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*N*th-order multifrequency coherence functions: A functional path integral approach[a)]

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A functional (or path) integral applicable to a broad class of randomly perturbed media is constructed for the nth-order multifrequency coherence function (a quantity intimately linked to nth-order pulse statistics). This path integral is subsequently carried out explicitly in the case of a nondispersive, deterministically homogeneous medium, with a simplified (quadratic) Kolmogorov spectrum, and a series of new results are derived. Special cases dealing with the two-frequency mutual coherence function for plane and beam pulsed waves are considered, and comparisons are made with previously reported findings.

1. INTRODUCTION

There has been recently a mounting interest in the subject of propagation of pulsed signals through randomly perturbed media. Proposed high data rate communication systems at millimeter and optical frequencies, remote sensing schemes, low-frequency underwater sound signaling and detection, interpretation of signals emitted by extraterrestrial radio sources such as pulsars, all require a quantitative assessment of stochastic pulse broadening. The latter leads to an irreversible degradation of a signal, in contradistinction to dispersive pulse spreading, which is a reversible phenomenon. (The receiver is usually equipped with "built-in dispersion" in order to make optimal use of the additional signal bandwidth).

Earlier contributions in this area (cf., e.g., Refs. 1–3) were confined to weakly turbulent media and/or short propagation paths, for which methods such as the Born and Rytov approximations were adequate. More recent theoretical analyses which account for multiple scattering, large-scale inhomogeneities and long propagation distances are based for the most part on the parabolic equation for the complex field amplitude and the Markov random process approximation.[4] Within the framework of this formalism, it has been recognized that complete information about transient signal propagation in random media requires the solutions of moment equations for the wave field at different frequencies and different positions. Although a complete set of such equations can be derived (cf. e.g., Ref. 5), solutions are available only for the two-frequency mutual coherence function, and these results (both analytical and numerical) are generally restricted to plane wave calculations.[6–15]

There exist physical situations (e.g., laser beam propagation in the atmosphere and in lightguides, and underwater sound wave propagation) where the restriction to spatially planar sources must be lifted. The problem of propagation of pulsed beam waves in randomly perturbed environment has been investigated by the method of "temporal moments."[16] Since this technique is based on the two-frequency mutual coherence function, it yields only information at the level of second-order pulse statistics (e.g., mean arrival time, mean square pulse width, etc.). Pulse shapes cannot be obtained directly by this method; however, they can be synthesized by superimposing individual temporals moments. An alternative technique which, at least in principle, enables one to compute approximately multifrequency coherence functions for arbitrary beam waves propagating in a random medium was proposed recently by Fante.[17] It is based on the phase-screen approximation along with the extended Huygens–Fresnel principle. So far, only results pertaining to the two-frequency mutual coherence function have been reported in the literature.[18]

One of the reasons why the aforementioned approaches (with the exception, perhaps, of the one suggested by Fante[17]) have not yielded sufficient information in connection with multifrequency coherence functions for beams in a random medium is that the study of the asymptotic (or even exact in special cases) behavior of these functions based on their governing local equations is a nontrivial problem. In contradistinction, recently formulated methods based on functional path integration (cf. Refs. 19–21) have the distinct advantage that they work on a *global* rather than a *local* level, thus making the algorithmic derivation of asymptotic solutions to higher-order moments easier.

It is our specific intent in this exposition to use the functional path integration approach in order to evaluate nth-order coherence functions for arbitrary source distributions. These evaluations will be performed within the confines of a simplified (quadratic) Kolmogorov spectrum. (Similar re-

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sults, for a single frequency and a specific source distribution, have been determined by Furutsu[22] using a different technique.)

The structure of the paper can be outlined as follows: Several preliminary concepts pertaining to pulse propagation within the domain of validity of the quasioptics (or parabolic) approximation are developed in Sec. 2. A functional path integral applicable to a broad class of random media is constructed at the beginning of Sec. 3 for the nth-order multifrequency coherence function (a quantity intimately linked to nth-order pulse statistics). This path integral is subsequently carried out explicitly in the case of a nondispersive, deterministically homogeneous medium, with a simplified (quadratic) Kolmogorov spectrum, and a series of new results are derived. Special cases dealing with the two-frequency mutual coherence function for plane and beam pulsed waves are considered in Sec. 4, where comparisons are also made with previously reported findings. Finally, the possibility of asymptotic expansions of our general exact results in the partially and fully saturated regimes are briefly discussed in Sec. 5.

2. PRELIMINARY CONCEPTS

The preliminary analysis, as well as the notation, in this section will be specialized to the case of electromagnetic wave propagation in a random channel. It should be emphasized, however, that the resulting stochastic complex parabolic equation is equally applicable in other physical areas (e.g., random underwater sound propagation).

A. The quasioptics approximation

Ignoring depolarization effects, the transverse, complex, electric field of radiation is governed by the stochastic Helmholtz equation

$$\nabla^2 E(\mathbf{r},\omega;\alpha) + (\omega/c)^2\epsilon_r(\mathbf{r},\omega;\alpha)E(\mathbf{r},\omega;\alpha) = 0, \quad \mathbf{r}\in R^3. \qquad (2.1)$$

Here, ω is the angular frequency, c is the speed of light in *vacuo*, and $\epsilon_r(\mathbf{r},\omega;\alpha)$—the relative permittivity provided the medium is nonmagnetic—is a dimensionless, scalar, random function depending on a parameter $\alpha\in A$, (A,F,P) being an underlying probability measure space. If, in addition to dispersion, the medium is characterized by either gain or loss, the relative permittivity $\epsilon_r(\mathbf{r},\omega;\alpha)$ is complex. In the sequel, we shall restrict our attention to physical situations where this quantity is real.

Let $E\{\epsilon_r(\mathbf{r},\omega;\alpha)\}$ and $\delta\epsilon_r(\mathbf{r},\omega;\alpha)$ denote respectively the average and fluctuating parts of the relative permittivity. Let, furthermore, $\epsilon_0(\omega)$ be a convenient "reference" quantity. It may, for example, coincide with the average relative permittivity if the latter is independent of the position variable $\mathbf{r}$. We introduce, next, three new quantities as follows:

$$k^2(\omega) = (\omega/c)^2\epsilon_0(\omega), \qquad (2.2)$$

$$\epsilon_1(\mathbf{r},\omega;\alpha) = \delta\epsilon_r(\mathbf{r},\omega;\alpha)/\epsilon_0(\omega), \qquad (2.3)$$

$$\epsilon_2(\mathbf{r},\omega) = [E(\epsilon_r(\mathbf{r},\omega;\alpha)) - \epsilon_0(\omega)]/\epsilon_0(\omega). \qquad (2.4)$$

With these definitions, (2.1) assumes the form

$$\nabla^2 E(\mathbf{r},\omega;\alpha) + k^2(\omega)[1 + \epsilon_2(\mathbf{r},\omega) + \epsilon_1(\mathbf{r},\omega;\alpha)]E(\mathbf{r},\omega;\alpha) = 0. \qquad (2.5)$$

It is clear that $\epsilon_2(\mathbf{r},\omega)$ accounts for deterministic inhomogeneities in the medium and, in light of a statement made earlier, it vanishes in the absence of such background profiles. On the other hand, $\epsilon_1(\mathbf{r},\omega;\alpha)$, a zero-mean random function, is directly associated with the superimposed random effects.

