ATTENUATION OF ELECTROMAGNETIC RADIATION BY WATER DROPLETS IN THE ATMOSPHERE

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SYMBOLS

$\bar{A}$	Magnetic vector potential.
A_r^i	Incident radial component of the magnetic vector potential.
A_r^s	Scattered radial component of the magnetic vector potential.
A_r^+	Radial component of the magnetic vactor potential outside the droplet of water.
A_r^-	Radial-component of the magnetic vector potential inside the droplet of water.
$\bar{E}^i$	Incident electric-field vector.
E_x^i	x-component of the incident electric field.
$\bar{E}^+$	Electric-field vector outside the droplet of water.
$\bar{E}^-$	Electric-field vector inside the droplet of water.
$\bar{E}^s$	Scattered electric-field vector.
$\bar{F}$	Electric vector potential.
F_r^i	Radial-component of the incident electric vector potential.
F_r^s	Radial-component of the scattered electric vector potential.
F_r^+	Radial-component of the electric vector potential outside the droplet of water.
F_r^-	Radial-component of the electric vector potential inside the droplet of water.
$\bar{H}$	Magnetic field vector.
H_y^i	y-component of the incident magnetic-field.

$\bar{H}^+$	Magnetic-field vector outside the droplet of water.
$\bar{H}^-$	Magnetic-field vector inside the droplet of water.
$\bar{H}^s$	Scattered magnetic-field vector.
$\bar{H}^*$	Complex conjugate of the magnetic-field vector.
$\hat{H}_n^{(2)}(x)$	Alternative spherical Hankel function of the second kind.
$\hat{H}_n^{(2)\prime}(x)$	First derivative of $\hat{H}_n^{(2)}(x)$ with respect to x.
$\hat{J}_n(x)$	Alternative spherical Bessel function of the first kind.
$\hat{J}_n^{\prime}(x)$	First derivative of $\hat{J}_n(x)$ with respect to x.
P_a	Power absorbed by the droplet of water.
$P_n(x)$	Legendre polynomial of degree n.
$P_n^1(x)$	Associated Legendre function of the first order.
$P_n^{\prime}(x)$	First derivative of $P_n(x)$ with respect to x.
P_s	Scattered power.
a	Radius of the droplet of water.
$j_n(x)$	Spherical Bessel function of the first kind.
k	Wave number.
k_o	Wave number in free space.

k_d Wave number in water $(2\pi f \sqrt{u_o \epsilon_d} = \frac{2\pi}{\lambda_d})$

$\overline{U}_r, \overline{U}_\theta, \overline{U}_\varphi$ Unit vectors along the spherical-coordinate axes.

$\overline{U}_x, \overline{U}_y, \overline{U}_z$ Unit vectors along the rectangular-coordinate axes.

v Terminal velocity of the rain drop.

α Attenuation constant.

ν Viscosity of air.

σ_a Absorption cross-sectional area.

σ_s Scattering cross-sectional area.

σ Conductivity.

λ_o Wave length in free space

λ_d Wave length in water.

ϵ_o Permittivity of free space.

ϵ_d Permittivity of water

u_o Permeability of free space

η Intrinsic wave impedance

η_o Intrinsic wave impedance in free space

ϵ' Real component of the complex permittivity (a-c Capacitivity)

ϵ'' Imaginary component of the complex permittivity (dielectric loss factor).

ϵ_r Relative permittivity.

$[\bar{E} \times \bar{H}^*]_r$ Radial component of $\bar{E} \times \bar{H}^*$.

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CHAPTER I

INTRODUCTION

1.1 Sources of Attenuation of Electromagnetic Radiation in the Atmosphere

The attenuation of radio waves through propagation in the atmosphere is due chiefly to the following:

1. Rainfall and suspended water droplets (including clouds)

2. Oxygen

3. Water vapor

This paper deals with the theoretical analysis of the effects of suspended water droplets and rain drops on the propagation of a linearly polarized uniform plane wave through the atmosphere in the frequency range from 10 M Hz. up to the frequency that makes $k_d a \leq 0.7$ at which this analysis is valid, only water droplets of radius less than or equal to 0.11 λ_d is considered.

The presence of water droplets in the atmosphere effects electromagnetic propagation in two ways: (1) Scattering - reflection of electromagnetic energy due to the discontinuity of the medium at the boundary of the droplets. (2) Absorption - which can be classified into two forms; (a) That due to the conductivity of water which causes conductivity loss. (b) That due to the dielectric loss factor of the water droplet (, i.e., ϵ_d'') which causes the dielectric loss.

The author uses the same procedure used by Harrington in the reference "Time Harmonic Electromagnetic fields". Harrington derived the scattering cross-sectional area for a perfect conducting sphere and then mentioned briefly the procedure for solving the boundary value problem of a loss-less dielectric. The author in this thesis extended this to the case of a water droplet in the atmosphere and derived the fields inside the droplet as well as the scattered fields. From these fields approximate expressions for the scattered power, scattering cross-sectional area, the power absorbed by the droplet and absorption cross sectional area are derived.

In this paper we find that the scattered power changes very rapidly with the change in frequency and increases very rapidly with the increase of the size of the droplet. In fact the power scattered is a function of $a^6 f^4$.

It is shown that the absorption loss due to the water conductivity is very small compared with the loss due to the dielectric loss factor of water (dielectric loss) in the frequency range of interest. The major portion of the absorption loss is due to the θ-component of the electric field inside the water droplet. It is shown that absorption loss is much greater than the loss due to scattering. Because the scattered power is very small compared with the dissipated power, we can safely neglect multiple scattering in this analysis.

Finally an expression is derived for the attenuation constant of a homogeneous distribution of rain droplets.

1.2 The Procedure for Analyzing the Scattering and Absorption by a Water Droplet

In this paper, the incident wave is assumed to have the following characteristics:

(1) It is a linearly polarized plane wave in the x-direction, which means that the electric-field $\bar{E}$ has only an x-component ; i.e., $\bar{E} = E_x \bar{U}_x$. Furthermore, the equiphase surface of the instantaneous electric field is a plane.

(2) It is a uniform plane wave, which means that the magnitude of the electric field intensity is constant over any equiphase plane.

(3) It is propagating in the z-direction.

(4) It is sinusoidally time varying.

The medium through which the electromagnetic wave propagates exclusive of the water droplets is linear, homogeneous, isotropic and source free. In this medium we can write Maxwell's equations in the form

$$- \nabla \times \overline{E} = j \omega \mu \overline{H} \qquad (1\cdot2\cdot1)$$

$$\nabla \times \overline{H} = (\sigma + j \omega \epsilon)\overline{E} \qquad (1\cdot2\cdot2)$$

In Eq. (1·2·1); μ is the permeability of the medium, which is free space in this analysis; i.e., $\mu = \mu_o = 4\pi(10^{-7})$. From Equation (1·2·1) and the relation $\overline{E} = E_x \overline{U}_x$, it follows that

$$j \omega \mu_o \overline{H} = - \frac{\partial E_x}{\partial z} \overline{U}_y$$

from which we get

$$\overline{H} = H_y \overline{U}_y = - \frac{\partial E_x}{\partial z} \cdot \frac{1}{j\omega\mu_o} \overline{U}_y \qquad (1\cdot2\cdot2a)$$

Therefore $\overline{H}$ has only a y-component and neither the electric field nor the magnetic field has a component in the z-direction; i.e., the wave is transverse electromagnetic to z as was pointed out before; i.e., $E_x = E_o e^{-jk_o z}$

where

$$k_o = \omega \sqrt{\mu_o \epsilon_o}$$

Substituting for E_x into equation (1·2·2a) we get,

$$H_y = \frac{k_o}{\omega\mu_o} E_o e^{-jk_o z}$$

therefore

$$\frac{E_x}{H_y} = \frac{\omega\mu_o}{k_o} = \sqrt{\frac{\mu_o}{\epsilon_o}} = \eta_o = 120\pi$$

Generally for a uniform plane travelling wave in a lossless dielectric we have

$$\eta = \frac{|\bar{E}|}{|\bar{H}|}$$

Let us now examine equation (1·2·2). For a good dielectric we have the following two conditions

(1) $\sigma \cong 0$

(2) $\epsilon = \epsilon' - j\epsilon''$ where $\epsilon' >> \epsilon''$ [1]

i.e., equation (1·2·2) can be written as

$$\nabla \times \bar{H} \cong j \omega \epsilon' \bar{E}$$

Let us now consider the problem of finding the effects of the suspended water droplets in the atmosphere on the propagation of the given transverse electromagnetic wave. The approach to the problem is as follows.

Any field can be resolved into two components, one being transverse electric to any particular direction (no electric field in that direction) and the other being transverse magnetic to the same direction[2]. We can regard the incident field as composed of two components, one transverse electric to the unit vector $\bar{U}_r$ and the other transverse magnetic to the same unit vector. We have complete freedom in choosing the direction of the unit vector $\bar{U}_r$, therefore, we choose the radial vector in a spherical-coordinate system to coincide with the spherical configuration of a water droplet. This direction provides simpler results than any other direction in the spherical system.

The procedure for developing the relations for radio frequency scattering and absorption by a water droplet is as follows:

(1) From the incident electric and magnetic field, we find the radial-components of the incident fields (i.e., E_r^i and H_r^i)

(2) From E_r^i and H_r^i we can find A_r^i and F_r^i (A_r^i produces a field T. M. to $\bar{U}_r$ and F_r^i produces a field T. E. to $\bar{U}_r$) using the equations (see Appendix B Eqs. (B-22) and (B-25))

$$E_r^i = \frac{1}{j\omega\epsilon_o}\left(\frac{\partial^2}{\partial r^2} + k_o^2\right) A_r^i \qquad (1.2.3)$$

and

$$H_r^i = \frac{1}{j\omega\mu_o}\left(\frac{\partial^2}{\partial r^2} + k_o^2\right) F_r^i \qquad (1.2.4)$$

(3) The forms for A_r^s and F_r^s can be obtained using the configuration of the droplet as will be shown later. Both forms of A_r^s and F_r^s contain a multiplicative constant which must be evaluated by applying the boundary conditions. Note that the centre of the droplet is taken as the origin for the spherical-coordinate

system; i.e., $r = a$ identifies the boundary of the droplet.

(4) We find the forms of A_r^- and F_r^-, the magnetic and electric vector potentials inside the droplet, making use of the fact that the point $r = 0$ is inside the droplet. Now we are in a position that enables us to find the total vector potentials outside the droplet, A_r^+ and F_r^+, where $A_r^+ = A_r^i + A_r^s$ and $F_r^+ = F_r^i + F_r^s$, the sum of the incident and scattered vector potentials.

(5) After all components of the electric and magnetic vector potentials are known inside and outside the water droplet, we can determine the transverse components of the electric and magnetic fields inside and outside the water droplet using these equations (see Appendix B Eqs. (B-23), (B-24), (B-26), & (B-27))

$$E_\theta = \frac{-1}{r \operatorname{Sin}\theta} \frac{\partial F_r}{\partial \varphi} + \frac{1}{\hat{y}\, r} \frac{\partial^2 A_r}{\partial r \partial \theta},$$

$$\text{where } \hat{y} = \sigma + j\,\omega\,\epsilon \tag{1.2.5}$$

$$E_\varphi = \frac{1}{r} \frac{\partial F_r}{\partial \theta} + \frac{1}{\hat{y}\, r \operatorname{Sin}\theta} \frac{\partial^2 A_r}{\partial r \partial \varphi} \tag{1.2.6}$$

$$H_\theta = \frac{1}{r \operatorname{Sin} \theta} \frac{\partial Ar}{\partial \phi} + \frac{1}{\hat{z}\, r} \frac{\partial^2 Fr}{\partial r \partial \theta},$$

$$\text{where } \hat{z} = j \omega u \qquad (1{\cdot}2{\cdot}7)$$

$$H_\phi = -\frac{1}{r} \frac{\partial Ar}{\partial \theta} + \frac{1}{\hat{z}\, r \operatorname{Sin} \theta} \frac{\partial^2 Fr}{\partial r \partial \phi} \qquad (1{\cdot}2{\cdot}8)$$

It is to be noted that each expression for A_r^s, A_r^-, F_r^s and F_r^- contain a constant and these constants are to be determined by matching the transverse fields at the boundary $r = a$

(6) The above 5-steps (a, b, c, d, e) enable us to determine the fields, and from these fields we can find

(a) The scattered power and the scattering cross-sectional area.

(b) The power dissipated within the water droplet and the absorption cross-sectional area.

From these steps (a, b) we can find the attenuation constant.

<u>Summary:</u>

In this section we briefly discussed the procedure we are going to use for developing the scattering and absorpton of an electromagnetic wave by a droplet of water in the atmosphere. The procedure is divided into 8 major sections. These sections are

(1) Determination of E_r^i and H_r^i.

(2) Determination of A_r^i and F_r^i.

(3) Determination of A_r^s and F_r^s.

(4) Determination of A_r^- and F_r^-.

(5) Determination of the scattering cross-sectional area.

(6) Determination of the power loss due to absorption.

(7) Comparison between the power loss due to scattering and that due to absorption.

(8) Determination of the attenuation constant.

The Second Chapter (Scattering of Electromagnetic wave by a droplet of water in the atmosphere) covers the first 5 sections and the third chapter (Absorption of Electromagnetic wave by a droplet of water in the atmosphere) covers the last 3 sections.

CHAPTER II

SCATTERING OF AN ELECTROMAGNETIC WAVE BY A DROPLET OF WATER IN THE ATMOSPHERE

2.1 Introduction

In this chapter we are going to solve the boundary value problem of a droplet of water in the atmosphere that causes scattering and absorption of an incident electromagnetic wave. From the solution we get the fields inside and outside the droplet, then we determine the scattering power and the scattering cross-sectional area.

2.2 Determination of the Radial Components of the Incident Fields

From the given incident fields we know that

$$\bar{E}^i = E_o \; e^{-jk_oz} \; \bar{U}_x \tag{2.2.1}$$

and

$$\bar{H}^i = \frac{E_o}{\eta_o} \; e^{-jk_oz} \; \bar{U}_y \tag{2.2.2}$$

Since $z = r \cos\theta$, $E_r^i = E_x^i \cos\varphi \sin\theta$ and $H_r^i = H_y^i \cdot \sin\varphi \sin\theta$

we get

$$E_r^i = E_o \; e^{-jk_or\cos\theta} \cos\varphi \sin\theta \tag{2.2.3}$$

and

$$H_r^i = \frac{E_o}{\eta_o} e^{-jk_o r \cos\theta} \sin\varphi \sin\theta \qquad (2.2.4)$$

Since $e^{-jk_o r \cos\theta}$ can be written as

$$\frac{1}{jk_o r \sin\theta} \frac{\partial}{\partial\theta} (e^{-jk_o r \cos\theta})$$

therefore E_r^i can be written as

$$E_r^i = \frac{E_o \cos\varphi}{jk_o r} \frac{\partial}{\partial\theta} (e^{-jk_o r \cos\theta}) \qquad (2.2.5)$$

Similarly we can find

$$H_r^i = \frac{E_o \sin\varphi}{j k_o r \eta_o} \frac{\partial}{\partial\theta} (e^{-jk_o r \cos\theta}) \qquad (2.2.6)$$

[3] Using the relation $e^{jk_o r \cos\theta} = \sum_{n=o}^{\infty} j^n (2n + 1) \, j_n (k_o r) P_n (\cos\theta)$, we can write the expression for E_r^i in term of the spherical Bessel function of the first kind $[j_n (k_o r)]$ and the Legendre polynomials $[P_n (\cos\theta)]$; i.e.,

$$E_r^i = \frac{E_o \cos\varphi}{jk_o r} \frac{\partial}{\partial\theta} \sum_{n=o}^{\infty} j^{-n} (2n + 1) \, j_n(k_o r) P_n(\cos\theta)$$

Using the relation $\frac{\partial Pn (\cos\theta)}{\partial\theta} = P_n^1 (\cos\theta)$

and letting $(k_o r)\, j_n\, (k_o r) = \hat{J}_n\, (k_o r)$, the relation for E_r^i becomes

$$E_r^i = -j\, E_o \frac{\cos \varphi}{(k_o r)^2} \sum_{n=o}^{\infty} j^{-n}\, (2n + 1)\, \hat{J}_n\, (k_o r)\, P_n^1\, (\cos \theta)$$

$$= -j\, E_o \frac{\cos \varphi}{(k_o r)^2} \left[\hat{J}_o\, (k_o r)\, P_o^1\, (\cos \theta) + \sum_{n=1}^{\infty} j^{-n}\, (2n + 1)\, \hat{J}_n\, (k_o r)\, P_n^1\, (\cos \theta) \right]$$

and since $P_o^1\, (\cos \theta) = 0$ for all values of θ therefore

$$E_r^i = \frac{-j\, E_o \cos \varphi}{(k_o r)^2} \sum_{n=1}^{\infty} j^{-n}\, (2n + 1)\, \hat{J}_n\, (k_o r)\, P_n^1\, (\cos \theta) \tag{2·2·7}$$

Similarly

$$H_r^i = \frac{-j\, E_o \sin \varphi}{\eta_o\, (k_o r)^2} \sum_{n=1}^{\infty} j^{-n}\, (2n + 1)\, \hat{J}_n\, (k_o r)\, P_n^1\, (\cos \theta) \tag{2·2·8}$$

Having found the radial components of the incident field (Eq. 2·2·7, 2·2·8), we now determine the vector potential functions.

