Soliton Propagation in Nonlinear Optical Fibers:

Theory and Application

by

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(ABSTRACT)

A survey of research in nonlinear optical fibers is given. Important background concepts are introduced and explained. Present and future applications of nonlinear optical fibers are reviewed. A mathematical model of a nonlinear optical fiber is developed using a coupled-mode theory approach, and methods of solving nonlinear partial differential equations are discussed. A detailed history of research in the field is given, and recommendations for future research are made.

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1.0 Introduction

This is an unconventional thesis in that no original work, experimental or theoretical, has been done by the author. This thesis is a survey of research in the field of nonlinear fiber optics. The justification for writing such a thesis can be understood after the following discussion.

The field of nonlinear fiber optics grew out of developments in mathematical physics in the late 1960's and early 1970's. The birth of the field as an engineering discipline occured with the publication of a pair of papers by Akira Hasegawa and Frederick Tappert of AT&T Bell Laboratories in 1973 [1, 2]. Further progress in the field was slow until 1980, when experimental results [3] gave further impetus for research in the field. Since that year research in nonlinear fiber optics has accelerated, and today several research groups around the world are working in the area.

Despite all this activity, *experimental* research in the field is still very limited, and is conducted almost entirely by groups in industry. Academia for the most part has not yet developed expertise in the experimental techniques involved in nonlinear fiber optics, due to the newness and difficulty of the field, and the expense of both equipment and computer time required. Nevertheless, the development of the theory and the demonstrated economic value of the results now make it possible, and very desireable, for large research universities to play a more active role in the experimental investigation of nonlinear fiber optics.

Therefore, the purpose of this thesis is to provide a background for, and generate interest in, further research in the area of nonlinear fiber optics at the Department of Electrical Engineering at Virginia Polytechnic Institute and State University.

This purpose requires an unconventional approach, rather than the traditional thesis format. Chapter 2 is intended to introduce and define terms which may be unfamiliar to the reader. Chapter 3 provides motivation for the rest of the work by introducing the exciting applications of the research. Chapter 4 presents a full derivation of a mathematical model of a nonlinear optical fiber, using the relatively simple formalism of coupled-mode theory. Chapter 5 explains a few basic topics in the solution of nonlinear dispersive partial differential equations. By Chapter 6 it is hoped that all mathematical questions have been answered, so that this chapter may give a detailed and critical review of work in the field. Finally, several specific recommendations for future research are made in the last chapter.

2.0 Some Basic Concepts

This chapter defines several terms in order to provide a background and establish a vocabulary for the following chapters. For the sake of brevity, historical comments are mostly excluded, and topics are discussed in terms of fiber optics. Two comments are necessary about this approach.

The fascinating history of soliton studies is the story of several interesting and important characters, not the least of whom is a British marine engineer named John Scott-Russell, who, along with a few other researchers, has become a legend among modern-day scientists. The story of his work, which regrettably must be excluded here, is summarized in several excellent historical reviews listed in the References [4, 5, 6]. The reader is urged to consult some of these references to gain a better understanding and appreciation of the development of soliton studies.

The second comment is that limiting the discussion to optical fibers should not be taken to imply that optical fibers can and should be considered by themselves. The study of nonlinear dispersive systems covers many branches of the physical and mathematical sciences, and advances in one branch can often be applied to another branch by observant researchers. Therefore, it is adviseable that the researcher in nonlinear fiber optics search the literature in terms of the underlying mathematics, and not limit himself or herself to research in optics. By the same token, research in nonlinear fiber optics can be useful to researchers in other fields, since optical fibers can often serve as easily-handled prototypes for many other nonlinear dispersive systems for which experimentation is not so easy [3].

With these things in mind, we begin a short review of *dispersion, nonlinearity,* and *solitons,* discussed in the context of fiber optics.

2.1 Dispersion

Pulses in optical fibers become less and less distinct as they propagate due to pulse broadening caused by *group velocity dispersion*, the separation of complex light into its components (frequencies or modes) traveling at different velocities. Pulse spreading results because components which form a sharp pulse at the input end of a fiber will arrive at the output end at different times and so will form a broadened version of the input pulse. There are three types of dispersion in optical fibers [8, 9]:

1. *Material dispersion* M is due to the variation of the index of refraction n with the optical wavelength λ . Material dispersion is a characteristic of the material, and is dependent on the spectral width of the optical source, but independent of the waveguide configuration. In pure silica M is zero at about $\lambda = 1.27$ μm.

2. *Waveguide dispersion* occurs because the group velocity v_g is a complicated function of frequency and the fiber dimensions, and can, to some degree, be designed to meet specifications by choice of waveguide dimensions. Like material dispersion, it is dependent on the spectral width of the optical source.

3. *Intermodal dispersion* is due to each *mode* at a particular frequency having a different group delay. Thus it is found only in multimode fibers and is independent of the spectral width of the optical source.

Since we will be concerned mainly with single-mode fibers, intermodal dispersion will generally not enter into further discussion in this thesis. Dispersion due to material and waveguide effects is discussed further in the next section.

2.1.1 Chromatic dispersion

The sum of material dispersion and waveguide dispersion is called *chromatic dispersion*, since it is dependent on the spectral width of the optical source. The chromatic dispersion D is given by [3]

$$D = -\frac{1}{L}\frac{dt_g}{d\lambda} = v_g^{-2}\frac{dv_g}{d\lambda} \qquad [2.1.1]$$

where v_g is the *group velocity* of the pulse envelope, $t_g = \frac{L}{v_g}$ is the *group delay*, and L is the length of the fiber. For convenience, D is typically given in units of ps/nm-km. Positive dispersion, called *normal* [1], occurs when the group velocity increases with increasing λ, *i.e.*, $\frac{dv_g}{d\lambda} > 0$. Negative dispersion, called *anomalous*, occurs when the group velocity decreases with increasing λ. Typical single-mode optical fibers now in use have zero dispersion near a wavelength of 1.3 μm, with anomalous dispersion for longer wavelengths and normal dispersion for shorter wavelengths.

Chromatic dispersion, whether anomalous or normal, results in pulse broadening. An optical pulse with initial spectral width σ , propagating in a fiber with length L and dispersion D, will increase in pulse width by an amount τ, given by [7]

$$\tau = |D|\sigma L. \qquad [2.1.2]$$

Dispersion can be treated mathematically by specifying the *dispersion relation*, an expression of the propagation constant β as a function of ω:

$$\beta = \beta(\omega). \qquad [2.1.3]$$

A convenient expression of the dispersion relation in a system with a quasimonochromatic source is the Taylor series expansion of β about the source freqency ω_o:

$$\beta(\omega) = \beta(\omega_o) + (\omega - \omega_o)\frac{d\beta}{d\omega} + \frac{1}{2}(\omega - \omega_o)^2\frac{d^2\beta}{d\omega^2} \quad \ldots \qquad [2.1.4]$$

where all the derivatives are evaluated at $\omega = \omega_o$. The n^{th}-order chromatic dispersion is proportional to the n^{th} ω-derivative of β. For instance, in single-mode fibers, where intermodal effects are absent, the total second-order group velocity dispersion can be expressed simply as [3]:

$$D = v_g^{-2}\frac{dv_g}{d\lambda} = \frac{2\pi c}{\lambda^2}\frac{d^2\beta}{d\omega^2}. \qquad [2.1.5]$$

If 2τ is the pulse width, then in the presence of second-order dispersion (proportional to $\frac{d^2\beta}{d\omega^2}$) alone, the pulse will double its width in a distance proportional to τ^2. In the presence of third-order dispersion (proportional to $\frac{d^3\beta}{d\omega^3}$) alone, the pulse width will double in a distance proportional to τ^3, and so on [10]. The ω –derivatives of β are related to the λ –derivatives of n by [10,11]:

$$\frac{d\beta}{d\omega} = \frac{n}{c}\left(1 - \frac{\lambda}{n}\frac{\partial n}{\partial \lambda}\right) \qquad [2.1.6]$$

$$\frac{d^2\beta}{d\omega^2} = \frac{\lambda}{2\pi c^2}\left(\lambda^2\frac{\partial^2 n}{\partial \lambda^2}\right) \qquad [2.1.7]$$

$$\frac{d^3\beta}{d\omega^3} = -\frac{\lambda^2}{4\pi^2 c^3}\frac{\partial}{\partial \lambda}\left(\lambda^3\frac{\partial^2 n}{\partial \lambda^2}\right). \qquad [2.1.8]$$

2.2 Nonlinearity

A nonlinear system is one for which superposition does not apply. This means that the output of the system contains frequency components which are not present in the input. A medium is nonlinear if its propagation characteristics are dependent on the amplitude of the signal propagating through it. A pulse propagating in a nonlinear medium will develop steeper edges as it propagates, due to harmonic generation. The pulse can steepen until it is multi-valued, at which point a *shock wave* is said to have formed. Shock waves are formed in systems where the steepening effect of nonlinearity dominates over the pulse-broadening effect of dispersion. Shock wave formation occurs in many familiar physical systems, such as water waves on the seashore, and the sonic boom of a supersonic aircraft.

Optical fibers are nonlinear due to the nonlinear polarizability of silica, which can be expressed as [12]:

$$\overline{P} = \kappa\overline{E}(\bar{r}, t) + \chi^{(3)}\overline{E}^3(\bar{r}, t) + \chi^{(5)}\overline{E}^5(\bar{r}, t) \ \ldots \qquad [2.2.1]$$

where $\overline{E}(\bar{r}, t)$ is the time-domain electric field in the fiber, κ is the linear susceptibility, and the χ's are tensors defining the nonlinear susceptibilities. The nonlinear polarizability results in an index of refraction of the form

$$n^2(\bar{r}, t) = n_o^2(\bar{r}) + n_2^2|\overline{E}(\bar{r}, t)|^2 + n_4^2|\overline{E}(\bar{r}, t)|^4 \ldots \qquad [2.2.2]$$

In linear fiber optics the signal amplitude is sufficiently small that the nonlinear terms can be neglected. The n_4 term and higher terms in [2.2.2] are generally negligible in all optical propagation applications [13]. The n_2 term is called the *Kerr nonlinearity,* and plays an important role in nonlinear optics [14].

2.3 Solitons

One might expect that in systems that are both dispersive *and* nonlinear, the pulse-broadening effect of dispersion and the pulse-steepening effect of nonlinearity can be precisely balanced to allow pulses to propagate without change in shape. Such propagation has been demonstrated in several nonlinear dispersive systems, including optical fibers. The stable pulses which result are known as *solitons*. The definition of soliton can only be properly explained in terms of the inverse scattering transform, which will be presented in Chapter 5. For the moment, we can state a few necessary (but not sufficient) conditions for a pulse to be classified as a soliton [4, 5, 15]:

1. A soliton is a *traveling wave, i.e.*, its dependence on the time t and longitudinal coordinate z is of the form $(z - vt)$, where v is the velocity of propagation.

2. A soliton is a *solitary wave, i.e.*, a traveling wave which is *localized*, meaning that its energy does not disperse, but remains concentrated as the wave propagates.

3. Solitons survive collisions with other solitons with their shape, height, and velocity intact, suffering no more than a phase shift.

There are pulses which have all of these three properties which cannot be strictly classified as solitons by the definition to be given in Chapter 5 [5]. Such "soliton-like" or "quasi-soliton" pulses, often loosely referred to as solitons, are still important, since sometimes only the three properties above, or even just the first two, are of interest.

Solitons are found as solutions of certain nonlinear dispersive wave equations, such as the Korteweg-deVries (KdV) equation

$$\psi_t + \alpha\psi\psi_z + \psi_{zzz} = 0, \qquad \alpha \textit{ constant} \qquad [2.3.1]$$

and the nonlinear Schrödinger (NLS) equation

$$j\psi_t + \psi_{zz} + k|\psi|^2\psi = 0, \qquad k \text{ constant}, \; j = \sqrt{-1} \qquad [2.3.2]$$

where the subscripts denote partial differentiation. Solitons are not found in systems which are nonlinear *and* dispersionless, or linear *and* dispersive. The familiar one-dimensional wave equation

$$\psi_{tt} - v^2\psi_{zz} = 0 \qquad [2.3.3]$$

which is linear and dispersionless, has the general solution

$$\psi(z,t) = F(z - vt) + G(z + vt) \qquad [2.3.4]$$

which consists of two solitary waves, *not solitons,* of arbitrary shape [5].

To illustrate several concepts, we will consider a prototype nonlinear dispersive partial differential equation: the Korteweg-deVries (KdV) equation, given in [2.3.1]. The KdV equation was originally derived to describe the unidirectional propagation of water waves in shallow channels, but can be used to model many other nonlinear dispersive systems [4]. The term $\alpha\psi\psi_z$ represents nonlinearity and ψ_{zzz} represents dispersion. It can be shown that the KdV equation has the single-soliton solution [4]

$$\psi(z - vt) = \frac{3v}{\alpha}\,sech^2\left[\frac{\sqrt{v}}{2}(z - vt)\right]. \qquad [2.3.5]$$

The height, width, and velocity of the KdV soliton are proportional to v , $\sqrt{v}$, and v, respectively. Thus a higher-amplitude KdV soliton will have a narrower width and a faster velocity than a lower-amplitude one. The KdV equation also has the two-soliton solution [4]

$$\psi(z,t) = \frac{216 + 288\cosh(2z - 8t) + 72\cosh(4z - 64t)}{\alpha[3\cosh(z - 28t) + \cosh(3z - 36t)]^2} \qquad [2.3.6]$$

which, for large time, separates into two single solitons of the form [4]

$$\psi(z,t) = \frac{12i^2}{\alpha}\,sech^2[i(z - 4i^2t) + \delta_i], \qquad i = 1, 2; \;\; \delta_i \text{ constant}. \qquad [2.3.7]$$

As seen from [2.3.7], the two-soliton solution of the KdV equation consists of two solitons traveling in the same direction. Figure 1 shows two KdV solitons, a tall one initially behind a shorter one. Since the taller soliton propagates faster, it eventually overtakes the other and passes through it. Both solitons retain their shapes and velocities through the collision.

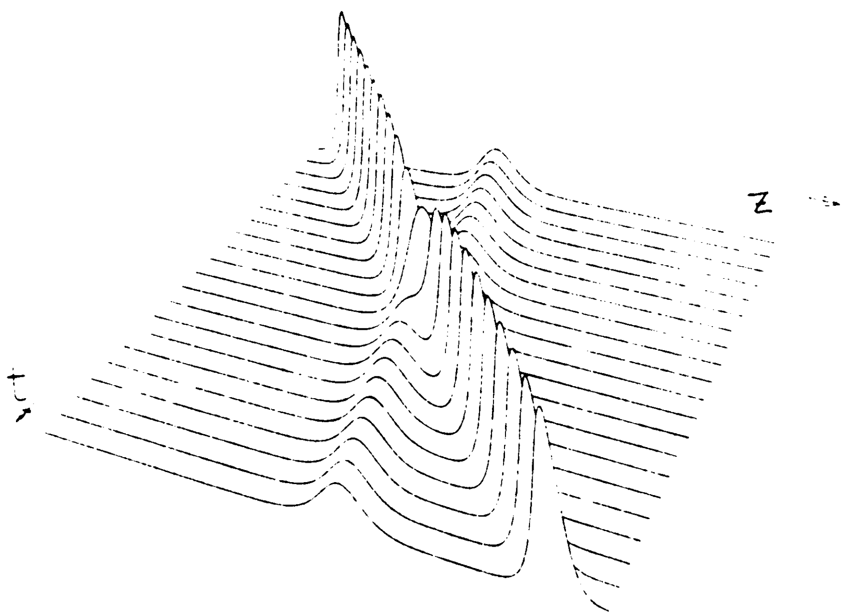

Figure 1. Two KdV solitons in a unidirectional collision. From Dodd *et al.* [6].

To study head-on collisions between solitons, it is necessary to consider an equation that allows bidirectional propagation, such as the Boussinesq equation, which is the bidirectional counterpart of the KdV [4].

2.3.1 Baseband and Envelope Solitons

Solitons are classified as *baseband* or *envelope* according to the form of the pulse [16]. A *baseband* soliton has a frequency spectrum centered on zero, whereas an envelope soliton has a shifted frequency spectrum, *i.e.*, an envelope soliton is a pulse shape which modulates some sinusoidal carrier wave. Solutions of the KdV equation given in Eqs. [2.3.5-7] are baseband solitons, whereas solutions of the NLS equation, important in nonlinear optics, are envelope solitons [4].

2.3.2 Fundamental and Higher-Order Solitons

Fundamental solitons propagate with no change in pulse shape, while *higher-order* solitons change periodically in pulse shape as they propagate [4]. The distance of propagation in which a higher-order soliton undergoes one and only one period is called the *soliton period* z_o. The soliton period is a property of the underlying wave equation; for instance, the canonic NLS equation has a dimensionless soliton period $z_o = \frac{\pi}{2}$. Solitons of the KdV equation do not exhibit periodic behavior.