For plane and beam propagation in the z-direction, it is convenient to resort to the transformation

$$E(\mathbf{r},\omega;\alpha) = \Psi(\mathbf{x},z,\omega;\alpha)\exp(ikz);\mathbf{r} = (\mathbf{x},z), k = k(\omega). \qquad (2.6)$$

In the quasioptical description, the slowly varying complex random amplitude function $\Psi(\mathbf{x},z,\omega;\alpha)$ is described exceedingly well by the stochastic complex parabolic equation

$$\frac{i}{k}\frac{\partial}{\partial z}\Psi(\mathbf{x},z,\omega;\alpha) = -\frac{1}{2k^2}\nabla_{\mathbf{x}}^2\Psi(\mathbf{x},z,\omega;\alpha) - \tfrac{1}{2}[\epsilon_2(\mathbf{x},z,\omega) + \epsilon_1(\mathbf{x},z,\omega;\alpha)]\Psi(\mathbf{x},z,\omega;\alpha), \quad z>0. \qquad (2.7a)$$

In the presence of a deterministic profile ($\epsilon_2\neq 0$), the parabolic equation (2.7a) constitutes a valid approximation to (2.5) if the normals to the wavefronts in the unperturbed problem, where $\epsilon_1 = 0$, remain close to the z axis.

Corresponding to the boundary condition $E(\mathbf{x},0,\omega;\alpha)\equiv E_0(\mathbf{x},\omega;\alpha)$ for (2.5), one has the "initial" condition

$$\Psi(\mathbf{x},0,\omega;\alpha)\equiv\Psi_0(\mathbf{x},\omega;\alpha) = E_0(\mathbf{x},\omega;\alpha), \qquad (2.7b)$$

which incorporates all the information concerning the temporal frequency spectrum and the spatial distribution of the source at the initial plane $z = 0$.

In our subsequent formulation based on the functional path integral technique we shall require the fundamental solution (referred to alternatively as the propagator or Green's function) of (2.7). This quantity, denoted here by $G(\mathbf{x},\mathbf{x}',z,\omega;\alpha)$, provides a link between the wavefunction $\Psi(\mathbf{x},z,\omega;\alpha), z>0$, and the boundary condition $\Psi_0(\mathbf{x},\omega;\alpha)$, viz.,

$$\Psi(\mathbf{x},z,\omega;\alpha) = \int_{R^2} d\mathbf{x}' G(\mathbf{x},\mathbf{x}',z,\omega;\alpha)\Psi_0(\mathbf{x}',\omega;\alpha), \qquad (2.8)$$

and satisfies the equation

$$\frac{i}{k}\frac{\partial}{\partial z}G(\mathbf{x},\mathbf{x}',z,\omega;\alpha) = -\frac{1}{2k^2}\nabla_{\mathbf{x}}^2 G(\mathbf{x},\mathbf{x}',z,\omega;\alpha) - \tfrac{1}{2}[\epsilon_2(\mathbf{x},z,\omega) + \epsilon_1(\mathbf{x},z,\omega;\alpha)]G(\mathbf{x},\mathbf{x}',z,\omega;\alpha), \quad z>0, \qquad (2.9a)$$

$$G(\mathbf{x},\mathbf{x}',0,\omega;\alpha) = \delta(\mathbf{x} - \mathbf{x}'). \qquad (2.9b)$$

B. *N*th-order pulse statistics

Consider next the situation where a receiver at range z is characterized by a temporal spectrum $F_r(\omega)$—usually a bandpass function of frequency. It follows, then, from our work so far together with the existing linearity, that the wavefunction of interest at the receiver site is $E_r(\mathbf{x},z,\omega;\alpha) \equiv E(\mathbf{x},z,\omega;\alpha)F_r(\omega)$. The corresponding time-dependent, real, random signal can be expressed as

$$e_r(\mathbf{x},z,t;\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \int_{R^2} d\mathbf{x}'\, G(x,\mathbf{x}',z,\omega;\alpha) \times F_r(\omega)E_0(\mathbf{x}',\omega;\alpha)\exp\{-i[\omega t - k(\omega)z]\}, \tag{2.10}$$

provided that the parabolic approximation is valid. The signal $e_r(\mathbf{x},z,t;\alpha)$ itself is not an observable quantity. However, a substantial amount of information associated with physically measurable pulse statistics is contained in the nth-order moments

$$E\left(\prod_{p=1}^{n} e_r(\mathbf{x}_p,z,t_p;\alpha)\right) = \frac{1}{(2\pi)^n}\int_{R^n} d\boldsymbol{\omega}\int_{R^{2n}} d\,\mathbf{X}'E\{G^{(n)}(\mathbf{X},\mathbf{X}',z,\boldsymbol{\omega};\alpha)\} \times F_r^{(n)}(\boldsymbol{\omega})E\{E_0^{(n)}(\mathbf{X}',\boldsymbol{\omega};\alpha)\} \times \exp\left(\sum_{p=1}^{n}(-i)\xi_p[\omega_p t_p - k(\omega_p)z]\right), \tag{2.11}$$

where n is assumed to be an even integer; $\xi_p = 1$, p odd, $\xi_p = -1$, p even, and the following notation is used:

$$\boldsymbol{\omega} = (\omega_1,\omega_2,\ldots,\omega_n)\in R^n, \tag{2.12a}$$

$$\mathbf{X} = (\mathbf{x}_1,\mathbf{x}_2,\ldots,x_n)\in R^{2n}, \tag{2.12b}$$

$$\mathbf{X}' = (\mathbf{x}_1'\mathbf{x}_2',\ldots,\mathbf{x}_n')\in R^{2n}; \tag{2.12c}$$

$$F_r^{(n)}(\boldsymbol{\omega}) = \prod_{p=1}^{n/2} F_r^*(\omega_{2p})F_r(\omega_{2p-1}), \tag{2.12d}$$

$$E_0^{(n)}(\mathbf{X}',\boldsymbol{\omega};\alpha) = \prod_{p=1}^{n/2} E_0^*(\mathbf{x}_{2p}',\omega_{2p};\alpha)E_0(\mathbf{x}_{2p-1}',\omega_{2p-1};\alpha), \tag{2.12e}$$

$$G^{(n)}(\mathbf{X},\mathbf{X}',z,\boldsymbol{\omega};\alpha) = \prod_{p=1}^{n/2} G^*(\mathbf{x}_{2p},\mathbf{x}_{2p}',z,\omega_{2p};\alpha) \times G(\mathbf{x}_{2p-1},\mathbf{x}_{2p-1}',z,\omega_{2p-1};\alpha). \tag{2.12f}$$

Several specific remarks are in order: (1) The derivation of (2.11) presupposes statistical independence between source incoherencies and random fluctuations in the medium; moreover, the receiver is assumed to be coherent; (2) The choice of n even is made on the strength of physical evidence that moments $E\{G^{(n)}\}$ with n odd—unequal number of conjugated and unconjugated terms in (2.12f)—decay relatively fast with increasing range z; (3) At the receiver site, further processing of the nth-order moments given in (2.11), such as averaging over the coordinates $\mathbf{x}_p$, may be necessary.