2.3 Determination of the Incident Vector Potential Functions (A_r^i, F_r^i)

Equations (1·2·3) and (1·2·4), that give A_r^i and F_r^i are based upon the following relations;

The general equations for vector potentials are

(a) $\nabla \times \nabla \times \bar{A} - k^2\bar{A} = -\hat{y}\, \nabla \phi^a$ (See Eq. (B-7))

where $\hat{y} = \sigma + j\,\omega\,\epsilon$ in general and $= j\,\omega\,\epsilon$ in lossless medium.

(b) $\nabla \times \nabla \times \bar{F} - k^2\bar{F} = -\hat{z}\, \nabla \phi^b$ (See Eq. (B-8))

where $\hat{z} = j\,\omega\,u_o$ in non magnetic medium.

ϕ^a, ϕ^b are arbitrary scalars. If we choose ϕ^a such that $-\hat{y}\,\phi^a = \dfrac{\partial Ar}{\partial r}$ and ϕ^b such that $-\hat{z}\,\phi^b = \dfrac{\partial Fr}{\partial r}$,

In Appendix (B) it is shown that

(a) A_r satisfies the relation $(\nabla^2 + k^2)\, \dfrac{A_r}{r} = 0$

(See Eq. (B-14))

(b) F_r satisfies the relation $(\nabla^2 + k^2)\, \dfrac{F_r}{r} = 0$

(c) $\bar{A}$ and $\bar{F}$ are defined as having only radial components; i.e., $\bar{A} = \bar{U}_r\, A_r$ and $\bar{F} = \bar{U}_r\, F_r$

(d) $\bar{A}$ gives a field transverse magnetic to $\bar{U}_r$ while $\bar{F}$ gives a field transverse electric to $\bar{U}_r$

Equation (1.2.4) is

$$E_r^1 = \frac{1}{j\omega\epsilon_o} \left(\frac{\partial^2}{\partial r^2} + k_o^2\right) A_r^1 \qquad (1.2.4)$$

Using equation (B- 16) in Appendix B, we have

$$A_r^1 = \sum_{n=o}^{\infty} \sum_{m=o}^{\infty} G_{mn}\, \hat{B}_n (k_o r)\, L_n^m (\cos\,\theta)\; h(m\phi)$$

where G_{mn} are constants

Since the incident field is finite at $\theta = 0$ and $\theta = \pi$, then $L_n^m (\cos \theta) = P_n^m (\cos \theta)$ (See Eq. (A-6b) in Appendix A and the accompanying discussions.) Also $h(m\varphi)$ must be $e^{\pm jm\varphi}$ (See Eq. (A-6a) in Appendix A and the accompanying discussions.) $\hat{B}_n (k_o r)$ are spherical Bessel's functions. Thus Eq. (B-16) becomes

$$A_r^i = \sum_{n=o}^{\infty} \sum_{m=o}^{\infty} G_{mn} \hat{B}_n (k_o r) P_n^m (\cos \theta) e^{\pm jm\varphi} \qquad (2\cdot3\cdot1)$$

Substituting the above expression for A_r^i into Eq. (1·2·4), we get

$$E_r^i = \frac{1}{j\omega\epsilon_o} \sum_{n=o}^{\infty} \sum_{m=o}^{\infty} G_{mn} P_n^m (\cos \theta) e^{\pm jm\varphi} \cdot \left\{ \frac{d^2 \hat{B}_n (k_o r)}{dr^2} + k_o^2 \hat{B}_n (k_o r) \right\} \qquad (2\cdot3\cdot2)$$

From Eq. (B-17), we have

$$\frac{d^2}{dr^2} \hat{B}_n (k_o r) + k_o^2 \hat{B}_n (k_o r) = \frac{n(n+1)}{r^2} \hat{B}_n (k_o r)$$

Substituting the above relation into Eq. (2·3·2), we get

$$E_r^i = \frac{1}{j\omega\epsilon_o r^2} \sum_{n=o}^{\infty} \sum_{m=o}^{\infty} G_{mn} P_n^m (\cos \theta) e^{\pm jm\varphi} \cdot \hat{B}_n (k_o r) n(n+1). \qquad (2\cdot3\cdot3)$$

Comparing Eq. (2·3·3) with Eq. (2·2·7), we get

$m = 1$, $E_o \dfrac{(2n+1)\ j^{-n}}{n(n+1)} = \omega\mu_o\ G_{mn}$, $\hat{B}_n\ (k_o r) = \hat{J}_n\ (k_o r)$,

$e^{\pm jm\phi} = \cos\phi$ and $n \geq 1$ From the above relations and Eq. (2.3.1), we get

$$A_r^i = \frac{E_o}{\omega\mu_o} \sum_{n=1}^{\infty} \cos\phi \quad a_n \quad \hat{J}_n\ (k_o r)\ P_n^1\ (\cos\theta) \quad (2.3.4)$$

where

$$a_n = \frac{j^{-n}\ (2n+1)}{n(\ n+1)} \quad (2.3.4a)$$

Equation (2.3.4) gives the value of the radial component of the incident magnetic vector potential (A_r^i) in terms of the magnitude of the incident electric field (E_o) and the wave number in free space (k_o).

Repeating the same procedure for H_r^i and F_r^i we get

$$F_r^i = \frac{E_o \sin\phi}{k_o} \sum_{n=1}^{\infty} a_n \quad \hat{J}_n\ (k_o r)\ P_n^1\ (\cos\theta) \quad (2.3.5)$$

where a_n is given by Eq. (2.3.4a)

Summary:

In this section we derived the relations for A_r^i, the incident radial magnetic vector potential, and F_r^i, the incident radial component of the electric vector potential. Note that $\bar{A}^i = \bar{U}_r\ A_r^i$ and $\bar{F}^i = \bar{U}_r\ F_r^i$. The equations for A_r^i and F_r^i are

$$A_r^i = \frac{E_o}{\omega\mu_o} \cos \varphi \sum_{n=1}^{\infty} j^{-n} \frac{(2n+1)}{n(n+1)} \hat{J}_n (k_o r) P_n^1 (\cos \theta) \qquad (2.3.4)$$

and

$$F_r^i = \frac{E_o}{k_o} \sin \varphi \sum_{n=1}^{\infty} j^{-n} \frac{(2n+1)}{n(n+1)} \hat{J}_n (k_o r) P_n^1 (\cos \theta) \qquad (2.3.5)$$

2.4 Determination of the Scattered Magnetic and Electric Vector Potentials (A_r^s and F_r^s) Forms

Using the above results we may now find the scattered electric and magnetic vector potentials, and consequently, the scattered electric and magnetic fields.

Since the scattered field is an outward travelling wave and the field must be finite as $r \longrightarrow \infty$, the scattered electric and magnetic vector potentials have the same forms as the incident ones with $\hat{J}_n$ (kr) replaced by $H_n^{(2)}$(kr), differing by a multiplying constant (See Eq. (B-21)). Thus we can write A_r^s and F_r^s as

$$A_r^s = \frac{E_o}{\omega\mu_o} \cos \varphi \sum_{n=1}^{\infty} b_n \hat{H}_n^{(2)} (k_o r) P_n^1 (\cos \theta) \qquad (2.4.1)$$

where b_n are constants to be determined from the boundary conditions and

$$F_r^s = \frac{E_o}{k_o} \sin \varphi \sum_{n=1}^{\infty} c_n \hat{H}_n^{(2)} (k_o r) P_n^1 (\cos \theta) \qquad (2.4.2)$$

where c_n are constants to be determined from the boundary conditions.

Outside the droplet we have $A_r^+ = A_r^i + A_r^s$, the total radial component of the magnetic vector potential,

$$A_r^+ = \frac{E_o}{\omega\mu_o} \cos\phi \sum_{n=1}^{\infty} [a_n \hat{J}_n (k_o r) + b_n \hat{H}_n^{(2)} (k_o r)] P_n^1 (\cos\theta) \tag{2.4.3}$$

and F_r^+ is the total radial component of the electric vector potential outside the droplet as,

$$F_r^+ = \frac{E_o}{k_o} \sin\phi \sum_{n=1}^{\infty} [a_n \hat{J}_n (k_o r) + c_n \hat{H}_n^{(2)} (k_o r)] P_n^1 (\cos\theta) \tag{2.4.4}$$

Summary

In this section we derived the expressions for the scattered electric and magnetic vector potentials; i.e., F_r^s and A_r^s and using these expressions along with the expressions for A_r^i and F_r^i, the incident radial components of the magnetic and electric vector potentials we get the total external vector potentials, A_r^+ and F_r^+. These expressions are

$$A_r^s = \frac{E_o}{\omega\mu_o} \cos\phi \sum_{n=1}^{\infty} b_n \hat{H}_n^{(2)} (k_o r) P_n^1 (\cos\theta), \tag{2.4.1}$$

$$F_r^s = \frac{E_o}{k_o} \sin\varphi \sum_{n=1}^{\infty} c_n \hat{H}_n^{(2)}(k_o r)\, P_n^1(\cos\theta), \qquad (2\cdot4\cdot2)$$

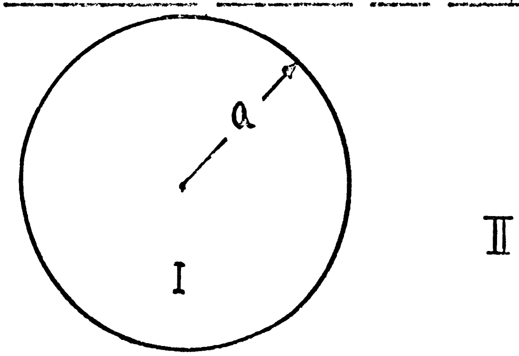
$$A_r^+ = \frac{E_o}{\omega\mu_o} \cos\varphi \sum_{n=1}^{\infty} [a_n \hat{J}_n(k_o r) + b_n \hat{H}_n^{(2)}(k_o r)]\, P_n^1(\cos\theta), \qquad (2\cdot4\cdot3)$$

and

$$F_r^+ = \frac{E_o}{k_o} \sin\varphi \sum_{n=1}^{\infty} [a_n \hat{J}_n(k_o r) + c_n \hat{H}_n^{(2)}(k_o r)]\, P_n^1(\cos\theta). \qquad (2\cdot4\cdot4)$$

Having found the vector potentials outside the droplet of water, we now find the forms of the potentials inside the droplet.

2.5 Determination of the Scattered Vector Potentials and the Vector Potentials Outside the Droplet



Consider region I (inside the droplet), $r < a$. The magnetic vector potential will have the form (See Eq. (B-19)).

$$A_r^- = \frac{E_o}{\omega\mu_o} \cos\varphi \sum_{n=1}^{\infty} d_n \hat{J}_n(k_d r)\, P_n^1(\cos\theta) \qquad (2\cdot5\cdot1)$$

d_n are constants depending only on n which are to be determined from the boundary conditions. Similarly, the electric vector potential F_r^- has the form

$$F_r^- = \frac{E_o}{k_o} \sin \varphi \sum_{n=1}^{\infty} e_n \hat{J}_n (k_d r) P_n^1 (\cos \theta) \qquad (2 \cdot 5 \cdot 2)$$

e_n are constants depending only on n which are to be determined from the boundary conditions.

Because the droplet of water is not a perfect conductor, the boundary condition require that the tangential components of both the electric and magnetic fields are continuous across the surface, r = a. Also since the field is finite inside the droplet including the center of it; i.e., r = 0, thus Eq. (B-19) is chosen to represent the vector potentials inside the droplet [See Appendix (A) Eq. (A-8) and the accompanying discussions, also see Eq. (B-19).]

These conditions can be written mathematically as,

$$\begin{aligned} E_\theta^+ &= E_\theta^- \quad \text{at } r = a \\ E_\varphi^+ &= E_\varphi^- \quad \text{at } r = a \\ H_\theta^+ &= H_\theta^- \quad \text{at } r = a \\ H_\varphi^+ &= H_\varphi^- \quad \text{at } r = a \end{aligned} \qquad (2 \cdot 5 \cdot 3)$$

The above four equations are sufficient to find the four constants b_n, c_n, d_n and e_n. Let us now find E_θ^-, E_φ^-, H_θ^- and H_φ^-

at r = a, we have

$$E_\theta^- = -\frac{1}{r \sin\theta} \frac{\partial F_r^-}{\partial \varphi} + \frac{1}{j\omega\epsilon_d r} \frac{\partial^2 A_r^-}{\partial r \partial \theta} \qquad (2.5.4)$$

(See Eq. (B-23)). Using Eqs. (2.5.1) and (2.5.2), we substitute for A_r^- and F_r^- into the preceding equation therefore,

$$E_\theta^- = -\frac{E_0}{k_0 r \sin\theta} \cos\varphi \sum_{n=1}^{\infty} e_n \hat{J}_n(k_d r) P_n^1 (\cos\theta) +$$

$$\frac{k_d E_0 \cos\varphi}{j\omega^2 \epsilon_d r \mu_0} \sum_{n=1}^{\infty} d_n \hat{J}_n'(k_d r) \cdot \frac{d}{d\theta} [P_n^1 (\cos\theta)]$$

where $k_d = \omega \sqrt{\mu_0 \epsilon_d}$

and $k_0 = \omega \sqrt{\mu_0 \epsilon_0}$

Evaluating at the boundary, r = a,

$$E_\theta^- \Big|_{r=a} = -\frac{E_0}{k_0 a \sin\theta} \cos\varphi \sum_{n=1}^{\infty} e_n \hat{J}_n (k_d a) P_n^1 (\cos\theta) -$$

$$j\frac{E_0}{a k_d} \cos\varphi \sum_{n=1}^{\infty} d_n \hat{J}_n'(k_d a) \frac{d}{d\theta} [P_n^1 (\cos\theta)] \qquad (2.5.4a)$$

Repeating the same procedure using Eq. (B-24) we find

$$E_\varphi^- = \frac{1}{r} \frac{\partial F_r^-}{\partial \theta} + \frac{1}{j\omega\epsilon_d r \sin\theta} \frac{\partial^2 A_r^-}{\partial r \partial \varphi} \qquad (2.5.5)$$

From which we get, at r = a

$$E_\varphi^- \Big|_{r=a} = \frac{E_o}{ak_o} \sin\varphi \sum_{n=1}^{\infty} e_n \hat{J}_n (k_d a) \frac{d}{d\theta} [P_n^1 (\cos\theta)] +$$

$$\frac{jE_o \sin\varphi}{k_d a \sin\theta} \sum_{n=1}^{\infty} d_n \hat{J}_n'(k_d a) \frac{d}{d\theta} [P_n^1 (\cos\theta)].$$

(2.5.5a)

Similarly for H_θ^- using Eq. (B-26) we have,

$$H_\theta^- = \frac{1}{r \sin\theta} \frac{\partial A_r^-}{\partial\varphi} + \frac{1}{j\omega\mu_o r} \frac{\partial^2 F_r^-}{\partial r \partial\theta} \qquad (2.5.6)$$

From which we get, at r = a

$$H_\theta^- \Big|_{r=a} = \frac{-1}{a \sin\theta} \frac{E_o}{\omega\mu_o} \sin\varphi \sum_{n=1}^{\infty} d_n \hat{J}_n (k_d a) P_n^1 (\cos\theta) +$$

$$\frac{E_o k_d \sin\varphi}{j\omega\mu_o a \ k_o} \sum_{n=1}^{\infty} e_n \hat{J}_n' (k_d a) \frac{d}{d\theta} [P_n^1 (\cos\theta)]$$

(2.5.6a)