The behavior of a higher-order soliton is predictable if the soliton period is known [3]. A second-order soliton first contracts to reach minimum width at $z = \frac{z_o}{2}$, and then expands to recover the original pulse shape (and spectrum) at $z = z_o$. A fourth-order soliton contracts to reach minimum width at $\frac{z_o}{4}$, then splits into 2 pulses at $\frac{z_o}{2}$, and then reverses this pattern to return to the original pulse shape at z_o . Solitons of even higher order go through even more complicated behavior, but the first action is always to *contract* to reach minimum pulse width at $\frac{z_o}{N}$, where N is the soliton order, and eventually return to the original pulse shape and spectrum at z_o .

For equations which have higher-order soliton solutions, the soliton order N can be defined in terms of pulse energy: The order of a soliton is equal to its area divided by the area of the fundamental soliton. For solitons of the same shape, the order is proportional to the amplitude [3].

A more complete explanation of higher-order solitons will be given in Chapter 5 after the introduction of the inverse scattering transform.

2.4 *Solitons in Optical Fibers*

As mentioned earlier, optical fibers are dispersive and, given sufficient signal amplitude, can be nonlinear. The conditions under which an optical fiber can be considered nonlinear depend on the pulse power and pulse width of the propagating signal. An optical fiber, like any physical medium, is nonlinear if the signal amplitude is large enough. Designing a nonlinear optical fiber is a matter of choosing the signal amplitude and designing the dispersion characteristic to obtain the desired balance. For conventional fibers used today, the magnitude of the nonlinear Kerr effect becomes comparable to the group velocity dispersion when the propagating pulse in the fiber has a peak power of about 30 mW or greater and an envelope pulse width of about 10 ps or less [15]. By decreasing the dispersion, the power required to form a soliton can also be decreased [32]. It will be shown in Chapter 4 that in the nonlinear regime the pulse envelope satisfies a form of the nonlinear Schrödinger equation, with a number of higher-order terms added depending on the pulse power and pulse width.

The interesting properties of soliton propagation just explained have several applications in fiber optics. These applications are discussed in the next chapter.

3.0 Applications of Nonlinear Optical Fibers

Optical fibers operated in a nonlinear regime have already been applied in many areas, and other applications have been proposed and are under development. These applications are reviewed here, first the existing ones, and then the proposed.

3.1 Existing Applications

3.1.1 Pulse Compression

Higher-order solitons undergo a periodic series of compressions, expansions, and splittings as they propagate. An N^{th}-order soliton reaches maximum compression at $z = \frac{z_o}{N}$, where z_o is the soliton period discussed in Chapter 2. If z_o is known, then a length of fiber can be selected so that the output pulse is a greatly compressed version of the input pulse. Researchers have reported pulse compression as great as 2700 to 1, and output pulses as narrow as 33 femtoseconds [17], using the soliton effect. By using a nonlinear optical fiber and an anomalously dispersive grating pair, researchers have produced the shortest optical pulses ever reported, with pulse width of 8 fs [18].

High-power, ultrashort optical pulses could find application in such areas as laser surgery and cauterization, and micro-cutting and spot welding [19].

3.1.2 Soliton Laser

A length of nonlinear fiber is used as a feedback element in the *soliton laser*, a device used as a source of intense, short soliton pulses [20, 21]. The width of the output pulse is selectable from picoseconds down to tens of femtoseconds by the choice of the feedback fiber length. The output pulses are solitons with intensity of the hyperbolic-secant-squared shape, which are often required for soliton propagation experiments.

3.1.3 Optical Kerr Medium

A nonlinear optical fiber is used as an optical Kerr medium in a nondemolitional interferometer for the measurement of photon number [22]. The optical beam to be measured interacts with a test beam in the nonlinear fiber, and the photon number can be inferred by measurements on the test beam, without effecting the value of photon number.

3.2 *Proposed Applications Being Developed*

3.2.1 Harmonic Generation

Nonlinear optical fibers are being developed specifically for use in sum-frequency [23] or second-harmonic [24, 25] generation, functions which have usually been performed by nonlinear crystals. The advantages of fibers are their low cost, ready availability, and low loss, allowing long interaction distances. Österberg and Margulis [25] have reported second-harmonic frequency generation efficiency as high as 5%, following a period of "preparation" of the fiber by high-intensity light. The efficiency of fiber harmonic generators is an order of magnitude below that of crystals, but the cost of a fiber is *five* orders of magnitude less than that of a crystal. Further improvements in the efficiency of fiber harmonic generators can be expected, since the technology is not yet fully developed, or even well understood. Terhune and Weinberger [26] have offered a theory of harmonic generation, which unfortunately does not explain the results of Österberg and Margulis. Recent work by Stolen and Tom [27] gives some improvements on the theory.

3.2.2 Optical Switching

Kaplan [28] has shown how a generalized nonlinear Schrödinger equation with certain nonlinearities can support multistable soliton states. A beam propagating in the appropriate medium could be switched from one soliton state to another, *i.e.*, from one soliton pulse shape and set of propagation parameters to another, by adding or removing a certain amount of energy from the beam. Numerical studies by Enns and Rangnekar [29, 30] support Kaplan's work. Unfortunately, the types of nonlinearity required do not include the Kerr nonlinearity which is currently exploited in nonlinear fiber optics, so developments in multistable fiber optic switching devices will require a search for new materials (new dopants) or fiber geometries which will provide the appropriate nonlinearities.

3.2.3 Distortionless Signal Propagation

The most exciting future application of nonlinear optical fibers, and the main motivation for their development, is to use them as signal transmission media in long-distance, high-bit-rate optical systems. This topic is introduced here, and is covered in more detail in Chapter 6. To understand the need for nonlinear fibers, a short history of linear fiber optics must be given [31, 9].

The first commercial fiber optic communication systems used multimode fibers and operated near a wavelength of 0.82 μm, the wavelength of sources available at the time. The multimode fibers used had a loss of 3 to 4 dB/km. Developments in fiber production and in optical sources allowed the second generation of optical systems to operate near 1.3 μm, the wavelength of zero dispersion, with losses of about 0.5 dB/km. These systems generally used multimode fibers, since the larger core area of a multimode fiber allowed the use of less expensive LED's instead of laser sources.

It became clear that pulse broadening caused by dispersion was the most important factor limiting fiber bandwidth, so third-generation systems (currently in use) use single-mode fibers so that intermodal dispersion is eliminated. To increase the data rate further, fourth-generation systems (under development) must operate around 1.55 μm, where the fiber loss is minimized, and use single-mode fibers with the waveguide dispersion carefully tailored to cancel the material dispersion at that wavelength [9]. Thus these *dispersion-shifted* systems could operate with minimized loss and zero (second-order) dispersion at the center frequency of the optical source. However, since the dispersion cannot be made zero over the entire spectral width of the pulse, and since higher-order dispersion shows up even if second-order dispersion is zero [11, 70], dispersion will always be present and will still be a limiting factor. The theoretical (length)-(bit rate) product for optimized dispersion-shifted systems is on the order of 10^3 GHz-km, with 10^4 GHz-km as the upper limit [70]. Attaining this data rate would require careful control of fiber parameters and source spectral width. Similar data rates could possibly also be obtained by using wavelength division multiplexing (WDM) in *dispersion-flattened* fibers, which have complicated index profiles which provide low dispersion over a range of wavelengths [9, 71]. However, such systems would require complicated electronic repeaters which may not be economical for systems which must span long distances.

In 1972 Hasegawa and Tappert of AT&T Bell Laboratories first proposed that nonlinear optical fibers be used as signal transmission media, with the signals transmitted in the form of soliton pulses [1]. Since dispersion can never be totally eliminated by design of the fiber in the linear regime, the fiber would be operated in a nonlinear regime so that nonlinearity would cancel the effect of dispersion. The signal propagation would then be described by a nonlinear dispersive partial differential equation (a form of the nonlinear Schrödinger equation), and the signal pulses would be envelope solitons which would propagate without distortion. These researchers calculated that the length-rate product could be increased beyond the limit for linear systems by using nonlinear systems [1, 2].

Characteristics of soliton propagation would also allow the use of simpler, all-optical repeaters which would not limit data rate like conventional electronic repeaters. Thus the use of solitons is especially attractive for undersea optical cable systems.

Optical communications systems using soliton pulses will be discussed in greater detail in Chapter 6. In the next two chapters, we will consider the problem of modeling a nonlinear optical fiber, and solving the equations which result from the modeling.

4.0 Mathematical Model of a Nonlinear Optical Fiber

It was shown in Chapter 2 that optical fibers are dispersive and can be nonlinear. In the last chapter several ways of exploiting these qualities were presented.

A long-standing impediment to exploiting the properties of nonlinear optical fibers has been the difficulty of nonlinear mathematics, both in *modeling* (the derivation of useful equations that govern the system), and the *solution* of the model equations.

It is generally agreed that some form of the nonlinear Schrödinger (NLS) equation is the best model describing the evolution of the envelope of a pulse propagating in a nonlinear optical fiber. However, several different derivations are used, resulting in equations with different terms and different coefficients. Controversy has arisen between a few research groups over the importance of different terms in the derivation, and it was only recently that Kodama and Hasegawa published the first rigorous derivation for a single-mode fiber [15].

In this chapter, coupled-mode theory will be used to develop a mathematical model of envelope soliton propagation in an optical fiber. While the coupled-mode approach is not as rigorous as the paper just mentioned, and does not supply all the terms of the more rigorous theory, the coupled-mode approach is sufficient for many purposes and has the advantages of being more understandable, and extendable to cover multimode fibers [33, 34].

In the first section of this chapter, a general coupled-mode theory will be developed and explained, following the work of Marcuse [35, 36]. The theory will result in equations for the fields in a non-ideal fiber in terms of the known modes of an ideal fiber. The theory will be general and applicable to any dielectric waveguide with arbitrary index of refraction distribution differing from the index distribution of an ideal waveguide with known modes. Page 35 shows a summary of the coupled-mode equations.

In the second section of this chapter, the coupled-mode theory will be specialized to the case of a nonlinear optical fiber, following the work of Crosignani *et al.* [37]. The theory will be simplified to the case of a single mode fiber in order to compare the resulting equation to an equation derived by other means.

4.1 Derivation of Ideal Mode Equations

We assume that the fields of the waveguide to be studied (here, the nonlinear fiber) can be expressed as expansions in terms of some known reference modes. Here the reference modes will be chosen to be the modes that would propagate in an ideal (linear) optical fiber.

We begin by applying Maxwell's equations to a general fiber optic waveguide in order to derive a set of equations which the transverse and longitudinal components of the electric and magnetic fields must satisfy. These equations will then be used to determine the modes which would propagate in an ideal (linear) fiber, and will also later be applied to the non-ideal (nonlinear) fiber. Therefore, we must do an analysis that is sufficiently general to apply to the nonlinear fiber. For this purpose, we assume a fiber optic waveguide with *arbitrary* index of refraction $n(\bar{r}, \omega)$, and take the permeability and permittivity to be

$$\mu = \mu_o \quad [4.1.1]$$

$$\varepsilon = \varepsilon_o n^2(\bar{r}, \omega) \quad [4.1.2]$$

respectively. Then Maxwell's curl equations in the frequency domain are

$$\nabla \times \bar{E}(\bar{r}, \omega) = -j\omega\mu_o \bar{H}(\bar{r}, \omega) \quad [4.1.3]$$

$$\nabla \times \bar{H}(\bar{r}, \omega) = j\omega\varepsilon_o n^2(\bar{r}, \omega)\bar{E}(\bar{r}, \omega) \quad [4.1.4]$$

Using $\hat{a}_z$ as the unit vector in the z (longitudinal) direction, we can decompose $\bar{E}$, $\bar{H}$, and ∇ into their transverse and longitudinal parts:

$$\bar{E}(\bar{r}, \omega) = \bar{E}_t(\bar{r}, \omega) + \bar{E}_z(\bar{r}, \omega) \quad [4.1.5]$$

$$\bar{H}(\bar{r}, \omega) = \bar{H}_t(\bar{r}, \omega) + \bar{H}_z(\bar{r}, \omega) \quad [4.1.6]$$

$$\nabla = \nabla_t + \hat{a}_z \frac{\partial}{\partial z} \quad [4.1.7]$$

Substituting [4.1.5], [4.1.6], and [4.1.7] into [4.1.3] and [4.1.4], we obtain

$$\left[\nabla_t + \hat{a}_z \frac{\partial}{\partial z}\right] \times [\bar{E}_t + \bar{E}_z] = -j\omega\mu_o[\bar{H}_t + \bar{H}_z] \quad [4.1.8]$$

$$\left[\nabla_t + \hat{a}_z \frac{\partial}{\partial z}\right] \times [\bar{H}_t + \bar{H}_z] = j\omega\varepsilon_o n^2[\bar{E}_t + \bar{E}_z] \quad [4.1.9]$$

After expanding [4.1.8],

$$-j\omega\mu_o[\bar{H}_t + \bar{H}_z] = (\nabla_t \times \bar{E}_t) + (\nabla_t \times \bar{E}_z) + (\hat{a}_z \times \frac{\partial}{\partial z}\bar{E}_t) + (\hat{a}_z \times \frac{\partial}{\partial z}\bar{E}_z) \quad [4.1.10]$$

we observe that the first term on the right-hand side is a longitudinal component, the next two terms are transverse components, and the fourth term is zero. So for the transverse component of $\overline{H}$ we have

$$-j\omega\mu_o\overline{H}_t(\bar{r},\omega) = \nabla_t \times \overline{E}_z(\bar{r},\omega) + \hat{a}_z \times \frac{\partial}{\partial z}\overline{E}_t(\bar{r},\omega) \qquad [4.1.11]$$

For the longitudinal component of $\overline{H}$ we have

$$-j\omega\mu_o\overline{H}_z(\bar{r},\omega) = \nabla_t \times \overline{E}_t(\bar{r},\omega) \qquad [4.1.12]$$

or

$$\overline{H}_z(\bar{r},\omega) = \left(\frac{-1}{j\omega\mu_o}\right)\nabla_t \times \overline{E}_t(\bar{r},\omega) \qquad [4.1.13]$$

So the longitudinal component of $\overline{H}$ can be found if the transverse component of $\overline{E}$ is known.

Similarly expanding [4.1.9] gives

$$j\omega\varepsilon_o n^2[\overline{E}_t + \overline{E}_z] = (\nabla_t \times \overline{H}_t) + (\nabla_t \times \overline{H}_z) + (\hat{a}_z \times \frac{\partial}{\partial z}\overline{H}_t) + (\hat{a}_z \times \frac{\partial}{\partial z}\overline{H}_z) \qquad [4.1.14]$$

By identifying the transverse and longitudinal parts, we have the equations for $\overline{E}$:

$$j\omega\varepsilon_o n^2\overline{E}_t(\bar{r},\omega) = \nabla_t \times \overline{H}_z(\bar{r},\omega) + \hat{a}_z \times \frac{\partial}{\partial z}\overline{H}_t(\bar{r},\omega) \qquad [4.1.15]$$

$$\overline{E}_z(\bar{r},\omega) = \left(\frac{1}{j\omega\varepsilon_o n^2}\right)\nabla_t \times \overline{H}_t(\bar{r},\omega) \qquad [4.1.16]$$

We now eliminate the longitudinal components $\overline{H}_z$ and $\overline{E}_z$ in [4.1.15] and [4.1.11] by using [4.1.13] and [4.1.16]:

$$-j\omega\mu_o\overline{H}_t(\bar{r},\omega) = \nabla_t \times \left[\frac{1}{j\omega\varepsilon_o n^2}\nabla_t \times \overline{H}_t(\bar{r},\omega)\right] + (\hat{a}_z \times \frac{\partial}{\partial z}\overline{E}_t(\bar{r},\omega)) \qquad [4.1.17]$$

$$j\omega\varepsilon_o n^2(\bar{r},\omega)\bar{E}_t(\bar{r},\omega) = \nabla_t \times \left[\frac{-1}{j\omega\mu_o}\nabla_t \times \bar{E}_t(\bar{r},\omega)\right] + (\hat{a}_z \times \frac{\partial}{\partial z}\bar{H}_z(\bar{r},\omega)) \qquad [4.1.18]$$

Keeping in mind that the index of refraction is a function of $\bar{r}$, we obtain

$$-j\omega\mu_o\bar{H}_t(\bar{r},\omega) = \left(\frac{1}{j\omega\varepsilon_o}\right)\nabla_t \times \left[(\frac{1}{n^2})\nabla_t \times \bar{H}_t(\bar{r},\omega)\right] + \hat{a}_z \times \frac{\partial}{\partial z}\bar{E}_t(\bar{r},\omega) \qquad [4.1.19]$$

$$j\omega\varepsilon_o n^2(\bar{r},\omega)\bar{E}_t(\bar{r},\omega) = \left(\frac{-1}{j\omega\mu_o}\right)\nabla_t \times \nabla_t \times \bar{E}_t(\bar{r},\omega) + \hat{a}_z \times \frac{\partial}{\partial z}\bar{H}_t(\bar{r},\omega) \qquad [4.1.20]$$

Equations [4.1.19] and [4.1.20] are general and apply to any optical fiber, since they are merely restatements of Maxwell's equations for the transverse fields of the fiber. Since we must first find the modes of the ideal fiber, we now specialize and let the index of refraction be linear (independent of $\bar{E}$), and homogeneous in the longitudinal direction:

$$n(\bar{r},\omega) = n_o(x,y,\omega) \qquad [4.1.21]$$

For this choice of linear n we assume that the solutions of [4.1.19] and [4.1.20] are *modes* of the form

$$\bar{H}_t(\bar{r},\omega) = \bar{H}_{mt}(\bar{\rho})e^{-j\beta_m z} \qquad [4.1.22]$$

$$\bar{E}_t(\bar{r},\omega) = \bar{E}_{mt}(\bar{\rho})e^{-j\beta_m z} \qquad [4.1.23]$$

where $\bar{\rho}$ is the transverse coordinate (x,y). Substituting [4.1.22] and [4.1.23] into [4.1.19] and [4.1.20] we obtain

$$-j\omega\mu_o\bar{H}_{mt} = (\frac{1}{j\omega\varepsilon_o})\nabla_t \times \left[\frac{1}{n_o^2}\nabla_t \times \bar{H}_{mt}\right] - j\beta_m(\hat{a}_z \times \bar{E}_{mt}) \qquad [4.1.24]$$

$$j\omega\varepsilon n_o^2\bar{E}_{mt} = \frac{-1}{j\omega\mu_o}\nabla_t \times \nabla_t \times \bar{E}_{mt} - j\beta_m(\hat{a}_z \times \bar{H}_{mt}). \qquad [4.1.25]$$

In the last two equations (and in the next two), the z dependence, entering through the term $e^{-j\beta z}$, has been canceled out so that the equations are independent of z. The propagation constants β_m are the eigenvalues of the last two equations, and the quantities $\overline{H}_{mt}(\overline{\rho})$ and $\overline{E}_{mt}(\overline{\rho})$ are called the *transverse modal eigenfunctions.* The subscript m is shorthand for the pair of integers needed to denote the mode numbers of an optical fiber, and the subscript t denotes 'transverse', as before. The eigenvalue problem represented by equations [4.1.24] and [4.1.25] implicitly contains boundary conditions in the specification of $n_o(\overline{\rho}, \omega)$ for all $\overline{\rho}$. In general, the last two equations represent an eigenvalue problem which is difficult, or impossible, to solve exactly, and which is not considered here. In practice, we would use the well-known modal function solutions which have been tabulated for several specific cases, such as the step-index profile. Therefore we assume that the eigenpair problem is solved and that the modal functions $\overline{H}_{mt}$ and $\overline{E}_{mt}$ and the propagation constants β_m of the ideal fiber are known.

