PERFECT NUMBERS AND OTHER NUMBERS DEFINED BY THE SUM OF THEIR DIVISORS

by

Kenneth Eugene Hawkes

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APPROVED:

E. A. Brown, Chairman

C. L. Mett

D. P. Roselle

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Chapter 1

INTRODUCTION

By a "number", we will mean a positive integer, unless otherwise noted, and numbers will be represented by lower case Latin letters. A number n is divisible by a number d if there exists a number c such that $n = dc$. If d divides n, we write $d|n$, while if d does not divide n, we write $d\nmid n$. If d^k is the largest power of d which divides n, then we write $d^k||n$. The greatest common divisor of m and n, (m, n), is the largest d such that $d|m$ and $d|n$. A number d is called a proper divisor of n if $d|n$ and d is not equal to 1 or n. If $n > 1$ and n has no proper divisors, then n is said to be prime.

The fundamental theorem of arithmetic states that any integer greater than 1 can be factored into a product of primes; this product is unique up to the order of the prime factors. Thus an integer $n > 1$ can be written as $n = p_1^{\alpha_1} p_2^{\alpha_2}...p_r^{\alpha_r}$ where $p_1, p_2, ..., p_r$ are distinct primes and $\alpha_1, \alpha_2, ..., \alpha_r$ are positive integers. If d is a divisor of n, then $d = p_1^{\beta_1} p_2^{\beta_2}...p_r^{\beta_r}$ where $0 \leq \beta_i \leq \alpha_i$ for $i = 1, 2, ..., r$. It is seen that there are $\alpha_1 + 1$ choices for β_1, $\alpha_2 + 1$ choices for β_2, ..., and $\alpha_r + 1$ choices for β_r. Thus the number of distinct divisors of n is equal to the product $(\alpha_1 + 1)(\alpha_2 + 1)...(\alpha_r + 1)$.

The sum of the divisors function, $\sigma(n)$, is defined by

$$\sigma(n) = \sum_{d|n} d .$$

When considering the divisors of a number n, the Greeks excluded n itself as a divisor. The positive divisors of n which are less than n are referred to as its aliquot parts; defining $\sigma_0(n)$ as the sum of the aliquot parts of n, we have $\sigma_0(n) = \sigma(n) - n$.

Lemma 1: If $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$ where the p_i's are distinct primes and $\alpha_i \geq 1$, then

$$\sigma(n) = (p_1^{\alpha_1})(p_2^{\alpha_2})\ldots(p_r^{\alpha_r}) = \prod_{i=1}^{r} \frac{p_i^{\alpha_i + 1} - 1}{p_i - 1}.$$

Proof: Let $n = ab$ where $(a, b) = 1$ and let $1, a_1, a_2, \ldots, a$ be the divisors of a, so that $\sigma(a) = 1 + a_1 + a_2 + \ldots + a$. Similarly, let $1, b_1, b_2, \ldots, b$ be the divisors of b. If d is a divisor of n, then $d = a_i b_j$ where a_i is one of the divisors of a and b_j is one of the divisors of b. Fixing a_i, we obtain the divisors $a_i.1, a_i b_1, \ldots, a_i b$ and summing these divisors yields $a_i(1 + b_1 + \ldots + b) = a_i\sigma(b)$. Taking the sum over all possible a_i, we obtain $(1 + a_1 + a_2 + \ldots + a)\sigma(b) = \sigma(a)\sigma(b)$. All the possible divisors of n are included in this sum, and each of these divisors is distinct. For suppose $a_i b_j = a_k b_1$ with $b_j \neq b_1$. Without loss of generality, assume $b_j < b_1$. Then $a_i > a_k$ and $a_i \mid a_k b_1$; hence there exists an integer $c > 1$ such that $(a_i, b_1) = c$. This contradicts the fact that $(a, b) = 1$. Thus $b_j = b_1$ and $a_i = a_k$. Therefore we have that $\sigma(n) = \sigma(a)\sigma(b)$.

Applying this result to n, we have

$$\sigma(n) = \sigma(p_1^{\alpha_1})\sigma(p_2^{\alpha_2}) \ldots \sigma(p_r^{\alpha_r}).$$

Further we know that

$$\sigma(p_i^{\alpha_i}) = 1 + p_i + p_i^2 + \ldots + p_i^{\alpha_i} = \frac{p_i^{\alpha_i + 1} - 1}{p_i - 1}.$$

Thus $\sigma(n) = \prod_{i=1}^{r} \frac{p_i^{\alpha_i + 1} - 1}{p_i - 1}$.

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Now $\sigma_0(n)$ is not multiplicative, for let p be a prime. Then $\sigma_0(p) = 1$, but $p + 1 = \sigma_0(p^2) \neq \sigma_0(p)\sigma_0(p) = 1$. Thus $\sigma(n)$ is generally easier to work with than $\sigma_0(n)$, although all the early results dealing with the sum of the divisors were obtained using $\sigma_0(n)$.

Example: Let $n = 12 = 2^2 \cdot 3$. The divisors of 12 are 1, 2, 3, 4, 6, and 12.

$\sigma(12) = \sigma(2^2)\sigma(3) = 7 \cdot 4 = 28$. $\sigma_0(12) = 28 - 12 = 16$.

We are now in a position to define deficient, perfect, and abundant numbers. A number n is called deficient if $\sigma(n) < 2n$, perfect if $\sigma(n) = 2n$, and abundant if $\sigma(n) > 2n$. Similarly, n is deficient if $\sigma_0(n) < n$, perfect if $\sigma_0(n) = n$, and abundant if $\sigma_0(n) > n$. So every integer greater than 1 is either deficient, perfect, or abundant.

Examples: (i) $\sigma(75) = \sigma(3)\sigma(5^2) = 4 \cdot 31 = 124 < 150$, and $\sigma_0(75) = 49 < 75$. Thus 75 is deficient. (ii) $\sigma(6) = 12$ and $\sigma_0(6) = 6$. Thus 6 is perfect. (iii) We know that $\sigma(12) = 28$ so that 12 is abundant. In fact, 12 is the first abundant number. There are 21 abundant numbers less than 100 and all of them are even numbers. 945 is the first odd abundant number.

The following two lemmas prove useful in determining when a number is abundant or deficient, while Lemma 4 gives a property of perfect numbers.

Lemma 2: A multiple of an abundant or perfect number is abundant. A proper divisor of a perfect number is deficient.

Proof: Let n be abundant or perfect and let $1, d_1, d_2, \ldots, d_k$ be the aliquot parts of n. Then we have $1 + d_1 + d_2 + \ldots + d_k = \sigma_0(n) \geq n$. Take any multiple tn of n, where $t > 1$. Now $1, t, td_1, td_2, \ldots, td_k$ are all distinct divisors of tn. We have $t(1 + d_1 + d_2 + \ldots + d_k) \geq tn$, so that $1 + t + td_1 + td_2 + \ldots + td_k > tn$. Therefore a multiple of an abundant or perfect number is abundant.

Let m be a perfect number and let $c|m$ with $1 < c < m$. Thus m is a multiple of c. If c were not deficient, then m would be abundant by the first assertion of this lemma. Therefore any proper divisor of a perfect number is deficient.

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Lemma 3: The product of two primes, other than $2 \cdot 3$, is deficient.

Proof: Let p_1 and p_2 be primes. If $p_1 = p_2$, then $\sigma(p_1p_2) = \sigma(p_1^2) = 1 + p_1 + p_1^2 < 2p_1^2$, so that p_1^2 is deficient. If $p_1 \neq p_2$, then assume $p_2 > p_1$. Since p_1 and p_2 are not 2 and 3, $p_2 \geq p_1 + 2$. Now $\sigma(p_1p_2) = 1 + p_1 + p_2 + p_1p_2$ and $1 + p_1 + p_2 < 2p_2 \leq p_1p_2$. Thus $\sigma(p_1p_2) < 2p_1p_2$, and so p_1p_2 is deficient.

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Lemma 4: If n is perfect, then $\sum_{d|n} \frac{1}{d} = 2$.

Proof: Let n be a perfect number. If d divides n, then $\frac{n}{d}$ divides n, so we have that

$$2n = \sum_{d|n} d = \sum_{d|n} \frac{n}{d} .$$

Dividing by n, we have that $\sum_{d|n} \frac{1}{d} = 2$. ¶

A primitive non-deficient number is a non-deficient number that is not a multiple of a smaller non-deficient number. By Lemma 2, we know that any perfect number is a primitive non-deficient number. The five primitive non-deficient numbers less than 100 are 6, 20, 28, 70, and 88.

L. E. Dickson [18], [19] proved that there is only a finite number of primitive non-deficient numbers having a given number n of distinct odd prime factors and a given number $m \geq 0$ of factors of 2. Dickson's results have been generalized for k-nondeficient numbers by H. N. Shapiro [102], H. A. Bernhard [5], H.-J. Kanold [64], [66], and C. Pomerance [94]. An integer n is said to be k-deficient if $\sigma(n) < kn$, otherwise n is k-nondeficient, where k is a rational number.

Chapter 2

PERFECT NUMBERS AND MERSENNE PRIMES

The first recorded mention of perfect numbers is found in Euclid's Elements, book VII, definition 22. Euclid defines a perfect number as a number which is "equal to its own parts;" that is, a number is perfect if it is equal to the sum of its aliquot parts. Euclid was able to prove that $2^{n-1}(2^n - 1)$ is perfect if $2^n - 1$ is a prime, in the Elements, book IX, proposition 36. Even today the only existent perfect numbers are those of Euclid's type, i.e., of the form $2^{n-1}(2^n - 1)$ where $2^n - 1$ is a prime.

The first four perfect numbers were known to Nicomachus [45] about 100 A.D. (The history of perfect numbers is in chapter I of [17]). They are 6, 28, 496, and 8128. Using an empirical type of induction argument on a very small data base, it was reasoned that perfect numbers end alternately in 6 and 8, and that for all k there exists exactly one perfect number with k digits. The fifth perfect number, 33550336, is given correctly in a fifteenth century manuscript. The sixth perfect number also ends in 6, and so both conjectures proved false.

Perfect numbers have had a place in numerology, the number 6 being given special properties since it is the first perfect number. God was said to have taken 6 days to create the earth, since 6 is a perfect number. The second beginning of mankind from the 8 people on Noah's ark was not as perfect as the creation since 8 is a deficient

number. The second perfect number, 28, represents the number of days in the moon's orbit of the earth (actually this number is closer to 29.5).

At this point in our discussion, it may be worthwhile to skip ahead and prove a result characterizing all even perfect numbers. The first proof of this theorem was due to Leonard Euler near the end of the eighteenth century.

