Using the Trans-log Expenditure Function to Endogenize New Market Access in Partial Equilibrium Models*

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I. Introduction

While Constant Elasticity of Substitution (CES) utility functions are a common choice in empirical partial and general equilibrium models because they are parsimonious in the number of model parameters, they are not well suited for cases where the consumption of a given could equal zero. This is because the uncompensated demand functions derived from a CES utility function will equal zero only if the price of that good is equal to infinity or the shift parameter in the utility function for that good is equal to zero (e.g., consumers do not wish to consume that good). In order to allow for the possibility of zero consumption, a preference structure must allow for the underlying "demand curve" to intersect the price axis. This will be important when assessing the effect of new market access for a particular good. In that case, initial consumption is equal to zero because of the existing policy. After a change in policy, the good may or may not be sold in a particular region or season if its price is less than the consumers' reservation price for that good. The purpose of this draft is to develop a preference structure that allows for zero consumption and how it could be implemented to assess the impact of new market access.

II. Translog Expenditure Function

Following Bergin and Feenstra (2009), one functional form that can allow for zero consumption is the trans-log expenditure function. The unit-expenditure function for the trans-log is defined by:

$$ln\, e(p) = \alpha_0 + \sum_{j=1}^{\tilde{N}} \alpha_j \, ln\, p_j + \frac{1}{2} \sum_{i=1}^{\tilde{N}} \sum_{j=1}^{\tilde{N}} \gamma_{ij} \, ln\, p_i \, ln\, p_j \,, \qquad (1)$$

where $\tilde{N}$ is the number of possible products. If a good is not available or not consumed, its price would equal its reservation price. In order for the trans-log expenditure function to be homogeneous of degree one in prices, the following parametric restrictions are required:

$$\sum_{j=1}^{\tilde{N}} \alpha_j = 1, \text{ and } \sum_{j=1}^{\tilde{N}} \gamma_{jk} = 0 \;\; \forall k = 1, \ldots, \tilde{N} \,. \tag{2}$$

Note that because of symmetry (due to Young's theorem), $\gamma_{ij} = \gamma_{ji}$, which then implies that:

$$\sum_{k=1}^{\tilde{N}} \gamma_{jk} = 0 \;\; \forall j = 1, \ldots, \tilde{N} \,. \tag{3}$$

Using Shephard's lemma, the compensated demand functions are:

$$s_i = \alpha_i + \sum_{j=1}^{\tilde{N}} \gamma_{ij} \, ln \, p_{ij} \,. \tag{4}$$

If preferences are homothetic, then equation (4) is also the uncompensated demand function.

Now assume that only N of the $\tilde{N}$ goods are available to consumers. Bergin and Feenstra (2009) show that one may solve for the reservation prices for the $\left(\tilde{N} - N\right)$ goods that are not available in terms of the parameters of the expenditure function in equation (1) and observed prices for the N goods that are available. By substituting these reservation prices back into the expenditure function in equation (1), it is possible to derive a "reduced-form" expenditure function that is valid for the N available goods. The parameters of this reduced-form expenditure function, which are determined based on observed prices and shares, are then used to determine the parameters of the "full" expenditure function for a given set of reservation prices.

III. Deriving the "Reduced-Form" Expenditure Function

In matrix form, the trans-log expenditure function in equation (1) can be written as:

$$ln\, e(p) = \alpha_0 + \alpha' \,ln\, p + 0.5 * ln\, p' \Gamma \,ln\, p\,, \tag{5}$$

where α_0 is a (1x1) scalar, α and p are $(\tilde{N}x1)$ vectors, and Γ is a $(\tilde{N}x\tilde{N})$ matrix. The share equations can then be expressed:

$$s = \alpha + \Gamma \,ln\, p\,, \tag{6}$$

where *s* is a $(\tilde{N}x1)$ vector of budget shares. Next, one can partition all vectors and matrices:

