APPROXIMATION OF INTEGRO-PARTIAL DIFFERENTIAL EQUATIONS OF HYPERBOLIC TYPE

by

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DEDICATION

This work is dedicated to my parents,

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TABLE OF CONTENTS

CHAPTER I

INTRODUCTION AND NOTATION

1.1 Introduction

In this paper we develop an approximation scheme for application to a class of partial functional differential equations (PFDEs) of hyperbolic type. Equations of this type often arise in models that describe the motion of materials which exhibit "fading-memory" behavior. In the underlying constitutive equations for these models, the stress is assumed to be a function of the past history of the strain. Viscoelastic materials, for example, have this property.

The basic theme of our approach is to pose the problem abstractly in a Hilbert space setting, the so-called state space formulation. This leads to an infinite dimensional problem, for which the dynamics are governed by an abstract Cauchy problem. In this setting, semigroup techniques are used to address questions of well-posedness and convergence of approximation schemes.

For example, the abstract linear quadratic cost optimal control problem is to minimize the cost functional

$$J(z_0,u) = \int_0^T (\langle Qz(s),z(s)\rangle + \langle Ru(s),u(s)\rangle)ds, \qquad (1.1.1)$$

subject to dynamics governed by an abstract Cauchy problem of the form

$$\dot{z}(t) = Az(t) + Bu(t) \qquad (1.1.2)$$
$$z(0) = z_0$$

in a Hilbert space H. Here A is the infinitesimal generator of a C_0-semigroup T(t) on H. Typically $u(t) \in \mathbb{R}^m$ and $B \in L(\mathbb{R}^m, H)$. The optimal control is characterized by a feedback law associated with the solution of an operator Riccati equation involving both the operator A and its adjoint A^*. A fundamental result due to Gibson [11] is that in order to approximate the feedback gain for (1.1.1)-(1.1.2), one must approximate both T(t) and $T^*(t)$ in the strong operator topology.

In chapter two, we review the relevant results on approximation from linear semigroup theory, including a more detailed discussion of Gibson's results on approximation of abstract optimal control problems. We also survey the application of these ideas to control problems governed by ordinary functional differential equations (FDEs). In the last section of chapter two, we give new results on a general framework for constructing satisfactory approximation schemes for these abstract problems.

In chapter three, we study a class of PFDEs of hyperbolic type. In particular, we develop a state space formulation to which the previously developed theory may be applied. The state space formulation is shown to be well-posed, an approximation scheme is developed, and convergence results are established for this scheme.

In Chapter IV we present the results of some numerical experiments for the viscoelastic system described in Chapter III.

1.2 Notation

The following notation will be used throughout the text. The norm of a linear space X will be denoted by $\|\cdot\|_X$, while $\langle\cdot,\cdot\rangle_X$ shall denote an inner product on X. The norm and inner product will be written $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ respectively, when the associated space and particular inner product are clear from the context. The space of all bounded linear operators from X to a linear space Y will be denoted L(X,Y). For $-\infty\leq a\leq b\leq\infty$, the space of all operator valued functions $B(t):(a,b) \longrightarrow L(X,Y)$ which are bounded on (a,b) is denoted by B(a,b;X,Y). For a Hilbert space Z, the space of all square integrable functions defined on (a,b) or [a,b] with values in Z will be denoted $L_2(a,b;Z)$. The space of absolutely continuous functions $f\in L_2(a,b;Z)$ with jth derivative $f^{(j)}$ absolutely continuous for $j=1,2,\ldots,K-1$ and $f^{(K)}\in L_2(a,b;Z)$ is denoted by $H^K(a,b;Z)$. When $Z=\mathbb{R}^\nu$ and the space is clear from the context, we shall write $L_2(a,b)$ and $H^K(a,b)$ for $L_2(a,b;\mathbb{R}^\nu)$ and $H^K(a,b;\mathbb{R}^\nu)$, respectively. We will denote by $L_2^0(a,b;\mathbb{R})$ the subspace of $L_2(a,b;\mathbb{R})$ of functions with integral mean equal to zero, i.e.

$$L_2^0(a,b) = \{f\in L_2(a,b): \int_a^b f(x)dx = 0\}.$$

Furthermore, when an interval is not specified for one of the above function spaces, the interval is assumed to be the interval [a,b]. That is, $L_2 = L_2(a,b)$, $H^1(-r,0;L_2) = H^1(-r,0;L_2(a,b))$, etc. For a function $x:[-r,\alpha)\longrightarrow X$, $r,\alpha>0$, the symbol x_t for $t\epsilon[0,\alpha)$ will represent the function x_t: $[-r,0)\longrightarrow X$ defined by $x_t(s) = x(t+s)$. We will not use the subscript notation for partial differentiation, thus our use of x_t to denote the above defined translation function should not cause confusion. By $A \epsilon G(M,\beta)$ we shall mean that the operator A is the infinitesimal generator of a C_0-semigroup $T(t)$ on a Hilbert space Z satisfying $\|T(t)\|_Z \leqslant Me^{\beta t}$, $t\geqslant 0$. The symbols $z^N \xrightarrow{s} z$ and $z^N \xrightarrow{w} z$ are used to denote strong and weak convergence, respectively.

Chapter II

APPROXIMATION

2.1 The Cauchy Problem

In this section, we discuss a method for approximating an abstract Cauchy problem by a sequence of finite-dimensional ODEs. Included in this discussion is the statement of one version of the Trotter-Kato Theorem, a semigroup approximation result which is used as a standard tool for proving convergence results in this framework.

Let Z be a Hilbert space and assume that an operator A generates a C_0-semigroup T(t) on Z. We shall consider the Cauchy problem

$$\dot{z}(t) = Az(t) + f(t) \qquad t>0$$
$$z(0) = z_0. \tag{2.1.1}$$

The associated homogeneous problem ($f\equiv 0$) is well-posed in Z if and only if A is the infinitesimal generator of a C_0-semigroup on Z. In this case, a mild solution of (2.1.1) is given by the variation of constants formula

$$z(t) = T(t)z_0 + \int_0^t T(t-s)f(s)ds, \quad t\geq 0. \tag{2.1.2}$$

Under suitable assumptions on f and z_0, (2.1.1) is equivalent to (2.1.2) (see [15], [18] for a complete discussion of this Cauchy problem).

The approximation problem is to construct a convergent sequence of systems of differential equations which can be solved numerically and whose solutions approximate the solution of (2.1.1). One proceeds in standard fashion by constructing an approximation scheme consisting of (typically finite-dimensional) subspaces $Z^N \subset Z$, corresponding projections $P^N: Z \to Z^N$ (often P^N is an orthogonal projection), and operators $A^N: \mathcal{D}(A^N) \subset Z \to Z^N$. We shall denote such an approximation scheme by (Z^N, P^N, A^N), $N=1,2,\ldots$. Consider now the problem

$$\dot{z}^N(t) = A^N z^N(t) + P^N f(t) \quad t>0 \qquad (2.1.3)$$

$$z^N(0) = P^N z_0$$

on the space Z^N. When Z^N is finite-dimensional, this ODE is equivalent to the variation of constants formula

$$z^N(t) = T^N(t) P^N z_0 + \int_0^t T^N(t-s) P^N f(s) ds \qquad (2.1.4)$$

where $T^N(t) = e^{tA^N}$ is the semigroup generated by A^N on Z^N.

If the approximation scheme (Z^N, P^N, A^N) has the property that Z^N approximates Z and A^N approximates A, in the sense that $P^N z \to z$ for all $z \epsilon Z$ and $T^N(t) \xrightarrow{s} T(t)$ (uniform in t for t in compact intervals), then the right-hand side of (2.1.4) converges to the right-hand side of (2.1.1). In this case, solutions of (2.1.3) converge to solutions of (2.1.1).

In this framework, a Trotter-Kato type of theorem is the tool which is often used to show the desired convergence of the approximating semigroups $T^N(t)$ to the semigroup $T(t)$. The version of the theorem which we shall employ is given below (see [18]).

<u>THEOREM</u> <u>2.1.1</u> Let $A \epsilon G(M,B)$ be the infinitesimal generator of a C_0-semigroup $T(t)$ on a Hilbert space Z and suppose

H1) $A^N \epsilon G(M,\beta)$ for $N=1,2,...,$

H2) $A^N z \to Az$ for $z \epsilon \mathcal{D}$, $\mathcal{D}$ a dense subset of Z,

H3) there exists λ_0 with $Re(\lambda_0) > \beta$ such that $(A-\lambda_0 I)\mathcal{D}$ is a dense subset of Z.

If $T^N(t)$ denotes the C_0-semigroup generated by A^N, then $T^N(t)z \to T(t)z$ for every $z \epsilon Z$, $t \geq 0$, and the convergence is uniform in compact t-intervals. ∎

We remark that the concept of dissipativeness together with the Hilbert space inner product often facilitate the proof that the approximation scheme satisfies the stability hypothesis H1) of the theorem.

2.2 The linear quadratic optimal control problem

In this section, we discuss the approximation theory for linear quadratic optimal control problems in a Hilbert space setting. We are particularly concerned with the problem of approximating the feedback law which determines the solution to these problems. Our approach is to use a sequence of finite dimensional control problems which "converges" to the original infinite dimensional problem. For a further discussion on the approximation theory presented here, see [6], [10], and [11].

Let Z and U be real Hilbert spaces. Assume that the operator A is the infinitesimal generator of a C_0-semigroup of bounded linear operators $T(t)$ on Z. Assume that $B \in L(U,Z)$ and $Q \in L(Z,Z)$, with Q self-adjoint and nonnegative, and that $R \in L(U,U)$ is a self-adjoint operator which satisfies $R \geqslant cI > 0$ for some positive real number c.

The quadratic cost optimal control problem is to find a $u \in L_2(0,T;U)$ which minimizes

$$J(z_0,u) = \int_0^T (\langle Qz(s),z(s)\rangle + \langle Ru(s),u(s)\rangle)ds, \qquad (2.2.1)$$

where $z(t)$ is the mild solution to (1.1.2), i.e.

$$z(t) = T(t)z_0 + \int_0^t T(t-\eta)Bu(\eta)d\eta. \qquad (2.2.2)$$

It is known (see [11]) that under the above assumptions on Q and R, the optimal control $\bar{u}(t)$ exists and is given in feedback form by

$$\bar{u}(t) = -R^{-1}B^{*}\Pi(t)\bar{z}(t) \tag{2.2.3}$$

where the operator $\Pi(t)$ is defined on $[0,T]$ by the Riccati integral equation

$$\Pi(t)z = \int_t^T T^{*}(\eta-t)[Q-\Pi(\eta)BR^{-1}B^{*}\Pi(\eta)]T(\eta-t)z\,d\eta. \tag{2.2.4}$$

The operator $\Pi(t)$ is uniformly bounded and self-adjoint on $[0,T]$. The operator $R^{-1}B^{*}\Pi(t)$ is called the gain operator.

Let (Z^N,P^N,A^N), $N=1,2,\ldots$ be a sequence consisting of subspaces $Z^N \subset Z$, the corresponding orthogonal projections P^N of Z onto Z^N, and operators A^N which generate a sequence of C_0-semigroups $(T^N(t))_{N=1}^{\infty}$ on Z^N, satisfying $\|T^N(t)\| \leqslant Me^{\omega t}$ for $N=1,2,\ldots$. We assume that the operators $B^N \epsilon L(U,Z^N)$ and $Q^N \epsilon L(Z^N,Z^N)$ are uniformly bounded in N, with each Q^N self-adjoint and nonnegative, and that

$$T^N(t)P^N \xrightarrow{s} T(t) \text{ uniformly on } [0,T] \tag{2.2.5}$$

$$T^{N*}(t)P^N \xrightarrow{s} T^*(t) \text{ uniformly on } [0,T] \tag{2.2.6}$$

$$B^N u \longrightarrow Bu \tag{2.2.7}$$

$$B^{N*}z \longrightarrow B^*z \tag{2.2.8}$$

and

$$Q^N P^N z \longrightarrow Qz \tag{2.2.9}$$

as $N \longrightarrow \infty$ for all $z \epsilon Z$ and $u \epsilon U$.

The Nth approximate control problem is to find u which minimizes

$$J^N(P^N z_0, u) = \int_0^T (\langle Q^N z^N(s), z^N(s)\rangle + \langle Ru(s), u(s)\rangle) ds, \tag{2.2.10}$$

where

$$\dot{z}^N(t) = A^N z^N(t) + B^N u(t) \quad t>0 \tag{2.2.11}$$

$$z^N(0) = P^N z_0.$$

The optimal control $\bar{u}^N(t)$ for (2.2.10)-(2.2.11) is given in feedback form by

$$\bar{u}^N(t) = -R^{-1}B^{N*}\Pi^N(t)\bar{z}^N(t) \qquad (2.2.12)$$

where $\Pi^N(t)$ is the unique solution to the Riccati equations for the Nth approximate problem. We now recall a fundamental convergence result (see Gibson [11]).

<u>THEOREM</u> <u>2.2.1</u>. If (2.2.5)-(2.2.9) hold, then $\Pi^N(t)P^Nz \longrightarrow \Pi(t)z$ for all $z \in Z$, uniformly in t on [0,T]. ■

An important observation to be made is that, for purposes of approximating the feedback gain operator for an infinite dimensional control problem, an approximation scheme $\{Z^N, P^N, A^N\}$ should satisfy both (2.2.5) and (2.2.6). We remark that if (2.2.5) holds, then $T^{N*}(t)P^N \xrightarrow{w} T^*(t)$, and one can conclude (see [11]) that $\Pi^N(t)P^N \xrightarrow{w} \Pi(t)$. However, in order to get the strong convergence given in Theorem 2.2.1, one must also verify (2.2.6), i.e. that $T^{N*}(t)P^N \xrightarrow{s} T^*(t)$ uniformly on compact t-intervals.

<u>REMARK</u> <u>2.2.1</u> In the special case that $A = \pm A^*$, if the approximation scheme is chosen so that $Z^N \subset \mathcal{D}(A)$ and $A^N \triangleq P^NAP^N$, then (2.2.5) implies (2.2.6). However, when $A \neq \pm A^*$, if the approximation scheme has the property that

$Z^N \subset \mathcal{D}(A)$ and $A^N \stackrel{\Delta}{=} P^N A P^N$, then it may happen that $Z^N \not\subset \mathcal{D}(A^*)$ so that $P^N A^* P^N$ is not well-defined. In this case, (2.2.5) may not imply (2.2.6), and in fact, (2.2.6) may not hold. We shall present an example in Section 2.3 which shows that this situation does indeed occur.

2.3 Hereditary control systems

In this section, we review some of the schemes which have appeared in the literature for approximation and control of linear hereditary systems via the semigroup techniques described in Sections 2.1 and 2.2. We consider an ordinary FDE of the form

$$\dot{x}(t) = A_0x(t) + A_1x(t-r) + \int_{-r}^{0} D(s)x(t+s)ds + B_0u(t) \quad (2.3.1)$$

$$x(0) = \eta, \qquad x(s) = \varphi(s) \qquad -r \leqslant s < 0, \qquad (2.3.2)$$

where A_0 and A_1 are nxn matrices, $D(\cdot) \in L_2(-r,0;L(\mathbb{R}^n \times \mathbb{R}^n))$, and B_0 is an nxm matrix. Let $\tilde{Q}$ be a real, symmetric, and nonnegative nxn matrix, and R a real, symmetric, and positive mxm matrix. The optimal control problem is to find $u \in L_2(0,T;\mathbb{R}^m)$ which minimizes

$$\tilde{J}(x(0),u) = \int_0^T (\langle \tilde{Q}x(s),x(s)\rangle + \langle Ru(s),u(s)\rangle)ds, \qquad (2.3.3)$$

where x(t) is the solution to (2.3.1)-(2.3.2) corresponding to u(t).

To develop a state space formulation, define the Hilbert space $Z \overset{\Delta}{=} \mathbb{R}^n \times L_2(-r,0;\mathbb{R}^n)$, and the operator $A:\mathcal{D}(A) \subset Z \longrightarrow Z$ on the domain

$$\mathcal{D}(A) = \{(\eta,\varphi)\epsilon Z : \varphi \epsilon H^1(-r,0),\ \varphi(0)=\eta\} \tag{2.3.4}$$

by

$$A(\varphi(0),\varphi) = (A_0\varphi(0) + A_1\varphi(-r) + \int_{-r}^{0} D(s)\varphi(s)ds,\ \dot{\varphi}). \tag{2.3.5}$$

It is well known (see [2]) that A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geqslant 0}$. The problem (2.3.1)-(2.3.2) is formulated as an abstract evolution equation on Z by

$$\dot{z}(t) = Az(t) + Bu(t), \tag{2.3.6}$$

$$z(0) = z_0 \overset{\Delta}{=} (\eta,\varphi),$$

where $Bu = (B_0u,0)$. Mild solutions of (2.3.6) are given by

$$z(t) = T(t)z_0 + \int_0^t T(t-s)Bu(s)ds. \tag{2.3.7}$$

The following theorem gives the equivalence of the FDE (2.3.1)-(2.3.2) and its abstract formulation (2.3.6). The proof may be found in [2].

