BUCKLING OF AN EQUATORIAL SEGMENT OF A SPHERICAL SHELL LOADED BY ITS OWN WEIGHT

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Thesis submitted to the Graduate Faculty of the

Virginia Polytechnic Institute

in candidacy for the degree of

MASTER OF SCIENCE

in

Engineering Mechanics

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March 1966

Blacksburg, Virginia

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IV. INTRODUCTION

The lunar orbit and landing approach simulator now being built at the Langley Research Center incorporates a large spherical shell which is to be supported at either the top or bottom or both. This shell is loaded by its own weight (fig. 1). To properly design this shell, it is necessary to consider its stability. The solution to the buckling problem of a narrow spherical shell segment centered at the equator and loaded by its own weight is thus of value and will be considered here. The weight of the shell is taken to be supported by line loads at the upper and lower edges.

An elementary approach to the problem is to assume that the spherical segment will buckle if the maximum compressive stress, as predicted by linear membrane theory, is greater than the critical compressive stress for a complete sphere loaded by uniform external pressure. The purpose of this paper is to obtain a more exact solution to the problem of the buckling of the spherical segment.

In the present paper a set of general nonlinear shell equations and linear buckling equations are derived for shells having the shape of a narrow segment of a toroidal shell centered at the equator. The derivation of the general equations is based on a method presented in reference 1 where shallow shell equations are derived by considering a flat plate with an initial deformation. This method is employed in reference 2 to study the buckling of segments of toroidal shells under uniform external pressure. It is noted in reference 2 that the resulting toroidal shell equations reduce to large deflection Donnell equations for the case of a circular cylindrical shell, and to Marquerre large deflection equations for the case of a shallow spherical cap. In the present paper, nonlinear shell and linear buckling equations are derived by considering a cylinder, described by Donnell equations, with an initial radial deformation. As expected, the resulting equations agree with reference 2 for the case of normal pressure loading.

The geometry of the toroidal shell is specialized to that of a spherical segment and the Galerkin method is used to solve the resulting buckling equations for a critical thickness parameter. Plots are presented which compare the present results with the results of the elementary approach stated above.

The author wishes to express his thanks to the National Aeronautics and Space Administration for permitting him to write his thesis at the Langley Research Center. The author also thanks Dr. Robert L. Armstrong, Engineering Mechanics Department, Virginia Polytechnic Institute, for his comments and guidance.

V. SYMBOLS

A_m, A_q	constants (A_1, A_2, etc.)
$\left[a_{qm}\right], \left[b_{qm}\right]$	$(M \times M)$ matrices
$\{A\}, \{B\}$	$(M \times 1)$ column matrices
C	constant of integration
E	Young's modulus
F_x, F_y, F_n	components of edge loading resultants acting in the meridional, circumferential, and normal directions, respectively
I	unity matrix
k	the proportion of the weight of the spherical segment supported at the upper edge
L	length of the toroidal shell
M	number of terms in assumed truncated series solution
M_x, M_y, M_{xy}	meridional circumferential and twisting moment resultants, respectively
m, q	integers
N	integer
N_x, N_y, N_{xy}	meridional, circumferential, and shear stress resultants, respectively
n	number of circumferential waves
p_x, p_y, p_n	components of surface loading acting in the meridional, circumferential, and normal directions, respectively
R	radius of spherical segment
$r(x)$	radius which defines the generator of the toroidal shell
r_x	radius of curvature of the generator of the toroidal shell

r_y	$r(x)$ at $x = 0$
T_x, T_{xy}	bending and twisting moments resultants acting at the edges of the toroidal shell, respectively
t	shell thickness
U, V, W	meridional, circumferential, and normal displacements of the toroidal shell, respectively
u, v, w	axial, circumferential, and radial displacements of the cylindrical shell, respectively
x, y	axial and circumferential coordinates, respectively
α	$\rho L/E$
β	R/L
∇^2, ∇^4	Laplacian and bi-Laplacian operators, respectively
δ	variational operator
$\epsilon_x, \epsilon_y, \epsilon_{xy}$	meridional, circumferential, and shear strains, respectively
λ	$t^2/\left[12(1 - \nu^2)R^2\right]$
λ^*	critical value of λ from elementary approach
ν	Poisson's ratio
Π_1, Π_2, Π_3	strain energy of shell, and work done by body forces and externally applied forces, respectively
ρ	specific weight of shell wall
ϕ	Airy stress function
$\chi_x, \chi_y, \chi_{xy}$	meridional bending, circumferential bending, and twisting strains, respectively
ω_x, ω_y	meridional and circumferential rotations, respectively

Superscripts

o	initial displacements and strains
—	displacements and strains due to loading
^	prebuckling terms
~	perturbation terms

Subscripts

,	denotes partial differentiation
—	denotes terms which drop out when prebuckling rotations are neglected
cr	critical value

VI. NONLINEAR SHELL EQUATIONS

In this section, general nonlinear equations are derived for shells having the shape of a narrow segment of a toroidal shell centered at the equator. The equations are derived by assuming the shell to be a cylinder, described by nonlinear Donnell equations, with an initial radial deformation, $\overset{o}{w}(x)$, but having no initial stresses. In this manner, nonlinear strain displacement equations are derived starting from nonlinear Donnell cylinder strains. Equilibrium equations and boundary conditions are then derived by utilizing the principle of virtual work.

Geometric Considerations

The equation for the generator of a segment of a toroidal shell centered at the equator is (see fig. 2):

$$r(x) = r_y - r_x \left\{ 1 - \left[1 - \left(\frac{x}{r_x} \right)^2 \right]^{1/2} \right\}; \; -\frac{L}{2} \leq x \leq \frac{L}{2} \qquad (1)$$

That is, the generator is a part of a circle with radius $|r_x|$. For $r_x > 0$, equation (1) defines a shell of positive Gaussian curvature and for $r_x < 0$ a shell of negative Gaussian curvature. Expanding the right side of equation (1) in powers of x/r_x yields

$$r(x) = r_y - r_x \left[\frac{1}{2}\left(\frac{x}{r_x}\right)^2 + \frac{1}{8}\left(\frac{x}{r_x}\right)^4 + \ldots \right]; \; -\frac{L}{2} \leq x \leq \frac{L}{2} \qquad (2)$$

For the shells considered in this development, the generator is shallow, that is, $\left(\frac{x}{2r_x}\right)^2 \ll 1$. Therefore, an approximate equation for the generator of the shell is

$$r(x) = r_y - \frac{x^2}{2r_x}; \; -\frac{L}{2} \leq x \leq \frac{L}{2} \qquad (3)$$

Assuming the toroidal shell to be a cylinder with an initial radial deformation, it follows from equation (3) that the radius of the undeformed cylinder should be taken as r_y, and the initial deformation should be given by

$$\overset{\circ}{w}(x) = -\frac{x^2}{2r_x} \tag{4}$$

where $\overset{\circ}{w}(x)$ is taken positive outwards (see fig. 3).

Strain-Displacement Equations

Donnell nonlinear strain-displacement equations for a cylindrical shell are (see, for example, ref. 3):

$$\epsilon_x = u_{,x} + \frac{1}{2}(w_{,x})^2 \tag{5a}$$

$$\epsilon_y = v_{,y} + \frac{w}{r_y} + \frac{1}{2}(w_{,y})^2 \tag{5b}$$

$$\epsilon_{xy} = \frac{1}{2}(v_{,x} + u_{,y} + w_{,x}w_{,y}) \tag{5c}$$

$$\chi_x = -w_{,xx} \tag{5d}$$

$$\chi_y = -w_{,yy} \tag{5e}$$

$$\chi_{xy} = -w_{,xy} \tag{5f}$$

where the positive sense of u, v, and w are shown in figure 3.

Let the displacements and strains be composed of a part due to the initial deformation, $\overset{\circ}{w}(x)$, and an additional part due to loading. The strains due to the initial deformation are:

$$\overset{o}{\epsilon}_x = \frac{1}{2}(\overset{o}{w}_{,x})^2 \tag{6a}$$

$$\overset{o}{\epsilon}_y = \frac{\overset{o}{w}}{r_y} \tag{6b}$$

$$\overset{o}{\chi}_x = -\overset{o}{w}_{,xx} \tag{6c}$$

$$\overset{o}{\epsilon}_{xy} = \overset{o}{\chi}_y = \overset{o}{\chi}_{xy} = 0 \tag{6d,6e,6f}$$

The total strains due to the initial deformation and loading are:

$$\bar{\epsilon}_x + \overset{o}{\epsilon}_x = \bar{u}_{,x} + \frac{1}{2}\left[(\bar{w} + \overset{o}{w})_{,x}\right]^2 \tag{7a}$$

$$\bar{\epsilon}_y + \overset{o}{\epsilon}_y = \bar{v}_{,y} + \frac{\bar{w} + \overset{o}{w}}{r_y} + \frac{1}{2}(\bar{w}_{,y})^2 \tag{7b}$$

$$\bar{\epsilon}_{xy} + \overset{o}{\epsilon}_{xy} = \frac{1}{2}\left[\bar{v}_{,x} + \bar{u}_{,y} + (\bar{w} + \overset{o}{w})_{,x}\bar{w}_{,y}\right] \tag{7c}$$

$$\bar{\chi}_x + \overset{o}{\chi}_x = -(\bar{w} + \overset{o}{w})_{,xx} \tag{7d}$$

$$\bar{\chi}_y + \overset{o}{\chi}_y = -\bar{w}_{,yy} \tag{7e}$$

$$\bar{\chi}_{xy} + \overset{o}{\chi}_{xy} = -\bar{w}_{,xy} \tag{7f}$$

Subtracting the initial strains, equations (6a) - (6f), from the total strains, equations (7a) - (7f), yields the equations for the additional strains due to loading in terms of the initial deformation, $\overset{o}{w}(x)$, and the additional displacements due to loading. With $\overset{o}{w}(x)$ given by equation (4), the strain-displacement equations are:

$$\bar{\epsilon}_x = \bar{u}_{,x} - \frac{x}{r_x}\,\bar{w}_{,x} + \frac{1}{2}(\bar{w}_{,x})^2 \tag{8a}$$

$$\bar{\epsilon}_y = \bar{v}_{,y} + \frac{\bar{w}}{r_y} + \frac{1}{2}(\bar{w}_{,y})^2 \tag{8b}$$

$$\bar{\epsilon}_{xy} = \frac{1}{2}\left(\bar{v}_{,x} + \bar{u}_{,y} - \frac{x}{r_x}\bar{w}_{,y} + \bar{w}_{,x}\bar{w}_{,y}\right) \tag{8c}$$

$$\bar{\chi}_x = -\bar{w}_{,xx} \tag{8d}$$

$$\bar{\chi}_y = -\bar{w}_{,yy} \tag{8e}$$

$$\bar{\chi}_{xy} = -\bar{w}_{,xy} \tag{8f}$$

Equations (8a) - (8f) are written in terms of displacements tangent to and normal to the undeformed cylinder. Let U, V, and W be displacements tangent to and normal to the initially deformed cylinder, that is, toroidal shell (see fig. 3). Then, the following expressions hold where $\frac{x}{r_x}U$ is neglected in the expression for $\bar{w}$ and $\frac{1}{2}\left(\frac{x}{r_x}\right)^2$ is neglected in comparison with one.

$$\bar{u} = U + \frac{x}{r_x}W \tag{9a}$$

$$\bar{v} = V \tag{9b}$$

$$\bar{w} = W \tag{9c}$$

Eliminating $\bar{u}$, $\bar{v}$, and $\bar{w}$ between equations (8a) - (8f) and (9a) - (9c) yields the strain-displacement equations for the toroidal shell in terms of tangential and normal displacements.

