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Semiclassical wave-packet scattering in one and two dimensions

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We prove that under short range potentials a semiclassical wave packet's propagation is accurate for infinite times in the $\hbar \to 0$ limit. © *2004 American Institute of Physics.* [DOI: 10.1063/1.1780613]

I. INTRODUCTION

Semiclassical analysis is the study of the connections between the quantum dynamics and the corresponding classical dynamics in the $\hbar \to 0$ limit. Consider the quantum dynamics determined by the time dependent Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}\psi(x,t) = \left\{ -\frac{\hbar^2}{2}\Delta_x + V(x) \right\} \psi(x,t). \tag{1}$$

Following the prescription defined in Refs. 1–5 one can construct approximate solutions to this equation whose time propagation is determined by the corresponding classical mechanics. These wave packets depend explicitly on the corresponding classical dynamics of the system, $\hbar$, and position. These semiclassical wave packets can be used to approximate the quantum dynamics.[1–5] Here we present a result for the semiclassical wave packets that is uniform in time. This result is an extension of a known result[1] to one and two space dimensions.

The organization of the paper is as follows: In Sec. II we present the construction of the semiclassical wave packets. In Sec. III we introduce the needed results from classical scattering theory. In Sec. IV we state and prove Theorem 1, the main result of the paper for $n=1$, referring the reader to some technical lemmas from Sec. V. In Sec. VI we provide the necessary tools needed to extend the proof to two dimensions.

Throughout we adopt standard multi-index notation.[7] The inner products are linear in the second term, conjugate linear in the first. Furthermore, we assume that our potential is "short range," i.e., $V(x)$ satisfies the short-range assumption (D) if for any multi-index α such that $|\alpha| = 0,1,2,3$, there exists $C_{|\alpha|} > 0$, $0 < \nu < 1$, such that

$$|(D^\alpha V)(x)| \le C_{|\alpha|}(1+|x|)^{-1-|\alpha|-\nu}.$$

Notice that if a potential is short range then

$$V(x) \in L^p(\mathbb{R}^n) \text{ for } p > \max\left\{ \frac{n}{1+\nu}, 1 \right\}$$

and

$$V(x)(1+x)^{-[(n/2)-1]+(\nu/2)} \in L^2(\mathbb{R}^n).$$

So, for $n=1,2$ our potentials are in $L^2(\mathbb{R}^n)$.

II. SEMICLASSICAL WAVE PACKETS

Here we present a definition of the semiclassical wave packets. Our construction is analogous to the standard construction of the harmonic oscillator eigenstates using raising and lowering operators. Greater detail on the construction presented here can be found in Ref. 4. Let a, $\eta \in \mathbb{R}^n$, and $\hbar > 0$. Furthermore assume that A and B are complex $n \times n$ matrices satisfying

$$A^t B - B^t A = 0, \tag{2}$$

$$A^* B + B^* A = 2I. \tag{3}$$

Conditions (2) and (3) are known to be equivalent to the following four conditions assumed in Ref. 1:

(i) A and B are invertible;
(ii) the real and imaginary parts of BA^{-1} are both real symmetric;
(iii) $\mathrm{Re}\, BA^{-1}$ is strictly positive definite;
(iv) $(\mathrm{Re}\, BA^{-1})^{-1} = AA^*$.

Let $p = -i\hbar\nabla_x$ be the momentum operator. For any $v \in \mathbb{C}^n$ we define associated raising and lowering operators by

$$\mathcal{A}(A,B,\hbar,a,\eta,v)^* = \frac{1}{\sqrt{2\hbar}}[\langle B\overline{v},(x-a)\rangle - i\langle A\overline{v},(p-\eta)\rangle]$$

and

$$\mathcal{A}(A,B,\hbar,a,\eta,v) = \frac{1}{\sqrt{2\hbar}}[\langle \overline{B}v,(x-a)\rangle + i\langle \overline{A}v,(p-\eta)\rangle].$$

