with a line bundle. Define $\mathcal{E} := E \otimes \mathcal{O}_E(1)$, a vector bundle over $G/B = \mathbb{P}(E)$. Then we have the Euler sequence of the projective bundle $\mathbb{P}(E)$

$$
0 \longrightarrow \mathcal{O}_{\mathbb{P}(E)} \longrightarrow \mathcal{E} \longrightarrow \mathcal{Q} \longrightarrow 0
$$

where $\mathcal{Q}$ is the relative tangent bundle $T_{p_{i_k}}$. Note that $\mathcal{E}$ is independent of the specific choice of E, and $pr_2 : G/B \times_{G/P_{i_k}} G/B \to G/B$, that is, the pull-back of $\mathbb{P}(E)$ via p_{i_k}, may be identified with $\mathbb{P}(\mathcal{E})$. Let $\mathcal{E}' := (\theta')^*\mathcal{E}$ and $\mathcal{Q}' := (\theta')^*\mathcal{Q}$, and pull-back the previous sequence via θ' to get an exact sequence

$$0 \longrightarrow \mathcal{O}_{Z'} \longrightarrow \mathcal{E}' \longrightarrow \mathcal{Q}' \longrightarrow 0 \,.$$

The inclusion $\mathcal{O}_{Z'} \hookrightarrow \mathcal{E}'$ gives a section $\sigma : Z' \to Z$ of π and therefore a divisor $D_k := \sigma(Z') = \mathbb{P}(\mathcal{O}_{Z'})$ in $\mathbb{P}(\mathcal{E}') = Z$. The SNC divisor on Z is defined by

$$D_Z = \pi^{-1}(D_{Z'}) \cup D_k$$

where $D_{Z'}$ is the inductively constructed SNC divisor on Z'. The following result is well known, see e.g., [BK05, §2.2].

Proposition 2.1. *If $s_{i_1} \dots s_{i_k}$ is a* reduced *word for w, then the image of the composition $\theta = pr_1 \circ \theta_1 : Z_{i_1,\dots,i_k} \to G/B$ is the Schubert variety $X(w)$. Moreover, $\theta^{-1}(X(w) \smallsetminus X(w)^\circ) = D_{Z_{i_1,\dots,i_k}}$ and the restriction map*

$$\theta : Z_{i_1,\dots,i_k} \smallsetminus D_{Z_{i_1,\dots,i_k}} \to X(w)^\circ$$

is an isomorphism.

The proposition implies that the Bott-Samelson variety $Z_{i_1,\dots,i_k}$ is a log-resolution for the Schubert variety $X(w)$.

The group G acts diagonally on the fiber product $G/B \times_{G/P_{i_k}} G/B$ with two orbits; the fiber product itself is the closure of the G-orbit of $(id.B, s_{i_k}.B)$ under this action. The closed orbit is the diagonal $\mathcal{D} \subseteq G/B \times_{G/P_{i_k}} G/B$, which is naturally isomorphic to G/B. The divisor D_k is the pull-back of $\mathcal{D}$ to Z. We will need the following result.

Lemma 2.2. *There is a natural isomorphism of G-equivariant vector bundles between the normal bundle $N_{\mathcal{D}}(G/B \times_{G/P_{i_k}} G/B)$ and the restriction $(T_{pr_2})|_{\mathcal{D}}$.*

Proof. Denote by $\mathcal{O}(\mathcal{D})$ the line bundle $\mathcal{O}_{G/B \times_{G/P_{i_k}} G/B}(\mathcal{D})$; it is naturally isomorphic to the normal bundle in the statement, as equivariant bundles. By [AM16, Proposition 6.2(b)] there is an isomorphism of equivariant bundles $\mathcal{O}(\mathcal{D}) \simeq T_{pr_2} \otimes \mathcal{O}_{\mathcal{E}}(1)$. Part (a) of the same result shows that the restriction $(\mathcal{O}_{\mathcal{E}}(1))|_{\mathcal{D}}$ is equivariantly trivial, since pr_1 and pr_2 agree along the diagonal. The statement follows from this. $\square$

3. Equivariant K-theory of flag manifolds and Demazure-Lusztig operators

In this section we recall the definition and basic properties of equivariant K-theory of flag manifolds, and of certain Demazure-Lusztig operators acting on equivariant K-theory. This setup is well-known from the theory of Hecke algebras, see e.g., [Lus85] and [Gin98].

3.1. **Equivariant K-theory.** Let X be a smooth, quasi-projective algebraic variety endowed with a T-action. The (algebraic) equivariant K-theory ring $K_T(X)$ is the ring generated by symbols $[E]$, where $E \to X$ is an equivariant vector bundle, modulo the relations $[E] = [E_1] + [E_2]$ for all short exact sequences $0 \to E_1 \to E \to E_2 \to 0$ of equivariant vector bundles. The ring addition is given by direct sums, and multiplication by tensor products. Since X is smooth, every (equivariant) coherent sheaf has a finite resolution by (equivariant) vector bundles [CG09, Proposition 5.1.28], and $K_T(X)$ coincides with the Grothendieck group of (equivariant) coherent sheaves on X. The ring $K_T(X)$ is an algebra

over the Laurent polynomial ring $K_T(pt) = \mathbb{Z}[e^{\pm t_1}, \ldots, e^{\pm t_r}]$ where e^{t_i} are characters corresponding to a basis of the Lie algebra of T; alternatively $K_T(pt)$ may be viewed as the representation ring $R(T)$ of T.

In our situation $X = G/B$ and T acts on X by left multiplication. Since X is smooth, the ring $K_T(X)$ coincides with the Grothendieck group of T-linearized coherent sheaves on X. There is a pairing

$$\langle -, - \rangle : K_T(X) \otimes K_T(X) \to K_T(pt) = R(T)$$

defined on classes $[E]$, $[F]$ of vector bundles by

$$\langle [E], [F] \rangle := \int_X E \otimes F = \chi(X; E \otimes F).$$

Here $\chi(X; -)$ is the (equivariant) Euler characteristic, i.e., the virtual character

$$\chi(X; -) = \sum (-1)^i \, ch_T(H^i(X; -)).$$

Let $\mathcal{O}_w := [\mathcal{O}_{X(w)}]$ be the Grothedieck class determined by the structure sheaf of $X(w)$ (a coherent sheaf), and similarly $\mathcal{O}^w := [\mathcal{O}_{Y(w)}]$. The equivariant K-theory ring has $K_T(pt)$-bases $\{\mathcal{O}_w\}_{w \in W}$ and $\{\mathcal{O}^w\}_{w \in W}$. Let $\partial X(w) := X(w) \setminus X(w)^\circ$ be the boundary of the Schubert variety $X(w)$, and similarly $\partial Y(w)$ the boundary of $Y(w)$. It is known that the dual bases of $\{\mathcal{O}_w\}$ and $\{\mathcal{O}^w\}$ are given by the classes of the ideal sheaves $\mathcal{I}^w := [\mathcal{O}_{Y(w)}(-\partial Y(w))]$, respectively $\mathcal{I}_w := [\mathcal{O}_{X(w)}(-\partial X(w))]$. I.e.,

$$\langle \mathcal{O}_u, \mathcal{I}^v \rangle = \langle \mathcal{O}^u, \mathcal{I}_v \rangle = \delta_{u,v}.$$

See e.g., [Bri05, Proposition 3.4.1] for the non-equivariant case; the same proof works equivariantly. See also [GK08, §2]. In fact,

$$(3.1) \qquad \mathcal{O}_w = \sum_{v \leq w} \mathcal{I}_v \quad \text{and} \quad \mathcal{I}_w = \sum_{v \leq w} (-1)^{\ell(w) - \ell(v)} \mathcal{O}_v$$

([Bri05, Proposition 4.3.2]). We will also need that

$$(3.2) \qquad \langle \mathcal{O}_u, \mathcal{O}^v \rangle = \begin{cases} 0 & \text{if } u < v \\ 1 & \text{if } u \geq v \end{cases};$$

this is proved in e.g., [Bri05, Theorem 4.2.1].

3.2. **Demazure-Lusztig (DL) operators.** Fix a simple root $\alpha_i \in \Delta$ and the corresponding minimal parabolic subgroup $P_i \subseteq G$. Consider the right-hand side of diagram (2.1):

$$FP := G/B \times_{G/P_i} G/B \xrightarrow{pr_1} G/B$$
$$\left\downarrow{pr_2} \qquad\qquad\qquad \right\downarrow{p_i}$$
$$G/B \xrightarrow{\qquad p_i \qquad} G/P_i$$

The Demazure operator $\partial_i : K_T(X) \to K_T(X)$ [Dem74] is defined by

$$\partial_i := (p_i)^* (p_i)_* = (pr_1)_* pr_2^*.$$

It satisfies (e.g., from [KK90, Lemma 4.12])

$$(3.3) \qquad \partial_i(\mathcal{O}_w) = \begin{cases} \mathcal{O}_{ws_i} & \text{if } ws_i > w; \\ \mathcal{O}_w & \text{otherwise}. \end{cases}$$

From this, one deduces that $\partial_i^2 = \partial_i$ and can also verify that for $v \in W$, represented by a reduced word $s_{i_1} \cdots s_{i_k}$, the operator

$$\partial_v := \partial_{i_1} \circ \cdots \circ \partial_{i_k}$$

is independent of the chosen reduced word; cf. [KK90, §3]. With this definition we have that if $\ell(uv^{-1}) = \ell(u) + \ell(v^{-1})$, then $\partial_v(\mathcal{O}_u) = \mathcal{O}_{uv^{-1}}$. Since p_i is G-equivariant and $Y(w) = w_0 X(w_0 w)$, it follows easily that $\partial_i(\mathcal{O}^w) = \mathcal{O}^{ws_i}$ if $ws_i < w$ and $\partial_i(\mathcal{O}^w) = \mathcal{O}^w$ otherwise.

Fix an indeterminate y. The λ_y-class of a vector bundle E is the class

$$\lambda_y(E) := \sum_k [\wedge^k E] y^k \in K_T(X)[y].$$

The λ_y-class is multiplicative, i.e., if $0 \to E_1 \to E \to E_2 \to 0$ is a short exact sequence of equivariant vector bundles, then $\lambda_y(E) = \lambda_y(E_1)\lambda_y(E_2)$ as elements in $K_T(X)[y]$. We refer to the books [FL85, Hir95] for details in the non-equivariant case. The equivariant case involves no additional subtleties.

We define next the main operators used in this paper.

Definition 3.1. Let $\alpha_i \in \Delta$ be a simple root. Define the operators

$$\mathcal{T}_i := \lambda_y(T_{p_i}^*)\partial_i - id; \quad \mathcal{T}_i^\vee := \partial_i \lambda_y(T_{p_i}^*) - id.$$

The operators $\mathcal{T}_i$ and $\mathcal{T}_i^\vee$ are $K_T(pt)[y]$-module endomorphisms of $K_T(X)[y]$. We will occasionally work in $K_T(X)[y^{\pm 1}]$ and regard these as $K_T(pt)[y^{\pm 1}]$-module endomorphisms.

Remark 3.2. The operator $\mathcal{T}_i^\vee$ was defined by Lusztig [Lus85, (4.2)] in relation to affine Hecke algebras and equivariant K-theory of flag varieties. (Lusztig worked in topological equivariant K-theory, but since $X = G/B$ has a T-invariant algebraic cell-decomposition by Schubert cells, the algebraic and topological equivariant K-theories of X are naturally isomorphic [CG09][Proposition 5.5.6, p. 272].) As we shall see below, the 'dual' operator $\mathcal{T}_i$ arises naturally in the study of motivic Chern classes of Schubert cells. In an algebraic form, $\mathcal{T}_i$ appeared recently in [LLL17, BBL15], in relation to Whittaker functions. ⌟

Lemma 3.3. *The operators $\mathcal{T}_i$ and $\mathcal{T}_i^\vee$ are adjoint to each other. That is, for every $a, b \in K_T(X)$,*

$$\langle \mathcal{T}_i(a), b \rangle = \langle a, \mathcal{T}_i^\vee(b) \rangle.$$

The same equality holds for $a, b \in K_T(X)[y^{\pm 1}]$, if one extends the pairing bilinearly in y.

Proof. The identity is self adjoint and ∂_i is also self adjoint. Indeed, by the projection formula

$$\langle \partial_i(a), b \rangle = \int_{G/B} p_i^*(p_i)_*(a) \cdot b = \int_{G/P_i} (p_i)_*(a) \cdot (p_i)_*(b),$$

and the last expression is symmetric in a, b. It remains to show that coefficient of y in both sides is the same, i.e., $\langle T_{p_i}^* \partial_i(a), b \rangle = \langle a, \partial_i T_{p_i}^*(b) \rangle$. We calculate

$$\langle T_{p_i}^* \partial_i(a), b \rangle = \int_{G/B} T_{p_i}^* p_i^*((p_i)_* a) \cdot b = \int_{G/P_i} (p_i)_*(a) \cdot (p_i)_*(T_{p_i}^* \cdot b)$$

$$= \int_{G/B} a \cdot p_i^*(p_i)_*(T_{p_i}^* \cdot b) = \langle a, \partial_i(T_{p_i}^* \cdot b) \rangle. \qquad \square$$

According to Lusztig's result ([Lus85, Theorem in §5]), the operators $\mathcal{T}_i^\vee$ satisfy the braid relations and the quadratic relations defining the Hecke algebra $H_W(-y)$ of the Weyl group W with parameter $-y$. (In the language of this paper, the variable q from [Lus85] satisfies $q = -y$.) Since the K-theoretic Poincaré pairing is non-degenerate, Lemma 3.3 implies that both sets of operators $\mathcal{T}_i$ and $\mathcal{T}_i^\vee$ satisfy the same algebraic formulas. We record this next.

Proposition 3.4 (Lusztig). *The operators $\mathcal{T}_i$ and $\mathcal{T}_i^\vee$ satisfy the braid relations for the Weyl group W. For each simple root $\alpha_i \in \Delta$ the following quadratic relations hold:*

$$(3.4) \qquad (\mathcal{T}_i + id)(\mathcal{T}_i + y) = (\mathcal{T}_i^\vee + id)(\mathcal{T}_i^\vee + y) = 0.$$

From these relations it follows that the operators $\mathcal{T}_i$ and $\mathcal{T}_i^\vee$ are invertible in $K_T(X)[y^{\pm 1}]$:

$$(3.5) \qquad \mathcal{T}_i^{-1} = -\frac{1}{y}\mathcal{T}_i - \frac{1+y}{y}id \quad ; \quad (\mathcal{T}_i^\vee)^{-1} = -\frac{1}{y}\mathcal{T}_i^\vee - \frac{1+y}{y}id.$$

Since the operators $\mathcal{T}_i, \mathcal{T}_i^\vee$ satisfy the braid relations, we may define

$$(3.6) \qquad \mathcal{T}_v := \mathcal{T}_{i_1} \cdots \mathcal{T}_{i_k} \quad , \quad \mathcal{T}_v^\vee := \mathcal{T}_{i_1}^\vee \cdots \mathcal{T}_{i_k}^\vee$$

for $v \in W$ represented by a reduced word $s_{i_1} \cdots s_{i_k}$.

The cohomological versions of the Demazure-Lusztig operators, which appear in the study of degenerate Hecke algebras [Gin98, AM16] are self inverse (i.e., $(\mathcal{T}_i^{coh})^2 = id$) and therefore satisfy the relations of the group algebra $\mathbb{Z}[W]$ of the Weyl group. Obviously, this is not true in K-theory, due to (3.5). However, the multiplication of Demazure-Lusztig operators behaves rather nicely, as shown in the following proposition.

Proposition 3.5. *Let $u, v \in W$ be two Weyl group elements. Then*

$$(3.7) \qquad \mathcal{T}_u \cdot \mathcal{T}_v^{-1} = c_{uv^{-1}}(y)\mathcal{T}_{uv^{-1}} + \sum_{w < uv^{-1}} c_w(y)\mathcal{T}_w,$$

where $c_{uv^{-1}}$ and $c_w(y)$ are rational functions in y determined by u, v. Further, if $\ell(uv^{-1}) = \ell(u) + \ell(v^{-1})$, then $c_{uv^{-1}}(y) = (-y)^{-\ell(v)}$. The same statements hold for the multiplication $\mathcal{T}_u^\vee \cdot (\mathcal{T}_v^\vee)^{-1}$ of the dual operators.

To prove Proposition 3.5 we need the following lemma.

Lemma 3.6. *Let $u, v \in W$ be two Weyl group elements, and let s be a root reflection such that $us > u$ and $vs < v$. Then $usv^{-1} < uv^{-1}$.*

Proof. (Cf. [Hum90, §5.7].) Let $s = s_\alpha$ for some positive root α. By hypothesis $us_\alpha > u$ and $vs_\alpha < v$, hence $u(\alpha) > 0$ and $v(\alpha) < 0$. Since $uv^{-1}(-v(\alpha)) < 0$, it follows that $uv^{-1}s_{v(\alpha)} < uv^{-1}$. On the other hand we have $s_{v(\alpha)} = vs_\alpha v^{-1}$; therefore $uv^{-1}s_{v(\alpha)} = us_\alpha v^{-1}$, and we are done. $\square$

Proof of Proposition 3.5. We use ascending induction on $\ell(v) \geq 0$. The statement is clear if $\ell(v) = 0$. For $\ell(v) > 0$ write $v = v's_k$ where $\ell(v') < \ell(v)$. By definition, $\mathcal{T}_u \cdot \mathcal{T}_v^{-1} = \mathcal{T}_u \cdot \mathcal{T}_k^{-1} \cdot \mathcal{T}_{v'}^{-1}$. We have two cases: $us_k < u$ and $us_k > u$. Consider first the situation $us_k < u$. Then u has a reduced decomposition ending in s_k, i.e., $u = u's_k$ and $\ell(u') < \ell(u)$. Then

$$\mathcal{T}_u \cdot \mathcal{T}_k^{-1} \cdot \mathcal{T}_{v'}^{-1} = \mathcal{T}_{u'} \cdot \mathcal{T}_k \cdot \mathcal{T}_k^{-1} \cdot \mathcal{T}_{v'}^{-1} = \mathcal{T}_{u'} \cdot \mathcal{T}_{v'}^{-1},$$

and since $v' < v$ the result is known by induction. Assume next that $us_k > u$. Using equation (3.5) we obtain

$$\mathcal{T}_u \cdot \mathcal{T}_v^{-1} = \mathcal{T}_u \cdot \mathcal{T}_k^{-1} \cdot \mathcal{T}_{v'}^{-1} = -\frac{1}{y}\mathcal{T}_u \cdot (\mathcal{T}_k + y + 1) \cdot \mathcal{T}_{v'}^{-1} = -\frac{1}{y}\mathcal{T}_{us_k} \cdot \mathcal{T}_{v'}^{-1} - \frac{1+y}{y}\mathcal{T}_u \cdot \mathcal{T}_{v'}^{-1}.$$

By induction, the leading term of $\mathcal{T}_{us_k} \cdot \mathcal{T}_{v'}^{-1}$ is $\mathcal{T}_{us_k v'^{-1}} = \mathcal{T}_{uv^{-1}}$ and the leading term of $\mathcal{T}_u \cdot \mathcal{T}_{v'}^{-1}$ is $\mathcal{T}_{uv'^{-1}}$. Observe that $vs_k = v' < v$. From Lemma 3.6 we obtain that $uv'^{-1} = us_k v^{-1} < uv^{-1}$, and this concludes the proof of the existence of an expression (3.7) as stated. If $\ell(u) + \ell(v^{-1}) = \ell(uv^{-1})$, then arguing again by ascending induction on $\ell(v)$ gives, with notation as above,

$$\mathcal{T}_u \cdot \mathcal{T}_v^{-1} = -\frac{1}{y}\mathcal{T}_{us_k} \cdot \mathcal{T}_{v'}^{-1} + \cdots = -\frac{1}{y}\frac{1}{(-y)^{\ell(v')}}\mathcal{T}_{uv^{-1}} + \cdots,$$

proving that $c(uv^{-1}) = (-y)^{-\ell(v)}$ as claimed.