$$s = \begin{bmatrix} s^1 \\ s^2 \end{bmatrix};\ \alpha = \begin{bmatrix} \alpha^1 \\ \alpha^2 \end{bmatrix};\ p = \begin{bmatrix} p^1 \\ p^2 \end{bmatrix};$$
$$s^1 = \begin{bmatrix} s_1 \\ \vdots \\ s_N \end{bmatrix};\ s^2 = \begin{bmatrix} s_{N+1} \\ \vdots \\ s_{\tilde{N}} \end{bmatrix}\ \alpha^1 = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix};\ \alpha^2 = \begin{bmatrix} \alpha_{N+1} \\ \vdots \\ \alpha_{\tilde{N}} \end{bmatrix}\ p^1 = \begin{bmatrix} p_1 \\ \vdots \\ p_N \end{bmatrix};\ p^2 = \begin{bmatrix} p_{N+1} \\ \vdots \\ p_{\tilde{N}} \end{bmatrix}$$
$$\Gamma = \begin{bmatrix} \Gamma^{11} & \Gamma^{12} \\ \Gamma^{21} & \Gamma^{22} \end{bmatrix}$$
$$\Gamma^{11} = \begin{bmatrix} \gamma_{11} & \cdots & \gamma_{1N} \\ \vdots & \ddots & \vdots \\ \gamma_{N1} & \cdots & \gamma_{NN} \end{bmatrix}\ \Gamma^{12} = \begin{bmatrix} \gamma_{1,\tilde{N}+1} & \cdots & \gamma_{1\tilde{N}} \\ \vdots & \ddots & \vdots \\ \gamma_{N,\tilde{N}+1} & \cdots & \gamma_{N\tilde{N}} \end{bmatrix}$$
$$\Gamma^{21} = \begin{bmatrix} \gamma_{\tilde{N}+1,1} & \cdots & \gamma_{\tilde{N}+1,N} \\ \vdots & \ddots & \vdots \\ \gamma_{\tilde{N}1} & \cdots & \gamma_{\tilde{N}N} \end{bmatrix}\ \Gamma^{22} = \begin{bmatrix} \gamma_{\tilde{N}+1,N+1} & \cdots & \gamma_{\tilde{N}+1,\tilde{N}} \\ \vdots & \ddots & \vdots \\ \gamma_{\tilde{N},\tilde{N}+1} & \cdots & \gamma_{\tilde{N}\tilde{N}} \end{bmatrix} \tag{7}$$

Note that Γ^{11} is a (*N*x*N*) matrix, Γ^{12} is a $\left[Nx\left(\tilde{N}-N\right)\right]$ matrix, Γ^{21} is a $\left[\left(\tilde{N}-N\right)xN\right]$ matrix, and Γ^{22} is a $\left[\left(\tilde{N}-N\right)x\left(\tilde{N}-N\right)\right]$ matrix.

By noting that s^2 equals zero for all elements, one can rewrite the share equations as:

$$s^1 = \alpha^1 + \Gamma^{11} \ln p^1 + \Gamma^{12} \ln p^2 , \tag{8}$$

$$0 = \alpha^2 + \Gamma^{21} \ln p^1 + \Gamma^{22} \ln p^2 , \tag{9}$$

From equation (9), one can solve for natural logarithm of the reservation prices:

$$\ln p^2 = -\left(\Gamma^{22}\right)^{-1}\left(\alpha^2 + \Gamma^{21} \ln p^1\right). \tag{10}$$

Rewriting equation (5) using the partitions defined in equation (7):

$$\begin{aligned} \ln e(p) &= \alpha_0 + \alpha^{1\prime} \ln p^1 + \alpha^{2\prime} \ln p^2 + \\ &\quad 0.5\left[\ln p^{1\prime}\Gamma^{11} \ln p^1 + \ln p^{1\prime}\Gamma^{12} \ln p^2 + \ln p^{2\prime}\Gamma^{21} \ln p^1 + \ln p^{2\prime}\Gamma^{22} \ln p^2 + \right], \end{aligned} \tag{11}$$

Rearranging terms in equation (11) and using equations (8) and (9):

$$\ln e(p) = \alpha_0 + 0.5\left(\alpha^1 + s^1\right)' \ln p^1 + 0.5\left(\alpha^2 + s^2\right)' \ln p^2 \tag{12}$$