Let $\{e_j\}$ be any orthonormal basis for $\mathbb{R}^n$, and define

$$\mathcal{A}_j(A,B,\hbar,a,\eta)^* = \mathcal{A}(A,B,\hbar,a,\eta,e_j)^*,$$

$$\mathcal{A}_j(A,B,\hbar,a,\eta) = \mathcal{A}(A,B,\hbar,a,\eta,e_j).$$

Then we can define

$$\mathcal{A}(A,B,\hbar,a,\eta)^* = \frac{1}{\sqrt{2\hbar}}[B^*(x-a) - iA^*(p-\eta)],$$

$$\mathcal{A}(A,B,\hbar,a,\eta) = \frac{1}{\sqrt{2\hbar}}[B^t(x-a) + iA^t(p-\eta)],$$

where the representation is in terms of the above basis. Define $\phi_0(A,B,\hbar,a,\eta,\cdot)$ to be a normalized vector with respect to $L^2(\mathbb{R}^n)$ such that

$$\mathcal{A}(A,B,\hbar,a,\eta)\phi_0(A,B,\hbar,a,\eta,\cdot) = 0.$$

It is seen that

$$\phi_0(A,B,\hbar,a,\eta,x) = (\pi\hbar)^{-n/4}(\det(A))^{-1/2}\exp\{-\langle(x-a),BA^{-1}(x-a)\rangle/(2\hbar) + i\langle\eta,(x-a)\rangle/\hbar\}.$$

Here a particular choice of phase is being made. For any multi-index k, we define

$$\phi_k(A,B,\hbar,a,\eta,x) = \frac{1}{\sqrt{k!}}(\mathcal{A}_1(A,B,\hbar,a,\eta)^*)^{k_1} \times \cdots \times (\mathcal{A}_n(A,B,\hbar,a,\eta)^*)^{k_n}\phi_0(A,B,\hbar,a,\eta,x).$$

Remark: The only ambiguity here is in the choice of sign on $(\det(A))^{-1/2}$, it is chosen depending on the initial conditions and continuity.

Remark: The functions $\phi_k(A,B,\hbar,a,\eta,\cdot)$ form an orthonormal basis of $L^2(\mathbb{R}^n)$.[4]

Let $S(t)$ be in $\mathbb{R}$, $a(t)$, $\eta(t)$ be vectors in $\mathbb{R}^n$, $A(t)$, $B(t)$ be complex $n\times n$ matrices all governed by the following system of ordinary differential equations:

$$\dot{a}(t) = \eta(t),$$

$$\dot{\eta}(t) = -\vec{\nabla} V(a(t)),$$

$$\dot{A}(t) = iB(t), \tag{4}$$

$$\dot{B}(t) = iV^{(2)}(a(t))A(t),$$

$$\dot{S}(t) = \frac{(\eta(t))^2}{2} - V(a(t)),$$

suppose the initial conditions given such that $A(0)$, $B(0)$ together satisfy (2) and (3) and $S(0)=0$. It is known that $A(t)$, $B(t)$ together still satisfy (2) and (3).[4]

Remark: Let

$$W_{a(t)}(x) = V(a(t)) + \langle V^{(1)}(a(t)),(x-a(t))\rangle + \tfrac{1}{2}\langle (x-a(t)),V^{(2)}(a(t))(x-a(t))\rangle,$$

the functions $\psi(x,t)=e^{iS(t)/\hbar}\phi_k(A(t),B(t),\hbar,a(t),\eta(t),x)$ provide exact solutions to the time dependent Schrödinger equation,

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2}\Delta_x\psi(x,t) + W_{a(t)}(x)\psi(x,t).$$

We state a result about the wave packets that will be used later. The reference is Ref. 4.

