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Scattering and inverse scattering in one-dimensional nonhomogeneous media

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The wave propagation in a one-dimensional nonhomogeneous medium is considered, where the wave speed and the restoring force depend on location. In the frequency domain this is equivalent to the Schrödinger equation $d^2\psi/dx^2 + k^2\psi = k^2P(x)\psi + Q(x)\psi$ with an added potential proportional to energy. The scattering and bound-state solutions of this equation are studied and the properties of the scattering matrix are obtained; the inverse scattering problem of recovering the restoring force when the wave speed and the scattering data are known are also solved.

I. INTRODUCTION

Consider the one-dimensional (1-D) Schrödinger equation,

$$\frac{d^2\psi(k,x)}{dx^2} + k^2\psi(k,x) = k^2P(x)\psi(k,x) + Q(x)\psi(k,x), \quad (1.1)$$

where $x \in \mathbb{R}$ is the space coordinate, $k^2 \in \mathbb{R}$ is energy, $k^2P(x)$ is the potential proportional to energy, and $Q(x)$ is also a potential. Both potentials $P(x)$ and $Q(x)$ are real. The Fourier transformation from the frequency k domain into the time t domain changes (1.1) into

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c(x)^2} \frac{\partial^2 u}{\partial t^2} = Q(x)u, \quad (1.2)$$

where $c(x) = 1/\sqrt{1-P(x)}$ is the wave speed and $Q(x)$ is the restoring force. The equation in (1.2) describes the propagation of waves in a medium, where the wave speed and the restoring force depend on location. We will let $H(x) = \sqrt{1-P(x)}$, and for a meaningful wave speed we will assume $P(x) < 1$. We also assume that $P(x)$ is bounded below, and thus

$$M = \sup_{x \in \mathbb{R}} H(x) \quad (1.3)$$

is a finite number.

The direct scattering problem for (1.1) consists of finding the scattering matrix $S(k)$ (which will be defined in Sec. II) when the potentials $P(x)$ and $Q(x)$ are known. There are three inverse scattering problems associated with (1.1). The first one is to recover the potential

$Q(x)$ when the scattering matrix $S(k)$ and the other potential $P(x)$ are known. The second inverse problem is to recover $P(x)$ when $S(k)$ and $Q(x)$ are given. The last one is to recover both $P(x)$ and $Q(x)$ when $S(k)$ is given, although its solution is, in general, not unique. In this paper we will only study the first inverse problem mentioned; physically, this inverse problem corresponds to the determination of the restoring force when the wave speed and the scattering data are known.

Letting

$$y = \int_0^x d\xi H(\xi)$$

and

$$\phi(k,y) = \sqrt{H(x)}\psi(k,x),$$

one can transform (1.1) into the Schrödinger equation,

$$\frac{d^2\phi}{dy^2} + k^2\phi = V(y)\phi, \quad (1.4)$$

where the new potential $V(y)$ is related to the potential of (1.1) as

$$V(y) = -G(x)/H(x), \quad (1.5)$$

where

$$G(x) = -\frac{H''(x)}{2H(x)^2} + \frac{3H'(x)^2}{4H(x)^3} - \frac{Q(x)}{H(x)}. \quad (1.6)$$

Note that throughout the paper we use the prime to denote the derivative with respect to x . $V(y)$ can be ob-

tained by solving the inverse scattering problem for (1.4) by using one of the inverse scattering methods for the 1-D Schrödinger equation. Inverting (1.5) we can obtain $Q(x)$ when $P(x)$ is known, thereby solving the inverse scattering problem for (1.1). However, in this paper we will use the spatial coordinate directly because this will enable us in the future to combine the results of the present paper with those of Ref. 1 in order to solve the second inverse problem and to study the third inverse problem mentioned earlier.

We formulate the first inverse scattering problem for (1.1) as a Riemann–Hilbert problem. Once the problem is posed as a Riemann–Hilbert problem, there are several methods to solve it, such as the Marchenko method,^{2–6} the Gel’fand–Levitan method,^{6,7} the Wiener–Hopf factorization method,⁸ and the Muskhelishvili–Vekua method,^{9,10} which is also known as the Newton–Jost method.¹¹ In this paper we will only use the Marchenko method to solve the first inverse problem.

All the results given in this paper hold for real potentials satisfying the conditions $Q \in L^1_+(\mathbf{R})$, $P(x) < 1$ and is bounded below, $P \in L^1_+(\mathbf{R})$, and $G \in L^1_+(\mathbf{R})$, where $L^1_j(\mathbf{R})$ is the class of Lebesgue-integrable potentials having a finite j th moment. Note that whenever $P \in L^1(\mathbf{R})$, we have $1 - H \in L^1(\mathbf{R})$ since $|1 - H| \leq |1 - H|(1 + H) = |P|$. This fact will be used throughout the paper. In the beginning of each section we will specify the sufficient conditions under which the results there hold.

This paper is organized as follows. In Sec. II we define the scattering solutions of (1.1), study their properties, and establish their asymptotics for small k . In Sec. III we study the large k asymptotics of the scattering solutions of (1.1). In Sec. IV we study the properties of the scattering matrix and establish its asymptotics for small and large k . In Sec. V we study the bound-state solutions of (1.1) and obtain a Levinson theorem that relates the number of bound states to the phase of the transmission coefficient. In Sec. VI, using the Marchenko method, we solve the inverse scattering problem by recovering $Q(x)$ from one of the reflection coefficients when $P(x)$ is known. In Sec. VII we obtain some properties of the scattering data when the Marchenko method leads to $G \in L^1_+(\mathbf{R})$, where $G(x)$ is the quantity defined in (1.6). In Sec. VIII we show that $G(x)$ obtained from the Marchenko method belongs to $L^1_+(\mathbf{R})$ when the scattering data satisfy the conditions obtained in Sec. VII, and we also show that the solution of each of the two Marchenko integral equations leads to a solution of the Schrödinger equation (1.1). Finally, in the Appendix we prove a lemma used in Sec. VI.

II. SCATTERING SOLUTIONS

In this section we study the scattering solutions of (1.1) and also establish their asymptotics for small k .

The sufficient conditions on the potentials in this section are $P(x) < 1$ and $P, Q \in L^1_+(\mathbf{R})$.

The physical solutions ψ_l from the left and ψ_r from the right satisfy

$$\psi_l(k, x) = \begin{cases} T_l(k)e^{ikx} + o(1), & x \rightarrow \infty, \\ e^{ikx} + L(k)e^{-ikx} + o(1), & x \rightarrow -\infty \end{cases} \quad (2.1)$$

and

$$\psi_r(k, x) = \begin{cases} e^{-ikx} + R(k)e^{ikx} + o(1), & x \rightarrow \infty, \\ T_r(k)e^{-ikx} + o(1), & x \rightarrow -\infty. \end{cases} \quad (2.2)$$

Here T_l and T_r are the transmission coefficients from the left and from the right, respectively, and L and R are the reflection coefficients from the left and from the right, respectively. The scattering matrix $S(k)$ is defined as

$$S(k) = \begin{bmatrix} T_l(k) & R(k) \\ L(k) & T_r(k) \end{bmatrix}. \quad (2.3)$$

We will study the properties of $S(k)$ in Sec. IV. The physical solutions ψ_l and ψ_r satisfy the Lippmann–Schwinger equation

$$\begin{aligned} \begin{bmatrix} \psi_l(k, x) \\ \psi_r(k, x) \end{bmatrix} &= \begin{bmatrix} e^{ikx} \\ e^{-ikx} \end{bmatrix} + \frac{1}{2ik} \int_{-\infty}^{\infty} dy e^{ik|x-y|} \\ &\times [k^2 P(y) + Q(y)] \begin{bmatrix} \psi_l(k, y) \\ \psi_r(k, y) \end{bmatrix}. \end{aligned} \quad (2.4)$$

The Jost solutions of (1.1), f_l from the left and f_r from the right, are defined as

$$\begin{aligned} f_l(k, x) &= [1/T_l(k)]\psi_l(k, x), \\ f_r(k, x) &= [1/T_r(k)]\psi_r(k, x). \end{aligned} \quad (2.5)$$

They satisfy the integral equations

$$\begin{aligned} f_l(k, x) &= e^{ikx} - \frac{1}{k} \int_x^{\infty} dy \sin k(x-y) \\ &\times [k^2 P(y) + Q(y)] f_l(k, y), \\ f_r(k, x) &= e^{-ikx} + \frac{1}{k} \int_{-\infty}^x dy \sin k(x-y) \\ &\times [k^2 P(y) + Q(y)] f_r(k, y), \end{aligned}$$

and the boundary conditions

$$f_l(k, x) = \begin{cases} e^{ikx} + o(1), & x \rightarrow \infty, \\ \frac{1}{T_l(k)} e^{ikx} + \frac{L(k)}{T_l(k)} e^{-ikx} + o(1), & x \rightarrow -\infty \end{cases}$$

and

$$f_r(k,x) = \begin{cases} \frac{1}{T_r(k)} e^{-ikx} + \frac{R(k)}{T_r(k)} e^{ikx} + o(1), & x \rightarrow \infty, \\ e^{-ikx} + o(1), & x \rightarrow -\infty. \end{cases}$$

Let us also define

$$\begin{aligned} m_l(k,x) &= [1/T_l(k)] e^{-ikx} \psi_l(k,x), \\ m_r(k,x) &= [1/T_r(k)] e^{ikx} \psi_r(k,x). \end{aligned} \tag{2.6}$$

Then from (1.1) and (2.6) it is seen that m_l and m_r satisfy the equations

$$\begin{aligned} m_l''(k,x) + 2ikm_l'(k,x) \\ = [k^2P(x) + Q(x)] m_l(k,x), \end{aligned} \tag{2.7}$$

$$\begin{aligned} m_r''(k,x) - 2ikm_r'(k,x) \\ = [k^2P(x) + Q(x)] m_r(k,x). \end{aligned} \tag{2.8}$$

We will call m_l and m_r the Faddeev solutions from the left and right, respectively; they satisfy the integral equations

$$\begin{aligned} m_l(k,x) &= 1 - \frac{1}{2ik} \int_x^\infty dy [1 - e^{2ik(y-x)}] \\ &\quad \times [k^2P(y) + Q(y)] m_l(k,y), \tag{2.9} \\ m_r(k,x) &= 1 - \frac{1}{2ik} \int_{-\infty}^x dy [1 - e^{2ik(x-y)}] \\ &\quad \times [k^2P(y) + Q(y)] m_r(k,y), \end{aligned} \tag{2.10}$$

and the boundary conditions

$$\begin{aligned} m_l(k,x) &= 1 + o(1), \quad x \rightarrow \infty, \\ m_l'(k,x) &= o(1), \quad x \rightarrow \infty, \\ m_r(k,x) &= 1 + o(1), \quad x \rightarrow -\infty, \\ m_r'(k,x) &= o(1), \quad x \rightarrow -\infty. \end{aligned}$$

Next we show that the Faddeev solutions defined in (2.6) can be extended analytically in k to the upper half complex plane \mathbf{C}^+ . We will use the notation \mathbf{C}^- for the lower half complex plane and use \mathbf{C}^\pm to denote $\mathbf{C}^\pm \cup \mathbf{R}$.

Theorem 2.1: If $Q \in L^1_+(\mathbf{R})$ and $P \in L^1(\mathbf{R})$, the Faddeev solutions $m_l(k,x)$ and $m_r(k,x)$ are analytic in k for $k \in \mathbf{C}^+$ and continuous in k for $k \in \overline{\mathbf{C}^+}$.

Proof: From (2.9) we have $m_l(k,x) = \sum_{j=0}^\infty n_j(k,x)$, where $n_0(k,x) = 1$ and

$$\begin{aligned} n_j(k,x) &= -\frac{1}{2ik} \int_x^\infty dy [1 - e^{2ik(y-x)}] \\ &\quad \times [k^2P(y) + Q(y)] n_{j-1}(k,y), \quad j \geq 1. \end{aligned}$$

Noting that $\int_0^{y-x} dt e^{2ikt} = -(1/2ik)[1 - e^{2ik(y-x)}]$, and using $|1 - e^{2ik(y-x)}| \leq 2$ when $y \geq x$ and $k \in \mathbf{C}^+$, we obtain

$$|n_j(k,x)| \leq \frac{1}{j!} \left| \int_x^\infty dy [|kP(y)| + (y-x)|Q(y)|] \right|^j.$$

Hence, we have

$$\begin{aligned} |m_l(k,x)| \leq \exp \left(\int_x^\infty dy [|kP(y)| \right. \\ \left. + (y-x)|Q(y)|] \right), \quad k \in \overline{\mathbf{C}^+}. \end{aligned} \tag{2.11}$$

Furthermore, each $n_j(k,x)$ is analytic in k for $k \in \mathbf{C}^+$ and continuous in k for $k \in \overline{\mathbf{C}^+}$, and thus by the Weierstrass theorem, $m_l(k,x)$, being the limit of a uniformly convergent sequence of analytic functions on compact subsets in \mathbf{C}^+ , is analytic in k for $k \in \mathbf{C}^+$ and continuous in k for $k \in \overline{\mathbf{C}^+}$ for each $x \in \mathbf{R}$ whenever $P \in L^1(\mathbf{R})$ and $Q \in L^1_+(\mathbf{R})$.

Repeating the above argument with (2.10), we obtain

$$\begin{aligned} |m_r(k,x)| \leq \exp \left(\int_{-\infty}^x dy [|kP(y)| \right. \\ \left. + (x-y)|Q(y)|] \right), \quad k \in \overline{\mathbf{C}^+}, \end{aligned} \tag{2.12}$$

and that $m_r(k,x)$ is analytic in k for $k \in \mathbf{C}^+$ and continuous in k for $k \in \overline{\mathbf{C}^+}$.

The solutions of (2.7) and (2.8) at $k = 0$ satisfy

$$m_l(0,x) = 1 + \int_x^\infty dy (y-x) Q(y) m_l(0,y), \tag{2.13}$$

$$m_r(0,x) = 1 + \int_{-\infty}^x dy (x-y) Q(y) m_r(0,y), \tag{2.14}$$

which can also be obtained from (2.9) and (2.10) in the limit $k \rightarrow 0$. Applying the analysis given in the beginning of the proof to (2.13) and (2.14), we see that (2.11) and (2.12) are valid also at $k = 0$. Hence, for each x , we have the continuity of $m_l(k,x)$ and $m_r(k,x)$ at $k = 0$. \blacksquare

Proposition 2.2: If $P, Q \in L^1_+(\mathbf{R})$, then for $k \in \overline{\mathbf{C}^+}$, the Faddeev solutions $m_l(k,x)$ and $m_r(k,x)$ satisfy the inequality $|m(k,x)| \leq C_1(k)[1 + |x|]$, where

$$C_1(k) = \exp\left(\int_{-\infty}^{\infty} dy [1 + |y|] |k^2 P(y) + Q(y)|\right) \\ \times \left[1 + \exp\left(\int_{-\infty}^{\infty} dz [|kP(z)| + |zQ(z)|]\right)\right] \\ \times \left[\int_{-\infty}^{\infty} dy |y| |k^2 P(y) + Q(y)|\right].$$

Proof: From the proof of Theorem 2.1 we see that the Lippmann–Schwinger equation for $m_l(k, x)$ can also be written as

$$m_l(k, x) = 1 + \int_x^{\infty} dy \int_0^{y-x} dt e^{2ikt} [k^2 P(y) + Q(y)] m_l(k, y),$$

and thus for $k \in \overline{\mathbf{C}^+}$ we have

$$|m_l(k, x)| \leq 1 + \int_x^{\infty} dy (y-x) |k^2 P(y) + Q(y)| |m_l(k, y)| \\ \leq 1 + \int_0^{\infty} dy y |k^2 P(y) + Q(y)| |m_l(k, y)| \\ - x \int_x^{\infty} dy |k^2 P(y) + Q(y)| |m_l(k, y)|.$$

Using (2.11) and letting

$$C_2(k) = 1 + \exp\left(\int_{-\infty}^{\infty} dz [|kP(z)| + |zQ(z)|]\right) \\ \times \left[\int_{-\infty}^{\infty} dy |y| |k^2 P(y) + Q(y)|\right],$$

we obtain

$$\frac{|m_l(k, x)|}{C_2(k) [1 + |x|]} \leq 1 + \int_x^{\infty} dy [1 + |y|] |k^2 P(y) + Q(y)| |m_l(k, y)| / (C_2(k) \times [1 + |y|]).$$

Hence, using iteration we obtain

$$|m_l(k, x)| / (C_2(k) [1 + |x|]) \leq \exp\left(\int_x^{\infty} dy [1 + |y|] |k^2 P(y) + Q(y)|\right),$$

which gives the result stated in the proposition with

$$C_1(k) = C_2(k) \exp\left(\int_{-\infty}^{\infty} dy [1 + |y|] |k^2 P(y) + Q(y)|\right).$$

In a similar way, we also obtain $|m_r(k, x)| \leq C_1(k) \times [1 + |x|]$. ■

From (2.9) and (2.10), we have

$$m'_l(k, x) = - \int_x^{\infty} dy e^{2ik(y-x)} \times [k^2 P(y) + Q(y)] m_l(k, y),$$

$$m'_r(k, x) = \int_{-\infty}^x dy e^{2ik(x-y)} \times [k^2 P(y) + Q(y)] m_r(k, y).$$

Hence, using Proposition 2.2 we obtain

$$|m'_l(k, x)| \leq C_1(k) \int_{-\infty}^{\infty} dy [1 + |y|] \times |k^2 P(y) + Q(y)|, \quad k \in \overline{\mathbf{C}^+},$$

and similarly

$$|m'_r(k, x)| \leq C_1(k) \int_{-\infty}^{\infty} dy [1 + |y|] \times |k^2 P(y) + Q(y)|, \quad k \in \overline{\mathbf{C}^+},$$

where $C_1(k)$ is as specified in Proposition 2.2. Thus, if $P, Q \in L^1_1(\mathbf{R})$, the functions $m'_l(k, x)$ and $m'_r(k, x)$ are analytic in $k \in \mathbf{C}^+$ and continuous in $k \in \overline{\mathbf{C}^+}$ for each $x \in \mathbf{R}$.