$$\bar{\epsilon}_x = U_{,x} + \frac{W}{r_x} + \frac{1}{2}(W_{,x})^2 \tag{10a}$$

$$\bar{\epsilon}_y = V_{,y} + \frac{W}{r_y} + \frac{1}{2}(W_{,y})^2 \tag{10b}$$

$$\bar{\epsilon}_{xy} = \frac{1}{2}(V_{,x} + U_{,y} + W_{,x}W_{,y}) \tag{10c}$$

$$\bar{\chi}_x = -W_{,xx} \tag{10d}$$

$$\bar{\chi}_y = -W_{,yy} \tag{10e}$$

$$\bar{\chi}_{xy} = -W_{,xy} \tag{10f}$$

Equilibrium Equations and Boundary Conditions

Nonlinear equilibrium equations and consistent boundary conditions are now developed by using the principle of virtual work. For a continuum, in equilibrium, the principle states that the virtual change in strain energy is equal to the virtual work done by the body forces and externally applied forces acting through a virtual displacement. Applying this principle to the toroidal shell leads to the following equation (see appendix A):

$$\int_{-L/2}^{L/2}\int_{o}^{2\pi r_y}\left[N_x\delta\bar{\epsilon}_x + 2N_{xy}\delta\bar{\epsilon}_{xy} + N_y\delta\bar{\epsilon}_y + M_x\delta\bar{\chi}_x + 2M_{xy}\delta\bar{\chi}_{xy} + M_y\delta\bar{\chi}_y\right]dy\ dx$$

$$= \int_{-L/2}^{L/2}\int_{o}^{2\pi r_y}\left[p_x\delta U + p_y\delta V + p_n\delta W\right]dy\ dx + \int_{o}^{2\pi r_y}\left[F_x\delta U\right.$$

$$\left. + F_y\delta V + F_n\delta W + T_x(\delta W)_{,x} + T_{xy}(\delta W)_{,y}\right]_{x=-L/2}^{x=L/2} dy \tag{11}$$

Using equations (10a) - (10f), the virtual strains are expanded in terms of virtual displacements as follows:

$$\delta\bar{\epsilon}_x = (\delta U)_{,x} + \frac{\delta W}{r_x} + W_{,x}(\delta W)_{,x} \tag{12a}$$

$$\delta\bar{\epsilon}_y = (\delta V)_{,y} + \frac{\delta W}{r_y} + W_{,y}(\delta W)_{,y} \tag{12b}$$

$$\delta\bar{\epsilon}_{xy} = \frac{1}{2}\left[(\delta V)_{,x} + (\delta U)_{,y} + W_{,x}(\delta W)_{,y} + W_{,y}(\delta W)_{,x}\right] \tag{12c}$$

$$\delta\bar{\chi}_x = -(\delta W)_{,xx} \tag{12d}$$

$$\delta\bar{\chi}_y = -(\delta W)_{,yy} \tag{12e}$$

$$\delta\bar{\chi}_{xy} = -(\delta W)_{,xy} \tag{12f}$$

Substituting the virtual strains given by equations (12a) - (12f) into the statement of the principle of virtual work, equation (11), and integrating by parts yields

$$\begin{aligned}
&\int_{-L/2}^{L/2}\int_{0}^{2\pi r_y}\Bigg\{\left[N_{x,x} + N_{xy,y} + p_x\right]\delta U + \left[N_{y,y} + N_{xy,y} + p_y\right]\delta V \\
&+ \left[M_{x,xx} + 2M_{xy,xy} + M_{y,yy} - \frac{N_x}{r_x} - \frac{N_y}{r_y} + (N_xW_{,x} + N_{xy}W_{,y})_{,x}\right. \\
&\left. + (N_{xy}W_{,x} + N_yW_{,y})_{,y} + p_n\right]\delta W\Bigg\}\, dy\, dx + \int_{0}^{2\pi r_y}\Bigg\{\left[N_x - F_x\right]\delta U \\
&+ \left[N_{xy} - F_y\right]\delta V + \left[M_{x,x} + 2M_{xy,y} + N_xW_{,x} + N_yW_{,y} + T_{xy,y} - F_n\right]\delta W \\
&+ \left[-M_x + T_x\right](\delta W)_{,x}\Bigg\}_{x=-L/2}^{x=L/2} dy = 0
\end{aligned} \tag{13}$$

Since the virtual displacements are arbitrary, except that they must satisfy the displacement boundary conditions, the following equilibrium equations and boundary conditions must hold:

Equilibrium equations

$$N_{x,x} + N_{xy,y} + p_x = 0 \tag{14a}$$

$$N_{y,y} + N_{xy,x} + p_y = 0 \tag{14b}$$

$$M_{x,xx} + 2M_{xy,xy} + M_{y,yy} - \frac{N_x}{r_x} - \frac{N_y}{r_y} + (N_x W_{,x} + N_{xy} W_{,y})_{,x}$$

$$+ (N_{xy} W_{,x} + N_y W_{,y})_{,y} + p_n = 0 \tag{14c}$$

Boundary conditions

On the edges, $x = \pm L/2$, prescribe

$$N_x = F_x \quad \text{or} \quad U \tag{15a}$$

$$N_{xy} = F_y \quad \text{or} \quad V \tag{15b}$$

$$M_{x,x} + 2M_{xy,y} + N_x W_{,x} + N_{xy} W_{,y} = F_n - T_{xy,y} \quad \text{or} \quad W \tag{15c}$$

$$M_x = T_x \quad \text{or} \quad W_{,x} \tag{15d}$$

Constitutive Equations

For a shell composed of a homogeneous isotropic material, the following constitutive equations are consistent with the statement of the principle of virtual work as given in equation (11).

$$N_x = \frac{Et}{1 - \nu^2}(\bar{\epsilon}_x + \nu\bar{\epsilon}_y) \tag{16a}$$

$$N_y = \frac{Et}{1 - \nu^2}(\bar{\epsilon}_y + \nu\bar{\epsilon}_x) \tag{16b}$$

$$N_{xy} = \frac{Et}{1 + \nu}\bar{\epsilon}_{xy} \tag{16c}$$

$$M_x = \frac{Et^3}{12(1 - \nu^2)}(\bar{\chi}_x + \nu\bar{\chi}_y) \tag{16d}$$

$$M_y = \frac{Et^3}{12(1 - \nu^2)}(\bar{\chi}_y + \nu\bar{\chi}_x) \tag{16e}$$

$$M_{xy} = \frac{Et^3}{12(1 + \nu)}\bar{\chi}_{xy} \tag{16f}$$

VII. BUCKLING EQUATIONS

In this section, classical buckling equations are derived by perturbing the nonlinear shell equations about a state of equilibrium which exists prior to buckling, that is, the prebuckling state. The perturbation is accomplished by separating the stresses, strains, and displacements into a prebuckling term and a small perturbation term. Let the prebuckling terms be indicated by a (^) superscript and perturbation terms by a (~) superscript. Then, the total stresses, strains, and displacements are:

$$N_x = \hat{N}_x + \tilde{N}_x \tag{17a}$$

$$N_y = \hat{N}_y + \tilde{N}_y \tag{17b}$$

$$N_{xy} = \hat{N}_{xy} + \tilde{N}_{xy} \tag{17c}$$

$$M_x = \hat{M}_x + \tilde{M}_x \tag{17d}$$

$$M_y = \hat{M}_y + \tilde{M}_y \tag{17e}$$

$$M_{xy} = \hat{M}_{xy} + \tilde{M}_{xy} \tag{17f}$$

$$\bar{\epsilon}_x = \hat{\epsilon}_x + \tilde{\epsilon}_x \tag{17g}$$

$$\bar{\epsilon}_y = \hat{\epsilon}_y + \tilde{\epsilon}_y \tag{17h}$$

$$\bar{\epsilon}_{xy} = \hat{\epsilon}_{xy} + \tilde{\epsilon}_{xy} \tag{17i}$$

$$\bar{\chi}_x = \hat{\chi}_x + \tilde{\chi}_x \tag{17j}$$

$$\bar{\chi}_y = \hat{\chi}_y + \tilde{\chi}_y \tag{17k}$$

$$\bar{\chi}_{xy} = \hat{\chi}_{xy} + \tilde{\chi}_{xy} \tag{17l}$$

$$U = \hat{U} + \tilde{U} \tag{17m}$$

$$V = \hat{V} + \tilde{V} \tag{17n}$$

$$W = \hat{W} + \tilde{W} \tag{17o}$$

Neglecting any change in the surface loads p_x, p_y, and p_n and prescribed boundary forces F_x, F_y, F_n, T_{xy}, and T_x during buckling, there follows:

$$p_x = \hat{p}_x \tag{18a}$$

$$p_y = \hat{p}_y \tag{18b}$$

$$p_n = \hat{p}_n \tag{18c}$$

$$F_x = \hat{F}_x \tag{18d}$$

$$F_y = \hat{F}_y \tag{18e}$$

$$F_n = \hat{F}_n \tag{18f}$$

$$T_{xy} = \hat{T}_{xy} \tag{18g}$$

$$T_x = \hat{T}_x \tag{18h}$$

Substituting equations (17a) - (17o) and (18a) - (18h) into the shell equations (12a) - (12f), (14a) - (14c), (15a) - (15d), and (16a) - (16f), dropping products of the perturbation terms, and noting that the prebuckling terms must also satisfy the shell equations, yields the following set of linear buckling equations:

<u>Strain displacement</u>

$$\tilde{\epsilon}_x = \tilde{U}_{,x} + \frac{\tilde{W}}{r_x} + \underline{\hat{W}_{,x}\tilde{W}_{,x}} \tag{19a}$$

$$\tilde{\epsilon}_y = \tilde{V}_{,y} + \frac{\tilde{W}}{r_y} + \underline{\hat{W}_{,y}\tilde{W}_{,y}} \tag{19b}$$

$$\tilde{\epsilon}_{xy} = \frac{1}{2}\left[\tilde{V}_{,x} + \tilde{U}_{,y} + \underline{\hat{W}_{,x}\tilde{W}_{,y}} + \underline{\hat{W}_{,y}\tilde{W}_{,x}}\right] \tag{19c}$$

$$\tilde{\chi}_x = -\tilde{W}_{,xx} \tag{19d}$$

$$\tilde{\chi}_y = -\tilde{W}_{,yy} \tag{19e}$$

$$\tilde{\chi}_{xy} = -\tilde{W}_{,xy} \tag{19f}$$

Equilibrium equations

$$\tilde{N}_{x,x} + \tilde{N}_{xy,y} = 0 \tag{20a}$$

$$\tilde{N}_{y,y} + \tilde{N}_{xy,x} = 0 \tag{20b}$$

$$\begin{aligned} &\tilde{M}_{x,xx} + 2\tilde{M}_{xy,xy} + \tilde{M}_{y,yy} - \frac{\tilde{N}_x}{r_x} - \frac{\tilde{N}_y}{r_y} + (\hat{N}_{x,x} + \hat{N}_{xy,y})\tilde{W}_{,x} \\ &+ (\hat{N}_{xy,x} + \hat{N}_{y,y})\tilde{W}_{,y} + (\tilde{N}_{x,x} + \tilde{N}_{xy,y})\hat{W}_{,x} + (\tilde{N}_{xy,x} + \tilde{N}_{y,y})\hat{W}_{,y} \\ &+ \hat{N}_x\tilde{W}_{,xx} + 2\hat{N}_{xy}\tilde{W}_{,xy} + \hat{N}_y\tilde{W}_{,yy} + \tilde{N}_x\hat{W}_{,xx} + 2\tilde{N}_{xy}\hat{W}_{,xy} + \tilde{N}_y\hat{W}_{,yy} = 0 \end{aligned} \tag{20c}$$