The statements for the dual operators are proved in the same way. $\square$

3.3. **Actions on Schubert and fixed point bases.** We will need several formulas concerning the action of the Demazure-Lusztig operators on Schubert classes and on the classes determined by the torus fixed points. For instance, the definition and a standard localization argument (cf. Lemma 3.7(a) below) imply that, for every $w \in W$,

$$\mathcal{T}_i(\mathcal{O}_w) = \begin{cases} (1 + ye^{-w\alpha_i})\mathcal{O}_{ws_i} + l.o.t. & \text{if } ws_i > w; \\ ye^{w\alpha_i}\mathcal{O}_w + l.o.t. & \text{if } ws_i < w. \end{cases}$$

where l.o.t. (lower order terms) stands for a sum of terms $P(y, e^t)\mathcal{O}_u$ with $u < ws_i$ on the first branch and $u < w$ on the second branch. A similar formula holds for the dual operator:

$$(3.8) \qquad \mathcal{T}_i^\vee(\mathcal{O}^w) = (1 + ye^{w_0 w(\alpha_i)})\mathcal{O}^{ws_i} + l.o.t. \quad \text{if } ws_i < w,$$

where now the l.o.t. consist of multiples of $\mathcal{O}^v$ for $v > ws_i$. Consider next the localized equivariant K-theory ring

$$K_T(G/B) \hookrightarrow K_T(G/B)_{loc} := K_T(G/B) \otimes_{K_T(pt)} \text{Frac}(K_T(pt)),$$

where Frac denotes the fraction field. The Weyl group elements $w \in W$ are in bijection with the torus fixed points $e_w \in G/B$. Let $\iota_w := [\mathcal{O}_{e_w}] \in K_T(G/B)_{loc}$ be the class of the structure sheaf of e_w. By the localization theorem, the classes ι_w form a basis for the localized equivariant K-theory ring; we call this the *fixed point basis*. For a weight λ consider the G-equivariant line bundle $\mathcal{L}_\lambda := G \times^B \mathbb{C}_\lambda$ with character λ in the fiber over $1.B$. For example, the relative cotangent bundle $T_{p_i}^*$ for the projection p_i is isomorphic to $\mathcal{L}_{\alpha_i}$. We need the following lemma.

Lemma 3.7. *The following formulas hold in $K_T(G/B)_{loc}[y^{\pm 1}]$:*

(a) *For every weight λ, $\mathcal{L}_\lambda \cdot \iota_w = e^{w\lambda}\iota_w$;*

(b) *For every simple root α_i,*

$$\partial_i(\iota_w) = \frac{1}{1 - e^{w\alpha_i}}\iota_w + \frac{1}{1 - e^{-w\alpha_i}}\iota_{ws_i};$$

(c) *The action of the operator $\mathcal{T}_i$ on the fixed point basis is given by the following formula*

$$\mathcal{T}_i(\iota_w) = -\frac{1 + y}{1 - e^{-w\alpha_i}}\iota_w + \frac{1 + ye^{-w\alpha_i}}{1 - e^{-w\alpha_i}}\iota_{ws_i}.$$

(d) *The action of the adjoint operator $\mathcal{T}_i^\vee$ is given by*

$$\mathcal{T}_i^\vee(\iota_w) = -\frac{1 + y}{1 - e^{-w\alpha_i}}\iota_w + \frac{1 + ye^{w\alpha_i}}{1 - e^{-w\alpha_i}}\iota_{ws_i}.$$

(e) *The action of the inverse operator $(\mathcal{T}_i^\vee)^{-1}$ is given by*

$$(\mathcal{T}_i^\vee)^{-1}(\iota_w) = -\frac{1 + y^{-1}}{1 - e^{w\alpha_i}}\iota_w - \frac{y^{-1} + e^{w\alpha_i}}{1 - e^{-w\alpha_i}}\iota_{ws_i}.$$

Proof. Part (a) is a standard localization calculation, based on the fact that the fiber of $\mathcal{L}_\lambda$ over the fixed point e_w is a one-dimensional T-module of weight $w(\lambda)$. For part (b), notice that $\partial_i(\iota_w) = [\mathcal{O}_{p_i^{-1}(e_w)}]$, where $p_i : G/B \to G/P_i$ is the projection, and (abusing notation) e_w also denotes the corresponding fixed point in G/P_i. The fiber $p_i^{-1}(e_w) \cong \mathbb{P}^1$ equals $w.[X(s_i)]$, the w-translate of the Schubert curve. It contains only two T-fixed points: e_w and e_{ws_i}. It follows that $[\mathcal{O}_{p_i^{-1}(e_w)}] = a\iota_w + b\iota_{ws_i}$ for suitable a, b. By the projection formula we can regard this as an expansion in the localized equivariant K-theory of the fiber itself. Here $(\iota_w)|_{e_w} = 1 - e^{w\alpha_i}$ (the Euler class of the cotangent space at w) and $(\iota_w)|_{e_{ws_i}} = 0$; similarly $(\iota_{ws_i})|_{e_{ws_i}} = 1 - e^{-w\alpha_i}$ and $(\iota_{ws_i})|_{e_w} = 0$. (For instance, this can be seen to hold at $w = id$; then one translates.) Since $[\mathcal{O}_{p_i^{-1}(e_w)}]$ is the identity in the K-theory ring of the fiber, $a(1 - e^{w\alpha_i}) = b(1 - e^{-w\alpha_i}) = 1$. The statement in (b) follows. Part (c) follows from (a) and (b), applied to $\mathcal{T}_i(\iota_w) = (1 + y\mathcal{L}_{\alpha_i})\partial_i(\iota_w) - \iota_w$. Formula (d) for the adjoint operator follows similarly. Finally, part (e) follows because $(\mathcal{T}_i^\vee)^{-1} = -y^{-1}(\mathcal{T}_i^\vee + y + 1)$ (cf. equation (3.5)). $\qquad\square$

We also record the action of several specializations of the Demazure-Lusztig operators.

Lemma 3.8. *(a) The specializations at $y = 0$ satisfy*

$$(\mathcal{T}_i)_{y=0} = (\mathcal{T}_i^\vee)_{y=0} = \partial_i - id.$$

Further, the following holds for every $w \in W$:

$$(\partial_i - id)(\mathcal{I}_w) = \begin{cases} \mathcal{I}_{ws_i} & \text{if } ws_i > w \\ -\mathcal{I}_w & \text{if } ws_i < w. \end{cases}$$

(b) Let $w \in W$. Then the specializations at $y = -1$ satisfy

$$(\mathcal{T}_i)_{y=-1}(\iota_w) = \iota_{ws_i}; \quad (\mathcal{T}_i^\vee)_{y=-1}(\iota_w) = \frac{1 - e^{w\alpha_i}}{1 - e^{-w\alpha_i}} \iota_{ws_i} = -e^{w\alpha_i}\iota_{ws_i}.$$

In other words, this specialization is compatible with the right Weyl group multiplication.

Proof. Part (a) is an easy exercise using the definition of $\mathcal{T}_i$ and identities (3.1) and (3.3). Part (b) is immediate from Lemma 3.7. $\qquad\square$

4. MOTIVIC CHERN CLASSES

We recall the definition of the motivic Chern classes, following [BSY10]. For now let X be a quasi-projective, complex algebraic variety, with an action of the complex torus T. First we recall the definition of the (relative) motivic Grothendieck group $G_0^T(var/X)$ of varieties over X, mostly following Looijenga's notes [Loo02] and Bittner's papers [Bit04, Bit05]. For simplicity, we only consider the T-equivariant quasi-projective context, which is enough for all applications in this paper. The group $G_0^T(var/X)$ is the free abelian group generated by symbols $[f : Z \to X]$ for isomorphism classes of T-equivariant morphisms $f : Z \to X$, where Z is a quasi-projective T-variety, modulo the usual additivity relations

$$[f : Z \to X] = [f : U \to X] + [f : Z \setminus U \to X]$$

for $U \subseteq Z$ an open invariant subvariety. If $X = pt$ then $G_0^T(var/pt)$ is a ring with the product given by the external product of morphisms, and the groups $G_0^T(var/X)$ have a module structure over $G_0^T(var/pt)$ also given by the external product. For every equivariant morphism $g : X \to Y$ of quasi-projective T-varieties there is a functorial push-forward

$g_! : G_0^T(var/X) \to G_0^T(var/Y)$ (given by composition) and pull-back $g^* : G_0^T(var/Y) \to G_0^T(var/X)$ (given by fiber product). Finally there are external products

$$\times : G_0^T(var/X) \times G_0^T(var/X') \to G_0^T(var/X \times X'); \quad [f] \times [f'] \mapsto [f \times f'],$$

which are $G_0^T(var/pt)$-bilinear and commute with push-forward and pull-back. In particular push-forward $g_!$ and pull-back g^* are $G_0^T(var/pt)$-linear.

Remark 4.1. (Cf. [BSY10, §0].) For every variety X, similar functors can be defined on the ring of constructible functions $\mathcal{F}(X)$, and the Grothendieck ring $G_0(var/X)$ may be regarded as a motivic version of $\mathcal{F}(X)$. In fact, there is a map $e : G_0(var/X) \to \mathcal{F}(X)$ sending $[f : Y \to X]$ to $f_!(\mathbb{1}_Y)$, where $f_!(\mathbb{1}_Y)$ is defined using compactly supported Euler characteristic of the fibers. The map e is a group homomorphism, and if $X = pt$ then e is a ring homomorphism. The constructions extend equivariantly to $G_0^T(var/X) \to \mathcal{F}^T(X)$, with $\mathcal{F}^T(X) \subseteq \mathcal{F}(X)$ the subgroup of T-invariant constructible functions. ⌟

Theorem 4.2. *Let X be a quasi-projective, non-singular, complex algebraic variety with an action of the torus T. There exists a unique natural transformation $MC_y : G_0^T(var/X) \to K_T(X)[y]$ satisfying the following properties:*

(1) *It is functorial with respect to T-equivariant proper morphisms of non-singular, quasi-projective varieties.*

(2) *It satisfies the normalization condition*

$$MC_y[id_X : X \to X] = \lambda_y(T_X^*) = \sum y^i[\wedge^i T_X^*]_T \in K_T(X)[y].$$

The transformation MC_y satisfies the following properties:

(3) *It commutes with external products:*

$$MC_y[f \times f' : Z \times Z' \to X \times X'] = MC_y[f : Z \to X] \boxtimes MC_y[f' : Z' \to X'].$$

(4) *For every smooth, T-equivariant morphism $\pi : X \to Y$ of quasi-projective and non-singular algebraic varieties, and any $[f : Z \to Y] \in G_0^T(var/Y)$, the following Verdier-Riemann-Roch (VRR) formula holds:*

$$\lambda_y(T_\pi^*) \cdot \pi^* MC_y[f : Z \to Y] = MC_y[\pi^* f : Z \times_Y X \to X].$$

Proof. In the non-equivariant case all these statements are proved in [BSY10, Theorem 2.1]. Fehér, Rimányi and Weber used similar ideas to extend the motivic Chern classes to the equivariant situation in [FRW18]. They proved that there is a well defined class

$$MC_y[f : Z \to X] \in K_T(X)[y]$$

for Z and X smooth, but omitted the details showing functoriality properties, referring instead back to [BSY10]. For completeness, we prove this theorem, using the definition (and well-definedness) from [FRW18]. First one gets as in [Bit04] a tautological isomorphism

$$G_0^T(sm/X) \to G_0^T(var/X) : [f : Z \to X] \mapsto [f : Z \to X],$$

where $G_0^T(sm/X)$ is the corresponding (relative) motivic Grothendieck group of *smooth* quasi-projective T-varieties mapping to X, with the corresponding 'additivity' relation only asked for T-invariant open subsets $U \subseteq Z$ with $Z \setminus U$ a smooth closed subvariety of Z. Here the inverse of $[f : Z \to X] \in G_0^T(var/X)$ in $G_0^T(sm/X)$ for Z possibly singular is defined as

$$[f : Z \to X] \mapsto \sum_i [f : Z_i \to X]$$

where $Z = \bigsqcup Z_i$ is a decomposition into a finite disjoint union of T-invariant (locally closed) smooth subvarieties $Z_i \subseteq Z$. Using this presentation, MC_y is determined by its

value on $G_0^T(sm/X)$: i.e., we have $MC_y[f : Z \to X] := \sum_i MC_y[f : Z_i \to X]$ with notation as above. If U is a smooth quasi-projective T-variety we consider the definition of $MC_y[f : U \to X]$ from [FRW18] and show that it satisfies the corresponding 'additivity' relation. Let $f : U \to X$ be an equivariant morphism with U quasi-projective and non-singular. Then there exists a non-singular quasi-projective algebraic variety $\overline{U}$ containing U such that the following are satisfied:

- $\overline{U}$ admits an action of T and the inclusion $i : U \to \overline{U}$ is an open, T-equivariant embedding,
- the boundary $D := \overline{U} \setminus U = \bigcup_{i=1,\ldots,s} D_i$ is a T-invariant simple normal crossing divisor with smooth irreducible components D_i,
- there exists a *proper* T-equivariant morphism $\overline{f} : \overline{U} \to X$ such that $f = \overline{f} \circ i$.

This is known in the non-equivariant case, using resolution of singularities; see e.g., results in Bittner [Bit04]. One can extend this equivariantly using the results of Sumihiro [Sum74] or [CG09, Theorem 5.1.25], which show the existence of a T-equivariant projective completion $\tilde{U}'$ of U. Taking the closure $\tilde{U}''$ of U in $\tilde{U}' \times X$, one gets a T-equivariant quasi-projectve extension $\tilde{U}''$ of U with a proper T-equivariant extension $\tilde{f} : \tilde{U}'' \to X$ of f. Using general arguments about equivariant resolution of singularities of $\tilde{U}''$ [BM97], one can finally construct a non-singular quasi-projective variety $\overline{U}$ and a *proper* T-equivariant morphism $\overline{f} : \overline{U} \to X$ as above (such a morphism is then also projective, by e.g., [Sta, Lemma 28.41.13]). Then 'additivity and normalization' forces

$$(4.1) \qquad MC_y[f : U \to X] := \sum_{I \subseteq \{1,\ldots,s\}} (-1)^{|I|} f_{I*} \lambda_y(T^* D_I),$$

with $D_I := \cap_{i \in I} D_i$ (and $D_\emptyset := \overline{U}$) and $f_I := \overline{f}|D_I$. By [FRW18, §2.4] the right-hand side is independent of all choices. In particular MC_y satisfies the normalization property from part (2) of the statement. We show next that the transformation MC_y satisfies the corresponding 'additivity' property. Let $Y \subseteq U$ be a closed T-invariant subvariety. By induction on the number of connected components of Y, we can assume Y is connected. Then one can find as in [Bit04] a partial compactification of U and f as before in such a way that the closure $\overline{Y}$ of Y in $\overline{U}$ is smooth and $\overline{Y}$ has normal crossing with D. Then $\overline{Y}$ and $\overline{f}|\overline{Y}$ is such a partial compactification of Y and $f|Y$ with the corresponding simple normal crossing divisor $\overline{Y} \cap D$. Let us now first assume that Y is hypersurface in U, so that $\overline{Y} \cup D = \overline{Y} \cup \bigcup_{i=1,\ldots,s} D_i$ is a simple normal crossing divisor in $\overline{U}$. Then one can use $\overline{U}$ and $\overline{f}$ as a partial compactification of $f : U \setminus Y \to X$, with the corresponding simple normal crossing divisor $\overline{Y} \cup D$. Using in this context the definition given in (4.1), one gets

$$MC_y[f : U \setminus Y \to X] = MC_y[f : U \to X] - MC_y[f : Y \to X],$$

which is precisely the sought-for 'additivity' property in the case of a hypersurface Y in U. In general, let $\widetilde{U}$ be the blow-up of $\overline{U}$ along $\overline{Y}$, with exceptional divisor E and $\widetilde{D}_i$ the strict transform of D_i ($i = 1, \ldots, s$), as well as $\tilde{f} : \widetilde{U} \to X$ the induced proper morphism. Then $\widetilde{D}_I$ is the blow-up of D_I along $\overline{Y}_I := \overline{Y} \cap D_I$ with exceptional divisor $E_I := E \cap \widetilde{D}_I$ for $I \subseteq \{1, \ldots, s\}$. As observed in [BSY10, Corollary 0.1 and p. 8] and [FRW18, §2.4], the key equality needed is the following 'blow-up relation',

$$MC_y[\tilde{f} : \widetilde{D}_I \to X] - MC_y[\tilde{f} : E_I \to X] = MC_y[\overline{f} : D_I \to X] - MC_y[\overline{f} : \overline{Y}_I \to X];$$

the heart of its proof relies on vanishing of certain sheaf cohomology groups proved in [GNA02, Proposition 3.3]. Using the blow-up relations and the additivity from the hypersurface case

to the partial compactification $\widetilde{X}$ of $U \backslash Y$ with corresponding simple normal crossing divisor $E \cup \widetilde{D}$ we obtain:

$$MC_y[f : U \backslash Y \to X] = MC_y[\tilde{f} : \widetilde{U} \backslash (E \cup \widetilde{D}) \to X]$$

$$= \sum_{I \subseteq \{1,\ldots,s\}} (-1)^{|I|} \left(MC_y[\tilde{f} : \widetilde{D}_I \to X] - MC_y[\tilde{f} : E_I \to X] \right)$$

$$= \sum_{I \subseteq \{1,\ldots,s\}} (-1)^{|I|} \left(MC_y[\overline{f} : D_I \to X] - MC_y[\overline{f} : \overline{Y}_I \to X] \right)$$

$$= MC_y[f : U \to X] - MC_y[f : Y \to X] \,.$$

This proves the 'additivity' property in general. This construction shows that the transformation MC_y is determined by its image on classes $[f : Z \to X]$ where Z is a *non-singular*, irreducible, quasi-projective algebraic variety and f is a T-equivariant *proper* morphism.

To prove part (1) of the statement, i.e., functoriality with respect to T-equivariant proper morphisms, observe that if $g : X \to X'$ is a proper equivariant morphism of smooth quasi-projective T-varieties, then $\overline{U}$ and $\overline{g \circ f} := g \circ \overline{f}$ is a partial compactification of $g \circ f : U \to X'$, with $(g \circ f)_I = g \circ f_I$ such that

$$MC_y[g \circ f : U \to X'] = \sum_{I \subseteq \{1,\ldots,s\}} (-1)^{|I|} g_* f_{I*} \lambda_y(T^* D_I) = g_* \left(MC_y[f : U \to X] \right) \,.$$

With the construction of MC_y given above, the proofs of parts (3) and (4) of the statement follow as in the non-equivariant case of [BSY10, Theorem 2.1], by making all K-theory classes and morphisms equivariant. □

If one forgets the T-action, then the equivariant motivic Chern class recovers the non-equivariant motivic Chern class from [BSY10], either by its construction, or by the properties (1)-(2) from Theorem 4.2 and the corresponding results from [BSY10]. Further, Theorem 4.2 and its proof work more generally for a possibly singular, quasi-projective T-equivariant base variety X, provided one works with the Grothendieck group $K_0^T(X)$ of T-equivariant coherent $\mathcal{O}_X$-modules; then one obtains $MC_y : G_0^T(var/X) \to K_0^T(X)[y]$.

Since most of the time the variety X will be understood from the context, for $Z \subseteq X$ a (not necessarily closed) subvariety we use the notation

$$MC_y(Z) := MC_y[Z \hookrightarrow X].$$

By functoriality, if $Z \subseteq X$ is a smooth closed subvariety, then $MC_y[i : Z \hookrightarrow X] = i_*(\lambda_y(T^*Z) \otimes [\mathcal{O}_Z])$ as elements in $K_T(X)$. We will often suppress the push-forward notation.

Remark 4.3. There are some differences between the definition of the relative equivariant Grothendieck group of varieties in [Loo02, Bit04, Bit05], and hypotheses used therein, and those used in this paper. For instance, [Bit04, Bit05] use finite groups G with a 'good' action; we use a torus T in the complex quasi-projective context, but can work similarly with a complex linear algebraic group G. Bittner also divides by an additional 'projective bundle relation', stating that for a G-equivariant projective bundle $\mathbb{P}(V) \to Z$ over a relative G-variety $Z \to X$:

$$[\mathbb{P}(V) \to Z \to X] = [\mathbb{P}^{\mathrm{rk}(V)-1} \times Z \to Z \to X] \,,$$

where on the right hand side G only acts on Z and X. This is not needed in this paper, but we will show in [AMSS] that the motivic Chern class also factorizes over this additional relation. Despite these differences, the proof of Theorem 4.2 applies to all these contexts, following the ideas from *loc. cit.* and [FRW18]. At the heart of the arguments

is the fact that $G_0^G(var/X) \simeq G_0^G(sm/X)$, together with results on equivariant completion, equivariant resolution of singularities and an equivariant weak factorization theorem [Sum74, BM97, AKMW02, Ber18] as used in [FRW18]. If one also divides by the 'projective bundle relation', then one can also define a *motivic duality* involution on the Grothendieck group (localized at the class of the affine line), which commutes under the motivic Chern class transformation with the Grothendieck-Serre duality involution. For a discussion of this involution see [Sch09, §5C] and our upcoming work [AMSS]; also cf. (8.9) below.   ⌟

We need the following general lemma.