C. General remarks

It is clear from the foregoing discussion that the study of pulse propagation in a random medium requires knowledge of the nth-order coherence functions $E\{G^{(n)}(\mathbf{X},\mathbf{X}',z,\boldsymbol{\omega};\alpha)\}$ at different frequencies and different transverse (with respect to z) coordinates.

Transport equations for the moments $E\{G^{(n)}(\mathbf{X},\mathbf{X}',z,\boldsymbol{\omega};\alpha)\}$ can be obtained using the Markovian random process approximation, even for dispersive media. In the absence of a deterministic profile ($\epsilon_2 = 0$), Lee,[5] for example, has derived such a set of transport equations for the special case of a randomly perturbed cold plasma.

The solution of these transport equations is difficult, in general. Nevertheless, as already mentioned in the introduction, some progress has been made in connection with the 2-frequency mutual coherence function $\Gamma(\mathbf{x}_2,\mathbf{x}_1,z,\omega_2,\omega_1) \equiv E\{\Psi^*(\mathbf{x}_2,z,\omega_2;\alpha)\Psi(\mathbf{x}_1,z,\omega_1;\alpha)\}$, albeit for planar source distributions, viz., $\Gamma(\mathbf{x}_2,\mathbf{x}_1,0,\omega_2,\omega_1) = \Gamma_0(\omega_2,\omega_1)$. This assumption gives rise to a great deal of simplification since, upon resorting to the center of mass and difference coordinates $\mathbf{x} = (\mathbf{x}_1 + \mathbf{x}_2)/2$ and $\mathbf{y} = \mathbf{x}_2 - \mathbf{x}_1$, one recognizes that Γ depends on $\mathbf{y}$ but not on $\mathbf{x}$, and the corresponding equation for $\Gamma(\mathbf{y},z,\omega_2,\omega_1)$ can be solved—analytically in the case of a simplified (quadratic) Kolmogorov spectrum (cf. Ref. 12), or numerically under more relaxed assumptions regarding the spectrum of the random inhomogeneities (cf. Refs. 7, 9, and 10).

The procedure outlined above in connection with the computation of $\Gamma(\mathbf{x}_2,\mathbf{x}_1,z,\omega_2,\omega_1)$ in the case of spatially planar source distributions cannot be extended easily to higher-order multifrequency moments. The complexity of the required generalizations is compounded in physical situations where nonplanar source distributions must be considered.

The aforementioned difficulties can be alleviated to some extent by using the functional path integral method. This formalism can be briefly outlined as follows: The propagator $G(\mathbf{x},\mathbf{x}',z,\omega;\alpha)$ and, in turn, $G^{(n)}(\mathbf{X},\mathbf{X}',z,\boldsymbol{\omega};\alpha)$, are first expressed as (continuous) functional path integrals. Upon ensemble averaging $E\{G^{(n)}(\mathbf{X},\mathbf{X}',z,\boldsymbol{\omega};\alpha)\}$, the required quantity, is also expressed as a path integral. The latter is finally evaluated—exactly in special cases, or asymptotically in general. Since this evaluation is based on a *global* expression for $E\{G^{(n)}(\mathbf{X},\mathbf{X}',z,\boldsymbol{\omega};\alpha)\}$, the underlying analysis (exact or asymptotic) is invariably simpler than the one required for the direct solution of the *local* transport equation for $E\{G^{(n)}(\mathbf{X},\mathbf{X}',z,\boldsymbol{\omega};\alpha)\}$.

3. *N*th-ORDER MULTIFREQUENCY COHERENCE FUNCTIONS: A FUNCTIONAL PATH INTEGRAL APPROACH

A. The Feynman path integral

The solution of the stochastic complex parabolic Eq. (2.9) for the propagator $G(\mathbf{x},\mathbf{x}',z,\omega;\alpha)$ can be expressed as a continuous functional path integral (cf. Refs. 23 and 24; see, also Ref. 25). Specifically,

$$G(\mathbf{x},\mathbf{x}',z,\omega;\alpha) = \int d\,[\mathbf{x}(\zeta)]\exp\left\{ik\int_0^z d\zeta\,\tfrac{1}{2}\left[\dot{\mathbf{x}}^2(\zeta) + \epsilon_2[\mathbf{x}(\zeta),\zeta,\omega] + \epsilon_1[\mathbf{x}(\zeta),\zeta,\omega;\alpha]\right]\right\}, \tag{3.1}$$

where $d[\mathbf{x}(\zeta)]$ is the usual Feynman path differential measure, and the integration is over "all" paths $\mathbf{x}(\zeta)$ subject to the boundary conditions $\mathbf{x}(0)=\mathbf{x}',\mathbf{x}(z)=\mathbf{x}$. The dot over $\mathbf{x}(\zeta)$ designates a derivative with respect to the argument ζ. As mentioned earlier, k is an abbreviation for $k(\omega)$. Finally $\dot{\mathbf{x}}^2(\zeta)$ denotes the square norm of the vector-valued function $\dot{\mathbf{x}}(\zeta)$.

Equation (3.1) can then be used as a basis for constructing a path integral represenation for the nth-order quantity $G^{(n)}(\mathbf{X},\mathbf{X}',z,\omega;\alpha)$ [cf. Eq. (2.12f)]. Specifically,

$$G^{(n)}(\mathbf{X},\mathbf{X}',z,\omega;\alpha) = \int d[\mathbf{X}(\zeta)]\exp\Big(\frac{i}{2}\sum_{p=1}^{n}\xi_p k_p\int_0^z d\zeta\Big[\dot{\mathbf{x}}_p^2(\zeta) + \epsilon_2[\mathbf{x}_p(\zeta),\zeta,\omega_p] + \epsilon_1[\mathbf{x}_p(\zeta),\zeta,\omega_p;\alpha]\Big]\Big), \quad (3.2)$$

where $d[\mathbf{X}(\zeta)] = d[\mathbf{x}_1(\zeta)]d[\mathbf{x}_2(\zeta)]\cdots d[\mathbf{x}_n(\zeta)]$; $\xi_p = 1$, p odd, $\xi_p = -1$, p even; $k_p = k(\omega_p)$; and the integration is over "all" paths $\mathbf{x}_p(\zeta)$, $p=1,2,\ldots,n$, subject to the boundary conditions $\mathbf{x}_p(0)=\mathbf{x}_p',\mathbf{x}_p(z)=\mathbf{x}_p$.