Theorem 1: An even number n is perfect if and only if $n = 2^{p-1}(2^p - 1)$ where p and $2^p - 1$ are primes.

Proof: Let $n = 2^{p-1}(2^p - 1)$ where p and $2^p - 1$ are primes. By Lemma 1, we have

$$\begin{aligned}\sigma(n) &= \sigma(2^{p-1})\sigma(2^p - 1)\\ &= (1 + 2 + 2^2 + \ldots + 2^{p-1})(2^p)\\ &= (2^p - 1)(2^p)\\ &= 2n\end{aligned}$$

Thus n is a perfect number.

Let $n = 2^k d$ be perfect where $(2, d) = 1$. Applying Lemma 1 again, $2n = \sigma(n) = \sigma(2^k)\sigma(d)$, and $\sigma(2^k) = 1 + 2 + \ldots + 2^k = 2^{k+1} - 1$. Now

$$d = \frac{n}{2^k} = \frac{2n}{2^{k+1}} = \frac{\sigma(n)}{2^{k+1}} = \frac{(2^{k+1} - 1)\sigma(d)}{2^{k+1}}$$

Hence we have that $\frac{d}{\sigma(d)} = \frac{2^{k+1} - 1}{2^{k+1}}$ is a fraction in its lowest terms, so that $d = (2^{k+1} - 1)c$ and $\sigma(d) = (2^{k+1})c$. If $c = 1$, then $d = 2^{k+1} - 1$ must be prime since $\sigma(d) = 2^{k+1}$.

Suppose $c > 1$; we obtain the inequality

$$\sigma(d) \geq d + (2^{k+1} - 1) + c + 1$$
$$= 2^{k+1}(c + 1), \text{ where } d = c(2^{k+1} - 1) .$$

Thus we have

$$\frac{d}{\sigma(d)} \leq \frac{c(2^{k+1} - 1)}{2^{k+1}(c + 1)} < \frac{2^{k+1} - 1}{2^{k+1}} = \frac{d}{\sigma(d)} .$$

This is a contradiction, so $c = 1$. Therefore we have that $n = 2^k(2^{k+1} - 1)$, where $2^{k+1} - 1$ is a prime.

Suppose $k+1$ is composite, that is, $k+1 = rs$ where $r, s > 1$. Then

$$2^{k+1} - 1 = 2^{rs} - 1 = (2^r - 1)(2^{r(s-1)} + 2^{r(s-2)} + \ldots + 2^r + 1) .$$

This is a contradiction since $2^{k+1} - 1$ is a prime.

Therefore we have that $n = 2^k(2^{k+1} - 1)$ where $k+1$ and $2^{k+1} - 1$ are primes.

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It is clear that the search for even perfect numbers amounts to the discovery of primes of the form $2^p - 1$, where p is a prime. As a result of Theorem 1, some interesting corollaries may be obtained.

Corollary 1: Every even perfect number is triangular.

Proof: A number m is triangular if $m = \sum_{i=1}^{k} i$. Now

$$\sum_{i=1}^{2^p-1} i = \frac{(2^p-1)2^p}{2} = 2^{p-1}(2^p - 1) .$$

Thus every number of the form $2^{p-1}(2^p - 1)$, and hence every even perfect number, is triangular.

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Corollary 2: Every even perfect number ends in 6 or 28.

Proof: Note that 6 satisfies the corollary. Let $n = 2^{p-1}(2^p - 1)$ be perfect, where p is an odd prime. If $p \equiv 1 \pmod 4$, then 2^{p-1} ends in 6 and $2^p - 1$ ends in 1. Thus we have that $2^{p-1}(2^p - 1)$ ends in 6.

Assume $p \equiv 3 \pmod 4$, so that $p \equiv 3, 7, 11$, or $19 \pmod{20}$.

$p \pmod{20}$	$2^{p-1} \pmod{100}$	$2^p - 1 \pmod{100}$
3	04	07
7	64	27
11	24	47
19	44	87

In each case, we have that $2^{p-1}(2^p - 1)$ ends in 28.

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Corollary 3: All even perfect numbers greater than 6 are of the form $9k + 1$.

Proof: Note that 28 satisfies the corollary. Let $n = 2^{p-1}(2^p - 1) = 2^{2p-1} - 2^{p-1}$ be an even perfect number, when p is an odd prime different from 3. Hence $p \equiv 1$ or $5 \pmod 6$. Now consider the powers of 2 modulo 9: $2^1 \equiv 2$, $2^2 \equiv 4$, $2^3 \equiv 8$, $2^4 \equiv 7$, $2^5 \equiv 5$ and $2^6 \equiv 1$.

Assume $p \equiv 1 \pmod 6$. Then $2p - 1 \equiv 1 \pmod 6$ and $p - 1 \equiv 0 \pmod 6$. Thus we have that $2^{2p-1} \equiv 2 \pmod 9$ and $2^{p-1} \equiv 1 \pmod 9$. Therefore $2^{2p-1} - 2^{p-1} \equiv 1 \pmod 9$.

Assume $p \equiv 5 \pmod 6$. Then $2p - 1 \equiv 3 \pmod 6$ and $p - 1 \equiv 4 \pmod 6$. Thus we have that $2^{2p-1} \equiv 8 \pmod 9$ and $2^{p-1} \equiv 7 \pmod 9$. Therefore $2^{2p-1} - 2^{p-1} \equiv 1 \pmod 9$.

Hence every even perfect number greater than 6 is of the form $9k + 1$.

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Corollary 4: Every even perfect number except 6 is a partial sum of the series of consecutive odd cubes $1^3 + 3^3 + 5^3 + \ldots$.

Proof: Now $\sum_{k=1}^{m} k^3 = \frac{m^2(m+1)^2}{4}$.

For m odd we have,

$$1^3 + 3^3 + \ldots + m^3 = \sum_{k=1}^{m} k^3 - 8 \sum_{j=1}^{\frac{m-1}{2}} j^3$$

$$= \frac{m^2(m+1)^2}{4} - \frac{8\left(\frac{m-1}{2}\right)^2\left(\frac{m+1}{2}\right)^2}{4}$$

$$= \frac{2m^4 + 4m^3 + 2m^2}{8} - \frac{m^4 - 2m^2 + 1}{8}$$

$$= \frac{m^4 + 4m^3 + 4m^2 - 1}{8}$$

Let $2^{p-1}(2^p - 1)$ be an even perfect number different from 6. Then p is an odd prime, $2^p - 1 = q$ is prime. Solving for m in terms of p, we have

$$\frac{m^4 + 4m^3 + 4m^2 - 1}{8} = 2^{p-1}(2^p - 1)$$

$$m^4 + 4m^3 + 4m^2 - 1 = 8(2^{p-1})(2^p - 1)$$

$$m^2(m + 2)^2 = (2^{p+2})(2^p - 1) + 1$$
$$= 4(q + 1)q + 1$$
$$= 4q^2 + 4q + 1$$
$$= (2q + 1)^2 .$$

Thus $m(m + 2) = 2q + 1$
$$= 2(2^p - 1) + 1$$
$$= 2^{p+1} - 1$$
$$= (2^{\frac{p+1}{2}} - 1)(2^{\frac{p+1}{2}} + 1) .$$

Thus $m = 2^{\frac{p+1}{2}} - 1$ which is an odd integer; hence

$$2^{p-1}(2^p - 1) = 1^3 + 3^3 + \ldots + m^3 = \sum_{k=1}^{\frac{m+1}{2}} (2k - 1)^3 ,$$

where $m = 2^{\frac{p+1}{2}} - 1$.

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The credit for the discovery of the sixth and seventh perfect numbers goes to P. A. Cataldi who verified in 1588 that $2^{17} - 1$ and $2^{19} - 1$ are primes by eliminating as possible divisors all primes less than their respective square roots.

Though Euler was yet to prove the converse to Euclid's theorem, the numbers given by various mathematicians as supposedly perfect numbers were of Euclid's type, and a number of mistakes were made. In 1644, Marin Mersenne commented that the list of perfect numbers published by P. Bungus was incorrect. Mersenne went on to state that there were eleven known perfect numbers, all of which were of Euclid's form, $2^{p-1}(2^p - 1)$, with $p = 2, 3, 5, 7, 13, 17, 19, 31, 67, 127,$ and

257. Numbers of the type $Mp = 2^p - 1$, where p is a prime, later became known as Mersenne numbers, while if Mp is a prime, it is called a Mersenne prime. Thus the discovery of an even perfect number is equivalent to the discovery of a Mersenne prime. It turns out that Mersenne actually made five mistakes in his list, being in error as to p = 67, 257, and omitting p = 61, 89, 107. Further it is interesting to note that Mersenne cannot be given credit for the discovery of a single Mersenne prime. The first seven values of p were already known, and it was Euler in 1750 who verified that $2^{31} - 1$ is prime. We should not be too harsh on Mersenne, though; it wasn't until 1947, or three hundred three years later that all fifty-five primes ≤ 257 could be checked.

The factorization of Mersenne numbers is thus related to the even perfect number problem. Some results concerning possible factors for Mp are known, but we shall, for the most part, omit them from our discussion. For instance, Pierre de Fermat and Euler showed that all factors of Mp must be of the form 2kp + 1 and simultaneously of the form $8m \pm 1$. Many mathematicians are interested in the factorization of Mersenne numbers as a problem in itself, and quite extensive tables have been published. For example, see S. Kravitz [69], J. Brillhart [11], [12], and D. Shanks and Kravitz [101].

We noted that Euler is credited with the discovery of the eighth even perfect number since he verified that $2^{31} - 1 = 2147483647$ is prime. Now $2^{31} - 1$ was also the largest known prime at the time. The largest known prime is a curiosity as such, and generally the largest

known Mersenne prime is the largest known prime. Euler's discovery led to the following comment in 1811 by Peter Barlow in his book entitled Theory of Numbers:

> The difficulty, therefore, of finding perfect numbers, arises from that of finding prime numbers, of the form $2^n - 1$, which is very laborious. Euler ascertained, that $2^{31} - 1 = 2147483647$ is a prime number; and this is the greatest at present known to be such, and, consequently, the last of the above perfect numbers, which depend upon this, is the greatest perfect number known at present, and probably the greatest that ever will be discovered; for, as they are merely curious without being useful, it is not likely that any person will attempt to find one beyond it.

From the discussion that follows, we will see that mathematicians following Euler did not take Barlow's advice and stop working on the even perfect number problem.