Using the first two equilibrium equations (20a) and (20b), and again noting that the prebuckling terms satisfy the shell equilibrium equations, equation (20c) becomes

$$\begin{aligned} &\tilde{M}_{x,xx} + 2\tilde{M}_{xy,xy} + \tilde{M}_{y,yy} - \frac{\tilde{N}_x}{r_x} - \frac{\tilde{N}_y}{r_y} - \hat{p}_x\tilde{W}_{,x} - \hat{p}_y\tilde{W}_{,y} \\ &+ \hat{N}_x\tilde{W}_{,xx} + 2\hat{N}_{xy}\tilde{W}_{,xy} + \hat{N}_y\tilde{W}_{,yy} + \underline{\tilde{N}_x\hat{W}_{,xx}} + \underline{2\tilde{N}_{xy}\hat{W}_{,xy}} \\ &+ \underline{\tilde{N}_y\hat{W}_{,yy}} = 0 \end{aligned} \tag{21}$$

Boundary conditions

On the edges, $x = \pm L/2$, prescribe

$$\tilde{N}_x = 0 \quad \text{or} \quad \tilde{U} = 0 \tag{22a}$$

$$\tilde{N}_{xy} = 0 \quad \text{or} \quad \tilde{V} = 0 \tag{22b}$$

$$\tilde{M}_{x,x} + 2\tilde{M}_{xy,y} + \hat{N}_x\tilde{W}_{,x} + \hat{N}_y\tilde{W}_{,y} + \tilde{N}_x\hat{W}_{,x} + \tilde{N}_y\hat{W}_{,y} = 0 \quad \text{or} \quad \tilde{W} = 0 \tag{22c}$$

$$\tilde{M}_x = 0 \quad \text{or} \quad \tilde{W}_{,x} = 0 \tag{22d}$$

Constitutive equations

$$\tilde{N}_x = \frac{Et}{1 - \nu^2}(\tilde{\epsilon}_x + \nu\tilde{\epsilon}_y) \tag{23a}$$

$$\tilde{N}_y = \frac{Et}{1 - \nu^2}(\tilde{\epsilon}_y + \nu\tilde{\epsilon}_x) \tag{23b}$$

$$\tilde{N}_{xy} = \frac{Et}{1 + \nu}\,\tilde{\epsilon}_{xy} \tag{23c}$$

$$\tilde{M}_x = \frac{Et^3}{12(1 - \nu^2)}(\tilde{\chi}_x + \nu\tilde{\chi}_y) \tag{23d}$$

$$\tilde{M}_y = \frac{Et^3}{12(1 - \nu^2)}(\tilde{\chi}_y + \nu\tilde{\chi}_x) \tag{23e}$$

$$\tilde{M}_{xy} = \frac{Et^3}{12(1 + \nu)}\,\tilde{\chi}_{xy} \tag{23f}$$

The underlined terms in equations (19) - (23) drop out when prebuckling rotations are neglected, that is, derivatives of $\hat{W}$ with respect to x and y are neglected.

When prebuckling rotations are neglected, equations (19) - (23) can be simplified by introducing a stress function and by formulating a compatibility equation. The first two equilibrium equations, (20a) and (20b), are identically satisfied by the Airy stress function, ϕ, defined by

$$\tilde{N}_x = \phi_{,yy} \tag{24a}$$

$$\tilde{N}_y = \phi_{,xx} \tag{24b}$$

$$\tilde{N}_{xy} = -\phi_{,xy} \tag{24c}$$

Introducing the stress function into the third equilibrium equation (eq. (21)) and neglecting prebuckling rotations yields

$$\tilde{M}_{x,xx} + 2\tilde{M}_{xy,xy} + \tilde{M}_{y,yy} - \frac{\phi_{,yy}}{r_x} - \frac{\phi_{,xx}}{r_y} - \hat{p}_x\tilde{W}_{,x} - \hat{p}_y\tilde{W}_{,y}$$

$$+ \hat{N}_x\tilde{W}_{,xx} + 2\hat{N}_{xy}\tilde{W}_{,xy} + \hat{N}_y\tilde{W}_{,yy} = 0 \tag{25}$$

Now, expanding the moments in terms of $\tilde{W}$ by using equations (19d) - (19f) and (23d) - (23f) gives the following equation relating $\tilde{W}$ and ϕ:

$$\frac{Et^3}{12(1 - \nu^2)} \nabla^4\tilde{W} + \frac{\phi_{,yy}}{r_x} + \frac{\phi_{,xx}}{r_y} + \hat{p}_x\tilde{W}_{,x} + \hat{p}_y\tilde{W}_{,y} - \hat{N}_x\tilde{W}_{,xx}$$

$$- 2\hat{N}_{xy}\tilde{W}_{,xy} - \hat{N}_y\tilde{W}_{,yy} = 0 \tag{26}$$

An additional equation relating $\tilde{W}$ and ϕ is obtained by using equations (19a) - (19c), with prebuckling rotations neglected, to formulate the following compatibility equation:

$$\tilde{\epsilon}_{x,yy} + \tilde{\epsilon}_{y,xx} - 2\tilde{\epsilon}_{xy,xy} = \frac{\tilde{W}_{,yy}}{r_x} + \frac{\tilde{W}_{,xx}}{r_y} \tag{27}$$

Now, using equations (23a) - (23c) and (24a) - (24c), equation (27) becomes

$$\nabla^4\phi = Et\left[\frac{\tilde{W}_{,yy}}{r_x} + \frac{\tilde{W}_{,xx}}{r_y}\right] \tag{28}$$

Eliminating ϕ between equations (26) and (28) yields

$$\frac{Et^3}{12(1 - \nu^2)} \nabla^4 \tilde{W} + \nabla^{-4} \left\{ Et \left[\frac{\tilde{W}_{,yyyy}}{r_x^{\,2}} + \frac{2\tilde{W}_{,xxyy}}{r_x r_y} + \frac{\tilde{W}_{,xxxx}}{r_y^{\,2}} \right] \right\}$$

$$+ \hat{p}_x \tilde{W}_{,x} + \hat{p}_y \tilde{W}_{,y} - \hat{N}_x \tilde{W}_{,xx} - 2\hat{N}_{xy} \tilde{W}_{,xy} - \hat{N}_y \tilde{W}_{,yy} = 0 \qquad (29)$$

VIII. BUCKLING OF THE SPHERICAL SEGMENT

A narrow segment of a thin-walled spherical shell is shown in figure 4. The shell is simply supported at both edges and the weight of the shell is supported by a vertical line load at the upper and lower edges. Also, the shell is supported such that no displacements in the circumferential direction can occur at the edges. It is of interest to determine how thick a spherical segment must be such that it will be able to support its weight.

For the spherical segment, the governing equations on the stress function, ϕ, and the displacement, W, are given by equations (28) and (29) with $r_x = r_y = R$.

$$\nabla^4\phi = \frac{Et}{R}\nabla^2\tilde{W} \tag{30a}$$

$$\frac{Et^3}{12(1-\nu^2)}\nabla^4\tilde{W} + \frac{Et}{R^2}\tilde{W} + \hat{p}_x\tilde{W}_{,x} + \hat{p}_y\tilde{W}_{,y} - \hat{N}_x\tilde{W}_{,xx}$$

$$- 2\hat{N}_{xy}\tilde{W}_{,xy} - \hat{N}_y\tilde{W}_{,yy} = 0 \tag{30b}$$

Since the shell is loaded symmetrically,

$$\hat{p}_y = \hat{N}_{xy} = 0 \tag{31}$$

The prebuckling quantities $\hat{p}_x$, $\hat{N}_x$, and $\hat{N}_y$ determined by a linear membrane solution are (see appendix B):

$$\hat{p}_x = -\rho t \tag{32a}$$

$$\hat{N}_x = \rho t\left[x + \left(k - \frac{1}{2}\right)L\right] \tag{32b}$$

$$\hat{N}_y = -\rho t\left[2x + \left(k - \frac{1}{2}\right)L\right] \tag{32c}$$

where ρ is the specific weight of the shell wall and k is the proportion of the weight of the shell supported at the upper edge.

Substituting equations (31) and (31a) - (32c) into equation (30b) yields

$$\frac{Et^3}{12(1-\nu^2)} \nabla^4\tilde{W} + \frac{Et}{R^2}\tilde{W} - \rho t\tilde{W}_{,x} - \rho t\left[x + \left(k - \frac{1}{2}\right)L\right]\tilde{W}_{,xx}$$

$$+ \rho t\left[2x + \left(k - \frac{1}{2}\right)L\right]\tilde{W}_{,yy} = 0 \qquad (33)$$

The boundary conditions at the upper and lower edges of the shell are:

At $x = \pm L/2$

$$\tilde{N}_x = 0, \implies \phi_{,yy} = 0 \qquad (34a)$$

$$\left.\begin{array}{l}\tilde{V} = 0\\ \tilde{W} = 0\end{array}\right\}, \implies \begin{array}{l}\phi_{,xx} = 0 \qquad (34b)\\ W = 0 \qquad (34c)\end{array}$$

$$\tilde{M}_x = 0, \implies W_{,xx} = 0 \qquad (34d)$$

Consider the following truncated series solution to equations (30a) and (33):

$$\tilde{W} = \cos\frac{ny}{R}\sum_{m=1}^{M} A_m \sin\frac{m\pi\left(x + \frac{L}{2}\right)}{L} \qquad (35a)$$

$$\phi = -\cos\frac{ny}{R}\sum_{m=1}^{M}\frac{\frac{Et}{R}A_m}{\left(\frac{n}{R}\right)^2 + \left(\frac{m\pi}{L}\right)^2}\sin\frac{m\pi\left(x + \frac{L}{2}\right)}{L} \qquad (35b)$$

Such a solution will satisfy equation (30a) and the boundary conditions given by equations (34a) - (34d) identically. Also, equation (33) is satisfied identically in the y-direction.

Substituting equations (35a) and (35b) into equation (33) and dividing through by $\cos \frac{ny}{R}$ yields

$$\sum_{m=1}^{M} \left[\left\{ \frac{Et^3}{12(1 - \nu^2)} \left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{R}\right)^2 \right]^2 + \rho t \left[x + \left(k - \frac{1}{2}\right) L \right] \left(\frac{m\pi}{2L}\right)^2 \right. \right.$$
$$\left. - \rho t \left[2x + \left(k - \frac{1}{2}\right) L \right] \left(\frac{n}{R}\right)^2 + \frac{Et}{R^2} \right\} A_m \sin \frac{m\pi\left(x + \frac{L}{2}\right)}{L}$$
$$\left. - \left\{ \rho t \left(\frac{m\pi}{L}\right) A_m \cos \frac{m\pi\left(x + \frac{L}{2}\right)}{L} \right\} \right] = 0 \qquad (36)$$