III. LARGE k ASYMPTOTICS OF THE SCATTERING SOLUTIONS

In this section, the sufficient assumptions are that $P(x) < 1$ and is bounded below, $P \in L^1(\mathbf{R})$, and $G \in L^1_1(\mathbf{R})$, where $G(x)$ is the quantity defined in (1.6). First, using techniques similar to those used in Refs. 12 and 1, we show the existence of two linearly independent solutions of (1.1) and establish their large k asymptotics. Then, relating these solutions to the scattering solutions of

(1.1) defined in Sec. II, we will establish the large k asymptotics of the scattering solutions of (1.1).

Assume a solution of (1.1) of the form $\psi(k,x) = Y(k,x)Z(k,x)$, where $Y(k,x)$ stands for either of the two functions defined by

$$Y_l(k,x) = \frac{\exp[ik\int_0^x dz H(z)]}{\sqrt{H(x)}} \tag{3.1}$$

and

$$Y_r(k,x) = \frac{\exp[-ik\int_0^x dz H(z)]}{\sqrt{H(x)}}. \tag{3.2}$$

Then $Z(k,x)$ satisfies

$$YZ'' + 2Y'Z' + [Y'' + k^2H^2Y - QY]Z = 0.$$

Multiplying the above equation by Y and rearranging terms, we have

$$(Y^2Z')' + Y^2[Y''/Y + k^2H^2 - Q]Z = 0. \tag{3.3}$$

Note that from (3.1) and (3.2) we have

$$Y''/Y + k^2H^2 - Q = G(x)H(x),$$

where $G(x)$ is the quantity in (1.6). Integrating (3.3) with the boundary condition $Z'(k,x_0) = 0$, we obtain

$$\begin{aligned} &Y(k,x)^2Z'(k,x) \\ &= - \int_{x_0}^x dz Y(k,z)^2G(z)H(z)Z(k,z), \end{aligned}$$

or equivalently

$$Z'(k,x) = - \int_{x_0}^x dz \frac{Y(k,z)^2}{Y(k,x)^2} G(z)H(z)Z(k,z). \tag{3.4}$$

Integrating (3.4) with the boundary condition $Z(k,x_0) = 1$, we obtain

$$\begin{aligned} Z(k,x) &= 1 - \int_{x_0}^x d\xi \int_{x_0}^\xi dz \frac{Y(k,z)^2}{Y(k,\xi)^2} \\ &\quad \times G(z)H(z)Z(k,z), \end{aligned} \tag{3.5}$$

and after changing the order of integration in (3.5), we obtain

$$Z(k,x) = 1 - \int_{x_0}^x dz \mathcal{L}(k;x,z)Z(k,z), \tag{3.6}$$

where

$$\mathcal{L}(k;x,z) = G(z)H(z) \int_z^x d\xi \frac{Y(k,z)^2}{Y(k,\xi)^2}.$$

From (3.6) choosing $x_0 = \pm \infty$, we will obtain two linearly independent solutions denoted by Z_l and Z_r , respectively, satisfying

$$Z_l(k,x) = 1 + \int_x^\infty dz \mathcal{L}_l(k;x,z)Z_l(k,z), \tag{3.7}$$

$$Z_r(k,x) = 1 - \int_{-\infty}^x dz \mathcal{L}_r(k;x,z)Z_r(k,z), \tag{3.8}$$

where

$$\mathcal{L}_l(k;x,z) = \frac{G(z)}{2ik} \left[1 - \exp\left(2ik \int_x^z d\xi H(\xi)\right) \right],$$

$$\begin{aligned} &\mathcal{L}_r(k;x,z) \\ &= - \frac{G(z)}{2ik} \left[1 - \exp\left(-2ik \int_x^z d\xi H(\xi)\right) \right]. \end{aligned}$$

Note that for $k \in \overline{\mathbf{C}^+} \setminus \{0\}$, we have

$$|\mathcal{L}_l(k;x,z)| \leq |G(z)|/|k| \quad \text{and}$$

$$|\mathcal{L}_r(k;x,z)| \leq |G(z)|/|k|,$$

in the domains of integration given in (3.7) and (3.8), respectively. Thus, iterating (3.7) and (3.8) we obtain

$$\begin{aligned} |Z_l(k,x)| &\leq \exp\left(\frac{1}{|k|} \int_x^\infty dz |G(z)|\right) \\ &\leq \exp\left(\frac{1}{|k|} \int_{-\infty}^\infty dz |G(z)|\right), \quad k \in \overline{\mathbf{C}^+} \setminus \{0\}, \end{aligned} \tag{3.9}$$

$$\begin{aligned} |Z_r(k,x)| &\leq \exp\left(\frac{1}{|k|} \int_{-\infty}^x dz |G(z)|\right) \\ &\leq \exp\left(\frac{1}{|k|} \int_{-\infty}^\infty dz |G(z)|\right), \quad k \in \overline{\mathbf{C}^+} \setminus \{0\}. \end{aligned} \tag{3.10}$$

Hence, by the Weierstrass theorem used before, when $G \in L^1(\mathbf{R})$, for each x both $Z_l(k,x)$ and $Z_r(k,x)$ have continuous extensions in k to $\overline{\mathbf{C}^+} \setminus \{0\}$, which are analytic on \mathbf{C}^+ . Furthermore, on estimating $Z_l(k,x) - 1$ and $Z_r(k,x) - 1$ by iterating (3.7) and (3.8), we obtain $Z_l(k,x) = 1 + O(1/k)$ and $Z_r(k,x) = 1 + O(1/k)$

as $k \rightarrow \infty$ in $\overline{\mathbf{C}^+}$. Using (3.4), (3.9), and (3.10), for $k \in \overline{\mathbf{C}^+} \setminus \{0\}$, we obtain

$$|Z'_l(k,x)| \leq M \exp\left(\frac{1}{|k|} \int_{-\infty}^{\infty} dz |G(z)|\right) \times \int_{-\infty}^{\infty} d\xi |G(\xi)|,$$

$$|Z'_r(k,x)| \leq M \exp\left(\frac{1}{|k|} \int_{-\infty}^{\infty} dz |G(z)|\right) \times \int_{-\infty}^{\infty} d\xi |G(\xi)|,$$

where M is the constant given in (1.3). Hence, by the Weierstrass theorem, both $Z'_l(k,x)$ and $Z'_r(k,x)$ have continuous extensions to $k \in \overline{\mathbf{C}^+} \setminus \{0\}$, which are analytic on \mathbf{C}^+ , and $Z'_l(k,x) = O(1)$ and $Z'_r(k,x) = O(1)$ as $k \rightarrow \infty$ in $\overline{\mathbf{C}^+}$.

Since $Z'_l(k,\infty) = 0$, $Z_l(k,\infty) = 1$, $Z'_r(k,-\infty) = 0$, and $Z_r(k,-\infty) = 1$, using

$$Y_l(k,x)Z_l(k,x) = \exp\left(ikx - ik \int_0^{\infty} [1-H]\right) + o(1), \quad x \rightarrow \infty,$$

$$Y_r(k,x)Z_r(k,x) = \exp\left(-ikx - ik \int_{-\infty}^0 [1-H]\right) + o(1), \quad x \rightarrow -\infty,$$

we see that the Jost solutions defined in (2.5) are given by

$$f_l(k,x) = \exp\left(ik \int_0^{\infty} [1-H]\right) Y_l(k,x) Z_l(k,x), \tag{3.11}$$

$$f_r(k,x) = \exp\left(ik \int_{-\infty}^0 [1-H]\right) Y_r(k,x) Z_r(k,x). \tag{3.12}$$

Hence, from (2.5) it is seen that the physical solutions of (1.1) are given by

$$\psi_l(k,x) = T(k) \exp\left(ik \int_0^{\infty} [1-H]\right) \times Y_l(k,x) Z_l(k,x),$$

$$\psi_r(k,x) = T(k) \exp\left(ik \int_{-\infty}^0 [1-H]\right) \times Y_r(k,x) Z_r(k,x),$$

and from (2.6) we obtain

$$m_l(k,x) = \frac{1}{\sqrt{H(x)}} \exp\left(ik \int_x^{\infty} [1-H]\right) Z_l(k,x), \tag{3.13}$$

$$m_r(k,x) = \frac{1}{\sqrt{H(x)}} \exp\left(ik \int_{-\infty}^x [1-H]\right) Z_r(k,x). \tag{3.14}$$

Thus, since $m_l(k,x)$ and $m_r(k,x)$ are continuous in k even at $k = 0$, it follows that $Z_l(k,x)$ and $Z_r(k,x)$ are also continuous in k as $k \rightarrow 0$ in $\overline{\mathbf{C}^+}$.

From (3.13) and (3.14), as $k \rightarrow \infty$ we obtain

$$m_l(k,x) = \frac{\exp(ik \int_x^{\infty} [1-H])}{\sqrt{H(x)}} \times [1 + O(1/k)], \quad k \in \overline{\mathbf{C}^+},$$

$$m_r(k,x) = \frac{\exp(ik \int_{-\infty}^x [1-H])}{\sqrt{H(x)}} \times [1 + O(1/k)], \quad k \in \overline{\mathbf{C}^+}.$$

Note that both $m_l(k,x)$ and $m_r(k,x)$ remain bounded as $k \rightarrow \pm \infty$ in \mathbf{R} .

IV. SCATTERING MATRIX

In this section we show that the scattering matrix $S(k)$ is unitary and continuous for $k \in \mathbf{R}$ and study its asymptotics for small and large k . In this section the sufficient assumptions on the potentials are $P(x) < 1$ and $P, Q \in L^1_1(\mathbf{R})$. Although we use $Q \in L^2_1(\mathbf{R})$ for mathematical simplicity in our proof to obtain the properties of $S(k)$ as $k \rightarrow 0$, the condition $Q \in L^1_1(\mathbf{R})$ suffices as in the scattering theory^{13,14} for the Schrödinger equation in (1.4). The proofs under the assumption $Q \in L^1_1(\mathbf{R})$ will be given in Ref. 15.

From (2.1), (2.2), and (2.4) we obtain the expressions for the transmission coefficients,

$$T_l(k) = 1 + \frac{1}{2ik} \int_{-\infty}^{\infty} dy e^{-iky} \times [k^2 P(y) + Q(y)] \psi_l(k,y), \tag{4.1}$$

$$T_r(k) = 1 + \frac{1}{2ik} \int_{-\infty}^{\infty} dy e^{iky} \times [k^2 P(y) + Q(y)] \psi_r(k,y), \tag{4.2}$$

and the reflection coefficients

$$L(k) = \frac{1}{2ik} \int_{-\infty}^{\infty} dy e^{iky} [k^2 P(y) + Q(y)] \psi_l(k, y), \quad (4.3)$$

$$R(k) = \frac{1}{2ik} \int_{-\infty}^{\infty} dy e^{-iky} [k^2 P(y) + Q(y)] \psi_r(k, y). \quad (4.4)$$

From (2.4) through differentiation, we obtain

$$\psi_l'(k, x) = \begin{cases} ikT_l(k)e^{ikx} + o(1), & x \rightarrow \infty, \\ ike^{ikx} - ikL(k)e^{-ikx} + o(1), & x \rightarrow -\infty \end{cases}$$

and

$$\psi_r'(k, x) = \begin{cases} -ike^{-ikx} + ikR(k)e^{ikx} + o(1), & x \rightarrow \infty, \\ -ikT_r(k)e^{-ikx} + o(1), & x \rightarrow -\infty. \end{cases}$$

Let $[f; g] = fg' - f'g$ denote the Wronskian of f and g ; it can be shown that the Wronskian of any two solutions of (1.1) is independent of x . Hence, as $x \rightarrow \pm\infty$, from the Wronskian $[\psi_l(-k, x); \psi_l(k, x)]$ we obtain

$$T_l(k)T_l(-k) + L(k)L(-k) = 1, \quad k \in \mathbf{R}, \quad (4.5)$$

from $[\psi_r(-k, x); \psi_r(k, x)]$ we obtain

$$T_r(k)T_r(-k) + R(k)R(-k) = 1, \quad k \in \mathbf{R}, \quad (4.6)$$

and from $[\psi_l(k, x); \psi_r(-k, x)]$ we find

$$T_l(k)R(-k) + L(k)T_r(-k) = 0, \quad k \in \mathbf{R}. \quad (4.7)$$

From (4.5), (4.6), and (4.7), it is seen that the scattering matrix $S(k)$ defined in (2.3) is unitary and that we have

$$S(-k)^t = \overline{S(k)}^t = S(k)^{-1}, \quad k \in \mathbf{R},$$

where $S(k)^t$ denotes the transpose and $S(k)^{-1}$ the inverse of the matrix $S(k)$. As a consequence, the transmission and reflection coefficients cannot exceed one in absolute value for $k \in \mathbf{R}$.

The Wronskian $[\psi_l(k, x); \psi_r(k, x)]$ can be computed using (2.1) and (2.2) to obtain

$$[\psi_l(k, x); \psi_r(k, x)] = -2ikT_l(k) = -2ikT_r(k).$$

Therefore the transmission coefficients from the right and left coincide, and this common value will be denoted by $T(k)$:

$$T(k) = T_l(k) = T_r(k).$$

The Wronskian of the Faddeev solutions can be computed from (2.7) and (2.8) to obtain

$$[m_r(k, x); m_l(k, x)] = -2ikm_r(k, x)m_l(k, x) + 2ik/T(k). \quad (4.8)$$

In Sec. II, we have shown that $m_l(k, x)$, $m_r(k, x)$, $m_l'(k, x)$, and $m_r'(k, x)$ are continuous in k for $k \in \mathbf{C}^+$ and analytic in k for $k \in \mathbf{C}^+$. Thus, $k/T(k)$ is continuous in \mathbf{C}^+ and analytic in \mathbf{C}^+ . We can write (4.8) as

$$T(k) = \frac{2ik}{2ikm_l(k, x)m_r(k, x) + [m_r(k, x); m_l(k, x)]},$$

from which it is seen that $T(k) \neq 0$ for $k \in \mathbf{R} \setminus \{0\}$. Hence, using the unitarity of $S(k)$, we see that the reflection coefficients $R(k)$ and $L(k)$ cannot be equal to 1 in absolute value when $k \in \mathbf{R} \setminus \{0\}$.