The next Mersenne prime was not determined until 1876, when E. Lucas verified, by hand, that $2^{127} - 1$ is prime. Now $2^{127} - 1$ has thirty-nine decimal digits, and remained the largest known prime until 1951 when A. Ferrier determined that $\frac{2^{148} + 1}{17}$ is a prime. Lucas determined that $2^{127} - 1$ is prime by employing a method which was later simplified by D. H. Lehmer and is usually referred to as the Lucas-Lehmer test. It is the method that has since been used in the several computer searches for Mersenne primes. As stated by Lehmer [74], the Lucas-Lehmer test is as follows:

> Theorem: The number $N = 2^n - 1$ is prime if and only if it divides the $(n - 1)^{st}$ term of the sequence 4, 14, 194, 37634, ..., S_k, ... where $S_k = S_{k-1}^2 - 2$. This sequence is known as a Lucas sequence.

Other values of S_1 besides $S_1 = 4$ may be used, with $S_1 = 10$ being another popular choice. The fact that one needs only to save the residue of S_k modulo $2^n - 1$ makes the test particularly adaptable to the computer. As a corollary to Lehmer's theorem we have that the number $2^n - 1$ is prime if and only if the $(n - 2)^{nd}$ term of the series is congruent to $\pm 2^{\frac{n+1}{2}}$ modulo $(2^n - 1)$.

The primality of $2^{61} - 1$ was verified independently by I. M. Pervouchine in 1883 and P. Seelhoff in 1886. In 1911, R. E. Powers established that $2^{89} - 1$ is prime by utilizing Lucas' test with the first term of the sequence being 4. The list of the twelve Mersenne primes discovered without the aid of computers was completed in 1914 when Powers and E. Fauquembergue determined, within six days of one another, that $2^{107} - 1$ is prime.

The first attempt to find Mersenne primes by the use of computers was made by A. M. Turing in 1951 and proved unsuccessful. In 1952 another attempt was made by D. H. Lehmer utilizing the National Bureau of Standards' Western Automatic Computer. The program employed the Lucas-Lehmer test and was programmed by R. Robinson [100]. The first day that the program actually ran, two new Mersenne primes were found, for $p = 521$ and 607 [119]. The program was run over a nine month span, testing all primes less than 2309; three additional Mersenne primes were found, for $p = 1279$, 2203, and 2281.

Evidently there are large gaps in the primes that yield Mersenne primes. The first twelve Mersenne primes are found in checking the first thirty-one primes, but between 127 and 521 there are sixty-six

primes which do not lead to a Mersenne prime. Table 1 gives the spacing of the primes for which Mp is prime, and summarizes this discussion on the discovery of Mersenne primes.

It is unknown whether there are an infinite number of Mersenne primes. One conjecture, which if true would yield an infinite number of them, was that $2^p - 1$ is always prime when p is a Mersenne prime. The conjecture is true for p = 3, 7, 31, and 127. The next value of p is 8191 = $2^{13} - 1$. In 1953, D. J. Wheeler, using the University of Illinois computer, was able to show that $2^{8191} - 1$ is composite. This result has since been verified by other computers working on the problem.

The eighteenth Mersenne prime and corresponding even perfect number was found for p = 3217 by H. Riesel [97], [98] in 1957. A Swedish electronic digital computer performed the calculators, checking all p in the range $2300 < p < 3300$. A. Hurwitz [50] increased the list of perfect numbers to twenty in 1961. An IBM 7090 computer used the Lucas-Lehmer test to check all p such that $3300 < p < 5000$. The values p = 4253 and 4423 yielded prime values for Mp.

D. B. Gillies [29] found three more Mersenne primes in 1963 with the aid of the ILLIAC II computer at the University of Illinois. The successful values of p were p = 9689, 9941, and 11213. Mp for p = 11213 assumed the title of the largest known prime, and the mathematics department at the University of Illinois was so proud of this accomplishment that the postage-meter stamp "$2^{11213} - 1$ is prime" was (and still is) used on outgoing mail.

The twenty-fourth and largest known Mersenne prime was found by B. Tuckerman [117] in 1971 for p = 19937. All the remaining primes less than 20,000 were tried using the Lucas-Lehmer test on an IBM 360/91 computer. $2^{19937} - 1$ is currently the largest known prime and has 6002 digits. The twenty-fourth perfect number is 12003 digits long, which is quite a big larger than the 19 digit perfect number found by Euler and predicted by Peter Barlow to be the greatest that would ever be discovered!

TABLE 1

KNOWN EVEN PERFECT NUMBERS $2^{p-1}(2^p - 1)$

	p	N(p) Where n=N(p) Means p is the n^{th} Prime	Number of Digits in $2^{p-1}(2^p - 1)$	Discovery
1.	2	1	1	Known by 100 A.D.
2.	3	2	2	"
3.	5	3	3	"
4.	7	4	4	"
5.	13	6	8	15th century manuscript
6.	17	7	10	P. A. Cataldi, 1588
7.	19	8	12	"
8.	31	11	19	L. Euler, 1750
9.	61	18	37	I. M. Pervouchine, 1883 P. Seelhoff, 1886
10.	89	24	54	R. E. Powers, 1911
11.	107	28	65	Powers, 1914 E. Fauquemberque, 1914
12.	127	31	77	E. Lucas, 1876
13.	521	98	314	D. H. Lehmer, 1952 SWAC computer
14.	607	111	366	"
15.	1279	207	770	"
16.	2203	328	1327	"
17.	2281	339	1373	"
18.	3217	455	1937	H. Riesel, 1957 BESK computer
19.	4253	583	2561	A. Hurwitz, 1961 IBM 7090
20.	4423	602	2663	"
21.	9689	1196	5834	D. B. Gillies, 1963 ILLIAC II computer
22.	9941	1226	5985	"
23.	11213	1357	6751	"
24.	19937	2254	12003	B. Tuckerman, 1971 IBM 360/91 computer

If we let A be a set of positive integers, then A(x) will denote the number of positive integers in the set A that are less than or equal to x. For example, if A is the set of even integers 2, 4, 6, ..., then A(3) = 1, A(6) = 3, and $A\left(\frac{19}{2}\right) = 4$. Let us now take A to be the set of perfect numbers. B. Hornfeck [46] showed that $A(x) \leq x^{\frac{1}{2}}$. H.-J. Kanold [65] improved this to $A(x) = o(x^{\frac{1}{2}})$, and this result was in turn improved by Hornfeck and E. Wirsing [48] who showed that $A(x) = O(x^{\frac{1}{4}} \ln x/\ln \ln x)$. Finally, Wirsing [126] has shown that there are constants $c > 0$ and x_0 such that $A_k(x) \leq \exp(c \ln x/\ln \ln x)$ for all $x \geq x_0$. This result of Wirsing's holds for all rational numbers k, where A_k is the set of integers m such that $\sigma(m) = km$.

The asymptotic density of a set A is defined as

$$\delta_1(A) = \liminf_{k \to \infty} \frac{A(k)}{k}.$$

In case the sequence $\frac{A(k)}{k}$ has a limit, we say that A has a natural density, $\delta(A)$. It is clear that if A is a finite set, that $\delta(A) = 0$. If $A = \{a_i\}$ is an infinite set, then

$$\delta_1(A) = \liminf_{k \to \infty} \frac{k}{a_k}$$

If $\delta(A)$ exists, then $\delta(A) = \lim_{k \to \infty} \frac{k}{a_k}$. For example, if A is the set of even integers 2, 4, 6, ..., then $\delta(A) = \frac{1}{2}$. If A is the set of positive integers k such that $k \equiv 2 \pmod 4$, then $\delta(A) = \frac{1}{4}$. If A is the set of primes, $\delta(A) = 0$. Thus an infinite set may have density 0.

Kanold [62] and others have shown that the asymptotic density of the set of perfect numbers is 0. In fact, the set

$$A = \{n : \sigma(n) \equiv 0 (\text{mod } n)\}$$

has density 0 [67]. It is unknown whether A has any odd members besides 1, or whether A is infinite.

Chapter 3

ODD PERFECT NUMBERS

The existence of odd perfect numbers is still an open question. However, we shall see that if an odd perfect number exists, then it must be quite large, and, in addition, must meet an increasing number of restrictions.

Mathematicians in the seventeenth century began expressing ideas as to the form an odd perfect number (if such exists) must take. René Descartes conjectured that an odd perfect number could exist, and stated that any odd perfect number would be of the form pr^2 where p is a prime. A few years later, Frenicle stated that odd perfect numbers, if they exist, must be of the form pr^2 where $p = 4m + 1$ is a prime. Once again, it was L. Euler who actually proved that an odd perfect number n must have the form $n = p^{4k+1}r^2$ where $p = 4m + 1$ is a prime. The proof of this theorem is included because it is the first important result concerning odd perfect numbers. The paper containing Euler's proof and the paper containing the proof that all even perfect numbers are of Euclid's type were both published after 1783, the year of Euler's death.

Theorem 2: Any odd perfect number n is of the form $n = p^{4k+1}r^2$ where p is a prime of the form $4m + 1$.

Proof: Assume $n = p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_s^{\alpha_s}$ is an odd perfect number, where the p_i's are distinct odd primes. By Lemma 1, we have that

$$2n = \sigma(n) = \sigma(p_1^{\alpha_1})\sigma(p_2^{\alpha_2})\ldots\sigma(p_s^{\alpha_s}) .$$

Now $2||2n$ so exactly one of the $\sigma(p_i^{\alpha_i})$ is even and the rest are odd. By renumbering, if necessary, we have that $\sigma(p_1^{\alpha_1})$ is even, while $\sigma(p_i^{\alpha_i})$ is odd for $2 \leq i \leq s$. We note that

$$\begin{aligned}\sigma(p_i^{\alpha_i}) &= 1 + p_i + p_i^2 + \ldots + p_i^{\alpha_i} \\ &\equiv 1 + 1 + 1 + \ldots + 1 \equiv \alpha_i + 1 \pmod 2 .\end{aligned}$$

Since $\sigma(p_i^{\alpha_i}) \equiv 1 \pmod 2$, then $\alpha_i \equiv 0 \pmod 2$ for $2 \leq i \leq s$. Thus we can rewrite n as

$$n = p_1^{\alpha_1} p_2^{2\beta_2} p_3^{2\beta_3} \ldots p_s^{2\beta_s} .$$

Now $2||\sigma(p_1^{\alpha_1})$, so $\sigma(p_1^{\alpha_1}) \equiv 2 \pmod 4$ and $\alpha_1 \equiv 1 \pmod 2$. If $p_1 \equiv 3 \pmod 4$, then $p_1^2 \equiv 1 \pmod 4$. As α_1 is odd,

$$\begin{aligned}\sigma(p_1^{\alpha_1}) &= 1 + p_1 + \ldots + p_1^{\alpha_1 - 1} + p_1^{\alpha_1} \\ &= (1 + p_1) + \ldots + (p_1^{\alpha_1 - 1} + p_1^{\alpha_1}) \\ &\equiv 4 + \ldots + 4 \equiv 0 \pmod 4 .\end{aligned}$$

Thus $p_1 \equiv 1 \pmod 4$. Now $\sigma(p_1^{\alpha_1})$ has $\alpha_1 + 1$ terms, each of which is congruent to 1 modulo 4. Since $\sigma(p_1^{\alpha_1}) \equiv 2 \pmod 4$, $\alpha_1 \equiv 1 \pmod 4$. Hence we may write n as

$$n = p_1^{\alpha_1}(p_2^{2\beta_2}\ldots p_s^{2\beta_s}) = p^{4k+1} r^2 ,$$

where p is a prime of the form $4m + 1$.