Noting that s^2 equals zero and

$$\begin{aligned} \ln e(p) &= \alpha_0 + 0.5\left(\alpha^1 + s^1\right)' \ln p^1 + 0.5\alpha^{2\prime} \ln p^2 \\ &= \alpha_0 + 0.5\left[\alpha^1 + \alpha^1 + \ln p^1\Gamma^{11} + \ln p^2\Gamma^{12}\right]\ln p^1 + 0.5\alpha^{2\prime} \ln p^2 \\ &= \alpha_0 + \alpha^{1\prime} \ln p^1 + 0.5\left[\ln p^{1\prime}\Gamma^{11} \ln p^1 + \ln p^{1\prime}\Gamma^{12} \ln p^2\right] + 0.5\alpha^{2\prime} \ln p^2 \end{aligned} \tag{13}$$

Next, substitute equation (10) into equation (13):

$$
\begin{aligned}
ln\, e(p) &= \alpha_0 + \alpha^{1\prime}\, ln\, p^1 + 0.5\, ln\, p^{1\prime}\Gamma^{11}\, ln\, p^1 - 0.5\, ln\, p^{1\prime}\Gamma^{12}\, ln\, p^2\left[\left(\Gamma^{22}\right)^{-1}\left(\alpha^2 + \Gamma^{21}\, ln\, p^1\right)\right] - \\
&\quad 0.5\alpha^{2\prime}\left[\left(\Gamma^{22}\right)^{-1}\left(\alpha^2 + \Gamma^{21}\, ln\, p^1\right)\right] \\
&= \alpha_0 + \alpha^{1\prime}\, ln\, p^1 + 0.5\, ln\, p^{1\prime}\Gamma^{11}\, ln\, p^1 - 0.5\left[ln\, p^{1\prime}\Gamma^{12}\left(\Gamma^{22}\right)^{-1}\alpha^2 + ln\, p^{1\prime}\Gamma^{12}\left(\Gamma^{22}\right)^{-1}\Gamma^{21}\, ln\, p^1\right] - \\
&\quad 0.5\left[\alpha^{2\prime}\left(\Gamma^{22}\right)^{-1}\alpha^2 + \alpha^{2\prime}\left(\Gamma^{22}\right)^{-1}\Gamma^{21}\, ln\, p^1\right] \\
&= \alpha_0 - 0.5\alpha^{2\prime}\left(\Gamma^{22}\right)^{-1}\alpha^2 + \left[\alpha^{1\prime} + \alpha^{2\prime}\left(\Gamma^{22}\right)^{-1}\Gamma^{21}\right] ln\, p^{1\prime} + \\
&\quad 0.5\, ln\, p^{1\prime}\left[\Gamma^{11} - \Gamma^{12}\left(\Gamma^{22}\right)^{-1}\Gamma^{21}\right] ln\, p^1
\end{aligned}
\tag{14}
$$

Using the following definitions:

$$a_0 = \alpha_0 - 0.5\alpha^{2\prime}\left(\Gamma^{22}\right)^{-1}\alpha^2, \tag{15}$$

$$a^1 = \alpha^1 - \alpha^{2\prime}\left(\Gamma^{22}\right)^{-1}\Gamma^{21}, \text{ and} \tag{16}$$

$$c^{11} = \Gamma^{11} - \Gamma^{12}\left(\Gamma^{22}\right)^{-1}\Gamma^{21} \tag{17}$$

the expenditure function in equation (14) can be written as:

$$ln\, e(p) = a_0 + a^{1\prime}\, ln\, p^1 + 0.5\, ln\, p^{1\prime}c^{11}\, ln\, p^1, \tag{18}$$

where it is only a function of the prices of the available goods.

