

Matter Effect on Neutrino Oscillations from the violation of Universality in Neutrino Neutral Current Interactions

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Abstract

The violation of lepton-flavor-universality in the neutrino- Z interactions can lead to extra matter effects on neutrino oscillations at high energies, beyond that due to the usual charged-current interaction of the electron-neutrino. We show that the dominant effect of the violation is a shift in the effective value of θ_{23} . This is in contrast to the dominant effect of the charged-current interaction which shifts θ_{12} and θ_{13} . The shift in θ_{23} will be difficult to observe if the value of $\sin^2(2\theta_{23})$ is too close to one. However, if the value of $\sin^2(2\theta_{23})$ is as small as 0.92, then a Fermilab \rightarrow Hyper-Kamiokande experiment can potentially place a constraint on universality violation at the 1% level after 5 years of data taking.

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I. INTRODUCTION

When considering matter effects on neutrino oscillations, it is customary to consider only the charged current interaction of the electron-neutrino mediated by W -exchange, and ignore the neutral current interactions of all three neutrino flavors mediated by Z -exchange. This is because the universality of the neutral current interaction ensures that the phases acquired by the three neutrino flavors through Z -exchange remain the same, and thereby do not lead to extra mixing effects beyond that due to W -exchange.

However, in many models beyond the Standard Model (SM), the universality of the $Z\nu_\ell\nu_\ell$ ($\ell = e, \mu, \tau$) couplings can be violated through radiative corrections, such as in Supersymmetric models with R-parity violating interactions [1], or through the mixing of the light active neutrinos with heavy sterile ones [2]. The existence of a Z' which couples to the three lepton flavors differently can also mimic the violation of universality in Z -exchange [3]. Though the violation of $Z\nu_\ell\nu_\ell$ coupling universality in the particular models considered in Refs. [1], [2], and [3] are strongly constrained by the universality of the $W\ell\nu_\ell$ and $Z\ell\ell$ couplings, they nevertheless provide existence proofs that the universality of neutral current interactions cannot be taken for granted.

The experimental bound on the violation of $Z\nu_\ell\nu_\ell$ coupling universality is also very weak. The sole constraint comes from CHARM and CHARM II [4, 5]:

$$\begin{aligned} g^{\nu_e} &= 0.528 \pm 0.085, \\ g^{\nu_\mu} &= 0.502 \pm 0.017, \\ g^{\nu_e}/g^{\nu_\mu} &= 1.05^{+0.15}_{-0.18} = 0.87 \sim 1.20, \end{aligned} \tag{1}$$

where g^{ν_ℓ} is the coupling of neutrino flavor ν_ℓ to the Z , normalized to 0.5 for the SM. These values were obtained from the measurements of the ratio R_μ and the double ratio R_e/R_μ , where

$$R_\ell \equiv \frac{\sigma(\nu_\ell N \rightarrow \nu_\ell X)}{\sigma(\nu_\ell N \rightarrow \ell^- X)}. \tag{2}$$

The constraint on g^{ν_μ} was obtained from R_μ , and the constraint on the ratio g^{ν_e}/g^{ν_μ} was obtained from the double ratio R_e/R_μ assuming charged current universality. The constraint on g^{ν_e} was obtained from those on g^{ν_μ} and g^{ν_e}/g^{ν_μ} .

As we can see from the above numbers, while g^{ν_μ} is fairly well constrained to the SM value of 0.5, g^{ν_e} is ill constrained and can deviate significantly from 0.5. Of course, the sum

of squares of the $Z\nu_\ell\nu_\ell$ couplings, namely

$$(g^{\nu_e})^2 + (g^{\nu_\mu})^2 + (g^{\nu_\tau})^2, \quad (3)$$

is well constrained to its SM value by the Z invisible width measured by LEP and SLD [6], so any deviation in g^{ν_e} must be accompanied by a corresponding deviation in g^{ν_τ} to maintain this agreement. However, as long as g^{ν_e} and g^{ν_τ} conspire to do so, large violations of universality are allowed.

In this paper, we investigate the effect of such violations of neutral current universality on neutrino oscillations in matter. If the violation is as large as that allowed by CHARM and CHARM II, then it could lead to new effects that are measurable by long baseline neutrino oscillation experiments. If such effects are not seen, it could then improve upon the CHARM/CHARM II universality constraint.

This paper is organized as follows: In section II, we derive the effective potentials due to the charged- and neutral-current interactions which enter the effective Hamiltonians that govern neutrino and anti-neutrino propagation in matter. In sections III and IV, we approximately diagonalize the effective Hamiltonians for neutrino (III) and anti-neutrino (IV) propagation using the method of Ref. [7], and show how the effective mass-squared differences and effective mixing angles are affected by the presence of neutral current universality violation. In particular, we will show that the effective mass-squared differences are little affected, while the shifts in the effective mixing angles are confined to just one angle; which angle this is depending on the mass hierarchy, and on whether the neutrino or anti-neutrino case is being considered. In section V, we discuss how these shifts in the effective mixing angles will manifest themselves in the neutrino and anti-neutrino oscillation probabilities, and point out that whether any effect can be seen or not depends crucially on the value of $\sin^2(2\theta_{23})$ in vacuum. In section VI, we present the results of a numerical calculation of the effective mass-squared differences, effective mixing angles, and oscillation probabilities, which validate the approximations used in the previous sections. In section VII, we consider a hypothetical experiment in which the Fermilab NUMI beam [8, 9] in its high-energy mode is aimed at a 1 Megaton class detector 9120 km away at Kamioka, Japan (the planned Hyper-Kamiokande [10]) and discuss the potential constraint such an experiment can place on neutral current universality violation. Section VIII concludes.

II. THE EFFECTIVE POTENTIALS DUE TO W AND Z EXCHANGE

Let us first derive the effective potentials for neutrino propagation in matter, which account for the W - and Z -exchange interactions between the neutrino and the matter fermions. The effective potential due to W -exchange is well known [11], but we will re-derive it in the following to provide a parallel to the Z -exchange case.

At momentum transfers much lower than the W and Z masses, the weak interaction Hamiltonian of the neutrinos is given by

$$H_{\text{weak}} = H_{CC} + H_{NC} , \quad (4)$$

where H_{CC} and H_{NC} are the charged and neutral current contributions, respectively:

$$\begin{aligned} H_{CC} &= \frac{G_F}{\sqrt{2}} [\bar{\nu}_\ell \gamma^\mu (1 - \gamma_5) \ell] [\bar{\ell} \gamma_\mu (1 - \gamma_5) \nu_\ell] , \\ H_{NC} &= \rho \frac{G_F}{\sqrt{2}} [\bar{\nu}_\ell \gamma^\mu (1 - \gamma_5) \nu_\ell] \left[\bar{f} \gamma_\mu \left(g_V^f - g_A^f \gamma_5 \right) f \right] . \end{aligned} \quad (5)$$

Here, ℓ is the lepton flavor ($\ell = e, \mu, \tau$); f denotes a generic fermion, and g_V^f and g_A^f are its vector, and axial-vector couplings to the Z :

$$\begin{aligned} g_V^f &= I_3^f - 2Q^f \sin^2 \theta_W , \\ g_A^f &= I_3^f . \end{aligned} \quad (6)$$

I_3^f and Q^f are, respectively, the isospin and electric charge of the fermion f . If lepton-flavor-universality is violated in the neutral current interaction, the product ρG_F in the expression for H_{NC} will depend on ℓ .

After a Fierz transformation, H_{CC} can be rewritten as

$$H_{CC} = \frac{G_F}{\sqrt{2}} [\bar{\ell} \gamma^\mu (1 - \gamma_5) \ell] [\bar{\nu}_\ell \gamma_\mu (1 - \gamma_5) \nu_\ell] . \quad (7)$$

The forward scattering amplitude of a neutrino ν_ℓ against a non-relativistic lepton ℓ via W -exchange is then

$$\begin{aligned} \mathcal{M}_{CC} &= \frac{G_F}{\sqrt{2}} \langle \ell | \bar{\ell} \gamma^0 (1 - \gamma_5) \ell | \ell \rangle \langle \nu_\ell | \bar{\nu}_\ell \gamma_0 (1 - \gamma_5) \nu_\ell | \nu_\ell \rangle \\ &= \sqrt{2} G_F \langle \ell | \ell^\dagger \ell | \ell \rangle \langle \nu_\ell | \nu_\ell^\dagger \left(\frac{1 - \gamma_5}{2} \right) \nu_\ell | \nu_\ell \rangle \\ &= \sqrt{2} G_F N_\ell (\phi_{\nu_\ell}^\dagger \phi_{\nu_\ell}) , \end{aligned} \quad (8)$$

where $N_\ell \equiv \langle \ell | \ell^\dagger \ell | \ell \rangle$ is the density of the charged lepton ℓ , and ϕ_{ν_ℓ} is the two-component wave-function of the left-handed neutrino ν_ℓ . This shows that the effective potential that the neutrino experiences as it travels through matter is

$$V_{CC} = \sqrt{2} G_F N_\ell . \quad (9)$$

In ordinary matter, $N_\mu = N_\tau = 0$. Therefore,

$$V_{CC} = \begin{cases} \sqrt{2} G_F N_e & (\text{for } \nu_e), \\ 0 & (\text{for } \nu_\mu, \nu_\tau). \end{cases} \quad (10)$$

Similarly, the forward scattering amplitude due to Z -exchange between a neutrino and a non-relativistic fermion f is given by

$$\begin{aligned} \mathcal{M}_{NC} &= \frac{G_F}{\sqrt{2}} \langle f | \bar{f} \gamma^0 (g_V^f - g_A^f \gamma_5) f | f \rangle \langle \nu_\ell | \bar{\nu}_\ell \gamma_0 (1 - \gamma_5) \nu_\ell | \nu_\ell \rangle \\ &= \sqrt{2} G_F \langle f | g_V^f (f^\dagger f) | f \rangle \langle \nu_\ell | \nu_\ell^\dagger \left(\frac{1 - \gamma_5}{2} \right) \nu_\ell | \nu_\ell \rangle \\ &= \sqrt{2} G_F g_V^f N_f (\phi_{\nu_\ell}^\dagger \phi_{\nu_\ell}) , \end{aligned} \quad (11)$$

where we have set the ρ -parameter to one, and $N_f = \langle f | f^\dagger f | f \rangle$ is the density of the fermion f . The effective potential due to the neutral current interaction is then

$$V_{NC} = \sqrt{2} G_F g_V^f N_f = \sqrt{2} G_F \left(I_3^f - 2Q^f \sin^2 \theta_W \right) N_f . \quad (12)$$

Since $N_e = N_p$ in electrically neutral matter, we find

$$\begin{aligned} V_{NC} &= \sqrt{2} G_F \left[\left(-\frac{1}{2} + 2s_W^2 \right) N_e + \left(\frac{1}{2} - 2s_W^2 \right) N_p + \left(-\frac{1}{2} \right) N_n \right] \\ &= -\frac{1}{2} \left(\sqrt{2} G_F N_n \right) . \end{aligned} \quad (13)$$

Assuming $N \equiv N_e = N_p \approx N_n$, which is valid for the lighter nuclei which constitutes most of the Earth, we can relate N (cm^{-3}) to the matter density ρ (g/cm^3) via the Avogadro number N_A :

$$2N = N_p + N_n = \rho N_A . \quad (14)$$

Then,

$$\sqrt{2} G_F N = \sqrt{2} \frac{G_F}{(\hbar c)^3} (\hbar c)^3 \frac{N_A}{2} \rho$$

$$= \frac{1}{2} (7.6324 \times 10^{-5} \text{ eV}^2) \times \left(\frac{\rho}{\text{g/cm}^3} \right) \times \left(\frac{1}{\text{GeV}} \right), \quad (15)$$

where we have used $G_F/(\hbar c)^3 = 1.16637(1) \times 10^{-5} \text{ GeV}^{-2}$, $N_A = 6.0221415(10) \times 10^{23} \text{ mol}^{-1}$, and $\hbar c = 0.197326968(17) \text{ GeV} \cdot \text{fm}$ [5]. Therefore,

$$\begin{aligned} a &\equiv 2EV_{CC} = (7.6324 \times 10^{-5} \text{ eV}^2) \times \left(\frac{\rho}{\text{g/cm}^3} \right) \times \left(\frac{E}{\text{GeV}} \right), \\ b &\equiv 2EV_{NC} = -\frac{1}{2} a. \end{aligned} \quad (16)$$

For anti-neutrinos, both V_{CC} and V_{NC} reverse their signs.

III. THE EFFECTIVE MIXING ANGLES, NEUTRINO CASE

A. Inclusion of Neutral Current Effects into the Effective Hamiltonian

The effective potentials derived above enter the effective Hamiltonian for neutrino oscillations (multiplied by $2E$) as follows:

$$H = \tilde{U} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tilde{U}^\dagger = U \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta m_{21}^2 & 0 \\ 0 & 0 & \delta m_{31}^2 \end{bmatrix} U^\dagger + \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} b_e & 0 & 0 \\ 0 & b_\mu & 0 \\ 0 & 0 & b_\tau \end{bmatrix}. \quad (17)$$

Here, U is the MNS matrix in vacuum [12], a comes from the W -exchange interaction of ν_e with the electrons in matter, while b_e , b_μ , and b_τ come from the Z -exchange interaction of each neutrino flavor with the neutrons. If $b_e = b_\mu = b_\tau = b$, then the b -matrix is proportional to the unit matrix, and it will not contribute to neutrino oscillations. However, if neutral current universality is broken, then $b_e \neq b_\mu \neq b_\tau$ in general and the b -matrix cannot be ignored.

The experimental constraints from CHARM/CHARM II, Eq. (1), allow b_e and b_τ to deviate significantly from $b = -a/2$, provided that $b_e + b_\tau = 2b$ to satisfy the Z invisible width constraint, Eq. (3). We therefore write

$$\frac{b_e}{b} = 1 + 2\xi, \quad \frac{b_\mu}{b} = 1, \quad \frac{b_\tau}{b} = 1 - 2\xi, \quad (18)$$

and use ξ to parametrize the violation of universality.

Notice that we can rewrite the matter-effect terms in Eq. (17) in several different ways:

$$\begin{aligned}
& \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} b_e & 0 & 0 \\ 0 & b_\mu & 0 \\ 0 & 0 & b_\tau \end{bmatrix} \\
&= \begin{bmatrix} (a + b_e - b_\tau) & 0 & 0 \\ 0 & (b_\mu - b_\tau) & 0 \\ 0 & 0 & 0 \end{bmatrix} + b_\tau \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \left(a + b_e - \frac{b_\mu + b_\tau}{2}\right) & 0 & 0 \\ 0 & \left(\frac{b_\mu - b_\tau}{2}\right) & 0 \\ 0 & 0 & -\left(\frac{b_\mu - b_\tau}{2}\right) \end{bmatrix} + \left(\frac{b_\mu + b_\tau}{2}\right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (19)
\end{aligned}$$

Since the unit matrix terms can be dropped, this shows that we can always reduce the problem to the case $b_\tau = 0$, or $b_\mu = -b_\tau$. We will use the latter replacement in the following.

For the case of Eq. (18), this entails making the replacement

$$\begin{bmatrix} a + b(1 + 2\xi) & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b(1 - 2\xi) \end{bmatrix} \Rightarrow \begin{bmatrix} (a + 3b\xi) & 0 & 0 \\ 0 & b\xi & 0 \\ 0 & 0 & -b\xi \end{bmatrix}. \quad (20)$$

Furthermore, we absorb the factor $3b\xi$ in the (1,1) element into a since we can expect $3b\xi \ll a$, and the uncertainty in the matter density ρ which enters into a can be expected to hide any such shift. Therefore, the effective Hamiltonian we will consider is

$$H = \tilde{U} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tilde{U}^\dagger = U \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta m_{21}^2 & 0 \\ 0 & 0 & \delta m_{31}^2 \end{bmatrix} U^\dagger + \begin{bmatrix} a & 0 & 0 \\ 0 & b\xi & 0 \\ 0 & 0 & -b\xi \end{bmatrix}. \quad (21)$$

The problem is to diagonalize H and find the eigenvalues λ_i ($i = 1, 2, 3$) and the diagonalization matrix \tilde{U} .

To this end, we use the method of Ref. [7] in which the λ_i 's and \tilde{U} were derived for the $\xi = 0$ case. The procedure followed in Ref. [7] was to approximately diagonalize the effective Hamiltonian, H , using the Jacobi method: 2×2 submatrices of H are diagonalized in the order which requires the the largest rotation angles until the off-diagonal elements are

negligibly small. As the order parameter to evaluate the size of these off-diagonal elements, we use

$$\varepsilon \equiv \sqrt{\frac{\delta m_{21}^2}{|\delta m_{31}^2|}}, \quad (22)$$

and consider H to be approximately diagonalized when the rotation angles required for further diagonalization are of order ε^3 or smaller. For $\delta m_{21}^2 = 8.2_{-0.5}^{+0.6} \times 10^{-5} \text{ eV}^2$ and $|\delta m_{31}^2| = (1.5 \sim 3.4) \times 10^{-3} \text{ eV}^2$, we have $\varepsilon = 0.15 \sim 0.24$ and $\varepsilon^3 = 0.0034 \sim 0.014$.

For the sizes of the mixing angles in vacuum, we assume $\theta_{13} = O(\varepsilon)$, $\cos(2\theta_{12})/2 = O(\varepsilon)$, and $\cos(2\theta_{23}) \leq O(\varepsilon)$ as in Ref. [7]. We also assume that the universality violation parameter ξ is of order $\varepsilon^2 = 0.02 \sim 0.06$, since the central value of the CHARM/CHARM II result translates to $\xi = 0.025$.

B. Diagonalization of the Effective Hamiltonian

For the neutral current term $b\xi$ in Eq. (21) to have a non-negligible effect on neutrino oscillations, we anticipate that it must be at least as large as, or larger than, the smaller mass-squared-difference δm_{21}^2 . Since we have assumed $\xi = O(\varepsilon^2)$, this requires $a = -2b = 2\sqrt{2}G_F N E$ to be at least as large as, or larger than, the larger mass-squared-difference $|\delta m_{31}^2|$. For the sake of concreteness, we will consider the case $a/|\delta m_{31}^2| = O(\varepsilon^{-1})$ in the following.