Lemma 4.4. *Let $X_1, X_2 \subseteq X$ be three T-equivariant varieties. The following equalities hold in $K_T(X)[y]$.*
(a) The inclusion exclusion formula:

$$MC_y[X_1 \cup X_2 \hookrightarrow X] = MC_y[X_1 \hookrightarrow X] + MC_y[X_2 \hookrightarrow X] - MC_y[X_1 \cap X_2 \hookrightarrow X].$$

(b) If X_1, X_2, X are smooth, and X_1, X_2 intersect transversally (so that $X_1 \cap X_2$ is also smooth), then

$$(4.2) \qquad\qquad MC_y[X_1 \cap X_2 \hookrightarrow X] = \frac{MC_y(X_1)MC_y(X_2)}{MC_y(X)}.$$

(c) More generally, (4.2) holds if X_1, X_2 are unions of smooth hypersurfaces such that $X_1 \cup X_2$ is a divisor with simple normal crossings.

Remark 4.5. In part (a), the scheme structure on the union $X_1 \cup X_2$ is irrelevant; in fact, $MC_y(Z \hookrightarrow X) = MC_y(Z_{red} \hookrightarrow X)$ since both classes equal $MC_y(X) - MC_y(X \smallsetminus Z)$. In parts (b) and (c) one formally inverts the class $MC_y(X) = 1 + \sum_{k>0} y^k[\wedge^k T^* X]$, and interprets the fraction $\frac{MC_y(X_1)MC_y(X_2)}{MC_y(X)}$ in an appropriate completion of $K_T(X)[y]$. The result must actually land in $K_T(X)[y]$, since it equals the left-hand side. When $X = G/B$, the inverse of $MC_y(X)$ can also be calculated from Remark 8.9.   ⌟

Proof. The statement in (a) is immediate from the additivity property in the Grothendieck group. Part (b) follows from standard exact sequences, using that the normal bundle $N_{X_1 \cap X_2} X$ is the restriction to $X_1 \cap X_2$ of $N_{X_1} X \oplus N_{X_2} X$. Finally, part (c) follows from repeated application of (a) and (b), using inclusion-exclusion and induction on $\dim X$ and on the number of components of $X_1 \cup X_2$.   □

Remark 4.6. In the (co)homological case, i.e. after replacing the motivic Chern classes by the CSM classes, it was proved in [Sch17] that this lemma holds for X_i possibly singular (under suitable transversality assumptions).   ⌟

5. Motivic Chern classes of Schubert cells via Demazure-Lusztig operators

In this section we calculate the motivic Chern classes of Schubert cells in $X = G/B$, using Demazure-Lusztig operators.

We use the definitions and notation from §2. Fix a word $(i_1, \ldots, i_k)$ and let $Z := Z_{i_1, \ldots, i_k}$ and $Z' := Z_{i_1, \ldots, i_{k-1}}$ be the corresponding Bott-Samelson varieties. Recall that we have determined a section $\sigma : Z' \to Z$ of the projection $\pi : Z \to Z'$, and let $D := D_k = \sigma(Z')$. The 'boundary' $\partial Z := \pi^{-1}(\partial Z') \cup D$ is a simple normal crossings divisor. The morphism

$\theta : Z \to G/B$ is the composition $pr_1 \circ \theta_1$ (cf. (2.1)).

$$
\begin{array}{ccccc}
& & & \theta & \\
Z & \xrightarrow{\theta_1} & G/B \times_{G/P_i} G/B & \xrightarrow{pr_1} & G/B \\
\sigma \Big(\Big\downarrow \pi & & \Big\downarrow pr_2 & & \Big\downarrow p_{i_k} \\
Z' & \xrightarrow{\theta'} & G/B & \xrightarrow{p_{i_k}} & G/P_{i_k}
\end{array}
$$

If $w = s_{i_1} \cdot \ldots \cdot s_{i_k}$ is a reduced decomposition, then Z is a resolution of the Schubert variety $X(w)$ (Proposition 2.1). But note that the construction of the Bott-Samelson variety for $(i_1, \ldots, i_k)$ can be carried out even if the word is non-reduced. Our main theorem is:

Theorem 5.1. *Let $(i_1, \ldots, i_k)$ be a (possibly non-reduced) word. Then*
$$
\theta_* MC_y(Z \setminus \partial Z) = \mathcal{T}_{i_k} \theta'_* MC_y(Z' \setminus \partial Z'),
$$
as elements in $K_T(G/B)[y]$. In particular, if $w \in W$ and s_i is a simple reflection such that $ws_i > w$, then
$$
MC_y[X(ws_i)^\circ \hookrightarrow G/B] = \mathcal{T}_i MC_y[X(w)^\circ \hookrightarrow G/B].
$$

Proof. The second claim follows from the first. Indeed, take any reduced word $(i_1, \ldots, i_{k-1})$ for $w \in W$, so that $(i_1, \ldots, i_{k-1}, i)$ is a reduced word for ws_i. The restrictions $\theta : Z \setminus \partial Z \to X(ws_i)^\circ$ and $\theta' : Z' \setminus \partial Z' \to X(w)^\circ$ are (equivariant) isomorphisms, and by functoriality $\theta_* MC_y(Z \setminus \partial Z) = MC_y[X(ws_i)^\circ \hookrightarrow G/B]$ and $\theta'_* MC_y(Z' \setminus \partial Z') = MC_y[X(w)^\circ \hookrightarrow G/B]$.

We now prove the first assertion. To shorten notation, let $\kappa := MC_y(Z \setminus \partial Z)$ and $\kappa' := MC_y(Z' \setminus \partial Z')$. Then, by additivity and Lemma 4.4(c)

$$
\kappa = MC_y(Z) - MC_y(\partial Z) = MC_y(Z) - MC_y(\pi^{-1}(\partial Z') \cup D)
$$
$$
= MC_y(Z) - MC_y(\pi^{-1}(\partial Z')) - MC_y(D) + \frac{MC_y(\pi^{-1}(\partial Z')) MC_y(D)}{MC_y(Z)}
$$
$$
= MC_y(Z) - MC_y(D) - MC_y(\pi^{-1}(\partial Z')) \left(1 - \frac{MC_y(D)}{MC_y(Z)} \right)
$$
$$
= MC_y(Z) - MC_y(D) - \lambda_y(T_\pi^*) \pi^* MC_y(\partial Z') \left(1 - \frac{MC_y(D)}{MC_y(Z)} \right),
$$

where the last equality follows from the VRR formula in Theorem 4.2. Write $MC_y(\partial Z') = MC_y(Z') - \kappa'$ and observe that $\lambda_y(T_\pi^*) \pi^* MC_y(Z') = MC_y(Z)$, because of the short exact sequence defining the relative cotangent bundle. Apply these to the last expression in the sequence above to deduce that

$$
\kappa = \lambda_y(T_\pi^*) \pi^*(\kappa') \left(1 - \frac{MC_y(D)}{MC_y(Z)} \right).
$$

The relative cotangent bundle is $T_\pi^* = \theta_1^* T_{pr_2}^*$, while D is the pull-back of the diagonal $\mathcal{D} \subseteq G/B \times_{G/P_i} G/B$. Since D is non-singular,

$$
\frac{MC_y(D)}{MC_y(Z)} \otimes [\mathcal{O}_D] = \frac{1}{\lambda_y(N_D^* Z)} \otimes [\mathcal{O}_D],
$$

and the conormal bundle $N_D^* Z = \theta_1^*(N_\mathcal{D}^*(FP))$, where $FP := G/B \times_{G/P_i} G/B$ is the fiber product in the diagram above. We obtained that

$$
\kappa = \lambda_y(T_\pi^*) \pi^*(\kappa') - \pi^*(\kappa') \theta_1^* \left(\frac{\lambda_y(T_{pr_2}^*)}{\lambda_y(N_\mathcal{D}^*(FP))} \otimes \mathcal{O}_D \right).
$$

Now Lemma 2.2 implies that $\frac{\lambda_y(T^*_{pr_2})}{\lambda_y(N^*_{\mathcal{D}}(FP))} \otimes [\mathcal{O}_{\mathcal{D}}] = [\mathcal{O}_{\mathcal{D}}]$, therefore

$$\kappa = \lambda_y(T^*_\pi)\pi^*(\kappa') - \pi^*(\kappa') \otimes \mathcal{O}_D.$$

A standard diagram chase, using the projection formula, and the base-change formula $(\theta_1)_*\pi^* = pr_2^*\theta'_*$, gives that

$$(\theta_1)_*(\kappa) = \lambda_y(T^*_{pr_2})pr_2^*\theta'_*(\kappa') - pr_2^*\theta'_*(\kappa') \otimes \mathcal{O}_{\mathcal{D}}.$$

Applying $(pr_1)_*$ to both sides, and observing that $T^*_{pr_2} = pr_1^*T^*_{p_{i_k}}$ yields

$$\theta_*(\kappa) = \lambda_y(T^*_{p_{i_k}})(pr_1)_*pr_2^*\theta'_*(\kappa') - (pr_1)_*(pr_2^*\theta'_*(\kappa') \otimes \mathcal{O}_{\mathcal{D}}) = \mathcal{T}_{i_k}(\theta'_*(\kappa')),$$

because $(pr_1)_*pr_2^* = \partial_{i_k}$ and the restriction $(pr_2)|_{\mathcal{D}}$ to the diagonal $\mathcal{D} \simeq G/B$ is the identity $G/B \to G/B$. This finishes the proof. $\qquad\square$

We record the following corollary.

Corollary 5.2. *Let $w \in W$ and let s_i be a simple reflection. Then*

$$\mathcal{T}_i(MC_y(X(w)^\circ)) = \begin{cases} MC_y(X(ws_i)^\circ) & \text{if } ws_i > w; \\ -(y+1)MC_y(X(w)^\circ) - yMC_y(X(ws_i)^\circ) & \text{if } ws_i < w. \end{cases}$$

Proof. The identity from the $ws_i > w$ branch was proved in Theorem 5.1. Assume that $ws_i < w$. Then the same result shows that $MC_y(X(w)^\circ) = \mathcal{T}_i(MC_y(X(ws_i)^\circ))$, thus

$$\mathcal{T}_i(MC_y(X(w)^\circ)) = \mathcal{T}_i^2(MC_y(X(ws_i)^\circ)).$$

By the quadratic relations from Proposition 3.4, $\mathcal{T}_i^2 = -(y+1)\mathcal{T}_i - y \cdot id$. Now we apply the right-hand side to $MC_y(X(ws_i)^\circ)$, using again Theorem 5.1 and that $(ws_i)s_i > ws_i$. $\quad\square$

Remark 5.3. In particular, setting $y = -1$:

$$\mathcal{T}_i|_{y=-1}(MC_{-1}(X(w)^\circ)) = MC_{-1}(X(ws_i)^\circ)$$

regardless of whether ws_i precedes or follows w in the Bruhat order. Combined with Lemma 3.8, this implies that

$$MC_{-1}(X(w)^\circ) = \iota_w,$$

the class of the fixed point e_w. The corresponding statement holds for the CSM class in (co)homology, cf. [AM16, Prop.6.5(d)]. $\qquad\lrcorner$

Remark 5.4. Recall that if $w = s_{i_1} \cdots s_{i_k}$ is a reduced decomposition, the operator $\mathcal{T}_w := \mathcal{T}_{i_1} \cdots \mathcal{T}_{i_k}$ is well-defined (cf. (3.6)). With this notation,

$$MC_y(X(w)^\circ) = \mathcal{T}_{w^{-1}}(\mathcal{O}_{id})$$

as a consequence of Theorem 5.1. $\qquad\lrcorner$

5.1. **Motivic Chern classes in G/P.**

Let $P \supset B$ be a parabolic subgroup containing B and let $W^P \subseteq W$ be the subset of minimal length representatives for W/W_P, the quotient of W by the subgroup W_P generated by the reflections in P. For $wW_P \in W/W_P$, $\ell(wW_P)$ denotes the length of the (unique) representative of wW_P in W^P. The Schubert cells in G/P are $X(wW_P)^\circ := BwP/P \subseteq G/P$; then $X(wW_P)^\circ \simeq \mathbb{A}^{\ell(wW_P)}$. The natural projection $\pi : G/B \to G/P$ sends $X(w)^\circ$ to $X(wW_P)^\circ$ and it is an isomorphism if $w \in W^P$. From this and the functoriality of motivic Chern classes it follows that

$$\pi_* MC_y[X(w)^\circ \hookrightarrow G/B] = MC_y[X(wW_P)^\circ \hookrightarrow G/P], \forall w \in W^P.$$

Remark 5.5. In fact, one can prove more: from [BCMP13, §2] one obtains that the restriction $\pi|_{X(w)^\circ} : X(w)^\circ \to X(wW_P)^\circ$ is an equivariantly trivial fibration with fiber a Schubert cell of dimension $\ell(w) - \ell(wW_P)$ in $\pi^{-1}(e_{wW_P}) \simeq P/B$, regarded as a homogeneous space for the Levi subgroup of P. It is not difficult to show that this implies that for all $w \in W$,

$$\pi_* MC_y[X(w)^\circ \hookrightarrow G/B] = (-y)^{\ell(w) - \ell(wW_P)} MC_y[X(wW_P)^\circ \hookrightarrow G/P].$$

Details of the proof and applications to point counting in characteristic p will be included in a continuation to this paper. ⌟

6. The Hecke duality for motivic Chern classes

It was proved in [AMSS17, §5] that the Poincaré duals of the CSM classes of Schubert cells are given by the operators which are adjoint to the Hecke-type operators which determine the CSM classes. The same phenomenon holds in the context of this paper, with the same idea of proof. However, in this context the DL operators satisfy the quadratic relations (3.4), while in the cohomological case studied in [AMSS17] the corresponding operators are self-inverse. This leads to somewhat more involved calculations for motivic Chern classes. In analogy with the dual CSM class from [AMSS17, Definition 5.3] we make the following definition.

Definition 6.1. Let $w \in W$. The *dual motivic Chern class* is defined by

$$MC_y^\vee(Y(w)^\circ) := (\mathcal{T}_{w_0 w}^\vee)^{-1}(MC_y(Y(w_0))) = (\mathcal{T}_{w_0 w}^\vee)^{-1}(\mathcal{O}^{w_0}) \in K_T(G/B)[y, y^{-1}].$$

The name of this class is explained by the following theorem, which is the K-theoretic analogue of [AMSS17, Theorem 5.7].

Theorem 6.2. *For every $u, v \in W$,*

$$\langle MC_y(X(u)^\circ), MC_y^\vee(Y(v)^\circ) \rangle = \delta_{u,v}(-y)^{\ell(u) - \dim G/B} \prod_{\alpha > 0}(1 + ye^{-\alpha}).$$

Remark 6.3. Geometrically, the quantity $\prod_{\alpha > 0}(1 + ye^{-\alpha})$ equals $\lambda_y(T_{w_0}^*(G/B))$, i.e. it is the λ_y class of the fiber of the cotangent bundle at w_0. ⌟

Proof of Theorem 6.2. Using the definition of both flavors of motivic classes, and the fact that $\mathcal{T}_i$ and $\mathcal{T}_i^\vee$ are adjoint to each other, we obtain that

$$\langle MC_y(X(u)^\circ), MC_y^\vee(Y(v)^\circ) \rangle = \langle \mathcal{T}_{u^{-1}}(\mathcal{O}_{\mathrm{id}}), (\mathcal{T}_{w_0 v}^\vee)^{-1}(\mathcal{O}^{w_0}) \rangle = \langle \mathcal{O}_{\mathrm{id}}, \mathcal{T}_u^\vee \cdot (\mathcal{T}_{w_0 v}^\vee)^{-1}(\mathcal{O}^{w_0}) \rangle.$$

By Proposition 3.5,

$$\mathcal{T}_u^\vee \cdot (\mathcal{T}_{w_0 v}^\vee)^{-1} = c_{uv^{-1}w_0}(y)\mathcal{T}_{uv^{-1}w_0}^\vee + \sum_{w < uv^{-1}w_0} c_w(y)\mathcal{T}_w^\vee.$$

Since $\mathcal{T}_w^\vee(\mathcal{O}^{w_0})$ is a combination of Schubert classes $\mathcal{O}^{w'}$ such that $w \leq w'$ and $\langle \mathcal{O}_{\mathrm{id}}, \mathcal{O}^w \rangle = \delta_{w,id}$, $\langle \mathcal{O}_{\mathrm{id}}, \mathcal{T}_u^\vee \cdot (\mathcal{T}_{w_0 v}^\vee)^{-1}(\mathcal{O}^{w_0}) \rangle$ is 0 unless $uv^{-1}w_0 = w_0$, i.e. $u = v$. In this case, by (3.8), the coefficient of $\mathcal{O}^{\mathrm{id}}$ in $\mathcal{T}_{w_0}^\vee(\mathcal{O}^{w_0})$ is $\prod_{\alpha > 0}(1 + ye^{-\alpha})$. By Proposition 3.5, the coefficient of

$\mathcal{T}_{w_0}^\vee$ in $\mathcal{T}_u^\vee \cdot (\mathcal{T}_{w_0 u}^\vee)^{-1}$ is $(-y)^{-\ell(w_0 u)}$. Therefore,

$$
\begin{aligned}
\langle MC_y(X(u)^\circ), MC_y^\vee(Y(u)^\circ)\rangle &= \langle \mathcal{O}_{\mathrm{id}}, \mathcal{T}_u^\vee \cdot (\mathcal{T}_{w_0 u}^\vee)^{-1}(\mathcal{O}^{w_0})\rangle \\
&= \langle \mathcal{O}_{\mathrm{id}}, (-y)^{-\ell(w_0 u)} \mathcal{T}_{w_0}^\vee(\mathcal{O}^{w_0})\rangle \\
&= \langle \mathcal{O}_{\mathrm{id}}, (-y)^{-\ell(w_0 u)} \prod_{\alpha>0}(1 + ye^{-\alpha})\mathcal{O}^{\mathrm{id}}\rangle \\
&= (-y)^{\ell(u)-\dim G/B} \prod_{\alpha>0}(1 + ye^{-\alpha}),
\end{aligned}
$$

concluding the proof. □

Remark 6.4. It is natural to consider the normalized class

(6.1) $$\widetilde{MC}_y(Y(w)^\circ) := (-y)^{\dim G/B - \ell(w)} MC_y^\vee(Y(w)^\circ).$$

The classes $\widetilde{MC}_y(Y(w)^\circ)$ are given by the normalized operator $\mathcal{L}_i := \mathcal{T}_i^\vee + (1 + y)id$; cf. equation (3.5). The coefficients in the Schubert expansion of this class are polynomial in y. ⌋

Example 6.5. The motivic Chern class for Schubert cells in Fl(3) were listed in the introduction. The normalized dual motivic classes $\widetilde{MC}_y(Y(w)^\circ)$ for the Schubert cells in Fl(3), computed using (6.1) and Definition 6.1, are:

$$
\begin{aligned}
\widetilde{MC}_y(Y(w_0)) &= \mathcal{O}^{w_0}; \\
\widetilde{MC}_y(Y(s_1 s_2)^\circ) &= (1 + y)\mathcal{O}^{s_1 s_2} + y\mathcal{O}^{w_0}; \\
\widetilde{MC}_y(Y(s_2 s_1)^\circ) &= (1 + y)\mathcal{O}^{s_2 s_1} + y\mathcal{O}^{w_0}; \\
\widetilde{MC}_y(Y(s_1)^\circ) &= (1 + y)^2\mathcal{O}^{s_1} + y(1 + y)\mathcal{O}^{s_1 s_2} + 2y(1 + y)\mathcal{O}^{s_2 s_1} + y^2\mathcal{O}^{w_0}; \\
\widetilde{MC}_y(Y(s_2)^\circ) &= (1 + y)^2\mathcal{O}^{s_2} + 2y(1 + y)\mathcal{O}^{s_1 s_2} + y(1 + y)\mathcal{O}^{s_2 s_1} + y^2\mathcal{O}^{w_0}; \\
\widetilde{MC}_y(Y(\mathrm{id})^\circ) &= (1 + y)^3\mathcal{O}^{\mathrm{id}} + y(1 + y)^2(\mathcal{O}^{s_1} + \mathcal{O}^{s_2}) + 2y^2(1 + y)(\mathcal{O}^{s_1 s_2} + \mathcal{O}^{s_2 s_1}) + y^3\mathcal{O}^{w_0}.
\end{aligned}
$$

An algebra verification, using the fact that $\langle \mathcal{O}_u, \mathcal{O}^v\rangle = 1$ if $u \geq v$ and $\langle \mathcal{O}_u, \mathcal{O}^v\rangle = 0$ otherwise (cf. (3.2)), shows that

$$\langle MC_y(X(u)^\circ), \widetilde{MC}_y(Y(v)^\circ)\rangle = (1 + y)^{\dim \mathrm{Fl}(3)}\delta_{u,v},$$

as prescribed by Theorem 6.2. (Here we are setting the equivariant variables e^α to 1.) At this time we note that the analogue of the positivity Conjecture 1 is false for the dual classes. For instance the coefficient of $\mathcal{O}^{s_3 s_1 s_2}$ in the expansion of $\widetilde{MC}_y(Y(\mathrm{id})^\circ) \in K(\mathrm{Fl}(4))$ equals $y^2(4y - 1)(1 + y)^3$. ⌋

In the next result we determine the action of the operators $\mathcal{T}_i^\vee$ on the dual motivic classes.