The longitudinal components of the ideal modes follow from the transverse components by using the ideal versions of equations [4.1.13] and [4.1.14]:

$$\overline{H}_{mz} = \frac{-1}{j\omega\mu_o}\nabla_t \times \overline{E}_{mt} \qquad [4.1.26]$$

$$\overline{E}_{mz} = \frac{1}{j\omega\varepsilon_o n_o^2}\nabla_t \times \overline{H}_{mt} \qquad [4.1.27]$$

We now know the modes of the ideal fiber, and next consider a non-ideal fiber with index profile n which differs from the ideal index n_o in some as yet unspecified way. We assume that the transverse fields $\overline{\mathbf{E}}_t(\overline{r}, \omega)$ and $\overline{\mathbf{H}}_t(\overline{r}, \omega)$ of the non-ideal fiber can be expressed as expansions in terms of the modes of the ideal fiber:

$$\overline{\mathbf{E}}_t(\overline{r}, \omega) = \sum_m a_m(z, \omega)\overline{E}_{mt}(\overline{\rho}) \qquad [4.1.28]$$

$$\overline{\mathbf{H}}_t(\bar{r}, \omega) = \sum_m b_m(z, \omega)\overline{H}_{mt}(\bar{\rho}) \qquad [4.1.29]$$

where the summation sign represents a summation over the discrete modes and an integration over the continuous (radiated) modes, i.e.,

$$\overline{\mathbf{E}}_t(\bar{r}, \omega) = \sum_{m=1}^{N} a_m(z, \omega)\overline{E}_{mt}(\bar{\rho}) + \int_0^{\infty} a_s(z, \omega)\overline{E}_{st}(\bar{\rho}, \omega)ds \qquad [4.1.30]$$

$$\overline{\mathbf{H}}_t(\bar{r}, \omega) = \sum_{m=1}^{N} b_m(z, \omega)\overline{H}_{mt}(\bar{\rho}) + \int_0^{\infty} b_s(z, \omega)\overline{H}_{st}(\bar{\rho}, \omega)ds \qquad [4.1.31]$$

Again we note that equations [4.1.19] and [4.1.20] are sufficiently general to be satisfied by the fields of the non-ideal fiber. Therefore, we can substitute the modal expansions represented by [4.1.28] and [4.1.29] into [4.1.19] and [4.1.20] to obtain

$$-j\omega\mu_o\left[\sum_m b_m\overline{H}_{mt}\right] = \frac{1}{j\omega\varepsilon_o}\nabla_t \times \left[\frac{1}{n^2}\nabla_t \times \sum_m b_m\overline{H}_{mt}\right] + \hat{a}_z \times \frac{\partial}{\partial z}\left[\sum_m a_m\overline{E}_{mt}\right] \qquad [4.1.32]$$

$$j\omega\varepsilon_o n^2(\bar{r}, \omega)\left[\sum_m a_m\overline{E}_{mt}\right] = \frac{-1}{j\omega\mu_o}\nabla_t \times \left[\nabla_t \times \sum_m a_m\overline{E}_{mt}\right] + \hat{a}_z \times \frac{\partial}{\partial z}\left[\sum_m b_m\overline{H}_{mt}\right] \qquad [4.1.33]$$

Keeping in mind the meaning of the summation notation, we can express equations [4.1.32] and [4.1.33] as single summations:

$$\sum_m \left[-j\omega\mu_o b_m\overline{H}_{mt} - \frac{1}{j\omega\varepsilon_o}\nabla_t \times \left(\frac{1}{n^2}\nabla_t \times b_m\overline{H}_{mt}\right) - \hat{a}_z \times \frac{\partial}{\partial z}\left(a_m\overline{E}_{mt}\right)\right] = 0 \qquad [4.1.34]$$

$$\sum_m \left[j\omega\varepsilon_o n^2 a_m \overline{E}_{mt} + \frac{1}{j\omega\mu_o} \nabla_t \times \left(\nabla_t \times a_m \overline{E}_{mt} \right) - \hat{a}_z \times \frac{\partial}{\partial z} \left(b_m \overline{H}_{mt} \right) \right] = 0 \qquad [4.1.35]$$

Consider the final term in the lefthand side of [4.1.35]. We have

$$\hat{a}_z \times \frac{\partial}{\partial z} \left[b_m \overline{H}_{mt}(\overline{\rho}, \omega) \right] = \hat{a}_z \times \overline{H}_{mt} \frac{\partial}{\partial z} b_m$$

$$= \left(\frac{\partial}{\partial z} b_m \right) \left(\hat{a}_z \times \overline{H}_{mt} \right) \qquad [4.1.36]$$

So [4.1.35] becomes

$$\sum_m \left[j\omega\varepsilon_o n^2 a_m \overline{E}_{mt} + \frac{1}{j\omega\mu_o} \nabla_t \times \nabla_t \times a_m \overline{E}_{mt} - \left(\frac{\partial}{\partial z} b_m \right) \left(\hat{a}_z \times \overline{H}_{mt} \right) \right] = 0 \qquad [4.1.37]$$

But from [4.1.25],

$$-\frac{1}{j\omega\mu_o} \nabla_t \times \nabla_t \times \overline{E}_{mt} = j\omega\varepsilon_o n_o^2 \overline{E}_{mt} + j\beta_m (\hat{a}_z \times \overline{H}_{mt}) \qquad [4.1.38]$$

Substituting [4.1.38] into [4.1.37],

$$\sum_m \left[j\omega\varepsilon_o n^2 a_m \overline{E}_{mt} - a_m \left(j\omega\varepsilon_o n_o^2 \overline{E}_{mt} + j\beta_m (\hat{a}_z \times \overline{H}_{mt}) \right) - \left(\frac{\partial}{\partial z} b_m \right) \left(\hat{a}_z \times \overline{H}_{mt} \right) \right] = 0 \qquad [4.1.40]$$

This can be rewritten as

$$\sum_m \left\{\left[\frac{\partial}{\partial z} b_m + j\beta_m a_m\right](\hat{a}_z \times \overline{H}_{mt}) - j\omega\varepsilon_o (n^2 - n_o^2) a_m \overline{E}_{mt}\right\} = 0 \qquad [4.1.41]$$

This is one of the desired equations relating the transverse modal fields to the mode amplitudes. To obtain the other, consider [4.1.34]. By symmetry with [4.1.36], the last term of [4.1.34] is

$$\hat{a}_z \times \frac{\partial}{\partial z}(a_m \overline{E}_{mt}) = (\frac{\partial}{\partial z} a_m)(\hat{a}_z \times \overline{E}_{mt}) \qquad [4.1.42]$$

Substituting this into [4.1.34] gives

$$\sum_m \left\{\left[\frac{\partial}{\partial z} a_m + j\beta_m b_m\right](\hat{a}_z \times \overline{E}_{mt}) + \frac{1}{j\omega\varepsilon_o}\nabla_t \times \left[(n^{-2} - n_o^{-2})\nabla_t \times \overline{H}_{mt}\right]\right\} = 0 \qquad [4.1.43]$$

Equations [4.1.41] and [4.1.43] relate the modal fields $\overline{E}_{mt}$ and $\overline{H}_{mt}$ to their mode amplitudes a_m and b_m. We now want to derive equations which give the relationship between a mode with subscript m and another mode with subscript n (not to be confused with the index of refraction; n as a subscript will always denote a mode). To do this, we will use the orthogonality relationship between modes [35, 36]

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{a}_z \cdot (\overline{E}_{mt} \times \overline{H}^*_{mt}) dxdy = 2s_n \frac{\beta_m^*}{|\beta_m|} P \delta_{mn} \qquad [4.1.44]$$

where δ_{mn} is unity if m and n denote the same mode and zero otherwise. The superscript * indicates complex conjugation. The factor s_n is plus or minus one as necessary to keep the righthand side of [4.1.44] positive. P is the power carried by the mode. Equation [4.1.44] will be used in the next steps.

We take the scalar product of [4.1.41] with $\overline{E}^*_{nt}$ and integrate over the infinite cross section of the fiber:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_m \left\{\left[\frac{\partial}{\partial z}b_m + j\beta_m a_m\right](\hat{a}_z \times \overline{H}_{mt}) - j\omega\varepsilon_o(n^2 - n_o^2)a_m\overline{E}_{mt}\right\} \cdot \overline{E}^*_{nt}dxdy = 0 \qquad [4.1.45]$$

This can be rewritten as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_m \left\{\left[\frac{\partial}{\partial z}b_m + j\beta_m a_m\right]\overline{E}^*_{nt} \cdot (\hat{a}_z \times \overline{H}_{mt})\right\}dxdy =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_m \left\{j\omega\varepsilon_o(n^2 - n_o^2)a_m\overline{E}_{mt} \cdot \overline{E}^*_{nt}\right\}dxdy \qquad [4.1.46]$$

Using the identity

$$\overline{E}^*_{nt} \cdot (\hat{a}_z \times \overline{H}_{mt}) = \hat{a}_z \cdot (\overline{H}_{mt} \times \overline{E}^*_{nt}) = -\hat{a}_z \cdot (\overline{E}_{nt} \times \overline{H}^*_{mt}), \qquad [4.1.47]$$

and interchanging the order of summation and integration, we obtain

$$-\sum_m \left[\frac{\partial}{\partial z}b_m + j\beta_m a_m\right]\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{a}_z \cdot (\overline{E}_{nt} \times \overline{H}^*_{mt})dxdy =$$

$$= j\omega\varepsilon_o\sum_m a_m\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(n^2 - n_o^2)\overline{E}_{mt} \cdot \overline{E}^*_{nt}dxdy \qquad [4.1.48]$$

The integral on the lefthand side of [4.1.48] is recognizeable as the orthogonality relationship [4.1.44], so we may write

$$-\left[\frac{\partial}{\partial z}b_n + j\beta_n a_n\right]2s_n\frac{\beta_n^*}{|\beta_n|}P = j\omega\varepsilon_o\sum_m a_m\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(n^2 - n_o^2)\overline{E}_{mt}\cdot\overline{E}_{nt}^*dxdy \qquad [4.1.49]$$

or

$$\frac{\partial}{\partial z}b_n + j\beta_n a_n = -\frac{j\omega\varepsilon_o}{2s_nP}\frac{|\beta_n|}{\beta_n^*}\sum_m a_m\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(n^2 - n_o^2)\overline{E}_{mt}\cdot\overline{E}_{nt}^*dxdy \qquad [4.1.50]$$

We can write this as

$$\frac{\partial}{\partial z}b_n + j\beta_n a_n = 2\sum_m \overline{K}_{mn}a_m \qquad [4.1.51]$$

where we have identified the coupling coefficient

$$\overline{K}_{mn}(z,\omega) = \frac{\omega\varepsilon_o}{j4s_nP}\frac{|\beta_n|}{\beta_n^*}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(n^2 - n_o^2)\overline{E}_{mt}\cdot\overline{E}_{nt}^*dxdy. \qquad [4.1.52]$$

To derive the other set of coupling coefficients, we take the dot product of [4.1.43] with $\overline{H}_{nt}^*$ and integrate over the infinite cross section:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_m \left\{\left[\frac{\partial}{\partial z}a_m + j\beta_m b_m\right]\overline{H}_{nt}^*\cdot(\hat{a}_z\times\overline{E}_{mt}) + \right.$$

$$\left. + \frac{1}{j\omega\varepsilon_o}b_m\overline{H}_{nt}^*\cdot\nabla_t\times\left[(n^{-2} - n_o^{-2})\nabla_t\times\overline{H}_{mt}\right]\right\}dxdy = 0 \qquad [4.1.53]$$

We can use the identity

$$\overline{H}_{nt}^{*} \cdot (\hat{a}_z \times \overline{E}_{mt}) = \hat{a}_z \cdot (\overline{E}_{mt} \times \overline{H}_{nt}^{*}) \qquad [4.1.54]$$

to obtain

$$\sum_m \left\{ \left[\frac{\partial}{\partial z} a_m + j\beta_m \right] \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{a}_z \cdot (\overline{E}_{mt} \times \overline{H}_{nt}^{*}) dxdy \right\} =$$

$$= - \sum_m \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{j\omega\varepsilon_o} b_m \overline{H}_{nt}^{*} \cdot \nabla_t \times \left[(n^2 - n_o^2) \nabla_t \times \overline{H}_{mt} \right] dxdy \qquad [4.1.55]$$

By the orthogonality relation [4.1.44], this becomes

$$\left[\frac{\partial}{\partial z} a_n + j\beta_n \right] 2s_n \frac{\beta_n^{*}}{|\beta_n|} P$$

$$= - \sum_m \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{j\omega\varepsilon_o} b_m \overline{H}_{nt}^{*} \cdot \nabla_t \times \left[(n^{-2} - n_o^{-2}) \nabla_t \times \overline{H}_{mt} \right] dxdy \qquad [4.1.56]$$

We can choose the coupling coefficients as

$$\overline{R}_{mn}(z, \omega) = - \frac{1}{j4\omega\varepsilon_o P s_n} \frac{|\beta_n|}{\beta_n^{*}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{H}_{nt}^{*} \cdot \nabla_t \times \left[(n^{-2} - n_o^{-2}) \nabla_t \times \overline{H}_{mt} \right] dxdy \qquad [4.1.57]$$

Then [4.1.56] becomes

$$\frac{\partial}{\partial z}a_n + j\beta_n b_n = 2\sum_m \overline{R}_{mn} b_m \qquad [4.1.58]$$

The coupling coefficients given by [4.1.57] are perfectly valid, but it is desireable to manipulate them into a form more like that appearing in [4.1.52]. To do this, we substitute [4.1.27] into [4.1.57]:

$$\overline{R}_{mn}(z,\omega) = -\frac{1}{4Ps_n}\frac{|\beta_n|}{\beta_n^*}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{H}_{nt}^* \cdot \nabla_t \times \left[n_o^2(n^{-2} - n_o^{-2})\overline{E}_{mz}\right]dxdy \qquad [4.1.59]$$

This can be rewritten as

$$\overline{R}_{mn}(z,\omega) = \frac{1}{4Ps_n}\frac{|\beta_n|}{\beta_n^*}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{H}_{nt}^* \cdot \nabla_t \times \left[\frac{1}{n^2}(n^2 - n_o^2)\overline{E}_{mz}\right]dxdy \qquad [4.1.60]$$

By writing out the vector expression in [4.1.60] in (x,y) -coordinates and performing the integration, we can show that [4.1.60] is equivalent to

$$\overline{R}_{mn}(z,\omega) = \frac{1}{4Ps_n}\frac{|\beta_n|}{\beta_n^*}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{n^2}(n^2 - n_o^2)\overline{E}_{mz} \cdot \left(\nabla_t \times \overline{H}_{nt}^*\right)dxdy \qquad [4.1.61]$$

But by [4.1.27],

$$\nabla_t \times \overline{H}_{nt}^* = -j\omega\varepsilon_o n_o^2 \overline{E}_{nz}^* \qquad [4.1.63]$$

Substituting this into [4.1.62], we obtain an expression for the $\overline{R}_{mn}$ coupling coefficients which is of the same form as equation [4.1.53]:

$$\overline{R}_{mn}(z,\omega) = \frac{\omega\varepsilon_o}{j4Ps_n}\,\frac{|\beta_n|}{\beta_n^*}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{n_o^2}{n^2}\right)(n^2 - n_o^2)(\overline{E}_{mz}\cdot\overline{E}_{nz})dxdy \qquad [4.1.64]$$

Summary

In terms of the mode amplitudes a_m and b_m, the coupled-mode equations are

$$\frac{\partial}{\partial z}b_n + j\beta_n a_n = 2\sum_m \overline{K}_{mn}a_m \qquad [4.1.51]$$

$$\frac{\partial}{\partial z}a_n + j\beta_n b_n = 2\sum_m \overline{R}_{mn}b_m \qquad [4.1.58]$$

with the coupling coefficients $\overline{K}_{mn}$ and $\overline{R}_{mn}$ given by

$$\overline{K}_{mn}(z,\omega) = \frac{\omega\varepsilon_o}{j4s_nP}\,\frac{|\beta_n|}{\beta_n^*}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(n^2 - n_o^2)\overline{E}_{mt}\cdot\overline{E}_{nt}^*dxdy \qquad [4.1.52]$$

$$\overline{R}_{mn}(z,\omega) = \frac{\omega\varepsilon_o}{j4Ps_n}\,\frac{|\beta_n|}{\beta_n^*}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{n_o^2}{n^2}\right)(n^2 - n_o^2)(\overline{E}_{mz}\cdot\overline{E}_{nz})dxdy \qquad [4.1.64]$$

Note that the $\overline{K}_{mn}$ coupling coefficients are given in terms of the transverse components of the ideal modes, and the $\overline{R}_{mn}$ coefficients are given in terms of the longitudinal components of the ideal modes.