Introducing the matrix

$$\mathcal{Q} = \text{diag}(1, 1, e^{i\delta}), \quad (23)$$

we begin by partially diagonalizing the Hamiltonian H as

$$\begin{aligned} H' &= \mathcal{Q}^\dagger U^\dagger H U \mathcal{Q} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta m_{21}^2 & 0 \\ 0 & 0 & \delta m_{31}^2 \end{bmatrix} + \mathcal{Q}^\dagger U^\dagger \begin{bmatrix} a & 0 & 0 \\ 0 & b\xi & 0 \\ 0 & 0 & -b\xi \end{bmatrix} U \mathcal{Q} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta m_{21}^2 & 0 \\ 0 & 0 & \delta m_{31}^2 \end{bmatrix} + a \mathcal{Q}^\dagger \begin{bmatrix} U_{e1}^* U_{e1} & U_{e1}^* U_{e2} & U_{e1}^* U_{e3} \\ U_{e2}^* U_{e1} & U_{e2}^* U_{e2} & U_{e2}^* U_{e3} \\ U_{e3}^* U_{e1} & U_{e3}^* U_{e2} & U_{e3}^* U_{e3} \end{bmatrix} \mathcal{Q} \end{aligned}$$

$$+b\xi \mathcal{Q}^\dagger \left\{ \begin{array}{c} \left[\begin{array}{ccc} U_{\mu 1}^* U_{\mu 1} & U_{\mu 1}^* U_{\mu 2} & U_{\mu 1}^* U_{\mu 3} \\ U_{\mu 2}^* U_{\mu 1} & U_{\mu 2}^* U_{\mu 2} & U_{\mu 2}^* U_{\mu 3} \\ U_{\mu 3}^* U_{\mu 1} & U_{\mu 3}^* U_{\mu 2} & U_{\mu 3}^* U_{\mu 3} \end{array} \right] - \left[\begin{array}{ccc} U_{\tau 1}^* U_{\tau 1} & U_{\tau 1}^* U_{\tau 2} & U_{\tau 1}^* U_{\tau 3} \\ U_{\tau 2}^* U_{\tau 1} & U_{\tau 2}^* U_{\tau 2} & U_{\tau 2}^* U_{\tau 3} \\ U_{\tau 3}^* U_{\tau 1} & U_{\tau 3}^* U_{\tau 2} & U_{\tau 3}^* U_{\tau 3} \end{array} \right] \end{array} \right\} \mathcal{Q}. \quad (24)$$

The matrix multiplying a is given by

$$M_a = \mathcal{Q}^\dagger \begin{bmatrix} U_{e1}^* U_{e1} & U_{e1}^* U_{e2} & U_{e1}^* U_{e3} \\ U_{e2}^* U_{e1} & U_{e2}^* U_{e2} & U_{e2}^* U_{e3} \\ U_{e3}^* U_{e1} & U_{e3}^* U_{e2} & U_{e3}^* U_{e3} \end{bmatrix} \mathcal{Q} = \begin{bmatrix} c_{12}^2 c_{13}^2 & c_{12} s_{12} c_{13}^2 & c_{12} c_{13} s_{13} \\ c_{12} s_{12} c_{13}^2 & s_{12}^2 c_{13}^2 & s_{12} c_{13} s_{13} \\ c_{12} c_{13} s_{13} & s_{12} c_{13} s_{13} & s_{13}^2 \end{bmatrix}. \quad (25)$$

Using $\theta_{13} = O(\varepsilon)$, we estimate the sizes of the elements of M_a to be

$$M_a = \begin{bmatrix} O(1) & O(1) & O(\varepsilon) \\ O(1) & O(1) & O(\varepsilon) \\ O(\varepsilon) & O(\varepsilon) & O(\varepsilon^2) \end{bmatrix}. \quad (26)$$

The matrix multiplying $b\xi$ is given by

$$M_b = \mathcal{Q}^\dagger \left\{ \begin{array}{c} \left[\begin{array}{ccc} U_{\mu 1}^* U_{\mu 1} & U_{\mu 1}^* U_{\mu 2} & U_{\mu 1}^* U_{\mu 3} \\ U_{\mu 2}^* U_{\mu 1} & U_{\mu 2}^* U_{\mu 2} & U_{\mu 2}^* U_{\mu 3} \\ U_{\mu 3}^* U_{\mu 1} & U_{\mu 3}^* U_{\mu 2} & U_{\mu 3}^* U_{\mu 3} \end{array} \right] - \left[\begin{array}{ccc} U_{\tau 1}^* U_{\tau 1} & U_{\tau 1}^* U_{\tau 2} & U_{\tau 1}^* U_{\tau 3} \\ U_{\tau 2}^* U_{\tau 1} & U_{\tau 2}^* U_{\tau 2} & U_{\tau 2}^* U_{\tau 3} \\ U_{\tau 3}^* U_{\tau 1} & U_{\tau 3}^* U_{\tau 2} & U_{\tau 3}^* U_{\tau 3} \end{array} \right] \end{array} \right\} \mathcal{Q} \\ = \begin{bmatrix} (s_{12}^2 - c_{12}^2 s_{13}^2) \cos(2\theta_{23}) + \sin(2\theta_{12}) \sin(2\theta_{23}) s_{13} \cos \delta \\ -(1 + s_{13}^2) s_{12} c_{12} \cos(2\theta_{23}) - (c_{12}^2 e^{-i\delta} - s_{12}^2 e^{+i\delta}) s_{13} \sin(2\theta_{23}) \\ -s_{12} c_{13} \sin(2\theta_{23}) e^{-i\delta} + c_{12} s_{13} c_{13} \cos(2\theta_{23}) \\ -(1 + s_{13}^2) s_{12} c_{12} \cos(2\theta_{23}) - (c_{12}^2 e^{+i\delta} - s_{12}^2 e^{-i\delta}) s_{13} \sin(2\theta_{23}) \\ (c_{12}^2 - s_{12}^2 s_{13}^2) \cos(2\theta_{23}) - \sin(2\theta_{12}) \sin(2\theta_{23}) s_{13} \cos \delta \\ c_{12} c_{13} \sin(2\theta_{23}) e^{-i\delta} + s_{12} s_{13} c_{13} \cos(2\theta_{23}) \\ -s_{12} c_{13} \sin(2\theta_{23}) e^{+i\delta} + c_{12} s_{13} c_{13} \cos(2\theta_{23}) \\ c_{12} c_{13} \sin(2\theta_{23}) e^{+i\delta} + s_{12} s_{13} c_{13} \cos(2\theta_{23}) \\ -c_{13}^2 \cos(2\theta_{23}) \end{bmatrix}. \quad (27)$$

Using $\cos(2\theta_{23}) \leq O(\varepsilon)$ and $\theta_{13} = O(\varepsilon)$, we estimate the sizes of the elements of M_b to be

$$M_b = \begin{bmatrix} O(\varepsilon) & O(\varepsilon) & O(1) \\ O(\varepsilon) & O(\varepsilon) & O(1) \\ O(1) & O(1) & O(\varepsilon) \end{bmatrix}. \quad (28)$$

Since we are only interested in the leading order effect in ξ , we neglect the $O(\varepsilon)$ terms in M_b and approximate

$$M_b \approx \begin{bmatrix} 0 & 0 & -s_{12}e^{+i\delta} \\ 0 & 0 & c_{12}e^{+i\delta} \\ -s_{12}e^{-i\delta} & c_{12}e^{-i\delta} & 0 \end{bmatrix}. \quad (29)$$

Under this approximation, the effective Hamiltonian that must be diagonalized is

$$\begin{aligned} H' &= \text{diag}(0, \delta m_{21}^2, \delta m_{31}^2) + a M_a + b\xi M_b \\ &= \begin{bmatrix} a c_{12}^2 c_{13}^2 & a c_{12} s_{12} c_{13}^2 & a c_{12} c_{13} s_{13} - b\xi s_{12} e^{+i\delta} \\ a c_{12} s_{12} c_{13}^2 & a s_{12}^2 c_{13}^2 + \delta m_{21}^2 & a s_{12} c_{13} s_{13} + b\xi c_{12} e^{+i\delta} \\ a c_{12} c_{13} s_{13} - b\xi s_{12} e^{-i\delta} & a s_{12} c_{13} s_{13} + b\xi c_{12} e^{-i\delta} & a s_{13}^2 + \delta m_{31}^2 \end{bmatrix}. \end{aligned} \quad (30)$$

At this point, we set $\delta = 0$ for the sake of simplicity. Then H' becomes

$$H' = \begin{bmatrix} a c_{12}^2 c_{13}^2 & a c_{12} s_{12} c_{13}^2 & a c_{12} c_{13} s_{13} - b\xi s_{12} \\ a c_{12} s_{12} c_{13}^2 & a s_{12}^2 c_{13}^2 + \delta m_{21}^2 & a s_{12} c_{13} s_{13} + b\xi c_{12} \\ a c_{12} c_{13} s_{13} - b\xi s_{12} & a s_{12} c_{13} s_{13} + b\xi c_{12} & a s_{13}^2 + \delta m_{31}^2 \end{bmatrix} = a \begin{bmatrix} O(1) & O(1) & O(\varepsilon) \\ O(1) & O(1) & O(\varepsilon) \\ O(\varepsilon) & O(\varepsilon) & O(\varepsilon) \end{bmatrix}. \quad (31)$$

B1. First Rotation

Applying the Jacobi method to H' , we first diagonalize the (1, 2) submatrix which requires a rotation by an angle of $O(1)$. Define the matrix V as:

$$V = \begin{bmatrix} c_\varphi & s_\varphi & 0 \\ -s_\varphi & c_\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (32)$$

where

$$c_\varphi = \cos \varphi, \quad s_\varphi = \sin \varphi, \quad \tan 2\varphi \equiv \frac{ac_{13}^2 \sin 2\theta_{12}}{\delta m_{21}^2 - ac_{13}^2 \cos 2\theta_{12}}, \quad \left(0 \leq \varphi < \frac{\pi}{2} - \theta_{12}\right). \quad (33)$$

Then,

$$H'' = V^\dagger H' V = \begin{bmatrix} \lambda'_1 & 0 & ac'_{12} c_{13} s_{13} - b\xi s'_{12} \\ 0 & \lambda'_2 & as'_{12} c_{13} s_{13} + b\xi c'_{12} \\ ac'_{12} c_{13} s_{13} - b\xi s'_{12} & as'_{12} c_{13} s_{13} + b\xi c'_{12} & as_{13}^2 + \delta m_{31}^2 \end{bmatrix}, \quad (34)$$

where

$$c'_{12} = \cos \theta'_{12}, \quad s'_{12} = \sin \theta'_{12}, \quad \theta'_{12} = \theta_{12} + \varphi, \quad (35)$$

and

$$\begin{aligned} \lambda'_1 &= \frac{(ac_{12}^2 c_{13}^2) c_\varphi^2 - (as_{12}^2 c_{13}^2 + \delta m_{21}^2) s_\varphi^2}{c_\varphi^2 - s_\varphi^2} = \lambda'_-, \\ \lambda'_2 &= \frac{(as_{12}^2 c_{13}^2 + \delta m_{21}^2) c_\varphi^2 - (ac_{12}^2 c_{13}^2) s_\varphi^2}{c_\varphi^2 - s_\varphi^2} = \lambda'_+, \end{aligned} \quad (36)$$

with

$$\lambda'_\pm = \frac{(ac_{13}^2 + \delta m_{21}^2) \pm \sqrt{(ac_{13}^2 - \delta m_{21}^2)^2 + 4ac_{13}^2 s_{12}^2 \delta m_{21}^2}}{2}. \quad (37)$$

As discussed in Ref. [7], in the region $a/|\delta m_{31}^2| = O(\varepsilon^{-1})$, we can expand θ'_{12} as

$$\theta'_{12} = \frac{\pi}{2} - \frac{\delta m_{21}^2}{2a} \sin(2\theta_{12}) + O(\varepsilon^5), \quad (38)$$

from which we can conclude

$$\begin{aligned} s'_{12} &\approx \sin\left(\frac{\pi}{2} - \frac{\delta m_{21}^2}{2a} \sin(2\theta_{12})\right) = \cos\left(\frac{\delta m_{21}^2}{2a} \sin(2\theta_{12})\right) = 1 - O(\varepsilon^6), \\ c'_{12} &\approx \cos\left(\frac{\pi}{2} - \frac{\delta m_{21}^2}{2a} \sin(2\theta_{12})\right) = \sin\left(\frac{\delta m_{21}^2}{2a} \sin(2\theta_{12})\right) = O(\varepsilon^3). \end{aligned} \quad (39)$$

Also, expanding λ'_\pm , we find

$$\begin{aligned} \lambda'_- &= \delta m_{21}^2 c_{12}^2 + a O(\varepsilon^6) = a O(\varepsilon^3), \\ \lambda'_+ &= ac_{13}^2 + \delta m_{21}^2 s_{12}^2 + a O(\varepsilon^6) = a O(1). \end{aligned} \quad (40)$$

Therefore, the sizes of the elements of H'' are evaluated to be

$$H'' = a \begin{bmatrix} O(\varepsilon^3) & 0 & O(\varepsilon^2) \\ 0 & O(1) & O(\varepsilon) \\ O(\varepsilon^2) & O(\varepsilon) & O(\varepsilon) \end{bmatrix}. \quad (41)$$

Unlike the $\xi = 0$ case considered in Ref. [7], both the (1, 3) and (2, 3) submatrices require rotations by angles of $O(\varepsilon)$ to diagonalize. Here, we diagonalize the (2, 3) submatrix next to maintain the parallel with the $\xi = 0$ case.

B2. *Second Rotation*

The matrix W which diagonalizes the (2, 3) submatrix is

$$W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\phi & s_\phi \\ 0 & -s_\phi & c_\phi \end{bmatrix}, \quad (42)$$

where $c_\phi = \cos \phi$, $s_\phi = \sin \phi$, and

$$\begin{aligned} \tan 2\phi &\equiv \frac{2(as'_{12}s_{13}c_{13} + b\xi c'_{12})}{\delta m_{31}^2 + as_{13}^2 - \lambda'_2} \\ &= \frac{2as'_{12}s_{13}c_{13}}{\delta m_{31}^2 + as_{13}^2 - \lambda'_2} \left\{ 1 - \xi \left(\frac{c'_{12}}{2s'_{12}s_{13}c_{13}} \right) \right\} \\ &= \tan 2\phi_0 \left\{ 1 - \xi \left(\frac{c'_{12}}{2s'_{12}s_{13}c_{13}} \right) \right\}. \end{aligned} \quad (43)$$

The angle ϕ is in the first quadrant when $\delta m_{31}^2 > 0$ (normal hierarchy), and in the fourth quadrant when $\delta m_{31}^2 < 0$ (inverted hierarchy). ϕ_0 is the rotation angle when $\xi = 0$. Taking the arc-tangent of both sides of the above equation, we find

$$\phi = \phi_0 - \xi \left(\frac{\sin 4\phi_0 \cot \theta'_{12}}{4 \sin 2\theta_{13}} \right) + O(\xi^2). \quad (44)$$

When $a/|\delta m_{31}^2| = O(\varepsilon^{-1})$, ϕ_0 is given by [7]

$$\phi_0 = \begin{cases} \left(\frac{\pi}{2} - \theta_{13} \right) - \frac{\delta m_{31}^2}{a} \theta_{13} + O(\varepsilon^3) & (\delta m_{31}^2 > 0), \\ -\theta_{13} - \frac{\delta m_{31}^2}{a} \theta_{13} + O(\varepsilon^3) & (\delta m_{31}^2 < 0). \end{cases} \quad (45)$$

Therefore, $\sin 4\phi_0 \approx -\sin(4\theta_{13})$ for both the $\delta m_{31}^2 > 0$ and $\delta m_{31}^2 < 0$ cases and using Eq. (38), we find

$$\phi \approx \phi_0 + \xi \left(\frac{\delta m_{21}^2}{4a} \right) \sin(2\theta_{12}) = \phi_0 + \frac{1}{4}O(\varepsilon^5). \quad (46)$$

Therefore, the difference between ϕ and ϕ_0 can be neglected in this range.

Using W , we obtain

$$\begin{aligned} H''' &= W^\dagger H'' W \\ &= \begin{bmatrix} \lambda'_1 & & & \\ -(ac'_{12}c_{13}s_{13} - b\xi s'_{12})s_\phi & \lambda''_2 & & 0 \\ (ac'_{12}c_{13}s_{13} - b\xi s'_{12})c_\phi & & 0 & \lambda''_3 \end{bmatrix}, \end{aligned} \quad (47)$$

where

$$\begin{aligned}\lambda_2'' &= \frac{\lambda_2' c_\phi^2 - (a s_{13}^2 + \delta m_{31}^2) s_\phi^2}{c_\phi^2 - s_\phi^2}, \\ \lambda_3'' &= \frac{(a s_{13}^2 + \delta m_{31}^2) c_\phi^2 - \lambda_2' s_\phi^2}{c_\phi^2 - s_\phi^2}.\end{aligned}\quad (48)$$

If we define

$$\lambda_\pm'' \equiv \frac{[\lambda_2' + (a s_{13}^2 + \delta m_{31}^2)] \pm \sqrt{[\lambda_2' - (a s_{13}^2 + \delta m_{31}^2)]^2 + 4(a s_{12}' c_{13} s_{13} + b \xi c_{12}')^2}}{2}, \quad (49)$$

then

$$\begin{aligned}\lambda_2'' &= \lambda_-'' , & \lambda_3'' &= \lambda_+'' , & \text{if } \delta m_{31}^2 > 0 , \\ \lambda_2'' &= \lambda_+'' , & \lambda_3'' &= \lambda_-'' , & \text{if } \delta m_{31}^2 < 0 .\end{aligned}\quad (50)$$

When $a/|\delta m_{31}^2| = O(\varepsilon^{-1})$, we can expand λ_\pm'' as

$$\begin{aligned}\lambda_-'' &= \delta m_{31}^2 c_{13}^2 + O(\varepsilon^3 |\delta m_{31}^2|) = a O(\varepsilon) , \\ \lambda_+'' &= a + \delta m_{31}^2 s_{13}^2 + \delta m_{21}^2 s_{12}^2 + O(\varepsilon^3 |\delta m_{31}^2|) = a O(1) .\end{aligned}\quad (51)$$

Also, from Eq. (45) and the fact that $\phi \approx \phi_0$, we conclude

$$\begin{aligned}s_\phi \approx c_{13} &= O(1) , & c_\phi \approx s_{13} &= O(\varepsilon) , & (\delta m_{31}^2 > 0) , \\ s_\phi \approx -s_{13} &= O(\varepsilon) , & c_\phi \approx c_{13} &= O(1) , & (\delta m_{31}^2 < 0) .\end{aligned}\quad (52)$$

In the $\xi = 0$ case considered in Ref. [7], H''' was already approximately diagonal and further diagonalization was not necessary. However, when $\xi = O(\varepsilon^2)$, the sizes of the elements of H''' are found to be

$$H''' = a \begin{bmatrix} O(\varepsilon^3) & O(\varepsilon^2) & O(\varepsilon^3) \\ O(\varepsilon^2) & O(\varepsilon) & 0 \\ O(\varepsilon^3) & 0 & O(1) \end{bmatrix} \quad (53)$$

when $\delta m_{31}^2 > 0$, and

$$H''' = a \begin{bmatrix} O(\varepsilon^3) & O(\varepsilon^3) & O(\varepsilon^2) \\ O(\varepsilon^3) & O(1) & 0 \\ O(\varepsilon^2) & 0 & O(\varepsilon) \end{bmatrix} \quad (54)$$

when $\delta m_{31}^2 < 0$. Therefore, for the $\delta m_{31}^2 > 0$ case (normal hierarchy) we must diagonalize the (1, 2) submatrix next, while for the $\delta m_{31}^2 < 0$ case (inverted hierarchy) we must diagonalize the (1, 3) submatrix next.