Proposition 6.6. *Let $w \in W$ be a Weyl group element and s_i a simple reflection. Then the following equalities hold:*

(a) $\quad \mathcal{T}_i^\vee(MC_y^\vee(Y(w)^\circ)) = \begin{cases} MC_y^\vee(Y(ws_i)^\circ) & \text{if } ws_i > w \\ -(y + 1)MC_y^\vee(Y(w)^\circ) - yMC_y^\vee(Y(ws_i)^\circ) & \text{if } ws_i < w \end{cases}$

(b) $\quad (\mathcal{T}_i^\vee)^{-1}(MC_y^\vee(Y(ws_i)^\circ)) = \begin{cases} MC_y^\vee(Y(w)^\circ) & \text{if } ws_i > w \\ -\frac{1}{y}MC_y^\vee(Y(w)^\circ) - \frac{y+1}{y}MC_y^\vee(Y(ws_i)^\circ) & \text{if } ws_i < w \end{cases}$

Proof. To prove part (a), consider first the case when $ws_i > w$. Then $w_0 ws_i < w_0 w$, thus $\mathcal{T}_{w_0 w}^\vee = \mathcal{T}_{w_0 ws_i}^\vee \mathcal{T}_i^\vee$. By Definition 6.1,

$$\mathcal{T}_i^\vee(MC_y^\vee(Y(w)^\circ)) = \mathcal{T}_i^\vee(\mathcal{T}_{w_0 w}^\vee)^{-1}(\mathcal{O}^{w_0}) = (\mathcal{T}_{w_0 ws_i}^\vee)^{-1}(\mathcal{O}^{w_0}) = MC_y^\vee(Y(ws_i)^\circ).$$

The situation when $ws_i < w$ is treated as in the proof of Corollary 5.2, using that $\mathcal{T}_i^\vee$ satisfies the quadratic relations from Proposition 3.4. Part (b) follows from (a) by applying $(\mathcal{T}_i^\vee)^{-1}$ to both sides. $\qquad\square$

7. Three recursions for localizations of motivic Chern classes

In this section, we use the Demazure Lusztig operators $\mathcal{T}_i$ to obtain recursive relations for the ordinary and dual motivic Chern classes of Schubert cells. These recursions will be used to compare the motivic Chern classes both with stable envelopes and with Casselman's basis. We also record a divisibility property for localizations of motivic classes, to be used later in the proof of Theorem 1.4.

7.1. **Recursions.** Consider the localized equivariant K-theory ring defined by

$$K_T(G/B) \hookrightarrow K_T(G/B)_{loc} := K_T(G/B) \otimes_{K_T(pt)} \mathrm{Frac}(K_T(pt)).$$

The K-theoretic analogue of the Bott localization theorem (see e.g., [CG09, § 5.10]) gives the expansion of the motivic Chern classes in terms of the fixed point classes ι_w, for every $w \in W$:

$$MC_y(X(w)^\circ) = \sum_{u \leq w} MC_y(X(w)^\circ)|_u \frac{\iota_u}{\lambda_{-1}(T_u^*(G/B))}$$

(7.1)
$$= \sum_{u \leq w} MC_y(X(w)^\circ)|_u \frac{\iota_u}{\prod_{\alpha > 0}(1 - e^{u\alpha})} \quad \in K_T(G/B)_{loc}[y].$$

The following three propositions give recursions formulas for various flavors of motivic Chern classes. These will be used later to make the connection with the Hecke algebra action on the principal series representation. The similarity of the recursions can be explained by the fact that they are related either by an automorphism of G/B or by the involution exchanging the Demazure-Lusztig operators.

Proposition 7.1. *The localizations $MC_y(X(w)^\circ)|_u$ are uniquely determined by the following conditions:*

(a) $MC_y(X(w)^\circ)|_u = 0$, *unless* $u \leq w$.
(b) *If* $u = w$:

$$MC_y(X(w)^\circ)|_w = \prod_{\alpha > 0, w\alpha < 0} (1 + ye^{w\alpha}) \prod_{\alpha > 0, w\alpha > 0} (1 - e^{w\alpha}).$$

(c) *If* $ws_i > w$, *then*

$$MC_y(X(ws_i)^\circ)|_u = -\frac{1+y}{1 - e^{-u\alpha_i}} MC_y(X(w)^\circ)|_u + \frac{1 + ye^{u\alpha_i}}{1 - e^{-u\alpha_i}} MC_y(X(w)^\circ)|_{us_i}.$$

Proof. Part (a) follows because the motivic class is supported on the Schubert variety $X(w)$. To prove part (b), observe that $MC_y(X(w)^\circ)|_w = MC_y(X(w))|_w$, by additivity and because $MC_y(X(v)^\circ)|_w = 0$ for $v < w$ by part (a). Then

$$MC_y(X(w)^\circ)|_w = MC_y(X(w))|_w = \lambda_y(T_w^* X(w))\lambda_{-1}(N_w^\vee),$$

where $T_w^* X(w)$ and $N_w^\vee$ are the fibers at the fixed point w of the dual of the cotangent, respectively the conormal bundle for $X(w)$. (A more general result is proved in Theorem 9.1

below.) Part (c) follows by applying the operator $\mathcal{T}_i$ to Equation (7.1) and taking the coefficients of ι_u; this requires the action of $\mathcal{T}_i$ on the fixed point basis described in Lemma 3.7. Finally, the uniqueness follows by induction on the length of w. $\qquad\square$

For further use, we also record the similar result for the motivic Chern class of the opposite Schubert cells.

Proposition 7.2. *The localizations $MC_y(Y(w)^\circ)|_u$ are uniquely determined by the following conditions:*

(a) $MC_y(Y(w)^\circ)|_u = 0$, *unless* $u \geq w$.

(b) *If* $u = w$:
$$MC_y(Y(w)^\circ)|_w = \prod_{\alpha>0,w(\alpha)>0} (1 + ye^{w\alpha}) \prod_{\alpha>0,w(\alpha)<0} (1 - e^{w\alpha}).$$

(c) *If* $ws_i > w$, *then*
$$MC_y(Y(w)^\circ)|_u = -\frac{1+y}{1-e^{-u\alpha_i}} MC_y(Y(ws_i)^\circ)|_u + \frac{1+ye^{u\alpha_i}}{1-e^{-u\alpha_i}} MC_y(Y(ws_i)^\circ)|_{us_i}.$$

Proof. The left multiplication by w_0 induces an automorphism of G/B sending $X(w)$ to $Y(w_0w)$. This is not T-equivariant, but it is equivariant with respect to the map $T \to T$ defined by $t \mapsto w_0 t w_0$. This induces an automorphism of $K_T(G/B)$ and its localized version, twisting the coefficients by w_0. Then the proposition follows from Proposition 7.1 above, by applying w_0. $\qquad\square$

Similar formulas hold for the dual classes $MC_y^\vee(Y(w)^\circ)$:

Proposition 7.3. *The localizations $MC_y^\vee(Y(w)^\circ)|_u$ are uniquely determined by the following conditions:*

(a) $MC_y^\vee(Y(w)^\circ)|_u = 0$, *unless* $u \geq w$.

(b) *If* $u = w$:
$$MC_y^\vee(Y(w)^\circ)|_w = (-1)^{\dim G/B - \ell(w)} \prod_{\alpha>0,w\alpha>0} (y^{-1} + e^{-w\alpha}) \prod_{\alpha>0,w\alpha<0} (1 - e^{w\alpha}).$$

(c) *If* $ws_i > w$, *then*
$$MC_y^\vee(Y(w)^\circ)|_u = \frac{1+y^{-1}}{e^{u\alpha_i}-1} MC_y^\vee(Y(ws_i)^\circ)|_u + \frac{y^{-1}+e^{-u\alpha_i}}{e^{-u\alpha_i}-1} MC_y^\vee(Y(ws_{\alpha_i})^\circ)|_{us_{\alpha_i}}.$$

Proof. These formulae are regarded in $K_T(pt)[y^{-1}] \hookrightarrow \mathrm{Frac}(K_T(pt))[y^{-1}]$. The uniqueness follows directly from induction. So we only need to show that $MC_y^\vee(Y(w)^\circ)|_u$ satisfies these properties. The support property follows because $(\mathcal{T}_i^\vee)^{-1}$ sends a Schubert class $\mathcal{O}^u$ into classes supported on $Y(u) \cup Y(us_i)$; then one applies Proposition 6.6. To calculate the localization at w, we use the duality from Theorem 6.2 and Bott localization to obtain
$$(-y)^{\ell(w)-\dim G/B} \prod_{\alpha>0}(1 + ye^{-\alpha}) = \langle MC_y(X(w)^\circ), MC_y^\vee(Y(w)^\circ) \rangle$$
$$= \sum_{u\in W} \frac{(MC_y(X(w)^\circ) \cdot MC_y^\vee(Y(w)^\circ))|_u}{\prod_{\alpha>0}(1-e^{u(\alpha)})} \cdot \int_{G/B} \iota_u .$$

The only non-zero contribution is for $u = w$, and the integral equals 1, thus
$$MC_y^\vee(Y(w)^\circ)|_w = \frac{(-y)^{\ell(w)-\dim G/B} \prod_{\alpha>0}(1 + ye^{-\alpha})(1 - e^{w(\alpha)})}{MC_y(X(w)^\circ)|_w}.$$

Part (b) follows from this and the localization from Proposition 7.1. Part (c) follows as in Proposition 7.1, using now Proposition 6.6 and part (e) of Lemma 3.7. $\qquad\square$

7.2. **A divisibility property for localization coefficients.** We record the following property which will be used in our applications to p-adic groups.

Theorem 7.4. *For every $w \le u \in W$, the polynomial $MC_y(Y(w)^\circ)|_u \in K_T(pt)[y]$ is divisible by*

$$\prod_{\alpha>0,u\alpha>0} (1 + ye^{u\alpha}) \prod_{\alpha>0,w\not\le us_\alpha<u} (1 - e^{u\alpha}).$$

Proof. By a general property of motivic classes proved in [FRW18, Theorem 5.3(ii)], the localization coefficient $MC_y(Y(w)^\circ)|_u$ is divisible by $\lambda_y(T_u^*Y(u)^\circ) = \prod_{\alpha>0,u\alpha>0}(1 + ye^{u\alpha})$. As α varies in the set of positive roots, the factors $1 - e^{u\alpha}$ and $1 + ye^{u\alpha}$ are relative prime to each other. Then it remains to show that for every $\alpha > 0$ such that $w \not\le us_\alpha < u$, the localization coefficient $MC_y(Y(w)^\circ)|_u$ is divisible by $1 - e^{u\alpha}$. Let $C \simeq \mathbb{P}^1$ denote the T-stable curve connecting the fixed points e_u and e_{us_α}. The T-weight of the tangent space T_uC is $-u\alpha$. By localization, the equivariant Euler characteristic is given by

$$\chi_T(C, MC_y(Y(w)^\circ)) = \frac{MC_y(Y(w)^\circ)|_u}{1 - e^{u\alpha}} + \frac{MC_y(Y(w)^\circ)|_{us_\alpha}}{1 - e^{-u\alpha}} = \frac{MC_y(Y(w)^\circ)|_u}{1 - e^{u\alpha}},$$

where the last equality holds because $MC_y(Y(w)^\circ)|_{us_\alpha} = 0$, as the hypothesis $w \not\le us_\alpha$ implies that $e_{us_\alpha} \notin Y(w)$. Since the equivariant Euler characteristic $\chi_T(C, MC_y(Y(w)^\circ))$ is a polynomial in $K_T(pt)[y] \hookrightarrow \mathrm{Frac}(K_T(pt))[y]$, the localization $MC_y(Y(w)^\circ)|_u$ must be divisible by $1 - e^{u\alpha}$. This finishes the proof.[2] $\qquad\square$

8. Motivic Chern classes and K-theoretic stable envelopes

In this section, we recall some basic properties of the K-theoretic stable basis of $T^*(G/B)$, including a recursive relation. Our main references are [SZZ17, Oko17, OS16]. We compare the recursive relation obtained in [SZZ17] to the one for motivic Chern classes, and we deduce that the two objects are closely related. This was also found by Féher and Rimányi and Weber in [FRW18] (see also [FRW]), using interpolation techniques for motivic Chern classes. In cohomology, the relation between stable envelopes and Chern-Schwartz-MacPherson classes was noticed in [RV18, AMSS17] and it was used in [AMSS17] to obtain a second 'stable basis duality'. In theorem 8.11 we generalize this duality to K-theory.

8.1. **K-theoretic stable envelopes.** The cotangent bundle of G/B is the homogeneous bundle $T^*(G/B) := G \times^B T_{1.B}^*(G/B)$, given by equivalence classes

$$\{[g,v] : (g,v) \in G \times T_{1.B}^*(G/B) \text{ and } (gb,v) \sim (g,b.v), \forall g \in G, b \in B\};$$

here $T_{1.B}^*(G/B)$ is the cotangent space at the identity with its natural B-module structure. As before, let T be the maximal torus in B, and consider the $\mathbb{C}^*$-module $\mathbb{C}$ with character $q^{1/2}$. We let $\mathbb{C}^*$ act trivially on G/B and we consider the $T \times \mathbb{C}^*$ action on the cotangent bundle defined by $(t,z).[g,v] = [tg, z^{-2}v]$. In other words, T acts via its natural left action; $\mathbb{C}^*$ acts such that the cotangent fibers get a weight q^{-1} and $K_{T\times\mathbb{C}^*}(pt) = K_T(pt)[q^{\pm1/2}]$.

The stable basis is a certain basis for the localized equivariant K-theory

$$K_{T\times\mathbb{C}^*}(T^*(G/B))_{loc} := K_{T\times\mathbb{C}^*}(T^*(G/B)) \otimes_{K_{T\times\mathbb{C}^*}(pt)} \mathrm{Frac}(K_{T\times\mathbb{C}^*}(pt)),$$

---

[2]We thank A. Okounkov for comments leading to this proof.

where Frac means taking the fraction field. The basis elements are called the *stable envelopes* $\{\text{stab}_{\mathcal{C},T^{\frac{1}{2}},\mathcal{L}}(w)|w \in W\}$ and were defined by Maulik and Okounkov in the cohomological case. We recall their definition in K-theory below, following mainly Okounkov's lectures [Oko17] and [SZZ17].

For a fixed Weyl group element, the definition of the stable envelope $\text{stab}_{\mathcal{C},T^{\frac{1}{2}},\mathcal{L}}(w)$ depends on three parameters:

- a chamber $\mathcal{C}$ in the Lie algebra of the maximal torus T, or equivalently, a Borel subgroup of G.
- a polarization $T^{\frac{1}{2}} \in K_{T\times\mathbb{C}^*}(T^*(G/B))$ of the tangent bundle $T(T^*(G/B))$, i.e., a Lagrangian subbundle $T^{\frac{1}{2}}$ of the tangent bundle $T(T^*(G/B)) \in K_{T\times\mathbb{C}^*}(T^*(G/B))$ which is a solution of the equation

$$T^{\frac{1}{2}} + q^{-1}(T^{\frac{1}{2}})^{\vee} = T(T^*(G/B))$$

  in the ring $K_{T\times\mathbb{C}^*}(T^*(G/B))_{loc}$. The most important solutions are $T(G/B)$ and $T^*(G/B)$. For every polarization $T^{\frac{1}{2}}$, there is an opposite polarization defined as $T^{\frac{1}{2}}_{\text{opp}} = q^{-1}(T^{\frac{1}{2}})^{\vee}$.
- A sufficiently general fractional equivariant line bundle on G/B, i.e. a general element $\mathcal{L} \in \text{Pic}_T(T^*(G/B)) \otimes_{\mathbb{Z}} \mathbb{Q}$, called the *slope* of the stable envelope. The dependence on the slope parameter is locally constant, in the following sense.

  The choice of a maximal torus $T \subseteq G$ determines a decomposition of $(\text{Lie}\,T)^* \otimes \mathbb{R}$ into *alcoves*; these are the complements of the affine hyperplanes $H_{\alpha^{\vee},n} = \{\lambda \in (\text{Lie}\,T)^* \otimes \mathbb{R} : \langle \lambda, \alpha^{\vee} \rangle = n\}$ as $\alpha^{\vee}$ varies in the set of positive coroots, and n over the integers. The alcove structure is independent the choice of a chamber (and hence of the Borel subgroup B), and the stable envelopes are constant for fractional multiples of (pull-backs of) line bundles $\mathcal{L}_{\lambda} = G \times^B \mathbb{C}_{\lambda}$ for weights λ in a given alcove.

The torus fixed point set $(T^*(G/B))^T = (G/B)^T$ is in one-to-one correspondence with the Weyl group W. For every $w \in W$, we still use w to denote the corresponding fixed point. For a chosen Weyl chamber $\mathfrak{C}$ in $\text{Lie}\,T$, pick any cocharacter $\sigma \in \mathfrak{C}$. Then the attracting set of $w \in W$ is defined as

$$\text{Attr}_{\mathfrak{C}}(w) = \left\{ x \in T^*(G/B) \mid \lim_{z\to 0} \sigma(z) \cdot x = w \right\}.$$

It is not difficult to show that $\text{Attr}_{\mathfrak{C}}(w)$ is the conormal space over the attracting variety in G/B for w; the latter attracting variety is a Schubert cell in G/B. Define a partial order on the fixed point set W as follows:

$$w \preceq_{\mathfrak{C}} v \quad \text{if} \quad \overline{\text{Attr}_{\mathfrak{C}}(v)} \cap w \neq \emptyset.$$

Then the order determined by the positive (resp., negative) chamber is the same as the Bruhat order (resp., the opposite Bruhat order).

Any chamber $\mathfrak{C}$ determines a decomposition of the tangent space $N_w := T_w(T^*(G/B))$ as $N_w = N_{w,+} \oplus N_{w,-}$ into T-weight spaces which are positive and negative with respect to $\mathfrak{C}$ respectively. For every polarization $T^{\frac{1}{2}}$, denote $N_w \cap T^{1/2}|_w$ by $N_w^{\frac{1}{2}}$. Similarly, we have $N_{w,+}^{\frac{1}{2}}$ and $N_{w,-}^{\frac{1}{2}}$. In particular, $N_{w,-} = N_{w,-}^{\frac{1}{2}} \oplus q^{-1}(N_{w,+}^{\frac{1}{2}})^{\vee}$. Consequently, we have

$$N_{w,-} - N_w^{\frac{1}{2}} = q^{-1}(N_{w,+}^{\frac{1}{2}})^{\vee} - N_{w,+}^{\frac{1}{2}}$$

as virtual vector bundles. The determinant bundle of the virtual bundle $N_{w,-} - N_w^{\frac{1}{2}}$ is a complete square and its square root will be denoted by $\left(\frac{\det N_{w,-}}{\det N_w^{\frac{1}{2}}}\right)^{\frac{1}{2}}$; cf. [Oko17, §9.1.5]. For instance, if we choose the polarization $T^{1/2} = T(G/B)$, the positive chamber, and $w = id$ then both $N_{\mathrm{id}}^{\frac{1}{2}}$ and $N_{id,-}$ have weights $-\alpha$, where α varies in the set of positive roots; in this case the virtual bundle $N_{id,-} - N_{\mathrm{id}}^{\frac{1}{2}}$ is trivial. To calculate the localizations at other fixed points w, one may use the equivariance; see e.g., [SZZ17, Lemma 2.2] for such calculations.

Let $f := \sum_\mu f_\mu e^\mu \in K_{T \times \mathbb{C}^*}(\mathrm{pt})$ be a Laurent polynomial, where $e^\mu \in K_T(\mathrm{pt})$ and $f_\mu \in \mathbb{Q}[q^{1/2}, q^{-1/2}]$. The *Newton Polygon* of f, denoted by $\deg_T f$, is

$$\deg_T f = Convex\ hull\ (\{\mu | f_\mu \neq 0\}) \subseteq X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The following theorem defines the K-theoretic stable envelopes.