<u>THEOREM</u> <u>2.3.1</u> Let $(\eta,\varphi)\epsilon Z$ be given. If $x(t;u)$ is the solution of (2.3.1)-(2.3.2) for $u\epsilon L_2(0,T)$, then $z(t)$

defined by (2.3.7) satisfies

$$z(t) = (x(t;u),\ x_t(\cdot,u)),\ \text{for}\ t \geq 0. \qquad (2.3.8)$$

Define the operator $Q: z \longrightarrow z$ by

$$Q = \begin{bmatrix} \tilde{Q} & \theta \\ \theta & \theta \end{bmatrix}$$

where θ stands for the appropriate zero operator. The control problem for (2.3.6) is to find $u \in L_2(0,T)$ which minimizes

$$J(z(0),u) = \int_0^T (\langle Qz(s),z(s)\rangle + \langle Ru(s),u(s)\rangle)ds, \qquad (2.3.9)$$

where $z(t)$ is the solution to (2.3.6) corresponding to $u(t)$. It is straightforward to verify that minimizing (2.3.9) is equivalent to minimizing (2.3.3), and hence the abstract control problem is equivalent to the original control problem.

It is clear from the above remarks that the adjoint system plays a key role in the convergence of approximation schemes. We note that the adjoint of A in Z is (see [11]) defined on

$$\mathcal{D}(A^*) = \{(\xi,\psi) \in Z: \psi \in H^1(-r,0),\ \psi(-r) = A_1^T\xi\} \qquad (2.3.10)$$

by

$$A^*(\xi,\psi) = (\psi(0) + A_0^T\xi,\ D^T(\cdot)\xi - \dot{\psi}).$$

Having recalled these facts on the state space formulation of hereditary control problems, we now turn to questions on approximation. A standard approach is to develop an approximation scheme (Z^N,P^N,A^N), N=1,2,..., and then to investigate the convergence properties of the semigroups $T^N(t)$ generated by the operators A^N on Z^N. For the hereditary control problem under consideration, constructing an appropriate sequence of finite dimensional subspaces Z^N of $Z = \mathbb{R}^n \times L_2(-r,0;\mathbb{R}^n)$ involves discretizing $L_2(-r,0;\mathbb{R}^n)$. We shall now briefly review and compare three of the approximation schemes found in the literature. These are the so-called AVE scheme (see [2]), which we denote by (Z_a^N,P_a^N,A_a^N), a spline-based scheme (see [3]), which we denote by (Z_s^N,P_s^N,A_s^N), and a new spline-based scheme, (see [14]), which we denote by $(Z_{ns}^N,P_{ns}^N,A_{ns}^N)$.

We partition the interval $[-r,0]$ into subintervals $[t_j^N,t_{j-1}^N]$ for j=1,2...N, where $t_j^N = \frac{-jr}{N}$.
Let χ_j^N denote the characteristic function on $[t_j^N,t_{j-1}^N)$,

$j=2,3,\ldots,N$, and let χ_1^N denote the characteristic function on $[t_1^N, t_0^N]$.

The AVE scheme is defined on the finite dimensional subspaces

$$Z_a^N = \{(\eta,\varphi)\in Z:\ \varphi=\sum_{j=1}^{N} v_j\chi_j^N, v_j\in\mathbb{R}^n\}.$$

The projections $P_a^N : Z \longrightarrow Z_a^N$ are defined by

$$P_a^N(\eta,\varphi) = (\eta,\ \sum_{j=1}^{N}\varphi_j^N\chi_j^N)$$

where

$$\varphi_j^N \overset{\Delta}{=} \frac{N}{r}\int_{t_j^N}^{t_{j-1}^N}\varphi(s)ds,\ \text{for } j=1,2,\ldots,N.$$

The operators $A_a^N : Z \longrightarrow Z_a^N$ are defined by

$$A_a^N(\eta,\varphi) \overset{\Delta}{=} (A_0\varphi_0^N + A_1\varphi_N^N + \sum_{j=1}^{N}\frac{r}{N}D_j^N\varphi_j^N,\ \sum_{j=1}^{N}\frac{N}{r}(\varphi_{j-1}^N-\varphi_j^N)\chi_j^N),$$

where

$$\varphi_0^N \overset{\Delta}{=} \eta,\qquad D_j^N \overset{\Delta}{=} \frac{N}{r}\int_{t_j^N}^{t_{j-1}^N}D(s)ds,\ j=1,2,\ldots,N.$$

We denote by $T_a^N(t)$ the C_0-semigroup generated by the operator A^N, N=1,2,... .

In order to define the next two schemes, it is convenient to let $\mathcal{S}_1^N$ denote the space of linear splines (i.e. piecewise linear continuous functions) with knots at t_j^N. The spline scheme is defined on the finite dimensional subspaces

$$Z_s^N = \{\varphi(0),\varphi)\epsilon Z:\ \varphi\epsilon\mathcal{S}_1^N\}.$$

Let P_s^N be the orthogonal projection of Z onto Z_s^N, and define the operators $A_s^N:Z \longrightarrow Z_s^N$ by $A_s^N \stackrel{\Delta}{=} P_s^N A P_s^N$. The operator A_s^N is well-defined since $Z_s^N \subseteq D(A)$. Again, $T_s^N(t)$ will denote the C_0-semigroup generated by the operator A_s^N.

The new spline scheme is defined on the finite dimensional subspaces

$$Z_{ns}^N = \{(\eta,\varphi(\cdot))\epsilon Z:\ \varphi\epsilon\mathcal{S}_1^N\} = \mathbb{R}^n x\mathcal{S}_1^N.$$

Let P_{ns}^N be the orthogonal projection of Z onto Z_{ns}^N and define the operators A_{ns}^N: $Z \longrightarrow Z_{ns}^N$ by $A_{ns}^N \stackrel{\Delta}{=} P_{ns}^N \tilde{A}\big|_{Z_{ns}^N}$,

where $\tilde{A}$ is the "formal extension of A" given by

$$\mathcal{D}(\tilde{A}) = \mathbb{R}^n \times H^1(-r,0;\mathbb{R}^n),$$

and

$$\tilde{A}(\eta,\varphi) = (A_0\eta + A_1\varphi(-r) + \int_{-r}^{0} D(s)\varphi(s)ds,\ \dot{\varphi} + [\eta - \varphi(0)]\delta_0).$$

Here δ_0 denotes the Dirac delta impulse at 0. Again $T^N_{ns}(t) \overset{\Delta}{=} e^{tA^N_{ns}}$ denotes the semigroup generated by $A^N_{ns} \epsilon L(Z^N_{ns}, Z^N_{ns})$.

We recall some of the properties of these schemes. In [2] and [11], it is shown that the AVE scheme has the property that $T^N_a(t) \overset{s}{\rightarrow} T(t)$ and $T^{N*}_a(t) \overset{s}{\rightarrow} T^*(t)$, uniformly in t on compact intervals. In [3], Banks and Kappel introduced the spline scheme in hopes of obtaining better convergence rates than those previously established for the AVE scheme. Banks and Kappel showed that $T^N_s(t) \overset{s}{\rightarrow} T(t)$, uniformly on compact t-intervals. As expected, the convergence rates for simulation purposes are better for the spline scheme than for the AVE scheme. However, when the spline scheme (Z^N_s, P^N_s, A^N_s) is used for the optimal control problem, numerical results indicate (see [4]) only weak convergence of the approximating feedback operators. That is, it does not appear that $T^{N*}_s(t) \overset{s}{\rightarrow} T^*(t)$. This result is not surprising, since $A \neq \pm A^*$ and the spline scheme has the property that $Z_s^N \subset \mathcal{D}(A)$ while $Z_s^N \not\subset \mathcal{D}(A^*)$ (recall Remark 2.2.2). Although the <u>numerical</u> <u>evidence</u> indicates that $T_s^{N*}(t)$ does not converge strongly to $T^*(t)$, it has not yet been established that this is in fact true. This interesting question remains unanswered.

One attempt to resolve this problem is to relax the definition of the finite dimensional subspaces Z^N so that Z^N contains "sufficiently many" elements of both $\mathcal{D}(A)$ and $\mathcal{D}(A^*)$. The new spline subspaces Z_s^N have this property, and the new spline scheme (Z_{ns}^N, P_s^N, A_s^N) does satisfy (2.2.5) and (2.2.6) (see [14]).

<u>Remark</u> <u>2.3.1</u> In general, when one defines an approximation scheme with $Z^N \subset \mathcal{D}(A)$, then the operators A^N can be defined by $A^N \stackrel{\Delta}{=} P^N A P^N$. Furthermore, if A is dissipative, then it follows immediately that each A^N is dissipative, and hence the stability condition H1) of Theorem 2.1.1 is easily verified. However, if $A \neq \pm A^*$, then Z^N may not be a subspace of $\mathcal{D}(A^*)$ and the scheme may not lead to an approximation (i.e. in the strong operator topology) of the adjoint semigroup. If one attempts to avoid this problem by

defining an approximation scheme with $Z^N \not\subset \mathcal{D}(A)$, then $P^N A P^N$ is not well-defined, and one is presented with the problem of defining the operators A^N appropriately (the AVE scheme falls into this category). In addition, it is no longer trivial to verify the stability condition H2) of Theorem 2.1.1. In the next section, we develop a general framework for the problem of defining the operators A^N so that even in the non self-adjoint case, conditions (2.2.5) and (2.2.6) hold.

2.4 A general framework for approximation

As mentioned in the closing remark of the previous section, we shall now present some results in the direction of a general framework for constructing satisfactory approximation schemes for the linear quadratic cost optimal control problem.

Let A be the infinitesimal generator of a C_0-semigroup $T(t)$ on a Hilbert space Z. The problem is to define an approximation scheme $\{Z^N, P^N, A^N\}$ which satisfies

i) $P^N \xrightarrow{s} I$

ii) $T^N(t) \xrightarrow{s} T(t)$, uniformly in compact t-intervals

iii) $T^{N*}(t) \xrightarrow{s} T^*(t)$, uniformly in compact t-intervals,

where $T^N(t) = e^{tA^N}$.

As indicated above, if $A = \pm A^*$, then an appropriate approach to this problem is to choose approximating subspaces $Z^N \subset \mathcal{D}(A)$. If A^N is defined by $A^N \triangleq P^N A P^N$, then under suitable conditions on Z^N and P^N, condition (2.2.5) will imply (2.2.6). When $A \neq \pm A^*$, then (as is seen for the hereditary control problem) it is possible that $Z^N \not\subset \mathcal{D}(A^*)$ and (2.2.6) may not hold. Our approach is to define the approximating subspaces $Z^N \subset Z$ so that Z^N "contains sufficiently many elements" of both $\mathcal{D}(A)$ and $\mathcal{D}(A^*)$. However, in this case typically $Z^N \not\subset \mathcal{D}(A)$, so that $P^N A P^N$ is no longer well-defined, and cannot be used to define A^N. Therefore, we will construct a bilinear form σ which extends A and A^* in an appropriate manner, and define A^N by restricting σ to the subspaces Z^N. We discuss a technique for constructing the bilinear form σ.

Assume that V is a real Hilbert space with the trace property (see [17]); that is,

$$V \subset Z=Z^* \subset V^*$$

and there exists $\gamma \in L(V,H)$ so that

$$\gamma: V \xrightarrow{\text{onto}} H.$$

Here, H is a Hilbert space and Z is a pivot space ([1],[17]) and is identified with its dual Z^*. Assume also that Ker $\gamma = V_0$, and the inclusions $V \subset Z$, $V_0 \subset Z$ are dense and continuous. That is, there are $c_1, c_2 > 0$ so that

$$\|x\|_Z \leqslant c_1 \|x\|_V \quad \forall x \in V \text{ and } \|x\|_Z \leqslant c_2 \|x\|_{V_0}.$$

<u>PROPOSITION</u> <u>2.4.1</u> In the above framework, let A be a closed, densely-defined linear operator in Z with $\mathcal{D}(A) = V_0 \subseteq V$. Assume that $\mathcal{D}(A^*) \subset V$ and there is a constant K such that $\|A^* u\|_Z \leqslant K \|u\|_V$ for all $u \in \mathcal{D}(A^*)$. If there is an

operator $\mathcal{Q}\epsilon L(V,Z)$ which extends A (i.e. $\mathcal{Q}|_{V_0} = A$), then there exists $\delta\epsilon L(V,H^*)$ so that the bilinear form σ on $V\times V$ given by

$$\sigma(u,v) \overset{\Delta}{=} \langle \mathcal{Q}u,v\rangle_Z + \langle \gamma u,\delta v\rangle_H$$

is bounded and has the property that

i) $\sigma(u,v) = \langle Au,v\rangle_Z$ for all $u\epsilon\mathcal{D}(A)$, $v\epsilon V$,

and

ii) $\sigma(u,v) = \langle u,A^*v\rangle_Z$ for all $u\epsilon V$, $v\epsilon\mathcal{D}(A^*)$.

<u>Proof</u>: Since $\mathcal{Q}\epsilon L(V,Z)$, it follows that there is a constant K_1 such that for all $u \in V$

$$\|\mathcal{Q}u\|_Z \leqslant K_1\|u\|_V.$$

Define the operator C: $\mathcal{D}(A^*)\subset V\longrightarrow V^*$ by

$$(Cu)(v) \overset{\Delta}{=} \langle v,A^*u\rangle_Z - \langle \mathcal{Q}v,u\rangle_Z.$$

Then C is bounded since

$$\|Cu\|_{V^*} \leqslant \sup_{\substack{v\epsilon V\\ \|v\|_V=1}} \left(\left|\langle v,A^*u\rangle_Z\right| + \left|\langle \mathcal{Q}v,u\rangle_Z\right| \right)$$

$$\leqslant c\|A^*u\|_Z + \|\mathcal{Q}v\|_Z\cdot\|u\|_Z$$

$$\leqslant c(K+K_1)\|u\|_V = K_3\|u\|_V.$$

Note that $\gamma\epsilon L(V,H)$ is onto, and hence $\gamma^* \in L(H^*,V^*)$ is 1-1, and $\mathcal{R}(\gamma^*) = [\mathrm{Ker}\ \gamma]^\perp \subset V^*$. Observe that since $[\mathrm{Ker}\ \gamma]^\perp$ is a Hilbert space with norm $\|\cdot\|_{V^*}$, then γ^* is a continuous linear bijective operator from H^* onto $[\text{Ker } \gamma]^{\perp}$. Hence, (see Theorem 4.3.3 in [1]),

$$(\gamma^*)^{-1} \in L([\text{Ker } \gamma]^{\perp}, H^*).$$

That is, $|(\gamma^*)^{-1}u|_H \leqslant K_4\|u\|_{V^*}$ for all $u \in [\text{Ker } \gamma]^{\perp}$. Now we define $\tilde{\delta}$ on $\mathcal{D}(A^*)$ into H^* by

$$\tilde{\delta}v = (\gamma^*)^{-1}Cv.$$

Note that this is well-defined since $\mathcal{R}(C) = [\text{Ker } \gamma]^{\perp} = \mathcal{D}((\gamma^*)^{-1})$. Also, $\tilde{\delta}$ is bounded since $(\gamma^*)^{-1}$ and C are bounded. Let δ be the extension of $\tilde{\delta}$ to a bounded operator on all of V, and define the bilinear form σ on $V \times V$ by

$$\sigma(u,v) \stackrel{\Delta}{=} \langle Qu, v\rangle_Z + \langle \gamma u, \delta v\rangle_H.$$

Property i) is clearly satisfied. To establish ii), observe that if $v \in \mathcal{D}(A^*)$, then

$\langle \gamma u, \delta v\rangle_H = \langle \gamma u, (\gamma^*)^{-1}Cv\rangle_H = (Cv)(u)$ for all $u \in V$. The result follows.

■

Although the previous result provides only the existence of δ, we have the following partial characterization of δ.

<u>LEMMA</u> <u>2.4.2</u> If $\delta \in L(V,H^*)$ is as in Proposition 2.4.1, then $\mathcal{D}(\mathcal{Q}^*) = \mathcal{D}(A^*) \cap \mathrm{Ker}\ \delta$.

<u>Proof</u>. Note that $A \subset \mathcal{Q}$ implies $\mathcal{D}(\mathcal{Q}^*) \subset \mathcal{D}(A^*)$. If $v \in \mathcal{D}(\mathcal{Q}^*)$, then $v \in \mathcal{D}(A^*)$ and for all $u \in V$ we have that

$$0 = \sigma(u,v) = \langle u, A^* v\rangle_Z = \langle \gamma u, \delta v\rangle_H.$$

Therefore, $\delta v \equiv 0$. On the other hand, if $v \in \mathcal{D}(A^*) \cap \mathrm{Ker}\ \delta$ and $u \in V$, then

$$0 = \sigma(u,v) - \langle u, A^* v\rangle_Z = \langle \mathcal{Q} u, v\rangle_Z - \langle u, A^* v\rangle_Z,$$

and hence $v \in \mathcal{D}(\mathcal{Q}^*)$. ∎

We apply these ideas in practice by investigating $\mathcal{D}(A^*)$ and $\mathcal{D}(\mathcal{Q}^*)$ to determine δ, and hence σ. We then choose appropriate approximating subspaces $Z^N \subset V$, and define the operators A^N by restricting σ to Z^N. That is, define $A^N\colon Z^N \longrightarrow Z^N$ by

$$\langle A^N u, v\rangle_Z \stackrel{\Delta}{=} \sigma(u,v)\Big|_{Z^N} \qquad \text{for } u,v \in Z^N.$$

We then must show convergence of the resulting approximation scheme. Let us consider an example.