The value of H_φ^- can be similarly found using Eq. (B-27); i.e.,

$$H_\varphi^- = \frac{-1}{r} \frac{\partial A_r^-}{\partial\theta} + \frac{1}{j\omega\mu_o r \sin\theta} \frac{\partial^2 F_r^-}{\partial r \partial\varphi} \qquad (2.5.7)$$

From which we get at r = a,

$$H_\varphi^- \Big|_{r=a} = \frac{-E_o}{a\omega\mu_o} \cos\varphi \sum_{n=1}^{\infty} d_n \hat{J}_n (k_d a) \frac{d}{d\theta} [P_n^1 (\cos\theta)] +$$

$$\frac{E_o \cos\varphi \ k_d}{j\omega\mu_o a \sin\theta \ k_o} \sum_{n=1}^{\infty} e_n \hat{J}_n' (k_d a) P_n^1 (\cos\theta)$$

(2.5.7a)

To complete the four equations that represent the boundary condition we need to find E_θ^+, E_φ^+, H_θ^+ and H_φ^+ evaluated at r = a. We have

$$E_\theta^+ = -\frac{1}{r \sin\theta} \frac{\partial F_r^+}{\partial \varphi} + \frac{1}{j\omega\epsilon_o r} \frac{\partial^2 A_r^+}{\partial r \partial \theta} \qquad (2\cdot5\cdot8)$$

(See Eq. (B-23)). From Equations (2·4·3) and (2·4·4), we substitute for A_r^+ and F_r^+ into the preceding equation to find $E_\theta^+\Big|_{r=a}$

$$E_\theta^+\Big|_{r=a} = -\frac{E_o \cos\varphi}{a \sin\theta\, k_o} \sum_{n=1}^{\infty} [a_n \hat{J}_n (k_o a) + c_n H_n^{(2)} (k_o a)] P_n^1(\cos\theta) -$$

$$j\frac{E_o}{ak_o} \cos\varphi \sum_{n=1}^{\infty} [a_n \hat{J}_n'(k_o a) + b_n \hat{H}_n^{(2)\prime}(k_o a)] \frac{d}{d\theta} [P_n^1 (\cos\theta)]$$

(2·5·8a)

Similarly for E_φ^+ using Eq. (B-24) we have

$$E_\varphi^+ = \frac{1}{r} \frac{\partial F_r^+}{\partial \theta} + \frac{1}{j\omega\epsilon_o r \sin\theta} \frac{\partial^2 A_r^+}{\partial r \partial \varphi} \qquad (2\cdot5\cdot9)$$

From which we get E_φ^+ evaluated at r = a, therefore

$$E_\varphi^+\Big|_{r=a} = \frac{E_o \sin\varphi}{ak_o} \sum_{n=1}^{\infty} [a_n \hat{J}_n(k_o a) + c_n \hat{H}_n^{(2)}(k_o a)] \cdot$$

$$\frac{d}{d\theta} [P_n^1 (\cos\theta)] + \frac{jE_o \sin\varphi}{k_o a \sin\theta} \sum_{n=1}^{\infty} [a_n \hat{J}_n'(k_o a) +$$

$$b_n \hat{H}_n^{(2)\prime}(k_o a)] [P_n^1 (\cos\theta)] \qquad (2\cdot5\cdot9a)$$

Similarly we find H_θ^+ using Eq. (B-26), we have

$$H_\theta^+ = \frac{1}{r \sin\theta} \frac{\partial A_r^+}{\partial\varphi} + \frac{1}{j\omega\mu_o r} \frac{\partial^2 F_r^+}{\partial r \partial\theta} \tag{2.5.10}$$

From which we get at r = a

$$H_\theta^+\Big|_{r=a} = \frac{-E_o \sin\varphi}{a \sin\theta\, \omega\mu_o} \sum_{n=1}^{\infty} \left[a_n \hat{J}_n (k_o a) + b_n H_n^{(2)} (k_o a)\right] \cdot$$

$$P_n^1 (\cos\theta) + \frac{E_o k_o \sin\varphi}{j\omega\mu_o a k_o} \sum_{n=1}^{\infty} \left[a_n \hat{J}_n (k_o a) +\right.$$

$$\left. c_n \hat{H}_n^{(2)} (k_o a)\right] \frac{d}{d\theta} \left[P_n^1 (\cos\theta)\right] \tag{2.5.10a}$$

H_φ^+ is found using Eq. (B-27),

$$H_\varphi^+ = \frac{-1}{r} \frac{\partial A_r^+}{\partial\theta} + \frac{1}{j\omega\mu_o r \sin\theta} \frac{\partial^2 F_r^+}{\partial r \partial\varphi} \tag{2.5.11}$$

From which we get, at r = a

$$H_\varphi^+\Big|_{r=a} = \frac{-E_o \cos\varphi}{a\omega\mu_o} \sum_{n=1}^{\infty} \left[a_n \hat{J}_n (k_o a) + b_n \hat{H}_n^{(2)} (k_o a)\right] \cdot$$

$$\frac{d}{d\theta} \left[P_n^1 (\cos\theta)\right] + \frac{E_o k_o \cos\varphi}{j\omega\mu_o r \sin\theta\, k_o} \left[\sum_{n=1}^{\infty} \left[a_n \hat{J}_n (k_o a) +\right.\right.$$

$$\left.\left. c_n \hat{H}_n^{(2)} (k_o a)\right] P_n^1 (\cos\theta)\right] \tag{2.5.11a}$$

Equations (2.5.8a), (2.5.9a), (2.5.10a) and (2.5.11a) give the required values for E_θ^+, E_φ^+, H_θ^+ and H_φ^+ evaluated at r = a. Substituting for E_θ^-, E_φ^-, H_θ^-, H_φ^-, E_θ^+, E_φ^+, H_θ^+ and H_φ^+ evaluated at r = a from Eqs. (2.5.4a), (2.5.5a), (2.5.6a), (2.5.7a), (2.5.8a), (2.5.9a), (2.5.10a) and (2.5.11a) into equations (2.5.3) we get

$$b_n = \frac{-\sqrt{\mu_o\epsilon_d}\,\hat{J}_n'(k_oa)\,\hat{J}_n(k_da) + \sqrt{\mu_o\epsilon_o}\,\hat{J}_n(k_oa)\,\hat{J}_n'(k_da)}{\sqrt{\mu_o\epsilon_d}\,\hat{H}_n^{(2)\prime}(k_oa)\,\hat{J}_n(k_da) - \sqrt{\mu_o\epsilon_o}\,\hat{H}_n^{(2)}(k_oa)\,\hat{J}_n'(k_da)}\, a_n$$

$$c_n = \frac{-\sqrt{\mu_o\epsilon_d}\,\hat{J}_n(k_oa)\,\hat{J}_n'(k_da) + \sqrt{\mu_o\epsilon_o}\,\hat{J}_n'(k_oa)\,\hat{J}_n(k_da)}{\sqrt{\mu_o\epsilon_d}\,\hat{H}_n^{(2)}(k_oa)\,\hat{J}_n'(k_da) - \sqrt{\mu_o\epsilon_o}\,\hat{H}_n^{(2)\prime}(k_oa)\,\hat{J}_n(k_da)}\, a_n$$

$$d_n = \frac{-j\sqrt{\mu_o\epsilon_d}}{\sqrt{\mu_o\epsilon_d}\,\hat{H}_n^{(2)\prime}(k_oa)\,\hat{J}_n(k_da) - \sqrt{\mu_o\epsilon_o}\,\hat{H}_n^{(2)}(k_oa)\,\hat{J}_n'(k_da)}\, a_n$$

$$e_n = \frac{j\sqrt{\mu_o\epsilon_o}}{\sqrt{\mu_o\epsilon_d}\,\hat{H}_n^{(2)}(k_oa)\,\hat{J}_n'(k_da) - \sqrt{\mu_o\epsilon_o}\,\hat{H}_n^{(2)\prime}(k_oa)\,\hat{J}_n(k_da)}\, a_n$$

(2.5.12)

where a_n is given by Eq. (2.3.4); i.e.,

$$a_n = \frac{j^{-n}(2n+1)}{n(n+1)}$$

Now we know completely A_r^s, A_r^-, F_r^s and F_r^- because b_n, c_n, d_n and e_n are now known. It is to be noted that each expression of A_r^-, A_r^s, F_r^- and F_r^s contains an infinite number of terms, however only the term corresponding to n = 1 predominates when $k_oa << 1$. (See Appendix C for values of k_oa in the frequency range of interest and for average drop size.)

Using the above approximation, we get[4]

$$b_1 \cong -(k_o a)^3 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)$$

$$c_1 \cong 0$$

$$d_1 \cong \frac{9}{2j(2 + \epsilon_r)} \qquad (2\cdot5\cdot12a)$$

$$e_1 \cong \frac{9}{8j\epsilon_r}$$

Thus we now have approximate expressions for A_r^s, A_r^-, F_r^s and F_r^- that are valid for the frequency range of interest and the average drop size.

Substituting the values of b_1, c_1, d_1 and e_1 from Eq. (2·5·12a) into Eq. (2·4·1), (2·4·2), (2·5·1) and (2·5·2) noting that P_1^1 (cos θ) = - sin θ, we get,

$$A_r^s \cong \frac{E_o \cos\varphi \; (k_o a)^3 \; (\epsilon_r - 1)}{\omega\mu_o \; (\epsilon_r + 2)} \left[\hat{H}_1^{(2)}(k_o r)\right] \sin\theta, \qquad (2\cdot5\cdot13)$$

$$F_r^s \cong 0, \qquad (2\cdot5\cdot14)$$

$$A_r^- \cong \frac{-E_o \cos\varphi \sin\theta}{\omega\mu_o} \left[\frac{9}{2j(2+\epsilon_r)}\right] \hat{J}_1(k_d r) \qquad (2\cdot5\cdot15)$$

and

$$F_r^- \cong \frac{-E_o \sin\varphi \sin\theta}{k_o} \left(\frac{9}{8j\epsilon_r}\right) J_1(k_d r). \qquad (2\cdot5\cdot16)$$

Equations (2·5·13), (2·5·14), (2·5·15) and (2.5.16) give the radial components of the scattered vector potentials inside the droplet.

2.6 Determination of the Scattering Cross-Sectional Area

The scattered fields can be determined completely because we have determined A_r^s and F_r^s in section (2.5). From the scattered fields we find the scattered power using the relation

$$P_s = \iint (\bar{E} \times \bar{H}^{s*}) \cdot \overline{ds} \quad [5]$$

where the integration is taken over a sphere of large radius, in other words the scattered fields, $\bar{E}^s$ and $\bar{H}^s$, are the far fields; i.e., a great distance from the scatterer, and $\overline{ds}$ is the element of area on the spherical surface, i.e., $\overline{ds} = r^2 \sin\theta \, d\theta d\varphi \, \bar{U}_r$. Therefore,

$$P_s = \int_0^{2\pi} \int_0^{\pi} [\bar{E}^s \times \bar{H}^{s*}]_r \; r^2 \sin\theta \, d\theta \, d\varphi \qquad (2\cdot6\cdot1)$$

where $[\bar{E} \times \bar{H}^{s*}]_r = [E_\theta^s H_\varphi^{s*} - E_\varphi^s H_\theta^{s*}]$

The scattering cross-sectional area is defined as the area of an ideal receiving antenna that intercepts an amount of the incident power equal to the scattered power. This can be written as

$$\sigma_s = \frac{P_s}{\left(\dfrac{E_o^2}{\eta_o}\right)} \qquad (2\cdot6\cdot2)$$

where (E_o^2/η_o) is the incident power density.

To find P_s, we need to find E_θ^s, E_φ^s, H_θ^s and H_φ^s. Using Eqs. (B-23) and (2·5·14) we have

$$E_\theta^s \approx \frac{1}{j\omega\epsilon_o r} \frac{\partial^2 A_r^s}{\partial r \partial \theta}$$

Substituting for A_r^s from equation (2·5·13) noting that for the far field, $k_o r \longrightarrow \infty$ and $H_1^{(2)\prime}(k_o r) = je^{-jk_o r}$ [6],

$$k_o r \longrightarrow \infty$$

we get,

$$E_\theta^s = \frac{E_o \cos\theta \cos\varphi\ (k_o a)^3 \cdot k_o\ (\epsilon r-1)}{j\omega\epsilon_o r\ (\omega\mu_o)\ (\epsilon_r+2)} \cdot je^{-jk_o r}$$

$$= \frac{E_o \cos\varphi \cos\theta\ (k_o a)^3\ (\epsilon_r-1)}{\omega r \sqrt{\mu_o \epsilon_o}\ (\epsilon_r+2)}\ e^{-jk_o r} \qquad (2\cdot6\cdot3)$$

Similarly for E_φ^s using Eqs. (B-24) and (2·5·14) we have

$$E_\varphi^s = \frac{1}{j\omega\epsilon_o r \sin\theta} \frac{\partial^2 A_r^s}{\partial r \partial \varphi},$$

from which we get using Eq. (2·5·13)

$$E_\varphi^s = \frac{-E_o \sin\varphi\ (k_o a)^3\ k_o\ (\epsilon_r-1) \sin\theta\ je^{-jk_o r}}{j\omega\epsilon_o r \sin\theta\ (\epsilon_r+2)\ \omega\mu}$$

$$= \frac{-E_o \sin\varphi\ (k_o a)^3\ (\epsilon_r-1)\ e^{-jk_o r}}{\omega r \sqrt{\mu_o \epsilon_o}\ (\epsilon_r+2)} \qquad (2\cdot6\cdot4)$$

To find H_θ^s we use Eqs. (B-26) and (2·5·14) to get

$$H_\theta^s = \frac{1}{r \sin\theta} \frac{\partial A_r^s}{\partial \varphi}$$

from which we get using Equation (2·5·13)

$$H_\theta^s = \frac{E_0 \sin\varphi\, (k_0 a)^3 (\epsilon_r - 1)}{\omega\mu_0\, r\, (\epsilon_r + 2)} \cdot e^{-jkr} \qquad (2\cdot6\cdot5)$$

Similarly for H_φ^s using Equations (B-27) and (2·5·14) we have,

$$H_\varphi^s = -\frac{1}{r}\,\frac{\partial A_r^s}{\partial\theta}$$

from which we get using Equation (2·5·13)

$$H_\varphi^s = \frac{E_0 \cos\theta \cos\varphi\, (k_0 a)^3 (\epsilon_r - 1)}{\omega\mu_0\, r\, (\epsilon_r + 2)} \cdot e^{-jkr} \qquad (2\cdot6\cdot6)$$

Now we have determined the required fields components for the evaluation of P_s, but before doing this let us examine the scattered electric field components E_θ^s and E_φ^s From Equation (2·6·3) we have

$$\left|E_\theta^s\right| = \frac{E_0 \cos\varphi \cos\theta\, (k_0 a)^3 (\epsilon_r - 1)}{\omega r \sqrt{\mu_0\epsilon_0}\, (\epsilon_r + 2)}$$

$$= \frac{L(a,f) \cos\varphi \cos\theta}{r}$$

At a particular frequency L(a,f) is a constant and equal to

$$\frac{E_0 (k_0 a)^3 (\epsilon_r - 1)}{\omega \sqrt{\mu_0\epsilon_0}\, (\epsilon_r + 2)}$$

The above equation for the far field magnitude $\left|E_\theta^s\right|$ shows that this magnitude varies sinusoidally with θ as well as φ. Thus for fixed value of θ and r the maximum value of $|E_\theta^s|$ occurs for $\varphi = 0$ and it is zero when $\varphi = \pi/2$. Similarly for fixed value of r and φ, the value of $|E_\theta^s|$ reaches maximum value when $\theta = 0$ and reaches zero when $\theta = \pi/2$. Of course the maximum value occurred when $\theta = \varphi = 0$. This is shown in Figure (2.1).

The Figure shows the variation of the magnitude of the θ component of the scattered electric field with θ for fixed values of $\varphi = 0°$, $\varphi = 45°$ and $\varphi = 60°$, at a fixed value of r.

A similar figure can be drawn for the variation of $|E_\theta^s|$ with φ for fixed values of θ and r.