Theorem 8.1. [Oko17] *For every chamber $\mathfrak{C}$, a sufficiently general $\mathcal{L}$, and a polarization $T^{1/2}$, there exists a unique map of $K_{T \times \mathbb{C}^*}(pt)$-modules*

$$\mathrm{stab}_{\mathfrak{C}, T^{\frac{1}{2}}, \mathcal{L}} : K_{T \times \mathbb{C}^*}((T^*(G/B))^T) \to K_{T \times \mathbb{C}^*}(T^*(G/B))$$

such that for every $w \in W$, the class $\Gamma := \mathrm{stab}_{\mathfrak{C}, T^{\frac{1}{2}}, \mathcal{L}}(w)$ satisfies:

(1) *(support)* $\mathrm{Supp}\,\Gamma \subseteq \cup_{z \preceq_{\mathfrak{C}} w} \overline{\mathrm{Attr}_{\mathfrak{C}}(z)}$;

(2) *(normalization)* $\Gamma|_w = (-1)^{\mathrm{rank}\,N_{w,+}^{\frac{1}{2}}} \left(\frac{\det N_{w,-}}{\det N_w^{\frac{1}{2}}}\right)^{\frac{1}{2}} \mathcal{O}_{\mathrm{Attr}_{\mathfrak{C}}(w)}|_w$;

(3) *(degree)* $\deg_T \Gamma|_v \subseteq \deg_T \mathrm{stab}_{\mathfrak{C}, T^{\frac{1}{2}}, \mathcal{L}}(v)|_v + \mathcal{L}|_v - \mathcal{L}|_w$, *for every $v \prec_{\mathfrak{C}} w$.*

The difference $\mathcal{L}|_v - \mathcal{L}|_w$ in the degree condition implies that the stable basis does not depend on the choice of the linearization of $\mathcal{L}$.

Let $+$ denote the chamber such that all the roots in B are positive on it, and let $-$ denote the opposite chamber. From now on we fix the 'fundamental slope' given by $\widetilde{\mathcal{L}} := \mathcal{L}_\rho \otimes 1/N$, where ρ is the sum of fundamental weights and N is a large enough positive integer. Recall that $\omega_{G/B} := \mathcal{L}_{2\rho}$ is the canonical bundle of G/B, therefore the slope $\widetilde{\mathcal{L}}$ can also be thought as a (fractional version of a) square root of the canonical line bundle. We will use the following notation:

$$\mathrm{stab}_+(w) := \mathrm{stab}_{+, T(G/B), (\widetilde{\mathcal{L}})^{-1}}(w), \quad and \quad \mathrm{stab}_-(w) := \mathrm{stab}_{-, T^*(G/B), \widetilde{\mathcal{L}}}(w).$$

The positive chamber and negative chamber stable basis are dual bases in the localized equivariant ring, i.e.,

(8.1) $$\langle \mathrm{stab}_+(w), \mathrm{stab}_-(u) \rangle_{T^*(G/B)} = \delta_{w,u},$$

where $\langle \cdot, \cdot \rangle_{T^*(G/B)}$ is the equivariant K-theory pairing on $T^*(G/B)$ defined via localization; see [Oko17, Example 9.1.17], [OS16, §2.2.1, Proposition 1], or [SZZ17, Remark 1.3]. We will study the pairing on $K_{T \times \mathbb{C}^*}(T^*(G/B))$ in more detail below, in §8.5.

8.2. **Automorphisms.** The stable envelopes for various triples of parameters can be related to each other by automorphisms of the equivariant K-theory ring $K_{T \times \mathbb{C}^*}(T^*(G/B))$. We will use the following types of automorphisms:

    a. the automorphism induced by the left Weyl group multiplication. Recall that this induces an automorphism of $K_{T \times \mathbb{C}^*}(T^*(G/B))$ which twists the coefficients

in $K_{T \times \mathbb{C}^*}(pt)$ by w. In terms of localization, for every $\mathcal{F} \in K_{T \times \mathbb{C}^*}(T^*(G/B))$, we have

$$(8.2) \qquad\qquad w(\mathcal{F})|_u = w(\mathcal{F}|_{w^{-1}u}).$$

b. The duality automorphism, mapping $[E] \mapsto [E^\vee]$, i.e., the class of a vector bundle to its dual. For $[\mathcal{F}] \in K_T(G/B)$, $[\mathcal{F}]^\vee$ denotes the class obtained by taking the alternating sum of duals in an equivariant resolution of $\mathcal{F}$ by vector bundles. This automorphism also acts on $K_{T \times \mathbb{C}^*}(pt)$ by taking $e^\lambda \mapsto e^{-\lambda}$ and $q^{\frac{1}{2}} \mapsto q^{-\frac{1}{2}}$.

c. The multiplication by the class of a line bundle. We can fix an integral weight $\lambda \in X^*(T)$ and a Borel subgroup B, and consider the equivariant line bundle $\mathcal{L}_\lambda = G \times^B \mathbb{C}_\lambda$. We will abuse notation and will denote with the same symbol a line bundle on G/B and on its cotangent bundle.

d. For the ring $K_{T \times \mathbb{C}^*}(G/B) = K_T(G/B)[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$, a composition of the previous two automorphisms gives the (equivariant) Grothendieck-Serre duality. This is an automorphism $\mathcal{D}$ of $K_T(G/B)[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ defined as follows: for every $[\mathcal{F}] \in K_T(G/B)$,

$$\mathcal{D}[\mathcal{F}] := [RHom(\mathcal{F}, \omega^\bullet_{G/B})] := \omega^\bullet_{G/B} \otimes [\mathcal{F}]^\vee \in K_T(G/B),$$

where $\omega^\bullet_{G/B} \simeq \omega_{G/B}[\dim G/B]$ is the (equivariant) dualizing complex of the flag variety; thus, $[\omega^\bullet_{G/B}] = (-1)^{\dim G/B}[\mathcal{L}_{2\rho}]$. Observe that

$$([\mathcal{F}]^\vee)^\vee = [\mathcal{F}]; \quad \mathcal{D}([\mathcal{F}] \otimes \omega^\bullet_{G/B}) = [\mathcal{F}]^\vee.$$

Extend the operation $\mathcal{D}$ to $K_T(G/B)[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ by sending $q^{\frac{1}{2}} \mapsto q^{-\frac{1}{2}}$.

The following lemma, proved in the Appendix, records the effect of these automorphisms on K-theoretic stable envelopes.

Lemma 8.2. *(a) Let $u, w \in W$. Under the left Weyl group multplication,*

$$w.stab_{\mathfrak{C}, T^{1/2}, \mathcal{L}}(u) = stab_{w\mathfrak{C}, wT^{1/2}, w.\mathcal{L}}(wu).$$

In particular, if both the polarization $T^{1/2}$ and the line bundle $\mathcal{L}$ are G-equivariant, then

$$w.stab_{\mathfrak{C}, T^{1/2}, \mathcal{L}}(u) = stab_{w\mathfrak{C}, T^{1/2}, \mathcal{L}}(wu).$$

(b) The duality automorphism acts by sending $q^{\frac{1}{2}} \mapsto q^{-\frac{1}{2}}$ and

$$(8.3) \qquad (stab_{\mathfrak{C}, T^{\frac{1}{2}}, \mathcal{L}}(w))^\vee = q^{-\frac{\dim G/B}{2}} stab_{\mathfrak{C}, T_{opp}^{\frac{1}{2}}, \mathcal{L}^{-1}}(w),$$

where $T_{opp}^{\frac{1}{2}} := q^{-1}(T^{\frac{1}{2}})^\vee$ is the opposite polarization; see [OS16, Equation (15)]. I.e., this duality changes the polarization and slope parameters to the opposite ones, while keeping the chamber parameter invariant.

(c) Let $\mathcal{L}, \mathcal{L}' \in \mathrm{Pic}_T(T^*(G/B))$ be any equivariant line bundles, let $w \in W$ and $a \in \mathbb{Q}$ a rational number. Then

$$stab_{\mathfrak{C}, T^{1/2}, a\mathcal{L} \otimes \mathcal{L}'}(w) = \frac{1}{\mathcal{L}'|_w} \mathcal{L}' \otimes stab_{\mathfrak{C}, T^{1/2}, a\mathcal{L}}(w),$$

as elements in $K_{T \times \mathbb{C}^*}(T^*(G/B))_{loc}$.

8.3. **Recursions for stable envelopes.** Because the $\mathbb{C}^*$-fixed locus of the cotangent bundle is the zero section (i.e. G/B), it follows that the torus fixed point locus $(T^*(G/B))^{T\times\mathbb{C}^*}$ coincides with the fixed locus $(G/B)^T$, a discrete set indexed by the Weyl group W. Therefore the equivariant K-theory classes associated to fixed points form a basis in the localized ring $K_{T\times\mathbb{C}^*}(T^*(G/B))_{loc}$. In order to compare motivic Chern classes to stable envelopes, we need the following result proved in [SZZ17, Proposition 3.6].

Proposition 8.3. *The restriction coefficients* $\mathrm{stab}_-(w)|_u$ *are uniquely characterized by*

(1) $\mathrm{stab}_-(w)|_u = 0$, *unless* $u \geq w$.
(2) $\mathrm{stab}_-(w)|_w = q^{\frac{\ell(w)}{2}} \prod_{\alpha>0, w\alpha<0}(1 - e^{-w\alpha}) \prod_{\alpha>0, w\alpha>0}(1 - qe^{-w\alpha})$.
(3) *If* $ws_i > w$, *then*

$$q^{\frac{1}{2}} \mathrm{stab}_-(w)|_u = \frac{(1-q)}{1 - e^{-u\alpha_i}} \mathrm{stab}_-(ws_i)|_u + \frac{1 - qe^{-u\alpha_i}}{1 - e^{u\alpha_i}} \mathrm{stab}_-(ws_i)|_{us_i}.$$

Applying parts (a) and (b) from Lemma 8.2, and from the definitions of the stable envelopes, we obtain that for every $u \in W$,

(8.4) $$w_0.(\mathrm{stab}_-(u))^\vee = q^{-\frac{\dim G/B}{2}} \mathrm{stab}_+(w_0u).$$

Then we immediately obtain the following analogue of Proposition 8.3:

Proposition 8.4 ([SZZ17])**.** *The localizations* $\mathrm{stab}_+(w)|_u$ *are uniquely characterized by the following properties:*

(1) $\mathrm{stab}_+(w)|_u = 0$, *unless* $u \leq w$.
(2) $\mathrm{stab}_+(w)|_w = q^{\frac{\ell(w)}{2}} \prod_{\alpha>0, w\alpha<0}(1 - q^{-1}e^{w\alpha}) \prod_{\alpha>0, w\alpha>0}(1 - e^{w\alpha})$.
(3) *If* $ws_i > w$, *then*

$$q^{\frac{1}{2}} \mathrm{stab}_+(ws_i)|_u = \frac{q-1}{1 - e^{u\alpha_i}} \mathrm{stab}_+(w)|_u - \frac{e^{u\alpha_i} - q}{1 - e^{-u\alpha_i}} \mathrm{stab}_+(w)|_{us_i}.$$

8.4. **Motivic classes are pull-backs of stable envelopes.** One of the key formulas in [AMSS17] shows that the dual CSM class equals the Segre-Schwartz-MacPherson (SSM) class, up to a normalization coefficient. The proof of that identity is based on a transversality argument, which can be expressed either in terms of (cohomological) stable basis elements or in terms of transversality of characteristic cycles. The same phenomenon occurs in K-theory. Let $i : G/B \hookrightarrow T^*(G/B)$ be the inclusion of the zero section into the cotangent bundle. Define

$$\mathrm{stab}'_+(w) := \mathcal{D}(i^* \mathrm{stab}_+(w)) = (-1)^{\dim G/B}(i^* \mathrm{stab}_+(w))^\vee \otimes [\mathcal{L}_{2\rho}] \in K_T(G/B)[q^{\frac{1}{2}}, q^{-\frac{1}{2}}].$$

The following result relates motivic Chern classes and stable envelopes and it is the K-theoretic analogue of the cohomological results from [AMSS17, Corollary 6.6] and [RV18]. It is also equivalent to results from [FRW18], where it is shown that motivic Chern classes satisfy the same localization properties as the stable envelopes for a certain triple of parameters; cf. Remark 8.7 below.

Theorem 8.5. *For every* $w \in W$, *we have*

$$q^{-\frac{\ell(w)}{2}} \mathrm{stab}'_+(w) = MC_{-q^{-1}}(X(w)^\circ) \in K_T(G/B)[q, q^{-1}].$$

Proof. We compare localization properties of the motivic Chern classes with those for the Grothedieck-Serre dual of $\mathrm{stab}_+(w)$. We have that

(8.5) $$\mathrm{stab}'_+(w)|_u = (-1)^{\dim G/B} e^{2u\rho}(\mathrm{stab}_+(w)|_u)\big|_{e^\lambda \to e^{-\lambda}, q^{\frac{1}{2}} \to q^{-\frac{1}{2}}}.$$

Then the corresponding result from Proposition 8.4 for $\text{stab}'_+(w)$ is that the localizations $\text{stab}'_+(w)|_u$ are uniquely characterized by the following properties

(1) $\text{stab}'_+(w)|_u = 0$, unless $u \le w$.

(2) $\text{stab}'_+(w)|_w = q^{\frac{\ell(w)}{2}} \prod_{\alpha>0, w\alpha<0}(1 - q^{-1}e^{w\alpha}) \prod_{\alpha>0, w\alpha>0}(1 - e^{w\alpha})$.

(3) If $ws_{\alpha_i} > w$, then

$$q^{-\frac{1}{2}} \text{stab}'_+(ws_{\alpha_i})|_u = \frac{q^{-1}-1}{1 - e^{-u\alpha_i}} \text{stab}'_+(w)|_u + \frac{1 - q^{-1}e^{u\alpha_i}}{1 - e^{-u\alpha_i}} \text{stab}'_+(w)|_{us_{\alpha_i}}.$$

Comparing this with localizations of motivic Chern classes from Proposition 7.1 finishes the proof. $\qquad\square$

A similar statement relates $\text{stab}_-(w)$ to the motivic Chern class of the opposite Schubert cells. We record the statement next; the proof is essentially the same, and details are left to the reader. Define

$$\text{stab}'_-(w) := q^{-\dim G/B} i^*(\text{stab}_-(w)) \otimes [\omega^\bullet_{G/B}] \in K_T(G/B)[q^{\frac{1}{2}}, q^{-\frac{1}{2}}].$$

Theorem 8.6. *For every $w \in W$,*

$$q^{\frac{\ell(w)}{2}} \text{stab}'_-(w) = MC_{-q^{-1}}(Y(w)^\circ) \in K_T(G/B)[q, q^{-1}].$$

Remark 8.7. Using Theorem 8.5 and Lemma 8.2, one can show that for every $w \in W$,

$$(8.6) \qquad q^{\frac{-\ell(w)}{2}} i^* \text{stab}_{+, T(G/B), \widetilde{\mathcal{L}}}(w) = MC_{-q^{-1}}(X(w)^\circ),$$

where $\widetilde{\mathcal{L}}$ is the fundamental slope. This is consistent with the choices of parameters for stable envelopes from [FRW18]. In fact, a direct check of the normalization and degree conditions shows that for any slope $\mathcal{L}$,

$$\mathcal{D}(i^*(\text{stab}_{+, T(G/B), \mathcal{L}})) = i^*(\text{stab}_{+, T(G/B), (\mathcal{L})^{-1}}).$$

Again we leave the proof details to the reader. $\qquad\lrcorner$

Remark 8.8. We prove in the upcoming paper [AMSS] that

$$(-q)^{-\dim G/B} i^*(gr(i_{w!}\mathbb{Q}^H_{Y(w)^\circ})) \otimes [\omega^\bullet_{G/B}] = MC_{-q^{-1}}(Y(w)^\circ),$$

where $i_w : Y(w)^\circ \to G/B$ is the inclusion, and $gr(i_{w!}\mathbb{Q}^H_{Y(w)^\circ})$ is the associated graded (or $\mathbb{C}^*$-equivariant) sheaf on $T^*(G/B)$ determined by the shifted mixed Hodge module $\mathbb{Q}^H_{Y(w)^\circ}$; see [Tan87, AMSS] (with our q^{-1} corresponding to the parameter q in Tanisaki's paper). Since i^* is an isomorphism, we deduce from Theorem 8.6 that:

$$(8.7) \qquad\qquad \text{stab}_-(w) = (-1)^{\dim G/B} gr(i_{w!}\mathbb{Q}^H_{Y(w)^\circ}).$$

Since the cycle associated to the coherent sheaf $gr((i_w)_!\mathbb{Q}_{Y(w)^\circ})$ is the characteristic cycle of the constructible function $\mathbb{1}_{Y(w)^\circ}$, this equation can be seen as the K-theoretic generalization of the coincidence between (cohomological) stable envelopes and characteristic cycles, indicated by Maulik and Okounkov [MO19, Remark 3.5.3]; see also [AMSS17, Lemma 6.5] for a proof. $\qquad\lrcorner$

8.5. **Stable basis duality.** As for the CSM classes, there are two sources for Poincaré type dualities of the motivic Chern classes. The first is a consequence of the existence of two adjoint Demazure-Lusztig operators. The second, which has a geometric origin, uses the duality from (8.1) for the stable envelopes, on the cotangent bundle. Given that the localization pairing on the cotangent bundle can also be expressed in terms of a twisted Poincaré pairing on the zero section, this leads to some remarkable identities among motivic Chern classes. Recall from the equation (8.1) that the 'opposite' stable envelopes are dual to each other with respect to the K-theoretic pairing on $T^*(G/B)$, defined as follows: for every $\mathcal{F}, \mathcal{G} \in K_{T \times \mathbb{C}^*}(T^*(G/B))$,

$$\langle \mathcal{F}, \mathcal{G} \rangle_{T^*(G/B)} := \sum_{w \in W} \frac{[\mathcal{F}]|_w \cdot [\mathcal{G}]|_w}{\prod_{\alpha > 0}(1 - e^{w\alpha})(1 - qe^{-w\alpha})}.$$

Recall that $i : G/B \hookrightarrow T^*(G/B)$ is the inclusion of the zero section. By localization, the pairing in $T^*(G/B)$ is related to the ordinary Poincaré pairing in equivariant K-theory of G/B:

$$(8.8) \qquad \langle \mathcal{F}, \mathcal{G} \rangle_{T^*(G/B)} = \left\langle i^* \mathcal{F}, \frac{i^* \mathcal{G}}{\lambda_{-q}(T(G/B))} \right\rangle.$$

Here the ordinary Poincaré pairing in equivariant K-theory of G/B is extended (by the equivariant projection formula) bilinearly over $\mathrm{Frac}(K_{T \times \mathbb{C}^*}(pt))$ to a pairing:

$$\langle -, - \rangle : K_{T \times \mathbb{C}^*}(G/B)_{loc} \times K_{T \times \mathbb{C}^*}(G/B)_{loc} \to \mathrm{Frac}(K_{T \times \mathbb{C}^*}(pt)).$$

Moreover, $\lambda_{-q}(T(G/B)) \in K_{T \times \mathbb{C}^*}(G/B) \hookrightarrow K_{T \times \mathbb{C}^*}(G/B)_{loc}$ is invertible in the localized ring by the following observation.

Remark 8.9. A K-theoretic analogue of [AMSS17, Lemma 8.1] gives

$$\lambda_{-q}(T^*(G/B))\lambda_{-q}(T(G/B)) = \prod_{\alpha > 0}(1 - qe^{\alpha})(1 - qe^{-\alpha}).$$

As in *loc. cit.*, this follows by localization, because for all $w \in W$,

$$\lambda_{-q}(T^*(G/B))|_w \cdot \lambda_{-q}(T(G/B))|_w = \prod_{\alpha > 0}(1 - qe^{\alpha})(1 - qe^{-\alpha})$$

as w permutes the set of roots. ⌟

We need the following lemma.

Lemma 8.10. *Let* $[\mathcal{F}], [\mathcal{G}] \in K_{T \times \mathbb{C}^*}(G/B)_{loc}$ *such that*

$$\langle [\mathcal{F}], [\mathcal{G}] \rangle = f(e^t, q^{\frac{1}{2}}) \in K_{T \times \mathbb{C}^*}(pt)_{loc}.$$

Then

$$\langle \mathcal{D}([\mathcal{F}]), [\mathcal{G}]^\vee \rangle = \langle [\mathcal{F}]^\vee, \mathcal{D}([\mathcal{G}]) \rangle = f(e^{-t}, q^{-\frac{1}{2}}) = (\langle [\mathcal{F}], [\mathcal{G}] \rangle)^\vee,$$

i.e., all weights are inverted by this operation.