Example 1.

We shall apply these ideas to the delay system considered in Section 2.3. Recall that the state space is given by $Z = \mathbb{R}^n \times L_2(-r,0)$. The state operator A is defined on the domain

$$\mathcal{D}(A) = \{(\eta,\varphi)\epsilon Z: \varphi\epsilon H^1(-r,0),\ \varphi(0)=\eta\}$$

by

$$A(\eta,\varphi) = (A_0\eta + A_1\varphi(-r) + \int_{-r}^{0} D(s)\varphi(s)ds,\ \dot{\varphi}).$$

The adjoint of A is defined on the domain

$$\mathcal{D}(A^*) = \{(\xi,\psi)\ \epsilon\ Z: \psi\epsilon H^1(-r,0),\ \psi(-r)=A_1^T\xi\}$$

by

$$A^*(\xi,\psi) = (\psi(0) + A_0^T\xi,\ D^T(\cdot)\xi - \dot{\psi}).$$

For our framework, we need a Hilbert space V satisfying $\mathcal{D}(A),\mathcal{D}(A^*) \subset V \subset Z$, with A bounded in the norm on V. To this end, let

$$V = \mathbb{R}^n \times H^1(-r,0)$$

with

$$\|(n,\varphi)\|_V^2 = |\eta|_{\mathbb{R}^n}^2 + \|\varphi\|_{H^1}^2.$$

We note that $A \neq \pm A^*$, and hence we seek approximating subspaces Z^N which contain elements of both $\mathcal{D}(A)$ and $\mathcal{D}(A^*)$. A natural choice is $Z^N \overset{\Delta}{=} Z_{ns}^N$ as in the spline scheme developed by Kappel and Salamon [14]. They approached the problem of defining the approximating operators A^N by first formally extending A to an operator $\tilde{A}$ defined on all of V. Then since $Z^N_{ns} \subset V$, A^N can be defined by $A^N \stackrel{\Delta}{=} P^N\tilde{A}P^N$. We proceed by defining (not formally) a bilinear form σ on V. Then the approximating operators A^N are defined by restricting σ to Z^N, i.e.

$$\langle A^N u,v\rangle_Z \stackrel{\Delta}{=} \sigma(u,v) \text{ for all } u,v\epsilon Z^N.$$

To effect this approach, we define the operator $\mathcal{Q}$ which extends A to V by

$$\mathcal{D}(\mathcal{Q}) = V$$

and

$$\mathcal{Q}(\eta,\varphi) = (A_0\eta+A_1\varphi(-r) + \int_{-r}^{0} D(s)\varphi(s)ds,\ \dot{\varphi}).$$

In order to apply Proposition 2.4.1, we set $H=\mathbb{R}^n$ and define $\gamma:V\longrightarrow H$ by $\gamma(\eta,\varphi) = \varphi(0)-\eta$.

Computing the adjoint of $\mathcal{Q}$ in Z leads to

$$\mathcal{D}(\mathcal{Q}^*) = \{(\xi,\psi)\epsilon V:\ \psi(-r)=A_1^*\xi,\ \psi(0)=0\}.$$

and

$$\mathcal{Q}^*(\xi,\psi) = (A_0^*\xi,\ D^*(\cdot)\xi-\dot{\psi}).$$

In view of Lemma 2.4.2, we define δ: $V\longrightarrow\mathbb{R}^n$ by

$$\delta(\xi,\psi) = \psi(0).$$

It follows that $\sigma:V\times V\longrightarrow\mathbb{R}^n$ should be defined by

$$\sigma((\eta,\varphi),(\xi,\varphi)) \overset{\Delta}{=} \langle Q(\eta,\varphi),(\xi,\psi)\rangle_Z + \langle(\varphi(0)-\eta),\ \psi(0)\rangle_{\mathbb{R}^n}.$$

We are now in a position to define an approximation scheme $\{Z^N,P^N,A^N\}$, $N=1,2,\ldots,$ with $Z^N \subset V$. Define the approximating subspaces Z^N by $Z^N \overset{\Delta}{=} Z^N_{ns}$ (see Section 2.3) and let P^N be the orthogonal projection of Z onto Z^N. We use σ to define A^N: $Z^N \to Z^N$ by

$$\langle A^N u,v\rangle_Z \overset{\Delta}{=} \sigma(u,v) \quad \text{for all } u,v \in Z^N.$$

A direct calculation verifies that

$$\langle A^N u,v\rangle = \sigma(u,v) = \langle A^N_{ns} u,v\rangle$$

for all $u,v \in Z^N$. Therefore, when the above framework is used to develop a spline-based approximation scheme for the hereditary control problem, it leads to a scheme (in this case, the new spline scheme $\{Z^N_{ns},\ P^N_{ns},\ A^N_{ns}\}$) which satisfies (2.2.5) and (2.2.6).

Chapter III

PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS

3.1 State Space Formulation

In this section, we consider the problem of developing a state space formulation for a PFDE of hyperbolic type. Well-posedness will be obtained by showing that the underlying operator is the infinitesimal generator of a C_0-semigroup. First we briefly discuss several recent and relevant papers (see [8], [9], [16], [22], [23]). In many of these investigations, the PFDE is treated as an initial value problem for an abstract ordinary functional differential equation of the form

$$\dot{u}(t) = A_0u(t) + A_1u(t-r) + \int_{-r}^{0} a(s)A_2u(t+s)ds + f(t) \quad (3.1.1)$$

$$u(0)=\xi, \qquad u(s)=\eta(s) \text{ a.e. in } [-r,0] \quad (3.1.2)$$

where $A_0:\mathcal{D}(A_0)\subset H\longrightarrow H$ is the infinitesimal generator of an analytic semigroup in a Hilbert space H and $A_1,A_2\in L(D_{A_0},H)$. Here, $D_{A_0}=\mathcal{D}(A_0)$ is endowed with the graph norm.

In order to use semigroup techniques, it is argued that (3.1.1)-(3.1.2) is well-posed with initial data from some space $Z = X \times L_2(-r,0;Y)$. A solution semigroup $T(t)$ is then defined on Z and the infinitesimal generator $\mathcal{A}$

of T(t) is characterized. This approach is used in [9] to study the stability of (3.1.1)-(3.1.2) via the spectral properties of A.

An important problem in this approach is the prescription of appropriate initial data so that (3.1.1)-(3.1.2) is well-posed. Kunisch and Schappacher have shown in [16] that if one makes the natural assumption that $(\xi,\eta) \in H \times L_2(-r,0;H)$, then (3.1.1)-(3.1.2) is generally not well-posed. It has been shown that the appropriate choice for initial data is

$$(\xi,\eta) \in F \times L_2(-r,0;D_{A_0}),$$

where F is a suitable interpolation space between D_{A_0} and H (see [7] and [8]).

However, when considering PFDEs of hyperbolic type, one does not have recourse to these well-posedness results for the corresponding abstract FDE. This is because in the hyperbolic case the operator A_0 in the corresponding abstract FDE typically is not the generator of an analytic semigroup. Since we wish to use semigroup techniques for approximation and control of hyperbolic PFDEs, we will use a different approach than that mentioned above. That is, we do not use the well-posedness of the abstract FDE to define a solution semigroup. Rather, we conjecture an equivalent state space formulation of the form

$$\dot{z} = Az(t) + F(t)$$
$$z(0) = z_0$$

on a state space Z. The operator A is constructed using the dynamics of the PFDE. The above-mentioned results of Kunisch and Schappacher [16] indicate that we must make a judicious choice of the state space in order that the problem be well-posed. Recall that the above problem is well-posed if and only if A is the infinitesimal generator of a C_0-semigroup on Z.

With this in mind, we consider the following hyperbolic PFDE which arises in viscoelasticity:

$$\rho\frac{\partial^2}{\partial t^2}y(t,x) = \alpha\frac{\partial^2}{\partial x^2}y(t,x) + \int_{-r}^{0} g(s)\frac{\partial^2}{\partial x^2}y(t+s,x)ds \qquad (3.1.3)$$

$$y(t,a) = 0 = y(t,b) \qquad (3.1.4)$$

$$y(0,x) = s(x), \quad \frac{\partial}{\partial t}y(0,x) = v(x) \text{ for } a\leqslant x\leqslant b \qquad (3.1.5)$$

$$y(t,x) = h(t,x) \text{ for } -r\leqslant t\leqslant 0. \qquad (3.1.6)$$

We assume that $g\in H^1(-r,0)$ and that there exists $\epsilon_0>0$ and a continuous function $g_0:\mathbb{R}^-\to\mathbb{R}^-$ satisfying

$$g_0(s)<0 \quad \text{for} \quad -r<s\leqslant 0$$

$$g_0(s)=0 \quad \text{for} \quad s\leqslant -r$$

and such that:

i) $\qquad g(s)\leqslant g_0(s) \quad \text{for} \quad -r<s\leqslant 0$

ii) $\qquad \epsilon_0\leqslant\epsilon$, where ϵ is defined by

$$\epsilon \overset{\Delta}{=} \alpha + \int_{-r}^{0} g(s)ds$$

iii) $\qquad \frac{d}{ds}g(s)\leqslant\mu g_0(s)$ for $-r<s\leqslant 0$, for some $\mu>0$.

The existence of ϵ_0 follows from general properties of elastic moduli. Condition iii) is a "decaying memory" assumption. For a further discussion of the physical basis for these assumptions, see [13] and [24].

Throughout the rest of this chapter, we shall often use the symbol ′ to denote differentiation with respect to the x variable. We now pose the above viscoelastic problem in the state space

$$Z=L_2(a,b) \times L_2^0(a,b) \times L_2(-r,0;L_2^0(a,b)), \qquad (3.1.7)$$

with norm defined by

$$\left\|\begin{matrix}\varphi\\ \psi\\ w\end{matrix}\right\|_Z^2 = \int_a^b (\rho\varphi^2+\epsilon\psi^2)dx - \int_{-r}^{0} g(s)\int_a^b w^2 dx ds.$$

The operator A is defined on the domain

$$\mathcal{D}(A)=\left\{\begin{pmatrix}\varphi\\ \psi\\ w\end{pmatrix}\epsilon Z:\varphi\epsilon H_0^1,\psi\epsilon H^1,w\epsilon H^1(-r,0;H^1),w(0)=\psi\right\}$$

by

$$A\begin{pmatrix}\varphi\\ \psi\\ w\end{pmatrix} = \left(\frac{\alpha}{\rho}\psi' + \frac{1}{\rho}\int_{-r}^{0} g(s)w'(s)ds,\varphi',\frac{dw}{ds}\right)^T. \qquad (3.1.8)$$

Note that A has the form

$$A\begin{pmatrix}\varphi\\ \psi\\ w\end{pmatrix} = \begin{pmatrix} A_0\begin{pmatrix}\varphi\\ \psi\end{pmatrix} + \int_{-r}^{0} G(s)A_0\begin{pmatrix}0\\ w\end{pmatrix}ds \\ \frac{dw}{ds}\end{pmatrix}, \qquad (3.1.9)$$

where

$$A_0\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = \begin{pmatrix}\frac{\alpha}{\rho}\ \psi'\\ \varphi'\end{pmatrix} = \begin{pmatrix}0 & \frac{\alpha}{\rho}\ \frac{d}{dx}\\ \frac{d}{dx} & 0\end{pmatrix}\begin{pmatrix}\varphi\\ \psi\end{pmatrix}$$

and

$$G(s) = \begin{pmatrix}\frac{1}{\alpha}g(s) & 0\\ 0 & 0\end{pmatrix}.$$

We shall establish that A generates a C_0-semigroup on Z. However, it is important to note that there are many possible state space formulations of the system (3.1.3)-(3.1.6). Walker [24] constructed a different state space model and used his model for a stability analysis. We shall make use of Walker's results; thus it is worthwhile to recall Walker's formulation. Let Z_w be the space

$$Z_w = H_0^1 \times L_2 \times L_2(-r,0;H_0^1)$$

with norm given by

$$\left\|\begin{pmatrix}\varphi\\ \psi\\ w\end{pmatrix}\right\|_{Z_w}^2 = \int_a^b [\varepsilon(\varphi')^2+\rho\psi^2]dx - \int_{-r}^0 g(s)\int_a^b (w')^2 dxds.$$

The operator A_w is defined on the domain

$$\mathcal{D}(A_w) = \left\{\begin{pmatrix}\varphi\\ \psi\\ w\end{pmatrix} \in Z_w:\ \psi \in H_0^1,\ \frac{\varepsilon}{\rho}\varphi' - \frac{1}{\rho}\int_{-r}^0 g(s)w'(s)ds \in H^1,\right.$$

$$\left. w \in H^1(-r,0;H^1),\ w(0)=0\right\},$$

by

$$A_w\begin{pmatrix}\varphi\\ \psi\\ w\end{pmatrix} = (\psi,[\tfrac{\varepsilon}{\rho}\varphi' - \tfrac{1}{\rho}\int_{-r}^0 g(s)w'(s)ds]',\ \psi+\tfrac{dw}{ds})^T. \qquad (3.1.11)$$

Walker showed that A_w generates a contraction semigroup on Z_w and discussed the asymptotic properties of this semigroup. The difference between these two formulations is the choice of "state" variable. Essentially, this is due to the fact that there are two ways to formulate a second order PDE as an equivalent first order system. For example, the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \tag{3.1.12}$$

may be written as

$$\frac{\partial}{\partial t}\begin{pmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial t}\end{pmatrix} = \begin{pmatrix}0 & \frac{\partial}{\partial x}\\ \frac{\partial}{\partial x} & 0\end{pmatrix}\begin{pmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial t}\end{pmatrix} \tag{3.1.13}$$

or

$$\frac{\partial}{\partial t}\begin{pmatrix}u\\ \frac{\partial u}{\partial t}\end{pmatrix} = \begin{pmatrix}0 & 1\\ \frac{\partial^2}{\partial x^2} & 0\end{pmatrix}\begin{pmatrix}u\\ \frac{\partial u}{\partial t}\end{pmatrix}. \tag{3.1.14}$$

The formulation (3.1.7)-(3.1.8) corresponds to (3.1.13) and a state variable

$$z(t) \sim \begin{pmatrix}\frac{\partial u}{\partial t}\\ \frac{\partial u}{\partial x}\\ \frac{\partial}{\partial x}(u_t)\end{pmatrix}$$

whereas (3.1.10)-(3.1.11) corresponds to (3.1.14) and a state variable

$$z_w(t) \sim \begin{pmatrix}u\\ \frac{\partial u}{\partial t}\\ u-u_t\end{pmatrix}.$$

Besides the presence of lower order spatial derivatives in our state operator A, a major motivation for our choice of state space formulation is the anticipated application to feedback control problems. For example, in the control of a flexible beam it is often desirable to feed back a measurement of strain. Thus a "natural" state would be the strain $\left(\frac{\partial^2 u}{\partial x^2}\right)$. When one applies our state space formulation to the problem of controlling a flexible beam, the strain is part of the state, which is not the case for other formulations. Besides this feature, another advantage of the formulation (3.1.7)-(3.1.9) is that the operator and state space structure is analogous to that for ordinary functional differential equations. Indeed, after discretizing the spatial variable in (3.1.7)-(3.1.9), we are led to a familiar FDE operator. This will be useful for approximation.

In the remainder of this section, we address the question of well-posedness of our state space formulation. That is, we show that A is the infinitesimal generator of a C_0-semigroup. We will make reference to some results in [8], where a state space formulation is developed for a PFDE of parabolic type. DiBlasio, Kunisch and Sinestrari showed that the underlying operator is the infinitesimal generator of a C_0-semigroup. We will use this fact to prove similar results for the operator A. Therefore, consider the operator B on $L_2(a,b)$ defined by

$$\mathcal{D}(B) = H^2 \cap H_0^1$$

and

$$Bu = \frac{d^2}{dx^2}u .$$

Let $D_B = \mathcal{D}(B)$ be endowed with the graph norm, and define the space Z_1 by

$$Z_1 = H_0^1(a,b) \times L_2(-r,0;D_B) \quad .$$

The norm on Z_1 is the usual product space norm

$$\left\| \begin{matrix} \psi \\ w \end{matrix} \right\|_{Z_1}^2 = \|\psi\|_{H^1}^2 + \|w\|_{L_2(-r,0;D_B)}^2 \quad .$$

Define the operator $A_1 : \mathcal{D}(A_1) \subset Z_1 \to Z_1$ on the domain

$$\mathcal{D}(A_1) = \Bigg\{ \begin{pmatrix} \psi \\ w \end{pmatrix} \epsilon Z_1 : \ w \epsilon H^1(-r,0;D_B), \ w(0)=\psi,$$

$$\frac{\alpha}{\rho} B\psi + \frac{1}{\rho}\int_{-r}^{0} g(s)Bw(s)ds \,\epsilon\, H_0^1(a,b) \Bigg\}$$

by

$$A_1 \begin{pmatrix} \psi \\ w \end{pmatrix} = \left(\frac{\alpha}{\rho} B\psi + \frac{1}{\rho}\int_{-r}^{0} g(s)Bw(s)ds, \ \frac{dw}{ds} \right) \quad .$$

The following result can be found in [8].