From (4.1) and (4.2) we have

$$\begin{aligned} 1 - \frac{1}{T(k)} &= \frac{1}{2ik} \int_{-\infty}^{\infty} dy [k^2 P(y) + Q(y)] m_l(k, y), \\ 1 - \frac{1}{T(k)} &= \frac{1}{2ik} \int_{-\infty}^{\infty} dy [k^2 P(y) + Q(y)] m_r(k, y). \end{aligned} \quad (4.9)$$

Similarly, from (4.3) and (4.4) we have

$$\begin{aligned} \frac{L(k)}{T(k)} &= \frac{1}{2ik} \int_{-\infty}^{\infty} dy e^{2iky} [k^2 P(y) + Q(y)] m_l(k, y), \\ \frac{R(k)}{T(k)} &= \frac{1}{2ik} \int_{-\infty}^{\infty} dy e^{-2iky} [k^2 P(y) + Q(y)] m_r(k, y). \end{aligned} \quad (4.10)$$

There are two cases to consider; namely, the case $\int_{-\infty}^{\infty} dy Q(y)m_l(0, y) \neq 0$, which is the generic case, and the case $\int_{-\infty}^{\infty} dy Q(y)m_l(0, y) = 0$, which is the exceptional case. In the generic case, as $k \rightarrow 0$ from \mathbf{C}^+ , using Proposition 2.2, from (4.9) we obtain

$$\frac{1}{T(k)} = 1 - \frac{1}{2ik} \int_{-\infty}^{\infty} dy Q(y)m_l(0, y) + o(1/k),$$

and hence

$$T(k) = \frac{-2ik}{\int_{-\infty}^{\infty} dy Q(y)m_l(0, y)} + o(k), \quad k \in \overline{\mathbf{C}^+}.$$

Thus, since $T(k)$ vanishes linearly as $k \rightarrow 0$, the quantity $k/T(k)$ does not have a zero at $k = 0$ in the generic case. From (4.9) and (4.10) we obtain

$$\frac{L(k) + 1}{T(k)} = 1 + \frac{1}{2ik} \int_{-\infty}^{\infty} dy [e^{2iky} - 1] \times [k^2 P(y) + Q(y)] m_l(k, y),$$

$$\frac{R(k) + 1}{T(k)} = 1 + \frac{1}{2ik} \int_{-\infty}^{\infty} dy [e^{-2iky} - 1] \times [k^2 P(y) + Q(y)] m_r(k, y),$$

and hence, as $k \rightarrow 0$ we have

$$\frac{L(k) + 1}{T(k)} = 1 + \int_{-\infty}^{\infty} dy y Q(y) m_l(0, y) + o(1), \quad k \in \mathbf{R},$$

$$\frac{R(k) + 1}{T(k)} = 1 - \int_{-\infty}^{\infty} dy y Q(y) m_r(0, y) + o(1), \quad k \in \mathbf{R},$$

where the convergence of the integrals above can be seen from Proposition 2.2 and from the assumption $Q \in L^1_2(\mathbf{R})$. Thus, in the generic case we have

$$L(k) = -1 + kc_l + o(k), \quad k \in \mathbf{R},$$

$$R(k) = -1 + kc_r + o(k), \quad k \in \mathbf{R},$$

where c_l and c_r are the constants given by

$$c_l = \frac{-2i[1 + \int_{-\infty}^{\infty} dy y Q(y) m_l(0, y)]}{\int_{-\infty}^{\infty} dy Q(y) m_l(0, y)},$$

$$c_r = \frac{-2i[1 - \int_{-\infty}^{\infty} dy y Q(y) m_r(0, y)]}{\int_{-\infty}^{\infty} dy Q(y) m_r(0, y)}.$$

Assuming $Q \in L^1_2(\mathbf{R})$, we obtain by differentiating (2.9) and (2.10) with respect to k that $m_l(k, x)$ and $m_r(k, x)$ are continuously differentiable with respect to k on \mathbf{R} . Letting $k \rightarrow 0$ we have

$$\begin{aligned} \frac{\partial m_l(0, x)}{\partial k} &= \int_x^{\infty} dy (y - x) Q(y) \frac{\partial m_l(0, y)}{\partial k} \\ &= i \int_x^{\infty} dy (y - x)^2 Q(y) m_l(0, y), \end{aligned}$$

$$\begin{aligned} \frac{\partial m_r(0, x)}{\partial k} &= \int_{-\infty}^x dy (x - y) Q(y) \frac{\partial m_r(0, y)}{\partial k} \\ &= i \int_{-\infty}^x dy (x - y)^2 Q(y) m_r(0, y). \end{aligned}$$

From (4.9) we then obtain

$$\begin{aligned} \left[\frac{d}{dk} \frac{k}{T(k)} \right]_{k=0} &= 1 - \frac{1}{2i} \int_{-\infty}^{\infty} dy Q(y) \frac{\partial m_l(0, y)}{\partial k} \\ &= 1 - \frac{1}{2i} \int_{-\infty}^{\infty} dy Q(y) \frac{\partial m_r(0, y)}{\partial k}. \end{aligned}$$

Thus, in the exceptional case we have, as $k \rightarrow 0$,

$$\begin{aligned} \frac{1}{T(k)} &= 1 - \frac{1}{2i} \int_{-\infty}^{\infty} dy Q(y) \frac{\partial m_l(0, y)}{\partial k} \\ &+ o(1), \quad k \in \overline{\mathbf{C}^+}, \end{aligned}$$

and hence $T(0) \neq 0$ and, as $k \rightarrow 0$,

$$\begin{aligned} T(k) &= \frac{1}{1 - (1/2i) \int_{-\infty}^{\infty} dy Q(y) [\partial m_l(0, y) / \partial k]} \\ &+ o(1), \quad k \in \overline{\mathbf{C}^+}. \end{aligned}$$

In the exceptional case, since $T(0) \neq 0$, the quantity $k/T(k)$ has a simple zero at $k = 0$.

From the preceding analysis it is seen that $L(k)$ and $R(k)$ are continuous for $k \in \mathbf{R}$ and $T(k)$ is continuous for $k \in \overline{\mathbf{C}^+}$. In fact, in the generic case $L(0) = R(0) = -1$ and $T(0) = 0$, while $R(0)$, $L(0)$, and $T(0)$ are nonzero in the exceptional case. Thus, in both the generic and exceptional cases, when $Q \in L^1_2(\mathbf{R})$, the continuity of $S(k)$ is also valid at the point $k = 0$. By using the method of Ref. 14 it is possible to prove¹⁵ the continuity of $S(k)$ at $k = 0$ under the weaker assumption $Q \in L^1_1(\mathbf{R})$, but for mathematical simplicity, in the above analysis we have assumed that $Q \in L^1_2(\mathbf{R})$.

Now let us study the large k asymptotics of the scattering matrix. From (3.7), (3.8), (3.11), and (3.12), we obtain

$$\begin{aligned}
& \sqrt{H(x)} f_l(k, x) \exp\left(-ik \int_x^\infty [1-H]\right) \\
&= e^{ikx} \left[1 + \int_x^\infty dz \frac{G(z)Z_l(k, z)}{2ik} \right] \\
&\quad - e^{-ikx} \exp\left(2ik \int_0^x [1-H]\right) \int_x^\infty dz \\
&\quad \times \frac{G(z)Z_l(k, z)}{2ik} \exp\left(2ikz - 2ik \int_0^z [1-H]\right)
\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}
& \sqrt{H(x)} f_r(k, x) \exp\left(-ik \int_{-\infty}^x [1-H]\right) \\
&= e^{-ikx} \left[1 + \int_{-\infty}^x dz \frac{G(z)Z_r(k, z)}{2ik} \right] \\
&\quad - e^{ikx} \exp\left(-2ik \int_0^x [1-H]\right) \\
&\quad \times \int_{-\infty}^x dz \frac{G(z)Z_r(k, z)}{2ik} \\
&\quad \times \exp\left(-2ikz + 2ik \int_0^z [1-H]\right).
\end{aligned} \tag{4.12}$$

Then from (4.11) and (4.12) the transmission and reflection coefficients are obtained as

$$\begin{aligned}
\frac{1}{T(k)} &= \exp\left(ik \int_{-\infty}^\infty [1-H]\right) \\
&\quad \times \left[1 + \int_{-\infty}^\infty dz \frac{G(z)Z_l(k, z)}{2ik} \right], \\
\frac{1}{\overline{T}(k)} &= \exp\left(ik \int_{-\infty}^\infty [1-H]\right) \\
&\quad \times \left[1 + \int_{-\infty}^\infty dz \frac{G(z)Z_r(k, z)}{2ik} \right], \\
\frac{L(k)}{T(k)} &= -\exp\left(ik \int_0^\infty [1-H] - ik \int_{-\infty}^0 \right) \\
&\quad \times [1-H] \int_{-\infty}^\infty dz \frac{G(z)Z_l(k, z)}{2ik} \\
&\quad \times \exp\left(2ikz - 2ik \int_0^z [1-H]\right),
\end{aligned}$$

$$\begin{aligned}
\frac{R(k)}{\overline{T}(k)} &= -\exp\left(ik \int_{-\infty}^0 [1-H] - ik \int_0^\infty \right) \\
&\quad \times [1-H] \int_{-\infty}^\infty dz \frac{G(z)Z_r(k, z)}{2ik} \\
&\quad \times \exp\left(-2ikz + 2ik \int_0^z [1-H]\right).
\end{aligned}$$

From the above expressions, as $|k| \rightarrow \infty$ we obtain

$$\begin{aligned}
T(k) &= \exp\left(-ik \int_{-\infty}^\infty [1-H]\right) \left[1 \right. \\
&\quad \left. - \int_{-\infty}^\infty dz \frac{G(z)}{2ik} + O(1/k^2) \right], \quad k \in \overline{\mathbf{C}^+}, \\
L(k) &= -\exp\left(-2ik \int_{-\infty}^0 [1-H]\right) \left[\int_{-\infty}^\infty dz \right. \\
&\quad \left. \times \frac{G(z)}{2ik} \exp\left(2ikz - 2ik \int_0^z [1-H]\right) \right] \\
&\quad + O(1/k^2), \quad k \in \mathbf{R}, \\
R(k) &= -\exp\left(-2ik \int_0^\infty [1-H]\right) \left[\int_{-\infty}^\infty dz \right. \\
&\quad \left. \times \frac{G(z)}{2ik} \exp\left(-2ikz + 2ik \int_0^z [1-H]\right) \right] \\
&\quad + O(1/k^2), \quad k \in \mathbf{R}.
\end{aligned} \tag{4.13}$$

V. BOUND STATES

In this section we study the bound-state solutions of (1.1). We assume that $P(x)$ and $Q(x)$ are bounded, $P, Q \in L^1_+(\mathbf{R})$, and $P(x) < 1$. Then, a multiplication by $Q(x)$ or by $1 - P(x)$ is a bounded operator on $L^2(\mathbf{R})$. By definition, a bound-state solution of (1.1) is a solution $\psi(k, x)$ belonging to $L^2(\mathbf{R})$ such that $\psi''(k, \cdot) + [k^2 \times (1 - P(\cdot)) - Q(\cdot)]\psi(k, \cdot)$ also belongs to $L^2(\mathbf{R})$. Due to the boundedness of $P(x)$ and $Q(x)$, the bound-state solutions of (1.1) thus belong to the domain of the Hamiltonian $\mathbf{H}_0 = -d^2/dx^2$.

Proposition 5.1: The bound-state energies for (1.1) correspond to the zeros of $k/T(k)$ in \mathbf{C}^+ and can only occur on the imaginary axis in \mathbf{C}^+ . There is never a bound state at zero energy.

Proof: Bound states when $k^2 > 0$ are ruled out owing to the asymptotic behavior of the Jost solutions and their complex conjugates [see (3.11) and (3.12)]. Note that $f_l(k, x)$ and $f_r(k, x)$ are linearly independent and that no

linear combination of them can lie in $L^2(\mathbf{R})$. Thus there cannot be any bound states when $k \in \mathbf{R} \setminus \{0\}$.

The Wronskian identity

$$[f_l(k,x); f_r(k,x)] = -2ik/T(k) \tag{5.1}$$

derived from (4.8) implies that, for $k \in \mathbf{C}^+$, the Jost solutions $f_l(k,x)$ and $f_r(k,x)$ are linearly dependent if and only if $T(k)$ has a pole at k . As seen from (3.11) and (3.12), $f_l(k,x)$ vanishes exponentially as $x \rightarrow +\infty$ and $f_r(k,x)$ vanishes exponentially as $x \rightarrow -\infty$. Thus, whenever $f_l(k,x)$ and $f_r(k,x)$ are linearly dependent, as $x \rightarrow \pm\infty$, we obtain an exponentially decaying solution of (1.1) and hence there is a bound state at k . On the other hand, if $T(k)$ does not have a pole at k , then since we are in the limit-point case at both endpoints $\pm\infty$, any solution that is square-integrable at $-\infty[+\infty]$ must be a multiple of $f_r(k,x)$ [$f_l(k,x)$]. Hence, if any nontrivial combination of the Jost solutions were in $L^2(\mathbf{R})$, then it would have to be a multiple of both $f_l(k,x)$ and $f_r(k,x)$. Since $f_l(k,x)$ and $f_r(k,x)$ are linearly independent, this is impossible and hence $-k^2$ cannot be a bound state.

As the analysis in Sec. IV shows, for $k \in \mathbf{R}$, the Wronskian in (5.1) is nonzero with only one exception; namely, at $k = 0$ in the exceptional case. In the generic case, the Wronskian in (5.1) is nonzero, even at $k = 0$. However, $k = 0$ in either case does not correspond to a bound state. This can be seen by noting that for $k = 0$, (1.1) reduces to the ordinary Schrödinger equation at $k = 0$, which is given by $\psi'' = Q(x)\psi$, and it is known that the ordinary Schrödinger equation does not have a bound-state solution at zero energy^{4,13,16} when $Q \in L^1_+(\mathbf{R})$.

In order to prove that the bound states can only occur on the imaginary axis in \mathbf{C}^+ , we proceed as follows. Since $P(x)$ is real, from (1.1) we have

$$\begin{aligned} \langle \psi, \mathbf{H}\psi \rangle &= \langle \psi, k^2(1 - P)\psi \rangle \\ &= k^2 \int_{-\infty}^{\infty} dy [1 - P(y)] |\psi(k,y)|^2, \end{aligned} \tag{5.2}$$

$$\begin{aligned} \langle \mathbf{H}\psi, \psi \rangle &= \langle k^2(1 - P)\psi, \psi \rangle \\ &= \bar{k}^2 \int_{-\infty}^{\infty} dy [1 - P(y)] |\psi(k,y)|^2, \end{aligned} \tag{5.3}$$

where $\mathbf{H} = -d^2/dx^2 + Q(x)$. Because $Q(x)$ is real, \mathbf{H} is a self-adjoint operator; using $P(x) < 1$, from (5.2) and (5.3), we see that at a bound state we have $k^2 = \bar{k}^2$, which can occur only when k is on the real axis or on the imaginary axis. However, above we have already excluded bound states for real k . ■

Proposition 5.2: Each zero of $k/T(k)$ in $\overline{\mathbf{C}^+}$ is a simple zero.

Proof: From the analysis in Sec. IV we know that $k/T(k)$ has either no zeros at $k = 0$ (generic case) or has a simple zero at $k = 0$ (exceptional case). Thus let us consider $k \in \overline{\mathbf{C}^+} \setminus \{0\}$. Let an overdot indicate the derivative with respect to k . Then, from (1.1), we obtain the identities

$$\begin{aligned} \frac{d}{dx} [\dot{f}_l(k,x); f_r(k,x)] \\ = 2k[1 - P(x)] f_l(k,x) f_r(k,x), \end{aligned} \tag{5.4}$$

$$\begin{aligned} \frac{d}{dx} [f_l(k,x); \dot{f}_r(k,x)] \\ = -2k[1 - P(x)] f_l(k,x) f_r(k,x). \end{aligned} \tag{5.5}$$

Adding (5.4) and (5.5) we see that

$$\begin{aligned} \frac{\partial}{\partial k} [f_l(k,x); f_r(k,x)] &= [\dot{f}_l(k,x); f_r(k,x)] \\ &\quad + [f_l(k,x); \dot{f}_r(k,x)] \end{aligned} \tag{5.6}$$

is independent of x . From (5.1) it is seen that the expression in (5.6) is equal to $-(d/dk)[2ik/T(k)]$. Hence, in order to show that each zero of $k/T(k)$ is a simple zero in $\overline{\mathbf{C}^+}$, it is enough to show that the right-hand side of (5.6) does not vanish at a bound-state energy. At the bound state $k = i\beta$, the functions $f_l(i\beta,x)$, $f_r(i\beta,x)$, $f'_l(i\beta,x)$, and $f'_r(i\beta,x)$ all vanish as $x \rightarrow \pm\infty$, and $f_r(i\beta,x) = c(\beta)f_l(i\beta,x)$ for a nonzero constant $c(\beta)$. Thus, from (5.4) and (5.5), using the fact that $[\dot{f}_l(k,x); f_r(k,x)]$ vanishes as $x \rightarrow +\infty$ and that $[f_l(k,x); \dot{f}_r(k,x)]$ vanishes as $x \rightarrow -\infty$, we obtain

$$\begin{aligned} [\dot{f}_l(i\beta,x); f_r(i\beta,x)] \\ = -2i\beta c(\beta) \int_x^{\infty} dy [1 - P(y)] f_l(i\beta,y)^2, \end{aligned}$$

$$\begin{aligned} [f_l(i\beta,x); \dot{f}_r(i\beta,x)] \\ = -2i\beta c(\beta) \int_{-\infty}^x dy [1 - P(y)] f_l(i\beta,y)^2, \end{aligned}$$

and hence

$$\begin{aligned} [\dot{f}_l(i\beta,x); f_r(i\beta,x)] + [f_l(i\beta,x); \dot{f}_r(i\beta,x)] \\ = -2i\beta c(\beta) \int_{-\infty}^{\infty} dy [1 - P(y)] f_l(i\beta,y)^2. \end{aligned} \tag{5.7}$$

Due to the fact that $f_l(-\bar{k},x) = \overline{f_l(k,x)}$ for $k \in \overline{\mathbf{C}^+}$, it follows that $f_l(i\beta,x)$, $f_r(i\beta,x)$, and $c(\beta)$ are all real. Thus

the integral in (5.7) cannot be zero. Hence, comparing (5.6) and (5.7), we see that $-(d/dk)[2ik/T(k)]$ is nonzero at a bound state and thus the zeros of $k/T(k)$ in \mathbb{C}^+ are simple. ■

Proposition 5.3: The number of bound states for (1.1) is finite.