4.1.1 Forward- and backward-traveling waves

We now convert [4.1.51] and [4.1.58] into coupled-mode equations for forward- and backward-traveling waves. To gain some insight into how to do this, we first consider the situation of no coupling, i.e.,

$$\overline{K}_{mn} = \overline{R}_{mn} = 0 \qquad [4.1.65]$$

Then the coupled-mode equations [4.1.51] and [4.1.58] become

$$\frac{\partial}{\partial z} a_n + j\beta_n b_n = 0 \qquad [4.1.66]$$

$$\frac{\partial}{\partial z} b_n + j\beta_n a_n = 0 \qquad [4.1.67]$$

Taking the derivative of each and substituting into the other gives the pair of second-order equations:

$$\frac{\partial^2}{\partial z^2} a_n + \beta_n^2 a_n = 0 \qquad [4.1.68]$$

$$\frac{\partial^2}{\partial z^2} b_n + \beta_n^2 b_n = 0 \qquad [4.1.69]$$

These equations have the solutions

$$a_n = c_n^+ e^{-j\beta_n z} + c_n^- e^{j\beta_n z} \equiv a_n^+ + a_n^- \qquad [4.1.70]$$

$$b_n = c_n^+ e^{-j\beta_n z} - c_n^- e^{j\beta_n z} \equiv a_n^+ - a_n^- \qquad [4.1.71]$$

This shows that for the case of no coupling the solution is a superposition of forward and backward traveling waves. We now assume that for the case of nonzero coupling the solution can likewise be separated into forward- and backward-traveling waves. We introduce the transformations represented by [4.1.70] and [4.1.71] into [4.1.51] and [4.1.58]:

$$\frac{\partial}{\partial z}[a_n^+ + a_n^-] + j\beta_n[a_n^+ - a_n^-] = 2\sum_m \overline{R}_{mn}[a_m^+ - a_m^-] \qquad [4.1.72]$$

$$\frac{\partial}{\partial z}[a_n^+ - a_n^-] + j\beta_n[a_n^+ + a_n^-] = 2\sum_m \bar{K}_{mn}[a_m^+ + a_m^-] \qquad [4.1.73]$$

Adding [4.1.72] to [4.1.73] gives

$$2\frac{\partial}{\partial z}a_n^+ + j2\beta_n a_n^+ = 2\sum_m \left[\bar{R}_{mn}(a_m^+ - a_m^-) + \bar{K}_{mn}(a_m^+ + a_m^-)\right] \qquad [4.1.74]$$

Subtracting [4.1.72] from [4.1.73] gives

$$2\frac{\partial}{\partial z}a_n^- - j2\beta_n a_n^- = 2\sum_m \left[\bar{R}_{mn}(a_m^+ - a_m^-) - \bar{K}_{mn}(a_m^+ + a_m^-)\right] \qquad [4.1.75]$$

We can write this in the form

$$\frac{\partial}{\partial z}a_n^+ = -j\beta_n a_n^+ + \sum_m \left[\bar{K}_{mn}^{++} a_m^+ + \bar{R}_{mn}^{+-} a_m^-\right] \qquad [4.1.76]$$

$$\frac{\partial}{\partial z}a_n^+ = j\beta_n a_n^+ + \sum_m \left[\bar{K}_{mn}^{-+} a_m^+ + \bar{R}_{mn}^{--} a_m^-\right] \qquad [4.1.77]$$

where

$$\bar{K}_{mn}^{pq} = p\bar{K}_{mn} + q\bar{R}_{mn} \qquad [4.1.78]$$

The symbols p and q denote + or - as superscripts and + 1 or -1 as factors. Finally, we note that a_n^+ and a_n^- are rapidly varying functions of z, since

$$a_n^+ = c_n^+ e^{-j\beta_n z} \qquad [4.1.79]$$

$$a_n^- = c_n^- e^{j\beta_n z} \qquad [4.1.80]$$

We can introduce these substitutions into [4.1.76] and [4.1.77] to find modal equations in terms of the slowly-varying functions c_n^+ and c_n^- : Replacing [4.1.79] and [4.1.80] into [4.1.76]:

$$\frac{\partial}{\partial z}[c_n^+ e^{-j\beta_n z}] = -j\beta_n[c_n^+ e^{-j\beta_n z}] + \sum_m \left\{\overline{K}_{mn}^{++}[c_m^+ e^{-j\beta_m z}] + \overline{K}_{mn}[c_m^- e^{+j\beta_m z}]\right\} \qquad [4.1.81]$$

But

$$\frac{\partial}{\partial z}[c_n^+ e^{-j\beta_n z}] = e^{-j\beta_n z}[\frac{\partial}{\partial z}c_n^+ - j\beta_n c_n^+] \qquad [4.1.82]$$

Substituting [4.1.82] into [4.1.81], we obtain

$$e^{-j\beta_n z}[\frac{\partial}{\partial z}c_n^+ - j\beta_n c_n^+] = -j\beta_n[c_n^+ e^{-j\beta_n z}] + \sum_m \left\{\overline{K}_{mn}^{++}[c_m^+ e^{-j\beta_m z}] + \overline{K}_{mn}^{+-}[c_m^- e^{j\beta_m z}]\right\} \qquad [4.1.83]$$

or

$$\frac{\partial}{\partial z}c_n^+ = \sum_m \left\{\overline{K}_{mn}^{++} c_m^+ e^{j(\beta_n - \beta_m)z} + \overline{K}_{mn}^{+-} c_m^- e^{-j(\beta_n + \beta_m)z}\right\} \qquad [4.1.84]$$

Similarly, replacing [4.1.79] and [4.1.80] into [4.1.77] results in

$$\frac{\partial}{\partial z}c_n^- = \sum_m \left\{\overline{K}_{mn}^{-+} c_m^+ e^{-j(\beta_n + \beta_m)z} + \overline{K}_{mn}^{--} c_m^- e^{-j(\beta_n - \beta_m)z}\right\} \qquad [4.1.85]$$

Summary of coupled-mode equations

Equations [4.1.84] and [4.1.85], repeated here:

$$\frac{\partial}{\partial z}c_n^+(z,\omega) = \sum_m \left\{ \overline{K}_{mn}^{++} c_m^+ e^{j(\beta_n - \beta_m)z} + \overline{K}_{mn}^{+-} c_m^- e^{-j(\beta_n + \beta_m)z} \right\} \qquad [4.1.84]$$

$$\frac{\partial}{\partial z}c_n^-(z,\omega) = \sum_m \left\{ \overline{K}_{mn}^{-+} c_m^+ e^{-j(\beta_n + \beta_m)z} + \overline{K}_{mn}^{--} c_m^- e^{-j(\beta_n - \beta_m)z} \right\} \qquad [4.1.85]$$

are infinite sets of partial differential equations for the mode amplitudes c_n^+ and c_n^- in terms of all the other amplitudes and the coupling coefficients, which can be expressed compactly as

$$\begin{aligned} \overline{K}_{mn}^{pq} &= p\overline{K}_{mn} + q\overline{R}_{mn} \\ &= \frac{\omega\varepsilon_o}{j4s_nP}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(n^2 - n_o^2)\left[\left(\frac{|\beta_n|}{\beta_n^p}\right)\overline{E}_{nt}^{p*}\cdot\overline{E}_{mt}^{q} + \left(\frac{n_o^2}{n^2}\right)\left(\frac{|\beta_n|}{\beta_n^{p*}}\right)\overline{E}_{nz}^{p*}\cdot\overline{E}_{mz}^{q}\right]dxdy \end{aligned} \qquad [4.1.90]$$

where

$$\beta_n^p = p\beta_n$$

$$\overline{E}_{nt}^- = \overline{E}_{nt}^+ = \overline{E}_{nt} \qquad [4.1.91]$$

$$\overline{E}_{nz}^- = -\overline{E}_{nz}^+ = -\overline{E}_{nz}.$$

Two comments should be made about the general coupled mode equations just derived.

1. The theory is exact, since no simplifying assumptions have been made. The equations derived amount to a restatement of Maxwell's equations. However, as was already pointed out, the problem of determining the transverse modal eigenfunctions given an arbitrary refractive index profile may be difficult or impossible, limiting the usefulness of the coupled-mode theory to refractive index profiles for which the eigenfunctions are known.

2. The work thus far has been entirely in the frequency domain, and the coupling coefficients and mode amplitudes are functions of z and ω only. The importance of this point will become clear in the next section.

4.2 Coupled Mode Theory Applied to a Nonlinear Optical Fiber

In the last section, Marcuse's general coupled-mode equations were derived using the modes of an ideal fiber. We now specialize and take the fiber to be studied to be a nonlinear one, for which the index of refraction is given by

$$n^2(\bar{r}, \omega) = n_o^2(\bar{r}, \omega) + n_2^2 |\overline{\mathrm{E}}(\bar{r}, t)|^2 \qquad [4.2.1]$$

where $\overline{\mathrm{E}}(\bar{r}, t)$ is the *time*-domain electric field in the nonlinear fiber. This expression can be derived from physical considerations [34]. Using the results of the previous section, we obtain the coupling coefficients from [4.1.90]:

$$\overline{K}_{mn}^{pq}(z, \omega) = \frac{\omega \varepsilon_o}{j4s_n P} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} n_2^2 |\overline{\mathrm{E}}(\bar{r}, t)|^2 \left[\left(\frac{|\beta_n|}{\beta_n^p} \right) \overline{E}_{nt}^{p*}(\bar{\rho}) \cdot \overline{E}_{mt}^{q}(\bar{\rho}) + \right.$$

$$\left. + \left(\frac{n_o^2}{n_o^2 + n_2^2 |\overline{\mathrm{E}}(\bar{r}, t)|^2} \right) \left(\frac{|\beta_n|}{\beta_n^{p*}} \right) \overline{E}_{nz}^{p*}(\bar{\rho}) \cdot \overline{E}_{mz}^{q}(\bar{\rho}) \right] dxdy. \qquad [4.2.2]$$

The coupled-mode equations for the mode amplitudes are [4.1.88] and [4.1.89].

We note immediately that we have violated one of the assumptions of the coupled-mode theory by introducing an index of refraction profile that is a function of time. This step is justified if the time variations of the instantaneous intensity $|\overline{E}(\bar{r}, t)|^2$ are slow compared to the period of the carrier [15, 34]. In other words, we consider the field to be a continuous wave of frequency ω_o with slow amplitude modulation giving a bandwidth $\delta\omega$ such that

$$\frac{\delta\omega}{\omega_o} << 1. \qquad [4.2.3]$$

For operation at a pulse center wavelength λ_o on the order of 1 μm, we obtain ω_o on the order of 10^{15} radians/second, so we require $\delta\omega << 10^{15}$ radians/second. This gives a pulse width $\tau >> 10^{-15}$ second. Therefore, this theory is valid only for pulses of width much greater than a femtosecond. Under this condition, the electric field can be expressed in terms of the slowly-varying envelope $\psi(z,t)$:

$$\overline{E}(\bar{r}, t) = E(\overline{\rho})e^{j[\omega_o t - \beta(\omega_o)z]}\psi(z,t). \qquad [4.2.4]$$

Equations [4.2.4], [4.1.88], and [4.1.89] represent the complete coupled-mode theory for the nonlinear optical fiber under the slowly-varying envelope condition [4.2.3]. Unfortunately, the completeness of these equations renders them unsolveable. Several simplifying assumptions must now be made in order to obtain equations which can be solved. In the next section, three assumptions will be made and justified, and the resulting simplified system of equations will be presented.

Assumption

The backward-traveling fields in the fiber are negligible, so that $\overline{E}_{nz}^{-} = \overline{E}_{nt}^{-} = 0$; this causes the three coupling coefficients with a (-) as a superscript to be zero.

Justification

The backward-traveling fields will be small compared to the forward-traveling fields if the refractive index profile varies smoothly with the longitudinal coordinate z. Since we have assumed a fiber which is homogeneous in the longitudinal direction, this assumption is justified.

Simplified equations

The only surviving coupling coefficient is $\overline{K}_{mn}^{++}$, which we will hereafter simply call $\overline{K}_{mn}$. The coupled-mode equations for the positive-going modes are given as

$$\frac{\partial}{\partial z}c_n(z, \omega) = \sum_{m} \{\overline{K}_{mn}c_m(z, \omega)e^{j(\beta_n - \beta_m)z}\} \qquad [4.2.5]$$

with the coupling coefficients given by

$$\overline{K}_{mn}(z, \omega) = \frac{\omega\varepsilon_o}{j4s_nP}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} n_2^2\,|\,\overline{\mathrm{E}}(\bar{r}, t)\,|^2\left[\left(\frac{|\,\beta_n\,|}{\beta_n}\right)\overline{E}_{nt}^{*}(\overline{\rho}) \cdot \overline{E}_{mt}(\overline{\rho}) + \right.$$

$$\left. + \left(\frac{n_o^2}{n_o^2 + n_2^2\,|\,\overline{\mathrm{E}}(\bar{r}, t)\,|^2}\right)\left(\frac{|\,\beta_n\,|}{\beta_n^{*}}\right)\overline{E}_{nz}^{*}(\overline{\rho}) \cdot \overline{E}_{mz}(\overline{\rho})\right]dxdy \qquad [4.2.6]$$

Assumption

The fiber is weakly guiding, so that the longitudinal components of all the fields involved are negligible compared to the transverse components.

Justification

Nearly all optical fibers are weakly guiding, *i.e.,* have small core-cladding index difference. For single-mode fibers this is required in order to have manageable core sizes. Since we will specialize to the case of a single-mode fiber, this assumption is not restrictive.

Simplified equations

The equation for the mode amplitudes remains the same:

$$\frac{\partial}{\partial z}c_n(z,\omega) = \sum_m \left\{\overline{K}_{mn}c_m(z,\omega)e^{j(\beta_n - \beta_m)z}\right\} \qquad [4.2.7]$$

but the coupling coefficients are simplified considerably:

$$\overline{K}_{mn}(z,\omega) = \frac{\omega\varepsilon_o}{j4s_nP}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} n_2^2\,|\mathbf{E}_t(\bar{r},t)|^2\left[\left(\frac{|\beta_n|}{\beta_n}\right)\overline{E}_{nt}{}^*(\bar{\rho})\cdot\overline{E}_{mt}(\bar{\rho})\right]dxdy. \qquad [4.2.8]$$

Note that the total field in the nonlinear fiber has been replaced by its transverse component.

Assumption

There is only one mode propagating through the fiber.

Justification

This is tantamount to assuming a single-mode fiber. A step-index optical fiber will be single-moded if the normalized frequency V satisfies [38]

$$V \equiv \frac{2\pi a}{\lambda}\sqrt{n_1^2 - n_2^2} < 2.405 \qquad [4.2.9]$$

where a is the core radius, and n_1 and n_2 are the core and cladding indices, respectively.

Simplified equations

The single mode amplitude coefficient, for the single mode traveling in the $+z$ direction, is now called simply $c(z, \omega)$. The equation for this coefficient is

$$\frac{\partial}{\partial z}c(z, \omega) = \overline{K}c(z, \omega) \qquad [4.2.10]$$

and the "coupling" coefficient is now called simply $\overline{K}$:

$$\overline{K}(z, \omega) = \frac{\omega \varepsilon_o}{j4P}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} n_2^2 |\overline{\mathbf{E}}_t(\bar{r}, t)|^2 |\overline{E}_t(\bar{\rho})|^2 dxdy. \qquad [4.2.11]$$

Note again that this coefficient is supposed to be a function of z and ω only, but is actually also a function of t. We must keep in mind that this equation is subject to the slowly-varying envelope condition of Eq. [4.2.3].