¶

If p is a prime, we have that $\sigma(p^\alpha) = \frac{p^{\alpha+1} - 1}{p - 1}$. Thus $p \nmid \sigma(p^\alpha)$, so that a power of a prime cannot be a perfect number. Thus any perfect number must have at least two distinct prime divisors. We have seen that for even perfect numbers there are exactly two distinct prime factors, and one method of attacking the odd perfect number problem is to decide whether there exist any odd perfect numbers with a given number of distinct prime factors.

In a paper published in 1844, V. A. Lebesque stated that he could prove that an odd perfect number must have at least four distinct prime divisors. About twenty years later, G. Nocco did prove that there is no odd perfect number with exactly two distinct prime factors. (A simpler proof of this fact will follow below.) J. J. Sylvester and Cl. Servais both were able to prove that there does not exist an odd perfect number with exactly three distinct prime divisors, so that an odd perfect number must indeed have at least four distinct prime divisors.

The modern work on odd perfect numbers can be said to stem from J. J. Sylvester. Sylvester [113] was convinced that odd perfect numbers were nonexistent:

> I hope, *numine favente*, sooner or later to discover a general principle which may serve as a key to a universal proof of the non-existence of any other than the Euclidean perfect numbers, for a prolonged meditation on the subject has satisfied me that the existence of any one such--its escape, so to say, from the complex web of conditions which hem it in on all sides--would be little short of a miracle.

[Intuition of great mathematicians versus rigor.]

Sylvester's method of proof was to show (he hoped) that there were no k-fold odd perfect numbers, a k-fold odd perfect number being one with exactly k distinct prime divisors, where k = 1, 2, 3, ... It was already known that an odd perfect number must have at least three distinct prime divisors. Sylvester was able to prove that there were neither three-fold nor four-fold perfect numbers, and that an odd perfect number not divisible by 3 must have at least eight distinct prime divisors. Sylvester also claimed to have a proof that no five-fold perfect numbers exist. However, he was unsuccessful in his quest to show the nonexistence of odd perfect numbers:

> This theory was called for to overcome certain difficulties which beset my phantom-chase in the chimerical region haunted by those doubtful or supposititious entities called odd perfect numbers. Whoever shall succeed in demonstrating their absolute non-existence will have solved a problem of the ages comparable in difficulty to that which previously to the labours of Hermite and Lindermann...environed the subject of the quadrature of the circle.

The following result is used in all the theorems involving the number of prime divisors of an odd perfect number.

Theorem 3: If $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$, where the p_i's are distinct primes, is a perfect number, then

$$\left(\frac{p_1}{p_1 - 1}\right)\left(\frac{p_2}{p_2 - 1}\right)\cdots\left(\frac{p_r}{p_r - 1}\right) > 2 .$$

Proof: Now n is perfect so we have that

$$2n = \sigma(n) = \sigma(p_1^{\alpha_1})\ldots\sigma(p_r^{\alpha_r})$$

$$= \left(\frac{p_1^{\alpha_1+1} - 1}{p_1 - 1}\right) \cdots \left(\frac{p_r^{\alpha_r+1} - 1}{p_r - 1}\right)$$

Dividing both sides by $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, we obtain

$$2 = \left(\frac{p_1 - p_1^{\frac{1}{\alpha_1}}}{p_1 - 1}\right) \cdots \left(\frac{p_r - p_r^{\frac{1}{\alpha_r}}}{p_r - 1}\right) .$$

Thus $2 < \left(\frac{p_1}{p_1 - 1}\right) \cdots \left(\frac{p_r}{p_r - 1}\right)$.

¶

Using this theorem, we can easily obtain some results concerning odd perfect numbers.

Theorem 4: Any odd perfect number must have at least three distinct prime divisors.

Proof: Let n be an odd perfect number. We have already observed that n cannot be a power of a prime since $p \nmid \sigma(p^\alpha)$. Suppose $n = p_1^{\alpha_1} p_2^{\alpha_2}$, where p_1 and p_2 are distinct odd primes. Then

$$\left(\frac{p_1}{p_1 - 1}\right)\left(\frac{p_2}{p_2 - 1}\right) \leq \frac{3}{2} \cdot \frac{5}{4} = \frac{15}{8} < 2 .$$

This contradicts Theorem 3. Thus n has at least three distinct prime divisors.

¶

Theorem 5: Any odd perfect number with less than seven distinct prime divisors must be divisible by 3.

Proof: Now $\frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{18} = \frac{323323}{165888} < 2$.

Thus if n is an odd perfect number and $3 \nmid n$, then n has at least seven distinct prime divisors. ¶

Theorem 5 cannot be extended to eight distinct divisors by Theorem 3 alone, since

$$\frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{18} \cdot \frac{23}{22} = \frac{7436429}{3649536} > 2 .$$

However Sylvester did extend it to eight distinct prime divisors, and later Kanold [54] extended it to nine different prime factors. K. Norton [88] utilized a computer to calculate the minimum number of distinct prime factors in an odd perfect number whose least prime factor is P_k, where P_k is the k^{th} prime. His table of results extends to $P_{100} = 541$.

We shall now prove Sylvester's result that there are no three-fold perfect numbers.

Theorem 6: Any odd perfect number must have at least four distinct prime divisors.

Proof: We need only consider the case where an odd perfect number would have exactly three distinct prime divisors since Theorem 4 says that an odd perfect number has at least three different prime divisors. Suppose $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$ is a perfect number where the p_i's are distinct primes. Then we have that

$$\left(\frac{p_1}{p_1 - 1}\right)\left(\frac{p_2}{p_2 - 1}\right)\left(\frac{p_3}{p_3 - 1}\right) > 2 .$$

Now $\frac{3}{2} \cdot \frac{7}{6} \cdot \frac{11}{10} = \frac{231}{120} < 2$, and $\frac{3}{2} \cdot \frac{5}{4} \cdot \frac{17}{16} = \frac{255}{128} < 2$. Thus two of the primes are 3 and 5, while the third prime is 7, 11, or 13.

$$\text{Case 1:} \quad n = 3^{\alpha_1}5^{\alpha_2}7^{\alpha_3}$$

Now 3 and 7 are not of the form 4k+1, hence by Theorem 2, any odd perfect number divisible by 3, 5, and 7 must be a multiple of $5 \cdot 3^2 \cdot 7^2$. However

$$\sigma(5 \cdot 3^2 \cdot 7^2) = 6 \cdot 13 \cdot 57 = 4446 > 4410 = 2(5 \cdot 3^2 \cdot 7^2),$$

so that $5 \cdot 3^2 \cdot 7^2$ is abundant. But any multiple of an abundant number is abundant by Lemma 2. As a corollary, we have that no perfect number, whatever the number of distinct prime divisors, is divisible by $3 \cdot 5 \cdot 7 = 105$.

$$\text{Case 2:} \quad n = 3^{\alpha_1}5^{\alpha_2}11^{\alpha_3}$$

Note that 5 is the only prime of the form 4k+1, so by Theorem 2, $\alpha_2 = 4j+1$, and α_1 and $\alpha_3 = 2i$ are even. Suppose j=0, so that $\alpha_2 = 1$. Now $9 \mid \sigma(5 \cdot 11^{2i})$ since $\alpha_1 \geq 2$. We have that

$$\sigma(5 \cdot 11^{2i}) = \frac{6(11^{2i+1} - 1)}{10} \quad .$$

But $11^{2i+1} \equiv 2 \pmod 3$, so that $9 \nmid \sigma(5 \cdot 11^{2i})$. Thus $j > 0$.

Now $\sigma(5^{4j+1}) = \frac{5^{4j+2} - 1}{5 - 1}$ which has as factors $\frac{5^{2j+1} - 1}{5 - 1}$, $\frac{5^{2j+1} + 1}{5 + 1}$, and $\frac{5 + 1}{2} = 3$. Note that $5 \equiv 2 \pmod 3$ and $5^2 \equiv 1 \pmod 3$, while $5 \equiv 5 \pmod 9$, $5^2 \equiv 7 \pmod 9$, $5^3 \equiv 8 \pmod 9$, $5^4 \equiv 4 \pmod 9$,

$5^5 \equiv 2 \pmod 9$, and $5^6 \equiv 1 \pmod 9$. Thus $3 \nmid \left(\frac{5^{2j+1} - 1}{5 - 1}\right)$ and $3 \nmid \left(\frac{5^{2j+1} + 1}{5 + 1}\right)$ if $3 \nmid (2j+1)$. Also $(5^{2j+1} - 1, 5^{2j+1} + 1) = 2$, so that $\left(\frac{5^{2j+1} - 1}{4}, \frac{5^{2j+1} + 1}{6}\right) = 1$. Thus we have that $\frac{5^{2j+1} - 1}{5 - 1}$, $\frac{5^{2j+1} + 1}{5 + 1}$, and 3 are relatively prime if $3 \nmid (2j+1)$. But none of these factors is divisible by 5, and n has only 3 distinct prime factors. Hence $3 \mid (2j+1)$.

Since $3 \mid (2j+1)$, 5^3+1 is a factor of $5^{2j+1} + 1$, and $5^3+1 = 126 = 7 \cdot 18$. Thus $7 \mid \sigma(5^{4j+1})$, so that $7 \mid 2n$. This is a contradiction since n has only 3, 5, and 11 as prime divisors.

Case 3: $n = 3^{\alpha_1}5^{\alpha_2}13^{\alpha_3}$

Suppose the exponent of 13 is of the form $2i + 1$. Now $13 \equiv -1 \pmod 7$ and $13^2 \equiv 1 \pmod 7$, so that $7 \mid \sigma(13^{2i+1})$ since $\sigma(13^{2i+1}) = \frac{13^{2i+2} - 1}{13 - 1}$. Thus the exponent of 13 must be even, since $7 \nmid 2n$.