Proof: Note that $1/T(k)$ cannot vanish on the real axis because $|T(k)| < 1$ for $k \in \mathbb{R}$. Due to the analyticity of $f_l(k,x), f_r(k,x), f_l'(k,x)$, and $f_r'(k,x)$ for $k \in \mathbb{C}^+$, as seen from (5.1), $k/T(k)$ is also analytic in \mathbb{C}^+ , and hence $k/T(k)$ can have only isolated zeros in \mathbb{C}^+ . As seen from (4.13), as $k \rightarrow \infty$ in \mathbb{C}^+ , the quantity

$$\frac{ik \exp(-ik \int_{-\infty}^{\infty} [1-H])}{T(k)}$$

grows like $|k|$ in absolute value and hence $k/T(k)$ cannot have zeros for large enough $|k|$. Furthermore, as seen from the analysis in Sec. IV, the quantity $k/T(k)$ either has an isolated simple zero at $k=0$ (exceptional case) or no zero at $k=0$ (generic case). Hence $k/T(k)$ can only have isolated zeros in a bounded region of \mathbb{C}^+ , and by Proposition 5.2 each zero of $k/T(k)$ in \mathbb{C}^+ is simple. Hence, by Proposition 5.1 and the analyticity of $k/T(k)$ in \mathbb{C}^+ , the number of bound states, which is equal to the number of zeros of $k/T(k)$ in \mathbb{C}^+ , must be finite. ■

When $Q(x) \geq 0$, we will show that there cannot be a bound state. Note that for $Q(x) \geq 0, 1 - P(x) > 0$, and $k^2 < 0$, using the fact the $H_0 = -d^2/dx^2$ is a non-negative self-adjoint operator, we see from (1.1) that

$$-\langle H_0 \psi, \psi \rangle + k^2 \|\psi \sqrt{1-P}\|^2 = \langle Q \psi, \psi \rangle. \tag{5.8}$$

Hence, the left-hand side of (5.8) is nonpositive and its right-hand side is non-negative, which can only occur when $\psi \equiv 0$. Thus, there cannot be any bound states when $Q(x) \geq 0$.

Let $\mathcal{N}(Q,P)$ denote the number of bound states for (1.1), which is the same as the number of discrete eigenvalues of (1.1). The next result shows that $\mathcal{N}(Q,P) = \mathcal{N}(Q,P=0)$, hence, if $Q(x)$ has a negative part and thus the possibility of bound states exists, the number of bound states of (1.1) does not depend on $P(x)$.

Proposition 5.4: The number of bound states for (1.1) is independent of $P(x)$.

Proof: In order to prove that $\mathcal{N}(Q,P) = \mathcal{N}(Q,P=0)$, we will use a variant of the Birman–Schwinger kernel.¹⁷ Let $Q(x) = Q_+(x) - Q_-(x)$, where $Q_+(x) = \max\{Q(x), 0\}$ and $Q_-(x) = \max\{-Q(x), 0\}$. Let $\varphi = \sqrt{Q_-} \psi$, and let $k = i\beta$ so that $k^2 = -\beta^2$; note that $\beta > 0$ at a bound state because, as shown in Proposition 5.1, the bound states can only occur when k is on the imaginary axis in \mathbb{C}^+ . Then, we can write (1.1) as

$$\varphi = \mathcal{K}_\beta \varphi, \tag{5.9}$$

where

$$\mathcal{K}_\beta = \sqrt{Q_-} \left[-\frac{d^2}{dx^2} + Q_+ + \beta^2(1-P) \right]^{-1} \sqrt{Q_-}$$

is a version of the Birman–Schwinger kernel. The operator \mathcal{K}_β is positive and self-adjoint. It is also compact, as the following inequalities show. Let $P_{\max} = \sup_{x \in \mathbb{R}} P(x) < 1$ and $P_{\min} = \inf_{x \in \mathbb{R}} P(x)$. Then, in the sense of operator inequalities, we have

$$\begin{aligned} \sqrt{Q_-} \left[-\frac{d^2}{dx^2} + Q_+ + \beta^2(1 - P_{\min}) \right]^{-1} \sqrt{Q_-} &\leq \mathcal{K}_\beta \\ &\leq \sqrt{Q_-} \left[-\frac{d^2}{dx^2} + Q_+ + \beta^2(1 - P_{\max}) \right]^{-1} \sqrt{Q_-}, \end{aligned} \tag{5.10}$$

and since $Q_+(x) \geq 0$, we have

$$\begin{aligned} \sqrt{Q_-} \left[-\frac{d^2}{dx^2} + Q_+ + \beta^2(1 - P_{\min}) \right]^{-1} \sqrt{Q_-} \\ \leq \sqrt{Q_-} \left[-\frac{d^2}{dx^2} + \beta^2(1 - P_{\max}) \right]^{-1} \sqrt{Q_-}. \end{aligned} \tag{5.11}$$

The operator appearing on the right-hand side in (5.11) contains the kernel $\sqrt{Q_-(x)}(2\alpha)^{-1}e^{-\alpha|x-y|} \times \sqrt{Q_-(y)}$, where $\alpha = \beta \sqrt{1 - P_{\max}}$, and, as a consequence, is Hilbert–Schmidt because $Q_- \in L^1(\mathbb{R})$. This implies via (5.11) and (5.10) that \mathcal{K}_β is compact. The latter follows from the fact that if A and B are positive operators such that $A \leq B$, then $\dim E_{(a,\infty)}(A) \leq \dim E_{(a,\infty)}(B)$ for every $a > 0$. Here $E_{(a,\infty)}(A)$ denotes the spectral projection of A for the interval (a, ∞) . For if $\dim E_{(a,\infty)}(A) > \dim E_{(a,\infty)}(B)$ then there would exist a unit vector Φ in the range of $E_{(a,\infty)}(A)$, which is perpendicular to the range of $E_{(a,\infty)}(B)$. Then $(\Phi, A\Phi) > a$ while $(\Phi, B\Phi) \leq a$, contradicting $A \leq B$. By the spectral theorem it follows that A is compact if B is compact.

Returning to (5.9) we see that \mathcal{K}_β has eigenvalue 1 if and only if $-\beta^2$ is an eigenvalue of (1.1). Moreover, as functions of β the eigenvalues of \mathcal{K}_β are strictly decreasing and approach zero as $\beta \rightarrow +\infty$. Hence, if $\beta_0 > 0$ is fixed, the number of eigenvalues of \mathcal{K}_{β_0} that are strictly greater than 1 is equal to the number of eigenvalues of (1.1) which are strictly less than $-\beta_0^2$. Since $\beta_0 > 0$ is arbitrary, (5.10) immediately translates into

$$\mathcal{N}(Q, P_{\min}) \leq \mathcal{N}(Q, P) \leq \mathcal{N}(Q, P_{\max}).$$

However, $\mathcal{N}(Q, P_{\min}) = \mathcal{N}(Q, 0) = \mathcal{N}(Q, P_{\max})$ for if $-\beta^2$ is an eigenvalue of (1.1) with $P = 0$, then $-\beta^2/[1 - P_{\max}]$ is an eigenvalue of (1.1) with $P = P_{\max}$ and $-\beta^2/[1 - P_{\min}]$ is an eigenvalue of (1.1) with $P = P_{\min}$. Thus the proof is complete. ■

For the ordinary Schrödinger equation, the Levinson theorem specifies the relation between the number of bound states and the phase of $T(k)$. The Levinson theorem for (1.1) can be stated for the phase of $T(k)\exp(ik\int_{-\infty}^{\infty}[1-H])$. As seen from (4.13), $T(k)\exp(ik\int_{-\infty}^{\infty}[1-H])$ converges to 1 as $k \rightarrow \infty$ in \mathbb{C}^+ , and hence, we can use the argument principle for the contour that consists of the real axis and the semicircle of infinite radius in \mathbb{C}^+ . Let \mathcal{N} be the number of bound states for (1.1) and let $\Theta(k)$ denote the phase of $T(k)\exp(ik\int_{-\infty}^{\infty}[1-H])$. One then obtains the Levinson theorem for (1.1)

$$\Theta(0+) - \Theta(+\infty) = \begin{cases} \pi(\mathcal{N} - \frac{1}{2}), & \text{generic case,} \\ \pi\mathcal{N}, & \text{exceptional case.} \end{cases}$$

As in the case of the regular Schrödinger equation in one dimension, since in the Levinson theorem the phases differ by $\frac{1}{2}\pi$ in the generic and exceptional cases, we will say that there is a half-bound state at zero energy in the exceptional case.

VI. RECOVERY OF $Q(x)$

In this section the sufficient assumptions are $P(x) < 1$ and $P, Q \in L^1_+(\mathbb{R})$. We will show that the potential $Q(x)$

can be recovered when the scattering matrix $S(k)$ and the other potential $P(x)$ are known. In fact, one of the reflection coefficients determines the potential $Q(x)$. When there are bound states, the norming constants must be specified for each bound state in order to obtain the potential uniquely.

Since k appears as k^2 in (1.1), $\psi_l(-k, x)$ and $\psi_r(-k, x)$ are also solutions of (1.1) whenever $\psi_l(k, x)$ and $\psi_r(k, x)$ are the physical solutions. Using (2.1) and (2.2) as well as (4.9) and (4.10), the solution vectors

$$\psi(-k, x) = \begin{bmatrix} \psi_l(-k, x) \\ \psi_r(-k, x) \end{bmatrix}$$

and

$$\psi(k, x) = \begin{bmatrix} \psi_l(k, x) \\ \psi_r(k, x) \end{bmatrix}$$

are found to be related to each other as

$$\psi(-k, x) = S(-k)'q\psi(k, x), \quad k \in \mathbb{R}, \quad (6.1)$$

where $q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Letting $Z(k, x) = \begin{bmatrix} Z_l(k, x) \\ Z_r(k, x) \end{bmatrix}$ and using (4.7), (3.4), (3.5), (3.22), and (3.23), we can write (6.1) as

$$Z(-k, x) = \Lambda(k, x)qZ(k, x), \quad k \in \mathbb{R}, \quad (6.2)$$

where

$$\Lambda(k, x) = \begin{bmatrix} T(k)\exp\left(ik\int_{-\infty}^{\infty}[1-H]\right) & -R(k)\exp\left(2ikx + 2ik\int_x^{\infty}[1-H]\right) \\ -L(k)\exp\left(-2ikx + 2ik\int_{-\infty}^x[1-H]\right) & T(k)\exp\left(ik\int_{-\infty}^{\infty}[1-H]\right) \end{bmatrix}.$$

Let $\hat{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. From Sec. III it is known that $Z(k, x)$ is continuous in $k \in \mathbb{C}^+$ and has an analytic extension in k to \mathbb{C}^+ for each x , while $Z(k, x) - \hat{1} = O(1/k)$ as $k \rightarrow \infty$ in \mathbb{C}^+ . The continuity of $Z(k, x)$ at $k = 0$ can be seen from (3.24) and (3.25) and the continuity of $m_l(k, x)$ and of $m_r(k, x)$. Similarly, $Z(-k, x)$ is continuous in $k \in \mathbb{C}^-$ and has an analytic extension in k to \mathbb{C}^- for each x , and $Z(-k, x) - \hat{1} = O(1/k)$ as $k \rightarrow \infty$ in \mathbb{C}^- . Hence, solving (6.2) for $Z(-k, x)$ and $Z(k, x)$ when $\Lambda(k, x)$ is known constitutes a Riemann–Hilbert problem.^{6,10,18} There are various methods to solve this Riemann–Hilbert problem, such as the Marchenko method,^{2,4} the Gel'fand–Levitan method,⁷ Newton's generalization of the Marchenko and Gel'fand–Levitan methods,⁶ the

Wiener–Hopf factorization method,⁸ and the Muskhelishvili–Vekua method.^{9–11}

Below we will solve (6.2) by the Marchenko method. From (6.2), letting $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, we obtain

$$Z(-k, x) - \hat{1} = [\Lambda(k, x) - \mathbf{I}]q[Z(k, x) - \hat{1}] + q[Z(k, x) - \hat{1}] + [\Lambda(k, x) - \mathbf{I}]\hat{1}, \quad (6.3)$$

and using the Fourier transform, we transform (6.3) into

$$B(x,y) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{iky} [\Lambda(k,x) - \mathbf{I}] q [Z(k,x) - \hat{1}] \quad Z_l(k,x) = 1 + \int_0^{\infty} dy B_l(x,y) e^{iky}, \quad (6.9)$$

$$+ \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{iky} [\Lambda(k,x) - \mathbf{I}] \hat{1} + qB(x, -y), \quad Z_r(k,x) = 1 + \int_0^{\infty} dy B_r(k,y) e^{iky}. \quad (6.10)$$

where we have defined

$$B(x,y) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-iky} [Z(k,x) - \hat{1}].$$

As in Sec. V, we will use \mathcal{N} to denote the number of bound states that occur at $k = i\beta_1, \dots, i\beta_{\mathcal{N}}$. Note that from the analysis in Sec. IV, we have $\Lambda(k,x) = \mathbf{I} + O(1/k)$ as $k \rightarrow \pm\infty$. From (4.18) and the properties of $T(k)$ in \mathbf{C}^+ that are established in Sec. IV, it follows that $T(k) \exp(ik \int_{-\infty}^{\infty} [1-H])$ is continuous for $k \in \mathbf{C}^+$, is meromorphic for $k \in \mathbf{C}^+$ with poles at $k = i\beta_1, \dots, i\beta_{\mathcal{N}}$, and behaves like $1 + O(1/k)$ as $k \rightarrow \infty$ in \mathbf{C}^+ . Let

$$u(y) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{iky} \left[T(k) \times \exp\left(2ikx + 2ik \int_x^{\infty} [1-H]\right) e^{iky}, \right. \\ \left. \times \exp\left(ik \int_{-\infty}^{\infty} [1-H]\right) - 1 \right]. \quad (6.11)$$

Then we have

$$u(y) = - \sum_{j=1}^{\mathcal{N}} E_j e^{-\beta_j y}, \quad y > 0, \quad (6.5)$$

where

$$E_j = \frac{\exp(-\beta_j \int_{-\infty}^{\infty} [1-H])}{\int_{-\infty}^{\infty} dz [1-P(z)] f_l(i\beta_j z) f_r(i\beta_j z)}. \quad (6.6)$$

Note that in order to obtain (6.6), we have used (5.1) and (5.7) and the residues of $T(k)$ at the bound states.

From the analytic and asymptotic properties studied in Sec. III, we see that the functions $Z_l(k,x) - 1$ and $Z_r(k,x) - 1$ belong to the Hardy space of L^2 functions in k with analytic continuations to \mathbf{C}^+ and hence their Fourier transforms are L^2 functions in y with support in $[0, \infty)$; i.e., $B_l(x,y) = B_r(x,y) = 0$ for $y < 0$, where

$$B_l(x,y) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} [Z_l(k,x) - 1] e^{-iky}, \quad (6.7)$$

$$B_r(x,y) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} [Z_r(k,x) - 1] e^{-iky}, \quad (6.8)$$

and hence

From (3.26) it is seen that $B_l(x,y)$ and $B_r(x,y)$ are real. Introducing

$$g(x,y) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{iky} [\Lambda(k,x) - \mathbf{I}],$$

$$g_l(x,y) = - \int_{-\infty}^{\infty} \frac{dk}{2\pi} R(k) \times \exp\left(2ikx + 2ik \int_x^{\infty} [1-H]\right) e^{iky},$$

$$g_r(x,y) = - \int_{-\infty}^{\infty} \frac{dk}{2\pi} L(k) \times \exp\left(-2ikx + 2ik \int_{-\infty}^x [1-H]\right) e^{iky},$$

from (6.4) we derive the 2×1 system of integral equations,

$$B(x,y) = g(x,y) \hat{1} + qB(x, -y) + \int_{-\infty}^{\infty} dz g(x,y+z) qB(x,z), \quad y \in \mathbf{R}, \quad (6.12)$$

where we have

$$g(x,y) \hat{1} = \begin{bmatrix} g_l(x,y) \\ g_r(x,y) \end{bmatrix} + u(y) \hat{1}.$$

Note that for $y > 0$, (6.12) gives us the two scalar equations

$$B_l(x,y) = g_l(x,y) + u(y) + \int_0^\infty dz g_l(x,y+z)B_l(x,z) + \int_0^\infty dz u(y+z)B_r(x,z), \quad y > 0, \tag{6.13}$$

$$B_r(x,y) = g_r(x,y) + u(y) + \int_0^\infty dz g_r(x,y+z)B_r(x,z) + \int_0^\infty dz u(y+z)B_l(x,z), \quad y > 0. \tag{6.14}$$

Although (6.13) and (6.14) seem to be coupled at a first glance, using (6.5) and (6.6) and the fact that $f_l(k,x)$ and $f_r(k,x)$ are linearly dependent if $k = i\beta_j$ we will show that (6.13) and (6.14) can actually be uncoupled. As in Sec. V, let us use $c(\beta_j)$ to denote the proportionality constants at bound states; i.e., let $f_r(i\beta_j,x) = c(\beta_j)f_l(i\beta_j,x)$. Then, using (3.4), (3.5), (3.20), and (3.21), we obtain $Z_r(i\beta_j,x) = C(\beta_j,x)Z_l(i\beta_j,x)$, where we have defined

$$C(\beta_j,x) = c(\beta_j)\exp\left(-2\beta_j \int_0^x H\right) \times \exp\left(-\beta_j \int_0^\infty [1-H]\right) \times \exp\left(\beta_j \int_{-\infty}^0 [1-H]\right). \tag{6.15}$$

From (6.5), we have, for $y > 0$,

$$u(y) + \int_0^\infty dz u(y+z)B_r(x,z) = - \sum_{j=1}^N e^{-\beta_j y} E_j \left[1 + \int_0^\infty dz e^{-\beta_j z} B_r(x,z) \right] = - \sum_{j=1}^N e^{-\beta_j y} E_j Z_r(i\beta_j,x), \tag{6.16}$$

as well as

$$u(y) + \int_0^\infty dz u(y+z)B_l(x,z) = - \sum_{j=1}^N e^{-\beta_j y} E_j \left[1 + \int_0^\infty dz e^{-\beta_j z} B_l(x,z) \right] = - \sum_{j=1}^N e^{-\beta_j y} E_j Z_l(i\beta_j,x). \tag{6.17}$$