B. Statistical analysis

Ensemble averaging (3.2) over the statistical realizations α results in the expression

$$E\{G^{(n)}(\mathbf{X},\mathbf{X}',z,\omega;\alpha)\} = \int d[\mathbf{X}(\zeta)]\exp\Big(\frac{i}{2}\sum_{p=1}^{n}\xi_p k_p\int_0^z d\zeta\Big[\dot{x}_p^2(\zeta) + \epsilon_2[\mathbf{x}_p(\zeta),\zeta,\omega_p]\Big]\Big)E\Big\{\exp\Big(\frac{i}{2}\sum_{p=1}^{n}\xi_p k_p\int_0^z d\zeta \times\epsilon_1[\mathbf{x}_p(\zeta),\zeta,\omega_p;\alpha]\Big)\Big\}. \quad (3.3)$$

To proceed further, we need to specify the structure of $\epsilon_1[\mathbf{x}_p(\zeta),\zeta,\omega_p;\alpha]$. We assume, first, that the dependence of the function ϵ_1 on ω_p (arising from the dispersive properties of the medium) enters multiplicatively,[26] viz.,

$$\epsilon_1[\mathbf{x}_p(\zeta),\zeta,\omega_p;\alpha] = \nu(\omega_p)\mu[\mathbf{x}_p(\zeta),\zeta;\alpha]. \quad (3.4)$$

All the information about the random fluctuations in the medium is now contained in the quantity $\mu[\mathbf{x}_p(\zeta),\zeta;\alpha]$. If the latter is assumed to be a Gaussian random process, the statistical averaging appearing in (3.3) can be carried out explicitly, with the result

$$I_1 \equiv E\Big\{\exp\Big(\frac{i}{2}\sum_{p=1}^{n}\xi_p k_p\int_0^z d\zeta\,\epsilon_1[\mathbf{x}_p(\zeta),\zeta,\omega_p;\alpha]\Big)\Big\} = \exp\Big(-\frac{1}{8}\sum_{p=1}^{n}\sum_{q=1}^{n}\xi_p\xi_q k_p k_q\nu_p\nu_q\int_0^z d\zeta \times\int_0^z d\zeta'\gamma[\mathbf{x}_p(\zeta),\mathbf{x}_q(\zeta'),\zeta,\zeta']\Big), \quad (3.5)$$

where $\nu_p=\nu(\omega_p)$ and γ is the correlation function of the random process μ, viz.,

$$\gamma[\mathbf{x}_p(\zeta),\mathbf{x}_q(\zeta'),\zeta,\zeta'] = E\{\mu[\mathbf{x}_p(\zeta),\zeta;\alpha]\mu[\mathbf{x}_q(\zeta'),\zeta';\alpha]\}. \quad (3.6)$$

It should be noted that the Gaussian assumption invoked in deriving (3.5) can be relaxed somewhat by using the theory of cumulants (cf. Ref. 27; see, also, Ref. 20).

We resort, next, to the usual Markovian approximation, i.e., we assume that the process μ is δ correlated along the longitudinal direction of propagation. We have, then, in the place of (3.6)

$$\gamma[\mathbf{x}_p(\zeta),\mathbf{x}_q(\zeta'),\zeta,\zeta'] = A[\mathbf{x}_p(\zeta),\mathbf{x}_q(\zeta')]\delta(\zeta-\zeta'). \quad (3.7)$$

With this simplification, the integration over ζ' in (3.5) can be performed trivially. The resulting expression for I_1 is given as follows:

$$I_1 = \exp\Big(-\frac{1}{8}\sum_{p=1}^{n}\sum_{q=1}^{n}\xi_p\xi_q k_p k_q\nu_p\nu_q \times\int_0^z d\zeta A[\mathbf{x}_p(\zeta),\mathbf{x}_p(\zeta)]\Big). \quad (3.8)$$

In many cases of physical interest, the "transverse" correlation $A[\mathbf{x}_p(\zeta),\mathbf{x}_q(\zeta)]$ is homogeneous and isotropic,[28] viz., $A[\mathbf{x}_p(\zeta),\mathbf{x}_q(\zeta)] = A[|\mathbf{x}_p(\zeta)-\mathbf{x}_q(\zeta)|]$, and of a power-law type, viz.,

$$A[|\mathbf{x}_p(\zeta)-\mathbf{x}_q(\zeta)|] = A(0)\Big\{1-\frac{1}{2}\Big[\frac{1}{L_0}|\mathbf{x}_p(\zeta)-\mathbf{x}_q(\zeta)|\Big]^\beta\Big\}. \quad (3.9)$$

Here, L_0 is a characteristic length, and the parameter β is usually within the range $1<\beta<4$ (cf. Refs. 9 and 20). For optical propagation through a turbulent medium such as the atmosphere, one has $\beta \leqslant 2$. This range includes the Gaussian spectrum and the Kolmogorov spectrum.

Introducing (3.9) into (3.8) and, in turn, the resulting expression for I_1 into (3.3), we obtain

$$E\{G^{(n)}(\mathbf{X},\mathbf{X}',z,\omega;\alpha)\} = \exp\Big[-\tfrac{1}{8}A(0)\Big(\sum_{p=1}^{n}\sum_{q=1}^{n}\xi_p\xi_q k_p k_q\nu_p\nu_q\Big)z\Big] \times\int d[\mathbf{X}(\zeta)]\exp\Big\{\frac{i}{2}\sum_{p=1}^{n}\xi_p k_p\int_0^z d\zeta\Big(\dot{\mathbf{x}}_p^2(\zeta) + \epsilon_2[\mathbf{x}_p(\zeta),\zeta,\omega_p] - \frac{i}{16}A(0)\sum_{q=1}^{n}\xi_q k_q\nu_p\nu_q \times\Big[\frac{1}{L_0}|\mathbf{x}_p(\zeta)-\mathbf{x}_q(\zeta)|\Big]^\beta\Big)\Big\}. \quad (3.10)$$

This expression is the starting point for all asymptotic evaluations of nth-order multifrequency coherence functions in the presence of dispersion, a deterministic inhomogeneous profile, and for a wide class of fluctuation spectra (cf. Ref. 20).

The versatility of the functional path integration technique will be illustrated below for a simple (but physically nontrivial) setting: a nondispersive, deterministically flat medium, characterized by a simplified (quadratic) Kolmogorov spectrum. In this case, the path integral in (3.10) can be carried out explicitly, yielding a series of presently unavailable results.