For the variation of $|E_\varphi^s|$ with θ and φ, we have from Equation (2·6·4)

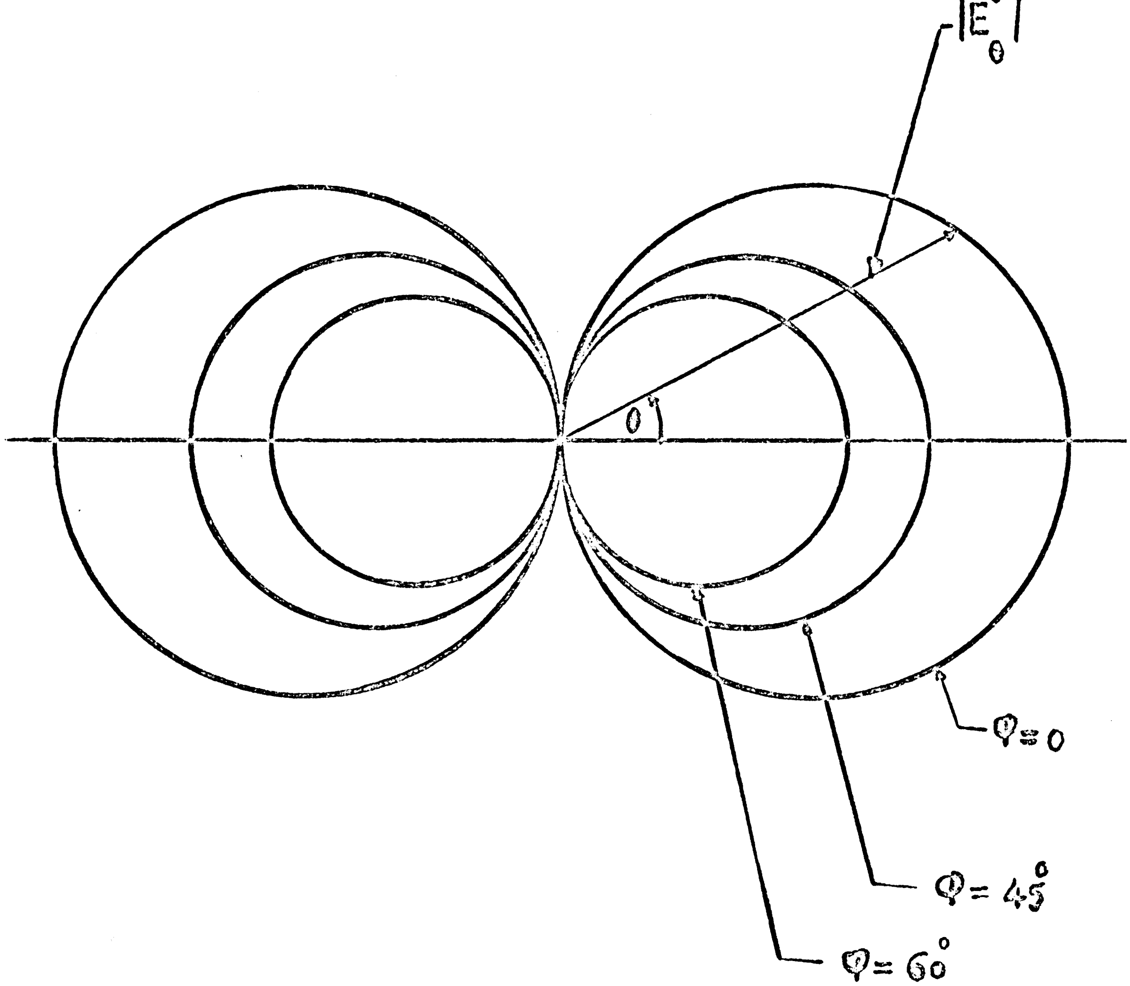
$$|E_\varphi^s| = \frac{E_0 \sin\varphi \, (k_0 a)^3 \, (\epsilon_r - 1)}{\omega r \sqrt{\mu_0 \epsilon_0} \, (\epsilon_r + 2)} = \frac{L(a,f) \sin\varphi}{r}$$

L(a,f) is the same constant used for $|E_\theta^s|$ above.

In this case the magnitude of E_φ^s does not depend on θ but varies sinusoidally with φ. As shown in Figure (2.2) the maximum value of $|E_\varphi^s|$ occurs when $\varphi = \frac{\pi}{2}$ and it is zero for $\varphi = 0$. Thus the magnitude of E_φ^s is zero when $|E_\theta^s|$ is maximum and vice versa.

Similarly, we may now find H_θ^s and H_φ^s. From Equation (2·6·5) we have,

$$|H_\theta^s| = \frac{E_0 \sin\varphi \, (k_0 a)^3 \, (\epsilon_r - 1)}{\omega \mu_0 r \, (\epsilon_r + 2)} = N(a,f) \, \frac{\sin\varphi}{r}$$



r = constant

Fig. (2.1) Variation of $\left|E_\theta^S\right|$ with θ and φ

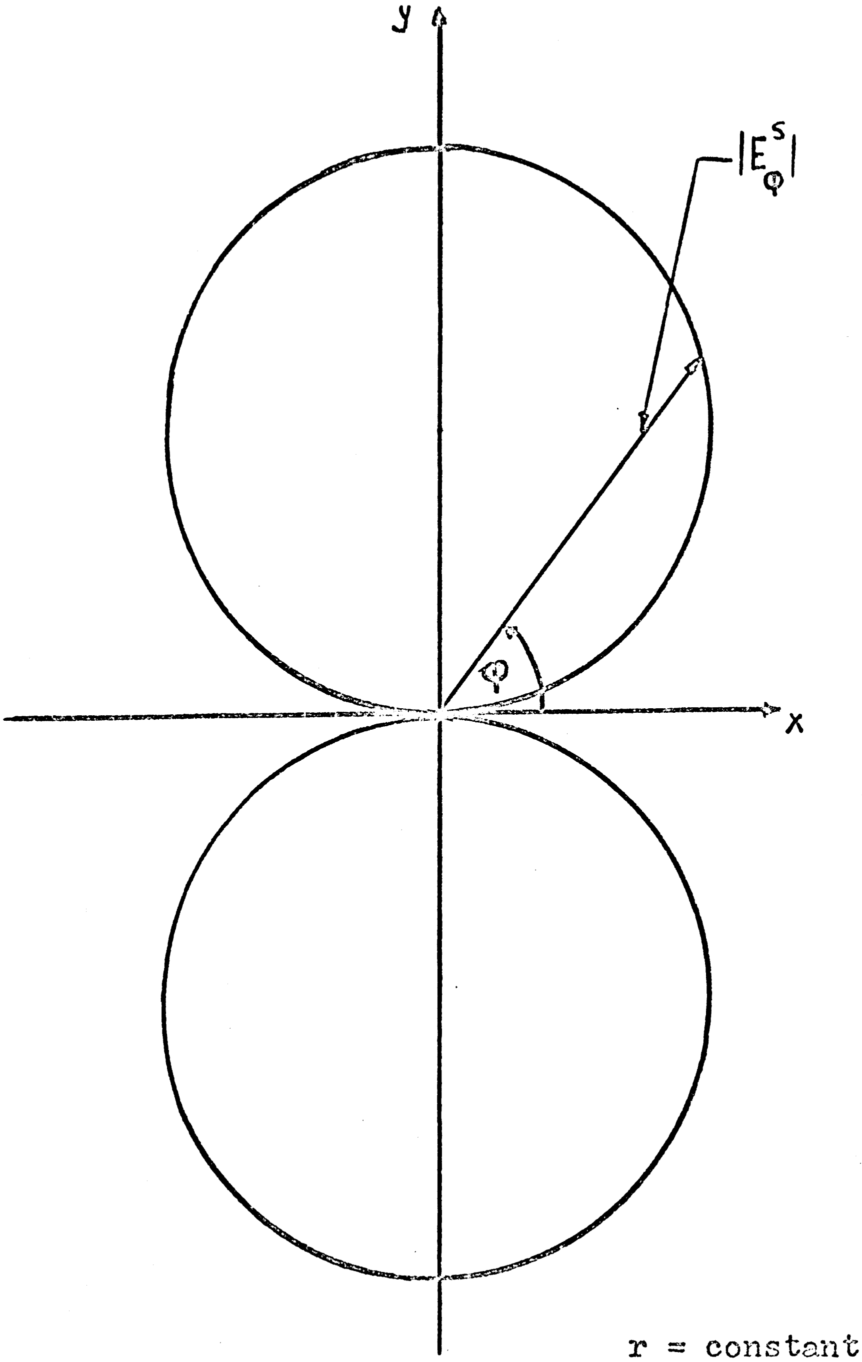


Fig. (2.2) Variation of $\left|E_\varphi^S\right|$ with φ

At a particular frequency N(a,f) is a constant and equal to

$$\frac{E_o\ (k_o a)^3\ (\epsilon_r - 1)}{\omega\mu_o\ (\epsilon_r + 2)}$$

therefore, $\left|H_\theta^s\right|$ does not depend on θ but varies sinusoidally with φ. From Equation (2·6·6) we have,

$$\left|H_\varphi^s\right| = \frac{E_o \cos\varphi \cos\theta\ (k_o a)^3\ (\epsilon_r - 1)}{\omega\mu_o\ r\ (\epsilon_r + 2)} = \frac{N \cos\varphi \cos\theta}{r}$$

therefore, $\left|H_\theta^s\right|$ varies with φ in the same manner $\left|E_\varphi^s\right|$ varies with θ and $\left|H_\varphi^s\right|$ varies with θ and φ in the same manner $\left|E_\theta^s\right|$ varies with θ and φ.

To find P_s we determine $(E_\theta^s H_\varphi^{s*} - E_\varphi^s H_\theta^{s*})$ therefore,

$$E_\theta^s H_\varphi^{s*} - E_\varphi^s H_\theta^{s*} = \frac{L}{r^2} e^{-jk_o r} N(\cos\varphi \cos\theta)^2 e^{jkr} +$$

$$\frac{Le^{-jkr}}{r^2} N \sin^2\varphi\ e^{jkr}$$

$$= LN \left[\frac{\cos^2\varphi \cos^2\theta + \sin^2\varphi}{r^2}\right]$$

Substituting this expression into Equation (2·6·1) we get,

$$P_s = \int_0^{2\pi} \int_0^{\pi} \frac{LN}{r^2} \left[\cos^2\varphi \cos^2\theta + \sin^2\varphi\right] r^2 \sin\theta\ d\theta\ d\varphi$$

$$= LN \int_0^{\pi} \int_0^{2\pi} (\sin\theta \cos^2\varphi \cos^2\theta + \sin\theta \sin^2\varphi)\ d\varphi\ d\theta$$

$$= \pi LN \left[\int_0^\pi \cos^2\theta \sin\theta \, d\theta + \int_0^\pi \sin\theta \, d\theta\right]$$

$$= \frac{8 \pi L N}{3} \qquad (2\cdot6\cdot7)$$

Substituting for L and N into Equation (2•6•7) we have

$$P_s = \frac{8 \pi E_0^2 (k_0 a)^6}{3 \omega^2 \mu_0 \sqrt{\mu_0 \epsilon_0}} \left[\frac{\epsilon_r - 1}{\epsilon_r + 2}\right]^2 \qquad (2\cdot6\cdot7a)$$

From Equations (2•6•2) and (2•6•7) we have,

$$\sigma_s = \frac{8\pi}{3} \left(\frac{LN}{E_0^2}\right) \eta_0$$

therefore,

$$\sigma_s = \frac{8\pi (k_0 a)^6}{3 \omega^2 \mu_0 \sqrt{\mu_0 \epsilon_0}} \left[\frac{\epsilon_r - 1}{\epsilon_r + 1}\right]^2 \sqrt{\frac{\mu_0}{\epsilon_0}}$$

$$= \frac{8 \pi}{3\omega^2} \frac{(k_0 a)^6}{\mu_0 \epsilon_0} \left[\frac{\epsilon_r - 1}{\epsilon_r + 2}\right]^2$$

Since $\frac{1}{\mu_0 \epsilon_0} = \lambda_0^2 \frac{\omega^2}{(2\pi)^2}$

therefore,

$$\sigma_s = \frac{2\lambda_0^2}{3\pi} (k_0 a)^6 \left[\frac{\epsilon_r - 1}{\epsilon_r + 2}\right]^2$$

In the frequency range of interest, $\epsilon_r = 78 \gg 1$ [7] and

$$\sigma_s \simeq \frac{2}{3\pi} \lambda_0^2 (k_0 a)^6,$$

which may be expressed in terms of square wave lengths

$$\bar{\sigma}_s = \frac{\sigma_s}{\lambda_o^2} \cong \frac{2}{3\pi} (k_o a)^6$$

$$\cong K \left(\frac{a}{\lambda}\right)^6$$

$$\text{where } K = \left(\frac{2^7}{3}\right) (\pi^5) = 1.29 \text{ X } 10^4$$

$$\bar{\sigma}_S = \frac{\sigma_S}{\lambda_o^2} \cong \; 1.29 \text{ X } 10^4 \left(\frac{a}{\lambda_o}\right) \qquad (2\cdot6\cdot8)$$

From the previous equation we see that the scattering cross sectional area σ_s varies as the sixth power of the radius, a; i.e., for two drops of water one of radius "a_1" and the other with a radius of "$2a_1$" we see that the scattered power varies from P_s to $64P_s$ respectively. So the scattered power is very sensitive to the size of the drop.

Concerning the effect of the frequency variation we see that the scattered power from a given drop varies as f^4 or the 4^{th} power of frequency. In other words increasing the frequency by a factor of two increases the scattered power by a factor of 16, for a given droplet size.

Note also that the scattered power is more sensitive to the radius of the drop than to the frequency of the incident plane polarized wave.

For the specific frequency of 0.3 G Hz, and a droplet size, a = 0.2 cm;

$$\lambda_o = \frac{3 \text{ X } 10^8}{3 \text{ X } 10^8} = 1 \text{ m}$$

and

$$\sigma_s = 1.29 \times 10^4 \cdot \left(\frac{2 \times 10^{-3}}{1}\right)^6 = 8.25 \times 10^{-13}\ m^2$$

$$= 8.25 \times 10^{-9}\ cm^2$$

For a smaller droplet, a = 0.1 cm

$$\sigma_s = 1.29 \times 10^4 \left(\frac{1 \times 10^{-3}}{1}\right)^6 = 1.29 \times 10^{-14}\ m^2$$

$$= 1.29 \times 10^{-10}\ cm^2$$

Summary

In section (2.5) we developed the expressions for the scattered field components E^s_θ, E^s_φ, H^s_θ and H^s_φ. These expressions are

$$E^s_\theta = \frac{E_o\ (k_o a)^3}{\omega r \sqrt{\mu_o \epsilon_o}} \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right) e^{-jk_o r} \cos\varphi \cos\theta. \tag{2.6.3}$$

$$E^s_\varphi = \frac{-E_o\ (k_o a)^3}{\omega r \sqrt{\mu_o \epsilon_o}} \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right) e^{-jk_o r} \sin\varphi. \tag{2.6.4}$$

$$H^s_\theta = \frac{E_o\ (k_o a)^3}{\omega \mu_o r} \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right) e^{-jk_o r} \sin\varphi \tag{2.6.5}$$

$$H^s_\varphi = \frac{E_o\ (k_o a)^3}{\omega \mu_o r} \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right) e^{-jk_o r} \cos\varphi \cos\theta \tag{2.6.6}$$

From the above scattered field components we derived the expression for the scattered power, P_s, as

$$P_s = \frac{8\pi E_o^2 (k_o a)^6}{3\omega^2 \mu_o \sqrt{\mu_o \epsilon_o}} \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)^2 \qquad (2\cdot6\cdot7a)$$

and from the scattered power, P_s, we developed the expression for the scattering cross-sectional area, σ_s, as

$$\sigma_s = \frac{2}{3\pi} \lambda_o^2 (k_o a)^6$$

The above expression for the scattering cross-sectional area shows that this cross-sectional area is proportional to the sixth power of "a" and that it is inversely proportional to the fourth power of the free space wavelength, λ_o.

CHAPTER III

ABSORPTION OF A LINEARLY POLARIZED PLANE WAVE BY A DROPLET OF WATER IN THE ATMOSPHERE

3.1 Introduction

So far we have solved the boundary value problem of a spherical rain-drop and we have determined the electric and magnetic fields inside and outside the rain drop.

Outside the rain drop we have found the scattered fields that exist due to the incident plane polarized wave. Finally we found the scattered power and from this we found the scattering cross-sectional area for the dielectric sphere of a rain-drop. The following assumptions have been made

(1) $k_o a = \frac{2\pi a}{\lambda_o} << 1$ which is valid for average size rain drops of radius "a) throughout the frequency range of interest [from 10 M Hz. up to 300 M Hz.] (see Appendix C).

(2) We assumed that the rain-drop is a lossless dielectric sphere of relative permittivity ϵ_r; this assumption as we shall see is valid for determining the field distribution.