On the other hand, from (6.6) and (6.15) we have, for $y > 0$,

$$\sum_{j=1}^N \left[C(\beta_j,x) E_j e^{-\beta_j y} + \int_0^\infty dz C(\beta_j,x) \times E_j e^{-\beta_j(y+z)} B_l(x,z) \right] = \sum_{j=1}^N C(\beta_j,x) e^{-\beta_j y} E_j \left[1 + \int_0^\infty dz e^{-\beta_j z} B_l(x,z) \right] = \sum_{j=1}^N C(\beta_j,x) e^{-\beta_j y} E_j Z_l(i\beta_j,x), \tag{6.18}$$

as well as

$$\sum_{j=1}^N \left[\frac{1}{C(\beta_j,x)} E_j e^{-\beta_j y} + \int_0^\infty dz \frac{1}{C(\beta_j,x)} \times E_j e^{-\beta_j(y+z)} B_r(x,z) \right] = \sum_{j=1}^N \frac{1}{C(\beta_j,x)} e^{-\beta_j y} E_j \left[1 + \int_0^\infty dz e^{-\beta_j z} B_r(x,z) \right] = \sum_{j=1}^N \frac{1}{C(\beta_j,x)} e^{-\beta_j y} E_j Z_r(i\beta_j,x). \tag{6.19}$$

Now let

$$\Omega_l(x,y) = g_l(x,y) - \sum_{j=1}^N C(\beta_j,x) E_j e^{-\beta_j y},$$

$$\Omega_r(x,y) = g_r(x,y) - \sum_{j=1}^N \frac{1}{C(\beta_j,x)} E_j e^{-\beta_j y},$$

or, using (6.6) and (6.15),

$$\Omega_l(x,y) = g_l(x,y) - \sum_{j=1}^N \frac{e^{-\beta_j y} \exp(-2\beta_j \int_0^\infty [1-H]) \exp(-2\beta_j \int_0^x H)}{\int_{-\infty}^\infty dz [1-P(z)] f_l(i\beta_j,z)^2}, \tag{6.20}$$

$$\Omega_r(x,y) = g_r(x,y) - \sum_{j=1}^N \frac{e^{-\beta_j y} \exp(-2\beta_j \int_{-\infty}^0 [1-H]) \exp(2\beta_j \int_0^x H)}{\int_{-\infty}^\infty dz [1-P(z)] f_r(i\beta_j,z)^2}. \tag{6.21}$$

Using $Z_r(i\beta_j, x) = C(\beta_j, x)Z_l(i\beta_j, x)$ as well as (6.16), (6.17), (6.18), and (6.19), we can write (6.13) and (6.14) as

$$B_l(x, y) = \Omega_l(x, y) + \int_0^\infty dz \Omega_l(x, y + z)B_l(x, z), \quad y > 0, \quad (6.22)$$

$$B_r(x, y) = \Omega_r(x, y) + \int_0^\infty dz \Omega_r(x, y + z)B_r(x, z), \quad y > 0. \quad (6.23)$$

The equations given in (6.22) and (6.23) are the Marchenko equations. Note that the scattering data given in (6.20) and (6.21) indicate that in the presence of bound states, for the unique solvability of the Marchenko equations, at each bound state the constants $\kappa_l(\beta_j)$ and $\kappa_r(\beta_j)$ must be specified, where

$$\kappa_l(\beta_j) = \sqrt{\int_{-\infty}^{\infty} dz [1 - P(z)] f_l(i\beta_j, z)^2},$$

$$\kappa_r(\beta_j) = \sqrt{\int_{-\infty}^{\infty} dz [1 - P(z)] f_r(i\beta_j, z)^2}.$$

This is the counterpart of specifying the norming constants for the inverse problem⁴⁻⁶ for the regular Schrödinger equation.

Let us write (3.16) as

$$Z_l(k, x) - 1 = \int_x^\infty dz \mathcal{L}_l(k; x, z) + \int_x^\infty dz \mathcal{L}_l(k; x, z) \times [Z_l(k, z) - 1].$$

Taking the Fourier transform of both sides and using (3.13), (3.15), (6.7) as well as the realness of $B_l(x, y)$, we obtain, for $y > 0$,

$$B_l(x, y) = \int_x^\infty dz \int_{-\infty}^\infty \frac{dk}{4\pi k} G(z) \left[-\sin(ky) - \sin\left(-ky + 2k \int_x^z H\right) \right] + \int_x^\infty dz \int_0^\infty d\xi \int_{-\infty}^\infty \frac{dk}{4\pi k} G(z) B_l(z, \xi) \left[\sin(k\xi - ky) - \sin\left(k\xi - ky + 2k \int_x^z H\right) \right]. \quad (6.24)$$

Using $\int_{-\infty}^\infty dk [\sin(\gamma k)/k] = \pi\theta(\gamma)$, where $\theta(\gamma)$ is the sign function, we can write (6.24) as

$$B_l(x, y) = -\frac{1}{4} \int_x^\infty dz \left[\theta(y) + \theta\left(-y + 2 \int_x^z H\right) \right] G(z) + \frac{1}{4} \int_x^\infty dz \int_0^\infty d\xi \times \left[\theta(\xi - y) - \theta\left(\xi - y + 2 \int_x^z H\right) \right] G(z) B_l(z, \xi). \quad (6.25)$$

Similarly, one obtains,

$$B_r(x, y) = \frac{1}{4} \int_{-\infty}^x dz \left[-\theta(y) - \theta\left(-y + 2 \int_z^x H\right) \right] G(z) + \frac{1}{4} \int_{-\infty}^x dz \int_0^\infty d\xi \times \left[\theta(\xi - y) - \theta\left(\xi - y + 2 \int_z^x H\right) \right] G(z) B_r(z, \xi). \quad (6.26)$$

Hence, from (6.25) and (6.26), we obtain

$$B_l(x, 0+) = -\frac{1}{2} \int_x^\infty dz G(z),$$

$$B_r(x, 0+) = -\frac{1}{2} \int_{-\infty}^x dz G(z),$$

and thus we have

$$G(x) = 2 \frac{dB_l(x, 0+)}{dx} = -2 \frac{dB_r(x, 0+)}{dx}, \quad (6.27)$$

where the derivative exists for almost every $x \in \mathbf{R}$. As a

consequence, using (1.6), $Q(x)$ can be recovered from the solution of either of the Marchenko equations (6.22) and (6.23).

Let us define

$$\sigma_l(x) = \int_x^\infty dz |G(z)|,$$

$$\sigma_r(x) = \int_{-\infty}^x dz |G(z)|, \tag{6.28}$$

$$\tau_l(x) = \int_x^\infty dz \sigma_l(z),$$

$$\tau_r(x) = \int_{-\infty}^x dz \sigma_r(z). \tag{6.29}$$

Note that from (6.28) and (6.29), we obtain

$$\tau_l(x) = \int_x^\infty dz (z-x) |G(z)|,$$

and

$$\tau_r(x) = \int_{-\infty}^x dz (x-z) |G(z)|.$$

Proposition 6.1: If $G \in L^1(\mathbf{R})$, the integral equations (6.25) and (6.26) are uniquely solvable for $B_l(x,y)$ and $B_r(x,y)$, respectively. Furthermore, the solutions satisfy

$$|B_l(x,y)| \leq \frac{1}{2} \sigma_l(x + y/(2M)) e^{M[\tau_l(x) - \tau_l(x + y/(2M))]}, \tag{6.30}$$

$$|B_r(x,y)| \leq \frac{1}{2} \sigma_r(x - y/(2M)) e^{M[\tau_r(x) - \tau_r(x - y/(2M))]},$$

where M is the constant in (1.3), and $\sigma_b, \sigma_r, \tau_b,$ and τ_r are the quantities defined in (6.28) and (6.29).

Proof: Note that from (6.28) we have $(d/dx)\tau_l(x)^n = -n\tau_l(x)^{n-1}\sigma_l(x)$ for $n \geq 1$. Note also that $\sigma_l(x)$ and $\tau_l(x)$ are decreasing functions of x , and if $G \in L^1(\mathbf{R})$, we

have $\sigma_l(-\infty) < +\infty$ and $\tau_l(x) < +\infty$ for each x . Let us omit the subscript from $B_l(x,y)$ for simplicity. From (6.25) we have

$$B(x,y) = B_0(x,y) + \sum_{j=1}^\infty B_j(x,y),$$

where

$$B_0(x,y) = -\frac{1}{4} \int_x^\infty dz \left[\theta(y) + \theta\left(-y + 2 \int_x^z H\right) \right] G(z),$$

$$B_{n+1}(x,y) = \frac{1}{4} \int_x^\infty dz \int_0^\infty d\xi \left[\theta(\xi - y) - \theta\left(\xi - y + 2 \int_x^z H\right) \right] G(z) B_n(z,\xi), \quad n \geq 0.$$

Then we have

$$|B_n(x,y)| \leq \frac{M^n}{2n!} \sigma_l\left(x + \frac{y}{2M}\right) \left[\tau_l(x) - \tau_l\left(x + \frac{y}{2M}\right) \right]^n$$

for $n \geq 0$. The proof is by induction and as follows. Since $H(x) \leq M$, we have $-y + 2 \int_x^z H \leq -y + 2M(z-x)$. Thus the term $\theta(y) + \theta(-y + 2 \int_x^z H)$ vanishes when $-y + 2M(z-x) < 0$. Hence the integration for $B_0(x,y)$ vanishes when $z < x + y/(2M)$ and

$$|B_0(x,y)| \leq \frac{1}{4} \int_{x+y/(2M)}^\infty dz \left| \theta(y) + \theta\left(-y + 2 \int_x^z H\right) \right| |G(z)|$$

$$\leq \frac{2}{4} \int_{x+y/(2M)}^\infty dz |G(z)| = \frac{1}{2} \sigma_l\left(x + \frac{y}{2M}\right).$$

Assume for n , the induction hypothesis holds. Since $0 < H(x) \leq M$, $\theta(\xi - y) - \theta(\xi - y + 2 \int_x^z H)$ vanishes if $\xi - y > 0$ or if $\xi - y + 2M(z-x) < 0$. Thus we have

$$|B_{n+1}(x,y)| \leq \frac{1}{2} \int_x^\infty dz \int_0^\infty d\xi |G(z)| |B_n(z,\xi)| \leq \frac{M^n}{4n!} \int_x^\infty dz \int_0^\infty d\xi |G(z)| \sigma_l\left(z + \frac{\xi}{2M}\right) \left[\tau_l(z) - \tau_l\left(z + \frac{\xi}{2M}\right) \right]^n$$

$$= \frac{M^n}{4n!} \int_0^y d\xi \int_{x+(y-\xi)/(2M)}^\infty dz |G(z)| \sigma_l\left(z + \frac{\xi}{2M}\right) \left[\tau_l(z) - \tau_l\left(z + \frac{\xi}{2M}\right) \right]^n$$

$$\begin{aligned}
 &= \frac{M^n}{4n!} \int_0^y d\xi \int_{x+y/(2M)}^\infty du \left| G\left(u - \frac{\xi}{2M}\right) \right| \sigma_l(u) \left[\tau_l\left(u - \frac{\xi}{2M}\right) - \tau_l(u) \right]^n \\
 &< \frac{M^n}{4n!} \sigma_l\left(x + \frac{y}{2M}\right) \int_0^y d\xi \int_{x+y/(2M)}^\infty du \left| G\left(u - \frac{\xi}{2M}\right) \right| \left[\tau_l\left(u - \frac{\xi}{2M}\right) - \tau_l(u) \right]^n \\
 &= \frac{M^n}{4n!} \sigma_l\left(x + \frac{y}{2M}\right) \int_{x+y/(2M)}^\infty du \int_0^y d\xi \left| G\left(u - \frac{\xi}{2M}\right) \right| \left[\tau_l\left(u - \frac{\xi}{2M}\right) - \tau_l(u) \right]^n \\
 &< \frac{M^n}{4n!} \sigma_l\left(x + \frac{y}{2M}\right) \int_{x+y/(2M)}^\infty du \left[\tau_l\left(u - \frac{y}{2M}\right) - \tau_l(u) \right]^n \int_0^y d\xi \left| G\left(u - \frac{\xi}{2M}\right) \right| \\
 &= \frac{M^{n+1}}{2n!} \sigma_l\left(x + \frac{y}{2M}\right) \int_{x+y/(2M)}^\infty du \left[\tau_l\left(u - \frac{y}{2M}\right) - \tau_l(u) \right]^n \left[\sigma_l\left(u - \frac{y}{2M}\right) - \sigma_l(u) \right] \\
 &= \frac{1}{2} \frac{M^{n+1}}{(n+1)!} \sigma_l\left(x + \frac{y}{2M}\right) \left[\tau_l(x) - \tau_l\left(x + \frac{y}{2M}\right) \right]^{n+1}.
 \end{aligned}$$

Hence the induction proof is complete and we obtain (6.30). Thus (6.25) is uniquely solvable and the solution can be obtained using iteration. The proof for the unique solvability for (6.26) and the bound on its solution can be obtained in a similar way. ■

Proposition 6.2: If $G \in L^1_1(\mathbf{R})$, the solution $B_l(x,y)$ of (6.25) satisfies the partial differential equation,

$$\frac{\partial}{\partial x} \left[\frac{\partial}{\partial y} - \frac{1}{2H(x)} \frac{\partial}{\partial x} \right] B_l(x,y) = \frac{1}{2} G(x) B_l(x,y) \tag{6.31}$$

and the solution $B_r(x,y)$ of (6.26) satisfies the partial differential equation,

$$\frac{\partial}{\partial x} \left[\frac{\partial}{\partial y} + \frac{1}{2H(x)} \frac{\partial}{\partial x} \right] B_r(x,y) = -\frac{1}{2} G(x) B_r(x,y). \tag{6.32}$$

Proof: Let us first work on $B_l(x,y)$ of (6.25). Since $d\theta(x)/dx = 2\delta(x)$, where $\delta(x)$ is the Dirac delta distribution, we have, in the sense of distributions,

$$\begin{aligned}
 \frac{\partial}{\partial y} B_l(x,y) &= -\frac{1}{2} \int_x^\infty dz \left[\delta(y) \right. \\
 &\quad \left. - \delta\left(-y + 2 \int_x^z H\right) \right] G(z) + \frac{1}{2} \int_x^\infty dz \\
 &\quad \times \int_0^\infty d\xi \left[-\delta(\xi - y) \right. \\
 &\quad \left. + \delta\left(\xi - y + 2 \int_x^z H\right) \right] G(z) B_l(z,\xi),
 \end{aligned}$$

or, equivalently,

$$\begin{aligned}
 \frac{\partial}{\partial y} B_l(x,y) &= -\frac{1}{2} \int_x^\infty dz \left[\delta(y) - \delta\left(-y + 2 \int_x^z H\right) \right] \\
 &\quad \times G(z) + \frac{1}{2} \int_x^\infty dz G(z) \left[-B_l(z,y) \right. \\
 &\quad \left. + B_l\left(z,y - 2 \int_x^z H\right) \right].
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 \frac{\partial}{\partial x} B_l(x,y) &= H(x) \int_x^\infty dz \delta\left(-y + 2 \int_x^z H\right) G(z) \\
 &\quad + H(x) \int_x^\infty dz G(z) B_l\left(z,y - 2 \int_x^z H\right).
 \end{aligned}$$

Thus we have

$$\left[\frac{\partial}{\partial y} - \frac{1}{2H(x)} \frac{\partial}{\partial x} \right] B_l(x,y) = -\frac{1}{2} \int_x^\infty dz G(z) B_l(z,y), \tag{6.33}$$

from which (6.31) follows. The proof for $B_r(x,y)$ is similar. By differentiating (6.26) we obtain

$$\left[\frac{\partial}{\partial y} + \frac{1}{2H(x)} \frac{\partial}{\partial x} \right] B_r(x,y) = -\frac{1}{2} \int_{-\infty}^x dz G(z) B_r(z,y),$$

from which we obtain (6.32). ■

From (6.33), using (6.30), we have the estimate

$$\begin{aligned} \left| \left[\frac{\partial}{\partial y} - \frac{1}{2H(x)} \frac{\partial}{\partial x} \right] B_l(x,y) \right| &\leq \frac{1}{2} \int_x^\infty dz |G(z)| |B_l(z,y)| \\ &\leq \frac{1}{4} \int_x^\infty dz |G(z)| \sigma_l \left(x + \frac{y}{2M} \right) e^{M[\tau_l(z) - \tau_l(x + y/(2M))]} \\ &\leq \frac{1}{2} \sigma_l \left(x + \frac{y}{2M} \right) e^{M[\tau_l(x) - \tau_l(x + y/(2M))]} \int_x^\infty dz |G(z)| \\ &\leq \frac{1}{2} \sigma_l(x) \sigma_l \left(x + \frac{y}{2M} \right) e^{M[\tau_l(x) - \tau_l(x + y/(2M))]} \end{aligned}$$

The next theorem shows that the Marchenko equations given in (6.22) and (6.23) are uniquely solvable.