Now, using the Galerkin method, multiply equation (36) by

$$\sin \frac{q\pi\left(x + \frac{L}{2}\right)}{L}; \quad q = 1, 2, 3, \ldots, M$$

and integrate the resulting equation over the interval

$$-\frac{L}{2} \leq x \leq \frac{L}{2}$$

to yield

$$\left\{\frac{Et^3}{12(1-\nu^2)}\left[\left(\frac{q\pi}{L}\right)^2+\left(\frac{n}{R}\right)^2\right]^2+\rho tL\left(k-\frac{1}{2}\right)\left[\left(\frac{q\pi}{L}\right)^2-\left(\frac{n}{R}\right)^2\right]\right.$$

$$\left.+\frac{Et}{R^2}\right\}\frac{L}{2}A_q-\sum_{\substack{m=1\\ \neq q}}^{M}\rho t\left\{\left[\left(\frac{m\pi}{L}\right)^2-2\left(\frac{n}{R}\right)^2\right]\left(\frac{L}{\pi}\right)^2\frac{2}{q^2-m^2}+1\right\}\frac{qm}{q^2-m^2}$$

$$\cdot\left[1-(-1)^{m+q}\right]A_m=0 \qquad q=1,2,3,\ldots,M \qquad (37)$$

Equation (37) represents M homogeneous equations in terms of the M unknowns,

$$A_1, A_2, A_3, \ldots, A_M$$

For determining the critical thickness of a shell, it is convenient to express equation (37) in terms of nondimensional parameters as follows:

$$\left\{\frac{\alpha\left(k-\frac{1}{2}\right)\left[n^2-(q\pi\beta)^2\right]-1}{\left[n^2+(q\pi\beta)^2\right]^2}-\lambda\right\}A_q$$

$$-\sum_{\substack{m=1\\ \neq q\\ m+q\ \text{odd}}}^{M}4\alpha\frac{\left[2n^2-(m\pi\beta)^2\right]\frac{2}{\pi^2(q^2-m^2)}-\beta^2}{\left[n^2+(q\pi\beta)^2\right]^2}\frac{qm}{q^2-m^2}A_m=0$$

$$q=1,2,3,\ldots,M \qquad (38)$$

where

$$\alpha = \frac{\rho L}{E}$$

$$\beta = \frac{R}{L}$$

$$\lambda = \frac{t^2}{12(1 - \nu^2)R^2}$$

Equation (38) represents M homogeneous algebraic equations. In matrix notation, these M equations are:

$$\left[\left[a_{qm} \right] - \lambda I \right] \left\{ A \right\} = 0 \qquad (39)$$

where, the elements of $\left[a_{qm} \right]$ are:

$$a_{qm} = \frac{\alpha\left(k - \frac{1}{2}\right)\left[n^2 - (q\pi\beta)^2\right] - 1}{\left[n^2 + (q\pi\beta)^2\right]^2} \qquad m = q$$

$$= -4\alpha \frac{\left[2n^2 - (m\pi\beta)^2\right]\frac{2}{\pi^2(q^2 - m^2)} - \beta^2}{\left[n^2 + (q\pi\beta)^2\right]^2} \cdot \frac{qm}{q^2 - m^2} \qquad m \neq q, \quad m + q \text{ odd}$$

$$= 0 \qquad m \neq q, \quad m + q \text{ even}$$

and

$$\left\{ A \right\} = \left\{ \begin{matrix} A_1 \\ A_2 \\ A_3 \\ \cdot \\ \cdot \\ \cdot \\ A_M \end{matrix} \right\}$$

Now, noting that

$$\left[2n^2 - (m\pi\beta)^2\right]\frac{2}{\pi^2(q^2 - m^2)} - \beta^2 = \left[4n^2 - (m\pi\beta)^2 - (q\pi\beta)^2\right]\frac{1}{\pi^2(q^2 - m^2)}$$

equation (39) can be put in the more convenient symmetric form

$$\left[\left[b_{qm}\right] - \lambda I\right]\left\{B\right\} = 0 \qquad (40)$$

where, the elements of $\left[b_{qm}\right]$ are:

$$b_{qm} = b_{mq} = \frac{\alpha\left(k - \frac{1}{2}\right)\left[n^2 - (q\pi\beta)^2\right] - 1}{\left[n^2 + (q\pi\beta)^2\right]^2} \qquad m = q$$

$$= -4\alpha\,\frac{\left[4n^2 - (m\pi\beta)^2 - (q\pi\beta)^2\right]}{\left[n^2 + (m\pi\beta)^2\right]\left[n^2 + (q\pi\beta)^2\right]} \cdot \frac{qm}{\pi^2(q^2 - m^2)^2} \qquad m \neq q,\ m + q \ \text{odd}$$

$$= 0 \qquad m \neq q,\ m + q \ \text{even}$$

and

$$\left\{B\right\} = \begin{Bmatrix} \left[n^2 + (\pi\beta)^2\right]A_1 \\ \left[n^2 + (2\pi\beta)^2\right]A_2 \\ \vdots \\ \left[n^2 + (m\pi\beta)^2\right]A_m \\ \vdots \\ \left[n^2 + (M\pi\beta)^2\right]A_M \end{Bmatrix}$$

Equation (40) is a standard eigenvalue problem, the nontrivial solutions of which yield M characteristic values of the thickness parameter, λ, as a function of α, β, k, and n. For a given α, β, and k, the limiting value of λ obtained by maximizing with respect to n as $M \to \infty$ yields the critical value λ_{cr}. That is, for λ greater than λ_{cr} the shell will not buckle. In practice, this critical value of λ can be determined by increasing M until λ converges to its limiting value.

IX. NUMERICAL RESULTS AND DISCUSSION

Equation (40) was programmed on a high-speed digital computer using a standard eigenvalue subroutine. The subroutine uses Householder's method (ref. 3) to find a similarity orthogonal transformation to reduce a real symmetric matrix to tridiagonal form. The eigenvalues of the tridiagonal matrix are then determined by using Ortega's extension of the theory of Sturm Sequences (ref. 4).

Critical values of the thickness parameters, λ_{cr}, have been determined over the range

$$10^{-7} \leq \frac{\rho L}{E} \leq 10^{-4}$$

for $R/L = 1$, 3, and 5, and $k = 0$, 1/4, 1/2, and 1. To gain some appreciation of the range of $\rho L/E$ considered, it should be noted that most metallic materials have a ρ/E of order 10^{-8} in^{-1}, while for plastics values of ρ/E of order 10^{-7} in^{-1} are common. Therefore, a value of $\rho L/E$ equal to 10^{-7} might indicate a metallic shell 10 inches long and a value of $\rho L/E$ equal to 10^{-4} might indicate a plastic shell 1,000 inches long.

The number of terms in the series solution used for convergence varied from 10 for $\rho L/E = 10^{-4}$ to 70 for $\rho L/E = 10^{-6}$. For the cases where $k = 0$ and 1/4, convergence was found to be extremely slow when the series solution was taken in the form of equation (35a). This difficulty was removed by using a modified truncated series solution of the form

$$\tilde{W} = \cos \frac{ny}{R} \sum_{m=N}^{N+M} A_m \sin \frac{m\pi\left(x + \frac{L}{2}\right)}{L}$$

The eigenvalue, λ, was then maximized with respect to both n and N. For both cases, λ maximized at $n = 0$ while the value of N varied from 1 with $\rho L/E = 10^{-4}$ to 1621 for $\rho L/E = 10^{-7}$. The axially symmetric buckling mode, $n = 0$, can be explained by noting that the maximum compressive stress is in the meridional direction and the corresponding circumferential stress is tensile (see eqs. (10-B) - (13-B)). The extremely large values of N for $\rho L/E = 10^{-7}$ indicates that the buckle wave lengths are extremely small.

The critical thickness parameter, λ_{cr}, is compared with λ^* where λ^*, the critical thickness parameter obtained by the elementary approach, is

$$\lambda^* = \left(\frac{1}{2} - \frac{k}{2}\right)^2 \left(\frac{\rho L}{E}\right)^2 \qquad 0 \leq k \leq 1/4$$

$$= \left(\frac{1}{4} + \frac{k}{2}\right)^2 \left(\frac{\rho L}{E}\right)^2 \qquad 1/4 \leq k \leq 1$$

Figures 6, 7, 8, and 9 are plots of the ratio of λ_{cr} to λ^* versus $\rho L/E$ with $k = 0$, 1/4, 1/2, and 1, respectively, for $R/L = 1$, 3, and 5. In figure 10, the ratio of λ^* to $(\rho L/E)^2$ and λ_{cr} to $(\rho L/E)^2$ for $R/L = 5$, $\rho L/E = 10^{-7}$, 10^{-6}, 10^{-5}, and 10^{-4} are plotted versus k over the range $0 \leq k \leq 1$.

From figures 6, 7, 8, and 9, it is noted that the ratio of λ_{cr} to λ^* increases as R/L decreases and as $\rho L/E$ decreases, the ratio always being less than 1. It is also noted that the ratios for $k = 1/2$ and 1 are significantly lower than the ratios for $k = 0$ and $1/4$. From figure 10, it is noted that the lowest value of $\frac{\lambda^*}{(\rho L/E)^2}$ occurs when $k = 1/4$, while the lowest values of $\frac{\lambda_{cr}}{(\rho L/E)^2}$ occur at values of k slightly greater than $1/4$. The optimum value of k is seen to increase from approximately $1/4$ to $3/8$ as $\rho L/E$ increases for 10^{-7} to 10^{-4}. Figures 6 through 10 and the physics of the problem also indicate that λ^* is an upper bound on λ_{cr} as $\frac{\rho L}{E}$ goes to zero for all values of k and $\frac{R}{L}$. However, it is not obvious as to whether or not λ^* is a least upper bound on λ_{cr}.

Using the results indicated above, the following design considerations for the spherical segment can be formulated:

(i) The elementary approach, which assumes that the shell will buckle if the maximum compressive stress is greater than the critical compressive stress for a complete sphere loaded by uniform external pressure, leads to conservative results. The results will be more conservative for higher values of R/L and $\rho L/E$ and the results for $k = 1/2$ and 1 will be more conservative than the results for $k = 0$ and $1/4$.

(ii) The optimum supporting conditions, that is, the conditions which minimize the critical thickness parameter with respect to k, depend on the geometry and material properties of the shell. For the range of parameters considered, the optimum values of k were slightly larger than $1/4$.

X. CONCLUDING REMARKS

Nonlinear shell equations and linear buckling equations have been derived for a shell having the shape of a narrow segment of a toroidal shell centered at the equator by considering a cylindrical shell with an initial radial deformation.

The linear buckling equations have been used to study the problem of buckling of a simply supported thin shell having the shape of a narrow segment of a spherical shell centered at the equator and loaded in the axial direction by its own weight. The results indicate that an elementary approach, which assumes that the shell will buckle if the maximum compressive stress is greater than the critical compressive stress for a complete sphere loaded by uniform external pressure, is conservative. It is also found for the range of parameters considered that one-fourth to three-eights of the weight of the shell should be supported at the upper edge for an efficient design.

XI. REFERENCES

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XIII. APPENDIX A

In this appendix, a statement of the principle of virtual work consistent with the basic Love-Kirchoff assumptions will be developed for the toroidal shell.

The principle of virtual work may be expressed by the equality

$$\delta\Pi_1 = \delta\Pi_2 + \delta\Pi_3 \tag{1-A}$$

where $\delta\Pi_1$ is the virtual change in strain energy, $\delta\Pi_2$ and $\delta\Pi_3$ the virtual work done by the body forces and externally applied forces, respectively, acting through the virtual displacements. The above equality will now be applied to the toroidal shell.

In reference 5, it is shown that an expression for the strain energy of a thin shell, Π_1, consistent with the basic Love-Kirchoff assumptions is obtained as the sum of the extensional strain and bending strain energy. Using the results of reference 5, the strain energy for the toroidal shell composed of a homogeneous isotropic material is

$$\Pi_1 = \int\!\!\int \left\{ \frac{Et}{2(1 - \nu^2)} \left[(\bar{\epsilon}_x + \bar{\epsilon}_y)^2 - 2(1 - \nu)(\bar{\epsilon}_x \bar{\epsilon}_y - \bar{\epsilon}_{xy}{}^2) \right] + \frac{Et^3}{24(1 - \nu^2)} \left[(\bar{\chi}_x + \bar{\chi}_y)^2 - 2(1 - \nu)(\bar{\chi}_x \bar{\chi}_y - \bar{\chi}_{xy}{}^2) \right] \right\} \frac{r(x)}{r_y} \sqrt{1 + \left(\frac{dr}{dx}\right)^2}\, dy\, dx \tag{2-A}$$

where the double integral is taken over the middle surface of the shell.