Proof. By the definition of the equivariant Grothendieck-Serre duality operator, it suffices to prove the equality

$$\langle [\mathcal{F}]^\vee, \mathcal{D}([\mathcal{G}]) \rangle = f(e^{-t}, q^{-\frac{1}{2}}).$$

Applying the K-theoretic Bott localization [CG09, § 5.10] we obtain

$$f(e^t, q^{\frac{1}{2}}) = \langle [\mathcal{F}], [\mathcal{G}] \rangle = \sum_w \frac{[\mathcal{F}]|_w \cdot [\mathcal{G}]|_w}{\prod_{\alpha > 0}(1 - e^{w\alpha})}.$$

Recall that $[\omega_{G/B}^\bullet] = (-1)^{\dim G/B}[\mathcal{L}_{2\rho}]$. Then

$$\langle \mathcal{D}([\mathcal{F}]), [\mathcal{G}]^\vee \rangle = (-1)^{\dim G/B} \langle [\mathcal{F}]^\vee, [\mathcal{G}]^\vee \otimes [\mathcal{L}_{2\rho}] \rangle$$

$$= (-1)^{\dim G/B} \sum_w \frac{([\mathcal{F}]|_w)^\vee ([\mathcal{G}]|_w)^\vee}{\prod_{\alpha>0}(1 - e^{w\alpha})} e^{2w\rho}$$

$$= \sum_w \frac{([\mathcal{F}]|_w)^\vee ([\mathcal{G}]|_w)^\vee}{\prod_{\alpha>0}(1 - e^{-w\alpha})}$$

$$= f(e^{-t}, q^{-\frac{1}{2}}).$$

The second-to-last equality holds because $2\rho = \sum_{\alpha>0} \alpha$, thus $e^{2w(\rho)} = \prod_{\alpha>0} e^{w(\alpha)}$, and the last equality follows since the effect of taking $(-)^\vee$ is to invert the T and $\mathbb{C}^*$ weights. $\square$

Theorem 8.11. *Let $u, w \in W$ and $y = -q^{-1}$. Then the following orthogonality relation holds:*

$$\left\langle MC_y(X(w)^\circ), \frac{\mathcal{D}(MC_y(Y(u)^\circ))}{\lambda_y(T^*(G/B))}(-y)^{\dim G/B - \ell(u)} \right\rangle = \delta_{w,u}.$$

Equivalently,

$$MC_y^\vee(Y(u)^\circ) = \prod_{\alpha>0}(1 + ye^{-\alpha}) \frac{\mathcal{D}(MC_y(Y(u)^\circ))}{\lambda_y(T^*(G/B))} \in K_{T\times\mathbb{C}^*}(G/B)_{loc}.$$

Proof. The idea is to use Theorem 8.5 to express $MC_y(X(u)^\circ)$ in terms of the Grothendieck-Serre dual, then use Lemma 8.10 to relate the pairing in the statement of the theorem to the pairing between orthogonal stable envelopes. We start by observing that

$$\lambda_y(T^*(G/B)) = (\lambda_{y^{-1}} T(G/B))^\vee \in K_T(G/B)[y, y^{-1}],$$

and that by Theorem 8.6

$$\mathcal{D}(MC_y(Y(u)^\circ)) = \mathcal{D}((-y)^{-\frac{\ell(u)}{2}}(-y)^{\dim G/B}\iota^*(\mathrm{stab}_-(u)) \otimes [\omega_{G/B}^\bullet])$$

$$= (-y)^{\frac{\ell(u)}{2} - \dim G/B}(\iota^*(\mathrm{stab}_-(u)))^\vee.$$

From this, the second term of the pairing equals

$$\frac{\mathcal{D}(MC_y(Y(u)^\circ))}{\lambda_y(T^*(G/B))}(-y)^{\dim G/B - \ell(u)} = \left(\frac{\iota^*(\mathrm{stab}_-(u))}{\lambda_{y^{-1}}(T(G/B))}(-y)^{\frac{\ell(u)}{2}} \right)^\vee.$$

Then Theorem 8.5, Lemma 8.10, and orthogonality of stable envelopes (Equation (8.1)) imply that

$$\left\langle MC_y(X(w)^\circ), \frac{\mathcal{D}(MC_y(Y(u)^\circ))}{\lambda_y(T^*(G/B))}(-y)^{\dim G/B - \ell(u)} \right\rangle$$

$$= \left\langle \mathcal{D}((-y)^{-\frac{\ell(w)}{2}}\iota^* \mathrm{stab}_+(w)), \left(\frac{\iota^*(\mathrm{stab}_-(u))}{\lambda_{y^{-1}}(T(G/B))}(-y)^{\frac{\ell(u)}{2}} \right)^\vee \right\rangle$$

$$= \left\langle (-y)^{\frac{\ell(u)-\ell(w)}{2}}\iota^* \mathrm{stab}_+(w), \frac{\iota^* \mathrm{stab}_-(u)}{\lambda_{y^{-1}}(T(G/B))} \right\rangle_{y\mapsto y^{-1}, e^t \mapsto e^{-t}}$$

$$= (-y)^{\frac{\ell(w)-\ell(u)}{2}} \langle \mathrm{stab}_+(w), \mathrm{stab}_-(u) \rangle_{T^*(G/B)}|_{y\mapsto y^{-1}, e^t \mapsto e^{-t}}$$

$$= \delta_{w,u},$$

where the second equality follows from Lemma 8.10, the third equality follows from Equation (8.8) and the last one follows from Equation (8.1). This proves the first assertion. The

second assertion follows from the 'Hecke orthogonality' of motivic Chern classes, proved in Theorem 6.2.   □

Theorem 8.11 justifies the definition of a dual motivic Chern class of a Schubert variety:

Definition 8.12. Let $w \in W$. Define the dual motivic Chern class of a dual Schubert variety by

$$MC_y^\vee(Y(w)) := \sum_{u \geq w} MC_y^\vee(Y(u)^\circ).$$

By Theorem 8.11,

$$MC_y^\vee(Y(w)) = \frac{\prod_{\alpha>0}(1 + ye^{-\alpha})}{\lambda_y(T^*(G/B))} \mathcal{D}(MC_y(Y(w))) \in K_{T\times\mathbb{C}^*}(G/B)_{loc},$$

where in the last term we use the ordinary motivic Chern class. The class

$$\frac{\mathcal{D}(MC_y(Y(w)))}{\lambda_y(T^*(G/B))} \in K_{T\times\mathbb{C}^*}(G/B)_{loc}$$

can be thought as a *motivic Segre class*, i.e., a K-theoretic analogue of the Segre-Schwartz-MacPherson class discussed in [Alu03, Ohm06, AMSS17, FR18]. More precisely, it will be the motivic Segre class of the *motivic dual* $\mathcal{D}_{mot}([Y(w)])$ of the dual Schubert variety $Y(w)$, for an equivariant motivic duality $\mathcal{D}_{mot}$ (extending [Bit04]) introduced in our forthcoming paper [AMSS], with

(8.9) $$\mathcal{D}(MC_y(Y(w))) = MC_y(\mathcal{D}_{mot}([Y(w)]))$$

as an equivariant extension of [Sch09, Corollary 5.19].

9. Smoothness of Schubert varieties and localizations of motivic Chern classes

Among the main applications of this paper are properties about the transition matrix between the standard and the Casselman's basis for Chevalley groups over nonarchimedean local fields. The matrix coefficients are rational functions, and of particular interest to us are certain factorizations and polynomial properties of these coefficients conjectured by Bump, Nakasuji, and Naruse; see section §10 below and [BN11, NN16, BN19]. We will prove in §10 that the transition matrix from the 'Casselman setting' corresponds to transition matrix between (dual) motivic Chern classes of Schubert varieties and an appropriate normalization of the fixed point basis. This motivates the study in this section of the underlying 'geometric' transition matrix between the motivic classes and fixed point basis. The main result of this section is Theorem 9.5 (Theorem 1.3 from the introduction) which is the geometric analogue of Theorem 10.1 about the principal series representation.

9.1. **A smoothness criterion.** In this section we prove a criterion for the smoothness of Schubert varieties in terms of the motivic Chern classes. The main theorem is the following:

Theorem 9.1. *Let $u, w \in W$ such that $u \leq w$. The opposite Schubert variety $Y(u)$ is smooth at the torus fixed point e_w if and only if*

(9.1) $$MC_y(Y(u))|_w = \prod_{\alpha>0, ws_\alpha \geq u} (1 + ye^{w\alpha}) \prod_{\alpha>0, u \not\leq ws_\alpha} (1 - e^{w\alpha}).$$

If $Y(u)$ is smooth at e_w then

$$MC_y(Y(u))|_w = \lambda_y(T_w^*Y(u)) \cdot \lambda_{-1}((N_{Y(u)}^\vee(G/B))_w).$$

Let $S'(u, w) := \{\alpha > 0 : u \leq ws_\alpha < w\}$. An immediate consequence of the theorem is that if $Y(u)$ is smooth at e_w, then the weights of the normal space $(N_{Y(w)}Y(u))_w = T_w(Y(u))/T_w(Y(w))$ of $Y(w)$ at e_w in $Y(u)$ are

$$(9.2) \qquad S(u, w) := \{\beta > 0 : u \leq s_\beta w < w\} = \{-w(\alpha) : \alpha \in S'(u, w)\}.$$

To prove the theorem we first prove two basic lemmas about localization of motivic Chern classes. (These are also implicit in the work of [FRW18].)

Lemma 9.2. *Let $i : X \subseteq M$ be a closed embedding of G-equivariant, non-singular, quasi-projective, algebraic varieties. Then $i^* MC_y[X \to M] = \lambda_y(T_X^*) \otimes \lambda_{-1}(N_X^\vee M)$, where $N_X^\vee M$ is the conormal bundle of X inside M.*

Proof. By the functoriality of motivic Chern classes and the self-intersection formula in K-theory [CG09, Proposition 5.4.10] we have

$$i^* MC_y[X \to M] = i^* i_* MC_y[id_X] = MC_y[id_X] \otimes \lambda_{-1}(N_X^\vee M) = \lambda_y(T_X^*) \otimes \lambda_{-1}(N_X^\vee M). \quad \square$$

Lemma 9.3. *Let $X \subseteq M$ be a closed embedding of T-equivariant, algebraic varieties, and assume that M is smooth. Let $p \in X$ be a smooth point, and $j : V \subseteq M$ any T-invariant open set such that $p \in V$ and $X' := V \cap X$ is smooth (e.g., $V := M \setminus X_{sing}$). Let $\iota_p : \{p\} \to M$ be the inclusion. Then*

$$\iota_p^* MC_y[X \to M] = \lambda_y(T_p^* X) \cdot \lambda_{-1}((N_X^\vee M)_p).$$

Proof. Let $\iota_p' : \{p\} \to V$ denote the embedding. Note that

$$\iota_p^* MC_y[X \to M] = (\iota_p')^* j^* MC_y[X \to M] = (\iota_p')^* MC_y(j^*[X \to M]) = (\iota_p')^* MC_y[X' \to V],$$

where the second equality follows from the Verdier-Riemann-Roch formula from Theorem 4.2, as j is an open embedding (thus a smooth morphism), with relative tangent bundle equal to 1. Applying Lemma 9.2 to the closed embedding $X' \hookrightarrow V$, we get

$$MC_y[X' \to V] = \lambda_y(T_{X'}^*) \otimes \lambda_{-1}(N_{X'}^\vee V).$$

The claim follows by pulling back via $(\iota_p')^*$, using that $(\iota_p')^*$ is a ring homomorphism in (equivariant) K-theory. $\qquad \square$

We will apply this lemma in the case when $Y(u) \subseteq G/B$ is a Schubert variety which is smooth at a torus fixed point e_w. We also need a variant of Kumar's cohomological criterion for smoothness of Schubert varieties.

Theorem 9.4 ([Kum96])**.** *Let u, w be two Weyl group elements such that $u \leq w$. Then the Schubert variety $Y(u)$ is smooth at e_w if and only if the localization of the equivariant fundamental class $[Y(u)] \in A_*^T(G/B)$ in the equivariant Chow group is given by:*

$$[Y(u)]|_w = \left(\prod_{\beta > 0, u \not\leq s_\beta w} \beta \right) \in A_*^T(pt) = \mathbb{Z}[\alpha_i \mid i = 1, \ldots, r].$$

If $Y(u)$ is smooth at e_w, then the torus weights of $T_w Y(u)$ are $\{-w\alpha \mid \alpha > 0, ws_\alpha \geq u\}$.

Proof. Consider the automorphism of the set R^+ of positive roots given by $\alpha \mapsto -w_0(\alpha)$. One checks that this is actually an automorphism of the Dynkin diagram. It induces the automorphism $w \mapsto w_0 w w_0$ of the Weyl group W, preserving the length and the Bruhat order.

It also induces an automorphism of G/B sending the Schubert cell $Y(w)^\circ$ to $Y(w_0ww_0)^\circ$. In particular, $Y(u)$ is smooth at e_w if and only if $Y(w_0uw_0)$ is smooth at w_0ww_0. Then

$$Y(w_0uw_0) \text{ is smooth at } e_{w_0ww_0}$$
$$\Longleftrightarrow X(uw_0) \text{ is smooth at } e_{ww_0}$$
$$\Longleftrightarrow [Y(u)]|_w = \prod_{\beta>0,\, u\not\leq s_\beta w} \beta,$$

where the last equivalence follows from the original version of Kumar's criterion, as stated in [BL00, Corollary 7.2.8][3], together with [BL00, Theorem 7.2.11] and Billey's formula for localization of equivariant Schubert classes [Bil99].

To show that the weights on the tangent space are those claimed, observe first that $T_{\mathrm{id}}(G/B) = \mathrm{Lie}(G)/\mathrm{Lie}(B)$ has weights $\{-\alpha : \alpha > 0\}$, thus after translation by w, $T_w(G/B)$ has weights $\{-w(\alpha) : \alpha > 0\}$. The tangent space $T_wY(u)$ is a T-submodule of $T_w(G/B)$, and its weights correspond to edges $w \mapsto ws_\alpha$ in the Bruhat graph of G/B where $ws_\alpha \in Y(u)$, i.e., $ws_\alpha \geq u$. $\qquad\square$

We are now ready to prove the Theorem 9.1.

Proof of Theorem 9.1. The second formula follows directly from Lemma 9.3, so we only need to verify the smoothness criterion. Assume first $Y(u)$ is smooth at e_w. Using the weights of the tangent bundle from Theorem 9.4 to obtain:

$$MC_y(Y(u))|_w = \lambda_y(T_w^*Y(u))\lambda_{-1}((N_{Y(u)}^\vee G/B)_w)$$

$$= \lambda_y(T_w^*Y(u))\frac{\lambda_{-1}(T_w^*(G/B))}{\lambda_{-1}(T_w^*Y(u))}$$

$$= \prod_{\alpha>0,\, ws_\alpha\geq u}(1+ye^{w\alpha})\frac{\prod_{\alpha>0}(1-e^{w\alpha})}{\prod_{\alpha>0,\, u\leq ws_\alpha}(1-e^{w\alpha})}$$

$$= \prod_{\alpha>0,\, ws_\alpha\geq u}(1+ye^{w\alpha})\prod_{\alpha>0,\, u\not\leq ws_\alpha}(1-e^{w\alpha}).$$

The converse follows immediately from Theorem 9.4, after setting $y = 0$ and taking leading terms in order to reduce to the cohomological case. Explicitly, note that clearly $MC_0(Y) = \mathcal{O}_Y \in \mathrm{K}_T(Y)$ if Y is smooth, and it follows from functoriality that $MC_0(Y) = \mathcal{O}_Y$ if Y has rational singularities. This is the case for Schubert varieties ([Bri05, Theorem 2.2.3]), so $MC_y(Y(u))|_{y=0} = \mathcal{O}^u$. Consider the *equivariant Chern character*

$$ch_T : K_T(G/B) \to \widehat{A}_*^T(G/B)$$

to an appropriate completion of the equivariant Chow group; see [EG00]. From the definition of ch_T and by [Ful84, Theorem 18.3] it follows that the top homological degree term of the equivariant Chern character $ch_T(\mathcal{O}^u)$ is the equivariant fundamental class $[Y(u)]_T$. Together with the fact that $ch_T(e^\lambda) = 1 + \lambda + $ higher degree *cohomological* terms, this implies that if (9.1) holds, then the top degree term of $ch_T(MC_y(Y(u))|_{y=0})$ localizes to

$$[Y(u)]|_w = \prod_{\alpha>0,\, u\not\leq ws_\alpha}(-w(\alpha)) = \prod_{\beta>0,\, u\not\leq s_\beta w} \beta\,.$$

This uses that $u \leq w$, therefore $u \not\leq ws_\alpha$ implies that $ws_\alpha < w$ and $w(\alpha) < 0$; further, $\beta = -w(\alpha) > 0$ and $ws_\alpha = s_\beta w$. By Theorem 9.4, $Y(u)$ is smooth at e_w. $\qquad\square$

---

[3]In *loc. cit.*, $d_{w,u} = [Y(u)]|_w$, see [BL00, Theorem 7.2.11].

9.2. **The geometric Bump-Nakasuji-Naruse conjecture.** Motivated by the applications to representation theory from §10, we study the following problem. Define the element $b_w \in K_T(G/B)_{loc}[y^{-1}]$ by the formula

$$(9.3) \qquad b_w := (-1)^{\dim G/B - \ell(w)} \prod_{\alpha > 0, w\alpha > 0} \frac{y^{-1} + e^{-w\alpha}}{1 - e^{w\alpha}} \iota_w.$$

Equivalently, b_w is the multiple of the fixed point basis element ι_w which satisfies $(b_w)|_w = MC_y^\vee(Y(w)^\circ)|_w$. Consider the expansion of the class $MC_y^\vee(Y(u))$ from Definition 8.12:

$$(9.4) \qquad MC_y^\vee(Y(u)) = \sum m_{u,w} b_w \quad \in K_T(pt)_{loc}[y^{-1}].$$

It is easy to see that $m_{u,w} = 0$ unless $u \le w$. For pairs $u \le w \in W$, recall that $S(u,w) := \{\beta \in R^+ | u \le s_\beta w < w\}$. The main result of this section is the following geometric analogue of a conjecture of Bump, Nakasuji, and Naruse [BN11, NN16, BN19].

Theorem 9.5 (Geometric Bump-Nakasuji-Naruse Conjecture). *For every $u \le w \in W$,*

$$(9.5) \qquad m_{u,w} = \prod_{\alpha \in S(u,w)} \frac{1 + y^{-1} e^\alpha}{1 - e^\alpha}$$

if and only if the Schubert variety $Y(u)$ is smooth at the torus fixed point e_w.

This is the geometric version of Theorem 10.1 in p-adic representation theory; the coefficients $m_{u,w}$ calculate the transition matrix between the 'standard basis' and 'Casselman's basis' for the Iwahori invariants of the principal series representation. The statement is a generalization of the original Bump-Nakasuji conjecture, communicated to us by H. Naruse; see [Nar14]. In fact, Naruse informed us that he obtained the implication of this theorem which assumes the factorization. Naruse's proof of this implication, and ours, are both based on Kumar's cohomological criterion for smoothness ([Kum96]; see Theorem 9.4). Naruse's proof is based on Hecke algebra calculations, while ours uses motivic Chern classes.

After harmonizing conventions between this note and [BN11, BN19], and passing to the 'geometric' version, the original conjecture states the following (see [BN11, Conjecture 1.2] and [BN19, p. 3]):

Corollary 9.6. *Let G be a complex, simply laced, reductive, linear algebraic group. Then the coefficient $m_{u,v}$ satisfies the factorization in (9.5) if and only if*

$$P_{w_0 w^{-1}, w_0 u^{-1}} = 1,$$

where $P_{w_0 w^{-1}, w_0 u^{-1}}$ denotes the Kazhdan-Lusztig polynomial.

We first prove this statement, assuming Theorem 9.5.