C. Specialization to a nondispersive, deterministically homogeneous medium, with a simplified (quadratic) Kolmogorov spectrum

In (3.10), let $\epsilon_2 = 0$, $\nu_p = 1, p = 1,2,...,n$, and $\beta = 2$. The corresponding path integral representation assumes the simpler form

$$E\{G^{(n)}(\mathbf{X},\mathbf{X}',z,\omega;\alpha)\} = \exp\left[-\frac{1}{8}A(0)\left(\sum_{p=1}^{n}\sum_{q=1}^{n}\xi_p\xi_q k_p k_q\right)z\right] \times \int d[\mathbf{X}(\zeta)]\exp(iS), \tag{3.11a}$$

$$S = \int_0^z d\zeta L[\dot{\mathbf{x}}_p(\zeta),\mathbf{x}_p(\zeta)], \tag{3.11b}$$

$$L[\dot{\mathbf{x}}_p(\zeta),x_p(\zeta)] = \frac{1}{2}\sum_{p=1}^{n}\xi_p k_p\left\{\dot{\mathbf{x}}_p^2(\zeta) - \frac{i}{4}D \times \sum_{q=1}^{n}\xi_q k_q[\mathbf{x}_p(\zeta)-\mathbf{x}_q(\zeta)]^2\right\}, \tag{3.11c}$$

$$D = A(0)/2L_0^2. \tag{3.11d}$$

Borrowing terminology from quantum mechanics (in connection with which the functional path integration method was originally developed by Feynman), we shall refer to L and S in (3.11) as the Lagrangian and action, respectively, and to $\mathbf{x}_p$ and $\dot{\mathbf{x}}_p$ as the coordinates and velocities (or momenta with respect to a "mass" normalized to unity), respectively.

For the problem under consideration here, S [cf. Eq. (3.11b)] is a quadratic action functional. It is well known in this case (cf. Ref. 24) that the path integral in (3.11a) becomes

$$I_2 \equiv \int d[\mathbf{X}(\zeta)]\exp(iS) = N(z)\exp(S_c). \tag{3.12}$$

Here, S_c is the "classical" action, i.e., the action S evaluated along the "classical" paths $\mathbf{x}_{cp}(\zeta)$ satisfying the Euler–Lagrange equations

$$\frac{d}{d\zeta}[\partial L/\partial\dot{\mathbf{x}}_{cp}(\zeta)] - \partial L/\partial\mathbf{x}_{cp}(\zeta) = 0, \tag{3.13a}$$

with the boundary conditions

$$\mathbf{x}_{cp}(0) = \mathbf{x}_p', \mathbf{x}_{cp}(z) = \mathbf{x}_p, \quad p = 1,2,...,n. \tag{3.13b}$$

In general, S_c is a function of $\mathbf{x}_p'$, $\mathbf{x}_p$, and z. The normalization quantity N in (3.12), however, depends only on z, and is related to the classical action as follows:

$$N(z) = \left(\det(i/2\pi)\left[\frac{\partial^2 S_c}{\partial\mathbf{x}_p\partial\mathbf{x}_q'}\right]\right)^{1/2}, \quad p,q,1,2,...,n. \tag{3.14}$$

The $2n\times 2n$ matrtix within the square brackets is referred to in the literature as the Van Vleck–Morette matrix (or "Hessian of the action").

Our attention will be directed next to the evaluation of the classical action S_c and the normalization factor $N(z)$ required in (3.12).

D. Evaluation of S_c

The Euler–Lagrange equations (3.13) corresponding to the Lagrangian $L[\dot{\mathbf{x}}_{cp}(\zeta),\mathbf{x}_{cp}(\zeta)]$ in (3.11c) yield the following equations for the classical paths $x_{cp}(\zeta)$:

$$\ddot{\mathbf{x}}_{cp}(\zeta) + g^2\sum_{q=1}^{n}\xi_q k_q[\mathbf{x}_{cp}(\zeta)-\mathbf{x}_{cq}(\zeta)] = 0, \tag{3.15a}$$

$$\mathbf{x}_{cp}(0) = \mathbf{x}_p', \mathbf{x}_{cp}(z) = \mathbf{x}_p, \tag{3.15b}$$

$$g^2 = \frac{i}{2}D\sum_{p=1}^{n}\xi_p k_p. \tag{3.15c}$$

We consider next the quadratic action functional S [cf. Eq. (3.11b)]. If the momentum-dependent term is integrated by parts, and the resulting expression for S is evaluated along the classical paths (3.15), we obtain

$$S_c = \frac{1}{2}\sum_{p=1}^{n}\xi_p k_p\mathbf{x}_{cp}(\zeta)\cdot[\dot{\mathbf{x}}_{cp}(\zeta)]_0^z. \tag{3.16}$$

The classical action given in the last equation can be manipulated into a form which is more suitable for further analysis. Specifically,

$$S_c = \tfrac{1}{2}\left(\sum_{p=1}^{n}\xi_p k_p\right)^{-1}\left[\left(\mathbf{v}(\zeta)\cdot\dot{\mathbf{v}}(\zeta) + \frac{1}{2}\sum_{p=1}^{n} \times \sum_{q=1}^{n}\xi_p\xi_q k_p k_q\mathbf{u}_{pq}(\zeta)\cdot\dot{\mathbf{u}}_{pq}(\zeta)\right)\right]_0^z, \tag{3.17}$$

where

$$\mathbf{v}(\zeta) = \sum_{p=1}^{n}\xi_p k_p\mathbf{x}_{cp}(\zeta), \tag{3.18}$$

and

$$\mathbf{u}_{pq}(\zeta) = \mathbf{x}_{cp}(\zeta) - \mathbf{x}_{cq}(\zeta). \tag{3.19}$$

We shall determine next all the quantities required in (3.17); that is, $\mathbf{v}(\zeta)$, $\mathbf{u}_{pq}(\zeta)$, and their derivatives. Toward this goal, we note that a relatively simple manipulation of Eqs. (3.15) for the classical paths yields[29]

$$\ddot{\mathbf{u}}_{pq}(\zeta) + g^2\mathbf{u}_{pq}(\zeta) = 0, \quad p\neq q, \tag{3.20a}$$

$$\mathbf{u}_{pq}(0) = \mathbf{u}_{pq}', \quad \mathbf{u}_{pq}(z) = \mathbf{u}_{pq}, \tag{3.20b}$$

and

$$\ddot{\mathbf{v}}(\zeta) = 0, \tag{3.21a}$$

$$\mathbf{v}(0) = \mathbf{v}', \mathbf{v}(z) = \mathbf{v}. \tag{3.21b}$$

The solutions to the two-point boundary value problems

(3.20) and (3.21) can be found in a straightforward manner—say, by a Laplace transformation. They are given as follows:

$$\mathbf{u}_{pq}(\zeta) = [\mathbf{u}_{pq}\operatorname{sing}\zeta - \mathbf{u}'_{pq}\operatorname{sing}(\zeta - z)]/\operatorname{sing}z, \qquad (3.22)$$

$$\mathbf{v}(\zeta) = (\mathbf{v} - \mathbf{v}')(\zeta/z) + \mathbf{v}'. \qquad (3.23)$$