The corresponding change in strain energy is

$$\delta\Pi_1 = \iint \left[\frac{\partial\Pi_1}{\partial\bar{\epsilon}_x} \delta\bar{\epsilon}_x + \frac{\partial\Pi_1}{\partial\bar{\epsilon}_y} \delta\bar{\epsilon}_y + \frac{\partial\Pi_1}{\partial\bar{\epsilon}_{xy}} \delta\bar{\epsilon}_{xy} + \frac{\partial\Pi_1}{\partial\bar{\chi}_x} \delta\bar{\chi}_x + \frac{\partial\Pi_1}{\partial\bar{\chi}_y} \delta\bar{\chi}_y \right.$$

$$\left. + \frac{\partial\Pi_1}{\partial\bar{\chi}_{xy}} \delta\bar{\chi}_{xy} \right] \frac{r(x)}{r_y} \sqrt{1 + \left(\frac{dr}{dx}\right)^2} \, dy \, dx \qquad (3\text{-A})$$

Now, defining the stress resultants and moments as partial derivates of the strain energy with respect to the middle surface strains and curvature changes (see ref. 5), there follows:

$$N_x = \frac{\partial\Pi_1}{\partial\bar{\epsilon}_x} = \frac{Et}{1 - \nu^2}(\bar{\epsilon}_x + \nu\bar{\epsilon}_y) \qquad (4\text{-A})$$

$$N_y = \frac{\partial\Pi_1}{\partial\bar{\epsilon}_y} = \frac{Et}{1 - \nu^2}(\bar{\epsilon}_y + \nu\bar{\epsilon}_x) \qquad (5\text{-A})$$

$$N_{xy} = \frac{\partial\Pi_1}{\partial\bar{\epsilon}_{xy}} = \frac{Et}{1 + \nu} \bar{\epsilon}_{xy} \qquad (6\text{-A})$$

$$M_x = \frac{\partial\Pi_1}{\partial\bar{\chi}_x} = \frac{Et^3}{12(1 - \nu^2)}(\bar{\chi}_x + \nu\bar{\chi}_y) \qquad (7\text{-A})$$

$$M_y = \frac{\partial\Pi_1}{\partial\bar{\chi}_y} = \frac{Et^3}{12(1 - \nu^2)}(\bar{\chi}_y + \nu\bar{\chi}_x) \qquad (8\text{-A})$$

$$M_{xy} = \frac{\partial\Pi_1}{\partial\bar{\chi}_{xy}} = \frac{Et^3}{12(1 + \nu)} \bar{\chi}_{xy} \qquad (9\text{-A})$$

The positive sense of the stress resultants is shown in figure 5. Substituting equations (4-A) - (9-A) into equation (3-A) gives

$$\delta\Pi_1 = \iint \Big[N_x \delta\bar{\epsilon}_x + 2N_{xy}\delta\bar{\epsilon}_{xy} + N_y\delta\bar{\epsilon}_y + M_x\delta\bar{\chi}_x + 2M_{xy}\delta\bar{\chi}_{xy}$$

$$+ M_y\delta\chi_y \Big] \frac{r(x)}{r_y} \sqrt{1 + \left(\frac{dr}{dx}\right)^2} \, dy \, dx \qquad (10\text{-A})$$

Neglecting body forces, or treating them as externally applied forces, the virtual work done by externally applied forces can be separated into two parts. The first part being the virtual work done by the externally applied forces distributed over the middle surface of the shell and the second part being the virtual work done by the externally applied forces and moments acting along the ends of the shell. Referring to figure 5, the expression for the virtual work done by the externally applied forces is

$$\delta\Pi_3 = \iint (p_x\delta U + p_y\delta V + p_n\delta W)\frac{r(x)}{r_y} \sqrt{1 + \left(\frac{dr}{dx}\right)^2} \, dy \, dx$$

$$+ \int \Big[F_x\delta U + F_y\delta V + F_n\delta W + T_x\delta\omega_x + T_{xy}\delta\omega_y \Big] \frac{r\left(\pm\frac{L}{2}\right)}{r_y} \, dy \qquad (11\text{-A})$$

where the line integral is taken over the ends of the shell. The virtual rotations $\delta\omega_x$ and $\delta\omega_y$ may be expressed in terms of δW as

$$\delta\omega_x = (\delta W)_{,x} \qquad (12\text{-A})$$

$$\delta\omega_y = (\delta W)_{,y} \qquad (13\text{-A})$$

Combining equations (1-A), (10-A), and (11-A) and neglecting terms of order $\frac{1}{2}\left(\frac{x}{r_x}\right)^2$ in comparison with unity, the principle of virtual work for the toroidal shell may be expressed by

$$\int_{-L/2}^{L/2}\int_{o}^{2\pi r_y}\left[N_x\delta\bar{\epsilon}_x + 2N_{xy}\delta\bar{\epsilon}_{xy} + N_y\delta\bar{\epsilon}_y + M_x\delta\bar{X}_x + 2M_{xy}\delta\bar{X}_{xy} + M_y\delta\bar{X}_y\right]dy\ dx$$

$$= \int_{-L/2}^{L/2}\int_{o}^{2\pi r_y}\left[p_x\delta U + p_y\delta V + p_n\delta W\right]dy\ dx + \int_{o}^{2\pi r_y}\left[F_x\delta U\right.$$

$$\left. + F_y\delta V + F_n\delta W + T_x(\delta W)_{,x} + T_{xy}(\delta W)_{,y}\right]_{x=-L/2}^{x=L/2} \qquad (14\text{-A})$$

XIV. APPENDIX B

The linear membrane equilibrium equations for the symmetrically loaded spherical shell shown in figure 4 are obtained from the shell equilibrium equations (14a) - (14b) by dropping moments and nonlinear terms. Also, due to symmetry, $N_{xy} = 0$, and derivative with respect to y are zero. The resulting equations are:

$$N_{x,x} + p_x = 0 \tag{1-B}$$

$$\frac{N_x + N_y}{R} - p_n = 0 \tag{2-B}$$

The loading functions p_x and p_n are obtained by projecting the external loads, in this case the weight of the shell, onto a direction tangent to and normal to the shell. Neglecting $\frac{1}{2}\left(\frac{x}{R}\right)^2$ in comparison with unity gives the following expressions for p_x and p_n:

$$p_x = -\rho t \tag{3-B}$$

$$p_n = -\frac{\rho t}{R} x \tag{4-B}$$

Substituting equation (3-B) into equation (1-B) and integrating yields

$$N_x = \rho t x + C \tag{5-B}$$

and from equations (2-B) and (4-B)

$$N_y = -2\rho t x - C \tag{6-B}$$

where C is a constant to be determined. The constant C is determined by the value of N_x at the upper or lower edge of the shell. By neglecting $\frac{1}{4}\left(\frac{L}{R}\right)^2$ in comparison with unity, the values of N_x at the upper and lower edges are:

$$N_x = \rho t L k; \quad x = L/2 \tag{7-B}$$

$$N_x = \rho t L(k - 1); \quad x = -L/2 \tag{8-B}$$

where k is the proportion of the weight of the shell supported at the upper edge. Substituting either of the two values for N_x into equation (5-B) and solving for C gives

$$C = \rho t L\left(k - \frac{1}{2}\right) \tag{9-B}$$

Substituting equation (9-B) into equations (5-B) and (6-B) yields the required membrane stress resultants

$$N_x = \rho t\left[x + \left(k - \frac{1}{2}\right)L\right] \tag{10-B}$$

$$N_y = -\rho t\left[2x + \left(k - \frac{1}{2}\right)L\right] \tag{11-B}$$

The maximum compressive stress resultants are:

$$N_x\Big|_{x=-L/2} = \rho t L(k - 1); \quad 0 \leq k \leq 1/4 \tag{12-B}$$

$$N_y\Big|_{x=L/2} = -\rho t L\left(k + \frac{1}{2}\right); \quad 1/4 \leq k \leq 1 \tag{13-B}$$

Assuming that the shell will buckle if the maximum compressive stress is greater than the critical compressive stress for a complete sphere loaded by uniform external pressure, that is,

$$N_x \text{ or } N_y = - \frac{Et^2}{R\sqrt{3(1 - \nu^2)}}$$

gives the following critical values for λ:

$$\lambda^* = \left(\frac{1}{2} - \frac{k}{2}\right)^2 \left(\frac{\rho L}{E}\right)^2; \quad 0 \leq k \leq 1/4 \tag{14-B}$$

$$\lambda^* = \left(\frac{1}{4} + \frac{k}{2}\right)^2 \left(\frac{\rho L}{E}\right)^2; \quad 1/4 \leq k \leq 1 \tag{15-B}$$

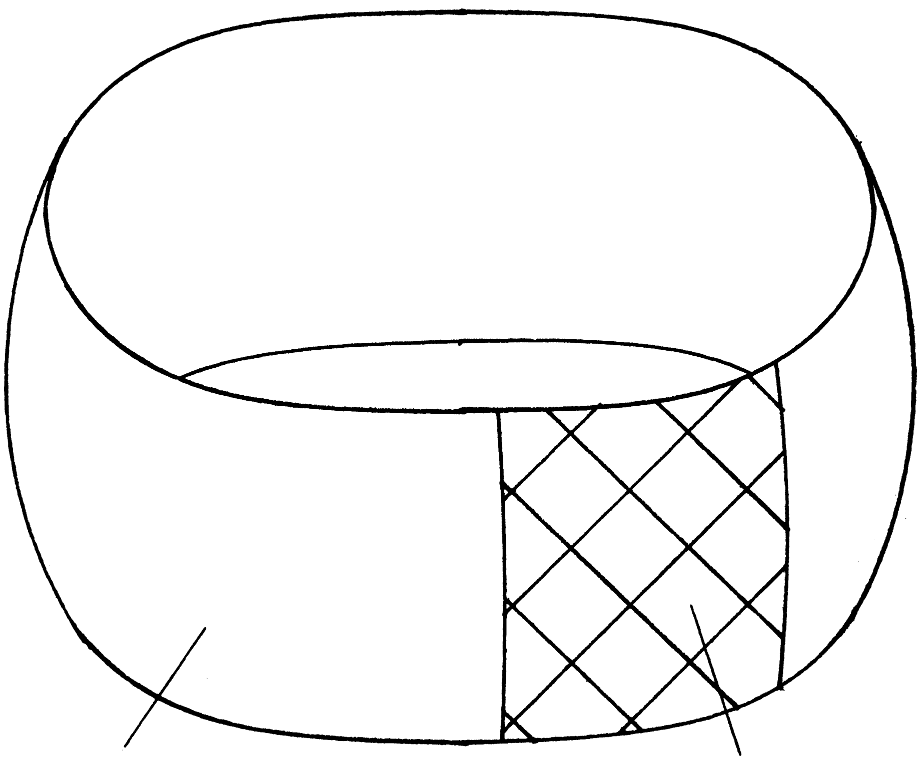

Figure 1.- Shell for the lunar orbit and landing approach simulator and a segment of a spherical shell.