By Theorem 2, the exponent of 3 must also be even, so that $\alpha_1 = 2i$, and $\alpha_3 = 2j$. We have that

$$\sigma(3^{2i} \cdot 13^{2j}) = \frac{(3^{2i+1} - 1)(13^{2j+1} - 1)}{2 \cdot 12}$$

Now $13 \equiv 3 \pmod 5$, $13^2 \equiv 3^2 \equiv 4 \pmod 5$, $13^3 \equiv 3^3 \equiv 2 \pmod 5$, and $13^4 \equiv 3^4 \equiv 1 \pmod 5$. Thus any odd power of 3 or 13 is congruent to 2 or 3 modulo 5. Hence we know that $3^{2i+1} - 1 \equiv 1, 2 \pmod 5$ and $13^{2j+1} - 1 \equiv 1, 2 \pmod 5$. Therefore $5 \nmid \sigma(5^{\alpha_1} \cdot 3^{2i} \cdot 13^{2j})$.

Thus we have shown that there is no odd perfect number with exactly three distinct prime divisors. ¶

Sylvester's proof that there is no four-fold odd perfect number is similar to Theorem 6. There are more cases to consider, but the methods are essentially the same. L. E. Dickson [18] was later able to give a different proof that there are no odd perfect numbers with less than five distinct primes by actually listing all primitive non-deficient odd numbers with four or fewer distinct prime factors. Recall that a primitive non-deficient number is a non-deficient number which is not a multiple of a smaller non-deficient number, and that all perfect numbers are primitive non-deficient. By showing that none of the numbers in his list are perfect, Dickson completed his proof. Recall that Dickson was able to prove that there is only a finite number of odd primitive non-deficient numbers having a given number of distinct odd prime factors. Thus there can be at most a finite number of k-fold odd perfect numbers for a given k.

The study of cyclotomic polynomials has been helpful in obtaining results concerning the number of prime divisors of an odd perfect number, because the sum of the divisors function can be considered in terms of cyclotomic polynomials. The polynomial

$$F_d(x) = \prod_i (x - \varepsilon_i)$$

where ε_i runs through all primitive d^{th} roots of unity, is called the cyclotomic polynomial of index d. It is of degree $\phi(d)$, where $\phi(d)$ is the number of positive integers less than or equal to d that are

relatively prime to d. $\phi(d)$ is called Euler's ϕ-function, or sometimes, the totient. If p is a prime, then

$$\sigma(p^{\alpha}) = 1 + p + \ldots + p^{\alpha} = \frac{p^{\alpha+1} - 1}{p - 1} = \prod_{\substack{d \mid \alpha+1 \\ d > 1}} F_d(p) .$$

Hence if $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ is an odd perfect number, then

$$2 \prod_{i=1}^{k} p_i^{\alpha_i} = \prod_{i=1}^{k} \sigma(p_i^{\alpha_i}) = \prod_{i=1}^{k} \prod_{\substack{d \mid \alpha_i+1 \\ d > 1}} F_d(p_i) .$$

Results concerning cyclotomic polynomials may be found in chapter V of Nagell [86].

I. S. Gradstein [31], U. Kühnel [70], and G. C. Webber [125] independently proved that an odd perfect number has at least six distinct prime divisors. In 1972, C. Pomerance [91] and N. Robbins [99] independently showed that there must be at least seven distinct prime factors. Recently, P. Hagis [40] has announced that he can show, using an extensive computer based case study, that an odd perfect number must have at least eight distinct prime divisors. The proofs of these results are somewhat similar to Theorem 6; they are more intricate, have more cases to consider, and employ a bit more mathematical machinery, but the essential idea remains to patiently exhaust all the possible cases.

After Sylvester's work in the late 1880's, not much of importance was done on the odd perfect number problem for the next fifty years. Since that time, however, quite a few results have been

obtained, including those mentioned in the preceding paragraph. Much of the remainder of this section shall be devoted to summarizing the more important results.

Suppose that n is an odd perfect number, and we write n as

$$n = p^{\alpha} q_1^{2\beta_1} q_2^{2\beta_2} \ldots q_t^{2\beta_t}$$

where p is prime, $p \equiv \alpha \equiv 1 \pmod 4$, and the q_i's are distinct primes. The problem is to restrict the values for α and the β_i's for which n can be perfect. R. Steuerwald [104] was able to show that for $\beta_1 = \beta_2 = \ldots = \beta_t = 1$, that n cannot be perfect. Kanold [52] proved that if $\beta_1 = \beta_2 = \ldots = \beta_t = 2$, then n is not perfect. Let d be the greatest common divisor of $2\beta_1 + 1, 2\beta_2 + 1, \ldots, 2\beta_t + 1$. Kanold showed that if n is perfect, then d is not divisible by 9, 15, 21 or 33. W. L. McDaniel [83] proved that n is not perfect if $2\beta_i + 1 \equiv 0 \pmod 3$ for $i = 1, \ldots, t$. Kanold [53] and A. Brauer [10] independently proved that for $\beta_1 = 2$, $\beta_2 = \beta_3 = \ldots = \beta_t = 1$ that n is not perfect. To obtain his result, Brauer improved some results of T. Nagell on Diophantive equations (results involving cyclotomic polynomials are also useful). Brauer's method of proof consists of disposing of the various cases by means of contradiction, the flavor of which is still like that of Theorem 6. Kanold [56], [61] later showed that n is not perfect for $\beta_1 = 3$ or 4, $\beta_2 = \beta_3 = \ldots = \beta_t = 1$, or for $\beta_1 = \beta_2 = 2$, $\beta_3 = \ldots = \beta_t = 1$. Hagis and McDaniel [42] proved that if $\beta_1 = \beta_2 = \ldots = \beta_t = 3$, then n is not perfect. McDaniel [84] proved that either n is divisible by at least the sixth power of a

prime or $n > 10^{9118}$. J. B. Muskat [85] was able to show that n must be divisible by a prime power greater than 10^{12}, which improved his earlier result of 10^{8}.

Using a somewhat different attack, R. J. Levit [79] proved that if $\frac{\sigma(p^{\alpha})}{2}$ and $\sigma(q_i^{2\beta_i})$, $i = 1, \ldots, t$, are prime powers, then n is not a perfect number. Thus at least two of the divisor sums $\frac{\sigma(p^{\alpha})}{2}$, $\sigma(q_i^{2\beta_i})$, $i = 1, \ldots, t$, must have a common factor greater than 1.

J. Touchard [116], utilizing a recurrence relation of Balth van der Pol, was able to show that odd perfect numbers, if they exist, must be of the form $12m + 1$ or $36m + 9$. D. Suryanarayana and Hagis [112] were able to place upper and lower bounds on $\sum_{p|n} \frac{1}{p}$ where the p's are distinct primes and n is an odd perfect number of the form $12m + 1$ or $36m + 9$.

O. Ore [90] defined the harmonic mean of a positive integer n as

$$H(n) = \frac{n\, d(n)}{\sigma(n)}$$

where d(n) is the number of divisors of n. If H(n) is a positive integer, then n is said to be harmonic. It is easily seen that if n is perfect, then n is harmonic. Ore has conjectured that there is no odd $n > 1$ which is harmonic. The proof of this conjecture, if it is true, should be quite difficult since its verification yields the corollary that there are no odd perfect numbers. Calculations by Ore showed that there are no odd harmonic numbers less than 10^{4} besides 1. M. Garcia [26] raised the bound to 10^{7}.

Results have been obtained concerning the size of certain of the prime factors of an odd perfect number. Cl. Servais proved the following.

Theorem 7: If an odd perfect number contains k distinct prime factors, then the least prime factor does not exceed k.

Proof: Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ be an odd perfect number where $p_1 < p_2 < \ldots < p_k$ are odd primes. Now $\frac{p_2}{p_2 - 1} < \frac{p_1 + 1}{p_1}$ since $p_2 \geq p_1 + 2$. Proceeding, we have $\frac{p_i}{p_i - 1} < \frac{p_1 + (i - 1)}{p_1 + (i - 2)}$. Thus we have that

$$2 < \left(\frac{p_1}{p_1 - 1}\right)\left(\frac{p_2}{p_2 - 1}\right)\cdots\left(\frac{p_k}{p_k - 1}\right) < \left(\frac{p_1}{p_1 - 1}\right)\left(\frac{p_1 + 1}{p_1}\right)$$
$$\cdots\left(\frac{p_1 + k\text{-}1}{p_1 + k\text{-}2}\right) = \frac{p_1 + k\text{-}1}{p_1 - 1}$$

Hence $2(p_1 - 1) < p_1 + k\text{-}1$ and $p_1 < k+1$. Therefore $p_1 \leq k$.

¶

O. Grün [32] improved Servais' result by showing that $p_1 < \frac{2k}{3} + 2$.

Kanold [54] showed that if p is the largest prime divisor of an odd perfect number, then $p \geq 61$. Hagis and McDaniel [43] proved that an odd perfect number is divisible by a prime greater than 11200. They have recently improved this result by showing that the largest prime factor p of an odd perfect number must satisfy the relation $p \geq 100129$ [44]. Pomerance [92] has considered the second largest prime factor q of an odd perfect number and has shown that $q \geq 139$.

Putting together the known results concerning odd perfect numbers, a lower bound can be established for the least possible odd

perfect number. In 1908, A. Turčaninov established a lower bound of 2×10^6. In papers published in 1949, H. A. Bernhard [4] and Kühnel [70] raised the bound to 10^{10} and 2.2×10^{12}, respectively. Kanold [54] raised the bound to 1.4×10^{14} and later to 10^{20}. B. Tuckerman [118] showed that an odd perfect number must be greater than 10^{36}. Hagis [39] raised the bound to 10^{50}. B. Stubblefield [105], [106] also obtained a bound of 10^{50}, and claims it can be raised to 10^{100}.

If Sylvester thought that an odd perfect number could not exist due to 'the complex web of conditions,' which were known when he was working on the problem, what would he think today?

Other types of numbers related to the perfect numbers have been defined and studied. They include, among others, amicable numbers, multiply perfect numbers, unitary perfect numbers, quasiperfect numbers, and super perfect numbers. The relation between these types of numbers stems from the fact that they are all defined in terms of the sum of their divisors. The most important of these relatives of the perfect numbers are the amicable numbers which will be discussed in the next section.