The desired expression for the classical action can be found by using the solutions (3.22) and (3.23) in conjunction with (3.17). One has, finally,

$$S_c = \frac{1}{2}\Big(\sum_{p=1}^{n}\xi_p k_p\Big)^{-1}\Big[\frac{1}{z}\Big(\sum_{p=1}^{n}\xi_p k_p(\mathbf{x}_p - \mathbf{x}'_p)\Big)^2 - \Big(\frac{g}{\operatorname{sing}z}\Big)\sum_{p=1}^{n}\sum_{q=1}^{n}\xi_p\xi_q k_p k_q(\mathbf{x}_p - \mathbf{x}_q)\cdot(\mathbf{x}'_p - \mathbf{x}'_q) + \tfrac{1}{2}g\cot gz\sum_{p=1}^{n}\sum_{q=1}^{n}\xi_p\xi_q k_p k_q[(\mathbf{x}_p - \mathbf{x}_q)^2 + (\mathbf{x}'_p - \mathbf{x}'_q)^2]\Big\}. \qquad (3.24)$$

E. Evaluation of *N(z)*

The normalization factor $N(z)$ defined in (3.14) is rewritten as

$$N(z) = (i/2\pi)^n\Big(\det\Big[\frac{\partial^2 S_c}{\partial\mathbf{X}\partial\mathbf{X}'}\Big]\Big)^{1/2}, \qquad (3.25)$$

where $\mathbf{X}$ and $\mathbf{X}'$ are the $2n$ vectors given in (2.12b) and (2.12c), respectively. The $2n\times 2n$ Van Vleck–Morette matrix appearing within the square brackets in (3.25) can be determined easily using the expression for the classical action which was derived in the previous subsection [cf. Eq. (3.24)]. Unfortunately, a direct evaluation of the determinant of this matrix is rather difficult for large n. In the following, we shall pursue an alternative procedure.

Consider the linear transformations,

$$T\mathbf{X} = \mathrm{R},\quad T\mathbf{X}' = \mathbf{R}', \qquad (3.26)$$

where T is a $2n\times 2n$ matrix and $\mathbf{R}$, $\mathbf{R}'$ are $2n$-vectors. Since the Van Vleck–Morette matrix is of the Hessian form, we have[30]

$$\Big[\frac{\partial^2 S_c}{\partial\mathbf{X}\partial\mathbf{X}'}\Big] = \overline{T}\Big[\frac{\partial^2 S_c}{\partial\mathbf{R}\partial\mathbf{R}'}\Big]T, \qquad (3.27)$$

where the overbar denotes the transpose of the matrix T. It follows, then, from (3.27) that

$$\det\Big[\frac{\partial^2 S_c}{\partial\mathbf{X}\partial\mathbf{X}'}\Big] = (\det T)^2\det\Big[\frac{\partial^2 S_c}{\partial\mathbf{R}\partial\mathbf{R}'}\Big]. \qquad (3.28)$$

A convenient change of variables is the following:

$$\mathbf{x}_1 - \mathbf{x}_r = \mathbf{u}_{1r},\quad r = 2,3,\ldots,n, \qquad (3.29a)$$

$$\sum_{p=1}^{n}\xi_p k_p\mathbf{x}_p = \mathbf{v}, \qquad (3.29b)$$

and similar relations for the primed coordinates. With this change of variables, the linear transformation $T\mathbf{X} = \mathbf{R}$ [cf. Eq. (3.26)] has the realization

$$\begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0\\ 1 & 0 & -1 & 0 & \cdots & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 1 & 0 & 0 & 0 & \cdots & 0 & -1\\ k_1 & -k_2 & k_3 & -k_4 & \cdots & k_{n-1} & -k_n \end{bmatrix} \times \begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\\ \vdots\\ \mathbf{x}_{n-1}\\ \mathbf{x}_n\end{bmatrix} = \begin{bmatrix}\mathbf{u}_{12}\\ \mathbf{u}_{13}\\ \vdots\\ \mathbf{u}_{1n}\\ \mathbf{v}\end{bmatrix}. \qquad (3.30)$$

A similar realization holds for the linear transformation $T\mathbf{X}' = \mathbf{R}'$. Because of the partitioning in the column matrices for $\mathbf{X}$ and $\mathbf{R}$, each entry of the $2n\times 2n$ matrix T in (3.30) must be understood as a 2×2 diagonal matrix.

The determinant of the matrix T can be easily evaluated from the realization of T shown in (3.30). Specifically,

$$\det T = \Big(\sum_{p=1}^{n}\xi_p k_p\Big)^2. \qquad (3.31)$$

Substituting this result into (3.28), we obtain

$$\det\Big[\frac{\partial^2 S_c}{\partial\mathbf{X}\partial\mathbf{X}'}\Big] = \Big(\sum_{p=1}^{n}\xi_p k_p\Big)^4\det\Big[\frac{\partial^2 S_c}{\partial\mathbf{U}\partial\mathbf{U}'}\Big]\det\Big[\frac{\partial^2 S_c}{\partial\mathbf{v}\partial\mathbf{v}'}\Big], \qquad (3.32)$$

where we have resorted to the obvious partitioning

$$\Big[\frac{\partial^2 S_c}{\partial\mathbf{R}\partial\mathbf{R}'}\Big] = \left[\begin{array}{c|c}\Big[\frac{\partial^2 S_c}{\partial\mathbf{U}\partial\mathbf{U}'}\Big] & 0\\ \hline 0 & \Big[\frac{\partial^2 S_c}{\partial\mathbf{v}\partial\mathbf{v}'}\Big]\end{array}\right], \qquad (3.33)$$

with $\mathbf{U} = (\mathbf{u}_{12},\mathbf{u}_{13},\ldots,\mathbf{u}_{1n})$ and $\mathbf{U}' = (\mathbf{u}'_{12},\mathbf{u}'_{13},\ldots,\mathbf{u}'_{1n})$.