B3. Third Rotation, $\delta m_{31}^2 > 0$ Case

Define the matrix X as

$$X = \begin{bmatrix} c_\chi & s_\chi & 0 \\ -s_\chi & c_\chi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (55)$$

where

$$c_\chi = \cos \chi, \quad s_\chi = \sin \chi, \quad \tan 2\chi \equiv \frac{-2(ac'_{12}c_{13}s_{13} - b\xi s'_{12})s_\phi}{\lambda''_2 - \lambda'_1}. \quad (56)$$

Then,

$$\begin{aligned} H_X''' &= X^\dagger H''' X \\ &= \begin{bmatrix} \lambda_{1X}''' & 0 & (ac'_{12}c_{13}s_{13} - b\xi s'_{12})c_\phi c_\chi \\ 0 & \lambda_{2X}''' & (ac'_{12}c_{13}s_{13} - b\xi s'_{12})c_\phi s_\chi \\ (ac'_{12}c_{13}s_{13} - b\xi s'_{12})c_\phi c_\chi & (ac'_{12}c_{13}s_{13} - b\xi s'_{12})c_\phi s_\chi & \lambda_3'' \end{bmatrix}, \end{aligned} \quad (57)$$

where

$$\begin{aligned} \lambda_{1X}''' &= \frac{\lambda'_1 c_\chi^2 - \lambda''_2 s_\chi^2}{c_\chi^2 - s_\chi^2} = \lambda_{X-}''', \\ \lambda_{2X}''' &= \frac{\lambda''_2 c_\chi^2 - \lambda'_1 s_\chi^2}{c_\chi^2 - s_\chi^2} = \lambda_{X+}''', \end{aligned} \quad (58)$$

with

$$\lambda_{X\pm}'' \equiv \frac{(\lambda''_2 + \lambda'_1) \pm \sqrt{(\lambda''_2 - \lambda'_1)^2 + 4(ac'_{12}c_{13}s_{13} - b\xi s'_{12})^2 s_\phi^2}}{2}. \quad (59)$$

Recalling that

$$\begin{aligned} ac'_{12}c_{13}s_{13} &\approx a \left(\frac{\delta m_{21}^2}{2a} \sin(2\theta_{12}) \right) c_{13}s_{13} = a O(\varepsilon^4), \\ b\xi s'_{12} &\approx -\frac{a}{2}\xi = -a O(\varepsilon^2), \\ \lambda'_1 &\approx \delta m_{21}^2 c_{12}^2 = a O(\varepsilon^3), \\ \lambda''_2 &\approx \delta m_{31}^2 c_{13}^2 = a O(\varepsilon), \end{aligned} \quad (60)$$

and $s_\phi \approx 1$, we find

$$\tan 2\chi = \frac{-2(ac'_{12}c_{13}s_{13} - b\xi s'_{12})s_\phi}{\lambda''_2 - \lambda'_1} = \frac{2b\xi}{\delta m_{31}^2} \{1 + O(\varepsilon^2)\}. \quad (61)$$

Therefore, the angle χ is given approximately by

$$\chi \approx \frac{b\xi}{\delta m_{31}^2} = O(\varepsilon), \quad (62)$$

from which we can conclude that $s_\chi = O(\varepsilon)$ and $c_\chi = O(1)$. The eigenvalues can also be expanded in ε and we find

$$\begin{aligned} \lambda_{1X}''' &= \delta m_{21}^2 c_{12}^2 - \frac{b^2 \xi^2}{\delta m_{31}^2} + a O(\varepsilon^5) = a O(\varepsilon^3), \\ \lambda_{2X}''' &= \delta m_{31}^2 c_{13}^2 + \frac{b^2 \xi^2}{\delta m_{31}^2} + a O(\varepsilon^4) = a O(\varepsilon). \end{aligned} \quad (63)$$

Note that these shifts of the eigenvalues are of order $aO(\varepsilon^3)$ and have negligible effect on $\delta\lambda_{31} = aO(1)$ or $\delta\lambda_{21} = aO(\varepsilon)$. Putting everything together, we evaluate the sizes of the elements of H_X'''' to find

$$H_X'''' = a \begin{bmatrix} O(\varepsilon^3) & 0 & O(\varepsilon^3) \\ 0 & O(\varepsilon) & O(\varepsilon^4) \\ O(\varepsilon^3) & O(\varepsilon^4) & O(1) \end{bmatrix}. \quad (64)$$

This shows that further diagonalization requires rotations by angles of $O(\varepsilon^3)$ or smaller, which we will neglect.

Thus, we have found that when $\xi = O(\varepsilon^2)$ and $\delta m_{31}^2 > 0$ (normal hierarchy), we need an extra (1,2)-rotation to diagonalize H , and the diagonalization matrix is $UVWX$, which we need to identify with

$$\tilde{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \tilde{c}_{23} & \tilde{s}_{23} \\ 0 & -\tilde{s}_{23} & \tilde{c}_{23} \end{bmatrix} \begin{bmatrix} \tilde{c}_{13} & 0 & \tilde{s}_{13} e^{-i\tilde{\delta}} \\ 0 & 1 & 0 \\ -\tilde{s}_{13} e^{i\tilde{\delta}} & 0 & \tilde{c}_{13} \end{bmatrix} \begin{bmatrix} \tilde{c}_{12} & \tilde{s}_{12} & 0 \\ -\tilde{s}_{12} & \tilde{c}_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (65)$$

to obtain the effective mixing angles and effective CP phase. From Ref. [7], we know that when $\delta m_{31}^2 > 0$ and $a/|\delta m_{31}^2| = O(\varepsilon^{-1})$, identification of \tilde{U} with UVW leads to

$$\begin{aligned} \tilde{\theta}_{13} &= \theta'_{13}, \\ \tilde{\theta}_{12} &= \frac{\pi}{2} - \frac{c_{13}}{c'_{13}} \left(\frac{\delta m_{21}^2}{2a} \right) \sin(2\theta_{12}), \\ \tilde{\theta}_{23} &= \theta_{23} + \frac{s_\phi}{c'_{13}} \left(\frac{\delta m_{21}^2}{2a} \right) \sin(2\theta_{12}), \\ \tilde{\delta} &= 0, \end{aligned} \quad (66)$$

where we have defined

$$\theta'_{13} \equiv \theta_{13} + \phi, \quad c'_{13} = \cos \theta'_{13}, \quad s'_{13} = \sin \theta'_{13}. \quad (67)$$

(Recall that we are considering the $\delta = 0$ case). Since X is an $(1, 2)$ -rotation, multiplication of $\tilde{U} = UVW$ from the right by X only shifts the value of $\tilde{\theta}_{12}$ by χ . Therefore, we can conclude that

$$\begin{aligned}\tilde{\theta}_{13} &= \theta'_{13} , \\ \tilde{\theta}_{12} &= \frac{\pi}{2} - \frac{c_{13}}{c'_{13}} \left(\frac{\delta m_{21}^2}{2a} \right) \sin(2\theta_{12}) + \chi , \\ \tilde{\theta}_{23} &= \theta_{23} + \frac{s_\phi}{c'_{13}} \left(\frac{\delta m_{21}^2}{2a} \right) \sin(2\theta_{12}) , \\ \tilde{\delta} &= 0 .\end{aligned}\tag{68}$$

In these expressions, non only χ , but also ϕ and $\theta'_{13} = \theta_{13} + \phi$ depend on ξ . However, the ξ -dependence of ϕ is very weak. The ξ -dependence of $\delta\lambda_{31}$ and $\delta\lambda_{21}$ are also weak, so the effect of a non-zero ξ will appear dominantly in $\tilde{\theta}_{12}$.

B4. Third Rotation, $\delta m_{31}^2 < 0$ Case

In this case, we need to diagonalize the $(1, 3)$ -submatrix of H''' . Define the matrix Y as

$$Y = \begin{bmatrix} c_\eta & 0 & s_\eta \\ 0 & 1 & 0 \\ -s_\eta & 0 & c_\eta \end{bmatrix} ,\tag{69}$$

where

$$c_\eta = \cos \eta , \quad s_\eta = \sin \eta , \quad \tan 2\eta \equiv \frac{2(ac'_{12}c_{13}s_{13} - b\xi s'_{12})c_\phi}{\lambda''_3 - \lambda'_1} .\tag{70}$$

Then,

$$\begin{aligned}H_Y''' &= Y^\dagger H''' Y \\ &= \begin{bmatrix} \lambda'''_{1Y} & -(ac'_{12}c_{13}s_{13} - b\xi s'_{12})s_\phi c_\eta & 0 \\ -(ac'_{12}c_{13}s_{13} - b\xi s'_{12})s_\phi c_\eta & \lambda''_2 & -(ac'_{12}c_{13}s_{13} - b\xi s'_{12})s_\phi s_\eta \\ 0 & -(ac'_{12}c_{13}s_{13} - b\xi s'_{12})s_\phi s_\eta & \lambda'''_{3Y} \end{bmatrix} ,\end{aligned}\tag{71}$$

where,

$$\lambda'''_{1Y} = \frac{\lambda'_1 c_\eta^2 - \lambda''_3 s_\eta^2}{c_\eta^2 - s_\eta^2} = \lambda'''_{Y+} ,$$

$$\lambda_{3Y}''' = \frac{\lambda_3'' c_\eta^2 - \lambda_1' s_\eta^2}{c_\eta^2 - s_\eta^2} = \lambda_{Y-}''' , \quad (72)$$

with

$$\lambda_{Y\pm}''' \equiv \frac{(\lambda_3'' + \lambda_1') \pm \sqrt{(\lambda_3'' - \lambda_1')^2 + 4(ac'_{12}c_{13}s_{13} + b\xi s'_{12})^2 c_\phi^2}}{2} . \quad (73)$$

Using

$$\begin{aligned} ac'_{12}c_{13}s_{13} &\approx a \left(\frac{\delta m_{21}^2}{2a} \sin(2\theta_{12}) \right) c_{13}s_{13} = a O(\varepsilon^4) , \\ b\xi s'_{12} &\approx -\frac{a}{2}\xi = -a O(\varepsilon^2) , \\ \lambda_1' &\approx \delta m_{21}^2 c_{12}^2 = a O(\varepsilon^3) , \\ \lambda_3'' &\approx \delta m_{31}^2 c_{13}^2 = a O(\varepsilon) , \end{aligned} \quad (74)$$

and $c_\phi \approx 1$, we find

$$\tan 2\eta = \frac{2(ac'_{12}c_{13}s_{13} - b\xi s'_{12})c_\phi}{\lambda_3'' - \lambda_1'} = -\frac{2b\xi}{\delta m_{31}^2} \{1 + O(\varepsilon^2)\} . \quad (75)$$

Therefore, the angle η is given approximately by

$$\eta \approx -\frac{b\xi}{\delta m_{31}^2} = \frac{b\xi}{|\delta m_{31}^2|} = O(\varepsilon) , \quad (76)$$

from which we can conclude that $s_\eta = O(\varepsilon)$ and $c_\eta = O(1)$. The eigenvalues can also be expanded in ε and we find

$$\begin{aligned} \lambda_{1Y}''' &= \delta m_{21}^2 c_{12}^2 - \frac{b^2 \xi^2}{\delta m_{31}^2} + a O(\varepsilon^5) = a O(\varepsilon^3) , \\ \lambda_{3Y}''' &= \delta m_{31}^2 c_{13}^2 + \frac{b^2 \xi^2}{\delta m_{31}^2} + a O(\varepsilon^4) = a O(\varepsilon) \end{aligned} \quad (77)$$

Again, these shifts in the eigenvalues are negligible. Putting everything together, we evaluate the sizes of the elements of H_Y'''' to find

$$H_Y'''' = a \begin{bmatrix} O(\varepsilon^3) & O(\varepsilon^3) & 0 \\ O(\varepsilon^3) & O(1) & O(\varepsilon^4) \\ 0 & O(\varepsilon^4) & O(\varepsilon) \end{bmatrix} . \quad (78)$$

This shows that further diagonalization requires rotations by angles of $O(\varepsilon^3)$ or smaller, which we will neglect.

Thus, we have found that when $\xi = O(\varepsilon^2)$ and $\delta m_{31}^2 < 0$ (inverted hierarchy), we need an extra (1, 3)-rotation to diagonalize H , and the diagonalization matrix is $UVWY$. We need to identify this product with

$$\tilde{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \tilde{c}_{23} & \tilde{s}_{23} \\ 0 & -\tilde{s}_{23} & \tilde{c}_{23} \end{bmatrix} \begin{bmatrix} \tilde{c}_{13} & 0 & \tilde{s}_{13}e^{-i\tilde{\delta}} \\ 0 & 1 & 0 \\ -\tilde{s}_{13}e^{i\tilde{\delta}} & 0 & \tilde{c}_{13} \end{bmatrix} \begin{bmatrix} \tilde{c}_{12} & \tilde{s}_{12} & 0 \\ -\tilde{s}_{12} & \tilde{c}_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (79)$$

to obtain the effective mixing angles and effective CP phase. From Ref. [7], we know that when $\delta m_{31}^2 < 0$ and $a/|\delta m_{31}^2| = O(\varepsilon^{-1})$, identification of \tilde{U} with UVW leads to

$$\begin{aligned} \tilde{\theta}_{13} &= \theta'_{13}, \\ \tilde{\theta}_{12} &= \theta'_{12}, \\ \tilde{\theta}_{23} &= \theta_{23}, \\ \tilde{\delta} &= 0. \end{aligned} \quad (80)$$

Furthermore, in the range $a/|\delta m_{31}^2| = O(\varepsilon^{-1})$ we have

$$\begin{aligned} \theta'_{12} &= \frac{\pi}{2} - \frac{\delta m_{21}^2}{2a} \sin(2\theta_{12}) + \dots = \frac{\pi}{2} + O(\varepsilon^3), \\ \theta'_{13} &= -\frac{\delta m_{31}^2}{a} \theta_{13} + \dots = O(\varepsilon^2), \end{aligned} \quad (81)$$

which implies that

$$UVW \approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & \tilde{c}_{23} & \tilde{s}_{23} \\ 0 & -\tilde{s}_{23} & \tilde{c}_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (82)$$

Then,

$$\begin{aligned} UVWY &\approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & \tilde{c}_{23} & \tilde{s}_{23} \\ 0 & -\tilde{s}_{23} & \tilde{c}_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\eta & 0 & s_\eta \\ 0 & 1 & 0 \\ -s_\eta & 0 & c_\eta \end{bmatrix} \\ &\approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & \tilde{c}_{23} & \tilde{s}_{23} \\ 0 & -\tilde{s}_{23} & \tilde{c}_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\eta & -s_\eta \\ 0 & s_\eta & c_\eta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned} \quad (83)$$

Therefore, η can be absorbed into $\tilde{\theta}_{23}$ as

$$\tilde{\theta}_{13} = \theta'_{13},$$

$$\begin{aligned}
\tilde{\theta}_{12} &= \theta'_{12}, \\
\tilde{\theta}_{23} &= \theta_{23} - \eta, \\
\tilde{\delta} &= 0.
\end{aligned}
\tag{84}$$

As in the $\delta m_{31}^2 > 0$ case, the ξ -dependence of $\theta'_{13} = \theta_{13} + \phi$ is very weak, so the effect of a non-zero ξ will appear dominantly in $\tilde{\theta}_{23}$.