This simplified model will be pursued further in the next section.

Derivation of Pulse Envelope Equation for a Single-mode Fiber

The coupled-mode equations just derived apply to single-mode fibers. These equations were used by Crosignani *et al.* [37] to derive an equation describing soliton propagation in nonlinear optical fibers. This work will be redone here, with the missing steps of Ref. 37 supplied and explained.

Under the assumptions just explained, the frequency-domain electric field can be obtained from [4.1.28] and [4.1.70] as

$$\overline{\mathbf{E}}_t(\bar{r}, \omega) = c(z, \omega)\overline{E}_t(\bar{\rho})e^{-j\beta z}. \qquad [4.2.12]$$

The frequency-domain field is of course related to the time-domain field by the inverse Fourier transform. We choose a symmetric transform pair, so that

$$\overline{\mathbf{E}}(\bar{r}, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{\mathbf{E}}(\bar{r}, \omega)e^{j\omega t}d\omega \qquad [4.2.13]$$

$$\overline{\mathbf{E}}(\bar{r}, \omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{\mathbf{E}}(\bar{r}, t)e^{-j\omega t}d\omega. \qquad [4.2.14]$$

So, using the inverse Fourier transform [4.1.13]

$$\overline{\mathbf{E}}_t(\bar{r}, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} c(z, \omega)\overline{E}_t(\bar{\rho})e^{j(\omega t-\beta z)}d\omega. \qquad [4.2.15]$$

Substituting into [4.2.11],

$$\overline{K}(z, \omega) = \frac{\omega\varepsilon_o n_2^2}{j4P(2\pi)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty} c(z, \omega)\overline{E}_t(\bar{\rho})e^{j(\omega t-\beta z)}d\omega\right|^2 |\overline{E}_t(\bar{\rho})|^2 dxdy \qquad [4.2.16]$$

or

$$\overline{K}(z,\omega) = \frac{\omega \varepsilon_o n_2^2}{j4P(2\pi)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} c(z,\omega) e^{j(\omega t - \beta z)} d\omega \right|^2 |\overline{E}_t(\overline{\rho})|^4 dxdy \qquad [4.2.17]$$

or

$$\overline{K}(z,\omega) = \frac{\omega \varepsilon_o n_2^2}{j4P(2\pi)} \left| \int_{-\infty}^{\infty} c(z,\omega) e^{j(\omega t - \beta z)} d\omega \right|^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\overline{E}_t(\overline{\rho})|^4 dxdy. \qquad [4.2.18]$$

For a single mode, the power P is given by [38]

$$P = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{a}_z \cdot (\overline{E}_t \times \overline{H}_t^*) dxdy. \qquad [4.2.19]$$

For weakly guiding fibers (as we have assumed) the transverse fields are related by [38]

$$\overline{H}_t = \left(\frac{\varepsilon_o}{\mu_o}\right)^{1/2} n_{co} \hat{a}_z \times \overline{E}_t \qquad [4.2.20]$$

where n_{co} is the index of the core of the ideal fiber. Substituting this into [4.2.19] we obtain the following:

$$P = \frac{1}{2}\left(\frac{\varepsilon_o}{\mu_o}\right)^{1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{a}_z \cdot \left[\overline{E}_t \times (n_{co} \hat{a}_z \times \overline{E}_t)^*\right] dxdy$$

$$= \frac{1}{2}\left(\frac{\varepsilon_o}{\mu_o}\right)^{1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{a}_z \cdot \left[(\overline{E}_t \cdot \overline{E}_t^*) n_{co} \hat{a}_z - (\overline{E}_t \cdot n_{co} \hat{a}_z) \overline{E}_t\right] dxdy$$

$$= \frac{1}{2}\left(\frac{\varepsilon_o}{\mu_o}\right)^{1/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} n_{co}\,|\,\overline{E}_t\,|^2 dxdy. \qquad [4.2.21]$$

The amplitude function $c(z, \omega)$ is symmetric in ω, so

$$\int_{-\infty}^{\infty} c(z, \omega)e^{j(\omega t-\beta z)}d\omega = 2\int_{0}^{\infty} c(z, \omega)e^{j(\omega t-\beta z)}d\omega \qquad [4.2.22]$$

Combining these last results in [4.2.18],

$$\overline{K}(z, \omega) = \frac{2\omega\varepsilon_o n_o^2}{j2\pi}\left(\frac{\varepsilon_o}{\mu_o}\right)^{-1/2}\left|\int_{0}^{\infty} c(z, \omega)e^{j(\omega t-\beta z)}d\omega\right|^2 \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\,\overline{E}_t(\overline{\rho})\,|^4 dxdy}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} n_{co}\,|\,\overline{E}_t(\overline{\rho})\,|^2 dxdy} \qquad [4.2.23]$$

or

$$\overline{K} = \frac{-j2\omega n_o^2}{2\pi c}\left|\int_{0}^{\infty} c(z, \omega)e^{j(\omega t-\beta z)}d\omega\right|^2 \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\,\overline{E}_t\,|^4 dxdy}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} n_{co}\,|\,\overline{E}_t\,|^2 dxdy} \qquad [4.2.24]$$

where $c = (\varepsilon_o\mu_o)^{-1/2}$ is the speed of light in vacuum. We now define the constant

$$\alpha \equiv \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\,\overline{E}_t(\overline{\rho})\,|^4 dxdy}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} n_{co}\,|\,\overline{E}_t(\overline{\rho})\,|^2 dxdy} \qquad [4.2.25]$$

and the function

$$R(\omega) \equiv \frac{-j\omega n_2^2\alpha}{2c}. \qquad [4.2.26]$$

With these two quantities defined, [4.2.24] becomes

$$\frac{\partial}{\partial z}c(z,\omega) = \frac{4}{2\pi}R(\omega)\left|\int_0^{\infty} c(z,\omega)e^{j(\omega t-\beta z)}d\omega\right|^2 \qquad [4.2.27]$$

Recall that we can express $\overline{\mathbf{E}}_t(\bar{r}, t)$ in terms of the pulse envelope $\psi(z, t)$:

$$\overline{\mathbf{E}}_t(\bar{r}, t) = \overline{E}_t(\bar{\rho})e^{j(\omega_o t-\beta(\omega_o)z)}\psi(z, t). \qquad [4.2.4]$$

Comparison on Eqs. 4.2.4, 4.2.12, and 4.2.13 reveals that

$$\psi(z,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} c(z,\omega)\exp\{j(\omega-\omega_o)t - j[\beta(\omega)-\beta(\omega_o)]z\}d\omega \qquad [4.2.28]$$

$$= \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} c(z,\omega)\exp\{j(\omega-\omega_o)t - j[\beta(\omega)-\beta(\omega_o)]z\}d\omega. \qquad [4.2.29]$$

We expand $\beta(\omega)$ in a Taylor's series about ω_o , keeping terms up to third order:

$$\beta(\omega) = \beta(\omega_o) + (\omega-\omega_o)\beta'(\omega_o) + \frac{(\omega-\omega_o)^2}{2}\beta''(\omega_o) + \frac{(\omega-\omega_o)^3}{6}\beta'''(\omega_o) \qquad [4.2.30]$$

where the primes denote differentiation with respect to ω. To simplify notation, we define

$$\beta_o' = \left.\frac{d\beta}{d\omega}\right|_{\omega=\omega_o}$$

$$\beta_o'' = \left.\frac{d^2\beta}{d\omega^2}\right|_{\omega=\omega_o} \qquad [4.2.31]$$

$$\beta_o''' = \left.\frac{d^3\beta}{d\omega^3}\right|_{\omega=\omega_o}$$

So [4.2.29] becomes

$$\psi(z,t)=\frac{2}{\sqrt{2\pi}}\int_0^{\infty}c(z,\omega)\exp\left\{j(\omega-\omega_o)t-j\left[(\omega-\omega_o)\beta_o'+(\omega-\omega_o)^2\frac{\beta_o''}{2}+(\omega-\omega_o)^3\frac{\beta_o'''}{6}\right]z\right\}d\omega$$

$$=\frac{2}{\sqrt{2\pi}}\int_0^{\infty}c(z,\omega)\exp\left\{j\left[(\omega-\omega_o)(t-\beta_o'z)-(\omega-\omega_o)^2\frac{\beta_o''z}{2}-(\omega-\omega_o)^3\frac{\beta_o'''z}{6}\right]\right\}d\omega. \quad [4.2.32]$$

For convenience, let

$$F(z,\omega)\equiv\exp\left\{j\left[(\omega-\omega_o)(t-\beta_o'z)-(\omega-\omega_o)^2\frac{\beta_o''z}{2}-(\omega-\omega_o)^3\frac{\beta_o'''z}{6}\right]\right\}. \quad [4.2.33]$$

We want to derive an equation which will be satisfied by $\psi(z,t)$. To do this, we take partial derivatives of ψ. The z-derivative is

$$\frac{\partial\psi}{\partial z}=\frac{2}{\sqrt{2\pi}}\int_0^{\infty}\left(c(z,\omega)\frac{\partial}{\partial z}F+F\frac{\partial}{\partial z}c(z,\omega)\right)d\omega$$

$$=\frac{2}{\sqrt{2\pi}}\int_0^{\infty}F\left\{jc(z,\omega)\left[(\omega-\omega_o)(-\beta_o')-(\omega-\omega_o)^2\frac{\beta_o''}{2}-(\omega-\omega_o)^3\frac{\beta_o'''}{6}\right]+\frac{\partial}{\partial z}c\right\}d\omega. \quad [4.2.34]$$

The first t-derivative is

$$\frac{\partial\psi}{\partial t}=\frac{2}{\sqrt{2\pi}}\int_0^{\infty}\left(c(z,\omega)\frac{\partial F}{\partial t}+F\frac{\partial c(z,\omega)}{\partial t}\right)d\omega. \quad [4.2.35]$$

We note that since the index of refraction is a function of time, the modal amplitude $c(z,\omega)$ also depends on time, so that $\frac{\partial}{\partial t}c(z,\omega)$ is not zero. However, since we have assumed that the pulse envelope varies slowly compared to the carrier, this derivative is negligible compared to $\frac{\partial F}{\partial t}$. Therefore, $\frac{\partial c}{\partial t}$ can be neglected, giving

$$\frac{\partial \psi}{\partial t} = \frac{2}{\sqrt{2\pi}} \int_0^\infty jFc(\omega - \omega_o) d\omega. \qquad [4.2.36]$$

Similarly,

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{-2}{\sqrt{2\pi}} \int_0^\infty Fc(\omega - \omega_o)^2 d\omega. \qquad [4.2.37]$$

and

$$\frac{\partial^3 \psi}{\partial t^3} = -\frac{1}{\pi} \int_0^\infty jFc(\omega - \omega_o)^3 d\omega \qquad [4.2.38]$$

We can write $\frac{\partial \psi}{\partial z}$ in terms of the other derivatives:

$$\frac{\partial \psi}{\partial z} = \frac{2}{\sqrt{2\pi}} \int_0^\infty jFc(z, \omega)(\omega - \omega_o)(-\beta_o{}')d\omega \;-\; \frac{2}{\sqrt{2\pi}} \int_0^\infty jFc(z, \omega)(\omega - \omega_o)^2 \frac{\beta_o{}''}{2} d\omega$$

$$- \frac{2}{\sqrt{2\pi}} \int_0^\infty jFc(\omega - \omega_o)^3 \frac{\beta_o{}'''}{6} d\omega \;+\; \frac{2}{\sqrt{2\pi}} \int_0^\infty F \frac{\partial}{\partial z} c(z, \omega) d\omega$$

$$= -\beta_o{}' \frac{\partial \psi}{\partial t} + j\frac{\beta_o{}''}{2} \frac{\partial^2 \psi}{\partial t^2} + \frac{\beta_o{}'''}{6} \frac{\partial^3 \psi}{\partial t^3} + \frac{2}{\sqrt{2\pi}} \int_0^\infty F \frac{\partial}{\partial z} c(z, \omega)\, d\omega \qquad [4.2.39]$$

To simplify the last term in [4.2.39], we consider [4.2.27]:

$$\frac{\partial}{\partial z} c(z, \omega) = \frac{4}{2\pi} R(\omega) \left| \int_0^\infty c(z, \omega) e^{j(\omega t - \beta z)} d\omega \right|^2 \qquad [4.2.27]$$

and [4.2.29], which can be written as

$$\psi(z,t) = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} c(z,\omega) \exp\{j(\omega t - \beta z)\} \exp\{-j[\omega_o t - \beta(\omega_o)z]\} d\omega$$

$$= \frac{2}{\sqrt{2\pi}} \exp\{-j[\omega_o t - \beta(\omega_o)z]\} \int_0^{\infty} c(z,\omega) \exp\{j(\omega t - \beta z)\} d\omega \qquad [4.2.40]$$

So

$$\int_0^{\infty} c(z,\omega) \exp\{j(\omega t - \beta z)\} d\omega = \psi \frac{\sqrt{2\pi}}{2} \exp\{j[\omega_o t - \beta(\omega_o)z]\} \qquad [4.2.41]$$

Replacing this in [4.2.27],

$$\frac{\partial}{\partial z} c(z,\omega) = \frac{2}{\pi} R \left| \psi \frac{\sqrt{2\pi}}{2} \exp\{j[\omega_o t - \beta(\omega_o)z]\} \right|^2$$

$$= Rc(z,\omega) |\psi|^2 \qquad [4.2.42]$$

So the last term in [4.2.39] is

$$\frac{2}{\sqrt{2\pi}} \int_0^{\infty} F \frac{\partial}{\partial z} c(z,\omega) d\omega = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} FRc(z,\omega) |\psi|^2 d\omega$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} F \left[\frac{-j\omega n_2^2 \alpha}{2c} \right] c(z,\omega) |\psi|^2 d\omega$$

$$= |\psi|^2 \left[\frac{-j\omega_o n_2^2 \alpha}{c} \right] \frac{1}{\sqrt{2\pi}} \int_0^{\infty} Fc(z,\omega) d\omega$$

$$= \frac{-j\omega_o n_2^2 \alpha}{2c} |\psi|^2 \psi \qquad [4.2.43]$$

by Eqs. [4.2.32] and [4.2.33]. Substituting this into [4.2.39], we obtain the equation for $\psi(z, t)$:

$$\frac{\partial \psi}{\partial z} + \beta_o' \frac{\partial \psi}{\partial t} - j\frac{\beta_o''}{2}\frac{\partial^2 \psi}{\partial t^2} - \frac{\beta_o'''}{6}\frac{\partial^3 \psi}{\partial t^3} + \frac{j\omega_o n_2^2 \alpha}{2c}|\psi|^2\psi = 0. \qquad [4.2.44]$$

This equation can be written in slightly different form:

$$j\left(\frac{\partial \psi}{\partial z} + \beta_o' \frac{\partial \psi}{\partial t}\right) + \frac{\beta_o''}{2}\frac{\partial^2 \psi}{\partial t^2} - j\frac{\beta_o'''}{6}\frac{\partial^3 \psi}{\partial t^3} - \frac{\omega_o n_2^2 \alpha}{2c}|\psi|^2\psi = 0. \qquad [4.2.45]$$

Eq. [4.2.45] is a form of the cubic nonlinear Schrödinger equation, with the addition of a third-order dispersion term. It can be compared to a model equation obtained by a different method which also invokes the slowly-varying envelope approximation [10]:

$$j\left(\frac{\partial \psi}{\partial z} + \gamma\psi + \beta_o' \frac{\partial \psi}{\partial t}\right) - \frac{\beta_o''}{2}\frac{\partial^2 \psi}{\partial t^2} - j\frac{\beta_o'''}{6}\frac{\partial^3 \psi}{\partial t^3} + \frac{\omega_o n_2}{2c}|\psi|^2\psi = 0 \qquad [4.2.46]$$

The differences between the two equations are explained as follows:

1. The addition of the loss term $\gamma\psi$ in [4.2.46] results from assuming an index of refraction with an imaginary part.

2. The two sign differences result from different conventions for the time dependence; Eq. [4.2.45] assumes the electrical engineering convention of $e^{j\omega t}$, whereas [4.2.46] assumes the physics convention of $e^{-j\omega t}$.

3. The n_2 instead of n_2^2 in [4.2.46] results from defining the nonlinear part of the refractive index as $n_2|\overline{E}|^2$ instead of $n_2^2|\overline{E}|^2$ in deriving [4.2.26].

4. The factor α in [4.2.45] is approximately unity and so creates only a small difference in the nonlinear coefficients. Note that while the coefficients of the other terms are widely accepted, the value of the nonlinear coefficient is disputed and different derivations give slightly different results.

Again we note that, as seen from the discussion before Eq. [4.2.4], the model equation [4.2.45] is valid only in the picosecond regime, and can not be expected to give the proper results for femtosecond pulses.

The solution of equations such as [4.2.45] is considered in the next chapter.

5.0 On the Solution of Nonlinear Equations

In the last chapter, a model equation for propagation in a single-mode nonlinear optical fiber was derived. In this chapter, analytical and numerical methods of solving such equations will be explained. An important topic in this chapter is the inverse scattering transform, in terms of which the definition of "soliton" may be understood.