The method used here to find the r-f energy dissipated in a rain drop is the perturbational method. The fields in the rain-drop are found by assuming a lossless dielectric. Using the lossless field distribution the loss due to the dielectric loss factor and the conductivity of water is calculated. If the conductivity loss is small compared to the dielectric loss and the dielectric loss tangent is very small, we conclude that our assumption of a lossless dielectric is fairly good. In other words we make an assumption and if this assumption leads to sufficiently accurate results we conclude that the assumption is valid.

We have found in section (2.5), the complete expression for both the magnetic and electric vector potentials inside the rain drop due to the incident plane polarized wave. These are equations (2•5•15) & (2•5•16) respectively.

For convenience we'll repeat equations (2•5•15) and (2•5•16)

$$A_r^- \cong -\frac{E_o}{\omega\mu_o}\left[\frac{9}{2j\,(2+\epsilon_r)}\right]\hat{J}_1\,(k_d r)\cos\varphi\sin\theta \qquad (2\cdot5\cdot15)$$

and

$$F_r^- = \frac{-E_o}{k_o}\left(\frac{9}{8j\epsilon_r}\right)\hat{J}_1\,(k_d r)\sin\varphi\sin\theta \qquad (2\cdot5\cdot16)$$

3.2 Determination of the Power Loss due to Absorption

To obtain the power loss due to absorption we have to find the electric field components inside the droplet. To find E_r^-, using Eq. (B-22) we have,

$$E_r^- = \frac{1}{j\omega\epsilon_d}\left(\frac{\partial^2 A_r^-}{\partial r^2} + k_d^2 A_r^-\right)$$

Substituting for A_r^- from equation (2·5·15) we get [It is to be noted that we are following the same assumption of lossless dielectric medium, we determine the fields as if the dielectric is lossless, then from these fields we'll find the power loss, if this power loss is such that the conductivity loss is very small compared with the dielectric loss and furthermore $\epsilon_d' >> \epsilon_d''$, where $\epsilon_d = \epsilon_d' - j\epsilon_d''$, throughout the frequency range of interest. We conclude that our assumption is reasonable.]

$$E_r^- = \frac{-9\, E_o \sin\theta \cos\varphi}{2j\omega\mu_o\,(2+\epsilon_r')\, j\omega\epsilon_d'}\left[\frac{d^2}{dr^2}\hat{J}_1(k_d r) + k_d^2\,\hat{J}_1(k_d r)\right]$$

From equation (B-17) we have

$$\left(\frac{d^2}{dr^2} + k_d^2\right)\hat{J}_1(k_d r) = \frac{2}{r^2}\hat{J}_1(k_d r)$$

therefore

$$E_r^- = \frac{9\, E_o \sin\theta \cos\varphi}{\omega^2 \mu_o\,(2+\epsilon_r')\,\epsilon_d' r^2}\hat{J}_1(k_d r) \qquad (3\cdot 2\cdot 1)$$

By a very good approximation, we modify the expression for E_r^- which contains $\hat{J}_1(k_d r)$ to obtain another expression which does not contain $\hat{J}_1(k_d r)$. It will be shown later that this approximation is mecessary to solve for the power loss of the rain-drop.

$\hat{J}_1(x)$ can be expanded in power series as follows

$$\hat{J}_1\,(x) = \frac{\sin x}{x} - \cos x \quad \text{(from definition)}$$

therefore

$$\hat{J}_1(x) = \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right) - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right)$$

$$= \left(\frac{x^2}{2!} - \frac{x^2}{3!}\right) + \left(\frac{x^5}{5!} - \frac{x^4}{4!}\right) + \left(\frac{x^6}{6!} - \frac{x^6}{7!}\right) + \left(\frac{x^8}{9!} - \frac{x^8}{8!}\right) + \ldots$$

$$= \frac{x^2}{3} - \frac{x^4}{30} + \frac{x^6}{840} - \frac{x^8}{840 \times 54} + \ldots$$

therefore

$$\hat{J}_1(k_d r) = \left[\frac{(k_d r)^2}{3} - \frac{(k_d r)^4}{30} + \frac{(k_d r)^6}{840} - \frac{(k_d r)^8}{840 \times 54} + \ldots\right]$$

therefore

$$\left|E_r^-\right|^2 = \left(\frac{9\, E_o \sin\theta \cos\omega}{\omega^2\mu_o\,(2+\epsilon_r')\,\epsilon_d'}\right)^2 \frac{1}{r^4}\left[\frac{(k_d r)^2}{3} - \frac{(k_d r)^4}{30} + \frac{(k_d r)^6}{840} - \frac{(k_d r)^8}{840 \times 54} + \ldots\right]^2$$

For water at the frequency range [10 M Hz - 300 M Hz.],

$\epsilon_r' \simeq 78$ at 25°C [7] where $\epsilon_r' = \dfrac{\epsilon_d'}{\epsilon_o}$

therefore

$$\left|E_r^-\right|^2 = (0.039\ E_o \sin\theta \cos\varphi)^2 \cdot$$

$$\left[1 - \frac{(k_d r)^2}{10} + \frac{(k_d r)^4}{280} - \frac{(k_d r)^6}{840 \times 18} + \ldots\right]^2$$

Approximating the above result, we get

$$\left|E_r^-\right|^2 \simeq 15.2 \times 10^{-4}\ E_o^2 \sin^2\theta \cos^2\varphi \left[1 - \frac{(k_d r)^2}{5}\right] \qquad (3\cdot2\cdot2)$$

The above approximating is valid to an accuracy of more than 98.6%, for $k_d a \leq 0.7$. The above expression for $\left|E_r^-\right|^2$ shows that for $k_d a \leq 0.7$, the variation of $\left|E_r^-\right|$ is sinusoidal with θ and also sinusoidal with φ.

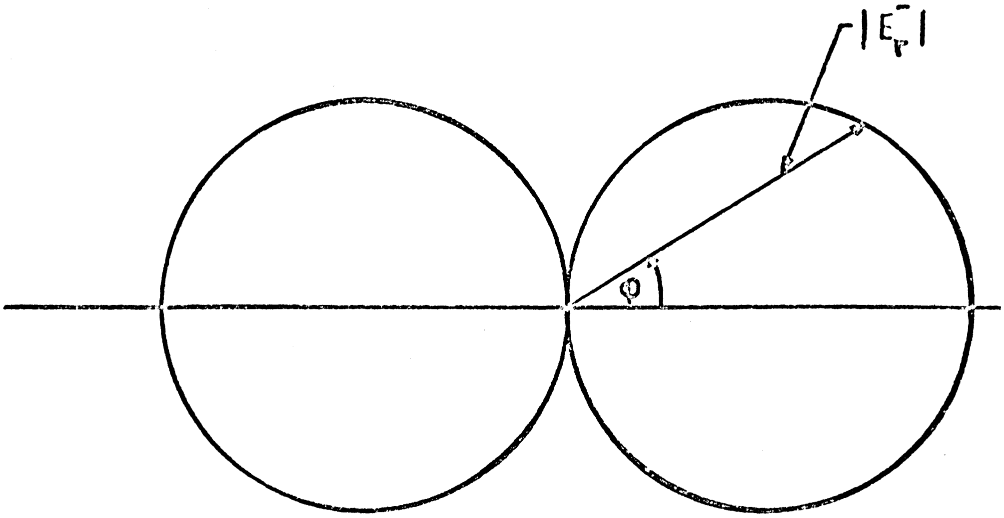
In other words, the value of $\left|E_r^-\right|$ is proportional to $\sin\theta \cos\varphi$, this means that for $\theta = 0$, $\left|E_r^-\right| = 0$, and for particular values of φ and r, $\left|E_r^-\right|$ reaches its maximum value for $\theta = \pi/2$. However for particular values of θ and r, $\left|E_r^-\right|$ is maximum for $\varphi = 0$. Figures (3·1a) and (3·1b) show the variation of $\left|E_r^-\right|$ with φ and θ.

Let us now consider E_θ^-, from equations (2·5·4), (2·5·15) and (2·5·16), we have

$$E_\theta^- = \frac{-E_o \cos\varphi}{k_o r \sin\theta}\ \frac{9}{8j\epsilon_r}\ \hat{J}_1\ (k_d r)\ P_1^1\ (\cos\theta) +$$

$$-j\ \frac{E_o}{\omega\mu_o r}\ \sqrt{\frac{\mu_o}{\epsilon_d}}\ \cos\varphi\ \frac{9}{2j\ (2+\epsilon_r)}\ \hat{J}_1'\ (k_d r)\ \frac{d}{d\theta}\ [P_1^1\ (\cos\theta)$$

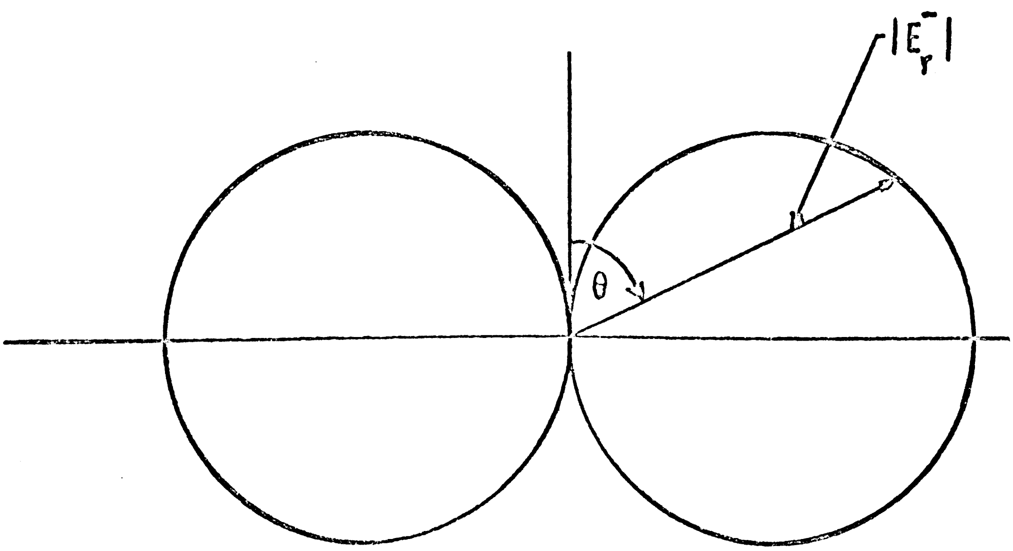
Substituting for $P_1^1\ (\cos\theta)$



r = constant

θ = constant

Fig. (3.1a) Variation of $|E_r^-|$ with φ



r = constant

φ = constant

Fig. (3.1b) Variation of $|E_r^-|$ with θ therefore

$$E_\theta^- = \frac{9\ E_o \cos\varphi\ \hat{J}_1(k_d r)}{8j\ k_o r\ \epsilon_r'} + \frac{9 \cos\varphi \cos\theta}{2(2+\epsilon_r')}\left[\frac{\hat{J}_1'(k_d r)}{k_d r}\right] \qquad (3\cdot2\cdot3)$$

Using approximations for $\hat{J}_1\ (k_d r)$ and $\hat{J}_1'(k_d r)$ in terms of their power series expansion (These approximations are necessary in finding the power absorbed within the droplet.)

Therefore

$$\left|E_\theta^-\right|^2 \simeq \Big[(1.68)^2\ E_o^{\ 2}\ \frac{\epsilon_d'}{\mu_o} \cos^2\varphi \cos^2\theta \left\{1 - \frac{2}{5}\ (k_d r)^2\right\} +$$

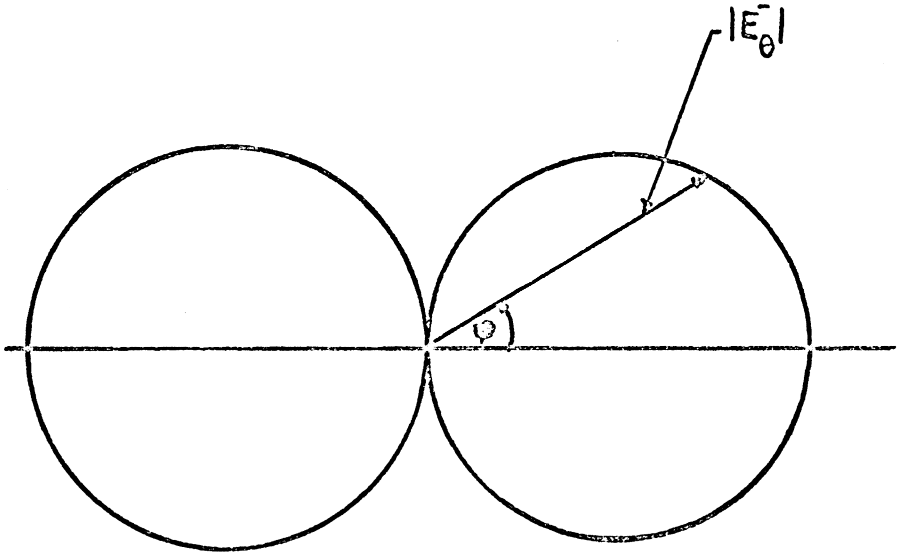
$$(0.043)^2\ k_d^2\ E_o^2\ r^2 \cos^2\varphi \left\{1 - \frac{(k_d r)^2}{5}\right\}\Big] \qquad (3\cdot2.4)$$

Note that k_d is equal to $\omega\sqrt{\mu_o \epsilon_d'}$ in all our calculations in this chapter of absorption, $\epsilon_d' >> \epsilon_d''$ for water in the frequency range of interest at 25°C.

Looking at equation (3·2·3) we see that $\left|E_\theta^-\right|$ is a function of $\cos\varphi$ for particular values of r and θ, i.e., for given values of r and θ, the value of $\left|E_\theta^-\right|$ is maximum at $\varphi = 0$, as shown in figure (3.2). However the variation of $\left|E_\theta^-\right|$ with respect to θ is not exactly sinusoidal because one term of Eq. (3·2·3), i.e., the term

$$\left(\frac{9\ E_o\ \hat{J}_1\ (k_d r)}{8j\ k_o r\ \epsilon_r'}\right)$$

does not depend on θ. The above approximation for $\left|E_\theta^-\right|^2$ is valid up to $k_d r = 0.7$



r = constant

θ = constant

Fig. (3.2) Variation of $|E_\theta^-|$ with φ

We now need only to find an expression for $\left|E_\varphi^-\right|^2$. Let us now evaluate the φ-component of the electric field inside the droplet of water; i.e., E_φ^- using Eq. (B-24)

$$E_\varphi^- = \frac{1}{r}\frac{\partial F_r^-}{\partial \theta} + \frac{1}{j\omega\epsilon_d' r \sin\theta}\frac{\partial^2 A_r^-}{\partial r \partial \varphi}$$

Using equations (2·5·15) and (2·5·16) for A_r^- and F_r^- respectively we get,

$$E_\varphi^- = j\,\frac{9\,E_o}{8\sqrt{\epsilon_r'}}\sin\varphi\cos\theta\left[\frac{\hat{J}_1(k_d r)}{k_d r}\right] - \frac{9\,E_o\sin\varphi}{2(\epsilon_r'+2)}\left[\frac{\hat{J}_1'(k_d r)}{k_d r}\right] \qquad (3\cdot2\cdot5)$$

For $k_d r \leq 0.7$, we approximate $\hat{J}_1(k_d r)$ and $\hat{J}_1'(k_d r)$ after expanding them in power series, using this approximation we get,

$$\left|E_\varphi^-\right|^2 \simeq (0.043)^2 k_d^2 r^2 E_o^2 \sin^2\varphi\cos^2\theta\left[1 - \frac{(k_d r)^2}{5}\right] + (0.039)^2 E_o^2 \sin^2\varphi\left[1 - \frac{2}{5}(k_d r)^2\right] \qquad (3\cdot2\cdot6)$$

The above expression for $\left|E_\varphi^-\right|$ shows that $\left|E_\varphi^-\right|$ is proportional to $\sin\varphi$ for particular values of r and θ, in other words the maximum value for $\left|E_\varphi^-\right|$ for particular values of r and θ occurs for $\varphi = \frac{\pi}{2}$, and for $\varphi = 0$ the value of $\left|E_\varphi^-\right|$ is zero.

This is shown in Figure (3.3)

The variation of $\left|\bar{E}_{\varphi}\right|$ with respect to θ is not exactly sinusoidal because the second term of equation (3·2·5), i.e., the term

$$\left[-\frac{9\,E_o}{2(\varepsilon'_r+2)}\;\left(\frac{\hat{J}_1'(k_d r)}{k_d r}\right)\;\sin\varphi\right]$$

does not depend on θ. So far we have determined the electric field components inside the rain drop, assuming that water is approximately a lossless medium.