Theorem 6.3: Suppose the potential $P(x)$ is bounded below, $1 - H \in L^1(\mathbf{R})$, $P(x) < 1$, and $G \in L^1_+(\mathbf{R})$, where G is the quantity defined in (1.6). Then the Marchenko integral operators are self-adjoint and compact on $L^2(0, \infty)$, and the Marchenko integral equations (6.22) and (6.23) are uniquely solvable.

Proof: Note that the reflection coefficients $R(k)$ and $L(k)$ are continuous for $k \in \mathbf{R}$, are of $O(1/k)$ as $k \rightarrow \pm \infty$ and belong to $L^2(\mathbf{R})$. We will only prove the unique solvability of (6.22); the proof for (6.23) is similar. Let us introduce the linear operators \mathcal{G}_l , \mathcal{H}_l and Ω_l by

$$\begin{aligned} (\mathcal{G}_l B)(y) &= \int_0^\infty dz g_l(x, y+z) B(z), \\ (\mathcal{H}_l B)(y) &= - \sum_{j=1}^{\mathcal{N}} C(\beta_j, x) E_j e^{-\beta_j y} \int_0^\infty dz e^{-\beta_j z} B(z), \\ (\Omega_l B)(y) &= \int_0^\infty dz \Omega_l(x, y+z) B(z). \end{aligned}$$

Then \mathcal{G}_l is compact on $L^2(0, \infty)$ as a result of the lemma in the Appendix and \mathcal{H}_l is compact as a result of the square integrability of its kernel. Thus $\Omega_l = \mathcal{G}_l + \mathcal{H}_l$ is compact on $L^2(0, \infty)$.

From (4.8) it follows that $g_l(x, y)$ is real and from (6.20) it is seen that $\Omega_l(x, y)$ is real. Since in (6.22) y and z appear as $y+z$ in the argument of the kernel, the

Marchenko integral operator Ω_l has a symmetric kernel. Hence, since Ω_l is also bounded, it is self-adjoint.

The proof of the unique solvability of (6.22) is similar to the proof given in Ref. 19. Since Ω_l is a compact operator, it suffices to show that the homogeneous Marchenko equation $\eta(y) = (\Omega_l \eta)(y)$ has no nontrivial solutions; i.e., if $\eta(y)$ is a solution of

$$\begin{aligned} \eta(y) &= \int_0^\infty dz g_l(x, y+z) \eta(z) \\ &+ \sum_{j=1}^{\mathcal{N}} C(\beta_j, x) E_j \zeta_j(y) \int_0^\infty dz \zeta_j(z) \eta(z), \end{aligned}$$

then $\eta(y)$ vanishes identically. Here $\zeta_j(z) = e^{-\beta_j z}$. Let $\langle \cdot, \cdot \rangle$ denote the usual inner product on $L^2(0, \infty)$. Then we have

$$\begin{aligned} 0 &= \langle (\mathbf{I} - \Omega_l) \eta, \eta \rangle \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty dk \left[1 + R(k) \right. \\ &\quad \times \exp \left(2ikx + 2ik \int_x^\infty [1 - H] \right) \left. \right] |\hat{\eta}(k)|^2 \\ &+ \sum_{j=1}^{\mathcal{N}} C(\beta_j, x) E_j |\langle \eta, \zeta_j \rangle|^2, \end{aligned} \tag{6.34}$$

where $\hat{\eta}(k)$ is the Fourier transform of $\eta(y)$ and

$$C(\beta_j, x) E_j = \frac{c(\beta_j) \exp(-2\beta_j \int_0^x H - \beta_j \int_0^\infty [1 - H] + \beta_j \int_{-\infty}^0 [1 - H] - \beta_j \int_{-\infty}^\infty [1 - H])}{\int_{-\infty}^\infty dz [1 - P(z)] f_l(i\beta_j z) f_r(i\beta_j z)} > 0,$$

in view of $f_l(i\beta_j z) = c(\beta_j) f_r(i\beta_j z)$ and $1 - P > 0$. As a result of these inequalities and $|R(k)| < 1$ for all nonzero real k , we obtain from (6.34),

$$\hat{\eta}(k) = 0 \quad \text{and} \quad \langle \eta, \xi_1 \rangle = \dots = \langle \eta, \xi_{N'} \rangle = 0,$$

whence $\eta(y) \equiv 0$. ■

VII. BOUNDS ON THE KERNELS OF THE MARCHENKO OPERATORS

In this section, under the assumption that the solution of each Marchenko equation through (6.27) leads to $G(x)$ satisfying $G \in L^1_+(\mathbf{R})$, we will obtain some estimates on the kernels being in (6.20) and (6.21).

Proposition 7.1: Assume $G(x)$ obtained from the solution of each Marchenko equation using (6.27) satisfies $G \in L^1_+(\mathbf{R})$. Then the kernels of the Marchenko equations (6.22) and (6.23) satisfy

$$|\Omega_l(x, y)| < \frac{1}{2} \sigma_l(x) e^{M\tau_l(x)} e^{M[e^{M\tau_l(x)} - 1]}, \tag{7.1}$$

$$|\Omega_r(x, y)| < \frac{1}{2} \sigma_r(x) e^{M\tau_r(x)} e^{M[e^{M\tau_r(x)} - 1]},$$

where M is the constant defined in (1.3) and $\sigma_l(x)$, $\sigma_r(x)$, $\tau_l(x)$, and $\tau_r(x)$ are the functions defined in (6.28) and (6.29).

Proof: We will give the proof for $\Omega_l(x, y)$ only; the proof for $\Omega_r(x, y)$ is similar. From (6.11) and (6.20) it is seen that $\Omega_l(x, y)$ is a function of $x + \int_x^\infty [1 - H] + y/2$. Let

$$\omega_l\left(x + \int_x^\infty [1 - H] + \frac{y}{2}\right) = \Omega_l(x, y). \tag{7.2}$$

From the Marchenko equation (6.22), letting $s = x + \int_x^\infty [1 - H] + y/2$, we obtain

$$\begin{aligned} \omega_l(s) &= B_l\left(x, 2s - 2x - 2 \int_x^\infty [1 - H]\right) \\ &\quad - 2 \int_s^\infty dt B_l(x, 2t - 2s) \omega_l(t). \end{aligned} \tag{7.3}$$

Similarly, from (6.11) and (6.21) it is seen that $\Omega_r(x, y)$ is a function of $-x + \int_{-\infty}^x [1 - H] + y/2$. Equation (7.3), being a Volterra equation, is uniquely solvable for $\omega_l(s)$. For simplicity, let us drop the subscript l in B_l , ω_l , σ_l , and τ_l . We then have $\omega(s) = \sum_{j=0}^\infty \omega_j(s)$, where $\omega_0(s) = B(x, 2s - 2x - 2 \int_x^\infty [1 - H])$ and

$$\omega_j(s) = -2 \int_s^\infty dt B(x, 2t - 2s) \omega_{j-1}(t), \quad j \geq 1.$$

Using (6.30) we obtain

$$|B(x, 2t - 2s)| < \frac{1}{2} \sigma(x + y/(2M)) e^{M\tau(x) - M\tau(x + y/(2M))}. \tag{7.4}$$

Thus we have $|\omega_0(s)| < \frac{1}{2} \sigma(x) e^{M\tau(x)}$. Assume

$$\begin{aligned} |\omega_n(s)| &< \frac{1}{2n!} \sigma(x) e^{M\tau(x)} \left[\int_s^\infty dt \right. \\ &\quad \left. \times \sigma\left(x + \frac{t-s}{M}\right) e^{M\tau(x) - M\tau(x + (t-s)/M)} \right]^n. \end{aligned}$$

Then by induction using (7.4) we obtain

$$\begin{aligned} |\omega_{n+1}(s)| &< \frac{\sigma(x) e^{M\tau(x)}}{2n!} \int_s^\infty dt \sigma\left(x + \frac{t-s}{M}\right) e^{M\tau(x) - M\tau(x + (t-s)/M)} \\ &\quad \times \left[\int_t^\infty dz \sigma\left(x + \frac{z-s}{M}\right) e^{M\tau(x) - M\tau(x + (z-s)/M)} \right]^n \\ &= \frac{1}{2(n+1)!} \sigma(x) e^{M\tau(x)} \left[\int_s^\infty dt \sigma\left(x + \frac{t-s}{M}\right) e^{M\tau(x) - M\tau(x + (t-s)/M)} \right]^{n+1}. \end{aligned}$$

Thus through summation we obtain

$$|\omega(s)| < \frac{1}{2} \sigma(x) e^{M\tau(x)} \exp\left[\int_s^\infty dt \sigma\left(x + \frac{t-s}{M}\right) e^{M\tau(x) - M\tau(x + (t-s)/M)} \right] = \frac{1}{2} \sigma(x) e^{M\tau(x)} e^{M[e^{M\tau(x)} - 1]}, \tag{7.5}$$

and hence, replacing s by $x + \int_x^\infty [1 - H] + y/2$ and using (7.2), we obtain (7.1). ■

Proposition 7.2: Assume $G(x)$ obtained from the solution of each Marchenko equation using (6.27) satisfies $G \in L^1_+(\mathbf{R})$. Then the derivatives of the kernels of the Marchenko equations (6.22) and (6.23) satisfy

$$\left| \frac{d}{dx} \Omega_l(x,0) - \frac{G(x)}{2H(x)} \right| \leq \frac{1+M}{2} \sigma_l(x)^2 e^{M\tau_l(x)} e^{M[e^{M\tau_l(x)} - 1]},$$

$$\left| \frac{d}{dx} \Omega_r(x,0) - \frac{G(x)}{2H(x)} \right| \leq \frac{1+M}{2} \sigma_r(x)^2 e^{M\tau_r(x)} e^{M[e^{M\tau_r(x)} - 1]},$$

where M is the constant defined in (1.3), and $\sigma_b, \sigma_r, \tau_b$ and τ_r are the quantities defined in (6.28) and (6.29).

Proof: We will give the proof only for $(d/dx)\Omega_l(x,0)$; the proof for $(d/dx)\Omega_r(x,0)$ is similar. For simplicity, we will again drop the subscript l in B_b, ω_b, σ_b and τ_b .

Let $y \rightarrow 0$ in the Marchenko equation, which is equivalent to letting $s \rightarrow x + \int_x^\infty [1-H]$ in (7.3). Let us use the notation $B_1(x,y)$ and $B_2(x,y)$ for $\partial B_l(x,y)/\partial x$ and $\partial B_l(x,y)/\partial y$, respectively. We then obtain

$$\omega\left(x + \int_x^\infty [1-H]\right) = B(x,0) - 2 \int_{x+\int_x^\infty [1-H]}^\infty dt B\left(x, 2t - 2x - 2 \int_x^\infty [1-H]\right) \omega(t). \tag{7.6}$$

Taking the derivative of both sides of (7.6) with respect to x , we obtain

$$H(x)\omega'\left(x + \int_x^\infty [1-H]\right) = B_1(x,0) + 2H(x)B(x,0)\omega\left(x + \int_x^\infty [1-H]\right) - 2 \int_{x+\int_x^\infty [1-H]}^\infty dt \times \left[B_1\left(x, 2t - 2x - 2 \int_x^\infty [1-H]\right) - 2H(x)B_2\left(x, 2t - 2x - 2 \int_x^\infty [1-H]\right) \right] \omega(t). \tag{7.7}$$

From (6.33) we have

$$B_1(x,y) - 2H(x)B_2(x,y) = H(x) \int_x^\infty dz G(z)B(z,y). \tag{7.8}$$

Thus, using $B_1(x,0) = \frac{1}{2}G(x)$ and (7.8) in (7.7), we obtain

$$\omega'\left(x + \int_x^\infty [1-H]\right) - \frac{G(x)}{2H(x)} = 2B(x,0)\omega\left(x + \int_x^\infty [1-H]\right) - 2 \int_{x+\int_x^\infty [1-H]}^\infty dt \omega(t) \int_x^\infty dz G(z) \times B\left(z, 2t - 2x - 2 \int_x^\infty [1-H]\right).$$

Thus we have

$$\left| \omega'\left(x + \int_x^\infty [1-H]\right) - \frac{G(x)}{2H(x)} \right| \leq 2|B(x,0)| \left| \omega\left(x + \int_x^\infty [1-H]\right) \right| + 2 \int_{x+\int_x^\infty [1-H]}^\infty dt |\omega(t)| \times \int_x^\infty dz |G(z)| \left| B\left(z, 2t - 2x - 2 \int_x^\infty [1-H]\right) \right|.$$

Note that using (7.4) and (7.5), we have

$$2|B(x,0)| \left| \omega\left(x + \int_x^\infty [1-H]\right) \right| \leq \frac{1}{2} \sigma(x)^2 e^{M\tau(x)} e^{M[e^{M\tau(x)} - 1]}$$

and

$$\begin{aligned}
 & 2 \int_{x+\int_x^\infty [1-H]}^\infty dt |\omega(t)| \int_x^\infty dz |G(z)| \left| B\left(z, 2t - 2x - 2 \int_x^\infty [1-H]\right) \right| \\
 & \leq \sigma(x) e^{M\tau(x)} e^{M[e^{M\tau(x)} - 1]} \int_x^\infty dz |G(z)| \int_{x+\int_x^\infty [1-H]}^\infty dt \frac{1}{2} \sigma\left(z + \frac{t-x-\int_x^\infty [1-H]}{M}\right) \\
 & \times \exp\left[M\tau(x) - M\tau\left(z + \frac{t-x-\int_x^\infty [1-H]}{M}\right)\right] \\
 & \leq \sigma(x) e^{M\tau(x)} e^{M[e^{M\tau(x)} - 1]} \left[\int_x^\infty dz \frac{M}{2} |G(z)| \right] = \left(\frac{M}{2}\right) \sigma(x)^2 e^{M\tau(x)} e^{M[e^{M\tau(x)} - 1]}.
 \end{aligned}$$

Hence the proof is complete. ■

Proposition 7.3: Assume $G \in L^1_1(\mathbb{R})$. Then the functions $\sigma_l(x)$ and $\sigma_r(x)$ defined in (6.28) satisfy

$$\int_a^\infty dx (1 + |x|) \sigma_l(x)^2 \leq C_l(a)$$

and

$$\int_{-\infty}^a dx (1 + |x|) \sigma_r(x)^2 \leq C_r(a),$$

where $C_l(a)$ is a decreasing function of a and $C_r(a)$ is an increasing function of a .

Proof: We will give the proof for $\sigma_l(x)$ only; the proof for $\sigma_r(x)$ is similar. For simplicity, let us drop the subscript l in σ_l . First note that

$$\begin{aligned}
 & \int_a^\infty dx (1 + |x|) \sigma(x)^2 \\
 & \leq \max\{0, -a\} (1 + |a|) \sigma(-|a|)^2 \\
 & \quad + \sigma(0) \int_0^\infty dx (1 + |x|) \sigma(x).
 \end{aligned}$$

Hence we only need to show that $\int_0^\infty dx (1 + x) \sigma(x)$ is finite.

We have

$$\int_0^\infty dx \sigma(x)^2 \leq \sigma(0) \int_0^\infty dx \sigma(x)$$

and

$$\int_0^\infty dx x \sigma(x)^2$$

$$\begin{aligned}
 & = \int_0^\infty dx x \int_x^\infty dy |G(y)| \int_x^\infty dz |G(z)| \\
 & \leq \int_0^\infty dx \int_x^\infty dy y |G(y)| \int_x^\infty dz |G(z)| \\
 & \leq \left(\int_0^\infty dy y |G(y)| \right) \int_0^\infty dx \int_x^\infty dz |G(z)| \\
 & \leq \left(\int_0^\infty dy y |G(y)| \right) \int_0^\infty dx \sigma(x).
 \end{aligned}$$

Hence we only need to prove that $\int_0^\infty dx \sigma(x)$ is finite. This follows from

$$\begin{aligned}
 \int_0^\infty dx \sigma(x) & = \int_0^\infty dy \int_0^y dx |G(y)| \\
 & = \int_0^\infty dy y |G(y)| < +\infty.
 \end{aligned}$$

Thus the proof is complete. ■

Proposition 7.4: Assume $G(x)$ obtained from the solution of each Marchenko equation using (6.27) satisfies $G \in L^1_1(\mathbb{R})$, and $\inf_{t \in \mathbb{R}} H(t) > 0$. We then have

$$\int_a^\infty dx (1 + |x|) \left| \frac{d\Omega_l(x,0)}{dx} \right| \leq c_l(a), \tag{7.9}$$

$$\int_{-\infty}^a dx (1 + |x|) \left| \frac{d\Omega_r(x,0)}{dx} \right| \leq c_r(a), \tag{7.10}$$

where $c_l(a)$ is a decreasing function of a and $c_r(a)$ is an increasing function of a .