Before explaining solution techniques, we consider the problem of converting a nonlinear equation to canonic form.

5.1 Conversion to Canonic Form

Generally the first step in solving an equation such as [4.2.45] is to convert it to a dimensionless form. This is done by introducing new coodinates which eliminate the dimensional quantities of the physical system and which also may be chosen to provide a convenient scaling between the real-world quantities and the dimensionless equation. Once the dimensionless equation is solved, the solution can be interpreted for many combinations of parameters just by changing the values in the transformations, without the need for doing the solution over again.

As an example, we can consider the conversion of the equation given in the last chapter [10]:

$$j\left(\frac{\partial\psi}{\partial z} + \gamma\psi + \beta_o'\frac{\partial\psi}{\partial t}\right) - \frac{\beta_o''}{2}\frac{\partial^2\psi}{\partial t^2} - j\frac{\beta_o'''}{6}\frac{\partial^3\psi}{\partial t^3} + \frac{\omega_o n_2}{2c}|\psi|^2\psi = 0. \qquad [5.1.1]$$

In the absence of loss (the γ term), dispersion (the β_o'' and β_o''' terms), and nonlinearity (the $|\psi|^2\psi$ term), Eq. [5.1.1] becomes

$$\frac{\partial\psi}{\partial z} + \beta_o'\frac{\partial\psi}{\partial t} = 0 \qquad [5.1.2]$$

which has the solution:

$$\psi = \psi(t - \beta_o' z) \qquad [5.1.3]$$

where

$$\beta_o' = \left.\frac{d\beta}{d\omega}\right|_{\omega=\omega_o} \qquad [5.1.4]$$

is the group velocity. Therefore, Eq. [5.1.1] is converted to dimensionless form by using a new coodinate τ which moves at the group velocity. The transforming relations are [10]

$$\tau = \frac{10^{-4.5}}{(-\lambda\beta_o'')^{1/2}}(t - \beta_o' z) \qquad [5.1.5]$$

$$\xi = 10^{-9}\frac{z}{\lambda} \qquad [5.1.6]$$

$$q = 10^{4.5}(\pi n_2)^{1/2}\psi \qquad [5.1.7]$$

With these transformations, the equation for the normalized, dimensionless electric field envelope $q(\tau, \xi)$ is [10]

$$j\frac{\partial q}{\partial \xi} + \frac{1}{2}\frac{\partial^2 q}{\partial \tau^2} + |q|^2 q = -j\Gamma q + jb\frac{\partial^3 q}{\partial \tau^3} \qquad [5.1.8]$$

where

$$\Gamma = 10^9 \lambda \gamma \qquad [5.1.9]$$

is the loss coefficient and

$$b = \frac{1}{6} \frac{\beta_o'''}{\beta_o''} \frac{10^{-4.5}}{(-\lambda\beta'')^{1/2}} \qquad [5.1.10]$$

is the third-order dispersion coefficient. The lefthand side of [5.1.8] is exactly the nonlinear Schrödinger operator (see Eq. [2.3.2]), and the right-hand side shows the loss and third-order dispersion terms as perturbations. The scaling factors were chosen so that the new variables τ, ξ, and q are of order unity for a pulse width of about 1 ps, a length of 1 km, and a peak electric field of 10^6 V/m [10].

5.2 *Analytical Methods*

5.2.1 The Inverse Scattering Transform and Definition of Soliton

The inverse scattering transform (IST) is a method of solving a nonlinear initial value problem (IVP), consisting of a nonlinear partial differential equation for a function $q(\xi, \tau)$ and an initial condition $q(\xi, 0)$. The IST was developed by Gardner, Greene, Kruskal, and Miura [39], who applied it to the KdV equation. The IST was first used to solve the NLS equation by Zakharov and Shabat [40] in 1970. The method consists of "going around" the nonlinear IVP by solving three related linear problems, called the direct problem, the time evolution of scattering data, and the inverse problem. Figure 2 shows a diagram of the method [4, 5, 15].

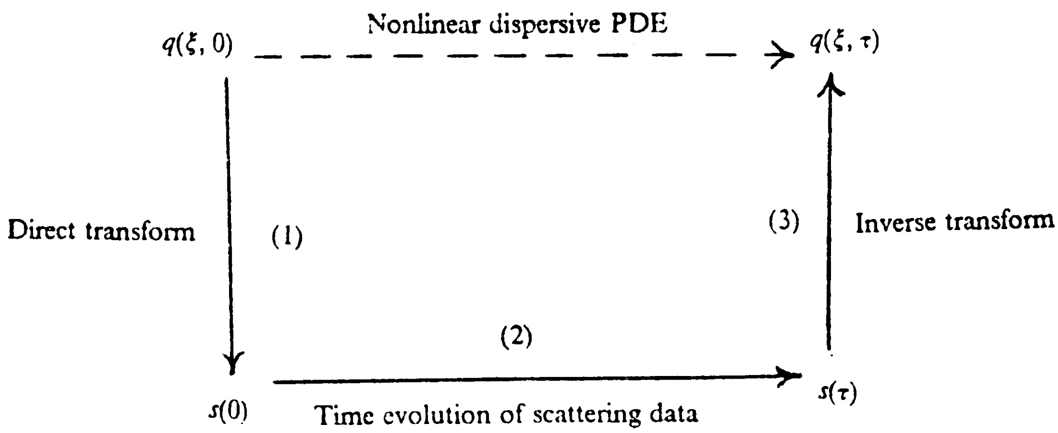

Figure 2. The inverse scattering transform.

In the direct problem, the initial condition $q(\xi, 0)$ is "transformed" to determine its eigenvalue spectrum. This transformation involves the solution of an eigenvalue problem. The collection of eigenvalues found is called $s(0)$, the scattering data at time $\tau = 0$. In the second problem, the spectral components found in the first problem are operated upon in order to find $s(\tau)$, the scattering data at time τ. In the inverse problem, the scattering data at time τ are inverted to obtain the desired function $q(\xi, \tau)$.

In the linear limit, the IST reduces to the familiar Fourier transform technique of solving differential equations, where the direct and inverse problems become the direct and inverse Fourier transforms, respectively [5].

Several important features of the solution are determined in the direct problem. The eigenvalue spectrum obtained may consist of discrete eigenvalues λ_n, along with a continuous spectrum. The number N of discrete eigenvalues is determined by the area A of the initial condition, where [10]

$$A = \int_{-\infty}^{\infty} |q(\xi, 0)|\, d\xi, \qquad [5.2.1]$$

and is called the *order* of the solution. The scattering data (the discrete and continuous components of the eigenvalue spectrum) are operated upon in the time evolution problem (step 2 in the diagram), and inverted (step 3). The discrete eigenvalues result in *solitons* in the final solution, and the continuous spectrum results in *linear dispersive waves,* which die off asymptotically as $\xi \to \infty$.

We can now give a crude definition of a soliton: *A soliton is a solution component corresponding to a discrete eigenvalue in a nonlinear dispersive initial value problem solvable by the inverse scattering transform* [15, 41].

The eigenvalues determine some of the characteristics of the solitons. The velocity of a soliton is a function of its eigenvalue. Thus, if all N eigenvalues are distinct, then the solution will consist of N first-order solitons, each traveling at a different velocity. If the eigenvalues are identical, then the solution will be a single N^{th}-order soliton, which is really N first-order solitons traveling together in a "bound state". Interference between the N solitons in a bound state causes the higher-order soliton to oscillate in the direction of propagation [15]. This is the reason for the complex compression, expansion, and splitting behavior of higher-order solitons. The bound state is found as a solution of the NLS equation, but not of the KdV equation [40].

To illustrate these concepts, we shall consider the canonic NLS equation:

$$j\frac{\partial q}{\partial \xi} + \frac{1}{2}\frac{\partial^2 q}{\partial \tau^2} + |q|^2 q = 0 \qquad [5.2.2]$$

with the initial condition

$$q(\xi,0) = a_o \; sech(\xi). \qquad [5.2.3]$$

The number N of discrete eigenvalues is given by [15, 69]

$$a_o - \frac{1}{2} \;<\; N \;\leq\; a_o + \frac{1}{2}. \qquad [5.2.4]$$

So if $\frac{1}{2} \leq a_o < \frac{3}{2}$, then N will be 1, and a fundamental soliton will be formed. For an initial pulse of arbitrary shape, the order N is defined as the area of the initial pulse divided by the area of the fundamental, where the area A is defined by [5.2.1]. The difference in energy between the initial condition and the soliton formed is radiated off as a linear dispersive wave. In a physical problem, the amplitude of the initial condition can be related to the field strength through the transforming relations used to convert the wave equation to canonic form, so we can determine the requirements for forming the fundamental and each higher-order soliton in a physical system such as an optical fiber.

For the pure soliton case (N = an integer), the IST gives the solution in the form of $2N$ simultaneous algebraic equations, which are solved numerically [42].

5.2.2 Perturbation Theory

When the equation to be solved can be recognized as a solveable equation with some added, sufficiently "small", terms, the equation can sometimes be solved by treating the added terms as perturbations of the solveable equation, using the known solution. Perturbation techniques have been used effectively for some variations of the nonlinear Schrödinger equation [41].

5.3 Numerical Methods

Equations that cannot be solved analytically, such as some wave equations with higher-order dispersion or nonlinear terms, can be solved by numerical methods such as the variations of the split-step Fourier approach [13, 43]. These numerical techniques divide the nonlinear dispersive wave equation into nonlinear and dispersive operators, and treat nonlinearity in the time domain and dispersion in the frequency domain. The fast Fourier transform (FFT) is used to transform between domains. The solution is advanced in steps through a thin section at a time, and in each section the appropriate transforms and operators are applied to the pulse envelope.

As a specific example of this type of method, consider the propagating beam method used by Yevick and Hermansson [13] to solve the following equation for the scaled, dimensionless pulse envelope q:

$$j\frac{\partial q}{\partial \xi}+\frac{1}{2}\frac{\partial^2 q}{\partial \tau^2}+|q|^2q=-j\Gamma q+j\beta\frac{\partial^3 q}{\partial \tau^3}-C|q|^4q. \qquad [5.3.1]$$

Here the term with Γ represents loss, and the last two terms represent third-order dispersion and fourth-order nonlinearity, respectively. The nonlinear operator (which also includes the loss term) is

$$H(q)=\exp\left[js\Delta\xi\left(|q|^2+Cs|q|^4\right)-\Gamma\Delta\xi\right] \qquad [5.3.2]$$

where s is the loss factor, given by

$$s=1-\Gamma\Delta\xi. \qquad [5.3.3]$$

The dispersive operator is

$$G=\exp\left[\frac{1}{2}\Delta\xi\left(\frac{j}{2}\frac{\partial^2}{\partial\tau^2}+\beta\frac{\partial^3}{\partial\tau^3}\right)\right]. \qquad [5.3.4]$$

The algorithm is expressed symbolically as

$$q(\tau,\Delta\xi)=GH(\ q(\tau,\xi))Gq(\tau,\xi)+O(\Delta\xi^3). \qquad [5.3.5]$$

The algorithm is carried out by transforming into the frequency domain before application of each H operator, and transforming into the time domain before applying each G operator.

It should be noted that solving an equation such as the nonlinear Schrödinger equation with loss and higher-order dispersion terms by numerical methods generally requires supercomputer power. Numerical solutions that have been worked out at Bell Labs were done on a Cray-1 computer, and Mollenauer *et al.* have pointed out that solutions for solitons of order greater than 4 are prohibitively lengthy (and thus expensive) even when done on a Cray [42]. Of course, higher-order solitons are generally of interest mainly in pulse-compression experiments; communications applications would use fundamental solitons. Hasegawa has reported that for a fundamental soliton, the solution of the NLS equation with added loss and source terms required approximately 30 minutes of computer time on a Cray [44].

6.0 The Development of Nonlinear Fiber Optics

In previous chapters of this thesis we have presented tutorial information and some of the necessary mathematical background for the field of nonlinear fiber optics. In this chapter we shall give a detailed review of theoretical and experimental work in the field. While such a review is more traditionally included at the beginning of a thesis, it was delayed in this case for the sake of presenting a sufficient introduction first. Furthermore, it is intended that this section cover more than the traditional literature review; some mathematical points will have to be brought up here in order to explain conflicts between the theories developed by different groups of researchers.

The development of nonlinear fiber optics can be divided into two areas of interest: the picosecond and the femtosecond regimes, depending on the width of the soliton pulses used or assumed.

In the picosecond regime, it is generally agreed that the conventional NLS equation, perhaps with the addition of a third-order dispersion term and a loss term, is sufficient to describe the evolution of the soliton pulse. In this regime, the main topic of interest is the development of soliton communications systems, which, for reasons outlined in this chapter, would use pulses of width no less than about 20 picoseconds. The development of such systems has been done almost exclusively by AT&T Bell Laboratories. While several groups around the world are doing theoretical work, only Bell Labs is doing detailed simulation and *design* of nonlinear fiber optic communications systems. The claims of feasibility of such systems are the central issue in this regime. Therefore, the first section of this chapter will cover a bit of preliminary history, and then the second section will cover the development of picosecond soliton communications systems, and will use the series of papers published by Bell Labs as a convenient framework.

In the femtosecond regime, on the other hand, the narrow pulse widths (wide bandwidths) involved require the addition of higher-order dispersion and nonlinearity terms to the NLS model. While the design of picosecond soliton communications systems proceeded largely independently of the development of femtosecond-regime models, the development of such models was motivated by the desire to extend the design of soliton communications systems into the femtosecond regime, and to understand the higher-order effects which were showing up in soliton experiments. Therefore, the third section of this chapter will treat the higher-order models and soliton effects such as the soliton self-frequency shift [45]. The most important paper concerning the propagation of solitons in the femtosecond regime (and in the picosecond regime) is the recently published theory by Kodama and Hasegawa [15], which for the first time gives a rigorous derivation of the evolution equation for soliton pulses in single-mode fibers. The final section of this chapter will cover the Kodama-Hasegawa (KH) model, and the implications of the KH model will be explained.

6.1 *Pre-1973*

By the mid-1960's the theory of electromagnetic propagation in nonlinear media had become an important branch of physics. In 1966 Zakharov [46] derived the NLS equation

$$j\left(\frac{\partial\psi}{\partial t} + \omega_k'\frac{\partial\psi}{\partial z}\right) + \frac{\omega_k''}{2}\frac{\partial^2\psi}{\partial z^2} - q|\psi|^2\psi = 0 \qquad [6.1.1]$$

to describe the evolution of the complex amplitude of a one-dimensional quasi-monochromatic wave in an *isotropic* nonlinear dispersive medium. (Note that this equation reverses the roles of t

and z from what we have seen before.) In 1968 Karpman and Krushkal [47] numerically solved the NLS equation and showed that a plane wave propagating in a nonlinear medium would decay into packets (solitons) whose profiles could be expressed in terms of the *sech* function. Another single-soliton solution, in terms of the *tanh* function, was known from particle physics [48]. The development of the inverse scattering method by Gardner, Greene, Kruskal and Miura [39] gave theoreticians an important tool for analytically solving nonlinear dispersive equations. In 1971 Zakharov and Shabat [40] solved the NLS equation by the inverse scattering transform and obtained the general N-soliton solution. These developments paved the way for the work of Hasegawa and Tappert at Bell Labs.

6.2 *The Development of Soliton Communications Systems*

6.2.1 Initial Proposal: Hasegawa and Tappert, 1973

By 1973 the problem of dispersion in optical fibers was well understood [8], and a proposal had been made to minimize the effect of dispersion by cancelation of waveguide and material dispersion at the operating wavelength [49]. Before the details of any such scheme could be worked out, another way to eliminate the problem of dispersion was proposed.

Noting the work being done in particle physics, Hasegawa and Tappert (HT) of AT&T Bell Labs considered the case of nonlinear propagation in an optical fiber [1]. They assumed an index of refraction of the form

$$n = n_o(\omega) + j\chi(\omega) + n_2 |\bar{E}(\bar{r}, t)|^2 \qquad [6.2.1]$$

where χ represents dielectric loss and n_2 is the Kerr coefficient, and expressed the time-domain electric field in terms of a radial eigenfunction $R(\overline{\rho})$ and an envelope function $\psi(z, t)$ which varied slowly compared to the carrier frequency ω_o:

$$\overline{E}(\overline{r}, t) = R(\overline{\rho})Re\left\{\psi(z,t)e^{[j(\beta_o z - \omega_o t)]}\right\} \qquad [6.2.2]$$

Using these assumptions, HT showed that propagation in the three-dimensional optical fiber could be described by a one-dimensional nonlinear equation, which they gave as

$$j\left(\frac{\partial\psi}{\partial t} + \nu_o\psi + \omega_o{}'\frac{\partial\psi}{\partial z}\right) + \frac{\omega_o{}''}{2}\frac{\partial^2\psi}{\partial z^2} + \frac{\alpha\omega_o n_2}{n_o}|\psi|^2\psi = 0. \qquad [6.2.3]$$

This is a the same nonlinear Schrödinger equation derived by Zakharov (Eq.6.1.1), but with the addition of a loss term with coefficient ν_o arising from the χ term in [6.2.1], and with the nonlinear coefficient given explicitly in terms of the fiber parameters. In deriving the coefficients, HT did not include waveguide effects in a rigorous manner, but rather included them by "averaging" the radial variation of the guided wave. The coefficient α is of order unity and results from this averaging process.