Chapter 4

AMICABLE NUMBERS

Two numbers are said to be amicable, or friendly, if the sum of the divisors of each is equal to the sum of the two numbers. Expressing this in more mathematical terms, m and n are amicable if $\sigma(m) = \sigma(n) = m + n$, i.e., if $\sigma_0(m) = n$ and $\sigma_0(n) = m$. We take m and n to be distinct, for if $m = n$, then $\sigma(m) = 2m$ and m is perfect. Let m be the smaller member of the amicable pair [m, n]. Then m is abundant since $\sigma(m) = m + n > 2m$, while n is deficient since $\sigma(n) = m + n < 2n$.

The history of amicable numbers actually precedes that of perfect numbers. Pythagoras is generally credited with the discovery of amicable numbers around 540 B.C. When asked what is a friend, Pythagoras replied 'alter ego.' Thus the term friendly was applied to two numbers, each of which is equal to the sum of the aliquot parts of the other. This first amicable pair to be found was [220, 284], and it remained the only known pair for quite some time. Not unexpectedly, the numbers 220 and 284 were believed by many to have the ability to foster a friendship between two people. Biblical scholars commented on the wisdom of Jacob's gifts to Esau, the purpose of which was to gain Esau's friendship. Included in the gifts, described in Genesis 32:14, were 200 she-goats and 20 he-goats, 200 ewes and 20 rams.

Thâbit ben Korrah, writing in the ninth century (A.D.), gave a theorem which could be used to give amicable pairs. The numbers $2^n pq$

and $2^n r$ are amicable whenever $p = 3 \cdot 2^{n-1} - 1$, $q = 3 \cdot 2^n - 1$, and $r = 9 \cdot 2^{2n-1} - 1$ and p, q, and r are odd primes. Note that $220 = 2^2 \cdot 5 \cdot 11$ and $284 = 2^2 \cdot 71$ satisfy the rule with $n = 2$. However, there was no widespread knowledge of Thâbit ben Korrah's rule, and [220, 284] remained the only known amicable pair until the seventeenth century. Pierre de Fermat rediscovered Thâbit ben Korrah's rule in 1636 and was able to find a second amicable pair, $[2^4 \cdot 23 \cdot 47, 2^4 \cdot 1151]$. Two years later, René Descartes also obtained the rule and found a third amicable pair, $[2^7 \cdot 191 \cdot 383, 2^7 \cdot 73727]$. (We will shortly examine one of Euler's problems of which Thâbit ben Korrah's rule is a special case.)

These remained the only known amicable pairs until Leonard Euler attacked the problem, finding 59 new pairs during the period 1747-1750. Euler obtained his pairs by studying five problems, each consisting of finding amicable pairs of a certain form (notation due to Escott [23]).

> (1) To find Epq, Er amicable where p, q, r are distinct primes, all of which are relatively prime to the given number E.
> (2) To find Epq, Ers amicable where p, q, r, s are distinct primes which are relatively prime to the given number E.
> (3) To find Epq, Erf amicable where s in problem 2 is replaced by the number f which may be composite.
> (4) To find Egpq, Ehr amicable where p, q, r are distinct primes and E is given.
> (5) To find Eap, Ebq amicable where a and b are given, p and q are unknown primes and E is unknown but relatively prime to a, b, p, q.

Examining certain forms for possible amicable numbers also proved fruitful to later discoverers of amicable pairs, so let us

study one of Euler's forms to see how we might obtain an amicable pair.

Recall the fundamental relations that an amicable pair [m, n] must satisfy.

$$\sigma(m) = \sigma(n) \tag{1}$$

$$\sigma(m) = m + n \tag{2}$$

We will consider Euler's first problem, so that we have amicable pairs of the form $m = Epq$, $n = Er$ where E is the greatest common divisor of m and n, and p, q, r are distinct primes. Now $\sigma(Epq) = \sigma(E)(p + 1)(q + 1)$ and $\sigma(Er) = \sigma(E)(r + 1)$, so that from (1) and (2) we have:

$$\sigma(E)(p + 1)(q + 1) = \sigma(E)(r + 1) \tag{3}$$

$$\sigma(E)(p + 1)(q + 1) = E(pq + r) \tag{4}$$

Dividing both sides of (3) by $\sigma(E)$ we obtain:

$$(p + 1)(q + 1) = r + 1 \tag{5}$$

To reduce the problem to an equation in the unknowns p and q, we eliminate r from equations (4) and (5):

$$\sigma(E)(p + 1)(q + 1) = E(2pq + p + q) \tag{6}$$

Now E is usually some specified number, so solving Euclid's problem involves finding distinct primes p and q so that (6) is satisfied,

$p \nmid E$ and $q \nmid E$.

Assume now that we have $E = 2^k$, so that $\sigma(E) = \sigma(2^k) = 2^{k+1} - 1$. Working from equation (6):

$$(2^{k+1} - 1)(p + 1)(q + 1) = 2^k (2pq + p + q)$$
$$2^{k+1} - 1 = pq - [(2^{k+1} - 1) - 2^k] (p + q)$$
$$2^{k+1} - 1 = pq - (2^k - 1)(p + q) \qquad (7)$$

We now factor equation (7) in a convenient way:

$$[p - (2^k - 1)][q - (2^k - 1)] = 2^{2k} \qquad (8)$$

Also writing $2^{2k} = (2^{k-j})(2^{k+j})$ with $j < k$, we equate the factors as follows:

$$p - (2^k - 1) = 2^{k-j} \text{ and } q - (2^k - 1) = 2^{k+j} .$$

Now solving for p and q, then substituting these values into (5), we have

$$p = 2^{k-j} (2^j + 1) - 1$$
$$q = 2^k (2^j + 1) - 1$$
$$r = 2^{2k-j} (2^j + 1)^2 - 1 \qquad (9)$$

If $j = 1$, the solution takes the form

$$p = 3 \cdot 2^{k-1} - 1$$
$$q = 3 \cdot 2^k - 1$$
$$r = 9 \cdot 2^{2k-1} - 1 ,$$

which is the formula of Thâbit ben Korrah. With $k = 2$, we have $p = 5$, $q = 11$ and $r = 71$, giving the amicable pair [220, 284]. When $k = 4$, the amicable pair found by Fermat is obtained, and setting $k = 7$ gives Descartes' pair. These are the only known solutions to Thâbit ben Korrah's formula, i.e., for $k = 2, 4, 7$, and it is known that there are no further solutions for $k < 1000$ [7].

Now consider solution (9) when j is even, say $j = 2i$ where $i \geq 1$, so that we have

$$\begin{aligned} r &= 2^{2k-j}(2^j + 1)^2 - 1 \\ &= [2^{k-i}(2^j + 1)]^2 - 1 \\ &= [2^{k-i}(2^j + 1) - 1][2^{k-i}(2^j + 1) + 1] . \end{aligned}$$

Thus r cannot be prime since neither factor is equal to one.

If $j = 3$ then either p or q is composite. Assume k is even, so that $k = 2t$, then

$$\begin{aligned} q &= 2^k(2^3 + 1) - 1 \\ &= 2^{2t}(9) - 1 \\ &= (3 \cdot 2^t - 1)(3 \cdot 2^t + 1) . \end{aligned}$$

Neither factor is equal to one, so q is composite. Similarly, if k is odd, then p is composite.

There are no known solutions for $j = 5$. If $j = 7$ and $k = 8$ we have $p = 257$, $q = 33023$ and $r = 8520191$, all of which are prime. This gives the amicable pair $[2^8 \cdot 257 \cdot 33023, 2^8 \cdot 8520191]$, discovered by Legendre in 1830.

The four amicable pairs listed were the only known pairs for $E = 2^k$ until H. J. J. te Riele [115] discovered that the values $j = 11$ and $k = 40$ yield an amicable pair of this form. Te Riele utilized a computer search to test for $r < 10^{132}$ with $k > j > 1$.

Let us return to the general case where we are still considering the pair [Epq, Er]; recall that:

$$\sigma(E)(p + 1)(q + 1) = E\,(2pq + p + q) \tag{6}$$

Thus

$$\frac{\sigma(E)}{E} = 2 - \frac{(p + 1) + (q + 1)}{(p + 1)(q + 1)} \tag{10}$$

$$= 2 - \frac{1}{g}$$

where $g = \dfrac{(p + 1)(q + 1)}{(p + 1) + (q + 1)}$ is a positive rational number. (At this point it is advantageous to have a table of deficient numbers since $\frac{\sigma(E)}{E} < 2$.)

As an example, let $E = 3^2 \cdot 7 \cdot 13$. Then $\sigma(E) = 13 \cdot 8 \cdot 14$, so that $\frac{\sigma(E)}{E} = \frac{16}{9} = 2 - \frac{2}{9}$. Substituting this into (10), we have

$$\frac{2}{9} = \frac{(p + 1) + (q + 1)}{(p + 1)(q + 1)} .$$

Thus $2pq - 7(p + q) = 16$ which can be factored as $(2p - 7)(2q - 7) = 81$. Now $(2p - 7)$ and $(2q - 7)$ are unequal factors of 81, so we have either

$$\text{(i)} \quad 2p - 7 = 1 \,, \; p = 4$$

$$2q - 7 = 81 \,, \; q = 44$$

or

$$\text{(ii)} \quad 2p - 7 = 3 \,, \; p = 5$$

$$2q - 7 = 27 \,, \; q = 17 \,.$$

Hence (ii) is the only solution where p and q are primes. Substituting into (5) for r, we obtain the amicable pair $[3^2 \cdot 7 \cdot 13 \cdot 5 \cdot 17, 3^2 \cdot 7 \cdot 13 \cdot 107]$.

From studying Euler's first problem, it appears that patience and computational techniques were sufficient to find amicable numbers. In fact, the second smallest pair of amicable numbers, $[1184 = 2^5 \cdot 37, 1210 = 2 \cdot 5 \cdot 11^2]$, was discovered by the great violinist Nicolo Paganini when he was 16, probably by trial and error. L. E. Dickson [20] later proved that there are only five amicable pairs in which the smaller number is less than 6233; these pairs are Pythagoras' pair, Paganini's pair, and the following pairs due to Euler: $[2^2 \cdot 5 \cdot 131, 2^2 \cdot 17 \cdot 43]$, $[2^2 \cdot 5 \cdot 251, 2^2 \cdot 13 \cdot 107]$ and $[2^3 \cdot 19 \cdot 41, 2^5 \cdot 199]$.

After some practice in searching for amicable numbers, one learns to avoid certain pitfalls, by proving elementary but useful theorems such as the following.

Theorem 8: If m and n are an even amicable pair, then $3 \nmid m$ and $3 \nmid n$.

Proof: Let m and n be an even amicable pair; suppose first that $3|m$ and $3|n$. Then 6 divides both m and n since they are even numbers. Now 6 is a perfect number, and it was shown in Lemma 2 that a multiple of a perfect number is abundant. Thus m and n are both abundant, which is a contradiction.