Proof: We will give the proof only for $(d/dx)\Omega_l(x,0)$; the proof for $(d/dx)\Omega_r(x,0)$ is similar. For simplicity, let us drop the subscripts in Ω_b, σ_b , and τ_b . Using Proposition 7.2 we have

$$\begin{aligned} & \int_a^\infty dx(1+|x|) \left| \frac{d\Omega(x,0)}{dx} \right| \\ & < \int_a^\infty dx(1+|x|) \left[\left| \frac{G(x)}{2H(x)} \right| \right. \\ & \quad \left. + \frac{1}{2}\sigma(x)^2 e^{2\tau(x)} e^{\sigma\tau(x)} - 1 \right] \\ & \leq \frac{1}{2 \inf_{t \in \mathbb{R}} H(t)} \int_a^\infty dx(1+|x|) |G(x)| \\ & \quad + \frac{1}{2} e^{2\tau(-|a|)} e^{\sigma\tau(-|a|)} - 1 \int_a^\infty dx(1+|x|) \sigma(x)^2. \end{aligned}$$

The first integral is finite because $G \in L^1_+(\mathbb{R})$ and the second integral is bounded, as shown in Proposition 7.3. ■

VIII. PROPERTIES OF THE POTENTIAL

In this section, when the scattering data satisfy (7.9) and (7.10), we show that $G(x)$ obtained through (6.27) from the solution of each Marchenko equation satisfies $G \in L^1_+(\mathbb{R})$. We also show that the solution of each Marchenko equation leads to a solution of the Schrödinger equation (1.1).

Note that (7.9) is equivalent to $\omega'_l \in L^1_+(a, \infty)$, where $\omega_l(t)$ is the function defined in (7.2). Define

$$\gamma_l(x) = \int_x^\infty dt |\omega'_l(t)|. \tag{8.1}$$

It is seen that $\gamma_l(x)$ is a decreasing function of x that is bounded on $[a, \infty)$ for any real number a . Note that $|\omega_l(x)| \leq \int_x^\infty dt |\omega'_l(t)|$, and hence $|\omega_l(x)| \leq \gamma_l(x)$. Furthermore, $\gamma_l \in L^1_+(a, \infty)$ because

$$\begin{aligned} & \int_a^\infty dx \gamma_l(x) \\ & = \int_a^\infty dx \int_x^\infty dt |\omega'_l(t)| \leq \gamma_l(a) \max\{0, -a\} \\ & \quad + \int_0^\infty dx \int_x^\infty dt |\omega'_l(t)| = C_3(a), \end{aligned}$$

where we have defined

$$C_3(a) = \gamma_l(a) \max\{0, -a\} + \int_0^\infty dt t |\omega'_l(t)|. \tag{8.2}$$

Proposition 8.1: Any solution in $L^1(0, \infty)$ of each of the homogeneous Marchenko equations is bounded and hence also belongs to $L^2(0, \infty)$.

Proof: We will give the proof for the homogeneous version of the Marchenko equation (6.22) only. The corresponding proof for (6.23) is similar. Let $h \in L^1(0, \infty)$ be a solution of

$$\begin{aligned} & h(y) \\ & = \int_0^\infty dz \omega_l \left(x + \int_x^\infty [1-H] + \frac{y}{2} + \frac{z}{2} \right) h(z), \quad y > 0. \end{aligned}$$

By using (8.1) we then get

$$\begin{aligned} & |h(y)| \\ & < \int_0^\infty dz |h(z)| \int_{x+\int_x^\infty [1-H]+y/2+z/2}^\infty dt |\omega'_l(t)| \\ & = \int_0^\infty dz |h(z)| \gamma_l \left(x + \int_x^\infty [1-H] + \frac{y}{2} + \frac{z}{2} \right) \\ & < \gamma_l \left(x + \int_x^\infty [1-H] + \frac{y}{2} \right) \left[\int_0^\infty dz |h(z)| \right]. \end{aligned}$$

Hence $h(y)$ is bounded and since

$$\begin{aligned} & \int_0^\infty dy |h(y)|^2 < \int_0^\infty dy |h(y)| \gamma_l \left(x + \int_x^\infty [1-H] \right. \\ & \quad \left. + \frac{y}{2} \right) \left[\int_0^\infty dz |h(z)| \right] \\ & < \gamma_l \left(x + \int_x^\infty [1-H] \right) \left[\int_0^\infty |h(z)| \right]^2, \end{aligned}$$

the solution $h \in L^1(0, \infty)$ also belongs to $L^2(0, \infty)$. ■

Proposition 8.2: The Marchenko operators Ω_l and Ω_r are compact in $L^1(0, \infty)$. Furthermore, $\|(I - \Omega_l)^{-1}\|$ is uniformly bounded for $x \in [a, \infty)$ and $\|(I - \Omega_r)^{-1}\|$ is uniformly bounded for $x \in (-\infty, a]$ for any $a \in \mathbb{R}$.

Proof: We will give the proof for Ω_l only; the proof for Ω_r is similar. For any $h \in L^1(0, \infty)$ we have

$$\begin{aligned} & (\Omega_l h)(y) \\ & = \int_0^\infty dz \omega_l \left(x + \int_x^\infty [1-H] + \frac{y}{2} + \frac{z}{2} \right) h(z). \end{aligned}$$

Then, as in the proof of Proposition 8.1, we have

$$\int_0^\infty dy |(\Omega h)(y)| < \int_0^\infty dy \gamma_l \left(x + \int_x^\infty [1-H] + \frac{y}{2} \right) \times \left[\int_0^\infty dz |h(z)| \right] \tag{8.3}$$

and

$$\int_0^\infty dy |(\Omega h)(y + \epsilon) - (\Omega h)(y)| < \int_0^\infty dz |h(z)| \int_0^\infty dy \left| \omega_l \left(x + \int_x^\infty [1-H] + \frac{y}{2} + \frac{\epsilon}{2} + \frac{z}{2} \right) - \omega_l \left(x + \int_x^\infty [1-H] + \frac{y}{2} + \frac{z}{2} \right) \right| \tag{8.4}$$

and also

$$\int_N^\infty dy |(\Omega h)(y)| < \left[\int_0^\infty dz |h(z)| \right] \left[\int_N^\infty dy \times \gamma_l \left(x + \int_x^\infty [1-H] + \frac{y}{2} \right) \right]. \tag{8.5}$$

Then, for any $h \in L^1(0, \infty)$, using the properties of $\omega_l(t)$

$$\int_0^\infty dz |B_l(x, z)| < \|(\mathbf{I} - \Omega_l)^{-1}\| \int_0^\infty dy \left| \omega_l \left(x + \int_x^\infty [1-H] + \frac{y}{2} \right) \right| < \|(\mathbf{I} - \Omega_l)^{-1}\| \int_0^\infty dy \gamma_l \left(x + \int_x^\infty [1-H] + \frac{y}{2} \right) < 2 \|(\mathbf{I} - \Omega_l)^{-1}\| \int_{x + \int_x^\infty [1-H]}^\infty dt \gamma_l(t) < 2 \|(\mathbf{I} - \Omega_l)^{-1}\| C_3 \left(x + \int_x^\infty [1-H] \right),$$

where $C_3(x + \int_x^\infty [1-H])$ is the quantity given in (8.2). ■

Proposition 8.4: The solution $B_l(x, y)$ of the Marchenko equation (6.22) is bounded for each $x \in [a, \infty)$, and the solution $B_r(x, y)$ of the Marchenko equation (6.23) is bounded for each $x \in (-\infty, a]$, where a is any real number. Furthermore, these solutions vanish as $y \rightarrow +\infty$.

Proof: We will give the proof for (6.22) only. The proof for (6.23) is similar. From the Marchenko equation (6.22) we have

$$|B_l(x, y)| < \left| \omega_l \left(x + \int_x^\infty [1-H] + \frac{y}{2} \right) \right| + \int_0^\infty dz$$

implied by $\omega_l' \in L^1_1(a, \infty)$, we can conclude that the integral in (8.3) is finite, the integral in (8.4) vanishes as $\epsilon \rightarrow 0$, and the integral in (8.5) vanishes as $N \rightarrow +\infty$. Therefore, all the three conditions in the Fréchet–Kolmogorov compactness criterion²⁰ are satisfied. Hence the Marchenko operator Ω_l maps bounded sets into relatively compact sets, and thus Ω_l is a compact operator on $L^2(0, \infty)$.

From Theorem 6.3 it follows that 1 is not an eigenvalue of the operator Ω_l defined on $L^2(0, \infty)$, and hence from Proposition 8.1 it follows that 1 is not eigenvalue of Ω_l in $L^1(0, \infty)$. As a result, the operator $(\mathbf{I} - \Omega_l)^{-1}$ exists for each x . The norm continuity of Ω_l with respect to x and the fact that $\|\Omega_l\| \rightarrow 0$ as $x \rightarrow +\infty$ imply that for each $a \in \mathbf{R}$, the L^1 norm $\|(\mathbf{I} - \Omega_l)^{-1}\|$ is uniformly bounded for $x \in [a, \infty)$. ■

Proposition 8.3: The solution of the Marchenko equation (6.22) is unique in $L^1(0, \infty) \cap L^2(0, \infty)$ for each $x \in [a, \infty)$, and the solution of the Marchenko equation (6.23) is unique in $L^1(0, \infty) \cap L^2(0, \infty)$ for each $x \in (-\infty, a]$, where a is any real number.

Proof: We will give the proof for (6.22) only. The proof for (6.23) is similar. Let us write (6.22) as $(\mathbf{I} - \Omega_l)B_l = \omega_l$. We then obtain $B_l = (\mathbf{I} - \Omega_l)^{-1}\omega_l$. Hence, in terms of the norm of $(\mathbf{I} - \Omega_l)^{-1}$ on $L^1(0, \infty)$, we have

$$\begin{aligned} & \times \left| \omega_l \left(x + \int_x^\infty [1-H] + \frac{y}{2} \right) \right| |B_l(x, z)| \\ & < \gamma_l \left(x + \int_x^\infty [1-H] + \frac{y}{2} \right) \\ & \times \left[1 + \int_0^\infty dz |B_l(x, z)| \right] \\ & < C_4(x) \gamma_l \left(x + \int_x^\infty [1-H] + \frac{y}{2} \right), \end{aligned}$$

where

$$C_4(x) = 1 + 2\|(\mathbf{I} - \Omega_l)^{-1}\|C_3\left(x + \int_x^\infty [1 - H]\right), \tag{8.6}$$

where $C_3(x + \int_x^\infty [1 - H])$ is the quantity given in (8.2). Since $\gamma \in L^1(a, \infty)$ for any $a \in \mathbf{R}$ and $\gamma_l(\infty) = 0$, the proof is complete. ■

Proposition 8.5: Suppose that $\Omega_l(x, y)$ obeys (7.9). Then the solution $B_l(x, y)$ of the Marchenko equation (6.22) has first partial derivatives (a.e.) such that $(\partial/\partial x)B_l(x, \cdot) \in L^1(0, \infty)$ and $(\partial/\partial y)B_l(x, \cdot) \in L^1(0, \infty)$. Similarly, when $\Omega_r(x, y)$ satisfies (7.10), for the solution $B_r(x, y)$ of (6.23) we have $(\partial/\partial x)B_r(x, \cdot) \in L^1(0, \infty)$ and $(\partial/\partial y)B_r(x, \cdot) \in L^1(0, \infty)$.

Proof: We will give the proof only for $(\partial/\partial x)B_l(x, y)$. The proof for $(\partial/\partial y)B_l(x, y)$ and for the derivatives of $B_r(x, y)$ is similar. From (6.22), we obtain

$$\begin{aligned} & \frac{\partial B_l(x, y)}{\partial x} \\ &= H(x)\omega'_l\left(x + \int_x^\infty [1 - H] + \frac{y}{2}\right) + H(x) \\ & \quad \times \int_0^\infty dz \omega'_l\left(x + \int_x^\infty [1 - H] + \frac{y}{2} + \frac{z}{2}\right) \end{aligned}$$

$$\begin{aligned} & \times B_l(x, z) + \int_0^\infty dz \\ & \times \omega_l\left(x + \int_x^\infty [1 - H] + \frac{y}{2} + \frac{z}{2}\right) \frac{\partial B_l(x, z)}{\partial x}, \end{aligned} \tag{8.7}$$

$y > 0.$

Obtaining (8.7) from (6.22) through differentiation is justified, as shown by the following argument. Letting $\Delta_\epsilon B_l(x, y) = B_l(x + \epsilon, y) - B_l(x, y)$ and $\Delta_\epsilon \Omega_l(x, y) = \Omega_l(x + \epsilon, y) - \Omega_l(x, y)$, from (6.22) we obtain

$$\begin{aligned} & \Delta_\epsilon B_l(x, y) \\ &= \Delta_\epsilon \Omega_l(x, y) + \int_0^\infty dz \Delta_\epsilon \Omega_l(x, y + z) \\ & \quad \times B_l(x + \epsilon, z) + \int_0^\infty dz \Omega_l(x, y + z) \Delta_\epsilon B_l(x, z). \end{aligned} \tag{8.8}$$

In order to prove that the pointwise limit of $\epsilon^{-1} \Delta_\epsilon \Omega_l(x, y)$ exists a.e. and satisfies

$$\lim_{\epsilon \rightarrow 0} \epsilon^{-1} \Delta_\epsilon \Omega_l(x, y) = \omega'_l\left(x + \int_x^\infty [1 - H] + \frac{y}{2}\right) H(x),$$

and that, in fact, the same limit exists in the L^1 sense (with respect to y), we write down the inequalities

$$\begin{aligned} & \int_0^\infty dy \left| \epsilon^{-1} \Delta_\epsilon \Omega_l(x, y) - \omega'_l\left(x + \int_x^\infty [1 - H] + \frac{y}{2}\right) H(x) \right| \\ & \leq \int_0^\infty dy \int_x^{x+\epsilon} ds \epsilon^{-1} \left| \omega'_l\left(s + \int_s^\infty [1 - H] + \frac{y}{2}\right) H(s) - \omega'_l\left(x + \int_x^\infty [1 - H] + \frac{y}{2}\right) H(x) \right| \\ & \leq \epsilon^{-1} \int_0^\epsilon du \int_0^\infty dy \left| \omega'_l\left(x + u + \int_{x+u}^\infty [1 - H] + \frac{y}{2}\right) H(x + u) - \omega'_l\left(x + \int_x^\infty [1 - H] + \frac{y}{2}\right) H(x) \right| \\ & \leq \sup_{0 < u < \epsilon} \int_0^\infty dy \left| \omega'_l\left(x + u + \int_{x+u}^\infty [1 - H] + \frac{y}{2}\right) H(x + u) - \omega'_l\left(x + \int_x^\infty [1 - H] + \frac{y}{2}\right) H(x) \right|. \end{aligned}$$

In an analogous manner, using Proposition 8.4, it follows that

$$\lim_{\epsilon \rightarrow 0} \epsilon^{-1} \int_0^\infty dz \Delta_\epsilon \Omega_l(x, y + z) B_l(x + \epsilon, z),$$

exists pointwise a.e. as well as in the L^1 sense.

Now let us write (8.8) as

$$\begin{aligned} \Delta_\epsilon B_l(x, y) &= (\mathbf{I} - \Omega_l)^{-1} \left[\Delta_\epsilon \Omega_l(x, y) + \int_0^\infty dz \right. \\ & \quad \left. \times \Delta_\epsilon \Omega_l(x, y + z) B_l(x + \epsilon, z) \right]. \end{aligned} \tag{8.9}$$

Since $(\mathbf{I} - \Omega_l)^{-1}$ is a bounded operator on $L^1(0, \infty)$, we conclude that $\lim_{\epsilon \rightarrow 0} \epsilon^{-1} \Delta_\epsilon B_l(x, y)$ exists in the L^1 sense. Hence, since $\Omega_l(x, y + z)$ is bounded, from (8.8) we can conclude that

$$\lim_{\epsilon \rightarrow 0} \epsilon^{-1} \int_0^\infty dz \Omega_l(x, y + z) \Delta_\epsilon B_l(x, z)$$

exists pointwise a.e. as well as in the L^1 sense. Thus, from (8.8) we see that $\lim_{\epsilon \rightarrow 0} \epsilon^{-1} \Delta_\epsilon B_l(x, y) = \partial B_l(x, y) / \partial x$ a.e. From (8.8) and (8.9) it follows that this partial derivative satisfies (8.7) and is in $L^1(0, \infty)$. ■

Defining

$$p_l(x, y) = \frac{1}{H(x)} \frac{\partial B_l(x, y)}{\partial x} - \omega_l' \left(x + \int_x^\infty [1 - H] + \frac{y}{2} \right), \tag{8.10}$$

$$\begin{aligned} \mu_l(x, y) &= \int_0^\infty dz \omega_l' \left(x + \int_x^\infty [1 - H] + \frac{y}{2} + \frac{z}{2} \right) B_l(x, z) \\ &+ \int_0^\infty dz \omega_l \left(x + \int_x^\infty [1 - H] + \frac{y}{2} + \frac{z}{2} \right) \\ &\times \omega_l' \left(x + \int_x^\infty [1 - H] + \frac{z}{2} \right), \end{aligned} \tag{8.11}$$

we can write (8.7) as

$$\begin{aligned} p_l(x, y) &= \mu_l(x, y) + \int_0^\infty dz \\ &\times \omega_l \left(x + \int_x^\infty [1 - H] + \frac{y}{2} + \frac{z}{2} \right) \\ &\times p_l(x, z), \quad y > 0. \end{aligned} \tag{8.12}$$

Proposition 8.6: For each real constant a , the quantity $p_l(x, y)$ defined in (8.10) is bounded and $p_l(x, \cdot) \in L^1(0, \infty)$, uniformly in x on $[a, \infty)$.