C. Summary of Neutrino Case

To summarize what we have learned, the main effect of including the $b\xi$ terms, which come from neutral current universality violation, in the effective Hamiltonian is to shift $\tilde{\theta}_{12}$ in the $\delta m_{31}^2 > 0$ (normal hierarchy) case, and $\tilde{\theta}_{23}$ in the $\delta m_{31}^2 < 0$ (inverted hierarchy) case, beyond the shifts due to the charged current interaction term a . In the $\delta m_{31}^2 > 0$ case, the shift in $\tilde{\theta}_{12}$ is given by

$$\chi \approx \frac{b\xi}{\delta m_{31}^2} = -\frac{a\xi}{2\delta m_{31}^2},
\tag{85}$$

while for the $\delta m_{31}^2 < 0$ case, the shift in $\tilde{\theta}_{23}$ is given by

$$-\eta \approx \frac{b\xi}{\delta m_{31}^2} = -\frac{b\xi}{|\delta m_{31}^2|} = \frac{a\xi}{2|\delta m_{31}^2|}.
\tag{86}$$

IV. THE EFFECTIVE MIXING ANGLES, ANTI-NEUTRINO CASE

A. Inclusion of Neutral Current Effects into the Effective Hamiltonian

For the anti-neutrinos, the effective Hamiltonian is given by

$$\bar{H} = \tilde{U}^* \begin{bmatrix} \bar{\lambda}_1 & 0 & 0 \\ 0 & \bar{\lambda}_2 & 0 \\ 0 & 0 & \bar{\lambda}_3 \end{bmatrix} \tilde{U}^T = U^* \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta m_{21}^2 & 0 \\ 0 & 0 & \delta m_{31}^2 \end{bmatrix} U^T + \begin{bmatrix} -a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -b_e & 0 & 0 \\ 0 & -b_\mu & 0 \\ 0 & 0 & -b_\tau \end{bmatrix}.
\tag{87}$$

The differences from the neutrino case are the reversal of signs of the CP violating phase δ (and thus the complex conjugation of the MNS matrix U), and the matter interaction terms a , and b_ℓ ($\ell = e, \mu, \tau$). We denote the matter effect corrected diagonalization matrix as \tilde{U} (note the mirror image tilde on top) to distinguish it from that for the neutrinos. As in the

neutrino case, we make the replacement

$$\begin{bmatrix} -a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -b_e & 0 & 0 \\ 0 & -b_\mu & 0 \\ 0 & 0 & -b_\tau \end{bmatrix} \rightarrow \begin{bmatrix} -a & 0 & 0 \\ 0 & -b\xi & 0 \\ 0 & 0 & b\xi \end{bmatrix}. \quad (88)$$

B. Diagonalization of the Effective Hamiltonian

Using the matrix Q from Eq. (23), we begin by partially diagonalize the effective Hamiltonian as

$$\begin{aligned} \bar{H}' &= QU^T \bar{H} U^* Q^* \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta m_{21}^2 & 0 \\ 0 & 0 & \delta m_{31}^2 \end{bmatrix} - QU^T \begin{bmatrix} a & 0 & 0 \\ 0 & b\xi & 0 \\ 0 & 0 & -b\xi \end{bmatrix} U^* Q^* \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta m_{21}^2 & 0 \\ 0 & 0 & \delta m_{31}^2 \end{bmatrix} - a Q \begin{bmatrix} U_{e1}U_{e1}^* & U_{e1}U_{e2}^* & U_{e1}U_{e3}^* \\ U_{e2}U_{e1}^* & U_{e2}U_{e2}^* & U_{e2}U_{e3}^* \\ U_{e3}U_{e1}^* & U_{e3}U_{e2}^* & U_{e3}U_{e3}^* \end{bmatrix} Q^* \\ &\quad - b\xi Q \left\{ \begin{bmatrix} U_{\mu 1}U_{\mu 1}^* & U_{\mu 1}U_{\mu 2}^* & U_{\mu 1}U_{\mu 3}^* \\ U_{\mu 2}U_{\mu 1}^* & U_{\mu 2}U_{\mu 2}^* & U_{\mu 2}U_{\mu 3}^* \\ U_{\mu 3}U_{\mu 1}^* & U_{\mu 3}U_{\mu 2}^* & U_{\mu 3}U_{\mu 3}^* \end{bmatrix} - \begin{bmatrix} U_{\tau 1}U_{\tau 1}^* & U_{\tau 1}U_{\tau 2}^* & U_{\tau 1}U_{\tau 3}^* \\ U_{\tau 2}U_{\tau 1}^* & U_{\tau 2}U_{\tau 2}^* & U_{\tau 2}U_{\tau 3}^* \\ U_{\tau 3}U_{\tau 1}^* & U_{\tau 3}U_{\tau 2}^* & U_{\tau 3}U_{\tau 3}^* \end{bmatrix} \right\} Q^*. \quad (89) \end{aligned}$$

The matrix which multiplies a is given by

$$\bar{M}_a = Q \begin{bmatrix} U_{e1}U_{e1}^* & U_{e1}U_{e2}^* & U_{e1}U_{e3}^* \\ U_{e2}U_{e1}^* & U_{e2}U_{e2}^* & U_{e2}U_{e3}^* \\ U_{e3}U_{e1}^* & U_{e3}U_{e2}^* & U_{e3}U_{e3}^* \end{bmatrix} Q^* = \begin{bmatrix} c_{12}^2 c_{13}^2 & c_{12} s_{12} c_{13}^2 & c_{12} c_{13} s_{13} \\ c_{12} s_{12} c_{13}^2 & s_{12}^2 c_{13}^2 & s_{12} c_{13} s_{13} \\ c_{12} c_{13} s_{13} & s_{12} c_{13} s_{13} & s_{13}^2 \end{bmatrix}, \quad (90)$$

while the matrix which multiplies $b\xi$ is given by

$$\begin{aligned} \bar{M}_b &= Q \left\{ \begin{bmatrix} U_{\mu 1}U_{\mu 1}^* & U_{\mu 1}U_{\mu 2}^* & U_{\mu 1}U_{\mu 3}^* \\ U_{\mu 2}U_{\mu 1}^* & U_{\mu 2}U_{\mu 2}^* & U_{\mu 2}U_{\mu 3}^* \\ U_{\mu 3}U_{\mu 1}^* & U_{\mu 3}U_{\mu 2}^* & U_{\mu 3}U_{\mu 3}^* \end{bmatrix} - \begin{bmatrix} U_{\tau 1}U_{\tau 1}^* & U_{\tau 1}U_{\tau 2}^* & U_{\tau 1}U_{\tau 3}^* \\ U_{\tau 2}U_{\tau 1}^* & U_{\tau 2}U_{\tau 2}^* & U_{\tau 2}U_{\tau 3}^* \\ U_{\tau 3}U_{\tau 1}^* & U_{\tau 3}U_{\tau 2}^* & U_{\tau 3}U_{\tau 3}^* \end{bmatrix} \right\} Q^* = M_b^* \\ &= \begin{bmatrix} (s_{12}^2 - c_{12}^2 s_{13}^2) \cos(2\theta_{23}) + \sin(2\theta_{12}) \sin(2\theta_{23}) s_{13} \cos \delta \\ -(1 + s_{13}^2) s_{12} c_{12} \cos(2\theta_{23}) - (c_{12}^2 e^{+i\delta} - s_{12}^2 e^{-i\delta}) s_{13} \sin(2\theta_{23}) \\ -s_{12} c_{13} \sin(2\theta_{23}) e^{+i\delta} + c_{12} s_{13} c_{13} \cos(2\theta_{23}) \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
& -(1 + s_{13}^2)s_{12}c_{12} \cos(2\theta_{23}) - (c_{12}^2e^{-i\delta} - s_{12}^2e^{+i\delta})s_{13} \sin(2\theta_{23}) \\
& (c_{12}^2 - s_{12}^2s_{13}^2) \cos(2\theta_{23}) - \sin(2\theta_{12}) \sin(2\theta_{23})s_{13} \cos \delta \\
& c_{12}c_{13} \sin(2\theta_{23})e^{+i\delta} + s_{12}s_{13}c_{13} \cos(2\theta_{23}) \\
& \left. \begin{aligned}
& -s_{12}c_{13} \sin(2\theta_{23})e^{-i\delta} + c_{12}s_{13}c_{13} \cos(2\theta_{23}) \\
& c_{12}c_{13} \sin(2\theta_{23})e^{-i\delta} + s_{12}s_{13}c_{13} \cos(2\theta_{23}) \\
& -c_{13}^2 \cos(2\theta_{23})
\end{aligned} \right] . \tag{91}
\end{aligned}$$

Using $\cos(2\theta_{23}) \leq O(\varepsilon)$ and $\theta_{13} = O(\varepsilon)$, the sizes of the elements of \bar{M}_b are evaluated to be

$$\bar{M}_b = \begin{bmatrix} O(\varepsilon) & O(\varepsilon) & O(1) \\ O(\varepsilon) & O(\varepsilon) & O(1) \\ O(1) & O(1) & O(\varepsilon) \end{bmatrix} . \tag{92}$$

As in the neutrino case, we neglect the $O(\varepsilon)$ terms in \bar{M}_b and approximate

$$\bar{M}_b \approx \begin{bmatrix} 0 & 0 & -s_{12}e^{-i\delta} \\ 0 & 0 & c_{12}e^{-i\delta} \\ -s_{12}e^{+i\delta} & c_{12}e^{+i\delta} & 0 \end{bmatrix} . \tag{93}$$

The effective Hamiltonian which must be diagonalized is then

$$\begin{aligned}
\bar{H}' &= \text{diag}(0, \delta m_{21}^2, \delta m_{31}^2) - a \bar{M}_a - b\xi \bar{M}_b \\
&= \begin{bmatrix} -a c_{12}^2 c_{13}^2 & -a c_{12} s_{12} c_{13}^2 & -a c_{12} c_{13} s_{13} + b\xi s_{12} e^{-i\delta} \\ -a c_{12} s_{12} c_{13}^2 & -a s_{12}^2 c_{13}^2 + \delta m_{21}^2 & -a s_{12} c_{13} s_{13} - b\xi c_{12} e^{-i\delta} \\ -a c_{12} s_{13} s_{13} + b\xi s_{12} e^{+i\delta} & -a s_{12} c_{13} s_{13} - b\xi c_{12} e^{+i\delta} & -a s_{13}^2 + \delta m_{31}^2 \end{bmatrix} \tag{94}
\end{aligned}$$

From this point on, we set $\delta = 0$ for the sake of simplicity. Then, \bar{H}' becomes

$$\bar{H}' = \begin{bmatrix} -a c_{12}^2 c_{13}^2 & -a c_{12} s_{12} c_{13}^2 & -a c_{12} c_{13} s_{13} + b\xi s_{12} \\ -a c_{12} s_{12} c_{13}^2 & -a s_{12}^2 c_{13}^2 + \delta m_{21}^2 & -a s_{12} c_{13} s_{13} - b\xi c_{12} \\ -a c_{12} s_{13} s_{13} + b\xi s_{12} & -a s_{12} c_{13} s_{13} - b\xi c_{12} & -a s_{13}^2 + \delta m_{31}^2 \end{bmatrix} = a \begin{bmatrix} O(1) & O(1) & O(\varepsilon) \\ O(1) & O(1) & O(\varepsilon) \\ O(\varepsilon) & O(\varepsilon) & O(\varepsilon) \end{bmatrix} . \tag{95}$$

B1. *First Rotation*

Applying the Jacobi method on \bar{H}' , we begin by diagonalizing the (1, 2)-submatrix. Define the matrix \bar{V} as

$$\bar{V} = \begin{bmatrix} \bar{c}_\varphi & \bar{s}_\varphi & 0 \\ -\bar{s}_\varphi & \bar{c}_\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (96)$$

where

$$\bar{c}_\varphi = \cos \bar{\varphi}, \quad \bar{s}_\varphi = \sin \bar{\varphi}, \quad \tan 2\bar{\varphi} \equiv -\frac{ac_{13}^2 \sin 2\theta_{12}}{\delta m_{21}^2 + ac_{13}^2 \cos 2\theta_{12}}, \quad (-\theta_{12} < \bar{\varphi} \leq 0). \quad (97)$$

Then,

$$\bar{H}'' = \bar{V}^\dagger \bar{H}' \bar{V} = \begin{bmatrix} \bar{\lambda}'_1 & 0 & -a\bar{c}'_{12}c_{13}s_{13} + b\xi\bar{s}'_{12} \\ 0 & \bar{\lambda}'_2 & -a\bar{s}'_{12}c_{13}s_{13} - b\xi\bar{c}'_{12} \\ -a\bar{c}'_{12}c_{13}s_{13} + b\xi\bar{s}'_{12} & -a\bar{s}'_{12}c_{13}s_{13} - b\xi\bar{c}'_{12} & -as_{13}^2 + \delta m_{31}^2 \end{bmatrix}, \quad (98)$$

where

$$\bar{c}'_{12} = \cos \bar{\theta}'_{12}, \quad \bar{s}'_{12} = \sin \bar{\theta}'_{12}, \quad \bar{\theta}'_{12} = \theta_{12} + \bar{\varphi}, \quad (99)$$

and

$$\begin{aligned} \bar{\lambda}'_1 &= \frac{(-ac_{12}^2c_{13}^2)\bar{c}_\varphi^2 + (as_{12}^2c_{13}^2 - \delta m_{21}^2)\bar{s}_\varphi^2}{\bar{c}_\varphi^2 - \bar{s}_\varphi^2} = \bar{\lambda}'_-, \\ \bar{\lambda}'_2 &= \frac{(-as_{12}^2c_{13}^2 + \delta m_{21}^2)\bar{c}_\varphi^2 + (ac_{12}^2c_{13}^2)\bar{s}_\varphi^2}{\bar{c}_\varphi^2 - \bar{s}_\varphi^2} = \bar{\lambda}'_+, \end{aligned} \quad (100)$$

with

$$\bar{\lambda}'_\pm = \frac{(\delta m_{21}^2 - ac_{13}^2) \pm \sqrt{(\delta m_{21}^2 + ac_{13}^2)^2 - 4ac_{13}^2s_{12}^2\delta m_{21}^2}}{2}. \quad (101)$$

From Ref. [7], we know that in the region $a/|\delta m_{31}^2| = O(\varepsilon^{-1})$, we can expand $\bar{\varphi}$ as

$$\bar{\theta}'_{12} = \frac{\delta m_{21}^2}{2a} \sin(2\theta_{12}) + O(\varepsilon^5) = O(\varepsilon^3). \quad (102)$$

Therefore,

$$\begin{aligned} \bar{c}'_{12} &= \cos\left(\frac{\delta m_{21}^2}{2a} \sin(2\theta_{12})\right) = 1 - O(\varepsilon^6), \\ \bar{s}'_{12} &= \sin\left(\frac{\delta m_{21}^2}{2a} \sin(2\theta_{12})\right) = O(\varepsilon^3). \end{aligned} \quad (103)$$

The expansions of $\bar{\lambda}'_{\pm}$ are given by

$$\begin{aligned}\bar{\lambda}'_- &= -ac_{13}^2 + \delta m_{21}^2 s_{12}^2 + aO(\varepsilon^6) = aO(1), \\ \bar{\lambda}'_+ &= \delta m_{21}^2 c_{12}^2 + aO(\varepsilon^6) = aO(\varepsilon^3).\end{aligned}\tag{104}$$

Therefore, the sizes of the elements of \bar{H}'' can be evaluated to be

$$\bar{H}'' = a \begin{bmatrix} O(1) & 0 & O(\varepsilon) \\ 0 & O(\varepsilon^3) & O(\varepsilon^2) \\ O(\varepsilon) & O(\varepsilon^2) & O(\varepsilon) \end{bmatrix}.\tag{105}$$

As in the neutrino case, though we have a choice of whether we diagonalize the (1, 3) or the (2, 3) submatrix, since both require rotations by angles of $O(\varepsilon)$, we diagonalize the (1, 3) submatrix next to maintain the parallel with the $\xi = 0$ case.

B2. Second Rotation

Define the matrix \bar{W} as

$$\bar{W} = \begin{bmatrix} \bar{c}_\phi & 0 & \bar{s}_\phi \\ 0 & 1 & 0 \\ -\bar{s}_\phi & 0 & \bar{c}_\phi \end{bmatrix},\tag{106}$$

where $\bar{c}_\phi = \cos \bar{\phi}$, $\bar{s}_\phi = \sin \bar{\phi}$, and

$$\begin{aligned}\tan 2\bar{\phi} &\equiv \frac{-2(a\bar{c}'_{12}c_{13}s_{13} - b\xi\bar{s}'_{12})}{\delta m_{31}^2 - as_{13}^2 - \bar{\lambda}'_1} \\ &= -\frac{2a\bar{c}'_{12}c_{13}s_{13}}{\delta m_{31}^2 - as_{13}^2 - \bar{\lambda}'_1} \left\{ 1 + \xi \left(\frac{\bar{s}'_{12}}{2\bar{c}'_{12}c_{13}s_{13}} \right) \right\} \\ &= \tan 2\bar{\phi}_0 \left\{ 1 + \xi \left(\frac{\bar{s}'_{12}}{2\bar{c}'_{12}c_{13}s_{13}} \right) \right\}.\end{aligned}\tag{107}$$

The angle $\bar{\phi}$ is in the fourth quadrant when $\delta m_{31}^2 > 0$ (normal hierarchy), and in the first quadrant when $\delta m_{31}^2 < 0$ (inverted hierarchy). $\bar{\phi}_0$ is the rotation angle when $\xi = 0$. Taking the arc-tangent of both sides of the above equation, we find

$$\bar{\phi} = \bar{\phi}_0 + \xi \left(\frac{\sin 4\bar{\phi}_0 \tan \bar{\theta}'_{12}}{4 \sin 2\theta_{13}} \right) + O(\xi^2).\tag{108}$$

When $a/|\delta m_{31}^2| = O(\varepsilon^{-1})$, $\bar{\phi}_0$ is given by

$$\bar{\phi}_0 = \begin{cases} -\theta_{13} + \frac{\delta m_{31}^2}{a} \theta_{13} + O(\varepsilon^3) & (\delta m_{31}^2 > 0), \\ \left(\frac{\pi}{2} - \theta_{13}\right) + \frac{\delta m_{31}^2}{a} \theta_{13} + O(\varepsilon^3) & (\delta m_{31}^2 < 0). \end{cases}\tag{109}$$

Therefore, $\sin 4\bar{\phi}_0 \approx -\sin(4\theta_{13})$ for both the $\delta m_{31}^2 > 0$ and $\delta m_{31}^2 < 0$ cases and using Eq. (102), we find

$$\bar{\phi} \approx \bar{\phi}_0 - \xi \left(\frac{\delta m_{21}^2}{4a} \right) \sin(2\theta_{12}) = \bar{\phi}_0 - \frac{1}{4}O(\varepsilon^5). \quad (110)$$

Therefore, the difference between $\bar{\phi}$ and $\bar{\phi}_0$ can be neglected in this range of a , just as in the neutrino case.