To proceed further, we shall have to express the classical action given in (3.24) in terms of the new coordinates $\mathbf{U}$, $\mathbf{U}'$, $\mathbf{v}$, and $\mathbf{v}'$. Omitting intermediate steps, we present below the final result:

$$S_c = \frac{1}{2}\Big(\sum_{p=1}^{n}\xi_p k_p\Big)^{-1}\Big[\frac{1}{z}(\mathbf{v} - \mathbf{v}')^2 - \Big(\frac{g}{\operatorname{sing}z}\Big)\times\Big(2\xi_1 k_1\sum_{r=2}^{n}\xi_r k_r\mathbf{u}_{1r}\cdot\mathbf{u}'_{1r} + \sum_{r=2}^{n}\sum_{s=2}^{n}\times\xi_r\xi_s k_r\ k_s(\mathbf{u}_{1r} - \mathbf{u}_{1s})\cdot(\mathbf{u}'_{1r} - \mathbf{u}'_{1s})\Big) + \tfrac{1}{2}g\cot gz\times\Big(2\xi_1 k_1\sum_{r=2}^{n}\xi_r k_r(\mathbf{u}_{1r}^2 + \mathbf{u}_{1r}'^2) + \sum_{r=2}^{n}\sum_{s=2}^{n}\times\xi_r\xi_s\ k_r k_s[(\mathbf{u}_{1r} - \mathbf{u}_{1s})^2 + (\mathbf{u}'_{1r} - \mathbf{u}'_{1s})^2]\Big)\Big]. \qquad (3.34)$$

The individual matrices needed in (3.33) can now be determined by straightforward differentiation:

$$\left[\frac{\partial^2 S_c}{\partial \mathbf{U}\partial \mathbf{U}'}\right] = \left[\frac{\partial^2 S_c}{\partial u^i_{1r}\,\partial u^{i'}_{1r'}}\right]$$

$$= -\left(\frac{g}{\operatorname{sing}z}\right)\left(\sum_{p=1}^{n}\xi_p k_p\right)^{-1}\Big[\xi_1 k_1 \xi_r k_r \delta_{rr'} + \xi_r k_r \times\left(\sum_{s=2}^{n}\xi_s k_s\right)\delta_{rr'} - \xi_r k_r \xi_{r'} k_{r'}\Big]\delta_{ii'}, \qquad (3.35)$$

for all $r, r' = 2,3,\ldots,n$; $i,i' = 1,2$, and

$$\left[\frac{\partial^2 S_c}{\partial \mathbf{v}\partial \mathbf{v}'}\right] = \left[\frac{\partial^2 S_c}{\partial v_i \partial v'_{i'}}\right] = -\left(z\sum_{p=1}^{n}\xi_p k_p\right)^{-1}\delta_{ii'}, \qquad (3.36)$$

for $i, i' = 1,2$.

The determinants of the matrices given in (3.35) and (3.36) can be computed without difficulty. Omitting again intermediate steps, we present below the final results:

$$\det\left[\frac{\partial^2 S_c}{\partial \mathbf{U}\partial \mathbf{U}'}\right] = \left(\frac{g}{\operatorname{sing}z}\right)^{2(n-1)}\left(\prod_{p=1}^{n}\xi_p k_p \Big/ \sum_{p=1}^{n}\xi_p k_p\right)^2, \qquad (3.37)$$

$$\det\left[\frac{\partial^2 S_c}{\partial \mathbf{v}\partial \mathbf{v}'}\right] = \left(z\sum_{p=1}^{n}\xi_p k_p\right)^{-2}. \qquad (3.38)$$

Introducing (3.37) and (3.38) into (3.28), and the resulting expression into (3.25), we find that the desired normalization factor $N(z)$ is given by

$$N(z) = (i/2)^n z^{-1}(g/\operatorname{sing}z)^{n-1}\left(\sum_{p=1}^{n}\xi_p k_p\right). \qquad (3.39)$$

The solution to our problem is now complete. Using (3.39), (3.24), and (3.12), together with (3.11), the final expression for the nth-order multifrequency coherence function is given as follows:

$$E\{G^{(n)}(\mathbf{X},\mathbf{X}',z,\boldsymbol{\omega};\alpha)\}$$

$$= (i/2\pi)^n z^{-1}\left(\sum_{p=1}^{n}\xi_p k_p\right)(g/\operatorname{sing}z)^{n-1}$$

$$\times\exp\left[-\tfrac{1}{8}A(0)\left(\sum_{p=1}^{n}\sum_{q=1}^{n}\xi_p\xi_q k_p k_q\right)z\right]$$

$$\times\exp\Bigg\{\frac{i}{2}\left(\sum_{p=1}^{n}\xi_p k_p\right)^{-1}\Bigg[\frac{1}{z}\left(\sum_{p=1}^{n}\xi_p k_p(\mathbf{x}_p - \mathbf{x}'_p)\right)^2$$

$$-\left(\frac{g}{\operatorname{sing}z}\right)\sum_{p=1}^{n}\sum_{q=1}^{n}\xi_p\xi_q k_p k_q(\mathbf{x}_p - \mathbf{x}'_q)\cdot(\mathbf{x}'_p - \mathbf{x}'_q)$$

$$+\tfrac{1}{2}g\cot gz\sum_{p=1}^{n}\sum_{q=1}^{n}\xi_p\xi_q k_p k_q[(\mathbf{x}_p - \mathbf{x}_q)^2$$

$$+(\mathbf{x}'_p - \mathbf{x}'_q)^2]\Bigg]\Bigg\}. \qquad (3.40)$$

It should be noted that this result is exact under the restrictions specified earlier in this section. When used in conjunction with (2.10), nth order pulse statistics can be studied. A special case ($n = 2$) of (3.40) will be examined in Sec. 4, and comparisons will be made with previously reported results.

4. SPECIAL CASES

We consider (3.40) in the special case where $n = 2$. We resort, also, to the following center of mass and difference variables: $\mathbf{x} = (\mathbf{x}_1 + \mathbf{x}_2)/2, \mathbf{y} = \mathbf{x}_2 - \mathbf{x}_1$; $\mathbf{x}' = (\mathbf{x}'_1 + \mathbf{x}'_2)/2$, $\mathbf{y}' = \mathbf{x}'_2 - \mathbf{x}'_1$; $k_s = (k_1 + k_2)/2, k_d = k_2 - k_1$; $\omega_s = (\omega_1 + \omega_2)/2, \omega_d = \omega_2 - \omega_1$. (For a nondispersive medium, we have, in general, the relationship $k = \omega/v$, where v is a characteristic reference velocity; hence $k_s = \omega_s/v$ and $k_d = \omega_d/v$ for the sum and difference quantities.) With these specifications, (3.40) simplifies to

$$E\{G^{(2)}(\mathbf{x},\mathbf{y},\mathbf{x}',\mathbf{y}',z,\omega_s,\omega_d;\alpha)\}$$

$$= (2\pi)^{-2}\lambda z^{-1}(g/\operatorname{sing}z)\exp[-\tfrac{1}{8}A(0)k_d^2 z]$$

$$\times\exp\Bigg\{-\frac{i}{2}\frac{1}{k_d}\Bigg[\frac{1}{z}[k_d(\mathbf{x}'-\mathbf{x}) + k_s(\mathbf{y}'-\mathbf{y})]^2$$

$$+2\lambda\left(\frac{g}{\operatorname{sing}z}\right)\mathbf{y}\cdot\mathbf{y}' - \lambda g\cot gz(\mathbf{y}^2 + \mathbf{y}'^2)\Bigg]\Bigg\}, \qquad (4.1)$$

where $\lambda = k_1 k_2 = k_s^2 - (k_d^2/4)$ and $g^2 = -(iDk_d)/2$ [cf. Eq. (3.15c)].