IV. Calibrating the Trans-log Expenditure Function

One drawback of using a flexible functional form, like the trans-log, in empirical simulation models is the large number of parameters that must be chosen or calibrated. After imposing homogeneity and symmetry, there are $0.5*\left(\tilde{N}+1\right)\tilde{N}$ independent parameters in equation (1). To reduce the number of parameters that must be chosen by the modeler, we follow the assumption made in Bergin and Feenstra (2009) that the cross-price effects are the same for all goods, implying that:

$$\gamma_{ij} = \gamma \ \forall i \neq j \text{ and } \gamma_{ii} = -\gamma \big/ \left(\tilde{N} - 1\right). \tag{19}$$

Note that this assumption is the same employed when using a CES preference structure. This leaves only the parameters α_0 and $\left(\tilde{N} - 1\right)$ of the α_j to be calibrated. Also note that from equation (10), the calibrated values of γ, α_0, and α_j, along with the observed prices of the available goods will determine the value of the reservation price for the "unavailable" goods.

From equations (15) and (16), one can see that α parameters in equation (1) are related to the *a* parameters in the reduced-form trans-log expenditure function in equation (18). Thus, calibrating the reduced-form expenditure function is the first step in calibrating the full trans-log expenditure function. To begin the calibration process, start by choosing a value for γ. To aid in this choice, note that γ can be related to the own-price demand elasticity (ε_{ii}) using equation (19):

$$\varepsilon_{ii} = \frac{\gamma_{ii}}{s_i} - 1 \rightarrow \gamma_{ii} = \left(\varepsilon_{ii} + 1\right) s_i \rightarrow \gamma = \frac{-\left(\varepsilon_{ii} + 1\right) s_i \tilde{N}}{\left(\tilde{N} - 1\right)} \tag{20}$$

Once a value of γ has been chosen, then from equation (19), all elements of the Γ matrix are identified as well as all elements of the c^{11}matrix in equation (17).

To illustrate, consider an example where $\tilde{N}$ = 3 and $N = 2$. Using the definitions of Γ^{11}, Γ^{12}, Γ^{22}, and Γ^{21}:

$$c^{11} = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{12} & \gamma_{22} \end{bmatrix} - \frac{1}{\gamma_{33}} \begin{bmatrix} \gamma_{13} \\ \gamma_{23} \end{bmatrix} \begin{bmatrix} \gamma_{31} & \gamma_{32} \end{bmatrix} = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{12} & \gamma_{22} \end{bmatrix} - \frac{1}{\gamma_{33}} \begin{bmatrix} \gamma_{13}^2 & \gamma_{13}\gamma_{23} \\ \gamma_{13}\gamma_{23} & \gamma_{23}^2 \end{bmatrix}$$

$$c^{11} = \begin{bmatrix} -\gamma\left(\tilde{N}-1\right) & \gamma \\ \gamma & -\gamma\left(\tilde{N}-1\right) \end{bmatrix} - \frac{1}{-\gamma\left(\tilde{N}-1\right)} \begin{bmatrix} \gamma^2 & \gamma^2 \\ \gamma^2 & \gamma^2 \end{bmatrix} = \begin{bmatrix} -\gamma\left(\tilde{N}-1\right) & \gamma \\ \gamma & -\gamma\left(\tilde{N}-1\right) \end{bmatrix} + \begin{bmatrix} \frac{\gamma}{\left(\tilde{N}-1\right)} & \frac{\gamma}{\left(\tilde{N}-1\right)} \\ \frac{\gamma}{\left(\tilde{N}-1\right)} & \frac{\gamma}{\left(\tilde{N}-1\right)} \end{bmatrix}$$

Thus, the value of $\gamma/\left(\tilde{N}-1\right)$ is added to each element of Γ^{11} to obtain the c^{11}matrix.

Once the c^{11}matrix has been determined, one can use the share equations for the reduced-form expenditure function to determine the values of a^1. Specifically:

$$s_i = a_i + \sum_{j=1}^{N} c_{ij}\, ln\, p_j \rightarrow a_i = s_i - \sum_{j=1}^{N} c_{ij}\, ln\, p_j \,. \tag{21}$$

The final step is then to determine the value of a_0:

$$a_0 = ln\, e\left(p\right) - \sum_{j=1}^{N} a_j\, ln\, p_j - 0.5 \sum_{i=1}^{N} \sum_{j=1}^{N} c_{ij}\, ln\, p_i\, ln\, p_j \,. \tag{22}$$