Using \bar{W} , we obtain

$$\begin{aligned} \bar{H}''' &= \bar{W}^\dagger \bar{H}'' \bar{W} \\ &= \begin{bmatrix} \bar{\lambda}_1'' & (a\bar{s}'_{12}s_{13}c_{13} + b\xi\bar{c}'_{12})\bar{s}_\phi & 0 \\ (a\bar{s}'_{12}s_{13}c_{13} + b\xi\bar{c}'_{12})\bar{s}_\phi & \bar{\lambda}_2 & -(a\bar{s}'_{12}s_{13}c_{13} + b\xi\bar{c}'_{12})\bar{c}_\phi \\ 0 & -(a\bar{s}'_{12}s_{13}c_{13} + b\xi\bar{c}'_{12})\bar{c}_\phi & \bar{\lambda}_3'' \end{bmatrix} \end{aligned} \quad (111)$$

where

$$\begin{aligned} \bar{\lambda}_1'' &= \frac{\bar{\lambda}'_1 \bar{c}_\phi^2 + (as_{13}^2 - \delta m_{31}^2)\bar{s}_\phi^2}{\bar{c}_\phi^2 - \bar{s}_\phi^2}, \\ \bar{\lambda}_3'' &= \frac{(-as_{13}^2 + \delta m_{31}^2)\bar{c}_\phi^2 - \bar{\lambda}'_1 \bar{s}_\phi^2}{\bar{c}_\phi^2 - \bar{s}_\phi^2}. \end{aligned} \quad (112)$$

If we define

$$\bar{\lambda}_\pm'' \equiv \frac{[(\delta m_{31}^2 - as_{13}^2) + \bar{\lambda}'_1] \pm \sqrt{[(\delta m_{31}^2 - as_{13}^2) - \bar{\lambda}'_1]^2 + 4(a\bar{c}'_{12}s_{13}c_{13} + b\xi\bar{s}'_{12})^2}}{2}, \quad (113)$$

then

$$\begin{aligned} \bar{\lambda}_1'' &= \bar{\lambda}_-'' , & \bar{\lambda}_3'' &= \bar{\lambda}_+'' , & \text{if } \delta m_{31}^2 > 0 , \\ \bar{\lambda}_1'' &= \bar{\lambda}_+'' , & \bar{\lambda}_3'' &= \bar{\lambda}_-'' , & \text{if } \delta m_{31}^2 < 0 . \end{aligned} \quad (114)$$

When $a/|\delta m_{31}^2| = O(\varepsilon^{-1})$, we can expand $\bar{\lambda}_\pm''$ as

$$\begin{aligned} \bar{\lambda}_-'' &= -a + s_{13}^2 \delta m_{31}^2 + s_{12}^2 \delta m_{21}^2 + O(\varepsilon^3 |\delta m_{31}^2|) = aO(1), \\ \bar{\lambda}_+'' &= c_{13}^2 \delta m_{31}^2 + O(\varepsilon^3 |\delta m_{31}^2|) = aO(\varepsilon). \end{aligned} \quad (115)$$

Also, from Eq. (109) and the fact that $\bar{\phi} \approx \bar{\phi}_0$, we conclude

$$\begin{aligned} \bar{s}_\phi &\approx -s_{13} = O(\varepsilon), & \bar{c}_\phi &\approx c_{13} = O(1), & (\delta m_{31}^2 > 0), \\ \bar{s}_\phi &\approx c_{13} = O(1), & \bar{c}_\phi &\approx s_{13} = O(\varepsilon), & (\delta m_{31}^2 < 0). \end{aligned} \quad (116)$$

Putting everything together, we evaluate the sizes of the elements of \bar{H}''' and find

$$\bar{H}''' = a \begin{bmatrix} O(1) & O(\varepsilon^3) & 0 \\ O(\varepsilon^3) & O(\varepsilon^3) & O(\varepsilon^2) \\ 0 & O(\varepsilon^2) & O(\varepsilon) \end{bmatrix} \quad (117)$$

when $\delta m_{31}^2 > 0$, and

$$\bar{H}''' = a \begin{bmatrix} O(\varepsilon) & O(\varepsilon^2) & 0 \\ O(\varepsilon^2) & O(\varepsilon^3) & O(\varepsilon^3) \\ 0 & O(\varepsilon^3) & O(1) \end{bmatrix} \quad (118)$$

when $\delta m_{31}^2 < 0$. Therefore, for the $\delta m_{31}^2 > 0$ (normal hierarchy) case, we must diagonalize the (2, 3) submatrix next, while for the $\delta m_{31}^2 < 0$ (inverted hierarchy) case, we must diagonalize the (1, 2) submatrix next.

B3. Third Rotation, $\delta m_{31}^2 > 0$ Case

To diagonalize the (2, 3) submatrix of \bar{H}''' , we define the matrix \bar{X} as

$$\bar{X} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \bar{c}_\chi & \bar{s}_\chi \\ 0 & -\bar{s}_\chi & \bar{c}_\chi \end{bmatrix}, \quad (119)$$

where

$$\bar{c}_\chi = \cos \bar{\chi}, \quad \bar{s}_\chi = \sin \bar{\chi}, \quad \tan 2\bar{\chi} \equiv -\frac{2(a\bar{s}'_{12}s_{13}c_{13} + b\xi\bar{c}'_{12})\bar{c}_\phi}{\bar{\lambda}_3'' - \bar{\lambda}_2''}. \quad (120)$$

Then,

$$\begin{aligned} \bar{H}_X'''' &= \bar{X}^\dagger \bar{H}''' \bar{X} \\ &= \begin{bmatrix} \bar{\lambda}_1'' & (a\bar{s}'_{12}s_{13}c_{13} + b\xi\bar{c}'_{12})\bar{s}_\phi\bar{c}_\chi & (a\bar{s}'_{12}s_{13}c_{13} + b\xi\bar{c}'_{12})\bar{s}_\phi\bar{s}_\chi \\ (a\bar{s}'_{12}s_{13}c_{13} + b\xi\bar{c}'_{12})\bar{s}_\phi\bar{c}_\chi & \bar{\lambda}_{2X}''' & 0 \\ (a\bar{s}'_{12}s_{13}c_{13} + b\xi\bar{c}'_{12})\bar{s}_\phi\bar{s}_\chi & 0 & \bar{\lambda}_{3X}''' \end{bmatrix}, \end{aligned} \quad (121)$$

where

$$\bar{\lambda}_{2X}''' = \frac{\bar{\lambda}_2'\bar{c}_\chi^2 - \bar{\lambda}_3''\bar{s}_\chi^2}{\bar{c}_\chi^2 - \bar{s}_\chi^2} = \bar{\lambda}_{X-}''',$$

$$\bar{\lambda}_{3X}''' = \frac{\bar{\lambda}_3'' \bar{c}_X^2 - \bar{\lambda}_2' \bar{s}_X^2}{\bar{c}_X^2 - \bar{s}_X^2} = \bar{\lambda}_{X+}''' , \quad (122)$$

with

$$\bar{\lambda}_{X\pm}''' \equiv \frac{(\bar{\lambda}_3'' + \bar{\lambda}_2') \pm \sqrt{(\bar{\lambda}_3'' - \bar{\lambda}_2')^2 + 4(a\bar{s}'_{12}s_{13}c_{13} + b\xi\bar{c}'_{12})^2\bar{c}_\phi^2}}{2} . \quad (123)$$

Recalling that

$$\begin{aligned} a\bar{s}'_{12}s_{13}c_{13} &\approx a \left(\frac{\delta m_{21}^2}{2a} \sin(2\theta_{12}) \right) s_{13}c_{13} = a O(\varepsilon^4) , \\ b\xi\bar{c}'_{12} &\approx -\frac{1}{2}\xi = -a O(\varepsilon^2) , \\ \bar{\lambda}_2' &\approx \delta m_{21}^2 c_{12}^2 = a O(\varepsilon^3) , \\ \bar{\lambda}_3'' &\approx \delta m_{31}^2 c_{13}^2 = a O(\varepsilon) , \end{aligned} \quad (124)$$

and $\bar{c}_\phi \approx 1$, we find

$$\tan 2\bar{\chi} = -\frac{2(a\bar{s}'_{12}s_{13}c_{13} + b\xi\bar{c}'_{12})\bar{c}_\phi}{\bar{\lambda}_3'' - \bar{\lambda}_2'} = -\frac{2b\xi}{\delta m_{31}^2} \{1 + O(\varepsilon^2)\} . \quad (125)$$

Therefore, the angle $\bar{\chi}$ is given approximately by

$$\bar{\chi} \approx -\frac{b\xi}{\delta m_{31}^2} = O(\varepsilon) , \quad (126)$$

from which we can conclude that $\bar{s}_X = O(\varepsilon)$ and $\bar{c}_X = O(1)$. The eigenvalues can also be expanded in ε and we find

$$\begin{aligned} \bar{\lambda}_{2X}''' &= \delta m_{21}^2 c_{12}^2 - \frac{b^2 \xi^2}{\delta m_{31}^2} + a O(\varepsilon^5) = a O(\varepsilon^3) , \\ \bar{\lambda}_{3X}''' &= \delta m_{31}^2 c_{13}^2 + \frac{b^2 \xi^2}{\delta m_{31}^2} + a O(\varepsilon^4) = a O(\varepsilon) . \end{aligned} \quad (127)$$

As in the neutrino case, the shifts in the eigenvalues are of order $aO(\varepsilon^3)$ and their effects on $\delta\bar{\lambda}_{31} = aO(1)$ and $\delta\bar{\lambda}_{21} = aO(1)$ are negligible. Putting everything together, we evaluate the sizes of the elements of \bar{H}_X'''' to find

$$\bar{H}_X'''' = a \begin{bmatrix} O(1) & O(\varepsilon^4) & O(\varepsilon^3) \\ O(\varepsilon^4) & O(\varepsilon^3) & 0 \\ O(\varepsilon^3) & 0 & O(\varepsilon) \end{bmatrix} . \quad (128)$$

This shows that further diagonalization requires rotations by angles of $O(\varepsilon^3)$ or smaller, which we will neglect.

We have found that when $\xi = O(\varepsilon^2)$ and $\delta m_{31}^2 > 0$ (normal hierarchy), we need an extra (2, 3)-rotation to diagonalize \bar{H} , and the diagonalization matrix is $\bar{U}\bar{V}\bar{W}\bar{X}$, which we need to identify with

$$\tilde{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \tilde{c}_{23} & \tilde{s}_{23} \\ 0 & -\tilde{s}_{23} & \tilde{c}_{23} \end{bmatrix} \begin{bmatrix} \tilde{c}_{13} & 0 & \tilde{s}_{13}e^{-i\tilde{\delta}} \\ 0 & 1 & 0 \\ -\tilde{s}_{13}e^{i\tilde{\delta}} & 0 & \tilde{c}_{13} \end{bmatrix} \begin{bmatrix} \tilde{c}_{12} & \tilde{s}_{12} & 0 \\ -\tilde{s}_{12} & \tilde{c}_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (129)$$

From Ref. [7], we know that when $\delta m_{31}^2 > 0$ and $a/|\delta m_{31}^2| = O(\varepsilon^{-1})$, identification of \tilde{U} with $\bar{U}\bar{V}\bar{W}$ yields

$$\begin{aligned} \tilde{\theta}_{13} &\approx \bar{\theta}'_{13}, \\ \tilde{\theta}_{12} &\approx \bar{\theta}'_{12}, \\ \tilde{\theta}_{23} &\approx \theta_{23}, \\ \tilde{\delta} &\approx 0, \end{aligned} \quad (130)$$

where $\bar{\theta}'_{13} \equiv \theta_{13} + \bar{\phi}$. (Note that we are considering the $\delta = 0$ case.) Furthermore, in the range $a/|\delta m_{31}^2| = O(\varepsilon^{-1})$ we have

$$\begin{aligned} \bar{\theta}_{12} &= \frac{\delta m_{21}^2}{2a} \sin(2\theta_{12}) + \dots = O(\varepsilon^3), \\ \bar{\theta}_{13} &= \frac{\delta m_{31}^2}{a} \theta_{13} + \dots = O(\varepsilon^2), \end{aligned} \quad (131)$$

which implies

$$\bar{U}\bar{V}\bar{W} \approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & \tilde{c}_{23} & \tilde{s}_{23} \\ 0 & -\tilde{s}_{23} & \tilde{c}_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \tilde{c}_{23} & \tilde{s}_{23} \\ 0 & -\tilde{s}_{23} & \tilde{c}_{23} \end{bmatrix}. \quad (132)$$

Since \bar{X} is an (2, 3)-rotation matrix, multiplying $\bar{U}\bar{V}\bar{W}$ from the right with \bar{X} will only lead to a shift in $\tilde{\theta}_{23}$. Therefore,

$$\begin{aligned} \tilde{\theta}_{13} &\approx \bar{\theta}'_{13}, \\ \tilde{\theta}_{12} &\approx \bar{\theta}'_{12}, \\ \tilde{\theta}_{23} &\approx \theta_{23} + \bar{\chi}, \\ \tilde{\delta} &\approx 0. \end{aligned} \quad (133)$$

B4. Third Rotation, $\delta m_{31}^2 < 0$ Case

In this case, we diagonalize the (1, 2) submatrix of \bar{H}''' . Define the matrix \bar{Y} as

$$\bar{Y} = \begin{bmatrix} \bar{c}_\eta & \bar{s}_\eta & 0 \\ -\bar{s}_\eta & \bar{c}_\eta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (134)$$

where

$$\bar{c}_\eta = \cos \bar{\eta}, \quad \bar{s}_\eta = \sin \bar{\eta}, \quad \tan 2\bar{\eta} \equiv \frac{2(a\bar{s}'_{12}s_{13}c_{13} + b\xi\bar{c}'_{12})\bar{s}_\phi}{\bar{\lambda}'_2 - \bar{\lambda}''_1}. \quad (135)$$

Then,

$$\begin{aligned} \bar{H}_Y''' &= \bar{Y}^\dagger \bar{H}''' \bar{Y} \\ &= \begin{bmatrix} \bar{\lambda}_{1Y}''' & 0 & (a\bar{s}'_{12}s_{13}c_{13} + b\xi\bar{c}'_{12})\bar{c}_\phi\bar{s}_\eta \\ 0 & \bar{\lambda}_{2Y}''' & -(a\bar{s}'_{12}s_{13}c_{13} + b\xi\bar{c}'_{12})\bar{c}_\phi\bar{c}_\eta \\ (a\bar{s}'_{12}s_{13}c_{13} + b\xi\bar{c}'_{12})\bar{c}_\phi\bar{s}_\eta & -(a\bar{s}'_{12}s_{13}c_{13} + b\xi\bar{c}'_{12})\bar{c}_\phi\bar{c}_\eta & \bar{\lambda}_3''' \end{bmatrix}, \end{aligned} \quad (136)$$

where

$$\begin{aligned} \bar{\lambda}_{1Y}''' &= \frac{\bar{\lambda}_1''\bar{c}_\eta^2 - \bar{\lambda}_2''\bar{s}_\eta^2}{\bar{c}_\eta^2 - \bar{s}_\eta^2} = \bar{\lambda}_{Y-}''', \\ \bar{\lambda}_{2Y}''' &= \frac{\bar{\lambda}_2''\bar{c}_\eta^2 - \bar{\lambda}_1''\bar{s}_\eta^2}{\bar{c}_\eta^2 - \bar{s}_\eta^2} = \bar{\lambda}_{Y+}''', \end{aligned} \quad (137)$$

with

$$\bar{\lambda}_{Y\pm}''' \equiv \frac{(\bar{\lambda}'_2 + \bar{\lambda}''_1) \pm \sqrt{(\bar{\lambda}'_2 - \bar{\lambda}''_1)^2 + 4(a\bar{s}'_{12}s_{13}c_{13} + b\xi\bar{c}'_{12})^2\bar{s}_\phi^2}}{2}. \quad (138)$$

Using

$$\begin{aligned} a\bar{s}'_{12}s_{13}c_{13} &\approx a \left(\frac{\delta m_{21}^2}{2a} \sin(2\theta_{12}) \right) s_{13}c_{13} = a O(\varepsilon^4), \\ b\xi\bar{c}'_{12} &\approx -\frac{1}{2}\xi = -a O(\varepsilon^2), \\ \bar{\lambda}_1'' &\approx c_{13}^2 \delta m_{31}^2 = a O(\varepsilon), \\ \bar{\lambda}_2'' &\approx c_{12}^2 \delta m_{21}^2 = a O(\varepsilon^3), \end{aligned} \quad (139)$$

and $\bar{s}_\phi \approx 1$, we find

$$\tan 2\bar{\eta} = \frac{2(a\bar{s}'_{12}s_{13}c_{13} + b\xi\bar{c}'_{12})\bar{s}_\phi}{\bar{\lambda}'_2 - \bar{\lambda}''_1} = -\frac{2b\xi}{\delta m_{31}^2} \{1 + O(\varepsilon^2)\}. \quad (140)$$

Therefore, the angle $\bar{\eta}$ is given approximately by

$$\bar{\eta} \approx -\frac{b\xi}{\delta m_{31}^2} = \frac{b\xi}{|\delta m_{31}^2|} = O(\varepsilon), \quad (141)$$

from which we can conclude that $\bar{s}_\eta = O(\varepsilon)$ and $\bar{c}_\eta = O(1)$. The eigenvalues can also be expanded in ε and we find

$$\begin{aligned} \bar{\lambda}_{1Y}''' &= \delta m_{31}^2 c_{13}^2 + \frac{b^2 \xi^2}{\delta m_{31}^2} + a O(\varepsilon^4) = a O(\varepsilon), \\ \bar{\lambda}_{2Y}''' &= \delta m_{21}^2 c_{12}^2 - \frac{b^2 \xi^2}{\delta m_{31}^2} + a O(\varepsilon^5) = a O(\varepsilon^3). \end{aligned} \quad (142)$$

Again, the shifts are negligible. Putting everything together, we evaluate the sizes of the elements of \bar{H}_Y'''' to find

$$\bar{H}_Y'''' = a \begin{bmatrix} O(\varepsilon) & 0 & O(\varepsilon^4) \\ 0 & O(\varepsilon^3) & O(\varepsilon^3) \\ O(\varepsilon^4) & O(\varepsilon^3) & O(1) \end{bmatrix}, \quad (143)$$

which shows that further diagonalization requires rotations by angles of $O(\varepsilon^3)$ or smaller, which we will neglect.

Thus, we have found that when $\xi = O(\varepsilon^2)$ and $\delta m_{31}^2 < 0$, the diagonalization of \bar{H} requires an extra $(1, 2)$ rotation, and the diagonalization matrix is $\bar{U}\bar{V}\bar{W}\bar{Y}$. As in the $\delta m_{31}^2 > 0$ case, $\bar{U}\bar{V}\bar{W}\bar{Y}$ must be identified with

$$\tilde{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \tilde{c}_{23} & \tilde{s}_{23} \\ 0 & -\tilde{s}_{23} & \tilde{c}_{23} \end{bmatrix} \begin{bmatrix} \tilde{c}_{13} & 0 & \tilde{s}_{13}e^{-i\tilde{\delta}} \\ 0 & 1 & 0 \\ -\tilde{s}_{13}e^{i\tilde{\delta}} & 0 & \tilde{c}_{13} \end{bmatrix} \begin{bmatrix} \tilde{c}_{12} & \tilde{s}_{12} & 0 \\ -\tilde{s}_{12} & \tilde{c}_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (144)$$

Again, from Ref. [7], we know that the identification of $\bar{U}\bar{V}\bar{W}$ with \tilde{U} yields

$$\begin{aligned} \tilde{\theta}_{13} &\approx \bar{\theta}'_{13}, \\ \tilde{\theta}_{12} &\approx \frac{c_{13}}{\bar{c}'_{13}} \left(\frac{\delta m_{21}^2}{2a} \right) \sin(2\theta_{12}), \\ \tilde{\theta}_{23} &\approx \theta_{23} - \frac{\bar{s}_\phi}{\bar{c}'_{13}} \left(\frac{\delta m_{21}^2}{2a} \right) \sin(2\theta_{12}), \\ \tilde{\delta} &\approx 0, \end{aligned} \quad (145)$$

where $\bar{c}'_{13} = \cos \bar{\theta}'_{13}$. Since \bar{Y} is an $(1, 2)$ -rotation matrix, multiplying $\bar{U}\bar{V}\bar{W}$ from the right by \bar{Y} will only lead to a shift in $\tilde{\theta}_{12}$. Therefore,

$$\tilde{\theta}_{13} \approx \bar{\theta}'_{13},$$

	$\delta m_{31}^2 > 0$ (normal hierarchy)	$\delta m_{31}^2 < 0$ (inverted hierarchy)
Neutrino	$\tilde{\theta}_{12}$ is shifted by $-\frac{a\xi}{2\delta m_{31}^2}$	$\tilde{\theta}_{23}$ is shifted by $+\frac{a\xi}{2 \delta m_{31}^2 }$
Anti-neutrino	$\tilde{\theta}_{23}$ is shifted by $+\frac{a\xi}{2\delta m_{31}^2}$	$\tilde{\theta}_{12}$ is shifted by $-\frac{a\xi}{2 \delta m_{31}^2 }$

TABLE I: Matter effects from neutral current universality violation. The parameter ξ , defined in Eq. (18), gives the size of the violation.

$$\begin{aligned}
\tilde{\theta}_{12} &\approx \frac{c_{13}}{c'_{13}} \left(\frac{\delta m_{21}^2}{2a} \right) \sin(2\theta_{12}) + \bar{\eta} , \\
\tilde{\theta}_{23} &\approx \theta_{23} - \frac{\bar{s}_\phi}{c'_{13}} \left(\frac{\delta m_{21}^2}{2a} \right) \sin(2\theta_{12}) , \\
\tilde{\delta} &\approx 0 .
\end{aligned} \tag{146}$$

C. Summary of Anti-Neutrino Case

To summarize, in contrast to the neutrino case, the main effect of including the $b\xi$ terms in the effective Hamiltonian for the anti-neutrinos is to shift $\tilde{\theta}_{23}$ in the $\delta m_{31}^2 > 0$ (normal hierarchy) case, and $\tilde{\theta}_{12}$ in the $\delta m_{31}^2 < 0$ (inverted hierarchy) case. The mixing angle that is affected depending on the sign of δm_{31}^2 is the exact opposite of the neutrino case. In the $\delta m_{31}^2 > 0$ case, the shift in $\tilde{\theta}_{23}$ is given by

$$\bar{\chi} \approx -\frac{b\xi}{\delta m_{31}^2} = \frac{a\xi}{2\delta m_{31}^2} , \tag{147}$$

while for the $\delta m_{31}^2 < 0$ case, the shift in $\tilde{\theta}_{12}$ is given by

$$\bar{\eta} \approx -\frac{b\xi}{\delta m_{31}^2} = \frac{b\xi}{|\delta m_{31}^2|} = -\frac{a\xi}{2|\delta m_{31}^2|} . \tag{148}$$

Listing these results together with those for the neutrino case from the previous section, we obtain Table I. The accuracy of our approximation will be demonstrated later by comparing our conclusions with the exact numerical results. Let us now investigate how these shifts in the effective mixing angles affect the oscillation probabilities.