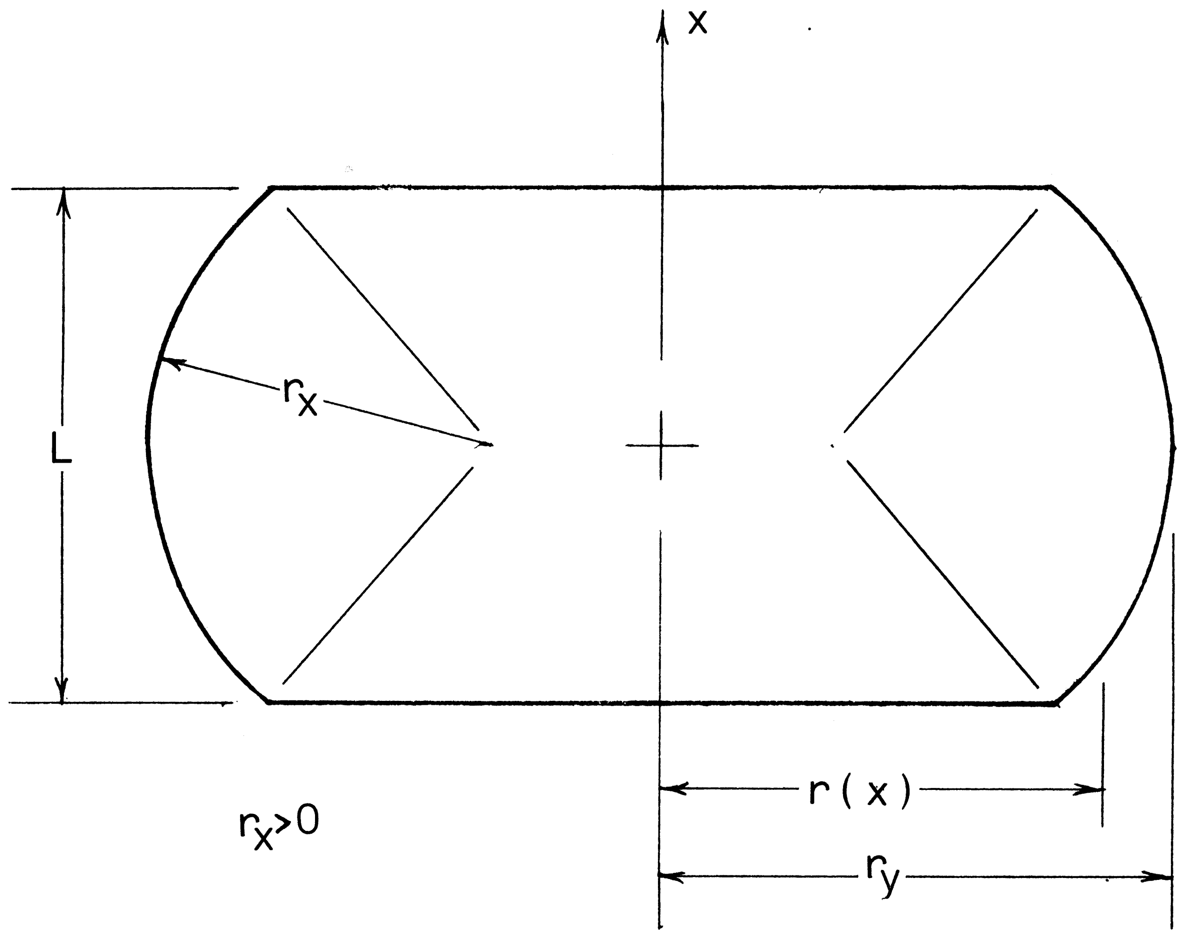

Figure 2.- Geometry of the toroidal shell.

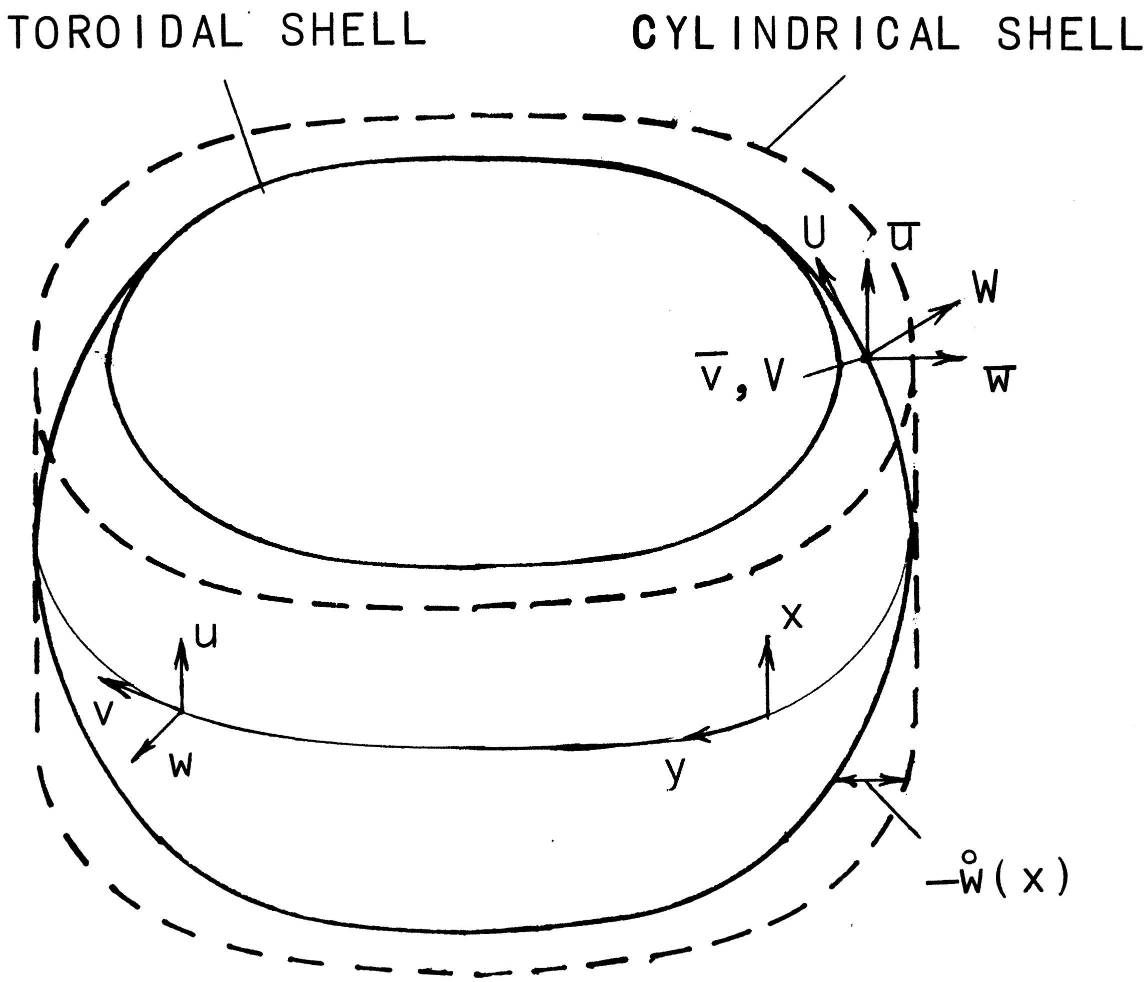

Figure 3.- Toroidal shell and undeformed cylinder.

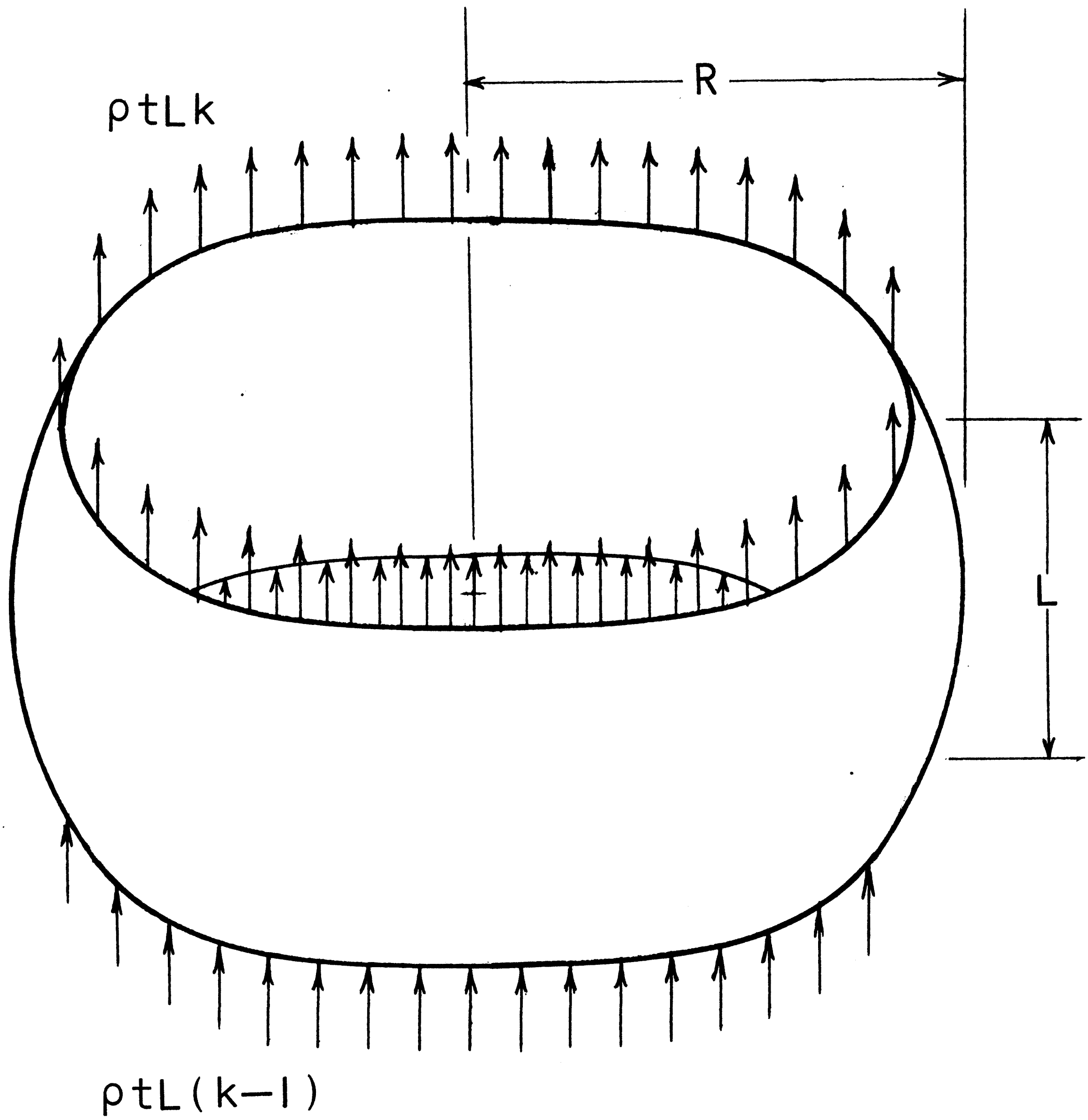

Figure 4.- Geometry and supporting conditions of the spherical segment.