LEMMA 3.1.1 The operator A_1 is closed, densely defined, and $\overline{R(I-A_1)}=Z_1$. ∎

Let $Z_e=Z$ denote the Hilbert space Z endowed with the norm defined by

$$\left\|\begin{matrix}\varphi\\ \psi\\ w\end{matrix}\right\|_e^2 = \int_a^b(\varphi^2+\psi^2)dx + \int_{-r}^0\int_a^b w^2 dx ds$$

$$= \|\varphi\|_{L_2}^2 + \|\psi\|_{L_2}^2 + \|w\|_{L_2(-r,0;L_2)}^2 .$$

Note that $\|\cdot\|_e$ and $\|\cdot\|_Z$ are equivalent norms on Z.

Heuristically, the relationship between A_1 and A is that A_1 arises from a parabolic PFDE and A arises from a hyperbolic PFDE. With this relationship in mind, we now state and prove a series of lemmas which will be used to show that A is the infinitesimal generator of a C_0-semigroup.

LEMMA 3.1.2. The domain of A is dense in Z_e.

Proof: We show that for an arbitrary $(\gamma,\beta,z)\epsilon Z_e$ and $\epsilon>0$, there exists $(\varphi,\psi,w)\epsilon\mathcal{D}(A)$ such that

$$\|(\varphi,\psi,w)-(\gamma,\beta,z)\|_e^2 < \epsilon.$$

Let $\epsilon>0$ and $(\gamma,\beta,z)\epsilon Z_e$. Since $L_2(-r,0;H^1\cap L_2^0)$ is dense in $L_2(-r,0;L_2^0)$, we can choose $\hat{z}\epsilon L_2(-r,0;H^1\cap L_2^0)$ such that

$$\|\hat{z}-z\|^2_{L_2(-r,0;L_2^0)} < \frac{\epsilon}{3}.$$

Also, since H_0^1 is dense in L_2, there exists $\varphi\epsilon H_0^1$ such that

$$\|\varphi-\gamma\|^2_{L_2} < \frac{\epsilon}{3}.$$

If we define β_1 and z_1 by $\beta_1=\int_a^x\beta$ and $z_1=\int_a^x\hat{z}$, respectively, then $(\beta_1,z_1)\epsilon Z_1$. The domain of A_1 is dense and hence there exists $(\psi_1,w_1)\epsilon\mathcal{D}(A_1)$ such that

$$\|(\psi_1,w_1)-(\beta_1,z_1)\|^2_{Z_1} < \frac{\epsilon}{3}.$$

If $\psi=\psi_1'$ and $w=w_1'$, then $(\varphi,\psi,w)\epsilon\mathcal{D}(A)$ and

$$\begin{aligned}
&\|(\varphi,\psi,w)-(\gamma,\beta,z)\|_e^2 \\
&\qquad\leqslant\|\varphi-\gamma\|^2_{L_2}+\|\psi_1'-\beta_1'\|^2_{L_2}+\|w_1'-z_1'\|^2_{L_2(-r,0;L_2^0)} \\
&\qquad+\|\hat{z}-z\|^2_{L_2(-r,0;L_2^0)} \\
&\qquad<\epsilon,
\end{aligned}$$

which establishes the lemma. ∎

<u>LEMMA</u> <u>3.1.3</u>. The range of I-A is dense in Z_e.

<u>Proof</u>: Let $S=\left\{\begin{pmatrix}\varphi\\ \psi\\ w\end{pmatrix}\epsilon Z_e:\ \varphi\epsilon H_0^1(a,b), w\epsilon L_2(-r,0;H^1)\right\}$. Observe that S is dense in Z_e and therefore it suffices to show that $S\subset\overline{\mathcal{R}(I-A)}$. Let $(\gamma,\beta,z)\epsilon S$, and $\epsilon>0$ be arbitrary. We will use the denseness of $\mathcal{D}(A_1)$ in Z_1 to find $(\varphi,\psi,w)\epsilon\mathcal{D}(A)$ such that $\|(I-A)(\varphi,\psi,w)-(\gamma,\beta,z)\|_e^2 < \epsilon$. Since $(\gamma+\int_a^x\beta, \int_a^x z)\epsilon Z_1$, there exists $(\psi_1,w_1)\epsilon\mathcal{D}(A_1)$ such that $\|(I-A_1)(\psi_1,w_1) - (\gamma+\int_a^x\beta, \int_a^x z)\|_{Z_1}^2 < \epsilon$. Define φ,ψ, and w by $\varphi=\int_a^x(\psi-\beta)$, $\psi=\psi_1'$, and $w=w_1'$, respectively, and note that $(\varphi,\psi,w)\epsilon Z_e$. Furthermore, it follows that $\psi\epsilon H^1$, $\varphi\epsilon H^1$, $\varphi(a)=0$, and $\varphi(b)=\int_a^b\psi - \int_a^b\beta = 0$. Moreover, $(\varphi,\psi,w)\epsilon\mathcal{D}(A)$ because $w\epsilon H^1(-r,0;H^1)$, and $w(0) = w_1'(0) = \psi_1' = \psi$. Finally, we have the estimate

$$\|(I-A)(\varphi,\psi,w)-(\gamma,\beta,z)\|_e^2$$

$$= \|\varphi - \frac{\alpha}{\rho}\psi' - \frac{1}{\rho}\int_{-r}^0 g(s)w'(s)ds - \gamma,\ \psi - \varphi' - \beta,\ w - \frac{dw}{ds} - z\|_e^2$$

$$= \|(\psi_1 - \frac{\alpha}{\rho}\psi_1'' - \frac{1}{\rho}\int_{-r}^{0} g(s)w''(s)ds) - (\gamma + \int_a^x \beta)\|_{L_2}^2$$

$$+ \|(w_1 - \frac{dw_1}{ds})' - (\int_a^x z)'\|_{L_2(-r,0;L_2^0)}^2$$

$$\leqslant \|(\psi_1 - \frac{\alpha}{\rho}\psi_1'' - \frac{1}{\rho}\int_{-r}^{0} g(s)w''(s)ds) - (\gamma + \int_a^x \beta)\|_{H_0^1}^2$$

$$+\|(w - \frac{dw_1}{ds}) - \int_a^x z\|_{L_2(-r,0;D_B)}^2$$

$$= \|(I-A_1)(\psi_1,w_1) - (\gamma + \int_a^x \beta, \int_a^x z)\|_{Z_1}^2$$

$$< \epsilon.$$ ∎

<u>LEMMA</u> <u>3.1.4</u>. The operator A is closed.

<u>PROOF</u>: Assume that the sequence (u_n,ψ_n,w_n) satisfies

$$(\varphi_n,\psi_n,w_n)\epsilon\mathcal{D}(A), \qquad (3.1.15)$$

$$(\varphi_n,\psi_n,w_n)\to(\varphi,\psi,w) \text{ in } Z_e, \qquad (3.1.16)$$

and

$$A(\varphi_n,\psi_n,w_n)\to(\gamma,\beta,z) \text{ in } Z_e. \qquad (3.1.17)$$

We must show that $(\varphi,\psi,w)\epsilon\mathcal{D}(A)$ and $A(\varphi,\psi,w) = (\gamma,\beta,z)$. We deduce from (3.1.16) and (3.1.17) that

$$w \in H^1(-r,0;L_2^0) \qquad (3.1.18)$$

and

$$\frac{dw}{ds} = z \ . \qquad (3.1.19)$$

This implies that $w_n \to w$ in $H^1(-r,0;L_2^0)$ and $\psi_n = w_n(0) \to w(0)$ in L_2^0. However, (3.1.15) implies that $\psi_n \to \psi$ in L_2^0, and hence

$$w(0)=\psi. \qquad (3.1.20)$$

Similarly, (3.1.16) implies that

$$\frac{\alpha}{\rho}\psi_n + \frac{1}{\rho}\int_{-r}^{0} g(s)w_n(s)ds \longrightarrow \frac{\alpha}{\rho}\psi + \frac{1}{\rho}\int_{-r}^{0} g(s)w(s)ds \quad \text{in } L_2^0$$

and (3.1.17) yields

$$\frac{\alpha}{\rho}\psi_n' + \frac{1}{\rho}\int_{-r}^{0} g(s)w_n'(s)ds \longrightarrow \beta \quad \text{in } L_2.$$

Consequently, we conclude that

$$\frac{\alpha}{\rho}\psi + \frac{1}{\rho}\int_{-r}^{0} g(s)w(s)ds \in H^1 \qquad (3.1.21)$$

and

$$\left[\frac{\alpha}{\rho}\psi + \frac{1}{\rho}\int_{-r}^{0} g(s)w(s)ds\right]' = \beta \ . \qquad (3.1.22)$$

Again, a similar argument yields

$$\varphi \epsilon H^1 \tag{3.1.23}$$

and

$$\varphi' = \gamma \ . \tag{3.1.24}$$

The identities, (3.1.16), (3.1.17) and (3.1.24) together imply that $\varphi_n \to \varphi$ in H^1, which implies that

$$\varphi(a)=0=\varphi(b). \tag{3.1.25}$$

Combining (3.1.18), (3.1.20), (3.1.21), (3.1.23), and (3.1.25) it follows that $(\varphi,\psi,w)\epsilon\mathcal{D}(A)$. Also (3.1.19), (3.1.22) and (3.1.24) imply that $A(\varphi,\psi,w)=(\gamma,\beta,z)$. Therefore, A is closed. ∎

We note that A does not satisfy a dissipative type of inequality on Z. We introduce a dissipative operator $\tilde{A}$ which is similar to A, and use the dissipativeness of $\tilde{A}$ and its similarity to A to facilitate proving some of the following results, including those on approximation in Section 3.2.

With this in mind, define the operator $L\epsilon L(Z,Z)$ by

$$L\begin{pmatrix}\varphi\\ \psi\\ w\end{pmatrix} = \begin{pmatrix}\varphi\\ \psi\\ \psi-w\end{pmatrix}. \tag{3.1.26}$$

Note that $L^{-1}=L$, and L is bijective. We shall need the following results.

<u>LEMMA</u> <u>3.1.5</u> Let H be a Hilbert space and let $L\epsilon L(H,H)$ be a bijective operator satisfying $L=L^{-1}$. Suppose B is a closed, densely defined operator in H and $\overline{\mathcal{R}(\omega I-B)} = H$. If the operator $\tilde{B}$ is defined on the domain

$$\mathcal{D}(\tilde{B}) = \{x\epsilon H\colon Lx\epsilon\mathcal{D}(B)\}$$

by $\tilde{B} \overset{\Delta}{=} L^{-1}BL = LBL$, then $\tilde{B}$ is closed and $\overline{\mathcal{R}(\omega I-\tilde{B})} = H$.

<u>Proof</u>: To show that $\tilde{B}$ is closed, assume that $x_n\epsilon\mathcal{D}(\tilde{B})$, $n=1,2,\ldots$, and $x_n \longrightarrow x$ and $\tilde{B}x_n \longrightarrow y$. It follows that $Lx_n\epsilon\mathcal{D}(B)$, $n=1,2,\ldots$, and $Lx_n \longrightarrow Lx$ and $BLx_n \longrightarrow Ly$. Therefore, since B is closed, $Lx\epsilon\mathcal{D}(B)$ and $BLx = Ly$. Hence, $x\epsilon\mathcal{D}(\tilde{B})$ and $\tilde{B}x = y$, which proves that $\tilde{B}$ is closed.

To show that $\overline{\mathcal{R}(\omega I-\tilde{B})} = H$, let $x\epsilon H$ and $\epsilon>0$ be arbitrary. Since $\overline{\mathcal{R}(\omega I-B)} = H$, there exists $y\epsilon\mathcal{D}(B)$ such that $\|(\omega I-B)y - Lx\|_H < \frac{\epsilon}{\|L\|}$. Consequently $Ly\epsilon\mathcal{D}(\tilde{B})$, and

$$\|(\omega I-\tilde{B})Ly - x\| = \|L(\omega I-B)y - LLx\|$$

$$\leqslant \|L\|\cdot\|(\omega I-B)y - Lx\| < \epsilon \ . \qquad \blacksquare$$

<u>LEMMA</u> <u>3.1.6</u> Assume that the hypotheses of the previous lemma hold. If $\tilde{B}-\lambda$ is dissipative for some $\lambda<\omega$, then B is the infinitesimal generator of a C_0-semigroup S(t) satisfying $\|S(t)\| \leqslant \|L\|^2 e^{\omega t}$.

<u>Proof</u>: If $\tilde{B}-\lambda$ is dissipative, then $(\omega I-\tilde{B})^{-1}$ is a closed, bounded operator, and it follows that $\mathcal{R}(\omega I-\tilde{B})=H$. Therefore, $\mathcal{D}(\tilde{B})$ is dense (see [18]). Hence by the Lumer-Phillips Theorem $\tilde{B}$ is the infinitesimal generator of a C_0-semigroup $\tilde{S}(t)$ satisfying $\|\tilde{S}(t)\| \leqslant e^{\omega t}$. Therefore, $L^{-1}\tilde{S}(t)L$ is also a C_0-semigroup, which we denote by S(t). Clearly $\|S(t)\| \leqslant \|L\|^2 e^{\omega t}$, and a straightforward calculation shows that

$$\lim_{t\downarrow 0} \frac{S(t)z-z}{t} = L^{-1}\left[\lim_{t\downarrow 0} \frac{\tilde{S}(t)Lz-Lz}{t}\right] = L^{-1}\tilde{B}Lz = B.$$

Thus, B is the infinitesimal generator of S(t), and the result follows. ∎

In order to apply these results to our problem, we define the operator $\tilde{A}$ on

$$\mathcal{D}(\tilde{A}) = \left\{z\epsilon Z:\ Lz\epsilon\mathcal{D}(A)\right\}$$

$$= \left\{ \begin{pmatrix} \varphi \\ \psi \\ w \end{pmatrix} \epsilon Z: \ \varphi \epsilon H_0^1, \ \psi \epsilon H^1, \ w \epsilon H^1(-r,0;H^1), w(0)=0 \right\},$$

by $\tilde{A} \triangleq L^{-1}AL$. In particular, it follows that

$$\tilde{A}\begin{pmatrix} \varphi \\ \psi \\ w \end{pmatrix} = (\frac{\epsilon}{\rho}\psi' - \frac{1}{\rho}\int_{-r}^{0} g(s)w'(s)ds, \varphi', \varphi' - \frac{dw}{ds}).$$

We now show that $\tilde{A}$ is dissipative and hence generates a C_0-semigroup of contractions.

<u>LEMMA</u> <u>3.1.7</u>. The operator $\tilde{A}$ generates a C_0-semigroup on Z.

<u>Proof</u>: If $\begin{pmatrix} \varphi \\ \psi \\ w \end{pmatrix} \epsilon \mathcal{D}(\tilde{A})$, then

$$2\langle \tilde{A}\begin{pmatrix} \varphi \\ \psi \\ w \end{pmatrix}, \begin{pmatrix} \varphi \\ \psi \\ w \end{pmatrix} \rangle_Z$$

$$=2\int_a^b [\epsilon\psi' - \int_{-r}^{0} g(s)w'(s)ds]\varphi dx + 2\int_a^b \epsilon\varphi'\psi dx$$

$$- 2\int_{-r}^{0} g(s)\int_a^b [\varphi' + \frac{dw}{ds}]w dx ds$$

$$= -2\int_{-r}^{0} g(s)\int_a^b w\frac{dw}{ds}dx ds$$

$$= -\int_{-r}^{0} \frac{\partial}{\partial s}\left[\int_a^b g(s)w^2(s)dx\right]ds + \int_{-r}^{0} \frac{d}{ds}g(s)\int_a^b w^2(s)dxds$$

$$\leqslant \int_{-r}^{0} \frac{d}{ds}g(s)\int_a^b w^2(s)dxds$$

$$\leqslant \mu\int_{-r}^{0} g_0(s)\int_a^b w^2(s)dxds \leqslant 0.$$

and the result follows. ∎

The next result is an immediate consequence of Lemma 3.1.6 and Lemma 3.1.7.

<u>Theorem</u> <u>3.1.8</u> The operator A defined by (3.1.8) is the infinitesimal generator of a C_0 semigroup T(t) satisfying $\|T(t)\| \leqslant \|L\|^2$. ∎

We have therefore shown that our state space formulation (2.1.7)-(3.1.9) is well-posed. We turn our attention in the next section to the problem of constructing an approximation scheme for this formulation.