V. THE OSCILLATION PROBABILITIES

The oscillation probability from neutrino flavor ν_α to neutrino flavor ν_β in vacuum is given by

$$P(\nu_\alpha \rightarrow \nu_\alpha) = 1 - 4|U_{\alpha 2}|^2(1 - |U_{\alpha 2}|^2)\sin^2\frac{\Delta_{21}}{2} - 4|U_{\alpha 3}|^2(1 - |U_{\alpha 3}|^2)\sin^2\frac{\Delta_{31}}{2} + 2|U_{\alpha 2}|^2|U_{\alpha 3}|^2\left(4\sin^2\frac{\Delta_{21}}{2}\sin^2\frac{\Delta_{31}}{2} + \sin\Delta_{21}\sin\Delta_{31}\right), \quad (149)$$

for the $\alpha = \beta$ case, and

$$P(\nu_\alpha \rightarrow \nu_\beta) = 4|U_{\alpha 2}|^2|U_{\beta 2}|^2\sin^2\frac{\Delta_{21}}{2} + 4|U_{\alpha 3}|^2|U_{\beta 3}|^2\sin^2\frac{\Delta_{31}}{2} + 2\Re(U_{\alpha 3}^*U_{\beta 3}U_{\alpha 2}U_{\beta 2}^*)\left(4\sin^2\frac{\Delta_{21}}{2}\sin^2\frac{\Delta_{31}}{2} + \sin\Delta_{21}\sin\Delta_{31}\right) + 4J_{(\alpha,\beta)}\left(\sin^2\frac{\Delta_{21}}{2}\sin\Delta_{31} - \sin^2\frac{\Delta_{31}}{2}\sin\Delta_{21}\right), \quad (150)$$

for the $\alpha \neq \beta$ case, where $J_{(\alpha,\beta)}$ is the Jarshkog invariant,

$$\begin{aligned} J_{(\alpha,\beta)} &= +\Im(U_{\alpha 1}^*U_{\beta 1}U_{\alpha 2}U_{\beta 2}^*) = +\Im(U_{\alpha 2}^*U_{\beta 2}U_{\alpha 3}U_{\beta 3}^*) = +\Im(U_{\alpha 3}^*U_{\beta 3}U_{\alpha 1}U_{\beta 1}^*) \\ &= -\Im(U_{\alpha 2}^*U_{\beta 2}U_{\alpha 1}U_{\beta 1}^*) = -\Im(U_{\alpha 1}^*U_{\beta 1}U_{\alpha 3}U_{\beta 3}^*) = -\Im(U_{\alpha 3}^*U_{\beta 3}U_{\alpha 2}U_{\beta 2}^*) \\ &= -J_{(\beta,\alpha)}, \end{aligned} \quad (151)$$

and

$$\Delta_{ij} \equiv \frac{\delta m_{ij}^2}{2E}L = 2.534 \frac{(\delta m_{ij}^2/eV^2)}{(E/\text{GeV})}(L/\text{km}), \quad \delta m_{ij}^2 \equiv m_i^2 - m_j^2. \quad (152)$$

The oscillation probabilities for anti-neutrinos can be obtained by replacing U with its complex conjugate, which amounts to flipping the sign of the CP violating phase δ .

The oscillation probabilities in matter are obtained by making the replacements

$$U_{\alpha i} \rightarrow \tilde{U}_{\alpha i}, \quad \Delta_{ij} \rightarrow \tilde{\Delta}_{ij} = \frac{\lambda_i - \lambda_j}{2E}L, \quad (153)$$

for the neutrinos, and

$$U_{\alpha i} \rightarrow \tilde{U}_{\alpha i}, \quad \Delta_{ij} \rightarrow \tilde{\Delta}_{ij} = \frac{\bar{\lambda}_i - \bar{\lambda}_j}{2E}L, \quad (154)$$

for the anti-neutrinos. For instance, the ν_μ and $\bar{\nu}_\mu$ survival probabilities in matter are given by

$$\tilde{P}(\nu_\mu \rightarrow \nu_\mu) = 1 - 4|\tilde{U}_{\mu 2}|^2(1 - |\tilde{U}_{\mu 2}|^2)\sin^2\frac{\tilde{\Delta}_{21}}{2} - 4|\tilde{U}_{\mu 3}|^2(1 - |\tilde{U}_{\mu 3}|^2)\sin^2\frac{\tilde{\Delta}_{31}}{2}$$

$$+ 2 |\tilde{U}_{\mu 2}|^2 |\tilde{U}_{\mu 3}|^2 \left(4 \sin^2 \frac{\tilde{\Delta}_{21}}{2} \sin^2 \frac{\tilde{\Delta}_{31}}{2} + \sin \tilde{\Delta}_{21} \sin \tilde{\Delta}_{31} \right), \quad (155)$$

$$\begin{aligned} \tilde{P}(\bar{\nu}_\mu \rightarrow \bar{\nu}_\mu) &= 1 - 4 |\tilde{U}_{\mu 2}|^2 \left(1 - |\tilde{U}_{\mu 2}|^2 \right) \sin^2 \frac{\tilde{\Delta}_{21}}{2} - 4 |\tilde{U}_{\mu 3}|^2 \left(1 - |\tilde{U}_{\mu 3}|^2 \right) \sin^2 \frac{\tilde{\Delta}_{31}}{2} \\ &+ 2 |\tilde{U}_{\mu 2}|^2 |\tilde{U}_{\mu 3}|^2 \left(4 \sin^2 \frac{\tilde{\Delta}_{21}}{2} \sin^2 \frac{\tilde{\Delta}_{31}}{2} + \sin \tilde{\Delta}_{21} \sin \tilde{\Delta}_{31} \right). \end{aligned} \quad (156)$$

Let us calculate these probabilities in the range $a/|\delta m_{31}^2| = O(\varepsilon^{-1})$, using the results of the previous sections.

A. Neutrino Oscillations

A1. $\delta m_{31}^2 > 0$ Case

From Eq. (155), we note that we need $\tilde{U}_{\mu 2}$ and $\tilde{U}_{\mu 3}$ to calculate $\tilde{P}(\nu_\mu \rightarrow \nu_\mu)$. When $\delta = 0$, these are given by

$$\begin{aligned} \tilde{U}_{\mu 2} &= \tilde{c}_{12} \tilde{c}_{23} - \tilde{s}_{12} \tilde{s}_{13} \tilde{s}_{23}, \\ \tilde{U}_{\mu 3} &= \tilde{c}_{13} \tilde{s}_{23}. \end{aligned} \quad (157)$$

For the $\delta m_{31}^2 > 0$ case, the effective mixing angles in the region $a/|\delta m_{31}^2| = O(\varepsilon^{-1})$ are well approximated by [7]

$$\begin{aligned} \tilde{\theta}_{13} &\approx \frac{\pi}{2} - \left(\frac{\delta m_{31}^2}{2a} \right) \sin(2\theta_{13}) = \frac{\pi}{2} - O(\varepsilon^2), \\ \tilde{\theta}_{12} &\approx \frac{\pi}{2} - \frac{c_{13}}{c'_{13}} \left(\frac{\delta m_{21}^2}{2a} \right) \sin(2\theta_{12}) + \chi = \frac{\pi}{2} - O(\varepsilon), \\ \tilde{\theta}_{23} &\approx \theta_{23} + \frac{s_\phi}{c'_{13}} \left(\frac{\delta m_{21}^2}{2a} \right) \sin(2\theta_{12}) = \theta_{23} + O(\varepsilon). \end{aligned} \quad (158)$$

Using $\tilde{s}_{13} = 1 - O(\varepsilon^4)$, $\tilde{c}_{13} = O(\varepsilon^2)$, we find

$$\begin{aligned} \tilde{U}_{\mu 2} &\approx \tilde{c}_{12} \tilde{c}_{23} - \tilde{s}_{12} \tilde{s}_{23} = \cos(\tilde{\theta}_{12} + \tilde{\theta}_{23}), \\ \tilde{U}_{\mu 3} &\approx 0. \end{aligned} \quad (159)$$

Therefore,

$$\tilde{P}(\nu_\mu \rightarrow \nu_\mu) \approx 1 - \sin^2 \{ 2(\tilde{\theta}_{12} + \tilde{\theta}_{23}) \} \sin^2 \frac{\tilde{\Delta}_{21}}{2}.$$

Note that

$$\tilde{\theta}_{12} + \tilde{\theta}_{23} \approx \frac{\pi}{2} + \theta_{23} + \left(\frac{s_\phi - c_{13}}{c'_{13}} \right) \left(\frac{\delta m_{21}^2}{2a} \right) \sin(2\theta_{12}) + \chi. \quad (160)$$

Using $s'_{13} = 1 - O(\varepsilon^4)$, $c'_{13} = O(\varepsilon^2)$, we find

$$s_\phi = \sin(\theta'_{13} - \theta_{13}) = s'_{13}c_{13} + c'_{13}s_{13} \approx c_{13} , \quad (161)$$

which shows that the $O(\varepsilon)$ terms in $\tilde{\theta}_{12}$ and $\tilde{\theta}_{23}$ other than χ cancel (this only happens for the $\delta = 0$ case considered here) and we can approximate

$$\tilde{\theta}_{12} + \tilde{\theta}_{23} \approx \frac{\pi}{2} + \theta_{23} + \chi . \quad (162)$$

Therefore,

$$\tilde{P}(\nu_\mu \rightarrow \nu_\mu) \approx 1 - \sin^2\{2(\theta_{23} + \chi)\} \sin^2 \frac{\tilde{\Delta}_{21}}{2} . \quad (163)$$

A2. $\delta m_{31}^2 < 0$ Case

For the $\delta m_{31}^2 < 0$ case, the effective mixing angles in the region $a/|\delta m_{31}^2| = O(\varepsilon^{-1})$ are well approximated by [7]

$$\begin{aligned} \tilde{\theta}_{13} &\approx - \left(\frac{\delta m_{31}^2}{a} \right) \theta_{13} = O(\varepsilon^2) , \\ \tilde{\theta}_{12} &\approx \frac{\pi}{2} - \left(\frac{\delta m_{21}^2}{2a} \right) \sin(2\theta_{12}) = \frac{\pi}{2} - O(\varepsilon^3) , \\ \tilde{\theta}_{23} &\approx \theta_{23} - \eta . \end{aligned} \quad (164)$$

Using $\tilde{s}_{13} = O(\varepsilon^2)$, $\tilde{c}_{13} = 1 - O(\varepsilon^4)$, $\tilde{s}_{12} = 1 - O(\varepsilon^6)$, $\tilde{c}_{12} = O(\varepsilon^3)$, we find

$$\begin{aligned} \tilde{U}_{\mu 2} &\approx 0 , \\ \tilde{U}_{\mu 3} &\approx \tilde{s}_{23} . \end{aligned} \quad (165)$$

Therefore,

$$\begin{aligned} \tilde{P}(\nu_\mu \rightarrow \nu_\mu) &\approx 1 - \sin^2(2\tilde{\theta}_{23}) \sin^2 \frac{\tilde{\Delta}_{31}}{2} \\ &\approx 1 - \sin^2\{2(\theta_{23} - \eta)\} \sin^2 \frac{\tilde{\Delta}_{31}}{2} . \end{aligned} \quad (166)$$

B. Anti-Neutrino Oscillations

B1. $\delta m_{31}^2 > 0$ Case

To calculate the $\bar{\nu}_\mu$ survival probability, we need $\tilde{U}_{\mu 2}$ and $\tilde{U}_{\mu 3}$ as can be seen from Eq. (156). When $\delta = 0$, we have

$$\tilde{U}_{\mu 2} = \tilde{c}_{12}\tilde{c}_{23} - \tilde{s}_{12}\tilde{s}_{13}\tilde{s}_{23} ,$$

$$\tilde{U}_{\mu 3} = \tilde{c}_{13} \tilde{s}_{23} . \quad (167)$$

For the $\delta m_{31}^2 > 0$ case, the effective mixing angles in the region $a/|\delta m_{31}^2| = O(\varepsilon^{-1})$ are well approximated by [7]

$$\begin{aligned} \tilde{\theta}_{12} &\approx \frac{\delta m_{21}^2}{2a} \sin(2\theta_{12}) = O(\varepsilon^3) , \\ \tilde{\theta}_{13} &\approx \frac{\delta m_{31}^2}{a} \theta_{13} = O(\varepsilon^2) , \\ \tilde{\theta}_{23} &\approx \theta_{23} + \bar{\chi} . \end{aligned} \quad (168)$$

Therefore, $\tilde{s}_{12} = O(\varepsilon^3)$, $\tilde{c}_{12} = 1 - O(\varepsilon^6)$, $\tilde{s}_{13} = O(\varepsilon^2)$, $\tilde{c}_{13} = 1 - O(\varepsilon^4)$, and we can approximate

$$\begin{aligned} \tilde{U}_{\mu 2} &\approx \tilde{c}_{23} , \\ \tilde{U}_{\mu 3} &\approx \tilde{s}_{23} , \end{aligned} \quad (169)$$

which yields

$$\begin{aligned} \tilde{P}(\bar{\nu}_\mu \rightarrow \bar{\nu}_\mu) &\approx 1 - \sin^2(2\tilde{\theta}_{23}) \sin^2 \frac{\tilde{\Delta}_{32}}{2} \\ &\approx 1 - \sin^2\{2(\theta_{23} + \bar{\chi})\} \sin^2 \frac{\tilde{\Delta}_{32}}{2} . \end{aligned} \quad (170)$$

B2. $\delta m_{31}^2 < 0$ Case

For the $\delta m_{31}^2 < 0$ case, the effective mixing angles in the region $a/|\delta m_{31}^2| = O(\varepsilon^{-1})$ are well approximated by [7]

$$\begin{aligned} \tilde{\theta}_{13} &\approx \frac{\pi}{2} + \left(\frac{\delta m_{31}^2}{a} \right) \theta_{13} = \frac{\pi}{2} + O(\varepsilon^2) , \\ \tilde{\theta}_{12} &\approx \frac{c_{13}}{\tilde{c}'_{13}} \left(\frac{\delta m_{21}^2}{2a} \right) \sin(2\theta_{12}) + \bar{\eta} = O(\varepsilon) , \\ \tilde{\theta}_{23} &\approx \theta_{23} - \frac{\bar{s}_\phi}{\tilde{c}'_{13}} \left(\frac{\delta m_{21}^2}{2a} \right) \sin(2\theta_{12}) = \theta_{23} + O(\varepsilon) . \end{aligned} \quad (171)$$

Using $\tilde{s}_{13} = 1 - O(\varepsilon^4)$, $\tilde{c}_{13} = O(\varepsilon^2)$, we can approximate

$$\begin{aligned} \tilde{U}_{\mu 2} &\approx \tilde{c}_{12} \tilde{c}_{23} - \tilde{s}_{12} \tilde{s}_{23} = \cos(\tilde{\theta}_{12} + \tilde{\theta}_{23}) , \\ \tilde{U}_{\mu 3} &\approx 0 , \end{aligned} \quad (172)$$

which yields

$$\tilde{P}(\bar{\nu}_\mu \rightarrow \bar{\nu}_\mu) \approx 1 - \sin^2\{2(\tilde{\theta}_{12} + \tilde{\theta}_{23})\} \sin^2 \frac{\tilde{\Delta}_{21}}{2} . \quad (173)$$

Note that

$$\tilde{\theta}_{12} + \tilde{\theta}_{23} \approx \theta_{23} - \left(\frac{\bar{s}_\phi - c_{13}}{\bar{c}'_{13}} \right) \left(\frac{\delta m_{21}^2}{2a} \right) \sin(2\theta_{12}) + \bar{\eta}. \quad (174)$$

Using $\bar{s}'_{13} = 1 - O(\varepsilon^4)$, $\bar{c}'_{13} = O(\varepsilon^2)$, we find

$$\bar{s}_\phi = \sin(\bar{\theta}'_{13} - \theta_{13}) = \bar{s}'_{13}c_{13} - \bar{c}'_{13}s_{13} \approx c_{13}, \quad (175)$$

which allows us to approximate

$$\tilde{\theta}_{12} + \tilde{\theta}_{23} \approx \theta_{23} + \bar{\eta}. \quad (176)$$

(Again, the cancellation of the $O(\varepsilon)$ terms other than $\bar{\eta}$ occurs only for the $\delta = 0$ case considered here.) Therefore,

$$\tilde{P}(\bar{\nu}_\mu \rightarrow \bar{\nu}_\mu) \approx 1 - \sin^2\{2(\theta_{23} + \bar{\eta})\} \sin^2 \frac{\tilde{\Delta}_{21}}{2}. \quad (177)$$

C. Summary of Oscillation Probabilities

To summarize what we have found, the ν_μ and $\bar{\nu}_\mu$ survival probabilities for the $\delta m_{31}^2 > 0$ (normal hierarchy) case are given by

$$\begin{aligned} \tilde{P}(\nu_\mu \rightarrow \nu_\mu) &\approx 1 - \sin^2\{2(\theta_{23} - \zeta)\} \sin^2 \frac{\tilde{\Delta}_{21}}{2}, \\ \tilde{P}(\bar{\nu}_\mu \rightarrow \bar{\nu}_\mu) &\approx 1 - \sin^2\{2(\theta_{23} + \zeta)\} \sin^2 \frac{\tilde{\Delta}_{32}}{2}, \end{aligned} \quad (178)$$

while for the $\delta m_{31}^2 < 0$ (inverted hierarchy) case, they are given by

$$\begin{aligned} \tilde{P}(\nu_\mu \rightarrow \nu_\mu) &\approx 1 - \sin^2\{2(\theta_{23} + \zeta)\} \sin^2 \frac{\tilde{\Delta}_{31}}{2}, \\ \tilde{P}(\bar{\nu}_\mu \rightarrow \bar{\nu}_\mu) &\approx 1 - \sin^2\{2(\theta_{23} - \zeta)\} \sin^2 \frac{\tilde{\Delta}_{21}}{2}, \end{aligned} \quad (179)$$

where we have defined

$$\zeta \equiv \frac{a\xi}{2|\delta m_{31}^2|}. \quad (180)$$

Therefore, though the effect of a non-zero ξ appears in different effective mixing angles depending on the mass hierarchy, and whether the particle considered is the neutrino or the anti-neutrino (cf. Table I), the net effect on the ν_μ and $\bar{\nu}_\mu$ survival probabilities for all cases is to shift θ_{23} in the oscillation amplitude.