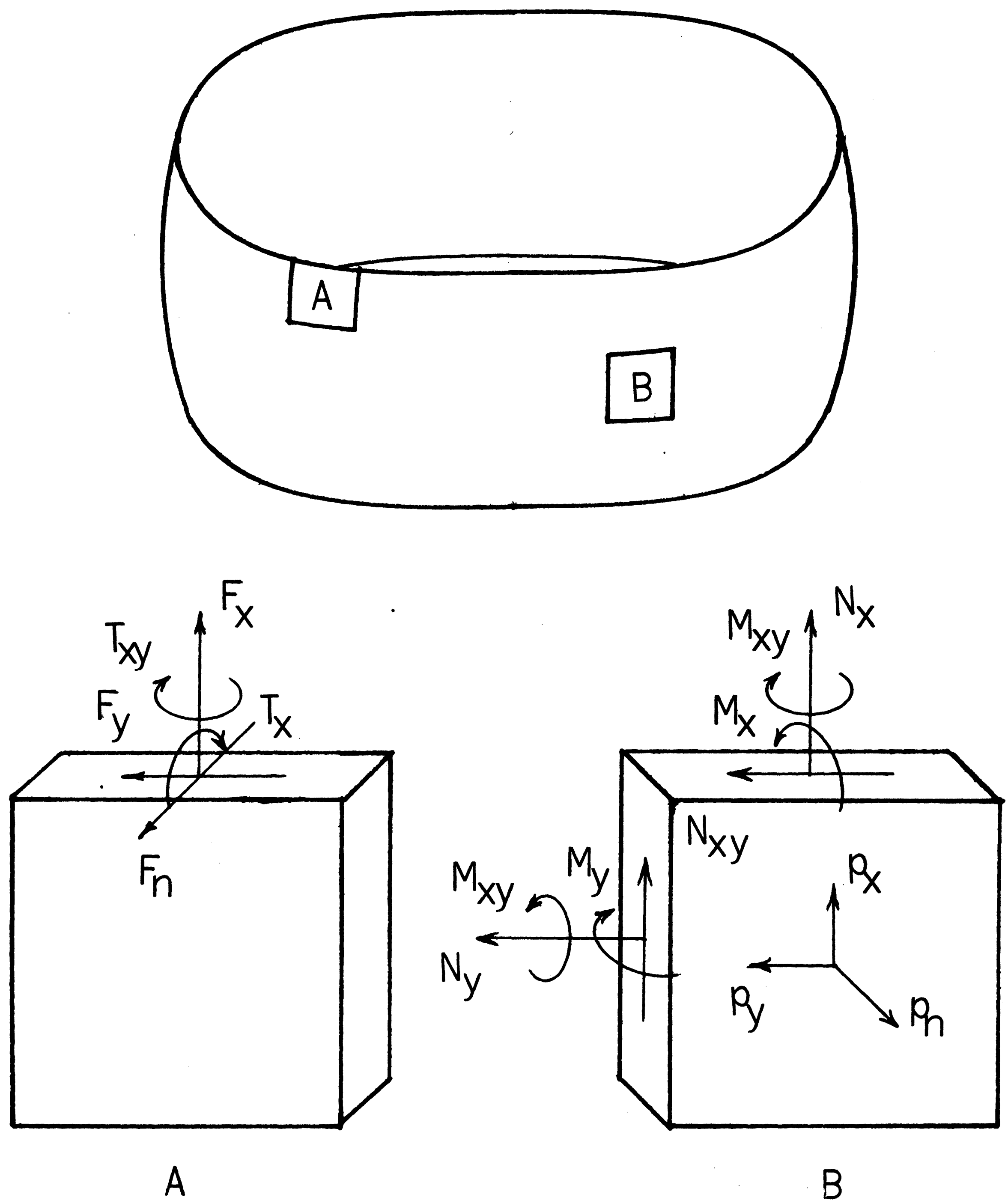

Figure 5.- Stress resultants and external forces acting on toroidal shell elements.

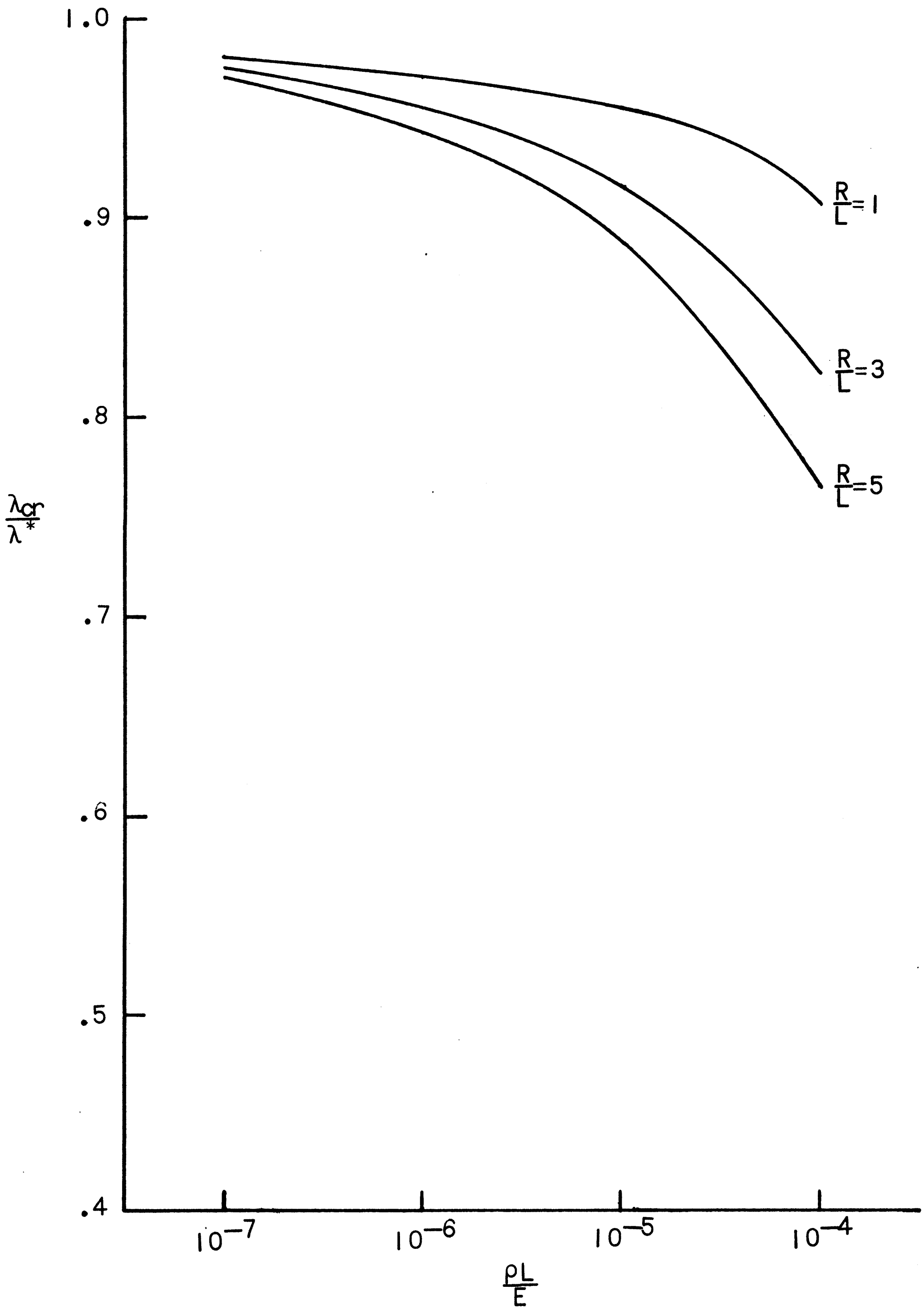

Figure 6.- Ratio of the critical values of the thickness parameters λ_{cr} and λ^* versus $\rho L/E$ with $k = 0$.

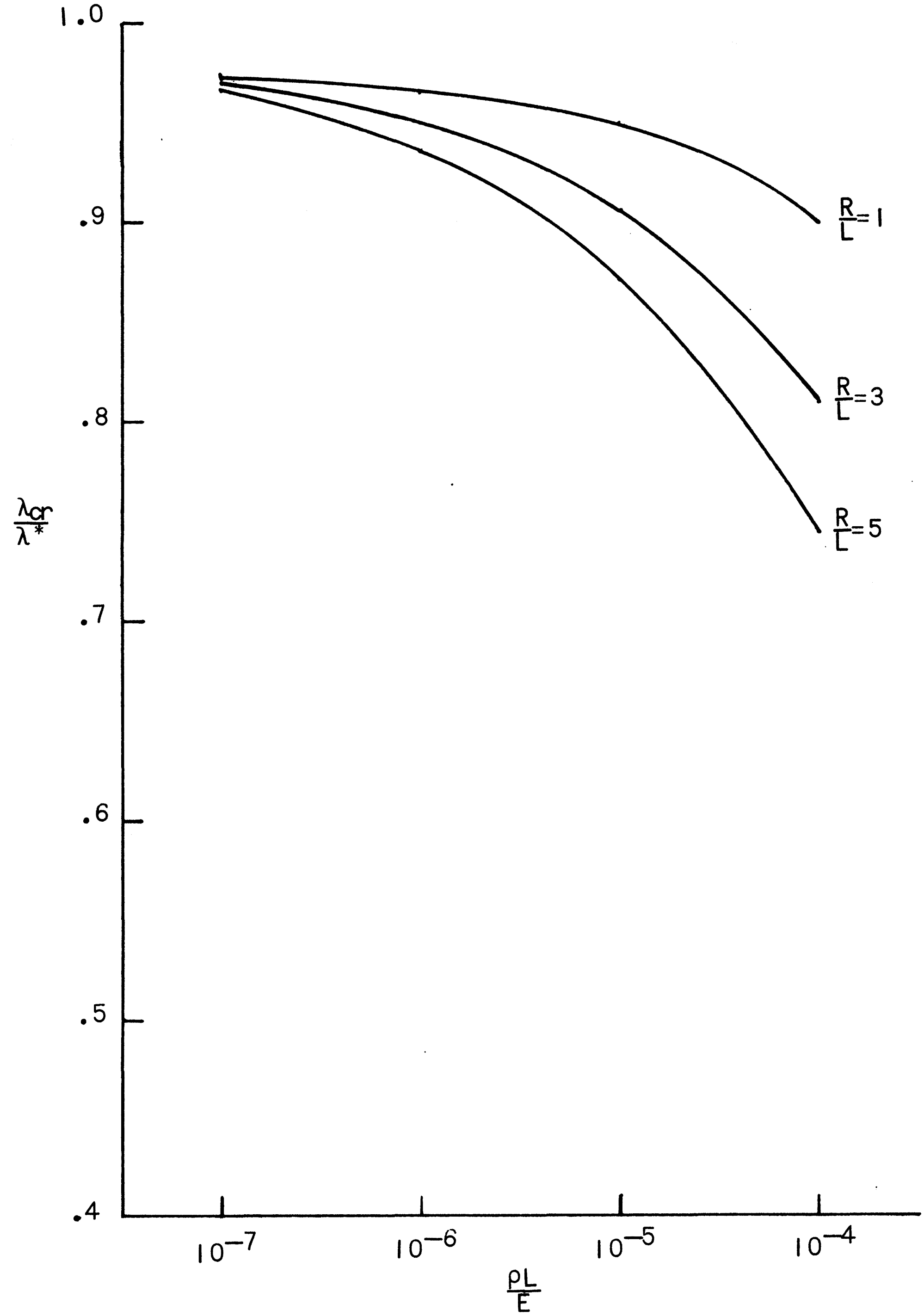

Figure 7.- Ratio of the critical values of the thickness parameters λ_{cr} and λ^* versus $\rho L/E$ with $k = 1/4$.