Once a^1 and a_0 are known, then one can use equations (15) and (16) to determine the values of α_0, and α_j,

V. Determining the Reservation Prices

When $\tilde{N} > N$, the value of the reservation prices are unknown and must be determined using equation (10). However, due to the linear homogeneity of the expenditure function, it is not possible to identify unique values for the α parameters and the reservation price. To illustrate this, again consider the case where $\tilde{N} = 3$ and $N = 2$. Note that the case where $\tilde{N} = N + 1$ will be a common occurrence when assessing the impact of a removal of an import ban. Ignoring the parameter α_0 for the moment, the three share equations constitute a system of three equations in three unknowns: α_1, α_2 and $\ln p_3$. In matrix notation, this system can be expressed as:

$$\begin{bmatrix} 1 & 0 & \gamma_{13} \\ 0 & 1 & \gamma_{23} \\ -1 & -1 & \gamma_{33} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ ln\, p_3 \end{bmatrix} = \begin{bmatrix} s_1 - \gamma_{11}\, ln\, p_1 - \gamma_{12}\, ln\, p_2 \\ s_2 - \gamma_{12}\, ln\, p_1 - \gamma_{22}\, ln\, p_2 \\ -\gamma_{13}\, ln\, p_1 - \gamma_{23}\, ln\, p_2 \end{bmatrix}$$

Note that if you added the first two rows of the above matrix together and multiplied by -1, that would be equivalent to the last row in the matrix. This occurs because by homogeneity, $-(\gamma_{13} + \gamma_{23})$ equals γ_{33}. Thus, the matrix is not of full rank and therefore there is no unique solution to this system of equations.

The following numerical example will illustrate the above discussion. Consider the case where $\tilde{N}$ = 3 and N = 2. Goods 1 and 2 are both available to the consumer while good 3 is unavailable. The observed prices and quantities for all goods are given in Table 1. Assuming that γ equals 0.5, and therefore γ_{ii} equals -1.0, from equation (9) (noting that the natural logarithm of one is zero) $ln\, p_3 = \alpha_3$. The share equations for goods 1 and 2 are:

$$s_1 = \alpha_1 + 0.5\alpha_3, \text{ and } s_2 = \alpha_2 + 0.5\alpha_3.$$

Note that because α_1, α_2, and α_3 must sum to one, from the share equation for good 1:

$$0.4 = \alpha_1 + 0.5(1 - \alpha_1 - \alpha_2) \rightarrow 0.2 = \alpha_1 - \alpha_2.$$

This shows that the relative difference between α_1 and α_2 is 0.2.

The last half of Table 1 provides the values of α_0, α_1, α_2, and α_3 for three different values of the natural logarithm of the reservation price. Once a reservation price is chosen, this implies a value of α_3. If the logarithm of the reservation price and α_3 equals zero, then α_1 and α_2 will equal the budget share for their respective good. If the logarithm of the reservation price and α_3 equals 0.1, then both α_1 and α_2 are scaled down by 0.05 while the opposite is true if α_3 equals -0.1. Thus, there are an infinite number of reservations prices that will satisfy the observed budget shares and prices of the available goods for the trans-log expenditure function.

Without a unique reservation price for a given set of observed prices and quantities for the available goods, the question is then how to choose an appropriate reservation price? One approach would be to set the reservation price equal to that of a comparable good that is available. For example, fresh lemons from Argentine are not currently eligible to be exported to the United States (unavailable good), but fresh lemons from Chile are exported (available good). Given the proximity and seasonal production patterns of these two regions, it is possible that consumers may have similar preferences for lemons from Argentina and Chile. The reservation price for Chilean lemons can be determined from its share equation from the "reduced-form" expenditure function, denoted as:

$$s_i = a_i + \sum_{j=1}^{N} c_{ij} \, ln \, p_j \, . \tag{23}$$

To solve for the reservation price, set equation (23) equal to zero and solve for logarithm of the price of Chilean (*CHL*) lemons:

$$ln \, p_{CHL} = \frac{s_{CHL} - a_{CHL} - \sum_{j \neq CHL}^{N} c_{CHL,j} \, ln \, p_j}{c_{CHL,CHL}} . \tag{24}$$