Unfortunately, this shift in θ_{23} may be difficult to observe. The current experimentally preferred value of $\sin^2(2\theta_{\text{atm}}) \approx \sin^2(2\theta_{23})$ is one, with the 90% lower limit given by [13, 14]

$$\sin^2(2\theta_{23}) > 0.92 . \quad (181)$$

Given the shape of the function $\sin^2(2\theta_{23})$ around $\theta_{23} = \pi/4$, $\sin^2(2\theta_{23})$ is insensitive to small shifts in θ_{23} . Indeed, because of this, the angle θ_{23} itself is ill constrained, the above limit translating into

$$\theta_{23} = (0.2 \sim 0.3)\pi . \quad (182)$$

However, our knowledge of the value of $\sin^2(2\theta_{23})$ is to be improved considerably in the near future. The long baseline neutrino oscillation experiments MINOS [8], T2K [15], NO ν A [16], and others [17–20] will measure $\sin^2(2\theta_{\text{atm}}) = 4|U_{\mu 3}|^2(1 - |U_{\mu 3}|^2) = 4s_{23}^2c_{13}^2(1 - s_{23}^2c_{13}^2)$ from $\nu_\mu \rightarrow \nu_\mu$ to better than 1%, while the reactor neutrino experiments Double-Chooz [21], KASKA [22], Braidwood [23], etc. [24] are expected to measure $\sin^2(2\theta_{\text{rct}}) = 4|U_{e 3}|^2(1 - |U_{e 3}|^2) = \sin^2(2\theta_{13})$ from $\bar{\nu}_e \rightarrow \bar{\nu}_e$ to an accuracy of ± 0.01 . These developments combined will determine $\sin^2(2\theta_{23})$ to better than 1%, albeit with a two-fold degeneracy. This degeneracy can be broken, in principle, by determining $4|U_{\mu 3}|^2|U_{e 3}|^2 = s_{23}^2 \sin^2(2\theta_{13})$ from the CP non-violating part of the $\nu_\mu \rightarrow \nu_e$ oscillation probability [16, 25–34]. Therefore, a unique and accurate value of $\sin^2(2\theta_{23})$, together with whether θ_{23} is larger or smaller than $\pi/4$, may be known. Furthermore, if the 1-Megaton Hyper-Kamiokande (HyperK) detector is ever constructed, a JPARC \rightarrow HyperK long-baseline experiment will improve the limits even further [10].

Even then, if the central value of $\sin^2(2\theta_{23})$ is too close to one, then the shift due to ζ will be invisible. Let us assume, for the sake of argument, that a 1% shift in $\sin^2(2\theta_{23})$ is detectable. Since

$$\sin^2\{2(\theta_{23} \pm \zeta)\} = \sin^2(2\theta_{23}) \pm 2 \sin(4\theta_{23})\zeta , \quad (183)$$

the shift due to ζ would be visible if

$$|2 \sin(4\theta_{23})\zeta| > 0.01 . \quad (184)$$

The size of ζ for $\rho = 4.6 \text{ g/cm}^3$, $E = 17 \text{ GeV}$, $\xi = 0.025$, and $|\delta m_{31}^2| = 2.5 \times 10^{-3} \text{ eV}^2$, for instance, is

$$\zeta = 0.03 . \quad (185)$$

For this shift to be visible, we must have

$$|\sin(4\theta_{23})| > \frac{1}{6}, \quad (186)$$

or

$$\sin^2(2\theta_{23}) < 0.993. \quad (187)$$

If we require a 2% shift, the limit will be $\sin^2(2\theta_{23}) < 0.97$, and a 3% shift would require $\sin^2(2\theta_{23}) < 0.93$. Therefore, whether the effect we are considering can be observed or not depends crucially on the value of $\sin^2(2\theta_{23})$.

VI. NUMERICAL RESULTS

The discussions up to this point were all based on approximate analytical calculations. To illustrate the accuracy of our analytical results, we presenting here the results of a numerical calculation of the effective mass-squared-differences, effective mixing angles, and oscillation probabilities.

As inputs, we use the following: For θ_{23} , we consider the two cases

$$\sin^2(2\theta_{23}) = 1, \quad \text{and} \quad \sin^2(2\theta_{23}) = 0.92 \quad \text{with} \quad \theta_{23} < \frac{\pi}{4}. \quad (188)$$

The values of θ_{23} for these cases are

$$\theta_{23} = \frac{\pi}{4}, \quad \text{and} \quad \theta_{23} = 0.204 \pi. \quad (189)$$

For ξ , we compare the two cases

$$\xi = 0, \quad \text{and} \quad \xi = 0.025. \quad (190)$$

$\xi = 0.025$ corresponds to the central value of the CHARM/CHARM II constraint, Eq. (1).

The remaining parameters are fixed to (see Ref. [7] and references therein) :

$$\begin{aligned} \delta m_{21}^2 &= 8.2 \times 10^{-5} \text{ eV}^2, \\ |\delta m_{31}^2| &= 2.5 \times 10^{-3} \text{ eV}^2, \\ \tan^2 \theta_{12} &= 0.4, \\ \sin^2(2\theta_{13}) &= 0.16, \\ \delta &= 0, \end{aligned}$$

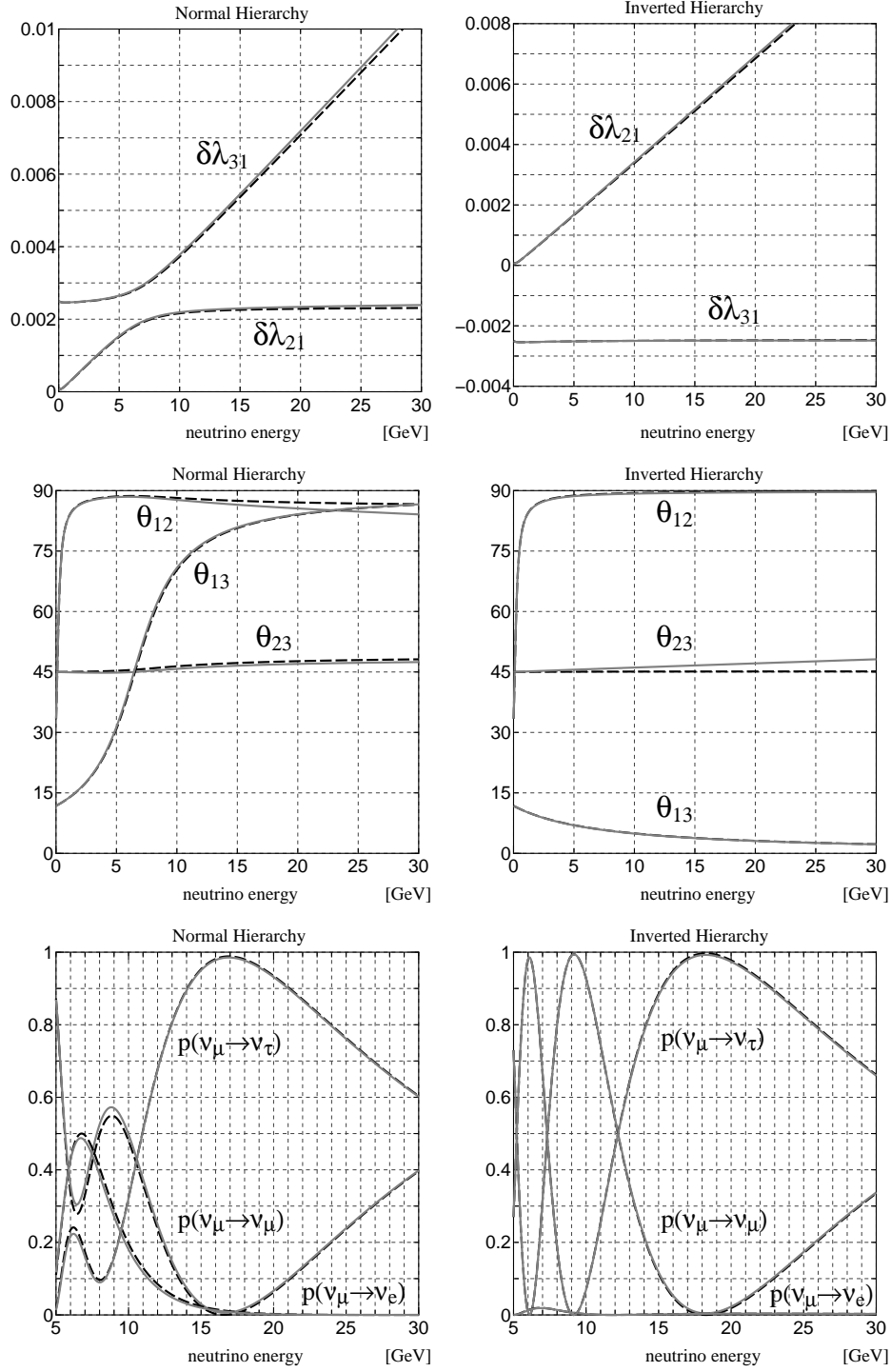


FIG. 1: The effective mass-squared-differences, effective mixing angles, and oscillation probabilities for the case $\sin^2(2\theta_{23}) = 1$. The other input parameters are given in Eq. (191). The $\xi = 0$ case is plotted with black dashed lines, while the $\xi = 0.025$ case is plotted with gray solid lines.

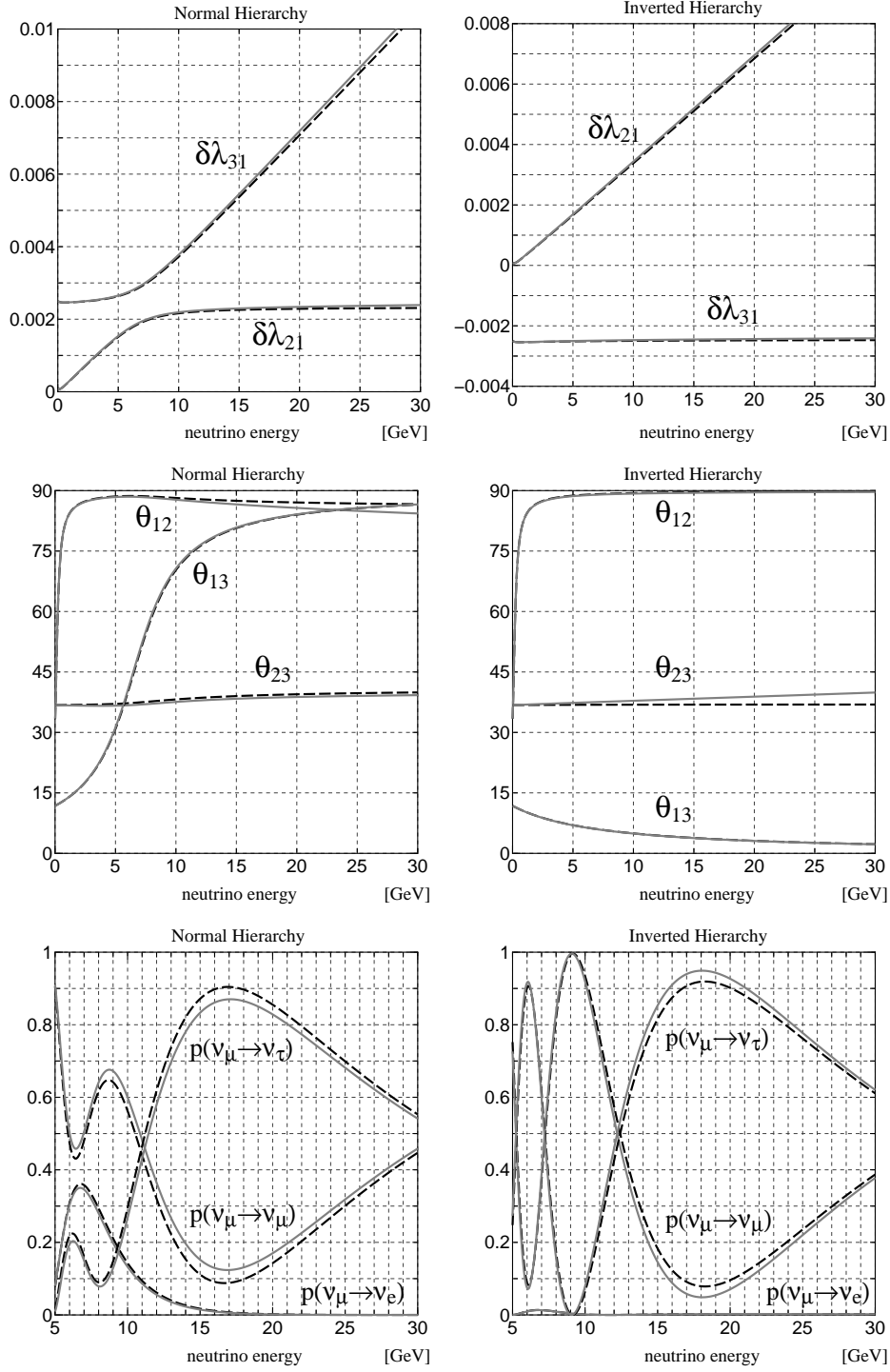


FIG. 2: The effective mass-squared-differences, effective mixing angles, and oscillation probabilities for the case $\sin^2(2\theta_{23}) = 0.92$ with $\theta_{23} < \pi/4$. The other input parameters are given in Eq. (191). The $\xi = 0$ case is plotted with black dashed lines, while the $\xi = 0.025$ case is plotted with gray solid lines.

$$\begin{aligned}\rho &= 4.6 \text{ g/cm}^3, \\ L &= 9120 \text{ km.}\end{aligned}\tag{191}$$

The baseline length of $L = 9120 \text{ km}$ is the distance from Fermilab to Kamioka, Japan, and the mass density of $\rho = 4.6 \text{ g/cm}^3$ is the average mass density along this baseline calculated from the Preliminary Earth Reference Model [35].

Figs. 1 and 2 show the energy dependence of the effective mass-squared differences, effective mixing angles, and oscillation probabilities of the neutrinos: Fig. 1 for the $\sin^2(2\theta_{23}) = 1$ case, and Fig. 2 for the $\sin^2(2\theta_{23}) = 0.92$ ($\theta_{23} < \pi/4$) case. In the figures, the $\xi = 0$ case is plotted in broken black lines while the $\xi = 0.025$ case is plotted in solid gray lines. As can be clearly seen from the graphs in the top rows of both figures, the effective mass-squared differences are little affected by ξ as expected. On the other hand, the graphs in the middle rows show that of the effective mixing angles, $\tilde{\theta}_{12}$ is shifted in the negative direction when $\delta m_{31}^2 > 0$ (normal hierarchy), while $\tilde{\theta}_{23}$ is shifted in the positive direction when $\delta m_{31}^2 < 0$ (inverted hierarchy), as tabulated in Table. I. However, the graphs on the bottom rows show that these shifts in the mixing angles are virtually invisible in the oscillation probabilities when $\sin^2(2\theta_{23}) = 1$, but quite visible when $\sin^2(2\theta_{23}) = 0.92$, again as expected.

Numerical calculations for the anti-neutrino case also confirm the accuracy of our analytical results, though we will not present them here.

VII. FERMILAB \rightarrow HYPER-KAMIOKANDE

If the value of $\sin^2(2\theta_{23})$ is not too close to one, then matter effects due to neutral current universality violation will lead to shifts in the oscillation probabilities, as shown in the bottom row of Fig. 2. Let us now ask whether such shifts are observable in long baseline neutrino oscillation experiments. In the following, we will assume that $\sin^2(2\theta_{23}) = 0.92$ ($\theta_{23} < \pi/4$), which was the value used in Fig. 2, and that it is accurately known.