3.2 An approximation scheme

In this section, we develop an approximation scheme for the state space formulation of the viscoelastic problem given in the previous section. Recall that for this formulation, the state space Z is given by

$$Z = L_2(a,b) \times L_2^0(a,b) \times L_2(-r,0;L_2^0(a,b)), \qquad (3.2.1)$$

with norm defined by

$$\left|\begin{matrix}\varphi\\ \psi\\ w\end{matrix}\right| = \int_a^b(\rho\varphi^2 + \epsilon\psi^2)dx - \int_{-r}^0 g(s)\int_a^b w^2 dx ds.$$

The state operator A is defined on the domain

$$\mathcal{D}(A) = \{\begin{pmatrix}\varphi\\ \psi\\ w\end{pmatrix}\epsilon Z\colon\ \varphi\epsilon H_0^1,\ \psi\epsilon H^1,\ w\epsilon H^1,\ w\epsilon H^1(-r,0;H^1),\ w(0) = \psi\},$$

by

$$A\begin{pmatrix}\varphi\\ \psi\\ w\end{pmatrix} = \left(\frac{\alpha}{\rho}\psi' + \frac{1}{\rho}\int_{-r}^0 g(s)w'(s)ds,\ \varphi',\ \frac{dw}{ds}\right)^T. \qquad (3.2.2)$$

The operator A is the infinitesimal generator of a C_0-semigroup T(t) and may be written in the form

$$A\begin{pmatrix}\varphi\\ \psi\\ w\end{pmatrix} = \begin{pmatrix} A_0\begin{pmatrix}\varphi\\ \psi\end{pmatrix} + \int_{-r}^{0} G(s)A_0\begin{pmatrix}0\\ w\end{pmatrix}ds \\ \frac{dw}{ds}\end{pmatrix},$$

where $A_0:H_0^1\times H^1 \longrightarrow L_2\times L_2^0$ is given by

$$A_0\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = \begin{pmatrix}\frac{\alpha}{\rho}\psi' \\ \varphi'\end{pmatrix}.$$

It can be shown that the operator A^* is defined on the domain

$$\mathcal{D}(A^*) = \{\begin{pmatrix}\varphi\\ \psi\\ w\end{pmatrix} \in Z\colon \varphi\in H_0^1,\ \psi\in H^1,\ w\in H^1(-r,0;H^1),\ w(-r)=0\}$$

by

$$A^*\begin{pmatrix}\varphi\\ \psi\\ w\end{pmatrix} = \left(-\frac{\epsilon}{\rho}\psi',\ -\frac{\alpha}{\epsilon}\varphi' - \frac{g(0)}{\epsilon}w(0),\ \varphi' - \frac{\frac{d}{ds}[g(s)w(s)]}{g(s)}\right)^T.$$

We shall often use the norm on $L_2 \times L_2^0$ defined by

$$\left\|\begin{matrix}\varphi\\ \psi\end{matrix}\right\|^2_{L_2\times L_2^0} \overset{\Delta}{=} \int_a^b(\rho\varphi^2 + \epsilon\psi^2)dx.$$

Similarly, we use the norm on $L_2(-r,0;L_2^0)$ defined by

$$\|w\|^2_{L_2(-r,0;L_2^0)} \overset{\Delta}{=} \int_{-r}^{0} -g(s) \int_a^b w^2dxds.$$

Before giving detailed results, we shall briefly outline our approach. We proceed by first discretizing the spatial variable. That is, we define finite dimensional subspaces S^N, S_0^N of $L_2(a,b)$, $L_2^0(a,b)$, respectively. We can then define approximating subspaces $Z^N \subset Z$ by

$$Z^N = S^N \times S_0^N \times L_2(-r,0;S_0^N).$$

To define the approximating operators A^N, we note that the action of the operator A on the spatial variable is determined by the operator A_0. Hence, we shall define an approximating operator A_0^N, and then define the operator A^N by

$$A^N\begin{pmatrix}\varphi\\ \psi\\ w\end{pmatrix} = \begin{pmatrix} A_0^N\begin{pmatrix}\varphi\\ \psi\end{pmatrix} + \int_{-r}^{0} G(s)A_0^N\begin{pmatrix}0\\ w\end{pmatrix}ds \\ \frac{dw}{ds}\end{pmatrix}.$$

The reader will note that for each N, the operator A^N and state space Z^N have the same structure as the operator and state spaassociated with a standard FDE on $\mathbb{R}^k$ (k=dimension (S^N) + dimension $\left(S_0^N\right)$). Therefore, the delay variable can be discretized according to known approximation schemes, such as those discussed in Section 2.3.

For purposes of this paper, we shall use the averaging approximation scheme (AVE) to discretize $L_2(-r,0;S_0^N)$. Therefore, let $\mathcal{S}^M$ denote the finite dimensional subspace of $L_2(-r,0;S_0^N)$ constructed using the AVE scheme, and let $Z^{N,M}$ denote the finite dimensional subspaces of Z defined by

$$Z^{N,M} \triangleq S^N \times S_0^N \times \mathcal{S}^M.$$

Let $Q^M:Z^N \longrightarrow Z^{N,M}$ be the operator constructed according to the AVE scheme, and define the operator

$$A^{N,M}:Z \longrightarrow Z^{N,M}$$

by

$$A^{N,M} \triangleq Q^M P^N z.$$

The operators $A^{N,M}$ and subspaces $Z^{N,M}$ form the basis of our approximation scheme.

Let us now be more specific in defining our approximation scheme. For any positive integer N, partition the interval [a,b] into subintervals $[x_j^N,x_{j+1}^N]$, where $x_j^N = a + j\frac{(b-a)}{N}$, $j=0,1,2,\ldots,N$. Define the familiar "hat" functions $h_i^N(x)$ to be piecewise linear functions on $[a,b]$ with knots at x_j^N, $j=0,1,\ldots,N$, and satisfying $h_i^N(x_j^N) = \delta_{ij}$, where δ_{ij} is the Kronecker delta ($\delta_{ij}=1$ if $i=j$, $\delta_{ij} = 0$ if $i\neq j$). Define the finite dimensional subspace S^N of $L_2(a,b)$ by

$$S^N \stackrel{\Delta}{=} \text{span } \{h_i^N(x)\}_{i=1}^{N-1}.$$

Note that $S^N \subset H_0^1(a,b)$. Also define the finite dimensional subspace S_0^N of $L_2^0(a,b)$ by

$$S_0^N \stackrel{\Delta}{=} \text{span } \{\tfrac{d}{dx}h_i^N(x)\}_{i=1}^{N-1}.$$

Note that $\frac{d}{dx}h_i^N(x)$ is a step function. For convenience in the ensuing discussion, we shall use the shorthand notation $h_i=h_i^N(x)$ and $h_i' = \frac{d}{dx}h_i^N(x)$. Therefore, $S^N = \text{span } \{h_i\}_{i=1}^{N-1}$ and $S_0^N = \text{span } \{h_i'\}_{i=1}^{N-1}$. We now define the subspaces $Z^N \subset Z$ by

$$Z^N \stackrel{\Delta}{=} S^N \times S_0^N \times L_2(-r,0;S_0^N). \tag{3.2.3}$$

Define P^N to be the orthogonal projection of Z onto Z^N.

For $\begin{pmatrix}\varphi\\ \psi\\ w\end{pmatrix} \epsilon Z$, we have

$$P^N\begin{pmatrix}\varphi\\ \psi\\ w\end{pmatrix} = \left(\sum_{i=1}^{N-1}\varphi_i h_i,\ \sum_{i=1}^{N-1}\psi_i h_i',\ \sum_{i=1}^{N-1} w_i(s) h_i'\right)^T, \tag{3.2.4}$$

where $\varphi_i, \psi_i \epsilon \mathbb{R}$ and $w_i(s) \epsilon L_2(-r,0)$ are determined by the equations

$$\sum_{i=1}^{N-1}\varphi_i\int_a^b h_i h_j = \int_a^b \varphi h_j \quad j = 1,2,\ldots,N-1, \tag{3.2.5}$$

$$\sum_{i=1}^{N-1}\psi_i\int_a^b h_i' h_j' = \int_a^b \psi h_j' \quad j = 1,2,\ldots, N-1, \tag{3.2.6}$$

and

$$\sum_{i=1}^{N-1} w_i(s)\int_a^b h_i' h_j' = \int_a^b w(s) h_j' \quad j = 1,2,\ldots,N-1. \tag{3.2.7}$$

respectively.

We denote by P_1^N the orthogonal projection of $L_2(a,b)$ onto S^N, by P_2^N the orthogonal projection of L_2^0 onto S_0^N, and by P_3^N the orthogonal projection of $L_2(-r,0;L_2^0)$ onto $L_2(-r,0;S_0^N)$. Therefore, we may write the projection P^N as

$$P^N\begin{pmatrix}\varphi\\ \psi\\ w\end{pmatrix} = \begin{pmatrix}P_1^N\varphi\\ P_2^N\psi\\ P_3^N w\end{pmatrix}.$$

Next we define operators A_0^N on $S^N \times S_0^N$ which "approximate" the operator A_0. Since $S^N \times S_0^N \not\subset \mathcal{D}(A_0)$, we cannot define A_0^N by restricting A_0 to $S^N \times S_0^N$. However, using the framework of Section 2.4 (which in this case is similar to the finite element method), we define the bilinear form σ_0 on $H^1 \times L_2^0$ by

$$\sigma_0\left(\begin{bmatrix}\varphi_1\\ \psi_1\end{bmatrix}, \begin{bmatrix}\varphi_2\\ \psi_2\end{bmatrix}\right) = -\int_a^b \frac{\alpha}{\rho}\varphi_2'\psi_1 + \int_a^b \varphi_1'\psi_2.$$

The operator A_0^N is defined as a restriction of σ_0 to $S^N \times S_0^N$ by

$$\langle A_0^N u, v\rangle \overset{\Delta}{=} \sigma(u,v) \text{ for } u,v \in S^N \times S_0^N.$$

Next, define the operator $A^N : \mathcal{D}(A^N) \subset Z^N \longrightarrow Z^N$ on the domain

$$\mathcal{D}(A^N) = \{\begin{pmatrix}\varphi\\ \psi\\ w\end{pmatrix} \in Z^N : w \in H^1(-r,0;S_0^N),\ w(0) = \psi\}, \tag{3.2.8}$$

by

$$A^N\begin{pmatrix}\varphi\\ \psi\\ w\end{pmatrix} = \begin{pmatrix} A_0^N\begin{pmatrix}\varphi\\ \psi\end{pmatrix} + \int_{-r}^0 G(s)A_0^N\begin{pmatrix}0\\ w\end{pmatrix}ds \\ \frac{dw}{ds}\end{pmatrix}. \qquad (3.2.9)$$

As mentioned above, for each N the operator A^N has the same structure as the reduced evolution operators associated with FDEs on $\mathbb{R}^k$. Therefore, each A^N, N=1,2,... is the infinitesimal generator of a C_0-semigroup $T^N(t)$ on Z^N (see [5], [19]), and we can use the AVE approximation ideas.

For each positive integer, M, partition the interval [-r,0] into subintervals $[t_j^M,t_{j-1}^M]$ for j=1,2,...,M, where $t_j^M = -\frac{jr}{M}$. Let $\chi_j^M \epsilon L_2(-r,0;S_0^N)$ denote the characteristic function on the interval $[t_j^M,t_{j-1}^M)$ for j=2,3,...,M, and let χ_0^M denote the characteristic function on $[t_1^M,t_0^M]$. Define the finite dimensional subspaces $\mathcal{S}^M$ of $L_2(-r,0;S_0^N)$ by

$$\mathcal{S}^M = \{\varphi\epsilon L_2(-r,0;S_0^N): \varphi = \sum_{j=1}^M v_j^M\chi_j^M,\ v_j^M\epsilon S_0^N\},$$

and define the finite dimensional subspaces $Z^{N,M}$ of Z by

$$Z^{N,M} \overset{\Delta}{=} S^N \times S_0^N \times \mathcal{S}^M. \tag{3.2.10}$$

Next define the operators $\mathcal{Q}^M : Z^N \longrightarrow Z^{N,M}$ by

$$\mathcal{Q}^M \begin{pmatrix} \varphi \\ \psi \\ w \end{pmatrix} = \begin{pmatrix} A_0^N \begin{pmatrix} \varphi \\ \psi \end{pmatrix} + \sum_{j=1}^{M} \frac{r}{M} G_j^M A_0^N \begin{pmatrix} 0 \\ w_j^M \end{pmatrix} \\ \sum_{j=1}^{M} \frac{M}{r}(w_{j-1}^M - w_j^M)\chi_j^M \end{pmatrix} \tag{3.2.11}$$

where

$$w_0^M \overset{\Delta}{=} \psi, \quad w_j^M \overset{\Delta}{=} \frac{M}{r} \int_{t_j^M}^{t_{j-1}^M} w(s)ds$$

and

$$G_j^M \overset{\Delta}{=} \frac{M}{r} \int_{t_j^M}^{t_{j-1}^M} G(s)ds \text{ for } j=1,2,\ldots,M.$$

Define the operators $A^{N,M} : Z \longrightarrow Z^{N,M}$ by

$$A^{N,M} \overset{\Delta}{=} \mathcal{Q}^M P^N. \tag{3.2.12}$$

<u>REMARK</u> <u>3.2.1</u> Our goal is to show that the subspaces $Z^{N,M}$ and operators $A^{N,M}$ approximate the space Z and operator A in the sense that the hypotheses of the Trotter-Kato Theorem are satisfied. In particular, this requires proving that as N and $M \longrightarrow \infty$

$$\|A^{N,M}z - Az\| \longrightarrow 0$$

for all z in some dense subset of $\mathcal{D}(A)$. From the triangle inequality, we see that

$$\|A^{N,M}z - Az\| \leqslant \|Q^M P^N z - A^N P^N z\| + \|A^N P^N z - Az\| = S_1 + S_2.$$

Elementary spline estimates imply that $S_2 \longrightarrow 0$ as $N \longrightarrow \infty$ for each z, but convergence of the AVE scheme implies only that $S_1 \longrightarrow 0$ as $M \longrightarrow \infty$ <u>for each fixed N and z</u>. In particular, the rate of convergence in M of $A^{N,M}$ to A^N is bounded by $\|A_0^N\|$. Although A_0 is an unbounded operator, we will show that $\|A_0^N\| = \mathcal{O}(N)$ (Lemma 3.2.5). We must then choose the index M <u>as a function of</u> N so that the convergence in M dominates the unbounded behavior of $\|A_0^N\|$. This will require the somewhat detailed error estimates found in the rest of this section. ∎

Let $f \in C(a,b)$. For each N, we denote the unique continuous piecewise linear interpolate of f by $f_1^N(x)$. That is, $f_1^N(x_i) = f(x_i)$ for $i = 0,1,\ldots,N$, and $f_1^N(x)$ is linear on each interval $[x_{i-1}, x_i]$ for $i = 1,2,\ldots,N$. We recall the following well-known convergence result for interpolating linear splines (see [21], Theorem 2.5).

<u>THEOREM 3.2.1</u> There exist fixed constants K_1, K_2 such that

if $f(x)\epsilon C^2(a,b)$, then

$$\|f_1^N - f\|_{L_2(a,b)} \leqslant \frac{1}{N^2}K_1\|\frac{d^2}{dx^2}f\|_{L_2(a,b)} \qquad (3.2.13)$$

and

$$\|\frac{d}{dx}(f_1^N-f)\|_{L_2(a,b)} \leqslant \frac{1}{N}K_2\|\frac{d^2}{dx^2}f\|_{L^2(a,b)}. \qquad (3.2.14)$$

■

The following lemma is an immediate consequence of the Schmidt inequality (see [21], page 7).

<u>LEMMA</u> <u>3.2.2</u>. There exists a constant K_3, independent of N, such that if $f\epsilon S^N$, then $\|f'\|_{L_2} \leqslant K_3N\|f\|_{L_2}$.

<u>Proof</u>: Since $f\epsilon S^N$, f is linear on each interval $[x_{i-1},x_i]$, i=1,2,...,N. Hence the Schmidt inequality applies on each interval and we have

$$\|f'\|_{L_2}^2 = \int_a^b |f'|^2 = \sum_{i=1}^{N}\int_{x_i}^{x_i}|f'|^2$$

$$\leqslant \sum_{i=1}^{N} 12N^2\int_{x_{i-1}}^{x_i}|f|^2$$

$$= 12N^2\|f\|_{L_2}^2,$$

and the result follows. ■

Let us now recall some convergence rates for the projection P_1^N defined above.