For a planar source distribution, the boundary condition $E_0(\mathbf{x},\omega;\alpha)$ [cf. Eq. (2.7d)] has the form

$$E_0(\mathbf{x},\omega;\alpha) = F_s(\omega), \qquad (4.2)$$

if we ignore source incoherencies. This initial distribution is introduced next into (2.11)—the latter must be specialized to $n = 2$ and (4.1) must be taken into consideration. In this case, the spatial integrations in (2.11) can be carried out explicitly, with the result

$$E\{e_r(\mathbf{x}+\tfrac{1}{2}\mathbf{y},t+\tfrac{1}{2}\tau;\alpha)e_r(\mathbf{x}-\tfrac{1}{2}\mathbf{y},t-\tfrac{1}{2}\tau;\alpha)\}$$

$$= (2\pi)^{-2}\int_{-\infty}^{\infty}d\omega_s\int_{-\infty}^{\infty}d\omega_d F_r^{(2)}(\omega_s,\omega_d)F_s^{(2)}(\omega_s,\omega_d)$$

$$\times\sec gz\,\exp\left(-\frac{1}{8v^2}A(0)\omega_d^2 z\right)\exp\left(-i\frac{\lambda}{2\omega_d}(g\tan gz)\mathbf{y}^2\right)$$

$$\times\exp[i\omega_s\tau + i\omega_d(t - z/v)], \qquad (4.3)$$

where $t = (t_1 + t_2)/2$, $\tau = t_2 - t_1$; $F^{(2)}_{r,s}(\omega_s,\omega_d) = F^*_{r,s}[\omega_s + (\omega_d/2)]F_{r,s}[\omega_s - (\omega_d/2)]$; $\lambda = [\omega_s^2 - (\omega_d^2/4)]/v^2$; and $g^2 = -(iD\omega_d)/2v$. As expected from physical considerations, the second-order moment in (4.3) is independent of the center of mass coordinate $\mathbf{x}$.

In the special case where $\mathbf{x}_2 = \mathbf{x}_1$ ($\mathbf{y} = 0$) and $t_2 = t_1$ ($\tau = 0$), Eq. (4.3) simplifies even further:

$$E\{e_r^2(z,t;\alpha)\}$$

$$= (2\pi)^{-2}\int_{-\infty}^{\infty}d\omega_s\int_{-\infty}^{\infty}d\omega_d F_r^{(2)}(\omega_s,\omega_d)F_s^{(2)}(\omega_s,\omega_d)$$

$$\times \sec gz \exp\left(-\frac{1}{8v^2}A(0)\omega_d^2 z\right)\exp\left[i\omega_d\left(t-\frac{z}{v}\right)\right]. \tag{4.4}$$

This is essentially the expression for the average pulse intensity reported by Sreenivasiah *et al.* (cf. Ref. 12). For a broadband receiver, i.e., $F_r^{(2)}(\omega)\simeq 1$, and an impulsive source intensity, the integrations over the sum and difference frequencies in (4.4) can be carried out. The resulting expression for the mean pulse intensity [cf., e.g., Eqs. (19)–(22) of Ref. 12] exhibits a smearing effect (broadening) caused by the random inhomogeneities in the medium.

Consider next a boundary condition $E_0(\mathbf{x},\omega;\alpha)$ of the form

$$E_0(\mathbf{x},\omega;\alpha) = F_s(\omega)\exp(-\mathbf{x}^2/2\sigma_0^2). \tag{4.5}$$

This initial distribution may represent, for example, the field of a coherent, pulsed, collimated, Gaussian laser beam having an aperture radius equal to σ_0. It turns out in this case that the spatial integrations in (2.11) can be performed exactly. As a specific illustration of such an operation, we present below the ensemble averaged pulsed intensity evaluated on the beam axis:

$$E\{e_r^2(z,t;\alpha)\} = (2\pi)^{-2}\int_{-\infty}^{\infty} d\omega_s \int_{-\infty}^{\infty} d\omega_d F_r^{(2)}(\omega_s,\omega_d)F_s^{(2)}(\omega_s,\omega_d)$$

$$\times H(\omega_s,\omega_d)\exp\left[-\frac{1}{8v^2}A(0)\omega_d^2 z\right]\exp\left[i\omega_d\left(t-\frac{z}{v}\right)\right]; \tag{4.6a}$$

$$H(\omega_s,\omega_d) = (\lambda/4z)(g/\sin gz)[\sigma_0^{-2} + i(\omega_d/2vz)]^{-1}[(2\sigma_0)^{-2} + (\omega_s/2vz)^2][\sigma_0^{-2}+i(\omega_d/2vz)]^{-1} + i(\omega_s^2/2vz\omega_d) - i(\lambda v g \cot gz/\omega_d)^{-1}. \tag{4.6b}$$

In contrast to the plane wave case [cf. Eq. (4.4)], the integrations over the sum and difference frequencies in (4.6) must be evaluated asymptotically (e.g., by the method of steepest descent), even for a broadband receiver and an impulsive source intensity.

5. CONCLUDING REMARKS

The two-frequency mutual coherence function [cf. Eq. (3.40) in the special case $n = 2$, or Eq. (4.1)] has also been derived by the authors independently by solving the transport equation for $E\{G^{(2)}\}$. The effort involved is considerable, and is expected to be even more substantial if nth-order multifrequency coherence functions are to be obtained by solving directly the associated local transport equations. On the other hand, it has been demonstrated in this paper that the path integration technique yields these results in a simple and straightforward manner.

The simplified (quadratic) Kolmogorov spectrum implicit in the derivation of (3.40) is valid in many physical situations (e.g., optical propagation through a turbulent atmosphere) provided propagation distances are large.[31] Under this assumption, the main results in this paper incorporated in (3.40) are exact, that is, *all* path contributions have been accounted for. However, one can proceed along the lines suggested by Dashen (cf. Ref. 20) in order to compute $E\{G^{(n)}(\mathbf{X},\mathbf{X}',z,\omega;\alpha)\}$ asymptotically with respect to a small parameter "a" whose order of magnitude is roughly $a\sim[6(L_0/z)^{3/2}/E\{\mu^2\}]^{1/2}$ in terms of the scale size, the range, and the rms fluctuations. Two regions are delineated: the fully and partially saturated regimes. To $O(a)$, the statistics of $G(\mathbf{x},\mathbf{x}',z,\omega;\alpha)$ in the fully saturated region is Gaussian. The situation is a little more complicated in the partially saturated regime. In both cases, however, the ensuing expressions for $E\{G^{(n)}(\mathbf{X},\mathbf{X}',z,\omega;\alpha)\}$ are expressible in terms of two-frequency mutual coherence functions. This simplification would facilitate considerably the computation of nth-order pulse statistics [cf. Eq. (2.11)] in physical situations where terms of $O(a)$ are negligible.

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