Proof: Using (8.11) and Proposition 8.3, we obtain

$$\begin{aligned} |\mu_l(x, y)| &< \int_0^\infty dz \left| \omega_l' \left(x + \int_x^\infty [1 - H] + \frac{y}{2} + \frac{z}{2} \right) B_l(x, z) \right| \\ &+ \int_0^\infty dz \left| \omega_l \left(x + \int_x^\infty [1 - H] + \frac{y}{2} + \frac{z}{2} \right) \right. \\ &\times \omega_l' \left(x + \int_x^\infty [1 - H] + \frac{z}{2} \right) \\ &\left. < C_5(x) \gamma_l \left(x + \int_x^\infty [1 - H] + \frac{y}{2} \right), \right. \end{aligned}$$

where we have defined

$$C_5(x) = 2[C_4(x) + 1] \gamma_l \left(x + \int_x^\infty [1 - H] \right),$$

and $C_4(x)$ is the quantity defined in (8.6). Hence we have the L^1 norm of μ_l satisfying

$$\begin{aligned} \|\mu_l\| &< C_5(x) \int_0^\infty dy \gamma_l \left(x + \int_x^\infty [1 - H] + \frac{y}{2} \right) \\ &= 2C_5(x) \eta_l(x), \end{aligned} \tag{8.13}$$

where

$$\eta_l(x) = \int_x^\infty dt |\omega_l(t)|.$$

Thus from (8.12) we obtain

$$\begin{aligned} |p_l| &< |\mu_l| + |\Omega_l p_l| \\ &< C_5(x) \gamma_l \left(x + \int_x^\infty [1 - H] + \frac{y}{2} \right) \\ &+ \gamma_l \left(x + \int_x^\infty [1 - H] + \frac{y}{2} \right) \|p_l\|. \end{aligned} \tag{8.14}$$

On the other hand, from (8.12) and (8.13) we have

$$\begin{aligned} \|p_l\| &< \|(\mathbf{I} - \Omega_l)^{-1}\| \|\mu_l\| \\ &< 2C_5(x) \eta_l(x) \|(\mathbf{I} - \Omega_l)^{-1}\|, \end{aligned} \tag{8.15}$$

and thus (8.14) gives us

$$\begin{aligned} |p_l| &< \gamma_l \left(x + \int_x^\infty [1 - H] + \frac{y}{2} \right) \\ &\times [C_5(x) + 2C_5(x) \eta_l(x) \|(\mathbf{I} - \Omega_l)^{-1}\|], \end{aligned}$$

and hence

$$\begin{aligned} |p_l| &< 2 \left[\gamma_l \left(x + \int_x^\infty [1 - H] + \frac{y}{2} \right) \right]^2 [1 + C_4(x)] \\ &\times [1 + 2\eta_l(x) \|(\mathbf{I} - \Omega_l)^{-1}\|]. \end{aligned} \tag{8.16}$$

Thus from (8.16) it follows that $p_l(x, y)$ is bounded in x and y on $[a, \infty) \times \mathbf{R}^+$, where a is any real constant; from (8.15) it follows that $p_l(x, \cdot) \in L^1(0, \infty)$, uniformly in x on $[a, \infty)$. ■

The next theorem shows that when the reflection coefficients are used as inputs to the Marchenko equations (6.22) and (6.23), the quantity $G(x)$ obtained by using (6.27) satisfies $G \in L^1_1(\mathbf{R})$ whenever the quantities in (6.20) and (6.21) obtained from the scattering data satisfy (7.9) and (7.10).

Theorem 8.7: Assume that the quantities in (6.20) and (6.21) that are obtained from the scattering data satisfy (7.9) and (7.10) and that $H(x)$ is bounded away from 0. Then the quantity $G(x)$ that is obtained from the solutions of the Marchenko equations using (6.27) satisfies $G \in L^1_+(\mathbf{R})$.

Proof: The proof will be given by proving that $\int_a^\infty dx(1 + |x|)|dB_l(x,0)/dx|$ and $\int_{-\infty}^a dx(1 + |x|)|dB_r(x,0)/dx|$ are finite for each $a \in \mathbf{R}$; we will only give the proof of the former since the proof of the latter is similar. From (8.10) it is seen that it suffices to prove that $\int_a^\infty dx(1 + |x|)|p_l(x,0)| < +\infty$. From (8.16) it is seen that this integral is finite if $\gamma_l^2 \in L^1_+(a, \infty)$, where $\gamma_l(x)$ is the quantity defined in (8.1). Since $\gamma_l(x)$ is bounded in $[a, \infty)$, it is enough for us to prove that $\gamma_l^2 \in L^1_+(0, \infty)$. The latter follows from the assumption $\omega_l \in L^1_+(a, \infty)$ and a repetition of the proof of Proposition 7.3. Thus we obtain $G/(2H) \in L^1_+(a, \infty)$ and $G/(2H) \in L^1_+(-\infty, a)$ for any real number a . Hence $G/(2H) \in L^1_+(\mathbf{R})$. Since $\inf_{x \in \mathbf{R}} H(x) > 0$, it follows that $G \in L^1_+(\mathbf{R})$. ■

Theorem 8.8: The solution of each of the Marchenko equations leads to the solution of the Schrödinger equation (1.1). As a result, the solutions of the Marchenko equations also lead to the solution of the Riemann-Hilbert problem (6.2).

Proof: We will give the proof only for the Marchenko equation (6.22); the proof for (6.23) is similar. Let $B_l(x,y)$ be a solution of (6.22) and set

$$G(x) = 2 \frac{dB_l(x,0+)}{dx}. \tag{8.17}$$

In the following steps we will make the assumption, in addition to the properties of $B_l(x,y)$ stated in Proposition 8.5, that

$$\frac{\partial}{\partial x} \left[-2 \frac{\partial}{\partial y} + \frac{1}{H(x)} \frac{\partial}{\partial x} \right] B_l(x,y) \in L^1(0 < y < \infty). \tag{8.18}$$

This assumption will be used to justify some of our calculations below, although it is not needed for the validity of the theorem; later in the proof, we will use an approximation argument in order to get rid of this extra assumption. Let

$$\eta(x,y) = \frac{\partial}{\partial x} \left[-2 \frac{\partial}{\partial y} + \frac{1}{H(x)} \frac{\partial}{\partial x} \right] B_l(x,y) + G(x)B_l(x,y). \tag{8.19}$$

We will show that $\eta(x,y)$ satisfies the homogeneous Marchenko equation

$$\eta(x,y) - \int_0^\infty dz \Omega_l(x,y+z)\eta(x,z) = 0, \quad y > 0, \tag{8.20}$$

and it will then follow from Theorem 6.3 that $\eta(x,y) = 0$, and thus

$$\frac{\partial}{\partial x} \left[-2 \frac{\partial}{\partial y} + \frac{1}{H(x)} \frac{\partial}{\partial x} \right] B_l(x,y) = -G(x)B_l(x,y). \tag{8.21}$$

In order to establish (8.20) we note that from (6.11) and (6.20), we have

$$\left[\frac{\partial}{\partial x} - 2H(x) \frac{\partial}{\partial y} \right] \Omega_l(x,y) = 0. \tag{8.22}$$

Using (6.22), (8.19), and (8.22), the left-hand side of (8.20) can be evaluated as

$$\begin{aligned} & \eta(x,y) - \int_0^\infty dz \Omega_l(x,y+z)\eta(x,z) \\ &= \frac{\partial}{\partial x} \left(-2 \frac{\partial}{\partial y} + \frac{1}{H(x)} \frac{\partial}{\partial x} \right) \Omega_l(x,y) + G(x)B_l(x,y) + \frac{\partial}{\partial x} \left(-2 \frac{\partial}{\partial y} + \frac{1}{H(x)} \frac{\partial}{\partial x} \right) \int_0^\infty dz \Omega_l(x,y+z)B_l(x,z) \\ & \quad + \int_0^\infty dz \Omega_l(x,y+z) \frac{\partial}{\partial x} \left(-2 \frac{\partial}{\partial z} + \frac{1}{H(x)} \frac{\partial}{\partial x} \right) B_l(x,z) + \int_0^\infty dz \Omega_l(x,y+z)G(x)B_l(x,z) \\ &= G(x)\Omega_l(x,y) + \frac{\partial}{\partial x} \int_0^\infty dz \Omega_l(x,y+z) \frac{1}{H(x)} \frac{\partial B_l(x,z)}{\partial x} - \int_0^\infty dz \Omega_l(x,y+z) \frac{\partial}{\partial x} \left(-2 \frac{\partial}{\partial z} + \frac{1}{H(x)} \frac{\partial}{\partial x} \right) B_l(x,z) \\ &= \left[G(x) - 2 \frac{dB_l(x,0+)}{dx} \right] \Omega_l(x,y). \end{aligned}$$

Hence, upon using (8.17), we see that $\eta(x,y)$ is a solution of (8.20), and since 1 is not an eigenvalue of Ω_p , we conclude that $\eta(x,y) = 0$. We will prove the assertion of the theorem by showing that $Z_l(k,x)$ defined by using (6.9) satisfies (3.3), or equivalently,

$$Z_l'' + [-H'/H + 2ikH]Z_l' + GHZ_l = 0,$$

which we write as

$$\frac{d}{dx} \left[\frac{Z_l'}{H} + 2ik(Z_l - 1) \right] + G[Z_l - 1] = -G. \quad (8.23)$$

In order to verify (8.23), we differentiate (6.9) and use integration by parts to obtain

$$\begin{aligned} \frac{Z_l'}{H} + 2ik(Z_l - 1) &= -2B_l(x,0+) - 2 \int_0^\infty dy \\ &\quad \times e^{iky} \left[\frac{\partial}{\partial y} - \frac{1}{2H(x)} \frac{\partial}{\partial x} \right] B_l(x,y), \end{aligned}$$

where we have used the fact that $B_l(x,\infty) = 0$, as seen from Proposition 8.4. Hence we can rewrite (8.23) as

$$\begin{aligned} \int_0^\infty dy e^{iky} \left[\frac{\partial}{\partial x} \left(-2 \frac{\partial}{\partial y} + \frac{1}{H(x)} \frac{\partial}{\partial x} \right) + G(x) \right] B_l(x,y) \\ = -G(x) + \frac{dB_l(x,0+)}{dx}. \end{aligned} \quad (8.24)$$

Each side of the equality in (8.24) vanishes; the left-hand side due to (8.21) and the right-hand side due to (8.17). This proves the theorem under the assumption given in (8.18).

In the general case without assuming (8.18), we can choose a sequence of functions $\omega_{l,n} \in C_0^\infty(0,\infty)$, satisfying

$$\lim_{n \rightarrow \infty} \int_0^\infty dz (1+z) |\omega_l'(z) - \omega_{l,n}'(z)| = 0,$$

and hence

$$\lim_{n \rightarrow \infty} \int_0^\infty dz |\omega_l'(z) - \omega_{l,n}'(z)| = 0.$$

Let $\Omega_{l,n}(x,y) = \omega_{l,n}(x + \int_x^\infty [1-H] + y/2)$ and let $B_{l,n}(x,y)$ denote the corresponding solution of (6.22). Also set $G_n(x) = 2[dB_{l,n}(x,0+)/dx]$ and denote by $Z_{l,n}(k,x)$ the corresponding solution of (8.23). Then it follows from (8.7) that $B_{l,n}(x,y)$ satisfies (8.18) so that the calculations above are justified for the approximating sequence. It also follows that $\|\Omega_l - \Omega_{l,n}\| \rightarrow 0$ and hence $\|(\mathbf{I} - \Omega_{l,n})^{-1} - (\mathbf{I} - \Omega_l)^{-1}\| \rightarrow 0$ as $n \rightarrow \infty$, where $\|\cdot\|$ denotes the operator norm in $L^1(0,\infty)$. By exploiting (8.7) it can then be shown that $|B_{l,n}(x,y) - B_l(x,y)| \rightarrow 0$

uniformly in x and y and that $\|B_{l,n}(x,\cdot) - B_l(x,\cdot)\| \rightarrow 0$ and $\|(\partial/\partial x)B_{l,n}(x,\cdot) - (\partial/\partial x)B_l(x,\cdot)\| \rightarrow 0$ uniformly in x ; we omit the details. From (6.9) we have

$$Z_{l,n}(k,x) = 1 + \int_{-\infty}^\infty dy B_{l,n}(x,y) e^{iky}. \quad (8.25)$$

The right-hand side in (8.25) tends to a limit uniformly in x , and thus so does $Z_{l,n}(k,x)$. Since by (3.7) we have

$$\begin{aligned} Z_{l,n}(k,x) &= 1 + \int_x^\infty dz \frac{G_n(z)}{2ik} \\ &\quad \times \left[1 - \exp\left(-2ik \int_x^z H\right) \right] Z_{l,n}(k,z) \end{aligned}$$

and $G_n(x) \rightarrow G(x)$ in $L^1(\mathbf{R})$, we conclude that $\lim_{n \rightarrow \infty} Z_{l,n}(k,x) = Z_l(k,x)$ is a solution of (8.23) with corresponding potential $G(x)$. The proof of Theorem 8.8 is now complete. ■

The equivalence of $2[dB_l(x,0+)/dx]$ and $-2[dB_l(x,0+)/dx]$ in (6.7) is assured because the solutions of the Marchenko integral equations (6.22) and (6.23) lead to the solution of the Riemann–Hilbert problem (6.1). The proof of this equivalence is similar to the corresponding proof in the inverse problem⁴ for the regular Schrödinger equation (1.4) and can be given in a straightforward manner by noting that the solution of (1.1) also satisfies (6.1).

APPENDIX

In this appendix we prove the following result used in the proof of Theorem 6.3.

Lemma A.1: Let $F(k)$ be continuous for $k \in \mathbf{R}$, vanish as $k \rightarrow \pm \infty$, and belong to $L^2(\mathbf{R})$. Put

$$f(y) = \int_{-\infty}^\infty \frac{dk}{2\pi} e^{iky} F(k), \quad y \in \mathbf{R}.$$

Then the operator \mathcal{G} defined on $L^2(0,\infty)$ by

$$(\mathcal{G}h)(y) = \int_0^\infty dz f(y+z)h(z), \quad y > 0,$$

is compact.

Proof: Let $j(x)$ be a non-negative C^∞ function for $x \in \mathbf{R}$ with support in $[-1,1]$ such that $\int_{-\infty}^\infty dx j(x) = 1$, and let $j_\epsilon(x) = (1/\epsilon)j(x/\epsilon)$. Define the mollification $\Phi_\epsilon = j_\epsilon * F$ by convolution. Then, from Lemma 2.18 of Ref. 21, we have (i) Φ_ϵ is a C^∞ function that vanishes at $\pm \infty$ together with all its derivatives; i.e., $\Phi_\epsilon \in C_0^\infty$; (ii) $\Phi_\epsilon \in L^2(\mathbf{R})$ with $\|\Phi_\epsilon\|_2 \leq \|F\|_2$ and $\lim_{\epsilon \rightarrow 0} \|\Phi_\epsilon - F\|_2 = 0$; and (iii) $\lim_{\epsilon \rightarrow 0} |\Phi_\epsilon(k) - F(k)| = 0$, uniformly in k on \mathbf{R} , due to the uniform continuity of F on \mathbf{R} . Now put

$$\varphi_\epsilon(y) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{iky} \Phi_\epsilon(k), \quad y \in \mathbf{R},$$

and define \mathcal{G}_ϵ on $L^2(0, \infty)$ by

$$(\mathcal{G}_\epsilon h)(y) = \int_0^\infty dz \varphi_\epsilon(y+z)h(z), \quad y \geq 0.$$

Since

$$\begin{aligned} \int_0^\infty dy y |\varphi_\epsilon(y)|^2 &\leq \frac{1}{2\pi} \|\Phi_\epsilon\|_2 \|\Phi'_\epsilon\|_2 \\ &\leq \frac{1}{2\pi} \|j_\epsilon\|_\infty \|j'_\epsilon\|_\infty \|F\|_2^2 < \infty, \end{aligned}$$

the operators \mathcal{G}_ϵ are Hilbert–Schmidt on $L^2(0, \infty)$. Also, noting that

$$[\mathcal{G} - \mathcal{G}_\epsilon]h = (f - \varphi_\epsilon) * h^*,$$

for $h^*(y) = h(-y)$, and putting

$$\hat{h}(k) = (1/\sqrt{2\pi}) \int_0^\infty dy e^{-iky} h^*(y),$$

we obtain

$$\begin{aligned} \|[\mathcal{G} - \mathcal{G}_\epsilon]h\|_2 &\leq \frac{1}{\sqrt{2\pi}} \|(F - \Phi_\epsilon)\hat{h}\|_2 \\ &\leq \frac{1}{\sqrt{2\pi}} \sup_{k \in \mathbf{R}} |F(k) - \Phi_\epsilon(k)| \|h\|_2, \end{aligned}$$

where we used $\|\hat{h}\|_2 = \|h\|_2$. As a result of (iii), $\|\mathcal{G} - \mathcal{G}_\epsilon\| \rightarrow 0$ as $\epsilon \downarrow 0$ in the operator norm of $L^2(0, \infty)$, which establishes the compactness of \mathcal{G} . ■

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