<u>LEMMA</u> <u>3.2.3</u> If $f \in L_2$, then

$$\|P_1^N f - f\|_{L_2} \longrightarrow 0 \text{ as } N \longrightarrow \infty.$$

If $g \in H^1$, then

$$\|P_1^N g' - g'\|_{L_2} \longrightarrow 0$$

and

$$\|(P_1^N g)' - g'\|_{L_2} \longrightarrow 0 \text{ as } N \longrightarrow \infty.$$

<u>Proof</u>: See, for example, [3] or [14]. ∎

<u>LEMMA</u> <u>3.2.4</u> i) If $f \in C^1(a,b) \cap L_2^0(a,b)$, then

$$\|P_2^N f - f\|_{L_2} \leqslant \frac{1}{N} K_2 \|\frac{d}{dx} f\|_{L_2}.$$

ii) If $w \in C(-r,0;C^1 \cap L_2^0)$, then

$$\|P_3^N w - w\|_{L_2(-r,0;L_2^0)} \longrightarrow 0 \text{ as } N \longrightarrow \infty.$$

<u>Proof</u>: Since P_2^N is the orthogonal projection of $L_2^0(a,b)$ onto S_0^N, it follows that for $f \in L_2^0(a,b)$,

$$\|P_2^N f - f\|_{L_2} = \min_{u \in S_0^N} \|u - f\|_{L_2}.$$

Define $F(x) \overset{\Delta}{=} \int_a^x f(s)ds$. Since the spline interpolate of F satisfies $\frac{d}{dx}(F_1^N) \in S_0^N$, it follows that

$$\|P_2^N f - f\|_{L_2} \leqslant \|\frac{d}{dx}(F_1^N) - f\| = \|\frac{d}{dx}(F_1^N - F)\|_{L_2} \leqslant \frac{1}{N}\cdot K_2 \|\frac{d}{dx} f\|_{L_2}$$

where the last inequality follows from (3.2.14). The result ii) follows from i) and the dominated convergence theorem. Thus the lemma is proved. ∎

<u>LEMMA</u> <u>3.2.5</u> If $\varphi \epsilon C^2 \cap C_0^1$ and $\psi \epsilon C^1 \cap L_2^0$, then

$$\|A_0^N \begin{pmatrix} P_1^N \varphi \\ P_2^N \psi \end{pmatrix} - A \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \|_{L_2 \times L_2^0} \longrightarrow 0 \text{ as } N \longrightarrow \infty .$$

<u>Proof</u>: Let $A_0^N \begin{pmatrix} P_1^N \varphi \\ P_2^N \psi \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{N-1} a_i h_i , & \sum_{i=1}^{N-1} b_i h_i' \end{pmatrix}$. From the definition of A_0^N it follows that

$$\sum_{i=1}^{N-1} a_i \int_a^b h_i h_j = -\frac{\alpha}{\rho} \int_a^b (P_2^N \psi) h_j'$$

$$= -\frac{\alpha}{\rho} \int_a^b \psi h_j'$$

$$= \frac{\alpha}{\rho} \int_a^b \psi' h_j , \text{ for all } j=1,2,\ldots,N-1 .$$

Hence, $\sum_{i=1}^{N-1} a_i h_i = P_1^N (\frac{\alpha}{\rho} \psi')$. It follows also from the definition of A_0^N that

$$\sum_{i=1}^{N-1} b_i \int_a^b h_i' \; h_j' = \int_a^b (P_1^N\varphi)' h_j' .$$

Therefore, since $(P_1^N\varphi)' \in S_0^N$, it follows that

$$\sum_{i=1}^{N-1} b_i \; h_i' = (P_1^N\varphi)' .$$

Hence,

$$\left\| A_0^N \begin{pmatrix} P_1^N\varphi \\ P_2^N\psi \end{pmatrix} - A_0 \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\|^2_{L_2 \times L_2^0}$$

$$= \| P_1^N(\psi') - \psi' \|^2_{L_2} + \| (P_1^N\varphi)' - \varphi' \|^2_{L_2},$$

which converges to 0 by Lemma 3.2.3. ∎

<u>LEMMA</u> <u>3.2.6</u> If $\varphi \epsilon C^2 \cap C_0^1$ and $\psi \epsilon C^1 \cap L_2^1$, then

$$\left\| A_0^N \begin{pmatrix} P_1^N\varphi \\ P_2^N\psi \end{pmatrix} \right\|^2_{L_2 \times L_2^0} \leqslant KN^2 \left\| \begin{matrix} \frac{\alpha}{\rho} P_1^N\psi \\ P_1^N\varphi \end{matrix} \right\|^2_{L_2 \times L_2^0} + e(N)$$

where $e(N) \longrightarrow 0$ as $N \longrightarrow \infty$, and K is independent of φ, ψ.

<u>Proof</u>: The same calculation used in Lemma 3.2.5 yields

$$A_0^N \begin{pmatrix} P_1^N\varphi \\ P_2^N\psi \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{\rho} P_1^N(\psi') \\ (P_1^N\varphi)' \end{pmatrix} .$$

Hence, we have the estimate

$$\left\| A_0^N \begin{pmatrix} P_1^N \varphi \\ P_2^N \psi \end{pmatrix} \right\|^2_{L_2 \times L_2^0} = \left\| \frac{\alpha}{\rho} P_1^N (\psi') \right\|^2 + \left\| (P_1^N \varphi)' \right\|^2$$

$$\leqslant 2 (\left\| \frac{\alpha}{\rho} (P_1^N \psi)' \right\|^2 + \left\| (P_1^N \varphi) \right\|^2$$

$$+ \left\| \frac{\alpha}{\rho} [(P_1^N \psi)' - \psi'] \right\|^2 + \left\| \frac{\alpha}{\rho} [\psi' - P_1^N (\psi') \right\|^2).$$

Now apply Lemma 3.2.2 to the first two terms, and Lemma 3.2.3 to the last two terms, and the result follows. ∎

Since $g \in H^1$, it follows from standard arguments (i.e. see the proof of Corollary 3.1 in [2]) that there is a constant K(g) such that

$$\int_{-r}^{0} | \sum_{i=1}^{M} g_i^M \chi_i^M (s) - g(s) |^2 ds \leqslant \frac{K}{M^2}.$$

Consequently, for each $N \geqslant 0$ there is an $M_1(N) = M_1$ such that

$$\lim_{N \to \infty} N^2 \int_{-r}^{0} | \sum_{i=1}^{M_1} g_i^{M_1} \chi_i^{M_1} (s) - g(s) |^2 ds = 0. \qquad (3.2.15)$$

It is important to note that if $M_1(N) = N^P$ for $P > 1$, then (3.2.15) follows from the previous estimate and $M_1(N) \to \infty$ as $N \to \infty$. On the other hand, for special forms of g(s),

$M_1(N)$ does not have to be unbounded. For example, if $g(s) \equiv c$, then $M_1(N) \equiv 1$ suffices to ensure (3.2.15).

LEMMA 3.2.7 There exist constants $\mathfrak{M},\beta$ such that for all N,M satisfying $M \geqslant M_1(N)$

$$\|T^{N,M}(t)\| \leqslant \mathfrak{M}e^{\beta t}.$$

Proof: By Lemma 3.1.5, it is sufficient to show that $\langle LA^{N,M}Lz,z\rangle \leqslant \beta\|z\|^2$ for all $z \in Z$. Therefore, let $\begin{pmatrix}\varphi\\ \psi\\ w\end{pmatrix} \in Z$ and

$$P^N\begin{pmatrix}\varphi\\ \psi\\ w\end{pmatrix} = \begin{pmatrix}\varphi_N\\ \psi_N\\ w_N\end{pmatrix}.$$

Consequently we have the equality

$$\langle LA^{N,M}Lz,z\rangle = \langle Q^M P^N Lz, P^N Lz\rangle$$

$$= -\alpha\int_a^b \varphi_N' \psi_N + \epsilon\int_a^b \varphi_N' [\psi_N - \frac{1}{\epsilon}\int_{-r}^0 g(s)w_N(s)ds]dx$$

$$+ \sum_{j=1}^{M} \frac{r}{M} G_j^M \int_a^b \varphi_N' [w_N - \psi_N]_j^M$$

$$- \int_{-r}^0 g(s)\int_a^b \left(\sum_{j=1}^{M} \frac{M}{r}\left[(\psi_N - w_N)_{j-1}^M - (\psi_N - w_N)_j^M\right]\chi_j^M\right) w_N dx ds$$

$$= \int_a^b \varphi_N' \left[\sum_{j=1}^{M} \int_{t_j^M}^{t_{j-1}^M} g_j^M w_N(s)ds - \int_{-r}^0 g(s)w_N(s)ds\right]dx$$

$$- \int_{-r}^0 g(s)\int_a^b \left[\sum_{j=1}^{M} \frac{M}{r}\left[(\psi_N - w_N)_{j-1}^M - (\psi_N - w_N)_j^M\right]\chi_j^M\right] w_N dx$$

$= S_1 + S_2.$

An argument analogous to the one used in the proof of Lemma 3.6 in [2], (in which the AVE scheme is shown to satisfy the stability hypotheses H1 of the Trotter-Kato Theorem), yields

$$S_2 \leqslant \tfrac{1}{2}\|\psi_N\|^2_{L_2} \leqslant \tfrac{1}{2}\|\psi\|^2_{L_2}. \tag{3.2.16}$$

Considering the term S_1 we have

$$S_1 = \int_a^b \varphi_N' \left[\int_{-r}^0 \left(\sum_{j=1}^M g_j^M \chi_j^M - g(s)\right) w_N(s)\,ds\right] dx$$

The Cauchy-Schwarz inequality yields

$$S_1 \leqslant \|\varphi_N'\|_{L_2} \cdot \int_{-r}^0 \left|\sum_{j=1}^M g_j^M \chi_j^M - g(s)\right| \cdot \|w_N(s)\|_{L_2} ds.$$

Since $M \geqslant M_1(N)$, it follows from Lemma 3.2.2, the Cauchy-Schwarz inequality, and the inequality $ab \leqslant \frac{1}{2}(a^2+b^2)$ that

$$\leqslant K_3 N \cdot \|\varphi_N\|_{L_2} \cdot \frac{1}{N} \cdot \left(\int_{-r}^0 \|w_N(s)\|^2_{L_2} ds\right)^{1/2}$$

$$\leqslant C_1\|\varphi_N\|\left(\int_{-r}^0 \|w_N(s)\|^2_{L_2}\right)^{1/2}$$

$$\leqslant C_2\left(r\|\varphi_N\|^2 + \int_{-r}^0 \|w_N(s)\|^2_{L_2} ds\right)$$

$$= C_2\left(\|P_1^N\varphi\|^2 + \int_{-r}^0 \|P_3^N w(s)\|\right)^2 \leqslant c\left\|\begin{matrix}\varphi\\ \psi\\ w\end{matrix}\right\|^2. \tag{3.2.17}$$

The equations (3.2.16) and (3.2.17) yield the result. ∎

In order to prove the remaining hypotheses of the Trotter-Kato Theorem, define the dense subset D of $\mathcal{D}(A)$ by

$$D = \left\{ \begin{pmatrix} \varphi \\ \psi \\ w \end{pmatrix} \in \mathcal{D}(A): \quad \varphi \in H^2, \ \psi \in H^1, \ w \in C^1(-r,0;L_2^0) \right\}. \qquad (3.2.18)$$

Clearly D is dense in $\mathcal{D}(A)$, hence in Z. The next lemma shows that the consistency condition H2) is satisfied.

<u>LEMMA</u> <u>3.2.8</u> Define the index M(N) such that

$$M(N) \geq M_1(N) \qquad (3.2.19)$$

and

$$M(N) \longrightarrow \infty \text{ as } N \longrightarrow \infty. \qquad (3.2.20)$$

If D is defined by (3.2.18), then $A^{N,M(N)} z \longrightarrow Az$ as $N \longrightarrow \infty$ for all $z \in D$.

<u>Proof</u>: If $z = \begin{pmatrix} \varphi \\ \psi \\ w \end{pmatrix} \in D$, then

$$\|A^{N,M(N)} z - Az\|^2$$

$$\leq \left\| A_0^N \begin{pmatrix} P_1^N \varphi \\ P_2^N \psi \end{pmatrix} - A_0 \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\|^2_{L_2 \times L_2^0}$$

$$+ \left\| \sum_{j=1}^{M} \frac{r}{M} G_j^M A_0^N \begin{pmatrix} 0 \\ [P_3^N w]_j^M \end{pmatrix} - \int_{-r}^{0} G(s) A_0 \begin{pmatrix} 0 \\ w \end{pmatrix} ds \right\|^2$$

$$- \int_{-r}^{0} g(s) \left\| \sum_{j=1}^{M} \frac{M}{r} \left[[P_3^N w]_{j-1}^M - [P_3^N w]_j^M \right] \chi_j - \frac{dw}{ds} \right\|^2_{L_2} ds$$

$$= S_1 + S_2 + S_3.$$

Lemma 3.2.5 implies $S_1 \longrightarrow 0$ as $N \longrightarrow \infty$. For the term S_2, we have that

$$S_2 = \left\| \sum_{j=1}^{M} G_j^M A_0^N \begin{pmatrix} 0 \\ \int_{t_j^M}^{t_{j-1}} P_3^N w \end{pmatrix} - \int_{-r}^{0} G(S) A_0 \begin{pmatrix} 0 \\ w \end{pmatrix} ds \right\|_{L_2}^2$$

$$\leqslant \left[\int_{-r}^{0} \left| \sum_{j=1}^{M} G_j^M \chi_j^M - G(s) \right| \; \left\| A_0^N \begin{pmatrix} 0 \\ P_3^N w \end{pmatrix} \right\| ds \right]^2$$

$$+ \int_{-r}^{0} |G(s)| \cdot \left\| A_0^N \begin{pmatrix} 0 \\ P_N^3 w(s) \end{pmatrix} - A_0 \begin{pmatrix} 0 \\ w(s) \end{pmatrix} \right\|^2 ds$$

$$= F_1 + F_2.$$

The dominated convergence theorem and Lemma 3.2.5 imply that $F_2 \longrightarrow 0$ as $N \longrightarrow \infty$. For the term F_1, applying first the Cauchy-Schwarz inequality and then Lemma 3.2.6 yields

$$F_1 \leqslant K \left[\int_{-r}^{0} \left| \sum_{j=1}^{M} g_j^M \chi_j^M - g(s) \right|^2 ds \right] \cdot N^2 \left(\|w_N\| + e(N) \right)$$

where $e(N) \longrightarrow 0$ as $N \longrightarrow \infty$. Hence, by (3.2.15) $F_1 \longrightarrow 0$ as $M \longrightarrow \infty$, from which we conclude that $S_2 \longrightarrow 0$ as $N \longrightarrow \infty$. For the term S_3, we have the estimate

$$S_3 \leqslant - \int_{-r}^{0} g(s) \left\| \sum_{j=1}^{M} \frac{M}{r} \left([P_3^N w]_{j-1}^M - [P_3^N w]_j^M \right) \chi_j - \frac{d}{ds} \left(P_3^N w \right) \right\|^2 ds$$

$$- \int_{-r}^{0} g(s) \left\| \frac{d}{ds} [P_3^N w - w] \right\|^2 ds$$

$$= E_1 + E_2.$$

Observing that $\frac{d}{ds}P_3^N w = P_3^N\left(\frac{dw}{ds}\right)$, Lemma 3.2.4 implies that E2 $\longrightarrow 0$ as $N \longrightarrow \infty$. For the term E_1, we follow the argument used by Banks and Burns in the proof of Corollary 3.1 in [2]. This leads to

$$E_1 \leqslant r \cdot \left\{\sup_{1\leqslant j\leqslant M} \tfrac{5}{2}\alpha_j^M\right\}^2 + \frac{r}{M}\left\{\tfrac{3}{2}\gamma_1^M + \tfrac{K}{2}\right\}^2$$

where

$$\gamma_j^M = \sup\left\{\left\|\left(\frac{d}{ds}P_3^N w\right)(\theta) - \left(\frac{d}{ds}P_3^N w\right)(\tau)\right\| : \theta,\ \tau\epsilon\left[t_j^M, t_{j-1}^M\right]\right\}$$

$$\leqslant \sup\left\{\left\|\left(\frac{dw}{ds}(\theta) - \frac{dw}{ds}(\tau)\right)\right\|_{L_2} : \theta,\tau\epsilon\left[t_j^M, t_{j-1}^M\right]\right\}$$

and

$$K = \sup\left\{\left\|\frac{dw}{ds}(\theta)\right\| : \theta\epsilon[-r,0]\right\}.$$

Since $N \longrightarrow \infty$ implies that $M(N) \longrightarrow \infty$, it follows from the uniform continuity of $\frac{dw}{ds}$ on $[-r,0]$ that $E_1 \longrightarrow 0$ as $N \longrightarrow \infty$. Therefore, $S_3 \longrightarrow 0$ as $N \longrightarrow \infty$ and the lemma is proved. ∎

The next lemma shows that hypothesis H3) of the Trotter-Kato Theorem is satisfied.

<u>LEMMA</u> <u>3.2.9</u> If D is defined by (3.2.18), then there exists a real number λ such that $\mathcal{R}(A-\lambda)D$ is dense in Z.