Proof. Since the group G is simply laced, an unpublished result of D. Peterson, re-proved by Carrell and Kuttler (see e.g., [CK03] or [BL00, Theorem 6.0.4]) shows that the condition that $Y(u)$ is smooth at e_w is equivalent to $Y(u)$ being rationally smooth at e_w. For arbitrary G, rational smoothness is equivalent to the fact that the Kazhdan-Lusztig polynomial $P_{w_0 u, w_0 w}$ equals 1, by a theorem Kazhdan and Lusztig [KL79, Theorem A2]. By Theorem 9.5, it remains to shows that $P_{w_0 w^{-1}, w_0 u^{-1}} = 1$ if and only if $P_{w_0 w, w_0 u} = 1$. In turn, this is equivalent to

$$P_{u,w} = 1 \iff P_{w_0 u^{-1} w_0, w_0 w^{-1} w_0} = 1.$$

This is proved in the next lemma below. $\qquad \square$

Lemma 9.7. *Let G be a complex reductive linear algebraic group of arbitrary Lie type. Then $P_{u,w} = 1$ if and only if $P_{w_0 u^{-1} w_0, w_0 w^{-1} w_0} = 1$.*

Proof. We use a characterization of the condition that the Kazhdan-Lusztig polynomials is equal to 1, proved in various generality by Deodhar, Carrell and Peterson; see [BL00, Theorem 6.2.10]. Let $\mathcal{R}$ be the set of (not necessarily simple) reflections in W. Then $P_{u,w} = 1$ if and only if

$$\#\{r \in \mathcal{R} : y < ry \leq w\} = \ell(w) - \ell(y), \quad \forall u \leq y \leq w.$$

It is well known that taking inverses, and conjugating by w_0 are bijections of W which preserve both the length and the Bruhat order of elements. Thus $y < w$ if and only if $w_0 y^{-1} w_0 < w_0 w^{-1} w_0$ and $\ell(w) - \ell(y) = \ell(w_0 w^{-1} w_0) - \ell(w_0 y^{-1} w_0)$. This finishes the proof. $\square$

We note that in general rational smoothness is different from smoothness, therefore the statement from Corollary 9.6 does not generalize to non-simply laced case. The statement of [BN11, Conjecture 1.2] is slightly different from the final version stated in Corollary 9.6 and in [BN19]. The initial statement was analyzed by Lee, Lenart and Liu in [LLL17], and they found that under certain conditions on the reduced words of w and z the factorization holds, but in general there are counterexamples. We refer to [Nar14, NN16, BN19] for work closely related to [BN11].

We now return to the proof of Theorem 9.5. The key part is the following result, which may be of interest in its own right.

Proposition 9.8. *(a) For every $w \geq u \in W$, the coefficient $m_{u,w}$ equals*

$$m_{u,w} = \left(\frac{MC_y(Y(u))|_w}{MC_y(Y(w)^\circ)|_w} \right)^\vee \in K_T(pt)_{loc}[y^{-1}].$$

(b) Assume that $Y(u)$ is smooth at e_w. Then

$$m_{u,w} = \frac{\lambda_{y^{-1}}((N_{Y(w)}Y(u))_w)}{\lambda_{-1}((N_{Y(w)}Y(u))_w)}.$$

In particular, we obtain a geometric analogue of the Langlands-Gindikin-Karpelevich formula [Lan71]:

$$m_{1,w} = \prod_{\alpha < 0, w^{-1}(\alpha) > 0} \frac{1 + y^{-1} e^\alpha}{1 - e^\alpha}.$$

Proof. Localizing both sides of (9.4) at the fixed point e_w gives $MC_y^\vee(Y(u))|_w = m_{u,w} \cdot MC_y^\vee(Y(w)^\circ)|_w$. Therefore, using Theorem 8.11 and Definition 8.12, we obtain:

$$m_{u,w} = \frac{MC_y^\vee(Y(u))|_w}{MC_y^\vee(Y(w)^\circ)|_w} = \frac{\mathcal{D}(MC_y(Y(u))|_w)}{\mathcal{D}(MC_y(Y(w)^\circ)|_w)} = \left(\frac{MC_y(Y(u))|_w}{MC_y(Y(w)^\circ)|_w} \right)^\vee.$$

This proves part (a). For part (b), we use part (a) and Theorem 9.1 to obtain

$$
m_{u,w} = \left(\frac{MC_y(Y(u))|_w}{MC_y(Y(w)^\circ)|_w} \right)^\vee
$$
$$
= \left(\frac{\lambda_y(T_w^* Y(u)) \cdot \lambda_{-1}((N_{Y(u)}^\vee(G/B))_w)}{\lambda_y(T_w^* Y(w)) \cdot \lambda_{-1}((N_{Y(w)}^\vee(G/B))_w)} \right)^\vee
$$
$$
= \frac{\lambda_{y^{-1}}(T_w Y(u)) \cdot \lambda_{-1}((N_{Y(u)}(G/B))_w)}{\lambda_{y^{-1}}(T_w Y(w)) \cdot \lambda_{-1}((N_{Y(w)}(G/B))_w)}
$$
$$
= \frac{\lambda_{y^{-1}}((N_{Y(w)}Y(u))_w)}{\lambda_{-1}((N_{Y(w)}Y(u))_w)}.
$$

The last equality follows from the multiplicativity of the λ_y class, and the short exact sequences

$$
0 \to T_w Y(w) \to T_w Y(u) \to (N_{Y(w)}Y(u))_w \to 0
$$

and

$$
0 \to N_{w,Y(u)}(G/B) \to N_{w,Y(w)}(G/B) \to (N_{Y(w)}Y(u))_w \to 0 \,.
$$

The case when $u = 1$ follows from the description of the weights from Theorem 9.1. $\square$

Proof of Theorem 9.5. If $Y(u)$ is smooth at e_w the claim follows from Proposition 9.8(b), using the description of appropriate weights from Theorem 9.4. Conversely, assume that

$$
m_{u,w} = \prod_{u \le s_\alpha w < w} \frac{1 + y^{-1}e^\alpha}{1 - e^\alpha}.
$$

Part (a) of Proposition 9.8, together with the localization result from Proposition 7.2 imply that

$$
MC_y(Y(u))|_w = \prod_{\alpha > 0, ws_\alpha \ge u} (1 + ye^{w\alpha}) \prod_{\alpha > 0, u \not\le ws_\alpha} (1 - e^{w\alpha}).
$$

Therefore, $Y(u)$ is smooth at e_w by Theorem 9.1. $\square$

10. MOTIVIC CHERN CLASSES AND THE PRINCIPAL SERIES REPRESENTATION

The goal of this section is to establish an isomorphism of Hecke modules between the Iwahori invariants of the unramified principal series representations of a group over a non archimedean local field and the (localized) equivariant K-theory of the flag variety for the complex Langlands dual group; see Theorem 10.2. A similar relation was established recently in [SZZ17], using the equivariant K-theory of the cotangent bundle and the stable basis. The advantage of using motivic Chern classes is that their functoriality properties will help get additional properties of this correspondence. For instance, we use functoriality to relate localization coefficients of the motivic Chern classes to coefficients in the transition matrix between the standard basis and *Casselman's basis* (defined below). This will be applied to solve some conjectures of Bump, Nakasuji and Naruse about the coefficients in the transition matrix between the standard and the Casselman's basis.

10.1. **Iwahori invariants of the principal series representation.** We recall below the definition and properties of the two bases in the Iwahori invariants of the principal series representation. The literature in this subject uses several normalization conventions. We will be consistent with the conventions used in the paper of Reeder [Ree92] and in [SZZ17], because they fit with our previous geometric calculations in this paper; these conventions differ from those in [BN11, BN19] or [BBL15], and when necessary we will explain the differences. Let $\mathscr{G}$ be a split, reductive, Chevalley group defined over $\mathbb{Z}$; see e.g., [Ste16]. Let $\mathscr{B} = TN \leq \mathscr{G}$ be a standard Borel subgroup containing a maximal torus $\mathscr{T}$ and its unipotent radical N. Let $W := N_{\mathscr{G}}(\mathscr{T})/\mathscr{T}$ be the Weyl group. We will also consider the Langlands dual G of $\mathscr{G}$; by definition this will be a *complex* reductive linear algebraic group, of type dual to the Lie type of $\mathscr{G}$. See e.g., [Bor79, §2.1][4]. Let F be a non archimedean local field, with ring of integers $\mathcal{O}$, uniformizer $\varpi \in \mathcal{O}$, and residue field $\mathbb{F}_{q'}$. Examples are finite extensions of the field of p-adic numbers, or of the field of Laurent series over $\mathbb{F}_p$. Since $\mathscr{G}$ is defined over $\mathbb{Z}$, we may consider $\mathscr{G}(F)$, the group of the F-points of $\mathscr{G}$, with maximal torus $\mathscr{T}(F)$ and Borel subgroup $\mathscr{B}(F) = \mathscr{T}(F)N(F)$. Let I be an Iwahori subgroup, i.e., the inverse image of $\mathscr{B}(\mathbb{F}_{q'})$ under the evaluation map $\mathscr{G}(\mathcal{O}) \to \mathscr{G}(\mathbb{F}_{q'})$. To simplify formulas, we let α, β denote the coroots of $\mathscr{G}$. Let R^+ and $R^{+\vee}$ denote the positive roots and coroots, respectively.

Let $\mathbb{H} = \mathbb{C}_c[I\backslash\mathscr{G}(F)/I]$ be the Iwahori Hecke algebra, consisting of compactly supported functions on $\mathscr{G}(F)$ which are bi-invariant under I. As a vector space, $\mathbb{H} = \Theta \otimes_{\mathbb{C}} H_W(q')$, where Θ is a commutative subalgebra isomorphic to the coordinate ring $\mathbb{C}[T]$ of the complex dual torus $T = \mathbb{C}^* \otimes X^*(\mathscr{T})$, and where $H_W(q')$ is the finite Hecke (sub)algebra with parameter q' associated to the (finite) Weyl group W. The finite Hecke algebra $H_W(q')$ is also a subalgebra of $\mathbb{H}$, and it is generated by elements T_w ($w \in W$) such that the following relations hold: $T_u T_v = T_{uv}$ if $\ell(uv) = \ell(u) + \ell(v)$, and $(T_{s_i} + 1)(T_{s_i} - q') = 0$ for a simple reflection s_i in W.

For every character τ of $\mathscr{T}$, and α a coroot define e^α by $e^\alpha(\tau) = \tau(h_\alpha(\varpi))$, where $h_\alpha : F^\times \to \mathscr{T}(F)$ is the one parameter subgroup. There is a pairing

$$\langle \cdot, \cdot \rangle : \mathscr{T}(F)/\mathscr{T}(\mathcal{O}) \times T \to \mathbb{C}^*$$

given by $\langle a, z \otimes \lambda \rangle = z^{val(\lambda(a))}$. This induces an isomorphism between $\mathscr{T}(F)/\mathscr{T}(\mathcal{O})$ and the group $X^*(T)$ of rational characters of T. It also induces an identification between T and unramified characters of $\mathscr{T}(F)$, i.e., characters which are trivial on $\mathscr{T}(\mathcal{O})$.

For simplicity, from now on we take τ to be an unramified character of $\mathscr{T}(F)$ such that $e^\alpha(\tau) \neq 1$ for all coroots α, and for which the stabilizer $W_\tau = 1$. The *principal series representation* is the induced representation $I(\tau) := \mathrm{Ind}_{\mathscr{B}(F)}^{\mathscr{G}(F)}(\tau)$. As a $\mathbb{C}$-vector space, $I(\tau)$ consists of locally constant functions f on $\mathscr{G}(F)$ such that $f(bg) = \tau(b)\delta^{\frac{1}{2}}(b)f(g)$ for every $b \in \mathscr{B}(F)$, where $\delta(b) := \prod_{\alpha>0} |\alpha^\vee(b)|_F$ is the modulus function on the Borel subgroup. The Hecke algebra $\mathbb{H}$ acts through convolution from the right on the Iwahori invariant subspace $I(\tau)^I$, so that the restriction of this action to $H_W(q')$ is a regular representation. One can pass back and forth between left and right $\mathbb{H}$-modules by using the standard anti-involution ι on $\mathbb{H}$ given by $\iota(h)(x) = h(x^{-1})$. If T_w denote the standard generators of the Hecke algebra $H_W(q')$, then $\iota(T_w) = T_{w^{-1}}$ and $\iota(q') = q'$, see [HKP10, Section 3.2]. This is of course consistent with the left $\mathbb{H}$-action on $I(\tau)$ described by Reeder in [Ree92, p. 325].

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[4]In the previous section we used flag varieties associated to complex simple Lie groups. However, all results extend without change to the semisimple case. Further, the flag varieties for the semisimple linear algebraic groups are the same as those for reductive groups. We will tacitly use these facts in this section.

We are interested in the Iwahori invariants $I(\tau)^I$ of the principal series representation for an unramified character. One reason to study the invariants is that as a $\mathscr{G}(F)$-module, the principal series representation $I(\tau)$ is generated by $I(\tau)^I$; cf. [Cas80, Proposition 2.7]. As a vector space, $\dim_{\mathbb{C}} I(\tau)^I = |W|$, the order of the Weyl group W. We will study the transition between two bases of $I(\tau)^I$. From the decomposition $\mathscr{G}(F) = \bigsqcup_{w \in W} \mathscr{B}(F)wI$ one obtains the basis of the *characteristic functions* on the orbits, denoted by $\{\varphi_w \mid w \in W\}$[5]. For $w \in W$, the element φ_w is characterized by the following two conditions [Ree92, p. 319]:

(1) φ_w is supported on $\mathscr{B}(F)wI$;
(2) $\varphi_w(bwg) = \tau(b)\delta^{\frac{1}{2}}(b)$ for every $b \in \mathscr{B}(F)$ and $g \in I$.

The (left) action of $\mathbb{H}$ on $I(\tau)^I$, denoted by π, was calculated e.g., by Casselman in [Cas80, Theorem 3.4]. With the conventions from Reeder [Ree92, p. 325], for every simple coroot α_i:

$$(10.1) \qquad \pi(T_{s_i})(\varphi_w) = \begin{cases} q'\varphi_{ws_i} + (q' - 1)\varphi_w & \text{if } ws_i < w; \\ \varphi_{ws_i} & \text{if } ws_i > w. \end{cases}$$

The second basis, called *Casselman's basis*, and denoted by $\{f_w \mid w \in W\}$, was defined by Casselman [Cas80, §3] by duality using certain intertwiner operators. We recall the relevant definitions, following again [Ree92]. For every character τ and $x \in W$, define $x\tau \in X^*(\mathscr{T})$ by the formula $x\tau(a) := \tau(x^{-1}ax)$ for every $a \in \mathscr{T}$. Since τ is unramified and it has trivial stabilizer under the Weyl group action, the space $\text{Hom}_{\mathscr{G}(F)}(I(\tau), I(x^{-1}\tau))$ is known to be one dimensional, spanned by an operator[6] $\mathcal{A}_x = \mathcal{A}_x^\tau$ defined by

$$\mathcal{A}_x(\varphi)(g) := \int_{N_x} \varphi(\dot{x}ng)dn,$$

where $\dot{x}$ is a representative of $x \in W$, $N_x = N(F) \cap \dot{x}^{-1}N^-(F)\dot{x}$ where N^- is the unipotent radical of the opposite Borel subgroup $\mathscr{B}^-$; the measure on N_x is normalized by the condition that $vol(N_x \cap \mathscr{G}(\mathcal{O})) = 1$ [Ree92]. If $x, y \in W$ satisfy $\ell(x) + \ell(y) = \ell(xy)$, then $\mathcal{A}_y^{x^{-1}\tau}\mathcal{A}_x^\tau = \mathcal{A}_{xy}^\tau$. Then there exist unique functions $f_w \in I(\tau)^I$ such that

$$(10.2) \qquad \mathcal{A}_x^\tau(f_w)(1) = \delta_{x,w}.$$

(Under our conventions f_w equals the element denoted $f_{w^{-1}}$ in [BN11].) For the longest element w_0 in the Weyl group, Casselman showed in [Cas80, Proposition 3.7] that

$$\varphi_{w_0} = f_{w_0}.$$

Reeder [Ree92] calculated the action of $\mathbb{H}$ on the functions f_w: he showed in [Ree92, Lemma 4.1] that the functions f_w are Θ-eigenvectors, and he calculated in [Ree92, Proposition 4.9] the action of $H_W(q')$. To describe the latter, let

$$(10.3) \qquad c_\alpha = \frac{1 - q'^{-1}e^\alpha(\tau)}{1 - e^\alpha(\tau)}.$$

For every simple coroot α_i and $w \in W$, write

$$J_{i,w} = \begin{cases} c_{w(\alpha_i)}c_{-w(\alpha_i)} & \text{if } ws_i > w; \\ 1 & \text{if } ws_i < w. \end{cases}$$

Then, we have

$$(10.4) \qquad \pi(T_{s_i})(f_w) = q'(1 - c_{w(\alpha_i)})f_w + q'J_{i,w}f_{ws_i}.$$

---

[5]Our φ_w is equal to $\phi_{w^{-1}}$ in [BN11, BN19]
[6]The intertwiner $\mathcal{A}_x$ is related to M_x from [Cas80, BN11] by the formula $\mathcal{A}_x = M_{x^{-1}}$.

10.2. **A conjecture of Bump, Nakasuji, and Naruse.** In this section we state a conjecture of Bump, Nakasuji, and Naruse regarding a factorization of certain coefficients of the transition matrix between the bases $\{\varphi_w\}$ and $\{f_w\}$. Let

$$\phi_u := \sum_{u \leq w} \varphi_w \in I(\tau)^I,$$

and consider the expansion in terms of the Casselman's basis:

$$\phi_u = \sum_w \tilde{m}_{u,w} f_w.$$

Then by the definition of f_w, $\tilde{m}_{u,w} = \mathcal{A}_w(\phi_u)(1)$. It is also easy to see that $\tilde{m}_{u,w} = 0$ unless $u \leq w$, see [BN11, Theorem 3.5]. We shall see below that $\tilde{m}_{u,w}$ equals the evaluation at τ of the coefficient $m_{u,w}$ from (9.4), defined for the Langlands dual flag variety. For every $u \leq w \in W$, recall the definition $S(u,w) := \{\beta \in R^{+,\vee} | u \leq s_\beta w < w\}$ (cf. (9.2)). Recall that G is the complex Langlands dual group, with the corresponding Borel subgroup B and the maximal torus T. The goal is to prove the following statement.

Theorem 10.1 (Bump-Nakasuji-Naruse Conjecture). *For every $u \leq w \in W$,*

$$(10.5) \qquad \tilde{m}_{u,w} = \prod_{\alpha \in S(u,w)} \frac{1 - q'^{-1} e^\alpha(\tau)}{1 - e^\alpha(\tau)},$$

if and only if the opposite Schubert variety $Y(u) := \overline{BuB/B}$ in the (dual, complex) flag manifold G/B is smooth at the torus fixed point e_w.

This is the representation theoretic counterpart of Theorem 9.5; its proof will be given in the next subsection.

We provide further historical context. Casselman [Cas80] asked for an expression of the basis f_w as a linear combination of the standard basis φ_w. Bump and Nakasuji found that the basis ϕ_w is better behaved for this question. Of course the original Casselman's basis can be obtained from the Möbius inversion

$$\varphi_u = \sum_{w \geq u} (-1)^{\ell(u)-\ell(w)} \phi_w.$$

The case $u = 1$ of the Bump-Nakasuji-Naruse conjecture is well known. In this case ϕ_1 is the spherical vector in $I(\tau)$, i.e., the vector fixed by the maximal compact subgroup $G(\mathcal{O})$, and

$$(10.6) \qquad \mathcal{A}_w(\phi_1)(1) = \tilde{m}_{1,w} = \prod_{\alpha \in S(1,w)} \frac{1 - q'^{-1} e^\alpha(\tau)}{1 - e^\alpha(\tau)}.$$

This is the *Gindikin–Karpelevich formula*, which in the non-archimedean setting was actually proved by Langlands [Lan71] after Gindikin and Karpelevich proved a similar statement for real groups. Casselman obtained another proof using his basis f_w, and this plays a crucial role in his computation of the Macdonald formula and the spherical Whittaker functions, see [Cas80, CS80]. See also [SZZ17] for an approach using the stable basis and the equivariant K-theory of the cotangent bundle $T^*(G/B)$. We will recover (10.6) below, as a consequence of Theorem 10.2. Other special cases of the conjecture follow from work of Reeder [Ree93].

10.3. **Casselman's problem and motivic Chern classes.** In this section, we construct the promised isomorphism between the $H_W(q')$-module of the Iwahori invariants of the principal series representation of $\mathscr{G}$ and the equivariant K group of the flag variety for the dual group G, regarded as an $H_W(-y)$-module via the action of the operators $\mathcal{T}_i^\vee$. This construction, together with the cohomological properties of the motivic Chern classes from §9, will be used to prove Theorem 10.1.