To fine the power absorbed by a droplet of water, we [8] know that the power dissipated within the spherical droplet is the real part of $(\sigma + j\omega\varepsilon_d)\iiint |E|^2\, d\tau$ plus the real part of $j\omega\mu_d \iiint |H|^2\, d\tau$

Since $\mu_d = \mu_o$ therefore the real part of

$$j\omega\mu_d \iiint |H|^2\, d\tau = 0$$

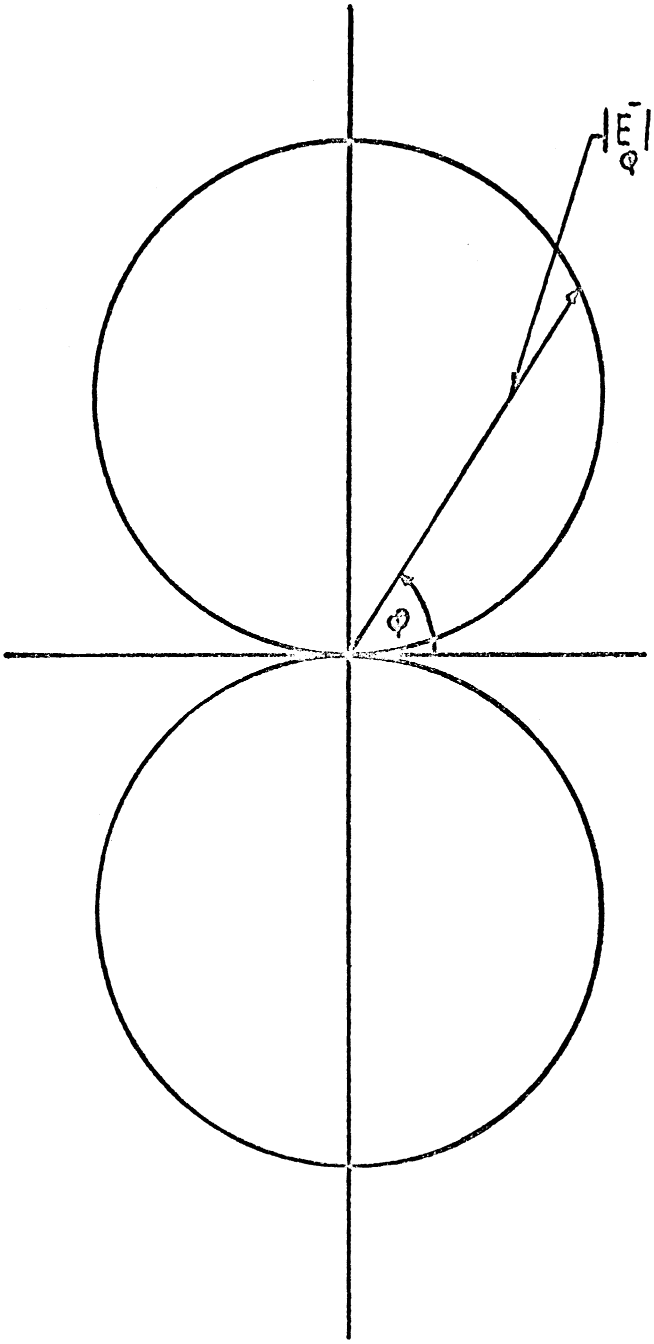
This gives,

$$P_a \equiv \text{Real}\left[\sigma + j\omega\,(\varepsilon'_d - j\varepsilon''_d)\right]\iiint |E|^2\, d\tau$$

where the integration is taken over the volume of the droplet. The power absorbed can be divided into two components:

(1) $\sigma\iiint |E|^2\, d\tau$ which is the loss due to conductivity of water.

(2) $\omega\varepsilon''_d \iiint |E|^2\, d\tau$ which is the dielectric loss.



r = constant

θ = constant

Fig. (3.3) Variation of $\left|E_{\varphi}^{-}\right|$ with φ

At the frequency of interest, $\sigma \ll \omega\epsilon''_d$ which enables us to write P_a as

$$P_a = \omega\epsilon''_d \iiint |E|^2 \, d\tau = \omega\epsilon''_d \iiint \left\{ |E_r^-|^2 + |E_\theta^-|^2 + |E_\varphi^-|^2 \right\} d\tau$$

therefore

$$P_a = \omega\epsilon''_d \, [I_1 + I_2 + I_3] \qquad (3\cdot2\cdot7)$$

where

$$I_1 = \iiint |E_r^-|^2 \, d\tau = \int_0^a \int_0^\pi \int_0^{2\pi} |E_r^-|^2 \, r^2 \sin\theta \, dr \, d\theta \, d\varphi$$

$$I_2 = \iiint |E_\theta^-|^2 \, r^2 \sin\theta \, dr \, d\theta \, d\varphi$$

and

$$I_3 = \iiint |E_\varphi^-|^2 \, r^2 \sin\theta \, dr \, d\theta \, d\varphi$$

We use the approximate expressions for $|E_r^-|^2$, $|E_\theta^-|^2$ and $|E_\varphi^-|^2$ to evaluate the preceding integrals. Substituting equation (3·2·2) into the expression for I_1 we get,

$$I_1 \cong (6.35 \times 10^{-3}) \left(\frac{a^3}{3} - \frac{a^5 k_d^2}{25} \right)$$

Note that $k_d = \omega\sqrt{u_o \epsilon'_d}$

Substituting equation (3·2·4) into the expression for I_2 we get,

$$I_2 \cong E_o^2\, a^3\, [1.07 + 3.86\, k_d^2] \times 10^{-3} -$$

$$E_o^2\, a^5\, k_d^2 \left\{0.256 + 0.4648\, k_d^2\right\} \times 10^{-3}$$

Similarly, substituting equation (3·2·6) into the expression of I_3, we get,

$$I_3 \cong 0.00387\, E_o^2\, k_d^2\, [\frac{a^5}{5} - \frac{k_d^2\, a^7}{35}] +$$

$$9.6 \times 10^{-3}\, E_o^2\, [\frac{a^3}{3} - \frac{2}{5}\, k_d^2\, a^5]$$

Note that I_1, I_2 and I_3 are proportional to the components of power loss associated with the individual spherical-coordinate components of the electric field inside the droplet of water; $\bar{E}_r$, $\bar{E}_\theta$ and $\bar{E}_\omega$ respectively.

Substituting the above values of I_1, I_2 and I_3 into equation (3·2·7), we get the power loss due to absorption as

$$P_a = E_o^2\, a^3\, \omega\epsilon_d''\, [0.639 \times 10^{-2} + 3.86 \times 10^{-3}\, k_d^2 -$$

$$(k_d a)^2\, (3.59 \times 10^{-3} + 0.465 \times 10^{-3}\, k_d^2 +$$

$$1.11 \times 10^{-4}\, k_d^2\, a^2)] \qquad (3·2·8)$$

$$k_d = \frac{2\pi}{\lambda_d} = \frac{2\pi}{\lambda_o}\sqrt{\frac{\epsilon'_d}{\epsilon_o}} = \frac{2\pi}{\lambda_o}\sqrt{\epsilon'_r} \quad \text{or} \quad k_d a = (k_o a)\sqrt{\epsilon'_r}$$

Equation (3·2·8) gives the power loss due to absorption in terms of the magnitude of the incident electric field, frequency and radius of the droplet.

<u>Evaluation of the Absorption Cross-Sectional area (σ_a)</u> - The definition of the absorption cross-sectional area is similar to the definition of the scattering cross sectional area with the scattered power replaced by the power absorbed. Thus,

$$\sigma_a = \frac{P_a}{E_o^2/\eta_o} \tag{3·2·9}$$

Substituting for P_a from equation (3·2·8), we get

$$\sigma_a = \eta_o a^3 \omega \epsilon''_d \,[0.639 \times 10^{-2} + 3.86 \times 10^{-3}\, k_d^2 - (k_d a)^2\,(3.59 \times 10^{-3} + 0.465 \times 10^{-3}\, k_d^2 + 1.11 \times 10^{-4}\, k_d^2\, a^2)] \tag{3·2·10}$$

Note that σ_a varies as the third power of the radius "a". As an example, $\epsilon''_r = 1.25$ [7] and $\epsilon'_r = 75.5$ at a frequency of 300 M Hz and temperature 25 °C. For a = 0.2 cm at the given frequency, $k_d = 55$ and $k_d a = 0.11$.

Since $k_d a < 0.7$ and the given frequency = 300 M Hz, we use equation (3·2·8) to find P_a, this gives

$$P_a = 19.8 \times 10^{-10} E_o^2$$

Note that the conductivity, σ, of water is neglected because $\sigma \ll \omega\epsilon_d''$ in the frequency of 300 M Hz.

Let us consider the same example and reduce "a" to one half; i.e., a = 0.1 cm, then

$$P_a = 2.45 \times 10^{-10} E_o^2$$

Thus reducing "a" to one half its original value decreases the power loss due to absorption to one eighth.

At the high frequency range (see next section), the most important electric-field-component that contributes to the power loss due to absorption is the θ-component.

On the basis of the above example, the absorption cross-sectional area is calculated.

f = 300 M Hz, a = 0.2, we get

$$\sigma_a = 7.42 \times 10^{-7} \text{ m}^2$$

For the second case; i.e., a = 0.1 cm, we get

$$\sigma_a = 9.3 \times 10^{-8} \text{ m}^2$$

Summary:

In this section we developed expressions for the electric field components inside the water droplet. These expressions are Eqs. (3·2·1), (3·2·3) and (3·2·5)

$$E_r^- = \frac{9\ E_o \sin\theta \cos\varphi}{\omega^2 \mu_o\ (2+\epsilon_r')\ \epsilon_d'\ r^2}\ \hat{J}_1\ (k_d r)\ , \qquad (3\cdot2\cdot1)$$

$$E_\theta^- = \frac{9\ E_o \cos\varphi\ \hat{J}_1\ (k_d r)}{8j\ k_o r\ \epsilon_r'} + \frac{9 \cos\varphi \cos\theta}{2(2+\epsilon_r')}\ \frac{\hat{J}_1'\ (k_d r)}{k_d r} \qquad (3\cdot2\cdot3)$$

and

$$E_\varphi^- = \frac{j\ 9\ E_o \sin\varphi \cos\theta}{8\sqrt{\epsilon_r'}}\ \left(\frac{\hat{J}_1\ (k_d r)}{k_d r}\right) -$$

$$\frac{9\ E_o \sin\varphi}{2(\epsilon_r'+2)}\ \frac{\hat{J}_1'\ (k_d r)}{k_d r} \qquad (3\cdot2\cdot5)$$

From the above expressions for the electric field components, we have approximate expressions for $|E_r^-|^2$, $|E_\theta^-|^2$ and $|E_\varphi^-|^2$, the approximation used is valid for $k_d a \leq 0.7$ and the error involved in these expressions is less than 1.4%. These expressions are Eqs. (3·2·2), (3·2·4) and (3·2·6)

$$|E_r^-|^2 \cong 15.2 \times 10^{-4}\ E_o^2 \sin^2\theta \cos^2\varphi\ \left[1 - \frac{(k_d r)^2}{5}\right], \qquad (3\cdot2\cdot2)$$

$$|E_\theta^-|^2 \cong \Big[(1.68)^2\ E_o^2\ \frac{\epsilon_d'}{\mu_o} \cos^2\varphi \cos^2\theta \left\{1 - \frac{2}{5}(k_d r)^2\right\} +$$

$$(0.043)^2\ k_d^2\ E_o^2\ r^2 \cos^2\varphi \left\{1 - \frac{(k_d r)^2}{5}\right\}\Big] \qquad (3\cdot2\cdot4)$$

and

$$\left|E_{\varphi}^{-}\right|^{2} \cong (0.043)^{2} \, (k_d r)^{2} \, E_o^2 \sin^2\varphi \cos^2\theta \left[1 - \frac{(k_d r)^2}{5}\right] +$$

$$(0.039)^{2} \, E_o^2 \sin^2\varphi \left[1 - \frac{2}{5} (k_d r)^2\right] \qquad (3{\cdot}2{\cdot}6)$$

From these expressions for $\left|E_r^-\right|^2$, $\left|E_\theta^-\right|^2$ and $\left|E_\varphi^-\right|^2$, we developed the expression for the power absorbed inside the droplet Eq.[(3·2·8)].

$$P_a = E_o^2 \, a^3 \, \omega\varepsilon_d'' \, [0.639 \times 10^{-2} + 3.86 \times 10^{-3} \, k_d^2 -$$

$$(k_d a)^2 \, (3.59 \times 10^{-3} + 0.465 \times 10^{-3} \, k_d^2 +$$

$$1.11 \times 10^{-4} \, k_d^2 \, a^2)] \qquad (3{\cdot}2{\cdot}8)$$

from which the expression for the absorption cross-sectional area has been obtained as,

$$\sigma_a = \eta_o a^3 \omega\varepsilon_d'' \, [0.639 \times 10^{-2} + 3.86 \times 10^{-3} \, k_d^2 -$$

$$(k_d a)^2 \, (3.59 \times 10^{-3} + 0.465 \times 10^{-3} \, k_d^2 +$$

$$1.11 \times 10^{-4} \, k_d^2 \, a^2)]$$

3.3 Comparison of the Power Loss due to Scattering with that due to Absorption.

The power scattered by a water droplet was found to be

$$P_s = \frac{2}{3\pi} \lambda_o^2 (k_o a)^6 \frac{E_o^2}{\eta_o}$$

$$= (3.8 \times 10^{-33}) E_o^2 f^4 a^6 \qquad (3.3.1)$$

The power absorbed within the droplet of water is

$$P_a = E_o^2 a^3 \omega\epsilon'' [0.639 \times 10^{-2} + 3.86 \times 10^{-3} k_d^2 -$$

$$(k_d a)^2 \left\{3.59 \times 10^{-3} + 0.465 \times 10^{-3} k_d^2\right\}]$$

In deriving the above two expressions for P_s and P_a we have assumed that water is approximately lossless. At 25°C this condition is only satisfied in the frequency range, 10 M Hz. up to 300 M Hz; [7] i.e., in this frequency range $\sigma \ll \omega\epsilon_d''$ and $\epsilon_d' \gg \epsilon_d''$. Furthermore $k_d a \leq 0.7$ for average rain-drops or condensed droplets. Thus the low frequency limit is 10 M Hz while the high frequency limit is the frequency that makes $k_d a = 0.7$ or 300 M Hz. whichever is smaller.

For $k_d \leq 12.9$; i.e., f = 70 M Hz, the term $(k_d a)^2 \left\{3.59 \times 10^{-3} + 0.465 \times 10^{-3} k_d^2\right\}$ in the expression for P_a can be neglected and we have the following expression for

P_a in the frequency range 10 M Hz. to 70 M Hz.

$$P_a \cong E_o^2\, a^3\, \omega\epsilon_d'' \left\{0.639 \times 10^{-2} + 3.86 \times 10^{-3}\, k_d^2\right\}$$

Using the relation $k_d = \omega\sqrt{\mu_o \epsilon_d'}$ where $\epsilon_r' = 78$ [7] as $\epsilon_d' = 78\ \epsilon_o$, we have

$$P_a = E_o^2\, a^3\, \epsilon_d'' \times 10^{-3} \left\{40\, f + 8.3 \times 10^{-13}\, f^3\right\} \qquad (3.3.2)$$

In the low frequency range (10 M Hz. up to 70 M Hz.), the ratio (P_a/P_s) is

$$\frac{P_a}{P_s} = 10.05 \times 10^{30}\, \frac{\epsilon_d}{f^3 a^3} + \frac{2.18\, \epsilon_d \times 10^{17}}{a^3 f}$$

Thus as the radius of the droplet increases, the above ratio decreases. However this ratio is very large even for the largest drop size (a = 0.01 meter).

We conclude that in the low frequency range the power absorbed by the droplet of water is much greater than the scattered power and the ratio of the scattered power to the absorbed power varies approximately as the ratio $\left(\frac{f}{\epsilon_r}\right)$ in the frequency range 40 M Hz up to 70 M Hz. This ratio $\left(\frac{f}{\epsilon_r}\right)$ increases with frequency. [7]

The expression for power absorbed within the droplet in the high frequency range is

$$P_a \cong E_o^2\, a^3\, \omega\epsilon_d'' \left\{3.86 \times 10^{-3}\, k_d^2 - k_d^4\, a^2 \times 4.65 \times 10^{-4}\right\} \qquad (3\cdot3\cdot3)$$

where the high frequency range is defined as the frequency range 70 M Hz up to the frequency that makes $k_d a = 0.7$ or 300 M Hz whichever is the smaller.