<u>Proof</u>: Since A is the infinitesimal generator of a C_0-semigroup, there exists a real number λ such that given

$\begin{pmatrix}\alpha\\ \beta\\ z\end{pmatrix} \epsilon Z$, the equation

$$(A-\lambda) \begin{pmatrix}\varphi\\ \psi\\ w\end{pmatrix} = \begin{pmatrix}\alpha\\ \beta\\ z\end{pmatrix} \tag{3.2.21}$$

has a unique solution $\begin{pmatrix}\varphi\\ \psi\\ w\end{pmatrix} \epsilon\ \mathcal{D}(A)$.

If $\begin{pmatrix}\alpha\\ \beta\\ z\end{pmatrix} \epsilon\ S \overset{\Delta}{=} C(a,b) \times C^1 \cap L_2^0 \times H^1(-r,0;L_2^0)$ (which is dense in Z), then the solution $\begin{pmatrix}\varphi\\ \psi\\ w\end{pmatrix} \epsilon\ \mathcal{D}(A)$ of (3.2.21) satisfies

$$\frac{\alpha}{\rho}\psi' + \frac{1}{\rho}\int_{-r}^{0} g(s)w'(s)ds - \lambda\,\varphi = \alpha \tag{3.2.22}$$

$$\varphi' - \lambda\psi = \beta \tag{3.2.23}$$

$$\frac{dw}{ds} - \lambda w = z. \tag{3.2.24}$$

Since $\psi\ \epsilon\ H^1$ and $\beta\ \epsilon\ C^1$, (3.2.23) implies that $\varphi \epsilon C^2$. Simialrly, since $w \epsilon H^1(-r,0;L_2^0)$ and $z \epsilon H^1(-r,0);L_2^0)$, (3.2.24) implies that $w \epsilon C^1(-r,0;L_2^0)$. Also, (3.2.22) implies that $\psi \epsilon H^1$ and hence $\begin{pmatrix}\varphi\\ \psi\\ w\end{pmatrix} \epsilon\ D$. Therefore, the range of $A-\lambda$ contains the dense subset S, and the result follows. ∎

The previous lemmas imply that our approximation scheme satisfies the hypotheses of the Trotter-Kato Theorem. We summarize the result in the following theorem.

<u>THEOREM</u> <u>3.2.10</u> Let the index M(N) be defined so that

(3.2.19) and (3.2.20) hold. If the sequence of operators $\mathcal{A}^N: Z \longrightarrow Z^{N,M(N)}$ is defined by $\mathcal{A}^N \overset{\Delta}{=} A^{N,M(N)}$, then the hypotheses of the Trotter-Kato Theorem hold; that is,

$$\mathcal{T}^N(t) \xrightarrow{s} T(t),$$

uniformly on compact t intervals, where $\mathcal{T}^N(t)$ is the semigroup generated by $\mathcal{A}^N$. ∎

Assuming that the approximation scheme $(Z^N,P^N,\mathcal{A}^N)$ and index M(N) are defined as in Theorem 3.2.10, let us consider the corresponding adjoint approximation scheme $(Z^N, P^N, \mathcal{A}^{N*})$. Since $\|T^{N*}(t)\| = \|T^N(t)\|$, it follows from Lemma 3.2.7 that hypothesis H1) of the Trotter-Kato Theorem is satisfied by the adjoint approximation scheme. We note that since $A_0^{N*} = -A_0^N$, it follows that the estimates in Lemma 3.2.6 apply to A_0^{N*}. Combining this observation with the convergence rates for the approximating adjoint operators in the AVE scheme (see [11]), it follows that

$$\mathcal{A}^{N*}z \longrightarrow A^*z$$

for all $z \in D^*$. Here D^* is the dense subset of $\mathcal{D}(A^*)$ defined by

$$D^* \ \{\begin{pmatrix}\varphi\\ \psi\\ \omega\end{pmatrix} \in \mathcal{D}(A^*): \ \varphi \in H^2, \ \psi \in H^1, \ \omega \in C^1(-r,0;L_2^0)\}.$$

Arguments completely analogous to those used in the proof of Lemma 3.2.9 yield that hypothesis H3) holds for the adjoint approximation scheme. From these observations we conclude the following.

Corollary 3.2.11 If the approximation scheme (Z^N, P^N, Q^N) is defined as in Theorem 3.2.10 then

$$T^{N*}(t) \xrightarrow{s} T^*(t),$$

uniformly in compact intervals. ∎

Therefore, we conclude (see Section 2.2) that our approximation scheme should prove reasonable for approximating the optimal control problem associated with the viscoelastic system (3.1.3)-(3.1.6). In the next chapter we discuss some of our numerical results for this problem.

CHAPTER IV

4.1 A viscoelastic model

In this chapter, we discuss some numerical results for the modeling of the viscoelastic system described in Chapter III. Although we have conducted a number of numerical experiments (primarily to test the convergence rates discussed in Chapter III), we shall present a small sample of these experiments to illustrate typical results.

All of the numerical results presented below are based on the approximation schemes developed in Chapter III for the equations of viscoelasticity. In particular, we considered the equation (for $0 \leqslant x \leqslant 1$ and $t \geqslant 0$)

$$\rho \frac{\partial^2}{\partial t^2} y(t,x) = \alpha \frac{\partial^2}{\partial x^2} y(t,x) + \int_{-r}^{0} g(s) \frac{\partial^2}{\partial x^2} y(t+s,x)\,ds + b(x)u(t) \qquad (4.1.1)$$

with boundary conditions

$$y(t,0) = 0 = y(t,1) \qquad (4.1.2)$$

and initial data

$$y(0,x) = s(x), \quad \frac{\partial}{\partial t} y(0,x) = v(x) \qquad (4.1.3)$$

$$y(s,x) = h(s,x) \qquad -r \leqslant s < 0 \quad 0 \leqslant x \leqslant 1 . \qquad (4.1.4)$$

The system parameters are defined to be $\rho=1$, $\alpha=1$, $r=1$ and the functions $b(x)$, $s(x)$, $v(x)$ and $h(s,x)$ are defined to be $b(x)\equiv 1$, $s(x)=\sin\pi x$, $v(x)=\pi\sin\pi x$ and $h(s,x)=\sin\pi x(\cos\pi s + \sin\pi s)$, respectively.

The selection of the function $g(s)$ is made to insure that the assumptions in Section 3.1 holds. We were particularly interested in the case where $g(s)$ is "singular" at $s=0$. However, the assumption that $g\in H^1(-r,0)$ does not allow for the inclusion of such kernel functions. Therefore, we constructed a function (depending on a parameter $p\geq 1$) that would still satisfy the assumptions needed to establish the well-posedness and convergence and yet become "nearly singular" as $p\to\infty$. In particular, for $1\leq p<+\infty$ let

$$g_p(s) \overset{\Delta}{=} -e^{5s}q_p(s) \overset{\Delta}{=} -e^{5s}\begin{cases}(-s)^{1/2} & -r\leq s\leq -\frac{r}{p}\\ M_p\left(s+\frac{r}{p}\right) + \left(\frac{p}{r}\right)^{1/2}, & -\frac{r}{p}\leq s\leq 0\end{cases}$$

where M is related to p by

$$M_p = \left[\frac{1+(e^{-5r}\left(\frac{p}{r}\right)^{1/2})/5}{((e^{\frac{-5r}{p}}-1) + \frac{5r}{p})/25}\right]. \tag{4.1.6}$$

Condition (4.1.6) implies that

$$\int_{-r}^{0} g_p(s)ds<1,$$

and it is easy to verify that $g_p(s)$ satisfies the remaining assumptions listed in Section 3.1. Moreover, $q_p(s) \longrightarrow \frac{1}{\sqrt{-s}}$ as $p \longrightarrow \infty$ and $q_1(s)$ is a linear function on $[-r,0]$. It is important to note that for $p=2^{10}$, $g_p(s)$ is "numerically singular" and hence the numerical results for this case should be indicative of a truly singular kernel.

In Section 4.2, we consider the approximation of the open-loop system (i.e. the location of the open-loop poles). We conclude this chapter with a short discussion of plans for future work.

4.2 The open-loop system

In this section we consider the open-loop approximating system

$$\dot{z}(t) = A^{N,M}z(t), \qquad (4.2.1)$$

where $A^{N,M}$ is the approximating operator constructed in Section 3.2 above. We are interested in the eigenvalues of $A^{N,M}$, and in particular, the eigenvalues of $A^{N,M}$ that correspond to the damping induced on each fundamental mode.

Recall that if $g(s)\equiv 0$, then the system (4.1.1)-(4.1.2) becomes the wave equation with fundamental frequencies $\omega_k = k\pi$, $k=1,2,\ldots$. There is no damping in the system and for this case, $A^{N,M} = A^N$ (there is no history term) is the standard finite element approximation for the wave equation. Consequently, one must be aware of the fact that the error induced by the finite element scheme increases with ω. For example, using 25 linear elements to estimate the first ten frequencies leads to the results given in Table 4.1. In Table 4.1, ω_k^{25} denotes the estimate for w_k obtained by using 25 linear elements.

TABLE 4.1

$\underline{K}$	$W_k = k\pi$	w_k^{25}
1	3.1416	3.1437
2	6.2833	6.2998
3	9.4248	9.4807
4	12.5664	12.6990
5	15.7080	15.9674
6	18.8496	19.2986
7	21.9911	22.7052
8	25.1327	26.1997
9	28.2743	29.7941
10	31.4159	33.4993

Observe that there is more than 5% error in the tenth frequency. This error provides at least a "lower bound" on the overall accuracy of the approximate system (4.2.1). Error estimates for the finite element method are well-known and we shall not dwell on this aspect of our approximation scheme. Our major concern is the nature of the damping induced by the history term in (4.1.1).

RUN 1 For this case we set $p=2^{10}$ (so that $g_p(s)$ is nearly singular) and constructed the approximate operator $A^{N,M}$. The IMSL routine EIGRF was used to compute the eigenvalues of $A^{N,M}$. Recall that $A^{N,M}$ is an (N-1)(M+2) dimensional square matrix and consequently has (N-1)(M+2) eigenvalues. Each figure contains only those eigenvalues in the upper half complex plane. For example, Figure 4.2 provides the eigenvalue location of $\lambda_i^{4,8}$, i=1,2,...30, for only those eigenvalues with $Im(\lambda) \geqslant 0$. Moreover, we are mainly interested in those eigenvalues that determine the damping of each fundamental frequency. These eigenvalues are denoted by $\lambda^{N,M}(k)$ and in Figure 4.2 the location of these eigenvalues are noted by ■. Therefore, in the remaining figures we shall display only these eigenvalues.

Figure 4.3 illustrates the behavior of $\lambda^{8,M}(k)$ for M = 4,8,16,32,64. The interesting feature here is that for low values of M the damping curve predicts near viscous damping and as M increases we see that the curve becomes quadratic (as to be expected for a "singular kernel"). Such results are not unexpected in view of the fact that the convergence of $A^{N,M}$ to the generator A was established only for N and $M(N) \geqslant M_1(N) \geqslant N^2$ (see Theorem 3.2.10). In particular, this scheme does not appear to be unconditionally stable.

Figure 4.4 illustrates the behavior of $\lambda^{N,8}(k)$ as N = 4,8,16. For $N \geqslant 16$, the results are identical (numerically) to the N=16 case. At high frequencies, the damping curves are nearly vertical straight lines, i.e.