The effect we would like to see only appears at mass densities and energies at which

$$|\delta m_{31}^2| < a = (7.6324 \times 10^{-5} \text{ eV}^2) \times \left(\frac{\rho}{\text{g/cm}^3} \right) \times \left(\frac{E}{\text{GeV}} \right), \tag{192}$$

or

$$(26 \sim 39) < \left(\frac{\rho}{\text{g/cm}^3} \right) \times \left(\frac{E}{\text{GeV}} \right), \tag{193}$$

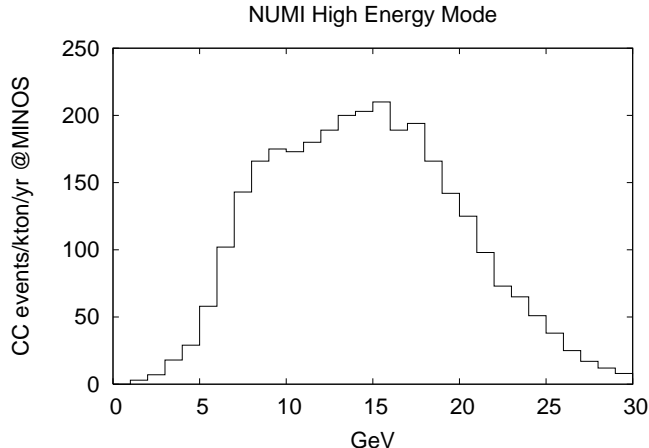


FIG. 3: The NUMI beam in its high energy mode. The vertical axis gives the number of expected charged current ν_μ events at the MINOS site (732 km away from Fermilab) per kiloton of detector material, per year, per GeV bin without any oscillation.

for $|\delta m_{31}^2| = (2 \sim 3) \times 10^{-3} \text{eV}^2$. Since the mass density of the Earth's crust and mantle are $3 \sim 5 \text{g/cm}^3$ [35], this requires the beam energy to be larger than $\sim 10 \text{GeV}$. At these energies, the position of the first oscillation peak (dip) is determined by the condition

$$\frac{a}{2E}L \sim \pi, \quad (194)$$

which translates to

$$\left(\frac{\rho}{\text{g/cm}^3}\right) \times \left(\frac{L}{\text{km}}\right) \sim 3 \times 10^4, \quad (195)$$

or

$$L \sim 10^4 \text{ km}. \quad (196)$$

Therefore, the experiment we need to consider is such that a neutrino beam of energy in excess of 10 GeV is aimed at a detector about 10,000 km away.

At this point, we note that a ν_μ beam with the required energies is already available at Fermilab. Fig. 3 is reproduced from the NUMI Technical Design Handbook [9] and shows the energy profile of the NUMI beam in its high energy mode. As we can see, the beam has considerable support in the $5 \sim 25 \text{GeV}$ range. The vertical axis is the expected number of charged current ν_μ events at MINOS per kiloton of detector material, per year, per GeV bin without any oscillation. If a similar beam were aimed at a detector $\sim 10^4 \text{km}$ away, which is more than 10 times the distance from Fermilab to MINOS, the ν_μ flux will be attenuated by

at least 2 orders of magnitude from what is available at MINOS. Therefore, a megaton class detector would be required if the number of observed events is to be statistically significant.

The planned Hyper-Kamiokande (HyperK) [10] is a megaton water-Cherenkov detector which would be at a distance of $L = 9120$ km from Fermilab. Aiming a NUMI-like beam from Fermilab at HyperK (the declination angle is 46 degrees) would provide the necessary energy, detector mass, and baseline length. So this is the setup we will consider. The average matter density along the baseline would be 4.6 g/cm^3 , and the oscillation probability to be measured will be that shown in Fig. 2.

In Fig. 4, we show the expected number of ν_μ events at HyperK for 5 years of data taking. The dotted line indicates the expected numbers without any oscillation, and was obtained by rescaling the numbers from Fig. 3 to take into account the difference in baseline length, detector mass, and the number of years of data taking. The solid line indicates the expected number of events with oscillation taken into account for the normal hierarchy case with $\xi = 0$. The input parameters are those listed in Eq. (191) with $\sin^2(2\theta_{23}) = 0.92$ ($\theta_{23} < \pi/4$). The $\xi = +0.025$ and $\xi = -0.025$ cases are shown with dashed, and dot-dashed lines, respectively. As one can see from the figure, the expected number of events is fairly large even at this distance, and even with oscillation.

Fig. 5 shows a blowup of Fig. 4. Even at the oscillation dip at 17 GeV, the expected number of events is in the hundreds. Due to this significant statistics, the $\xi = 0$ and $\xi = \pm 0.025$ cases are clearly distinguishable as indicated by the error bars on the $\xi = 0$ case. Therefore, this experiment can easily detect a violation in neutral current universality if it is as large as the CHARM/CHARM II central value.

To see what kind of constraint this experiment could place on ξ , we calculate the χ^2 between the $\xi = 0$ and the $\xi \neq 0$ cases, *i.e.*

$$\chi^2(\xi) \equiv \sum_{8 \text{ GeV} < i < 22 \text{ GeV}} \frac{[N_i(\xi) - N_i(0)]^2}{N_i(0)}, \quad (197)$$

where $N_i(\xi)$ is the expected number of events in the i -th GeV-wide bin, and plot the ξ -dependence of the $\chi^2(\xi)$ in Fig. 6. We have restricted the bins that enter into $\chi^2(\xi)$ to the 8 to 22 GeV range (14 bins), since that is the range in which the expected number of events fluctuates significantly with ξ . With 5 years of data taking, we can read off from the graph that the $\xi = 0$ and $\xi = \pm 0.005$ cases are distinguishable at the 99% confidence level. This corresponds to a limit on universality violation at the 1% level, which will be comparable to

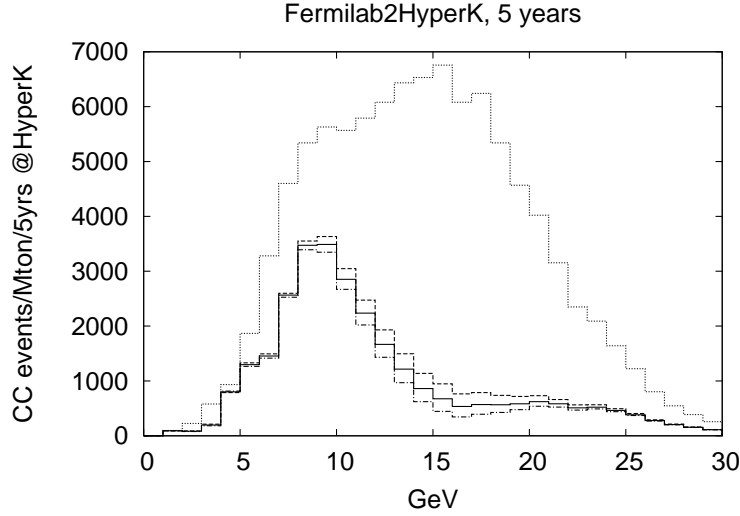


FIG. 4: The expected number of ν_μ events at a Fermilab \rightarrow HyperK experiment with 5 years of data taking. The dotted line indicates the number of events without any oscillation. The solid, dashed, and dot-dashed lines indicate the number of expected events with oscillation taken into account for the $\xi = 0$, $\xi = +0.025$, and $\xi = -0.025$ cases, respectively. The mass hierarchy assumed was the normal hierarchy, $\sin^2(2\theta_{23}) = 0.92$ ($\theta_{23} < \pi/4$), and the other input parameters were those listed in Eq. (191).

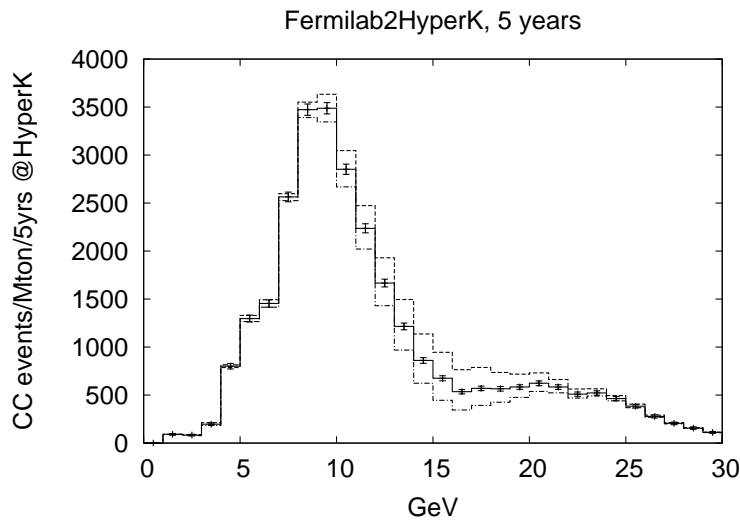


FIG. 5: Blowup of Fig. 4. Error bars have been added for the $\xi = 0$ case.

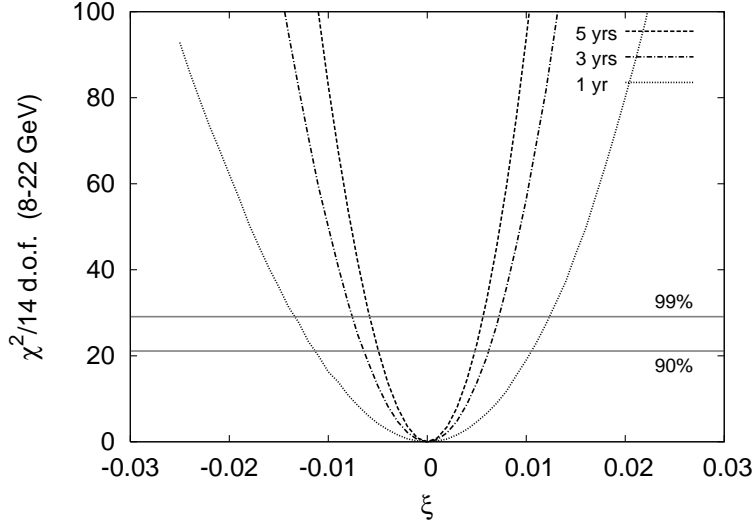


FIG. 6: The χ^2 for the 14 bins from 8 to 22 GeV between the $\xi = 0$ and $\xi \neq 0$ cases.

the constraints from LEP/SLD [1–3] but completely model independent. For fewer years of data taking, the limits will be correspondingly weaker.

We emphasize that the conclusions in this section are valid only for the $\sin^2(2\theta_{23}) = 0.92$ ($\theta_{23} < \pi/4$) case. The closer $\sin^2(2\theta_{23})$ is to one, the more difficult it will be to detect the presence or absence of ξ .

VIII. SUMMARY AND CONCLUSIONS

In this paper, we have considered the matter effect on neutrino oscillations due to neutral current universality violation. It was shown that the effect of the violation appears dominantly as a shift in the effective value of θ_{23} at high energies, while the other effective mixing angles and effective mass-squared-differences are virtually unaffected. As a result, the effect will manifest itself as changes in the amplitudes of the oscillation probabilities, while the locations of the oscillation peaks and dips in distance/energy remain the same. However, since the amplitudes of the ν_μ and $\bar{\nu}_\mu$ survival probabilities are proportional to $\sin^2(2\theta_{23})$, the shift in θ_{23} would be difficult to detect if $\sin^2(2\theta_{23})$ is too close to one.

If the value of $\sin^2(2\theta_{23})$ is as small as 0.92, the current 90% lower bound, then a 5-year measurement of the ν_μ survival spectrum by a Fermilab \rightarrow HyperK experiment could place a model-independent constraint on neutral current universality violation at the 1% level. This would be competitive with the model-dependent constraints extracted from LEP/SLD data

[1–3].

The analysis in this paper was restricted to the $\delta = 0$ case, in which the effective θ_{23} was unaffected by charged-current interactions. For the $\delta \neq 0$ cases, one needs to account for the charged-current shift discussed in Ref. [7], in addition to the neutral-current shift discussed in this paper, making the analysis somewhat more complicated. However, for the neutrino case with inverted hierarchy ($\delta m_{31}^2 < 0$), and the anti-neutrino case with normal hierarchy ($\delta m_{31}^2 > 0$), charged-current effects are always absent from θ_{23} , regardless of the value of δ . Therefore, using neutrinos if the hierarchy is inverted, and anti-neutrinos if the hierarchy is normal, can potentially provide a clean signal.

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- [1] O. Lebedev, W. Loinaz and T. Takeuchi, Phys. Rev. D **61**, 115005 (2000) [arXiv:hep-ph/9910435].
 - [2] W. Loinaz, N. Okamura, S. Rayyan, T. Takeuchi and L. C. R. Wijewardhana, Phys. Rev. D **70**, 113004 (2004) [arXiv:hep-ph/0403306].
 - [3] L. N. Chang, O. Lebedev, W. Loinaz and T. Takeuchi, Phys. Rev. D **63**, 074013 (2001) [arXiv:hep-ph/0010118].
 - [4] J. Dorenbosch *et al.* [CHARM Collaboration], Phys. Lett. B **180**, 303 (1986); P. Vilain *et al.* [CHARM-II Collaboration], Phys. Lett. B **320**, 203 (1994).
 - [5] *Review of Particle Physics*, Phys. Lett. B **592** (2004) 1.
 - [6] ALEPH, DELPHI, L3, OPAL, and SLD Collaborations, LEP Electroweak Working Group, SLD Electroweak Group, and SLD Heavy Flavour Group, arXiv:hep-ex/0509008.

- [7] M. Honda, Y. Kao, N. Okamura, and T. Takeuchi, VPI-IPPAP-06-01, hep-ph/0602115.
- [8] G. S. Tzanakos [MINOS Collaboration], AIP Conf. Proc. **721**, 179 (2004); See also <http://www-numi.fnal.gov/talks/postedtalks.html>.
- [9] Figure 2-5 in Section 2.3 of the NUMI Technical Design Handbook, available at http://www-numi.fnal.gov/numwork/tdh/tdh_index.html
- [10] Y. Itow *et al.*, arXiv:hep-ex/0106019; updated version available at <http://neutrino.kek.jp/jhfnu/>.
- [11] L. Wolfenstein, Phys. Rev. D **17**, 2369 (1978); R. R. Lewis, Phys. Rev. D **21**, 663 (1980); V. D. Barger, K. Whisnant, S. Pakvasa and R. J. N. Phillips, Phys. Rev. D **22**, 2718 (1980); S. P. Mikheyev and A. Yu. Smirnov, Yad. Fiz. **42**, 1441 (1985), [Sov. J. Nucl. Phys. **42**, 913 (1986)]; S. P. Mikheyev and A. Yu. Smirnov, Nuovo Cimento **C9**, 17 (1986).
- [12] Z. Maki, M. Nakagawa and S. Sakata, Prog. Theor. Phys. **28**, 870 (1962).
- [13] Y. Ashie *et al.* [Super-Kamiokande Collaboration], Phys. Rev. Lett. **93**, 101801 (2004) [arXiv:hep-ex/0404034]; Y. Ashie *et al.* [Super-Kamiokande Collaboration], arXiv:hep-ex/0501064.
- [14] M. H. Ahn *et al.* [K2K Collaboration], Phys. Rev. Lett. **93**, 051801 (2004) [arXiv:hep-ex/0402017], E. Aliu *et al.* [K2K Collaboration], Phys. Rev. Lett. **94**, 081802 (2005) [arXiv:hep-ex/0411038], C. Mariani [K2K Collaboration], arXiv:hep-ex/0505019.
- [15] Y. Itow *et al.*, arXiv:hep-ex/0106019; Y. Hayato [T2K Collaboration], Nucl. Phys. Proc. Suppl. **143**, 269 (2005), K. Kaneyuki [T2K Collaboration], Nucl. Phys. Proc. Suppl. **145**, 178 (2005), Y. Hayato [T2K Collaboration], Nucl. Phys. Proc. Suppl. **147**, 9 (2005). See also the JHF Neutrino Working Group's home page, <http://neutrino.kek.jp/jhfnu/>.
- [16] D. Ayres *et al.* [NOvA Collaboration], arXiv:hep-ex/0210005, D. S. Ayres *et al.* [NOvA Collaboration], arXiv:hep-ex/0503053,
- [17] M. V. Diwan *et al.*, Phys. Rev. D **68**, 012002 (2003) [arXiv:hep-ph/0303081].
- [18] N. Okamura, M. Aoki, K. Hagiwara, Y. Hayato, T. Kobayashi, T. Nakaya and K. Nishikawa, arXiv:hep-ph/0104220; M. Aoki, K. Hagiwara, Y. Hayato, T. Kobayashi, T. Nakaya, K. Nishikawa and N. Okamura, Phys. Rev. D **67**, 093004 (2003) [arXiv:hep-ph/0112338].
- [19] H. S. Chen *et al.* [VLBL Study Group H2B-1], arXiv:hep-ph/0104266, Y. F. Wang, K. Whisnant, Z. H. Xiong, J. M. Yang and B. L. Young [VLBL Study Group H2B-4], Phys. Rev. D **65**, 073021 (2002) [arXiv:hep-ph/0111317].

- [20] S. Antusch, P. Huber, J. Kersten, T. Schwetz and W. Winter, Phys. Rev. D **70**, 097302 (2004) [arXiv:hep-ph/0404268].
- [21] F. Ardellier *et al.* [Double-Chooz Collaboration], arXiv:hep-ex/0405032; F. Dalnoki-Veress [Double-Chooz Collaboration], arXiv:hep-ex/0406070.
- [22] F. Suekane [KASKA Collaboration], arXiv:hep-ex/0407016.
- [23] Braidwood Collaboration, <http://mwtheta13.uchicago.edu/>
- [24] K. Anderson *et al.*, arXiv:hep-ex/0402041.
- [25] O. Mena Requejo, S. Palomares-Ruiz and S. Pascoli, Phys. Rev. D **72**, 053002 (2005) [arXiv:hep-ph/0504015]; O. Mena and S. Parke, Phys. Rev. D **72**, 053003 (2005) [arXiv:hep-ph/0505202]; O. Mena, S. Palomares-Ruiz and S. Pascoli, arXiv:hep-ph/0510182.
- [26] H. Minakata and H. Nunokawa, Phys. Lett. B **413**, 369 (1997) [arXiv:hep-ph/9706281];
- [27] V. Barger, D. Marfatia and K. Whisnant, Phys. Lett. B **560**, 75 (2003) [arXiv:hep-ph/0210428].
- [28] H. Minakata, H. Sugiyama, O. Yasuda, K. Inoue and F. Suekane, Phys. Rev. D **68**, 033017 (2003) [Erratum-ibid. D **70**, 059901 (2004)] [arXiv:hep-ph/0211111].
- [29] M. Aoki, K. Hagiwara and N. Okamura, Phys. Lett. B **606**, 371 (2005) [arXiv:hep-ph/0311324].
- [30] K. B. McConnel and M. H. Shaevitz, arXiv:hep-ex/0409028.
- [31] P. Huber, M. Lindner, M. Rolinec, T. Schwetz and W. Winter, Phys. Rev. D **70**, 073014 (2004) [arXiv:hep-ph/0403068].
- [32] M. Ishitsuka, T. Kajita, H. Minakata and H. Nunokawa, Phys. Rev. D **72**, 033003 (2005) [arXiv:hep-ph/0504026].
- [33] K. Hagiwara, N. Okamura and K. i. Senda, arXiv:hep-ph/0504061.
- [34] K. Hiraide, H. Minakata, T. Nakaya, H. Nunokawa, H. Sugiyama, W. J. C. Teves and R. Z. Funchal, arXiv:hep-ph/0601258.
- [35] A. M. Dziewonski and D. L. Anderson, Physics of the Earth and Planetary Interiors, **25** (1981) 297–356.