For now we assume that the unramified character τ is in the open set in T such that $1 - q'e^\alpha(\tau) \neq 0$, for every (positive or negative) coroot α. Regard the representation ring $K_T(pt)$ as a subring of $\mathbb{C}[T]$ and let $\mathbb{C}_\tau$ denote the one dimensional $K_T(pt)$-module induced by evaluation at τ. Recall that the operators $\mathcal{T}_i^\vee$ from Definition 3.1 satisfy

$$(\mathcal{T}_i^\vee + 1)(\mathcal{T}_i^\vee + y) = 0,$$

and the braid relations (Proposition 3.4). Hence, they induce an action of the Hecke algebra $H_W(-y)$ with parameter $-y$ on the K-theory ring $K_T(G/B)[y, y^{-1}]$ by sending T_w to $\mathcal{T}_w^\vee$ (here the parameter $-y$ corresponds to the parameter q in [Lus85]). We use the symbol π to denote this action.

Define a $K_T(pt)$-module homomorphism

$$\Psi : K_T(G/B)[y, y^{-1}] \otimes_{K_T(pt)[y,y^{-1}]} \mathbb{C}_\tau \xrightarrow{\sim} I(\tau)^I,$$

by $\Psi(MC_y^\vee(Y(w)^\circ) \otimes 1) = \varphi_w$ and $y \mapsto -q'$, with q' the number of elements in the residue field $\mathbb{F}_{q'}$. Since both the motivic Chern classes and the elements φ_w form bases in the appropriate modules, Ψ is clearly a module isomorphism. As in (9.3), define the element $\tilde{b}_w \in K_T(G/B)_{loc}[y^{-1}]$ by the formula

$$\tilde{b}_w := (-1)^{\dim G/B - \ell(w)} \prod_{\alpha > 0, w\alpha > 0} \frac{y^{-1} + e^{-w\alpha}}{1 - e^{w\alpha}} \iota_w \otimes 1.$$

Equivalently, $\tilde{b}_w = b_w \otimes 1$, with b_w from equation (9.3). We now state the main comparison theorem.

Theorem 10.2. *The following hold:*
(a) Ψ is an isomorphism of left $H_W(q')$-modules;
(b) There is an equality $\Psi(\tilde{b}_w) = f_w$.
(c) The coefficients $\tilde{m}_{u,w}$ are represented by the meromorphic functions $m_{u,w}$ on T, defined in equation (9.4), for the Langlands dual complex flag variety G/B. More precisely, let $\tau \in T$ be any regular unramified character, i.e., with trivial stabilizer W_τ. Then

$$\tilde{m}_{u,w} = m_{u,w}(\tau).$$

Remark 10.3. There is an analogue of this theorem for the equivariant K-theory of $T^*(G/B)$ proved in [SZZ17]. This is also studied by Lusztig and Braverman–Kazhdan in [Lus98, BK99] from different points of view. ⌟

Proof. The fact that Ψ is a map of $H_W(q')$-modules follows from comparing Proposition 6.6(a) to equation (10.1); these describe the Hecke actions on the basis of dual motivic Chern classes $MC_y^\vee(Y(w)^\circ)$ and on the basis of characteristic functions φ_w.

To prove part (b), we argue by descending induction on $\ell(w)$. Recall that $f_{w_0} = \varphi_{w_0}$ and $b_{w_0} = \iota_{w_0} = MC_y^\vee(Y(w_0))$ (from Definition 6.1); therefore $\Psi(\tilde{b}_{w_0}) = f_{w_0}$. Now take any $w < w_0$ and assume that $\Psi(\tilde{b}_z) = f_z$ for all $\tilde{b}_z$ with $\ell(z) > \ell(w)$. Pick a simple root α_i such that $ws_i > w$. Then by induction, $\Psi(\tilde{b}_{ws_i}) = f_{ws_i}$. Since Ψ is a homomorphism of Hecke

modules,

$$\Psi(\mathcal{T}_i^\vee(\widetilde{b}_{ws_i})) = \pi(T_i)(\Psi(\widetilde{b}_{ws_i})) = \pi(T_i)(f_{ws_i}).$$

On one hand, Lemma 3.7 gives

$$\Psi(\mathcal{T}_i^\vee(\widetilde{b}_{ws_i})) = \Psi\left(\frac{1+y}{e^{w\alpha_i}(\tau)-1}\tilde{b}_{ws_i} - y\tilde{b}_w\right) = \frac{1-q'}{e^{w\alpha_i}(\tau)-1}f_{ws_i} - y\Psi(\tilde{b}_w).$$

Here we have used that $\{\alpha > 0 : w(\alpha) > 0\} = \{\alpha > 0 : ws_i(\alpha) > 0\} \cup \{\alpha_i\}$. On the other hand, equation (10.4) gives

$$\pi(T_i)(f_{ws_i}) = \frac{1-q'}{e^{w\alpha_i}(\tau)-1}f_{ws_i} - yf_w.$$

Therefore, $\Psi(\widetilde{b}_w) = f_w$. By induction, this finishes the proof of part (b).

For part (c), observe that the previous two parts, together with the definitions of $\tilde{m}_{u,w}$ and $m_{u,w}$ imply the equality $\tilde{m}_{u,w} = m_{u,w}(\tau)$ for all regular unramified characters τ satisfying $1 - q'e^\alpha(\tau) \neq 0$, for every coroot α. However, it is known that the intertwiners $\mathcal{A}_x$ depend holomorphically on regular characters $\tau \in T$; see e.g., [Cas80, §3] or [Cas, §6.4]. Then one can extend meromorphically the equality $\tilde{m}_{u,w} = m_{u,w}(\tau)$ to any regular unramified character τ. $\square$

Combining Theorem 10.2 with Proposition 9.8 above gives a formula for $\tilde{m}_{u,w}$ in terms of localizations of motivic Chern classes, and in particular it recovers the Langlands-Gindikin-Karpelevich formula from (10.6). Also, Theorem 10.1 follows now from Theorem 10.2(c) and the main theorem from §9.

Proof of Theorem 10.1. This follows from Theorem 9.5 above together with the equality $\tilde{m}_{u,w} = m_{u,w}(\tau)$ for all regular unramified characters τ. $\square$

10.4. **Analytic properties of transition coefficients.** In this section we prove a conjecture of Bump and Nakasuji [BN19, Conjecture 1] about analytic properties for the transition coefficients $\tilde{m}_{u,w}$ and the set of coefficients $\tilde{r}_{u,w}$ defined as follows (cf. [BN19, Theorem 3]). If $f = f(q')$ is a function, let $\bar{f}(q') := f(q'^{-1})$. Define

$$(10.7) \qquad\qquad \tilde{r}_{u,w} := \sum_{u \leq x \leq w} (-1)^{\ell(x)-\ell(u)}\overline{\tilde{m}}_{x,w}.$$

Since we are interested only in analytic properties, by the comparison Theorem 10.2(c) we can replace the coefficients $\tilde{m}_{u,w}$ by the 'geometric' ones $m_{u,w}$. We accordingly let $r_{u,w} := \sum_{u \leq x \leq w}(-1)^{\ell(x)-\ell(u)}\overline{m}_{x,w}$ be the corresponding coefficients, where $\bar{f}(y) := f(y^{-1})$. Therefore, $\tilde{r}_{u,w} = r_{u,w}(\tau)|_{y=-q'}$ for any regular unramified character τ. With this notation, we can prove the following statement, cf. [BN19, Conjecture 1].

Theorem 10.4. *Let $u \leq w$ be two Weyl group elements. Then the functions*

$$\prod_{\alpha \in S(u,w)}(1-e^\alpha)r_{u,w} \quad , \quad \prod_{\alpha \in S(u,w)}(1-e^\alpha)m_{u,w}$$

are holomorphic on the full dual torus T.

Proof. As observed by Bump and Nakasuji in *loc. cit.*, the conjecture for $r_{u,w}$ implies the conjecture for $m_{u,w}$. Further, using the formula for $m_{u,w}$ from Proposition 9.8,

$$(\overline{r}_{u,w})^\vee = \sum_{u \leq x \leq w} (-1)^{\ell(x)-\ell(u)}(m_{x,w})^\vee = \frac{1}{MC_y(Y(w)^\circ)|_w} \sum_{u \leq x \leq w} (-1)^{\ell(x)-\ell(u)} MC_y(Y(x))|_w$$

$$= \frac{1}{MC_y(Y(w)^\circ)|_w} \sum_{u \leq x} (-1)^{\ell(x)-\ell(u)} MC_y(Y(x))|_w = \frac{MC_y(Y(u)^\circ)|_w}{MC_y(Y(w)^\circ)|_w};$$

here the third equality holds because $MC_y(Y(x))|_w = 0$ for $x \not\leq w$, as $e_w \notin Y(x)$, and the last equality follows by Möbius inversion on the Bruhat poset W. It follows that

$$\Big(\prod_{\alpha \in S(u,w)} (1 - e^\alpha) r_{u,w} \Big)^\vee = \prod_{\alpha \in S(u,w)} (1 - e^{-\alpha}) \frac{MC_{y^{-1}}(Y(u)^\circ)|_w}{MC_{y^{-1}}(Y(w)^\circ)|_w},$$

therefore it suffices to show that the right-hand side is holomorphic in y. Using Proposition 7.2, the definition of $S(u,w)$ (9.2), and the description of $MC_y(Y(w)^\circ)|_w$ from Theorem 9.1 (and its proof), we obtain

$$\prod_{\alpha \in S(u,w)} (1 - e^{-\alpha}) \frac{MC_{y^{-1}}(Y(u)^\circ)|_w}{MC_{y^{-1}}(Y(w)^\circ)|_w} = \frac{MC_{y^{-1}}(Y(u)^\circ)|_w}{\lambda_{y^{-1}}(T_w^* Y(w))} \times \frac{\prod_{\alpha \in S(u,w)}(1 - e^{-\alpha})}{\prod_{\alpha>0,ws_\alpha<w}(1 - e^{w\alpha})}$$

$$= \frac{MC_{y^{-1}}(Y(u)^\circ)|_w}{\lambda_{y^{-1}}(T_w^* Y(w))} \times \frac{\prod_{\alpha>0,u\leq ws_\alpha<w}(1 - e^{w\alpha})}{\prod_{\alpha>0,ws_\alpha<w}(1 - e^{w\alpha})}$$

$$= \frac{MC_{y^{-1}}(Y(u)^\circ)|_w}{\lambda_{y^{-1}}(T_w^* Y(w)) \cdot \prod_{\alpha>0,u\not\leq ws_\alpha<w}(1 - e^{w\alpha})}.$$

The last expression is a holomorphic function by Theorem 7.4 and we are done. $\square$

We also record the following result, obtained in the proof of Theorem 10.4.

Proposition 10.5. *The coefficients $r_{u,w}$ are obtained from the expansion:*

$$MC_y^\vee(Y(u)^\circ) = \sum \overline{r}_{u,w} b_w$$

or, equivalently

$$(\overline{r}_{u,w})^\vee = \frac{MC_y(Y(u)^\circ)|_w}{MC_y(Y(w)^\circ)|_w} \in \mathrm{Frac}\left(K_T(pt)[y^{\pm 1}]\right) \quad .$$

Remark 10.6. The coefficients $r_{u,w}$ satisfy other remarkable properties, such as a certain duality and orthogonality; see [NN16, BN19]. We will study these properties using motivic Chern classes in an upcoming note.   ⌋

11. Appendix: proof of Lemma 8.2

Recall Lemma 8.2.

Lemma 11.1. *(a) Let $u, w \in W$. Under the left Weyl group multiplication,*

$$w.stab_{\mathfrak{C},T^{1/2},\mathcal{L}}(u) = stab_{w\mathfrak{C},wT^{1/2},w.\mathcal{L}}(wu).$$

In particular, if both the polarization $T^{1/2}$ and the line bundle $\mathcal{L}$ are G-equivariant, then

$$w.stab_{\mathfrak{C},T^{1/2},\mathcal{L}}(u) = stab_{w\mathfrak{C},T^{1/2},\mathcal{L}}(wu).$$

(b) The duality automorphism acts by sending $q \mapsto q^{-1}$ and

$$\tag{11.1} (\mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},\mathcal{L}}(w))^{\vee} = q^{-\frac{\dim G/B}{2}}\, \mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}}_{opp},\mathcal{L}^{-1}}(w),$$

where $T^{\frac{1}{2}}_{opp} := q^{-1}(T^{\frac{1}{2}})^{\vee}$ is the opposite polarization; see [OS16, Equation (15)]. I.e., this duality changes the polarization and slope parameters to the opposite ones, while keeping the chamber parameter invariant.

(c) Fix integral weights $\lambda, \mu \in X^*(T)$ and the equivariant line bundles $\mathcal{L}_\lambda = G \times^B \mathbb{C}_\lambda$ and $\mathcal{L}_\mu$. Let $a \in \mathbb{Q}$ be a rational number. Then

$$\mathrm{stab}_{\mathfrak{C},T^{1/2},a\mathcal{L}_\lambda \otimes \mathcal{L}_\mu}(w) = e^{-w(\mu)}\mathcal{L}_\mu \otimes \mathrm{stab}_{\mathfrak{C},T^{1/2},a\mathcal{L}_\lambda}(w),$$

as elements in $K_{T\times\mathbb{C}^*}(T^*(G/B))$

Proof. We sketch the proof of part (a). By the uniqueness of the stable basis, we only need to check that the expression on the left-hand side satisfies the defining properties of the stable basis $\mathrm{stab}_{w\mathfrak{C},w(T^{\frac{1}{2}}),w\mathcal{L}}(wu)$ from Theorem 8.1. The support condition follows from the fact that

$$\tag{11.2} u \preceq_{\mathfrak{C}} v \iff wu \preceq_{w\mathfrak{C}} wv.$$

The normalization condition follows from an explicit computation. We now check the degree condition. We need to show that

$$\deg_T\left(w(\mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},\mathcal{L}}(v))|_{wu}\right) \subseteq \deg_T\left(w(\mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},\mathcal{L}}(u))|_{wu}\right) + (w\mathcal{L})|_{wu} - (w\mathcal{L})|_{wv} \quad,$$

for every $wu \prec_{w\mathfrak{C}} wv$. By Equations (8.2) and (11.2), this is equivalent to

$$\deg_T w(\mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},\mathcal{L}}(v)|_u) \subseteq \deg_T w(\mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},\mathcal{L}}(u)|_u) + w(\mathcal{L}|_u) - w(\mathcal{L}|_v) \quad,$$

for every $u \prec_{\mathfrak{C}} v$, which is the defining property for the stable basis $\mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},\mathcal{L}}(v)$. Hence, the degree condition is satisfied.

Part (b) is [OS16, Equation (15)] with $\hbar^{-\frac{\dim X}{2}}$ replaced[7] by $q^{-\frac{\dim G/B}{2}}$. Since this equation is not proved in *loc. cit.*, we include a proof in the case when the fixed point set X^T satisfies $\dim X^T = 0$ (e.g., $X = T^*(G/B)$).[8] By the uniqueness of stable envelopes, we need to show $q^{\frac{\dim X}{4}}(\mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},\mathcal{L}})^{\vee}$ satisfies the defining properties of $\mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}}_{opp},\mathcal{L}^{-1}}$. The support condition is obvious, and the degree condition follows because $\deg_T$ remains unchanged after multiplication by powers of $q^{\pm\frac{1}{2}}$. We turn to the normalization condition. Denote by F a component of X^T (a point, in our case), and use the notation N_+, N_-, $N^{\frac{1}{2}}$, etc., for the appropriate normal subspaces to F, as in the paragraphs preceding Theorem 8.1. Since $N_- - T^{\frac{1}{2}} = q^{-1}(T^{\frac{1}{2}}_{>0})^{\vee} - T^{\frac{1}{2}}_{>0}$ (see [OS16, p. 13]), the normalization is (see [OS16, Equation (10)])

$$\mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},\mathcal{L}}|_F = (-1)^{\mathrm{rk}\,T^{\frac{1}{2}}_{>0}}\left(\frac{\det N_-}{\det T^{\frac{1}{2}}}\right)^{\frac{1}{2}}\mathcal{O}_{\mathrm{Attr}}|_F = (-1)^{\mathrm{rk}\,T^{\frac{1}{2}}_{>0}} q^{-\frac{\mathrm{rk}\,T^{\frac{1}{2}}_{>0}}{2}}(\det T^{\frac{1}{2}}_{>0})^{\vee}\mathcal{O}_{N_+}|_F.$$

The last equality holds because the normal bundle of Attr at F is spanned by the *non-attracting* weights at F; this is the same as the normal bundle of N_+ inside N, therefore $\mathcal{O}_{\mathrm{Attr}}|_F = \mathcal{O}_{N_+}|_F = \lambda^{T\times\mathbb{C}^*}_{-1}(N_-^{\vee})$.

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[7]In [OS16] the variety X is the symplectic resolution, and it corresponds to our $T^*(G/B)$, so $\hbar^{-\frac{\dim X}{2}}$ should be $q^{-\dim G/B}$; the missing factor of 2 is a typo.

[8]Our T is denoted by A in *loc. cit.* and this is the torus preserving the symplectic form of X.

Let $\{\gamma_j\}$ be the torus weights of $T^{\frac{1}{2}}_{>0}|_F$ and let $\{\beta_i\}$ be the torus weights of $T^{\frac{1}{2}}_{<0}|_F$. Since $T(X) = T^{\frac{1}{2}} + q^{-1}(T^{\frac{1}{2}})^\vee$, the torus weights of $N_-|_F$ are $\{\beta_i\}$ and $\{q^{-1}\gamma_j^{-1}\}$. We abuse notation and write λ for $e^\lambda \in R(T)$. Then,

$$\mathrm{stab}_{\mathfrak{C}, T^{\frac{1}{2}}, \mathcal{L}}|_F = (-1)^{\mathrm{rk}\, T^{\frac{1}{2}}_{>0}} q^{-\frac{\mathrm{rk}\, T^{\frac{1}{2}}_{>0}}{2}} \prod_j \gamma_j^{-1} \prod_i (1 - \beta_i^{-1}) \prod_j (1 - q\gamma_j)$$

$$(11.3) \qquad\qquad = q^{\frac{\mathrm{rk}\, T^{\frac{1}{2}}_{>0}}{2}} \prod_i (1 - \beta_i^{-1}) \prod_j (1 - q^{-1}\gamma_j^{-1}).$$

Since $T^{\frac{1}{2}}_{opp} = q^{-1}(T^{\frac{1}{2}})^\vee$, the torus weights of $T^{\frac{1}{2}}_{opp,>0}|_F$ are $\{q^{-1}\beta_i^{-1}\}$ and the torus weights of $T^{\frac{1}{2}}_{opp,<0}|_F$ are $\{q^{-1}\gamma_j^{-1}\}$. A similar calculation shows

$$\mathrm{stab}_{\mathfrak{C}, T^{\frac{1}{2}}_{opp}, \mathcal{L}^{-1}}|_F = q^{\frac{\mathrm{rk}\, T^{\frac{1}{2}}_{<0}}{2}} \prod_j (1 - q\gamma_j) \prod_i (1 - \beta_i).$$

Taking the dual of (11.3), we get

$$\left(\mathrm{stab}_{\mathfrak{C}, T^{\frac{1}{2}}, \mathcal{L}}\right)^\vee |_F = q^{-\frac{\mathrm{rk}\, T^{\frac{1}{2}}_{>0}}{2}} \prod_i (1 - \beta_i) \prod_j (1 - q\gamma_j).$$

Therefore,

$$\left(\mathrm{stab}_{\mathfrak{C}, T^{\frac{1}{2}}, \mathcal{L}}\right)^\vee |_F = q^{-\frac{\mathrm{rk}\, T^{\frac{1}{2}}}{2}} \mathrm{stab}_{\mathfrak{C}, T^{\frac{1}{2}}_{opp}, \mathcal{L}^{-1}}|_F = q^{-\frac{\dim X}{4}} \mathrm{stab}_{\mathfrak{C}, T^{\frac{1}{2}}_{opp}, \mathcal{L}^{-1}}|_F.$$

This proves the normalization condition, whence part (b). Part (c) follows directly from the uniqueness of the stable envelope. $\qquad\square$

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Mathematics Department, Florida State University, Tallahassee FL 32306
Email address: aluffi@math.fsu.edu

Department of Mathematics, Virginia Tech University, Blacksburg, VA 24061
Email address: lmihalce@vt.edu

Mathematisches Institut, Universität Münster, Germany
Email address: jschuerm@uni-muenster.de

Department of Mathematics, University of Toronto, Toronto, ON, M5S, 2E4, Canada
Email address: changjiansu@gmail.com