Suppose that $3|m$ and $3 \nmid n$; then $m < n$, since m is abundant. Thus we have that $m = 2^r 3^s a$ and $n = 2^t b$ where a and b are relatively prime

to 6. Since $n \not\equiv 0 \pmod 3$,

$$\sigma(m) = \sigma(n) = m + n \equiv 1, 2 \pmod 3 \qquad (1)$$

Note that

$$\begin{aligned}\sigma(2^{2k-1}) &= 2^{2k} - 1 \\ &= (2^2 - 1)(2^{2k-2} + 2^{2k-4} + \ldots + 2^2 + 1) \\ &= 3(2^{2k-2} + 2^{2k-4} + \ldots + 2^2 + 1) .\end{aligned}$$

Thus r and t must be even in order to satisfy (1); hence $r \geq 2$ and $t \geq 2$.

Now we write $m = 2^2 \cdot 3 \cdot c$ and $n = 2^2 d$, where $c \geq 1$ and $d > 3c$. Since $c \geq 1$, $\frac{m}{\sigma(m)} \leq \frac{12}{\sigma(12)} = \frac{3}{7}$ and since $d > 3c$, $\frac{n}{\sigma(n)} < \frac{2^2}{\sigma(2^2)} = \frac{4}{7}$. Thus $\frac{m}{\sigma(m)} + \frac{n}{\sigma(n)} < 1$; but this is a contradiction since

$$\frac{m}{\sigma(m)} + \frac{n}{\sigma(n)} = \frac{m}{m+n} + \frac{n}{m+n} = 1 .$$

¶

E. B. Escott [23] listed the 390 pairs of amicable numbers that were known in 1940. The discoverers not previously mentioned and the approximate date of their discoveries are P. Seelhoff (1884), L. E. Dickson (1911), T. E. Mason (1921), P. Poulet (1929), A. Geradin (1929), Escott (1934), and B. H. Brown (1939). Escott grouped the amicable pairs to as forms, and his classification system is generally accepted (Euler's five problems, mentioned above, were stated in Escott's notation). Further pairs were discovered, without computer aid, by Poulet [96] and M. Garcia [27].

There are currently over 1000 known amicable pairs. During the past fifteen years most pairs have been found with the aid of a friendly computer. E. J. Lee [71] was able to find 260 new amicable pairs by studying solutions of bilinear diophantine equations with computer help. Others have been less subtle in looking for new friendly numbers. An upper bound is set and the computer tests all numbers below that bound in looking for amicable pairs. H. L. Rolf was the first to discover an amicable pair by computer, finding one new pair by testing all numbers less than 10^5. J. Alanen, O. Ore, and J. Stemple [2] found that there are 42 amicable pairs in which the smaller member of the pair is less than 10^6, eight of which were new. P. Bratley and J. McKay [8] raised the bound to 10^7 finding 108 pairs less than 10^7, 14 of which were new. H. Cohen [16] found that there are 236 amicable pairs with the smaller member less than 10^8, 56 of which were new. The only limitation on finding more friendly numbers in this fashion seems to be access to, and capabilities of, computers.

Further amicable pairs have been found by E. J. Lee, W. Borho, A. Wulf, Cohen, R. David and H. L. Nelson. Te Riele [115] has found the largest amicable numbers. (After writing the section on amicable numbers, it was discovered that Lee and J. S. Madachy [73] have written a paper on the history and discovery of amicable numbers.)

As in many areas of research, the motivation of those seeking for amicable numbers is varied. We have seen that some people wish a complete list, and hence perform computer searches. Others want to see if attacking the problem in a particular manner will yield results;

in particular, Borho [7] was able to find rules similar to that of Thâbit ben Korrah to locate more friendly numbers. Another method of attacking the problem is to see if amicable numbers of a certain form can exist. B. H. Brown [14] noticed that in most pairs the smallest prime factors occur to the same power in each member, the only exceptions being pairs where both members are even. By studying odd pairs, Brown obtained the amicable pair $[3^3 \cdot 5 \cdot 7 \cdot 13, 3 \cdot 5 \cdot 7 \cdot 139]$. Finally there is T. E. Mason who decided to pursue the subject when he saw how many articles had been written on amicable numbers compared with the small number discovered. After discovering 14 new amicable pairs, Mason [80] concluded

> that any one with a little skill in manipulation of numbers, considerable patience, and access to Lehmer's List of Prime Numbers, can add to the known list of amicable number pairs, amicable k-tuples, or multiply amicable number sets.

Today the prime attribute would be computer access.

There remain some unanswered questions concerning amicable numbers. To date no amicable pair has been found where the members are relatively prime; there is also no known pair where one number is even and the other odd. Thus, for every known pair of amicable numbers $[m, n]$, $m \equiv n \pmod 2$ and $(m, n) > 1$. As in the case with odd perfect numbers, relatively prime amicable numbers, if they exist, must meet certain conditions. The case where amicable numbers m and n are of opposite parity, but not relatively prime, has drawn little or no study. In 1953 Kanold [59] proved that if m and n are amicable

numbers which are relatively prime then $m \cdot n$ has at least 21 distinct prime factors and $n > m > 10^{23}$. Studying relatively prime odd amicable numbers m and n, P. Hagis [34] raised these bounds to $n > m > 10^{33}$ and the number of distinct prime divisors of $m \cdot n$ to 22. Working with relatively prime amicable numbers of opposite parity, Hagis [36] obtained the bounds $m, n > 10^{60}$.

The question of whether there are an infinite number of pairs of amicable numbers is unanswered. P. Erdös [22] was able to prove that the asymptotic density of amicable numbers is 0.

As an offshoot of the subject of amicable numbers, the aliquot series of an integer has been studied. Recall that $\sigma_0(n)$ is the sum of the aliquot divisors of n. We define

$$\sigma_0^{(0)}(n) = n \ , \ \sigma_0^{(1)}(n) = \sigma_0(n) \ ,$$

$$\sigma_0^{(2)}(n) = \sigma_0(\sigma_0(n)), \ \ldots, \ \sigma_0^{(k)}(n) = \sigma_0(\sigma_0^{(k-1)}(n)) \ .$$

The sequence $\sigma_0^{(k)}(n)$ for $k = 1, 2, \ldots,$ is called the aliquot series, or chain, for n. If there exists a positive integer k such that $\sigma_0^{(k)}(n) = n$ then n is said to have a chain of period k, or equivalently n is called a sociable number of order k. When the chain for n is not periodic, it is either unbounded or it converges, that is, there exists k such that $\sigma_0^{(k)}(n) = p$ and $\sigma_0^{(k+1)}(n) = \sigma_0(p) = 1$. Now every perfect number, n, has a chain of period 1 since $\sigma_0(n) = n$, while amicable numbers have chains of period 2. For [m, n], an amicable pair, $\sigma_0(m) = n$ and $\sigma_0(n) = m$ so that

$$\sigma_0^{(2)}(m) = \sigma_0(\sigma_0(m)) = \sigma_0(n) = m .$$

Poulet discovered that n = 12496 has a chain of period 5, while 14316 has a chain of period 28. Other chains of varying lengths have been found, but there is no known triple of sociable numbers of order 3; Bratley, Lunnon, and KcKay [9] exhausted by computer to show that there is no crowd whose smallest member is less than 5×10^7.

E. Catalan conjectured that the aliquot series is never unbounded for any n, that is, every chain is periodic or contains a prime. The aliquot series is known to be bounded for $n \leq 275$. The behavior of $\sigma_0^{(k)}(276)$ is unknown, but has been calculated for 119 terms, Cohen [16]. D. H. Lehmer extended the aliquot series for 276 to 433 terms. Lehmer also verified that all sequences starting with $n \leq 552$ are bounded with the possible exceptions of $396 = \sigma_0(276) = \sigma_0(306)$. Te Riele [114] constructed an aliquot series with more than 5092 monotonically increasing terms beginning with $n_0 \cdot 27$ where n_0 is the twenty-fourth even perfect number. Te Riele remarked that if there are infinitely many even perfect numbers, it should be possible to construct an aliquot series whose first n terms are monotonically increasing, for any given $n > 0$. R. K. Guy and J. L. Selfridge [33] conjectured that an infinite number of aliquot series are unbounded.

There are other generalizations of amicable numbers which have been studied. Numbers such as 27 and 35 where $\sigma_0(27) = \sigma_0(35) = 13$ are sometimes referred to as imperfectly amicable. More than two numbers may be pairwise imperfectly amicable: for instance,

$\sigma_0(93) = \sigma_0(145) = \sigma_0(253) = 35$. Dickson [20] gives a list of such numbers less than 1600; he [21] also defined an amicable k-tuple to be a set of k numbers $n_1, n_2, \ldots, n_k$ such that

$$\sigma(n_1) = \sigma(n_2) = \ldots = \sigma(n_k) = n_1 + n_2 + \ldots + n_k .$$

When k = 2, the definition of an amicable pair results. Dickson was able to find amicable triples, while Mason [80] later obtained amicable k-tuples for k = 3, 4, 5, and 6.

R. D. Carmichael [15] defined two numbers, m and n, to be a multiply amicable pair of multiplicity k if $\sigma(m) = \sigma(n) = k(m + n)$. Some interesting examples may be obtained by considering perfect and amicable numbers. Now $m = 3^2 \cdot 5 \cdot 13 \cdot 11 \cdot 19$ and $n = 3^2 \cdot 5 \cdot 13 \cdot 239$ are amicable numbers; it turns out that rm and rn are multiply amicable of multiplicity 2 where r is any of the known perfect numbers except 6. Six must be excluded since it is not relatively prime to m and n. Another pair, $m = a \cdot 11 \cdot 41 \cdot 173$ and $n = a \cdot 71 \cdot 1217$, was obtained by Mason [80]. Now m and n are amicable for $a = 2^3$, they are multiply amicable of multiplicity 2 for $a = 2^{10} \cdot 3^2 \cdot 7 \cdot 13 \cdot 23 \cdot 89$, and they are multiply amicable of multiplicity 3 for $a = 2^{21} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 13^3 \cdot 17 \cdot 19^3 \cdot 23 \cdot 89 \cdot 181 \cdot 379 \cdot 683 \cdot 757$.