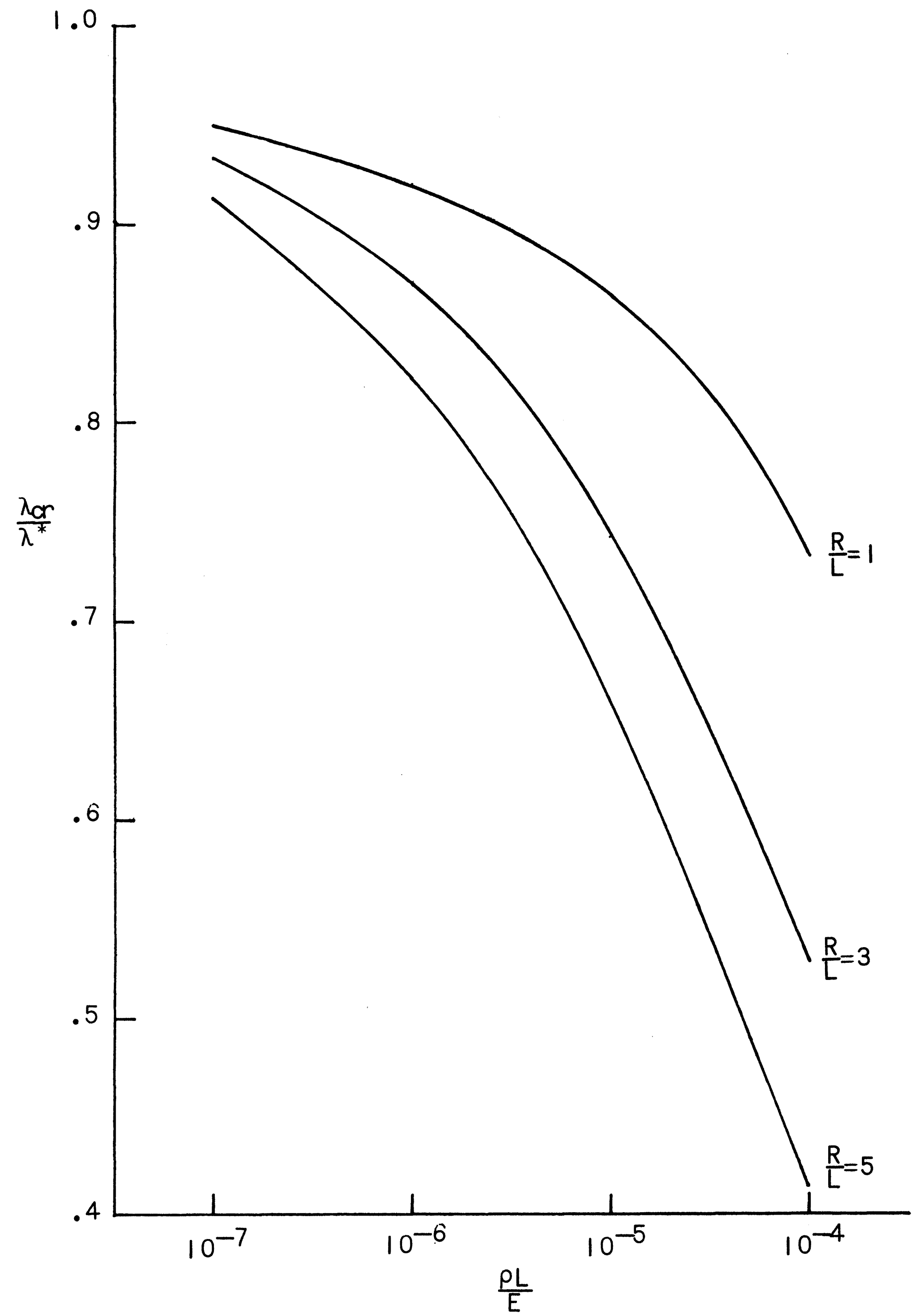

Figure 8.- Ratio of the critical values of the thickness parameters λ_{cr} and λ^* versus $\rho L/E$ with $k = 1/2$.

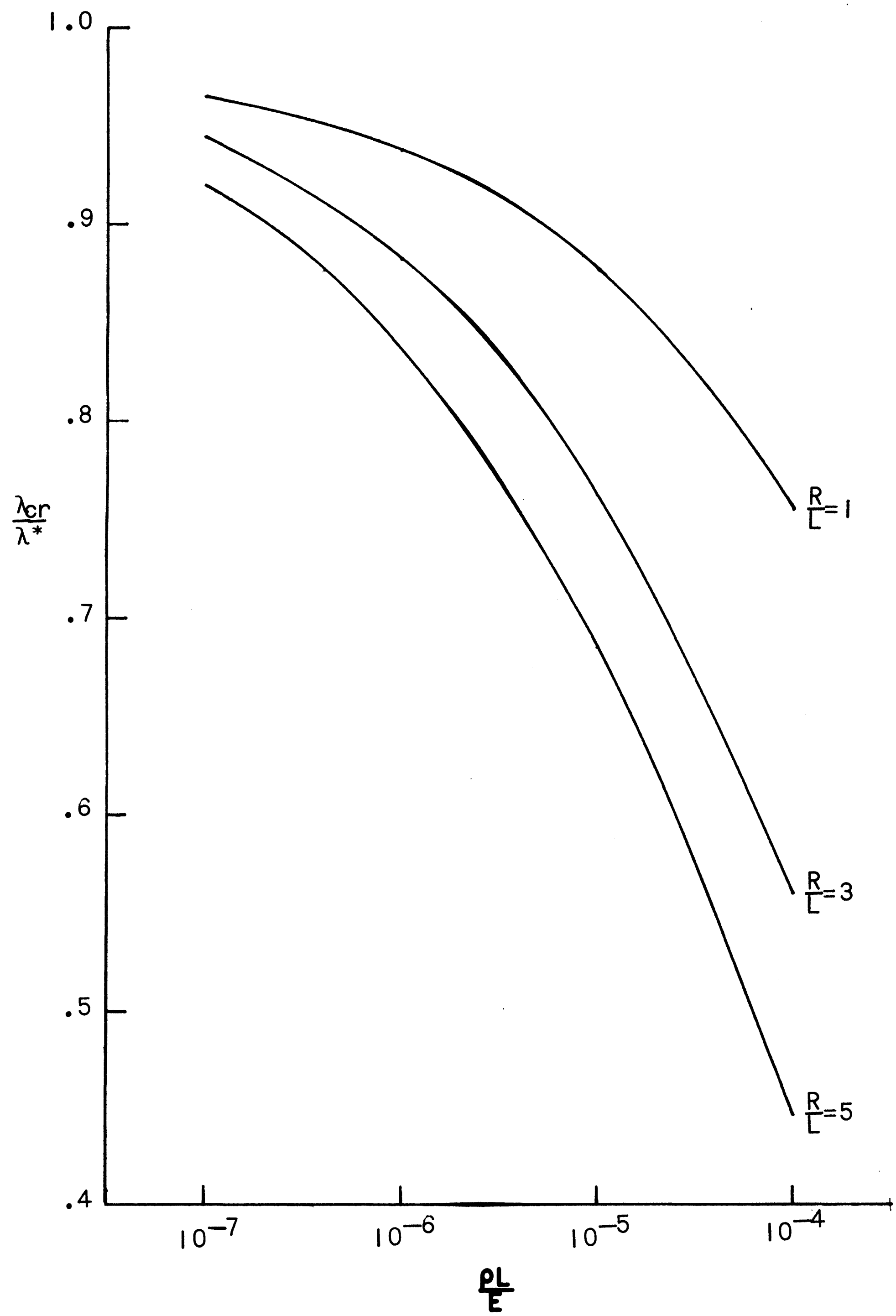

Figure 9.- Ratio of the critical values of the thickness parameters λ_{cr} and λ^* versus $\rho L/E$ with $k = 1$.

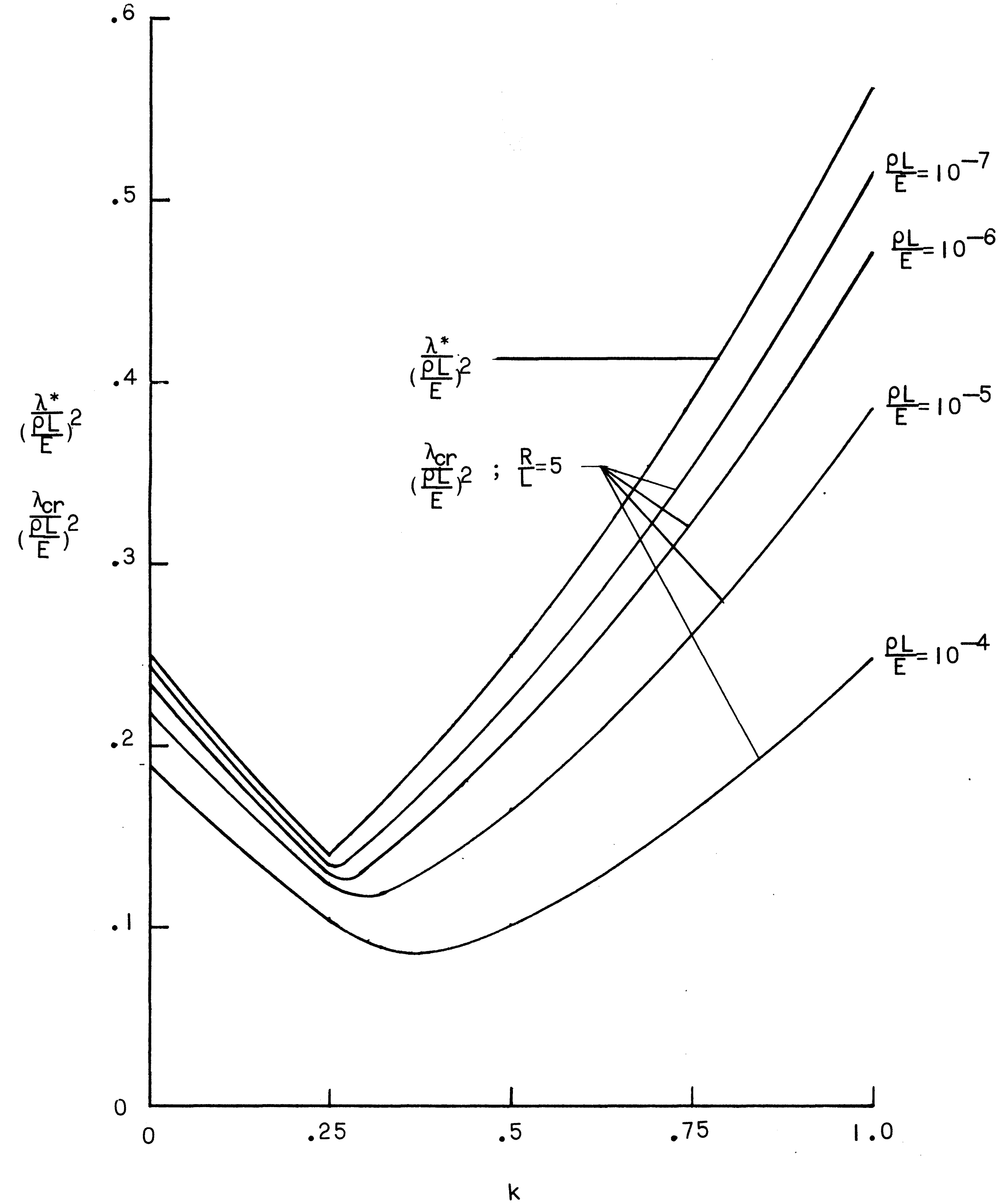

Figure 10.- Critical values of the thickness parameters λ_{cr} and λ^* versus the support condition, k.

BUCKLING OF AN EQUATORIAL SEGMENT OF A SPHERICAL SHELL LOADED BY ITS OWN WEIGHT

By

Robert E. Blum

ABSTRACT

Nonlinear shallow shell equations are derived for a thin shell of revolution having the shape of a narrow segment of a toroidal shell centered at the equator. The equations are derived by considering a cylindrical shell, described by nonlinear Donnell theory, with an initial radial deformation. Linear buckling equations are obtained by perturbing the nonlinear shell equations. The buckling equations are solved for the case of a simple supported equatorial segment of a spherical shell loaded in the axial direction by its own weight. Plots are presented which compare a critical thickness parameter with the results of an elementary approach. The elementary approach assumes that the shell will buckle if the maximum compressive stress is greater than the critical compressive stress for a complete sphere loaded by uniform external pressure.