The ratio $(\frac{P_a}{P_s})$ in the high frequency range is smaller than that in the low frequency range because the term

$$0.465 \times 10^{-3}\, k_d^2\, (k_d a)^2$$

is subtracted from the term

$$3.86 \times 10^{-3}\, k_d^2$$

in the expression for P_a in the high frequency range. The ratio $\frac{P_a}{P_s}$ in the high frequency range is

$$\frac{P_a}{P_s} = \frac{2.18\, \epsilon_d'' \times 10^{17}}{a^3 f} - \frac{8.9 \times 10^2\, f\, \epsilon_d''}{a}$$

Thus as the radius "a" of the droplet increases, (P_a/P_s) decreases and also $(\frac{P_a}{P_s})$ decreases with the increase of the frequency, this means that the smallest ratio of $(\frac{P_a}{P_s})$ occurs at the highest frequency and the largest radius. At $k_d a = 0.7$, the power absorbed within the droplet of water is

$$P_a = 6.9 \times 10^{-27}\, E_o^2\, \epsilon_r''\, a^3\, f^3$$

which makes the ratio

$$\frac{P_a}{P_s} = \frac{\epsilon_r''}{5.5 \times 10^{-5} f a^3}$$

Thus the ratio $(\frac{P_a}{P_s})$ is very large even for the highest frequency and the largest radius of rain-drops (a = 0.01 meter). We conclude that the power absorbed is much greater than the scattered power in the frequency range 10 M Hz up to the frequency that makes $k_d a = 0.7$ or 300 M Hz. whichever is the smaller.

Summary

In this section we divided the frequency range of interest (10 M Hz. to 300 M Hz.) into two ranges. [The conditions of validity of these relations are (1) The electric conductivity "σ" is much smaller than $(\omega\epsilon_d'')$ in this frequency range. (2) $k_o a << 1$. (3) $k_d a < 0.7$.] These two ranges are:

(1) The low-frequency range (10 M Hz. to 70 M Hz.). In this range the power absorbed "P_a" was found to be

$$P_a \cong E_o^2 a^3 \epsilon_d'' \times 10^{-3} (40f + 8.3 \times 10^{-3} f^3) \quad (3\cdot3\cdot2)$$

(2) The high-frequency range (70 M Hz to 300 M Hz) and the power absorbed in this range, P_a, was

found to be

$$P_a \cong E_o^2\, a^3\, \epsilon_d'' \,(3.86 \times 10^{-3}\, k_d^2 - 4.65 \times 10^{-4}\, k_d^4\, a^2) \qquad (3\cdot3\cdot3)$$

In the entire frequency range (10 M Hz. to 300 M Hz.) the power absorbed within the droplet was found to be much greater than the power scattered.

3.4 Determination of the Attenuation Constant of a Homogeneous Distribution of Rain Droplets

The total power loss from a radio frequency field caused by the presence of a rain-drop is the sum of the power scattered and the power absorbed by the rain-drop; i.e., $P_t = P_s + P_a$.

Using Eqs. (2·6·8) and (3·2·8), we find

$$P_t = E_o^2 \Big[\frac{1.29 \times 10^4\, a^6}{\lambda_o^4 \times 120\pi} + a^3\, \omega\, \epsilon_d'' \Big\{ 0.639 \times 10^{-2} + 3.86 \times 10^{-3}\, k_d^2 - (k_d a)^2\, (3.59 \times 10^{-3} + 0.465 \times 10^{-3}\, k_d^2 + 1.11 \times 10^{-4}\, k_d^2\, a^2) \Big\} \Big] \qquad (3\cdot4\cdot1)$$

We can now find the attenuation constant in decibels (db) per meter in the presence of rain falling at the rate of p mm per hour with a particular rain-drop size. The attenuation constant "α" is given by

α in db =

$$10 \log_{10} \frac{\text{Total power loss in a cube of volume one cubic meter}}{\text{Incident power density}} \qquad (3\cdot4\cdot2)$$

Thus we have to find the total power loss in a volume of one cubic meter. The radius "a" of each drop enables us to find the terminal falling velocity of the drop using the relation,

Drag force = weight of the drop

which gives

$$3\,\nu\,v\,\pi\,d = 32\,m$$

where m is the mass of the drop, d = 2a, ν is the viscosity of air which is equal to 0.18 $\frac{\text{kgm -sec}}{m^2}$ at 70 $^\circ$F and v is the terminal velocity from which we get

$$v = 3.9 \times 10^4\, a^2$$

where a is in meter and v is in meter per second. In other words, the terminal velocity is proportional to the square of the radius of the rain-drop.

The rain drops in a volume of one meter cube reaches the ground after $\frac{1}{v}$ seconds; i.e., $\frac{10^{-4}}{3.9\, a^2}$ seconds. From this period we get the volume of rain drops within a volume of one cubic meter as

$$\frac{p}{3.9\ a^2 \times 360}\ m^3 \tag{3.4.3}$$

From (3.4.1) we get the number of rain drops within the given volume of one cubic meter by dividing Eq. (3.4.1) by the volume of one drop, which yields

$$n = 1.7 \times 10^{-12}\ \frac{p}{a^5}\ \text{droplets}\ /m^3 \tag{3.4.4}$$

where "p" is the rainfall rate in mm per hour and "a" is in meter. From Eqs. (3.4.1), (3.4.2) and (3.4.4) we get the attenuation constant "α" as

$$\alpha = 10 \log_{10} 6.4\ p \times 10^{-10} \left[\frac{\omega \epsilon_d''}{a^2} \left\{ 0.639 \times 10^{-2} + 3.86 \times 10^{-3}\ k_d^2 - (k_d a)^2\ (3.59 \times 10^{-3} + 0.465 \times 10^{-3}\ k_d^2 + 1.11 \times 10^{-4}\ k_d^2\ a^2) \right\} + \frac{1.29 \times 10^4\ a}{120\ \pi\ \lambda_o^4} \right] \text{db loss per meter distance} \tag{3.4.5}$$

Summary

In this section we developed the attenuation constant "α" [Eq. 3.4.5] for a particular rainfall rate and particular radius of the rain-drops. From this expression for the attenuation constant we see that "α" increases very rapidly with frequency. Also for a particular rainfall rate and at particular frequency, the attenuation constant decreases as fast as $\frac{1}{a^2}$ with the increase of the size of the drops because the term corresponding to the scattered power in Eq. (3.4.5); i.e.,

$$\left[\frac{1.29 \times 10^4 \, a}{120 \, \pi \, \lambda_o^4} \right]$$

is very small compared with that of the power absorbed by the rain-drops.

CHAPTER IV

SUMMARY

In this thesis we derived the expressions for the incident vector potentials using the expressions for the incident field. These are Eqs. (2.3.4) and (2.3.5); i.e.,

$$\bar{A}^i = A_r^i \, \bar{U}_r = \left[\frac{E_o \cos \varphi}{\omega \mu_o} \sum_{n=1}^{\infty} a_n \hat{J}_n(k_o r) \, P_n^1(\cos \theta) \right] \bar{U}_r \quad (2.3.4)$$

where

$$a_n = j^{-n} \frac{(2n+1)}{n(n+1)}$$

and

$$\bar{F}^i = F_r^i \, \bar{U}_r = \left[\frac{E_o}{K_o} \sin \varphi \sum_{n=1}^{\infty} a_n \hat{J}_n(k_o r) \, P_n^1(\cos \theta) \right] \bar{U}_r \quad (2.3.5)$$

Also we derived the expressions for the scattered vector potentials and the vector potentials outside the droplet of water, these are Eqs. (2.4.1), (2.4.2), (2.4.3) and (2.4.4).

$$A_r^s = \frac{E_o}{\omega \mu_o} \cos \varphi \sum_{n=1}^{\infty} b_n \hat{H}_n^{(2)}(k_o r) \, P_n^1(\cos \theta), \quad (2.4.1)$$

$$F_r^s = \frac{E_o}{k_o} \sin \varphi \sum_{n=1}^{\infty} c_n \hat{H}_n^{(2)}(k_o r) \, P_n^1(\cos \theta), \quad (2.4.2)$$

$$A_r^+ = \frac{E_o}{\omega\mu_o} \cos\varphi \sum_{n=1}^{\infty} \left[a_n \hat{J}_n(k_o r) + b_n \hat{H}_n^{(2)}(k_o r) \right] P_n^1 (\cos\theta) \quad (2\cdot4\cdot3)$$

and

$$F_r^+ = \frac{E_o}{k_o} \sin\varphi \sum_{n=1}^{\infty} \left[a_n \hat{J}_n(k_o r) + c_n \hat{H}_n^{(2)}(k_o r) \right] P_n^1 (\cos\theta) \quad (2\cdot4\cdot4)$$

We developed the expressions for the vector potentials inside the droplet of water, these are Eqs. (2·5·1) and (2·5·2)

$$A_r^- = \frac{E_o}{\omega\mu_o} \cos\varphi \sum_{n=1}^{\infty} d_n \hat{J}_n(k_d r) P_n^1 (\cos\theta) \quad (2\cdot5\cdot1)$$

and

$$F_r^- = \frac{E_o}{k_o} \sin\varphi \sum_{n=1}^{\infty} e_n \hat{J}_n(k_d r) P_n^1 (\cos\theta) \quad (2\cdot5\cdot2)$$

The values of the constants b_n, c_n, d_n and e_n were found from the boundary conditions. These constants were evaluated on the assumption that $k_o a \ll 1$ in the frequency range considered in this thesis [roughly from 10 M Hz. to 300 M Hz.], i.e.,

$$b_1 \cong -(k_o a)^3 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)$$

$$c_1 \cong 0$$

$$(2\cdot5\cdot12a)$$

$$d_1 \cong \frac{9}{2j\,(2+\epsilon_r)}$$

$$e_1 \cong \frac{9}{8j\epsilon_r}$$

Eq. (2·5·12a) reduces Eqs. (2·4·1), (2·4·2), (2·5·1) and (2·5·2) to Eqs. (2·5·13), (2·5·14), (2·5·15) and (2·5·16)

$$A_r^s \cong \frac{E_o \cos\varphi \, (k_o a)^3 \, (\epsilon_r - 1)}{\omega\mu_o \, (\epsilon_r + 2)} \; \hat{H}_1^{(2)} \, (k_o r) \sin\theta \qquad (2\cdot5\cdot13)$$

$$F_r^s \cong 0 \qquad (2\cdot5\cdot14)$$

$$A_r^- \cong \frac{-E_o \cos\varphi \sin\theta}{\omega\mu_o} \left[\frac{9}{2j\,(2+\epsilon_r)}\right] \hat{J}_1 \, (k_d r) \qquad (2\cdot5\cdot15)$$

and

$$F_r^- \cong \frac{-E_o \sin\varphi \sin\theta}{k_o} \left(\frac{9}{8j\epsilon_r}\right) \hat{J}_1 \, (k_d r) \qquad (2\cdot5\cdot16)$$

Using Eqs. (2·5·13), (2.5.14), (B-23), (B-26) and (B-27) we derived the scattered field components, E_θ^s, E_φ^s, H_θ^s and H_φ^s, required to find the scattered power and the scattering cross-sectional area; i.e.,

$$E_\theta^s = \frac{E_o \, (k_o a)^3}{\omega r \sqrt{\mu_o \epsilon_o}} \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right) e^{-jk_o r} \cos\varphi \cos\theta \qquad (2\cdot6\cdot3)$$

$$E_\varphi^s = \frac{-E_o \, (k_o a)^3}{\omega r \sqrt{\mu_o \epsilon_o}} \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right) e^{-jk_o r} \sin\varphi \qquad (2\cdot6\cdot4)$$

$$H_\theta^s = \frac{E_o \, (k_o a)^3}{\omega\mu_o r} \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right) e^{-jk_o r} \sin\varphi \qquad (2\cdot6\cdot5)$$

$$H_\varphi^s = \frac{E_o \, (k_o a)^3}{\omega\mu_o r} \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right) e^{-jk_o r} \cos\varphi \cos\theta \qquad (2\cdot6\cdot6)$$

From these components the expression for the scattered power has been developed,

$$P_s = \frac{8\pi E_o^2 (k_o a)^6}{3\omega^2 \mu_o \sqrt{\mu_o \epsilon_o}} \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)^2 \qquad (2\cdot6\cdot7a)$$

and the scattering cross-sectional area

$$\sigma_s = \frac{2}{3\pi} \lambda_o^2 (k_o a)^6$$

The above expression for the scattering cross-sectional area shows that this cross-sectional area is proportional to the sixth power of "a" and that it is inversely proportional to the fourth power of the free space wavelength, λ_o.

Using Eqs. (2·5·15), (2·5·16), (B-22), (B-23) and (B-24) the expressions for the electric field components inside the droplets have been derived

$$E_r^- = \frac{9 E_o \sin\theta \cos\varphi}{\omega^2 \mu_o (2+\epsilon_r') \epsilon_d' r^2} \hat{J}_1(k_d r) \qquad (3\cdot2\cdot1)$$

$$E_\theta^- = \frac{9 E_o \cos\varphi \hat{J}_1(k_d r)}{8j k_o r \epsilon_r'} + \frac{9 \cos\varphi \cos\theta}{2(2+\epsilon_r')} \frac{\hat{J}_1'(k_d r)}{k_d r} \qquad (3\cdot2\cdot3)$$

and

$$E_\varphi^- = \frac{j 9 E_o \sin\varphi \cos\theta}{8\sqrt{\epsilon_r'}} \left(\frac{\hat{J}_1(k_d r)}{k_d r}\right) - \frac{9 E_o \sin\varphi}{2(\epsilon_r'+2)} \frac{\hat{J}_1'(k_d r)}{k_d r} \qquad (3\cdot2\cdot5)$$

From Eqs. (3·2·1), (3·2·3) and (3·2·5) approximate expressions for $|E_r^-|^2$, $|E_\theta^-|^2$ and $|E_\varphi^-|^2$ have been derived, the approximation used is valid for $k_d a \leq 0.7$ and the error involved in these approximations is less than 1.4%, these are equations (3·2·3), (3·2·4) and (3·2·6). From these expressions for $|E_r^-|^2$, $|E_\theta^-|^2$ and $|E_\varphi^-|^2$ the expression for the power absorbed inside the droplet has been derived, i.e.,

$$P_a = E_o^2 a^3 \omega \epsilon_d'' \Big[0.639 \times 10^{-2} + 3.86 \times 10^{-3} k_d^2 - (k_d a)^2 (3.59 \times 10^{-3} + 0.465 \times 10^{-3} k_d^2 + 1.11 \times 10^{-4} k_d^2 a^2)\Big] \qquad (3\cdot2\cdot8)$$

and from it the absorption cross-sectional area, σ_a

$$\sigma_a = \eta_o a^3 \omega \epsilon_d'' \Big[0.639 \times 10^{-2} + 3.86 \times 10^{-3} k_d^2 - (k_d a)^2 (3.59 \times 10^{-3} + 0.465 \times 10^{-3} k_d^2 + 1.11 \times 10^{-4} k_d^2 a^2)\Big]$$

A comparison between the power absorbed by the droplet of water and the scattered power shows that the power absorbed within the droplet is much greater than the power scattered because of the presence of a water droplet in the atmosphere in the frequency range (10 M Hz. to 300 M Hz.). In this frequency range the value of $\omega\epsilon_d''$ >> σ and ϵ_d' >> ϵ_d'', i.e., water is considered a lossless medium for the purpose of determining the field components inside the droplet of water.

At the end of the thesis an expression for the attenuation constant "α" has been developed in the case of rainfall at a particular rate "p" and a particular raindrop size. This is Eq. (3·4·5)

$$\alpha = 10 \log_{10} 6.4\, p \times 10^{-10} \Big[\frac{\omega\epsilon_d''}{a^2} \Big\{ 0.639 \times 10^{-2} + 3.86 \times 10^{-3} k_d^2 - (k_d a)^2 \, (3.59 \times 10^{-3} + 0.465 \times 10^{-3} k_d^2 + 1.11 \times 10^{-4} k_d^2 a^2) \Big\} + \frac{1.29 \times 10^4 a}{120 \pi \lambda_o^4} \Big] \text{ db loss per meter distance} \qquad (3\cdot4\cdot5)$$

From equation (3·4·5) we conclude that for a particular rainfall rate and particular radius of the raindrops the attenuation constant "α" increases very rapidly with frequency. Also for a particular rainfall rate and at particular frequency, the attenuation constant decreases as fast as $\frac{1}{a^2}$ with the increase of the size of the drops because the term corresponding to the scattered power in Eq. (3·4·5); i.e., $\left(\frac{1.29 \times 10^4 a}{120 \pi \lambda_o^4}\right)$ is very small compared with that of the power absorbed by the rain-drops.