With loss neglected ($\nu_o = 0$), Eq. [6.2.3] has exact single-soliton solutions, in terms of the *sech* and tanh functions, which had been found previously. With a knowledge of the general form of the solutions of the NLS equation, HT offered an argument concerning the expected form of the solution of [6.2.3] with loss neglected: Since the coefficient of the nonlinear term is always positive, then if $\omega_o{}'' > 0$ (anomalous dispersion), the nonlinear and dispersive terms will be opposed provided that $\frac{\partial^2\psi}{\partial z^2} < 0$. This specifies a pulse that is concave downward, like the *sech* function, which is the envelope of a known solution of the NLS equation. A different kind of solution would be expected for the case of normal dispersion.

Therefore, in their first paper, which considered the case of anomalous dispersion, HT assumed a solution which was a generalization of the known *sech* solution of the NLS equation:

$$\psi(z,t) = E_s sech\left(\frac{t - t_o - \frac{z}{v_g}}{\tau_o}\right)e^{j(\kappa z - \Omega t)} \qquad [6.2.4]$$

where E_s is the maximum field intensity, t_o is the pulse center, v_g is the transmission speed, τ_o is the pulse half-width, and κ and Ω are shifted wavenumber and frequency variables. The pulse envelope is of hyperbolic-secant shape, and the exponential function in [6.2.4] represents the carrier. Equations relating the different quantities were derived by substituting [6.2.4] into [6.2.3].

To determine the stability of the soliton solution under the influence of loss and noise, Eq. [6.2.3], including the loss term, was also numerically solved using the then recently-derived split-step Fourier method, a forerunner of the propagating beam method presented in the last chapter. HT found that an initial hyperbolic-secant pulse would remain stable under the influence of loss, noise, and even large perturbations. The effect of loss was to cause the amplitude of the pulse to decay, but with a corresponding broadening in width so that the area of the soliton was maintained as well as the soliton character. Ways of compensating this "adiabatic" decay were to be the subject of several subsequent papers [10, 50, 51, 52, 53]. Another feature HT noted about solitons was that they seemed to attract each other through some nonlinear effect. This would set a limit on the spacing of soliton pulses that could be allowed in a communications system. Numerical simulations reported in the original paper [1] showed that a separation of four pulse widths rendered the attractive effect negligible.

In the second paper of the pair [2], HT considered the case of normal (positive) dispersion, and showed that in this regime Eq. [6.2.3] has solutions of the form

$$\psi(z,t) = E_s \tanh\left(\frac{t - t_o - \frac{z}{v_g}}{\tau_o}\right)e^{j(\kappa z - \Omega t)} \qquad [6.2.5]$$

where the variables are as defined before. The pulse of Eq. [6.2.5] is called a "dark" pulse, since the power flux is proportional to

$$|\psi|^2 = E_s^2\left[1 - sech^2\left(\frac{t - t_o - \frac{z}{v_g}}{\tau_o}\right)\right] \qquad [6.2.6]$$

Thus the intensity envelope is concave upwards, with an absence of light when the pulse is present and a constant light background when the pulse is absent.

At the end of this second paper HT outlined several problems which would have to be solved in order to implement their proposal. These problems included development of schemes to generate and modulate soliton pulses.

6.2.2 First Experimental Observation: 1980

Limits on technology prevented further work on the problem for years. By 1980 new lasers and the development of low-loss (0.2 dB/km) optical fibers allowed experimental work in soliton propagation. In this year Mollenauer, Stolen, and Gordon (MSG) reported the first experimental evidence of soliton propagation in optical fibers [3], supporting the theory developed by Hasegawa and Tappert. In their experiments, MSG showed that an input pulse with sufficient power underwent the pulse compression and splitting which theory said was characteristic of higher-order solitons.

MSG knew that the canonic NLS equation has the dimensionless period $\frac{\pi}{2}$. Relating this to the real-world soliton period z_o through the transformations used to convert to the canonic equation, they derived the soliton period

$$z_o = \frac{0.322\pi^2\tau^2 c}{\lambda^2 |D|} \qquad [6.2.7]$$

and the power required for the fundamental soliton

$$P_1 = \frac{A_{eff}\lambda}{4n_2 z_o} \qquad [6.2.8]$$

where τ is the pulse width, D is the dispersion parameter of the fiber, A_{eff} is the effective area of the core, λ is the vacuum wavelength of the carrier, and n_2 is the optical Kerr coefficient.

They then studied the behavior of pulses in a fiber with large negative dispersion at the operating frequency of $\lambda_o = 1.55\ \mu m$. The fiber was of the step index type, with a central index dip and a GeP-doped silica core and P silica cladding. The length of the fiber was 700 m, approximately half of the soliton period of 1260 m.

Pulses of 7-ps full width were injected into the fiber, and the output pulse shape measured by autocorrelation. For low power, the pulse simply broadened due to dispersion. As the power was increased, the output pulse width decreased until the input pulse width was restored at an input power of 1.2 W, which compares well with the theoretical value of 1.0 W for the fundamental soliton power P_1. As the power was increased further, the output pulse narrowed and split according to the theoretical behavior, with generally good quantitative agreement with theory. Deviations from expected results, including asymmetry in the output pulses, were explained as effects of the asymmetry of the input pulses and their variation from the required *sech* shape, and the deviation of the fiber length from half a soliton period.

This experiment represented the first reported observation of soliton behavior in an optical fiber, and inspired new research in the field of soliton communications.

6.2.3 Initial System Development at Bell Labs

After the experimental observation of soliton effects, Hasegawa, with his new research partner Yuji Kodama, once again considered the problem of nonlinear optical fibers. Hasegawa and Kodama (HK) wrote four papers [10, 50, 51, 52] in 1981-3 in which they investigated more details of a fiber optic communication system using solitons.

In the first of these papers [10], HK considered the factors which limit the data rate in linear systems, and showed that for systems operating at the dispersion minimum at $\lambda = 1.3\mu m$, an optimistic limit for the data rate would be 0.13 Tbits/s for a length of 20 km, giving a length-rate product of 2600 GHz-km. They then went on the work out some details of a soliton-based system to find a theoretical data rate for such a system.

The original model was improved by including the effects of loss and higher-order dispersion as perturbations to the nonlinear Schrödinger equation used previously. The new model equation, in dimensionless form, was

$$j\frac{\partial q}{\partial \xi} + \frac{1}{2}\frac{\partial^2 q}{\partial \tau^2} + |q|^2 q = -j\Gamma q + jb\frac{\partial^3 q}{\partial \tau^3} \qquad [6.2.9]$$

where

$$\Gamma = 10^9 \lambda \gamma \qquad [6.2.10]$$

is the loss factor and

$$b = \frac{1}{6}\frac{\beta_o'''}{\beta_o''}\frac{10^{-4.5}}{(-\lambda\beta'')^{1/2}} \qquad [6.2.11]$$

is the third-order dispersion factor. (This is the same equation used as an example of conversion to canonic form in Chapter 5.) The left-hand side of [6.2.9] is exactly the nonlinear Schrödinger operator, and the right-hand side shows the loss and third-order dispersion terms as perturbations.

Eq. [6.2.9] was studied using perturbation theory [41]. It was found, as before, that the loss term causes the soliton to spread adiabatically, *i.e.*, retaining its soliton properties. The amplitude of the soliton decays as $\exp(-2\Gamma\xi)$ while the width increases as $\exp(+2\Gamma\xi)$. A loss of 0.2 dB/km would cause the pulse width to double in 20 km. Since the pulse retains its soliton shape, it could be reshaped just by amplification, without the need for detection and regeneration. The effect of the third-order dispersion term was to deform the pulse shape only to order b, a small number, and to modify the group velocity. Therefore, the loss and higher-order dispersion terms had no significantly detrimental effects on the stable soliton propagation.

In the remainder of the paper, HK derived equations necessary for the design of a communication system, and chose parameters to optimize bit rate. The dispersion constants in the equations were taken from linearized graphs of data given by Marcuse [11]. With their optimal system, they showed that the achievable bit rate was an order of magnitude better than that of the best linear system possible.

One main problem to be solved was to determine the gain mechanism to be used to compensate the soliton spreading due to loss. Two schemes for compensating the pulse spreading were proposed and studied in detail in 1982. These schemes are outlined here.

1. Kerr-Effect Amplification by Injection of cw Energy, June 1982

The first scheme was to use the Kerr nonlinearity itself as the gain mechanism [50]. The Kerr effect contracts the pulse in proportion to $|E|^2$, where E is the sum of the soliton field and any other field present, so in effect the soliton could be contracted by addition of more energy to the waveguide, if the added energy is of the same frequency and phase as the soliton carrier. The soliton would absorb the energy of the continuous wave, and a new soliton would be formed that was taller and narrower than the old. Any energy that was not absorbed by the soliton would propagate as a linear dispersive wave. These heuristic ideas were studied in detail using perturbation theory, and optimal distances for the injection of cw energy were worked out. The drawbacks of this system are that it would require sophisticated technologies: the detection and matching of the phase and frequency of the soliton carrier, and the production of input solitons without phase, frequency, or amplitude jitter. Therefore, while this scheme is a possibility, it is not a very attractive one.

2. Linear Amplification, July 1982

The second scheme was to use simple linear amplifiers [51]. Since the decay of solitons due to dissipation is adiabatic, the soliton is automatically reshaped by linear amplification. Since an amplifier can be made much less expensively that a repeater which must detect and regenerate a pulse, the proposed scheme gives a great advantage over a comparable linear system.

A drawback to this system is that amplification produces linear dispersive waves which interfere with the solitons. The energy of the linear waves which are created is equal to the difference in the energy of the newly amplified pulse and the energy of the soliton that is eventually formed.

Kodama and Hasegawa ran a simulation of a system 6000 km long, operating at $\lambda = 1.5\ \mu m$, with amplifiers with gain of 2 dB spaced every 10 km. It was found that a pair of soliton pulses of width 34.2 ps, separated by 8 pulse widths, would propagate with practically no distortion and no intersymbol interference. This gives a data rate of 2 Gbit/sec, or a length-rate product of 12000 GHz-km, an order of magnitude better than the linear limit. With increased power or decreased amplifier spacing, the data rate could be increased even further. Such systems with long lengths (on the order of continental distances) and simple, all-optical amplifiers would be especially attractive for transoceanic systems.

A concern with the system just described is the effect of random variation in the gain of the amplifiers used. Would the soliton pulses still remain stable and distinct if each of the amplifiers in the system had a slightly different gain? Kodama (now at The Ohio State University Department of Mathematics) and Hasegawa simulated a system [52] with amplifiers whose gain varied about some nominal value. It was found that the shorter system lengths and closer amplifier spacing was required, but even with large (100%) variation in gain, a system could be designed in which the soliton pulses would retain their shapes remarkably well even after propagating 1000 km. Therefore random variations in system parameters would degrade the system somewhat, but would not destroy the promise of higher data rates than were possible with linear systems.

6.2.4 Experimental and Theoretical Work, 1983

Researchers at Bell Labs published three papers in 1983 which had further implications for the use of the simple NLS model for nonlinear optical fibers, and encouraged the further development of nonlinear systems.

Observation of pulse restoration

Stolen, Mollenauer, and Tomlinson reported the first observation of higher-order soliton pulse restoration after propagation through a soliton period [54]. These researchers calculated the soliton period z_o and fundamental soliton power P_1 for a fiber, and then showed that at integer-squared multiples of P_1 the input pulse shape and spectrum were approximately restored at z_o, in agreement with theory. The experiment was carried out with pulses of 6.4 ps width at a wavelength of 1.55 μm. Discrepancies in results were explained as effects of neglecting loss and higher-order dispersion in the calculations, and the variation of the input pulse shape from a perfect hyperbolic secant.

Pulse compression by soliton effect

In May, the same group, with the addition of Gordon, demonstated pulse narrowing from 7 ps to 0.26 ps by using higher-order solitons propagating in a fiber length equal to $\frac{z_o}{N}$ [42].

However, the results did not always show good quantitative agreement with the simple NLS equation model, indicating that the short pulse widths (wide bandwidths) were causing the introduction of higher-order effects not considered previously. This was an indication of the need for a more detailed model.

Theory of interaction forces

The particle-like qualities of solitons were further emphasized when Gordon studied the simple, unperturbed NLS equation and found that "the motion of neighboring solitons of nearly equal amplitudes can be described in terms of interaction forces that decrease exponentially with their distance of separation and vary sinusoidally with their relative phase" [55]. This gave a quantitative measure to the qualitative observation made by Hasegawa and Tappert ten years before in their original proposal [1]. Gordon found that to prevent two adjacent 1-ps solitons from drifting toward each other by more than their pulse width in travelling 100 km, they would have to be spaced by more that 10.6 ps. This gave a better idea of the limit this effect would place on a communication system.

6.2.5 Raman gain and final system development

Because of limitations of the two aforementioned schemes which were proposed to overcome pulse spreading due to fiber loss, a third scheme was studied. This scheme used the stimulated Raman effect, which is a transfer of energy from a "pump" frequency to a lower frequency due to stimulated Raman emission [56, 57]. Figure 3 shows the gain spectrum for the Raman process in silica for a pump wavelength of 1 μm. (For other wavelengths, the gain coefficient would simply be scaled by the inverse of the pump wavelength in μm.) For maximum gain the pump should be at a frequency lower than the signal by an amount corresponding to a peak in the curve.

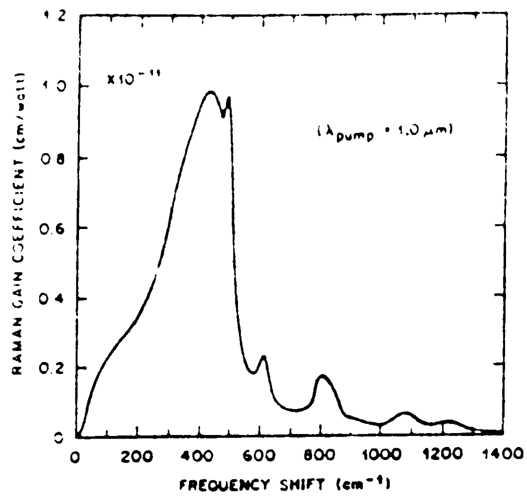

Figure 3. Raman gain for silica with pump wavelength of 1.0 μm. From Stolen [57].

In 1983 Hasegawa proposed to use the stimulated Raman process inherent in optical fibers as the gain mechanism to overcome pulse spreading due to absorption [53]. The scheme required the injection of cw Raman pump energy at regular intervals along the fiber. While this is similar to the procedure for amplification by the Kerr effect, the Raman gain scheme is better in that there is no need to detect the frequency and phase of the incoming soliton; the cw pump energy may be of relatively poor quality, with a bandwidth as large as 200 GHz centered on the nominal pump frequency. Thus this system is much less expensive than one with Kerr amplifiers.

Numerical study of soliton system with Raman gain

By mid-1984 Bell Labs had decided that Raman gain should be the mechanism for gain in a nonlinear fiber system. In that year Hasegawa worked out a detailed simulation of a *bidirectional* soliton communication system using periodically spaced amplifiers using Raman gain [44]. A main concern of the study was the loss rate effecting the Raman pump wave, so that the effect of gain decreases with distance from the Raman amplifiers. Thus a propagating soliton oscillates between maximum amplitude at the Raman amplifiers and minimum amplitude halfway between. The purpose of the study was to determine the optimal parameters, such as amplifier spacing and pump power, to prevent this amplitude variation from becoming unstable. A simulation demonstrated a stable transmission of a 10-Gbit soliton train over a distance of 4800 km by using Raman pumps every 34.4 km.

The Soliton Laser

A difficulty in experimental work had always been the production of narrow *sech* pulses of desired width. This problem was solved in the same year as the numerical study by the development of the soliton laser [20]. This device (see Fig. 4) uses a length of

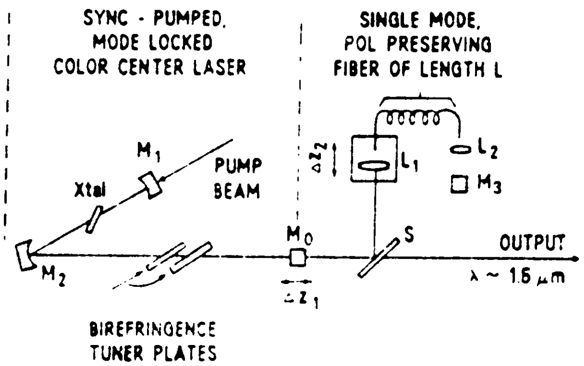

Figure 4. Schematic of the soliton laser. From Mollenauer [20].

polarization-preserving single-mode fiber in its feedback loop to control the width of the output pulses, which are sech-shaped solitons. The operation is as follows: The pump beam is deflected into the length of soliton fiber, whose parameters are chosen so that the pulse forms an $N=2$ soliton. This pulse contracts by an amount determined by the fiber length, and is eventually reinjected into the laser cavity, where it forces the laser to produce narrower pulses. The output power is about 1 W.