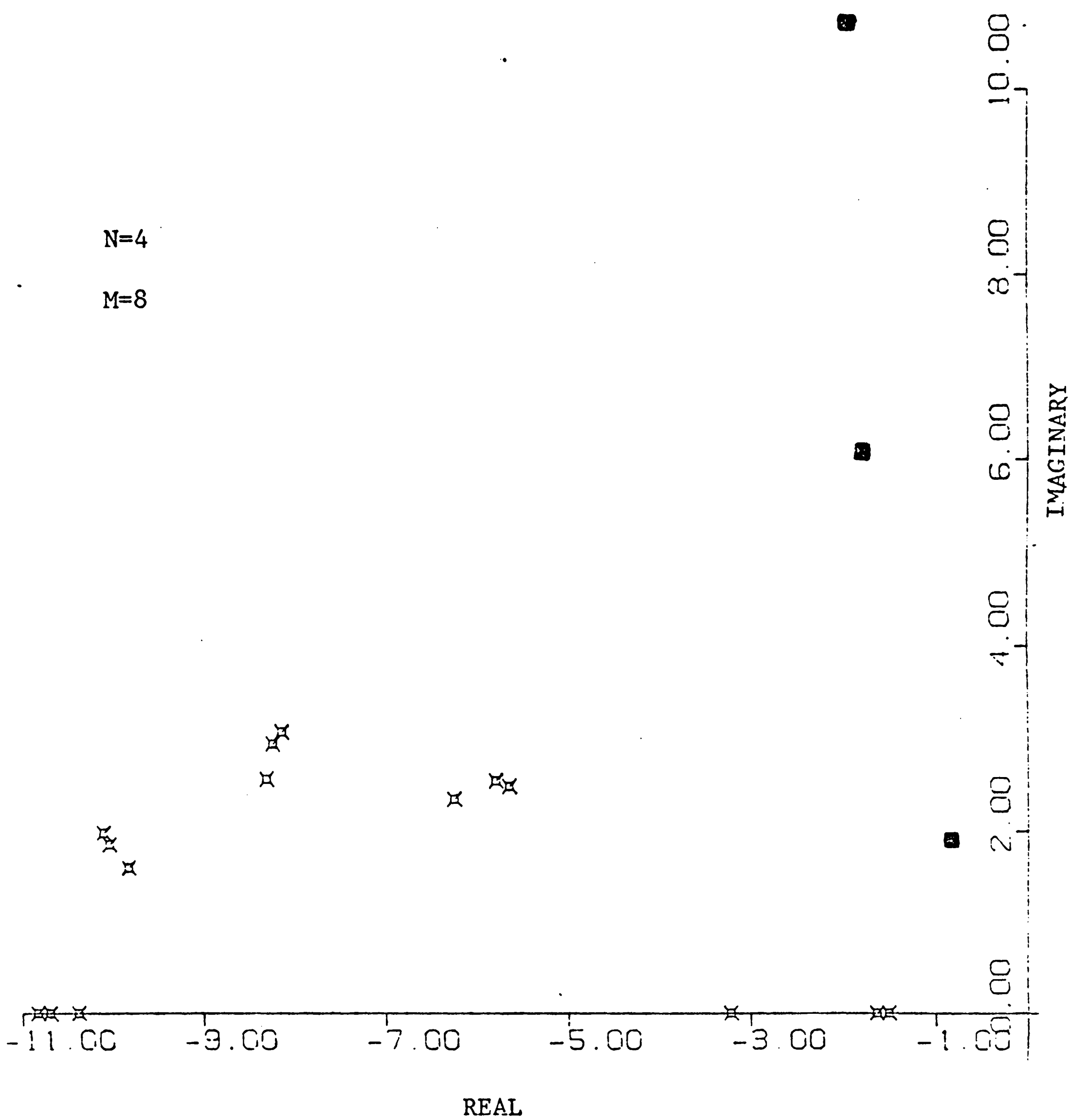

N=4
M=8
IMAGINARY
0.00
2.00
4.00
6.00
8.00
10.00
-11.00
-9.00
-7.00
-5.00
-3.00
-1.00
REAL

FIGURE 4.2

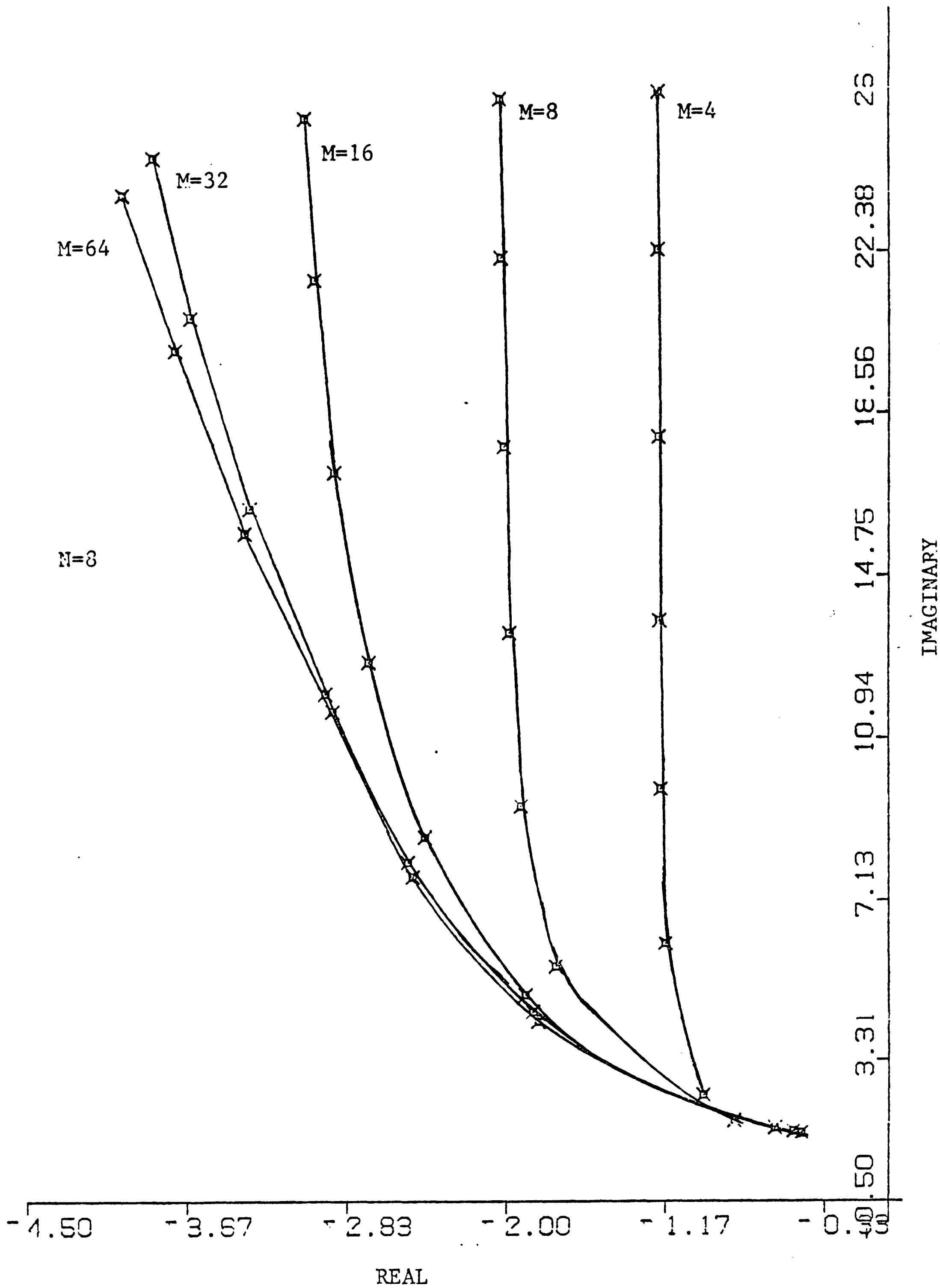
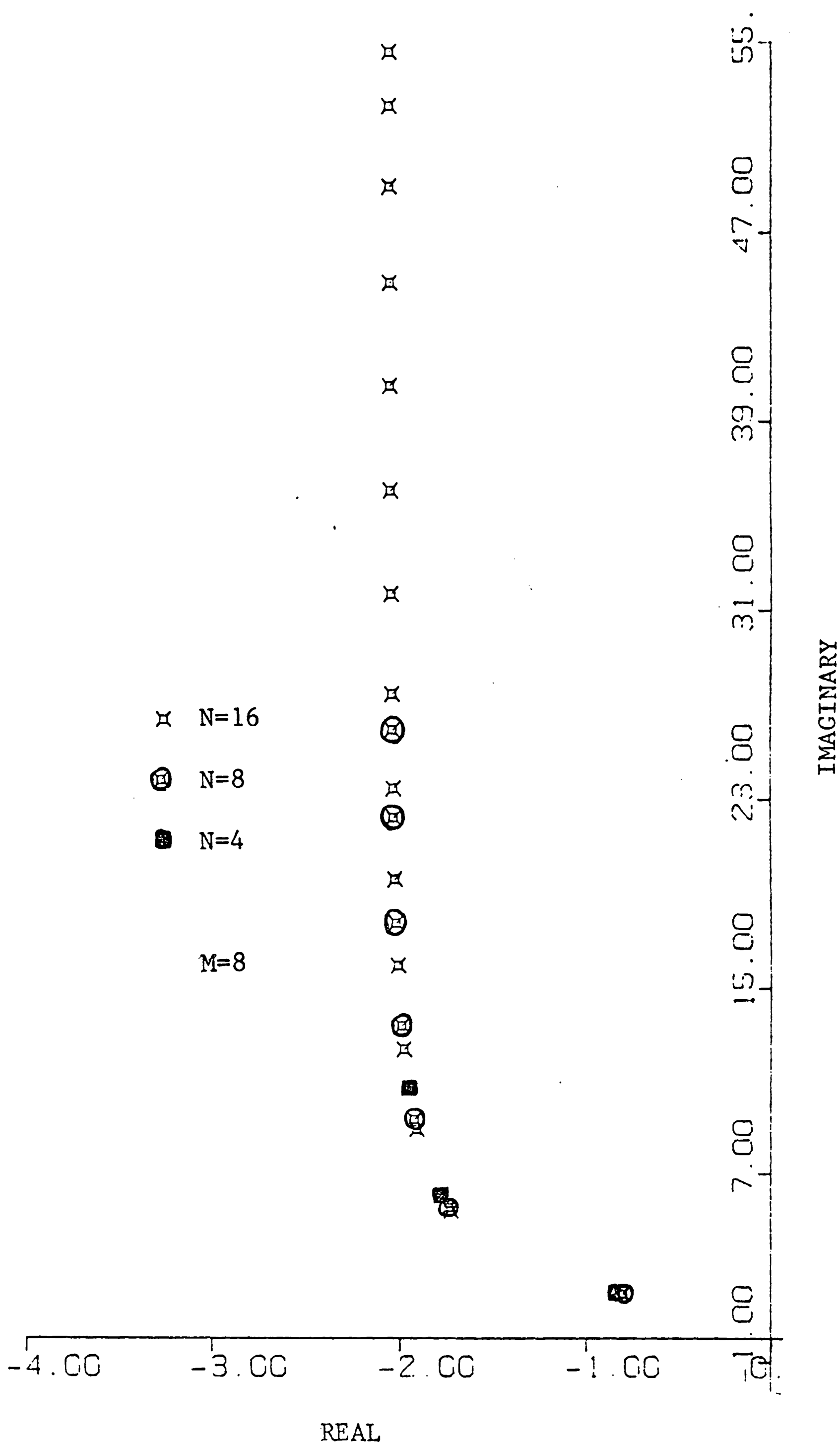

M=4
M=8
M=16
M=32
M=64
M=8
-4.50
-3.67
-2.83
-2.00
-1.17
-0.33
0.50
3.31
7.13
10.94
14.75
18.56
22.38
25
REAL
IMAGINARY

FIGURE 4.3



55.
47.00
39.00
31.00
23.00
15.00
7.00
1.00
IMAGINARY
N=16
N=8
N=4
M=8
-4.00
-3.00
-2.00
-1.00
.0
REAL

FIGURE 4.4

representative of pure viscous damping. Again, this illustrates the need to construct an approximate model $A^{N,M}$ with $M(N) \gg M_1(N) \gg N^2$ to insure that the finite dimensional model accurately predicts the damping produced by the history term.

RUN 2: The results illustrated in Figures (4.2)-(4.4) are typical of the many numerical experiments that we conducted using various parameters and kernels $g_p(s)$ for large p. In order to determine the effect of a non-singular kernel, we reproduced the N=8, M=64 results for the kernel function $g_2(s)$. Figure 4.5 contains the eigenvalue locations for this system. Again we see that the approximate model predicts purely viscous damping in the high frequency modes. This run illustrates the importance of the singular kernel in the modeling of structural damping at high frequency.

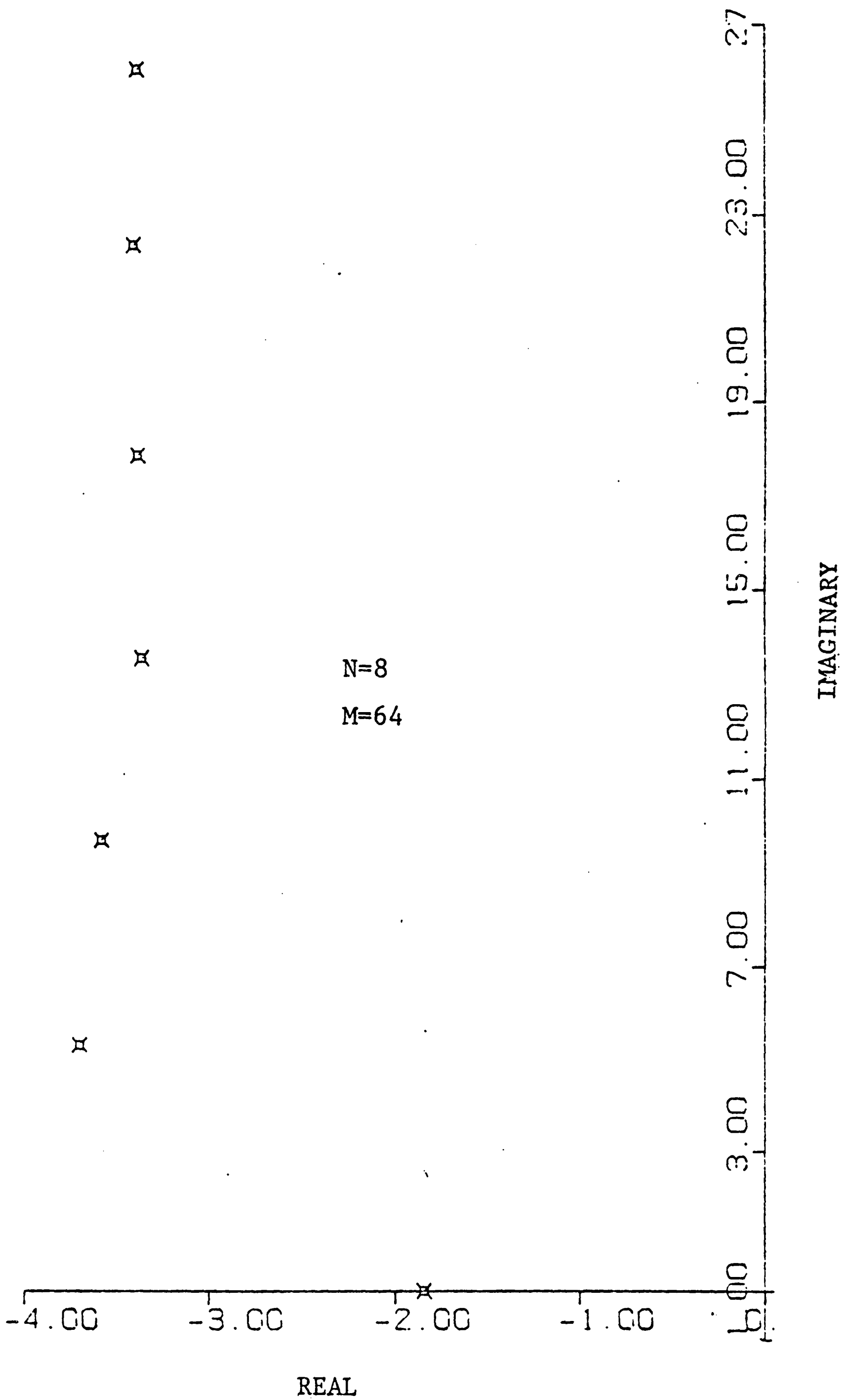

N=8
M=64
-4.00
-3.00
-2.00
-1.00
0
3.00
7.00
11.00
15.00
19.00
23.00
27
REAL
IMAGINARY

FIGURE 4.5

4.3 Concluding remarks

In this paper, we considered well-posedness and approximation schemes for a class of functional partial differential equations of hyperbolic type. Our study was motivated by the desire to construct state space models and convergent finite dimensional approximate models suitable for control design. Therefore, we first developed a general algorithm for constructing finite dimensional systems that approximate well-posed Cauchy problems and their adjoints. We then established the well-posedness of the Boltzmann model for viscoelasticity in the state space $L_2 \times L_2^0 \times L_2(-r,0;L_2^0)$ and used this model to develop a convergent numerical scheme. We have used this scheme to simulate the system and to study the effects of the kernel on the damping in the system.

Our numerical results are preliminary, and there is much room for further study. For example;

1) We plan to test the approximation scheme on optimal control problems for the viscoelastic model.
2) We plan to use the "new splines" for discretizing the hereditary variable in our approximation scheme, as compared to the "AVE" scheme which we used in this paper.
3) We plan to investigate the use of higher order elements (i.e. cubic splines) as a basis for the spatial discretization in our approximation scheme.

Preliminary runs indicate superior performance over linear splines (Re: Table 4.1).

4) We are interested in extending our ideas to fourth order (in the spatial variable) equations which are used to model beams. Preliminary applications to a clamped-clamped beam have been successful.

5) In our proof of well-posedness, we made the restriction $g \epsilon H^1(-r,0)$. This excludes singular Kernels, which are important. Thus we are investigating well-posedness results which allow for singular Kernels.

6) We plan to investigate boundary conditions other than the Dirichlet boundary conditions. For example, to model a slewing beam, other boundary conditions must be considered.

REFERENCES

1. Aubin, J.P., Applied Functional Analysis, Wiley - Intersciences, New York, (1979).

2. Banks, H.T. and Burns, J.A., Hereditary control problems: numerical methods based on averaging approximations, SIAM J. Control Optimization, Vol. 16, (1978), 169-208.

3. Banks, H.T. and Kappel, F., Spline approximation for functional differential equations, J. Differential Equations, Vol. 34 (1979), 496-522.

4. Banks, H.T., Rosen, I.G., and Ito, K., A spline based technique for computing Riccati operators and feedback controls in regulator problems for delay equations, SIAM J. Scientific Statistical Computing, Vol. 5 (1984), 830-855.

5. Cliff, E.M. and Burns, J.A., "Reduced approximations in parameter identification of hereditary systems", Proc. IFIP, N.Y., (1981).

6. Curtain, R.F. and Pritchard, A.J., Infinite Dimensional Linear Systems Theory, Springer, Berlin (1978).

7. DiBlasio, G., The linear-quadratic optimal control problem for delay differential equations, Rend. Accad. Naz. Lincei, 71 (1981), 156-161.

8. DiBlasio, G., Kunisch, K., and Sinestrari, E., L^2-Regularity for parabolic partial integrodifferential equations with delay in the highest-order derivatives, J. Math. Anal. Appl., 102 (1984), 38-57.

9. DiBlasio, G., Kunisch, K., and Sinestrari, E., Stability for abstract linear functional differential equations, preprint.

10. Gibson, J.S., The Riccati integral equations for optimal control problems on Hilbert spaces, SIAM J. on Control, Vol. 17, No. 4, (1979), 537-565.

11. Gibson, J.S., Linear quadratic optimal control of hereditary differential systems: Infinite dimensional Riccati equations and numerical approximations, SIAM J. Control Optim., 21(1983), 95-139.

12. Gibson, J.S., and Rosen, I.G., Shifting the closed-loop spectrum in the optimal linear quadratic regulator problem for hereditary systems, preprint.

13. Hrusa, W.J. and Nohel, J.A., Global existence and

asymptotics in one-dimensional nonlinear voscoelasticity, Proceedings of the 5th Symposium on applications of pure mathematics to mechanics, Springer lecture notes in physics, Vol, 195(1984), 165-187.

14. Kappel, F. and Salamon, D., Spline approximation for retarded systems and the Riccati equations, MRC Technical Report #2680, 1984.

15. Krein, S.G., Linear Differential Equations in Banach Spaces, Transl. Math. Mono., 29, American Mathematical Society, Providence, RI, 1971.

16. Kunisch, K. and Schappacher, W., Necessary conditions for partial differential equations with delay to generate C_0-semigroups, Journal of Differential Equations, 50(1983), 49-79.

17. Oden, J. Tinsley, Applied Functional Analysis, Prentice-Hall, New Jersey, (1979).

18. Pazy, A., Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer, New York (1983).

19. Powers, R.K., Chandrasekhar equations for distributed parameter systems, Ph.D Thesis, Mathematics Department, Virgina Polytechnic Institute and State University, Blacksburg, VA (1984).

20. Russell, D.L., Mathematics of Finite Dimensional Control Systems, Theory and Design, Marcel Dekker, N.Y. 1979.

21. Schultz, M.H., Spline Analysis, Prentice Hall, Englewood Cliffs, N.Y. (1973).

22. Travis, C. and Webb, G., Partial differential equations with deviating arguments in the time variable, J. Math. Anal. Appl. 56(1976), 397-409.

23. Travis, C. and Webb, G., Existence, stability and compactness in the α-norm for partial functional differential equations, Trans. Am. Math. Soc., 240(1978) 129-143.

24. Walker, J.A., Dynamical Systems and Evolution Equations, Plenum Press, New York (1980).

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APPROXIMATION OF

INTEGRO-PARTIAL DIFFERENTIAL EQUATIONS

OF HYPERBOLIC TYPE

by

Richard H. Fabiano, Jr.

(ABSTRACT)

A state space model is developed for a class of integro-partial differential equations of hyperbolic type which arise in viscoelasticity. An approximation scheme is developed based on a spline approximation in the spatial variable and an averaging approximation in the delay variable. Techniques from linear semigroup theory are used to discuss the well-posedness of the state space model and the convergence properties of the approximation scheme. We give numerical results for a sample problem to illustrate some properties of the approximation scheme.