APPENDIX A*

<u>Solution of the Scalar Helmholtz Equation $(\nabla^2 + k^2)\,\Psi = 0$ in Spherical-Coordinate System</u>

In spherical coordinates the scalar Helmholtz equation is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Psi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Psi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \Psi}{\partial \varphi^2} + k^2 \Psi = 0 \qquad \text{(A-1)}$$

where Ψ is a wave potential function. The solution of equation (A-1) is possible using the method of separation of variables and letting

$$\Psi = R(r)\,\Theta(\theta)\,\Phi(\varphi) \qquad \text{(A-2)}$$

Substituting this into Eq. (A-1), dividing by Ψ and multiplying by $r^2 \sin^2\theta$, we obtain

$$\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} + k^2 r^2 \sin^2\theta = 0 \qquad \text{(A-1a)}$$

* The material covered in Appendix A and that in Appendix B is the same as that presented in Harrington "Time Harmonic Electromagnetic Fields". It is presented here for reader convenience.

The φ dependence is now separated out, and we let

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = -m^2 \tag{A-3}$$

where m is constant. Substituting Eq. (A-3) into Eq. (A-1a) and dividing by $\sin^2\theta$ gives

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta} + k^2 r^2 = 0 \tag{A-1b}$$

This separates the r and θ dependence. An apparently strange choice of separation constant n is made according to

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = -n(n+1) \tag{A-4}$$

because the properties of the Θ functions depend upon whether or not n is an integer. With this choice Eq. (A-1b) becomes

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - n(n+1) + k^2 r^2 = 0 \tag{A-5}$$

which completes the separation procedure.

Collecting the above results, we have these three separate equations;

(1) From Eq. (A-3), $\frac{d^2\Phi}{d\varphi^2} + m^2\Phi = 0$

which gives $\Phi(\varphi) = h(m\varphi)$ (A-6a)

where $h(m\varphi)$ is the harmonic function and if a single-valued Ψ in the range 0 to 2π on φ is desired, we must choose $h(m\varphi)$ to be a linear combination of $\sin(m\varphi)$ and $\cos(m\varphi)$, or of $e^{jm\varphi}$ and $e^{-jm\varphi}$, with m an integer.

(2) Eq. (A-4) can be written as

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left[n(n+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0$$

The solutions of this equation are called associated Legendre functions. We denote solutions in general by $L_n^m(\cos\theta)$ where $L_n^m(\cos\theta)$ are combination of $P_n^m(\cos\theta)$ and $Q_n^m(\cos\theta)$. $P_n^m(\cos\theta)$ are the associated Legendre functions of the first kind and $Q_n^m(\cos\theta)$ are the associated Legendre functions of the second kind. Note that $Q_n^m(\cos\theta)$ are infinite at $\theta = 0$ and $\theta = \pi$. Thus, if Ψ is to be finite in the range $\theta = 0$ or $\theta = \pi$, then n must be an integer and $L_n^m(\cos\theta)$ must be $P_n^m(\cos\theta)$ and generally

$$\Theta(\theta) = L_n^m(\cos\theta) \qquad \text{(A-6b)}$$

(3) From Eq. (A-5) we have

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \left[(kr)^2 - n(n+1)\right] R = 0$$

This equation is closely related to Bessel's equation. Its solutions are called spherical Bessel functions, denoted by b_n (kr), which are related to ordinary Bessel functions by

$$b_n\,(kr) = \sqrt{\frac{\pi}{2kr}}\, B_{n+\frac{1}{2}}\,(kr)$$

where B_n (kr) are the Bessel's functions.

Thus

$$R(r) = b_n\,(kr) \qquad \text{(A-6c)}$$

b_n (kr) is a linear combination of j_n (kr) and n_n (kr) where j_n (kr) are the spherical Bessel's functions of the first kind and n_n (kr) are the spherical Bessel's functions of the second kind. Note that the only spherical functions finite at $r = 0$ are j_n (kr). Thus to represent a finite field inside a sphere, b_n (kr) must be j_n (kr). It is to be noted that $h_n^{(2)}$(kr) represent an outward-travelling wave, thus to represent a finite feild outside of a sphere, b_n (kr) must be $h_n^{(2)}$(kr).

From Equations (A-2), (A-6a), (A-6b) and (A-6c) we get

$$\Psi = b_n (kr) L_n^m (\cos \theta) h(m\varphi)$$

and generally we have

$$\Psi = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_{mn} b_n (kr) L_n^m (\cos \theta) h(m\varphi) \qquad \text{(A-7)}$$

C_{mn} is a constant

where we have the following restrictions on $b_n(kr)$, $L_n^m(\cos \theta)$ and $h(m\varphi)$

(a) When the field is finite at $r = 0$, $b_n(kr)$ must be $j_n(kr)$, if Ψ is to be single-valued in the range 0 to 2π on φ, $h(m\varphi)$ must be $e^{\pm jm\varphi}$ with m being integer, and if the field is to be finite at $\theta = 0$ or $\theta = \pi$, $L_n^m (\cos \theta)$ must be $P_n^m (\cos \theta)$ and Ψ takes the form

$$\Psi = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_{mn} j_n(kr) P_n^m(\cos \theta) e^{\pm jm\varphi} \qquad \text{(A-8)}$$

(b) When $r = \infty$ is considered and the restrictions on θ and φ are the same as in case (a), then $b_n(kr)$ must be changed to $h_n^{(2)}(kr)$ and Ψ takes the general form

$$\Psi = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_{mn} \, h_n^{(2)}(kr) \, P_n^m(\cos\theta) \, e^{\pm jm\varphi} \qquad \text{(A-9)}$$

APPENDIX B

Determination of the Field Equations in Terms of the Vector Potentials

In a homeogeneous source-free isotropic region the field satisfies

$$-\nabla \times \bar{E} = \hat{z}\,\bar{H} \quad \text{where } \hat{z} = j\,\omega\,\mu \tag{B-1}$$

$$\nabla \cdot \bar{H} = 0 \tag{B-2}$$

$$\nabla \times \bar{H} = \hat{y}\,\bar{E} \quad \text{where } \hat{y} = \sigma + j\,\omega\,\epsilon \tag{B-3}$$

$$\nabla \cdot \bar{E} = 0 \tag{B-4}$$

Since any divergenceless vector may be expressed as the curl of some other vector, from Eq. (B-2) we may define a magnetic vector potential, A, such that

$$\bar{H} = \nabla \times \bar{A} \tag{B-5}$$

Substituting Eq. (B-5) into Eq. (B-1), we have

$$\nabla \times (\bar{E} + \hat{z}\,\bar{A}) = 0$$

Any curl-free vector may be expressed as the gradient of some scalar, thus a scalar potential may be defined such that

$$\bar{E} + \hat{z}\,\bar{A} = -\nabla \phi^{a} \tag{B-6}$$

To obtain the equation for $\bar{A}$, Eqs. (B-5) and (B-6) are substituted into Eq. (B-3) giving

$$\nabla \times \nabla \times \bar{A} - k^2 \bar{A} = - \hat{y} \nabla \phi^a \qquad \text{(B-7)}$$

where k is the wave number and = $-\hat{z}\ \hat{y}$

A similar equation may be derived in terms of electric vector and scalar potentials,

$$\nabla \times \nabla \times \bar{F} - k^2 \bar{F} = - \hat{z} \nabla \phi^f, \qquad \text{(B-8)}$$

where ϕ^f is a magnetic scalar potential and $\bar{F}$ is the electric vector potential such that

$$- \nabla \times \bar{F} = \bar{E} \qquad \text{(B-9)}$$

and

$$\bar{H} + \hat{y} \bar{F} = - \nabla \phi^f$$

By superimposing the values of $\bar{E}$ and $\bar{H}$ due to both the magnetic and electric vector potentials using Eqs. (B-3), (B-5) and (B-9), we get

$$\bar{E} = - \nabla \times \bar{F} + \frac{\nabla \times \bar{H}}{\hat{y}} = - \nabla \times \bar{F} + \frac{\nabla \times \nabla \times \bar{A}}{\hat{y}} \qquad \text{(B-10)}$$

and using Eqs. (B 1), (B 5) and (B 9), we have

$$\bar{H} = \nabla \times \bar{A} - \frac{\nabla \times \bar{E}}{\hat{z}} = \nabla \times \bar{A} + \frac{\nabla \times \nabla \times \bar{F}}{z} \qquad \text{(B-11)}$$

Equations (B-7) and (B-8) are the general equations for the vector potentials while equations (B-10) and (B-11) are the field equations in terms of the vector potentials.

Suppose we attempt to construct the field as a super position of two parts, one T.M. to $\bar{U}_r$ and the other T.E. to $\bar{U}_r$. For this we choose $\bar{A} = \bar{U}_r A_r$ and $\bar{F} = \bar{U}_r F_r$ with the fields given by Equations (B-10) and (B-11). To determine the equations that A_r and F_r must satisfy, we return to the general equations for vector potentials (B-7) and (B-8). For the magnetic vector potential we let $\bar{A} = \bar{U}_r A_r$ and expand Eq. (B-7). The θ and φ components of the resulting equation are, respectively,

$$\frac{\partial^2 A_r}{\partial r \partial \theta} = -\hat{y} \frac{\partial \Phi^a}{\partial \theta} \quad ; \qquad \frac{\partial^2 A_r}{\partial r \partial \varphi} = -\hat{y} \frac{\partial \Phi^a}{\partial \varphi}$$

Note that the above two equations are satisfied identically if we choose

$$-\hat{y}\Phi^a = \frac{\partial A_r}{\partial r} \tag{B-12}$$

Substituting this into the r-component of Eq. (B-7), we have

$$\frac{\partial^2 A_r}{\partial r^2} + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial A_r}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 A_r}{\partial \varphi^2} + k^2 A_r = 0 \tag{B-13}$$

It readily can be shown that this equation is

$$(\nabla^2 + k^2) \ \frac{A_r}{r} = 0 \tag{B-14}$$

So $\frac{A_r}{r}$ is a solution to the scalar Helmholtz equation that we obtained in Appendix A. Therefore

$$\frac{A_r}{r} = \Psi \quad \text{or} \quad A_r = r\,\Psi \tag{B-15}$$

From Equation (A-7) and (B-15) we get

$$A_r = \frac{1}{k} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_{mn} \left[kr \cdot b_n\,(kr) \right] L_n^m\,(\cos\,\theta)\ h\,(m\varphi)$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} G_{mn}\ \hat{B}_n\,(kr)\ L_n^m\,(\cos\,\theta)\ h(m\varphi) \tag{B-16}$$

where G_{mn} are constants. $\hat{B}_n\,(kr) = kr \cdot b_n\,(kr)$, called the associated spherical Bessel functions. $\hat{B}_n\,(kr)$ satisfies the differential equation

$$\left[\frac{d^2}{dr^2} + k^2 - \frac{n(n+1)}{r^2} \right] \hat{B}_n\,(kr) = 0 \tag{B-17}$$

which can be obtained by substituting for b_n in terms of $\hat{B}_n$ in equation (A-5) noting that $R = b_n\,(kr)$. For case (a) in Appendix A, i.e., when the field is finite at $r = 0$, we get using Eqs. (B-15) and (A-8),

$$A_r = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} G_{mn} \hat{J}_n (kr) P_n^m (\cos \theta) e^{\pm jm\varphi} \quad \text{(B-18)}$$

If the terms corresponding to m = 1 are the only terms that exist, therefore,

$$A_r = \sum_{n=0}^{\infty} G_n \hat{J}_n (kr) P_n^1 (\cos \theta) \begin{Bmatrix} \cos \varphi \\ \sin \varphi \end{Bmatrix}$$

and since $P_0^1 (\cos \theta) = 0$, then

$$A_r = \sum_{n=1}^{\infty} G_n \hat{J}_n (kr) P_n^1 (\cos \theta) \begin{Bmatrix} \cos \varphi \\ \sin \varphi \end{Bmatrix} \quad \text{(B-19)}$$

Similarly for case (b) in Appendix A we get using equations (B-15) and (A-9)

$$A_r = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} G_{mn} H_n^{(2)} (kr) P_n^m (\cos \theta) e^{\pm jm\varphi} \quad \text{(B-20)}$$

Note that this case holds only if we are considering an outward travelling wave and the point $r = \infty$ is included. For m = 1, Eq. (B-20) becomes

$$A_r = \sum_{n=1}^{\infty} G_n H_n^{(2)} (kr) P_n^1 (\cos \theta) \begin{Bmatrix} \cos \varphi \\ \sin \varphi \end{Bmatrix} \quad \text{(B-21)}$$

A dual development applies to the electric vector potential $\overline{F}$ and we get a similar formula for F_r.

Letting $\overline{A} = \overline{U}_r A_r$ and $\overline{F} = \overline{U}_r F_r$ and expanding Eqs.

(B-10) and (B-11), we get

$$E_r = \frac{1}{\hat{y}} \left(\frac{\partial^2}{\partial r^2} + k^2\right) A_r, \qquad \hat{y} = \sigma + j\omega\epsilon \tag{B-22}$$

$$E_\theta = \frac{-1}{r \sin\theta} \frac{\partial F_r}{\partial \varphi} + \frac{1}{\hat{y} r} \frac{\partial^2 A_r}{\partial r \partial \theta} \tag{B-23}$$

$$E_\varphi = \frac{1}{r} \frac{\partial F_r}{\partial \theta} + \frac{1}{\hat{y}\, r \sin\theta} \frac{\partial^2 A_r}{\partial r \partial \varphi} \tag{B-24}$$

$$H_r = \frac{1}{\hat{z}} \left(\frac{\partial^2}{\partial r^2} + k^2\right) F_r, \qquad \hat{z} = j\omega\mu \tag{B-25}$$

$$H_\theta = \frac{1}{r \sin\theta} \frac{\partial A_r}{\partial \varphi} + \frac{1}{\hat{z} r} \frac{\partial^2 F_r}{\partial r \partial \theta} \tag{B-26}$$

$$H_\varphi = \frac{-1}{r} \frac{\partial A_r}{\partial \theta} + \frac{1}{\hat{z}\, r \sin\theta} \frac{\partial^2 F_r}{\partial r \partial \varphi} \tag{B-27}$$

The above six equations are the basic equations used throughout this paper.

APPENDIX C

C-1. Average Rain-Drop Sizes and the Corresponding Values of k_oa and k_da in the Frequency Range 10 M Hz to 300 M Hz

In the frequency range (10 M Hz to 300 M Hz) at a temperature of 25°C the value of $\epsilon_r' \cong 78$[7] for water droplets

a in cm	Frequency in M Hz	$k_oa = \frac{2\pi a}{\lambda_o}$	$k_da = k_oa \sqrt{\epsilon_r'}$
0.2	300	0.0125	0.11
0.4	300	0.025	0.22
0.5	300	0.031	0.272

Note that k_da is smaller than 0.7 for the given drop sizes and at a frequency of 300 M Hz. Note also that $k_oa \ll 1$ for the same drop sizes. For a frequency, f, between 10 M Hz and 300 M Hz. the value of $k_oa \lesssim$ the value of k_oa at frequency 300 M Hz.

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ATTENUATION OF ELECTROMAGNETIC RADIATION BY WATER DROPLETS IN THE ATMOSPHERE

by

Abdel Wahab Fayez Hussein

ABSTRACT

This thesis deals with the theoretical analysis of the effect of the water droplets in the atmosphere on the propagation of a linearly polarized plane wave. These effects are (1) scattering - it is found that the scattered power is proportional to the sixth power of the radius of the droplet, also the scattered power varies as the fourth power of the frequency. (2) Absorption - it is found that the dielectric loss is much greater than the conductivity loss in the frequency range 10 M Hz. to 300 M Hz. The absorbed power is found to be much greater than the scattered power in the frequency range 10 M Hz. to 300 M Hz. Multiple scattering is neglected because scattered power is very small compared with the power absorbed. At the end of the thesis an expression for the attenuation constant is derived for homogeneous distribution of rain-drops of particular size falling at a particular rainfall rate.