Long distance pulse propagation with Raman gain

In 1985, using a soliton laser source and Raman gain to compensate loss, Mollenauer, Stolen, and Islam [58] demonstrated distortionless propagation of 10-ps fundamental soliton pulses over a distance of 10 km at a repetition rate of 1 MHz. Raman pump light of proper frequency and intensity to just compensate loss was introduced into the fiber from the output end through a polarizing beam splitter. Implementation of this scheme in a practical communication system would require development of low-loss directional couplers to introduce pump energy into the middle of a segment, and faster sources and detectors.

This paper underscored the difficulty of experimental work in the field of nonlinear optics, and the lag of technology behind theory. While Kodama and Hasegawa were working out details of systems the length of a continent, with bit rates of several gigabits per second, the experimentalists were managing merely to produce and transmit solitons at a relatively low repetition rate (1 MHz) over lengths on the order of a kilometer. Obviously technology had a long way to develop, and there was still the possibility for unforeseen technological difficulties to interfere with the attainment of the proposed systems.

6.2.6 Optimal system simulation

In January 1986 the results of several years of research were accumulated and Mollenauer, Gordon, and Islam published a detailed simulation of an optimal soliton communication system which used Raman gain [32]. They showed that such a system could achieve a single-channel length-rate product of 29000 GHz-km, which could be increased by an order of magnitude by wavelength division multiplexing (WDM).

This *bidirectional* system used Raman gain to compensate loss by injecting pump power from cw laser diodes through wavelength-dependent directional couplers. The difference in photon energy between the cw pump and the soliton center frequency corresponded to the Raman gain peak for silica [56, 57]. The bit error rate was 10^{-9}, with signal powers of only a few mW and Raman pump powers less than 100 mW.

The range of possible pulse widths was determined by the choice of operating wavelength (1.55 μm), and the possible values of the dispersion parameter that can be obtained in an optical fiber. After simulation of the system with many choices of parameters, it was found that the stability of this periodically perturbed system was almost exclusively a function of $\frac{L}{z_o}$, where L is the amplification period and z_o is the soliton period. (Note that fundamental solitons are used, so that here the soliton period is only a characteristic length.) The system was found to be unstable for $\frac{L}{z_o} \cong 8$, and the most stable transmission occurred when $\frac{L}{z_o} << 8$. This requirement, along with the limits on attainable dispersion parameter and desired signal power, resulted in a chosen pulse width of about 20 ps.

Several advantages of the soliton system were outlined in the paper. These included:

1. The length-rate product for the proposed WDM soliton system is an order of magnitude better than the theoretical limit for linear systems, and two orders of magnitude better than the best linear limit that had actually been attained.

2. Electronic repeaters, which greatly limit bit-rate, are replaced by simple, fast, cw laser diode energy injectors. There is no need for expensive signal lasers at each repeater.

3. Electronics are eliminated except for simple servo control of Raman gain.

4. Multiplexing and demultiplexing only need to be done at the ends of the system. Complicated operations are eliminated in the middle of the system, allowing greater reliability and easier maintenance, qualities highly desireable for transoceanic systems.

5. Since only one signal source is required, at the input, this source could be an expensive, fast laser source.

6. The large number of photons per soliton pulse (on the order of 10^6) eliminates detector problems, makes the system easy to tap for data distribution, and allows detector sensitivity to be traded for speed.

Finally, Mollenauer, Gordon, and Islam pointed out that "in the context of long-distance, all-optical systems, it is not just that the use of solitons is more convenient or economical, but rather that such use represents the *only* viable solution".

Random walk

One negative note followed the publication of the optimal system simulation. A more detailed study of noise effects caused the researchers to re-estimate the bandwidth attainable with a soliton system.

Gordon of Bell Labs and Haus of MIT [59] showed that amplifier noise in the Raman amplifiers would cause a soliton's group velocity to undergo a random walk, which would result in timing errors in the output. This random-walk effect would limit the length-rate product of the proposed optimal system to 23600 GHz-km, which could still be increased by an order of magnitude by wavelength multiplexing.

To review: Linear systems using dispersion shift or WDM in dispersion-flattened fibers are expected to attain length-rate products of 10^4 GHz-km at most. Attainment of this limit would require careful control of fiber parameters, narrow bandwidth sources, and complicated electronic repeaters [31, 70, 71]. A reasonable theoretical limit for a single channel of a soliton system is 23600 GHz-km [59], with relative ease of wavelength multiplexing to increase this figure by a factor of ten.

The additional advantages of the optimal soliton system make it attractive for transoceanic systems, even if the attainable bit rate were only as low as for a linear system.

6.3 The Femtosecond Regime and Higher-Order Models

As the development of picosecond-regime communications systems was advancing, several theoretical research groups around the world set out to extend the theory of soliton propagation into the femtosecond regime, where the assumptions made in deriving the lower-order NLS equation models were no longer valid. For instance, the slowly-varying envelope approximation was no longer valid for femtosecond pulses, which would have pulse widths only a few optical periods long. New theories were developed to predict behavior of femtosecond solitons propagating in optical fibers. Some of these theories, and their predictions, are reviewed here.

6.3.1 The Shock Term

The most common correction factor added to the NLS model was the so-call ″shock term″, which was of the form $\frac{\partial}{\partial t}(|\psi|^2\psi)$. An equation containing this term apparently first appeared in a paper by Tzoar and Jain [60] in 1980, the same year that solitons were first observed in optical fibers. Tzoar and Jain gave their equation as

$$j\frac{\partial\psi}{\partial z} + \alpha\frac{\partial^2\psi}{\partial\tau^2} + \beta|\psi|^2\psi + j\gamma\frac{\partial}{\partial\tau}(|\psi|^2\psi) = 0 \qquad [6.3.1]$$

where α, β, and γ are constants, ψ is the complex amplitude, and $\tau = t - \frac{z}{v_g}$. (In the following discussion we will ignore the numerical value of the constants and instead concentrate on the *form* of the equations and the general physical effect of the various terms.) An equation identical in form to [6.3.1] was also given Anderson and Lisak [61] in 1982, and Golovchenko *et al.* [62] in 1985. The shock term also appeared in papers by Blow, Doran, and Cummings [63] in 1983 and Bourkoff *et al.* [64] in 1987. Both of these last two papers also included third-order dispersion terms.

The effect of the shock term was described variously.

Anderson and Lisak [61] attributed to the shock term the asymmetry of the output pulses observed by Mollenauer *et al.* [3] in their initial observation of solitons, and in pulse compression experiments by Nakatsuka *et al.* [65]. However, Anderson and Lisak's computations were done assuming *no dispersion*, which was hardly the case in these experiments. With no dispersion present, it is natural for the pulse to become asymmetrical and develop into a shock, due to the nonlinearity of the NLS equation itself.

Golovchenko *et al.* [62] theorized that the shock term would result in the instability of the bound state so that an N^{th}-order soliton would decay into N fundamental solitons, and would also result in a frequency shift of the solitons to *higher* frequencies in the negative (anomalous) dispersion region

Bourkoff *et al.*, in their numerical simulation [64], showed that in a fiber with *normal* dispersion, the combination of the shock term *and* cubic dispersion results in a shift in the soliton center frequency towards higher frequency. This effect in normal dispersion fibers had been reported by Knox *et al.* [18].

6.3.2 Exact Radial Dependence

Two researchers at Johns Hopkins published a theory in 1984-85 that represented a unique approach and which made some special claims. Christodoulides and Joseph (CJ) [66] pointed out that while the bright pulse solution given by Hasegawa and Tappert had been verified [3], the dark pulse solution [2] had not. CJ maintained that the dark pulse solution was not valid and would never be found due to a basic fault of the original model: the failure to consider the radial dependence of the field in an exact manner. To do a more rigorous analysis, CJ derived a partial differential equation for the scalar amplitude $A(r, \xi)$, where $\xi = z - v_g t$, and then made the substitution

$$A(r, \xi) = sech(p\xi) \sum_{n=0}^{\infty} R_n(r) sech^{2n}(p\xi) \qquad [6.3.2]$$

where $p = \frac{1}{\tau v_g}$. After substituting [6.3.2] into their equation, they obtained an infinite set of differential equations for the R_n, which could be solved recursively.

The result was the derivation of conditions required for bright pulse propagation. These conditions reduced to results obtained previously for weak dispersion. In addition, the previous dark pulse solitons were shown to be nonexistent, but *new* dark solitons were found. These new dark solitons were called "bactrian", or double-humped (as opposed to "dromedary", or single-humped, like the bright pulse solutions).

In another paper the following year, these two researchers derived a new evolution equation without invoking the slowly-varying envelope approximation, and then showed that this equation admitted femtosecond solitons at certain operating points [67]. These soliton solutions have not been verified experimentally.

6.3.3 The Soliton Self-Frequency Shift

Despite the development of several femtosecond models, experimentalists at Bell Labs were surprised in 1986 when they tried to extend soliton experiments into the femtosecond regime and discovered the "soliton self-frequency shift", a continuous downshift in the center frequency of a femtosecond pulse propagating in a nonlinear optical fiber [45]. The shift was observed to be 8 THz for 260 fs pulses, and 20 THz for 120 fs pulses, corresponding to 4% and 10% of the soliton carrier frequency, respectively. This effect was so surprising because it was particularly pernicious and had not been predicted by any of the new theories. In fact, the frequency shift predicted by

Golovchencko *et al.* for the anomalous region was in the direction *opposite* to that observed. (The frequency shift predicted by Bourkoff *et al.* was for the *normal* dispersion region, and was also in the opposite direction to the self-frequency shift, which was found in the *anomalous* dispersion region. Whether there is a direct relationship between the sign of the dispersion and the direction of the shift has not been verified.) The effect of the discovery was to destroy all hope of soliton communications systems using pulses of width of less than a few picoseconds. However, since the effect seemed to scale as τ^{-4}, it would be negligible for pulses in the picosecond regime. Since the optimal scheme [32] would use 20-ps pulses, the self-frequency shift would not be a problem. The researchers who discovered the shift immediately proposed that it could be used to produce soliton pulses of frequency outside the tuning range of the source laser. Gordon of Bell Labs [68] quickly developed a theory which approximately explained the results. The effect was not fully explained until the development of a rigorous theory by Kodama and Hasegawa in 1987. This research is reported in the next section.

6.3.4 The Kodama-Hasegawa Model

Following the discovery of the soliton self-frequency shift, it became clear that for further exploitation of the properties of solitons in optical fibers, and for rigorous justification of the claims of feasibility of the designed communications systems, a model valid in the femtosecond regime would have to be developed and carefully verified against experimental results. Such a theory was published by Kodama and Hasegawa in May of 1987 [15]. The derivation of this model is beyond the scope of this thesis. Therefore, the KH model, in terms of the simplified TE coefficients, will simply be given here:

$$j\left(\frac{\partial\psi}{\partial z} + \beta_o'\frac{\partial\psi}{\partial t}\right) - \frac{\beta_o''}{2}\frac{\partial^2\psi}{\partial t^2} + \nu|\psi|^2\psi$$

$$- j\frac{\beta_o'''}{6}\frac{\partial^3\psi}{\partial t^3} + ja_1\frac{\partial}{\partial t}(|\psi|^2\psi) + ja_2\psi\frac{\partial}{\partial t}|\psi|^2 = 0. \quad [6.3.3]$$

Formulas giving the values of the coefficients ν, a_1, and a_2 are given in the paper, and will be ignored here in order to concentrate on the general *qualitative* effect of the terms.

The terms on the first line constitute the nonlinear Schrödinger operator. These terms, with the addition of a loss term as in Eq. [5.1.1], form a sufficient model of propagation in the picosecond regime. The first term on the second line is the third-order dispersion term which had been included in many derivations. The second term on the second line is the shock term which had been added as a correction for propagation in the femtosecond regime. The final term, which we shall call the Raman term, had been neglected in all previous derivations, but is significant in the femtosecond regime, as will be explained shortly. After derivation of this equation, Kodama and Hasegawa studied the effect of the three higher-order terms (the ones on the last line) to determine their effects. These effects are explained here.

Third-order dispersion

It had already been shown the the third-order dispersion term has a negligible effect on the propagation of a fundamental soliton [10]. KH now studied the effects of the third-order dispersion on the propagation of a higher-order soliton, *i.e.,* a bound state of N fundamental solitons. It was found that the higher-order dispersion perturbs the eigenvalues, and thus the velocities, of the solitons in the bound state, so that the bound state decays into N fundamental solitons, each propagating at a slightly different velocity. Thus in a pulse compression experiment using the soliton effect, if the higher-order dispersion is significant for the pulse width selected, the higher-order soliton will not be stable but will decay if given sufficient propagation distance. This is one explanation of the deviation of experimental results from the theoretical results in soliton pulse-compression experiments. Since soliton communications systems would use fundamental solitons, this effect would not be present.

The shock term

KH pointed out that the NLS including this term is solvable by the IST, and showed that the shock term has no effect on single soliton propagation.

The Raman term

The final term in [6.3.3] was shown to be responsible for the soliton self-frequency shift. The effect of this term is to produce an induced Raman process that pumps energy from higher-frequency components to the lower. The amount of frequency shift is proportional to the distance of propagation as well as the inverse fourth power of the pulse width, consistent with experimental observation [45] and the earlier theory by Gordon [68]. This scaling makes this Raman effect negligible for picosecond solitons, as would be used in a communication system.

This completes the review of the Kodama-Hasegawa model, which represents the best theory of soliton propagation to date. Several conclusions and recommendations are made in the next chapter.

7.0 Conclusions and Recommendations

From a review of the information presented in previous chapters, several conclusions can be drawn:

1. Agreement between theory and experiment indicates that the phenomenon of soliton propagation in optical fibers is understood and can be quantitatively predicted well into the femtosecond regime. For picosecond pulses, as would be used in a communication system, propagation is adequately described by the simple nonlinear Schrödinger equation model, with the addition of the loss term, such as Eq. [5.1.1]. For pulses of width less than a picosecond, the femtosecond model (Eq. [6.3.1]) should be used to account for the higher-order effects, such as the soliton self-frequency shift and higher-order dispersion, which become significant in the femtosecond regime. The model equations can sometimes be solved by the inverse scattering transform or perturbation theory, and if not, by the propagating beam method.

2. Soliton propagation in optical fibers has been exploited to great benefit in several applications. Present applications of soliton effects include production of narrow pulses by the soliton laser, pulse compression by the periodic property of higher-order solitons, and various applications in devices and measurements.

3. An important foreseeable application is in optical communication systems. The properties of solitons allow the design of systems with length-rate products at least an order of magnitude higher than is attainable with conventional linear systems. It can probably be safely expected that the next generation of optical communication systems after the dispersion-shifted and dispersion-flattened systems now under development will use the properties of solitons to increase bandwidth. The economic advantages of nonlinear systems listed in Chapter 5 will make such systems very attractive for high bit-rate, long-distance (such as transoceanic) systems.

Therefore, several proposals for further research are made:

1. *The subsystems of the proposed soliton communication system should be developed.* The necessary devices include efficient wavelength-dependent couplers for Raman pump wave injection, high-speed sources and detectors, wavelength multiplexing techniques, and tapping devices which take advantage of the large number of photons per soliton pulse.

2. *Experimentation in both the picosecond and femtosecond regimes should continue in order to increase understanding and make further improvements on theory.* The recent history of nonlinear optics has shown the importance of comparing theory to experimental results. The recently developed Kodama-Hasegawa model should be rigorously verified by experiment.

3. *New applications of soliton propagation, particularly in the femtosecond regime, should be investigated.* Some promising applications which have been proposed are optical switching by bistable soliton propagation, and pulse frequency tuning by the soliton self-frequency shift.

4. *A coupled-mode model which does not require the slowly-varying envelope approximation should be developed.* The coupled-mode model developed by Crosignani *et al.* uses a mixed frequency- and time-domain expression for the index of refraction, in violation of the strictly frequency-domain nature of the coupled-mode theory. This requires invoking a slowly-varying envelope

approximation, thus limiting the usefulness of the theory to pulses of sufficient width. A more rigorous coupled-mode model, formulated entirely in the frequency domain, would be a natural tool for studying soliton propagation in multimode fibers, but would not be limited to picosecond regime as is the model of Crosignani *et al.*

5. *Soliton propagation in multimode fibers should be experimentally verified.* The theoretical results of Crosignani *et al.* [34] could be used a a guide to study nonlinear multimode fibers in the picosecond regime.

6. *The existence and usefulness of dark pulse solutions of the nonlinear Schrödinger equation should be investigated.* While the bright pulse solution has been verified and exploited in a number of applications, the dark (concave up) pulse solutions derived by Hasegawa and Tappert [2] and by Christodoulides and Joseph [66] have never been verified. Since these pulses would exist in the anomalous dispersion regime, they would not be useful as signal carriers, since a communication system would operate in the normal dispersion regime around the loss minimum at 1.55 μm. However, discovery of and experimentation with dark pulses could lead to the development of new applications and a better understanding of the theory of soliton propagation.

The exploitation of solitons in engineering gives the promise of many profitable new products, systems, and techniques for industry, and could provide worthwhile direction for many research projects in academia for years to come. If optical soliton communications systems can be developed as envisioned, there will be a profound effect on the optics industry and on other communications industries, such as satellite communications. Developments in soliton studies should be watched carefully over the next few years.

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