VI. Illustration of Calibration Process

To illustrate the above discussion, consider the following example where there are five existing supply regions of fresh lemons (N = 5) and consumers view lemons from each region as a heterogeneous product. There is also one supply region (Argentina) whose lemons are not available, due to existing SPS regulations (e.g., $\tilde{N}$ = 6). Table 2 provides the initial prices, quantities, and budget shares for the five available varieties of fresh lemons. If γ = 0.08, then γ_{ij} = 0.08, γ_{ii} = -0.4, c_{ij} = 0.096, (0.08+0.08/5) c_{ii} = -0.384. Using equations (21) and (22), the values of a_i and a_0 can be determined. These values are listed in the middle portion of Table 2. Given those values, the reservation price for Chilean lemons is \$1.646/kg or:

$$ln\ p_{CHL} = \frac{0.0218 - 0.0596 - 0.096 * \left(ln\ p_{MX} + ln\ p_{ESP} + ln\ p_{OTH} + ln\ p_{US}\right)}{-0.384} = 0.4983,$$

where *MX* = Mexico, *ESP* = Spain, *OTH* = other, and *US* = United States

Setting this equal to the reservation price for Argentine lemons, the value of α_{ARG} can be determined using equation (10):

$$\alpha_{ARG} = -\left[0.08 * \left(ln\ p_{MX} + ln\ p_{ESP} + ln\ p_{OTH} + ln\ p_{US}\right) - 0.4 * ln\ p_{ARG}\right] = 0.0542,$$

where *ARG* refers to Argentina. Then the remaining α parameters can be determined using equations (15) and (16):

$$\alpha_i = a_i + \frac{\gamma_{ARG,i}}{\gamma_{ARG,ARG}} \alpha_{ARG} = a_i + \frac{0.08}{-0.4} 0.0524 \quad \forall i = MX, ESP, OTH, US$$

$$\alpha_0 = a_0 + \frac{\alpha_{ARG}^2}{2 * \gamma_{ARG,ARG}} = a_0 + \frac{(0.0524)^2}{2 * -0.4}.$$

The calibrated values for all α parameters are listed at the bottom of Table 2. Using these parameters, the chosen values of γ_{ij} and the reservation price for Argentina, and the observed prices for all available lemon varieties in the share equation for the full trans-log expenditure function (see equation (4)) yields the observed budget shares listed in Table 2.

Table 1. Numerical Example of Non-uniqueness of Reservation Prices

Data	***Good* 1**	***Good* 2**	***Good* 3**
Price	1.0	1.0	
Quantity	4.0	6.0	0.0
Share	0.4	0.6	0.0
Parameters			
γ	0.5		
γ_{ii}	-1.0		
Scenarios	1	2	3
Logarithm of reservation price	-0.1	0.0	0.1
α_3	-0.1	0.0	0.1
α_1	0.45	0.40	0.35
α_2	0.65	0.60	0.55
α_0	ln(10)+0.005	ln(10)	ln(10)-0.005
s_1	0.4	0.4	0.4
s_2	0.6	0.6	0.6
s_3	0.0	0.0	0.0

Table 2. Calibration Example for Lemons

Data

Region	Price ($/kg)	Quantity (1,000 MT)	Share	Log Price
Mexico	1.089	1.93	0.0163	0.0853
Chile	1.555	1.81	0.0218	0.4415
Spain	1.583	0.66	0.0081	0.4593
Other	1.409	0.16	0.0017	0.3429
US	1.624	75.56	0.9520	0.4849
Argentina		0.0	0.0	

Parameters	a	α
Mexico	-0.1169	-0.1277
Chile	0.0596	0.0488
Spain	0.0545	0.0436
Other	-0.0078	-0.0186
US	1.0106	0.9998
Argentina		0.0542
γ	0.08	
a_0	4.3562	
α_0	4.3525	

References

Bergin, P.R. and R.C. Feenstra. (2009) "Pass-Through of Exchange Rates and Competition between Floaters and Fixers." *Journal of Money, Credit and Banking*. 41 (S1) (February 2009): 35-70.