Unitary amicable numbers have been defined by Hagis [37]. If n = ab and (a, b) = 1, then a is called a unitary divisor of n. Now the sum of the unitary divisors of n, $\sigma^*(n)$, is defined as

$$\sigma^*(n) = (1 + p_1^{\alpha_1})(1 + p_2^{\alpha_2})\ldots(1 + p_k^{\alpha_k})$$

where $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ and the p_i's are distinct primes. If $\sigma^*(m) = \sigma^*(n) = m + n$ then m and n are said to be unitary amicable. Note that if m and n are both square-free and amicable, then they are unitary amicable. The smallest unitary amicable pair is [114 = 2 · 3 · 19, 126 = 2 · 7 · 9]. At least the question of existence of a unitary amicable pair of opposite parity has been answered, in the negative.

Chapter 5

MULTIPLY PERFECT NUMBERS

A number n is said to be multiply perfect of multiplicity k if $\sigma(n) = kn$, where k is a positive integer. We shall designate a multiply perfect number of multiplicity k by P_k. Thus a perfect number is a P_2. A P_k is also said to be of class k. Multiply perfect, or pluperfect, numbers were first named and studied by seventeenth century French mathematicians, including Marin Mersenne, René Descartes and Pierre de Fermat. The early history of multiply perfect numbers can be found in chapter I of Dickson [17].

The first known multiply perfect number was 120 which is a P_3 since $\sigma(120) = 360$. The numbers $120 = 2^3 \cdot 3 \cdot 5$ and $672 = 2^5 \cdot 3 \cdot 7$, which is also a P_3, are the only multiply perfect numbers with exactly three distinct prime divisors. The only multiply perfect numbers with exactly four distinct prime factors are $523776 = 2^9 \cdot 3 \cdot 11 \cdot 31$ which is a P_3 and $30240 = 2^5 \cdot 3^3 \cdot 5 \cdot 7$ which is a P_4.

When considering the number of distinct prime divisors of a multiply perfect number, we can approach the problem in the same manner as we did when considering odd perfect numbers.

Theorem 9: If $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$, where the p_i's are distinct primes, is a multiply perfect number of multiplicity k, then

$$k < \left(\frac{p_1}{p_1 - 1}\right)\left(\frac{p_2}{p_2 - 1}\right)\cdots\left(\frac{p_r}{p_r - 1}\right)$$

Proof: The proof of this theorem is the same as the proof of Theorem 3, replacing $\sigma(n) = 2n$ by $\sigma(n) = kn$. ¶

Utilizing Theorem 9, we can obtain a number of results concerning the prime divisors of multiply perfect numbers.

Corollary 1: There do not exist any multiply perfect numbers of multiplicity $k \geq 3$ with fewer than three distinct prime divisors.

Proof: We know that $p \nmid \sigma(p)$ for p prime, so that a prime power cannot be multiply perfect. Suppose $n = p_1^{\alpha_1} p_2^{\alpha_2}$, where p_1 and p_2 are distinct primes, is such that $\sigma(n) = kn$, $k \geq 3$. Now

$$3 < \left(\frac{p_1}{p_1 - 1}\right)\left(\frac{p_2}{p_2 - 1}\right) \leq \left(\frac{2}{1}\right)\left(\frac{3}{2}\right) = 3 ,$$

which is a contradiction. ¶

Corollary 2: If $k > 4$, then a multiply perfect number of multiplicity k has at least k+1 distinct prime divisors. (It is known that a P_4 must have at least 4 distinct prime divisors.)

Proof: We use induction to show that

$$\prod_{i=1}^{k} \left(\frac{p_i}{p_i - 1}\right) < k \qquad (1)$$

where the p_i's are distinct primes. Actually we take $p_1, p_2, \ldots, p_k$ to be the first k primes since this maximizes the left hand side of (1). Let $k = 5$. Now

$$\left(\frac{2}{1}\right)\left(\frac{3}{2}\right)\left(\frac{5}{4}\right)\left(\frac{7}{6}\right)\left(\frac{11}{10}\right) = \frac{77}{16} < 5 .$$

Thus inequality (1) holds for $k = 5$.

Assume $\prod_{i=1}^{m-1}\left(\frac{p_i}{p_i - 1}\right) < m - 1$. Let p_m be the m^{th} prime. Now we know that $\frac{p_m}{p_m - 1} < \frac{m}{m - 1}$, so that

$$\left(\frac{p_m}{p_m - 1}\right)\prod_{i=1}^{m-1}\left(\frac{p_i}{p_i - 1}\right) < (m - 1)\left(\frac{m}{m - 1}\right) .$$

Hence $$\prod_{i=1}^{k}\left(\frac{p_i}{p_i - 1}\right) < m .$$

Thus $\prod_{i=1}^{k}\left(\frac{p_i}{p_i - 1}\right) < k$ for all $k > 4$. Therefore by Theorem 9, a multiply perfect number of multiplicity $k > 4$ has at least $k+1$ distinct prime divisors.

¶

The results of Corollary 2 can be improved for a particular choice of k. For instance, a P_6 must have at least nine distinct prime divisors, while a P_7 must have at least fourteen.

Corollary 3: There do not exist any odd P_k with only three distinct prime factors.

Proof: Now $\left(\frac{3}{2}\right)\left(\frac{5}{4}\right)\left(\frac{7}{6}\right) = \frac{35}{16} < 3$. Thus k must equal 2, but by Theorem 6 we saw that there are no odd perfect numbers with three distinct prime divisors.

¶

Not only are there no known odd perfect numbers, a P_2, but there are no known odd multiply perfect numbers of any multiplicity. The

existence of an odd multiply perfect number remains an open question.

P. Poulet [95] listed the 334 multiply perfect numbers known in 1929. New multiply perfect numbers have been found by Alan L. Brown [13] and B. Franqui and M. Garcia [24], [25].

Chapter 6

OTHER GENERALIZATIONS

Unitary perfect numbers have been defined by M. V. Subbarao and L. J. Warren [108]. A divisor d of n is said to be a unitary divisor of n if d and $\frac{n}{d}$ are relatively prime. The sum of the unitary divisors of n is denoted by $\sigma^*(n)$. Now $\sigma^*(n)$ is a multiplicative function with $\sigma^*(1) = 1$ and

$$\sigma^*(n) = (1 + p_1^{\alpha_1})(1 + p_2^{\alpha_2})\ldots(1 + p_k^{\alpha_k})$$

where $n = p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_k^{\alpha_k}$ with the p_i's distinct primes. Thus $\sigma^*(n)$ is even unless $n = 1$ or n is a power of 2. A number n is said to be unitary perfect if $\sigma^*(n) = 2n$. Unlike our experience in dealing with perfect numbers, it can be easily shown that there are no odd unitary perfect numbers.

Theorem 10: There are no odd unitary perfect numbers.

Proof: Suppose $n = p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_k^{\alpha_k}$ is an odd unitary perfect number. Then we have that $2 \| \sigma^*(n)$, where

$$\sigma^*(n) = (1 + p_1^{\alpha_1})(1 + p_2^{\alpha_2})\ldots(1 + p_k^{\alpha_k}) .$$

Thus $k = 1$ since $\sigma^*(i)$ is even unless $i = 1$ or i is a power of 2. Hence $1 + p_1^{\alpha_1} = 2p_1^{\alpha_1}$, which is a contradiction. ¶

The first four unitary perfect numbers are 6, 60, 90 and 87360. C. R. Wall [124] discovered a unitary perfect number of twenty-four

digits and was able to show that it was the next unitary perfect number following 87360. Wall [123] also extended the concept to bi-unitary perfect numbers. A divisor d of n is a bi-unitary divisor if the greatest common unitary divisor of d and $\frac{n}{d}$ is 1. If $\sigma^{**}(n)$ denotes the sum of the bi-unitary divisors of n, then n is said to be bi-unitary perfect if $\sigma^{**}(n) = 2n$. The only bi-unitary perfect numbers are the first three unitary perfect numbers, 6, 60 and 90. Thus 6 is the only number which is perfect, unitary perfect, and bi-unitary perfect.

A number n is said to be quasiperfect if $\sigma(n) = 2n + 1$. A difference in working with a quasiperfect number n is that $(\sigma(n), n) = 1$, while $(\sigma(m), m) = m$ if m is perfect or multiply perfect. The existence of a quasiperfect is still in question. It has been shown [1] that if n is quasiperfect, then $n > 10^{20}$. Also n is divisible by at least five distinct primes, while if $3 \nmid n$, then n has at least eight distinct prime factors. Thus quasiperfect numbers are in somewhat the same situation as odd perfect numbers. Pseudoperfect numbers have also been defined where n is pseudoperfect if $\sigma(n) = 2n - 1$. At least pseudoperfect numbers are known to exist, since $\sigma(2^k) = 2^{k+1} - 1 = 2(2^k) - 1$.

A number n is super perfect if $\sigma(\sigma(n)) = 2n$. D. Suryanarayana [111] proved that an even integer n is super perfect if and only if n is of the form 2^k, where $2^{k+1} - 1$ is a prime. Thus the number of known even super perfect numbers is the same as the number of known even perfect numbers. It is not known whether any odd super perfect

numbers exist. Various conditions have been placed on the existence of an odd super perfect number, including that any such must be greater than 7×10^{24} [49].

Much of the appeal of number theory stems from the fact that many difficult problems can be stated so as to be understood by the nonmathematician. The problem of perfect numbers is of this type; they are easily defined and with only a small amount of calculation the first few perfect numbers can be quickly determined. However, from what might seem an unassuming problem, a very difficult question appears: do any odd perfect numbers exist? When confronted with all the conditions to be met by an odd perfect number, we tend to agree with the previously quoted observations of Sylvester.

As for the usefulness of perfect numbers, consider the title of Martin Gardner's paper [28], "A short treatise on the useless elegance of perfect numbers and amicable pairs." However, we have seen that a large number of well-known mathematicians have tackled the problem of perfect numbers, and their efforts helped lead to the development of number theory. Another consideration, which cannot be measured, is the stimulation of the thought processes by problems of this type.

When it comes to all the generalizations of the perfect numbers, they are of some interest as long as they stay in hand. We do not wish to see a class of number defined for every possible value of the sum of the divisors function. As to the use of computers in looking for various types of numbers, they are indispensable in performing

calculations. On the other hand, some of the computer studies really do not open up any new avenues, for it can be easier to exhaust by computer, than to look for new theoretical results which might shed more light on the subject.

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PERFECT NUMBERS AND OTHER NUMBERS DEFINED BY THE SUM OF THEIR DIVISORS

by

Kenneth Eugene Hawkes

(ABSTRACT)

Perfect, abundant, and deficient numbers are defined in terms of the sum of the divisors function. The history and the discovery of perfect numbers is discussed, and the question of the existence of an odd perfect number is considered.

There is a discussion of amicable pairs and multiply perfect numbers. Other numbers which are defined by the sum of their divisors are also introduced.