Lemma 1: Suppose $V\in C^3(\mathbb{R}^n)$ satisfies $-C_1\leqslant V(x)\leqslant C_2e^{Mx^2}$ for some C_1, C_2 and M. Let $(A(t),B(t),a(t),\eta(t),S(t))$ be a solution to the system (4) with appropriate initial conditions. Let $H(\hbar)=-(\hbar^2/2)\Delta+V(x)$. Then there exists some $C(k,t)$ such that

$$\|e^{-itH(\hbar)/\hbar}\phi_k(A(0),B(0),\hbar,a(0),\eta(0),x) - e^{-iS(t)/\hbar}\phi_k(A(t),B(t),\hbar,a(t),\eta(t),x)\| \leqslant C(k,t)\hbar^{1/2}. \tag{5}$$

Using these semiclassical wave packets one can now attempt to provide a construction for approximate solutions to the Schrödinger equation and prove accuracy estimates. For details on this see Refs. 1–5.

III. CLASSICAL SCATTERING

Existence of scattering states in classical mechanics is crucial to our study.

Lemma 2: Let $V(x)$ satisfy the short-range assumption (D). Given any $(a_-,\eta_-)\in\mathbb{R}^{2n}$ such that $\eta_-\neq 0$. Let A_-, and B_- be complex $n\times n$ matrices satisfying conditions (2) and (3) then there exist a unique solution $[a(t),\eta(t),A(t),B(t),S(t)]$ to the system (4) such that

$$\lim_{t\to-\infty}|a(t)-a_- - \eta_- t| = 0,$$

$$\lim_{t\to-\infty}|\eta(t)-\eta_-| = 0,$$

$$\lim_{t\to-\infty}|S(t)-t\eta_-^2/2|=0, \tag{6}$$

$$\lim_{t\to-\infty}\|A(t)-A_--iB_-t\|=0,$$

$$\lim_{t\to-\infty}\|B(t)-B_-\|=0.$$

Moreover, there exists $n\times n$ complex matrices A_+,B_+ satisfying (2) and (3) and a closed set E of measure zero contained in $\mathbb{R}^{2n}$ such that $(a_-,\eta_-)\in\mathbb{R}^{2n}\backslash E$ implies the existence of $(a_+,\eta_+)\in\mathbb{R}^{2n}$ with $\eta_+\neq 0$, $S_+\in R$ such that

$$\lim_{t\to\infty}|a(t)-a_+-\eta_+t|=0,$$

$$\lim_{t\to\infty}|\eta(t)-\eta_+|=0,$$

$$\lim_{t\to\infty}\|A(t)-A_+-iB_+t\|=0, \tag{7}$$

$$\lim_{t\to\infty}\|B(t)-B_+\|=0,$$

$$\lim_{t\to\infty}|S(t)-S_+-t\eta_+^2/2|=0.$$

This result basically says that given an incoming free state we can find an interacting state that approaches it at infinite negative time. Then for almost any free incoming state there exists a free outgoing state that approximates the interaction state at infinite time. In the language of scattering, the above theorem is existence, uniqueness of scattering operators coupled with asymptotic completeness. The proof of this for the position and momentum variables $a(t)$, $\eta(t)$, is given in Ref. 8, the proof for the spreading variables $A(t)$, $B(t)$ and the action variable $S(t)$, is given in Ref. 1.

IV. STATEMENT AND PROOF OF THE MAIN RESULT FOR $N=1$

Let

$$H(\hbar)=-\frac{\hbar^2}{2}\Delta_x+V(x)$$

and

$$H_1(t,\hbar)=-\frac{\hbar^2}{2}\Delta_x+W_{a(t)}(x),$$

with corresponding unitary propagators $U(t)$ and $U_1(t,0)$, respectively. Recall

$$U_1(t,0)\phi_0(A(0),B(0),\hbar,a(0),\eta(0),x)=e^{iS(t)/\hbar}\phi_0(A(t),B(t),\hbar,a(t),\eta(t),x).$$

Theorem 1: *If $V(x)$ satisfies the short-range assumption (D), then there exists C, $\lambda>0$, both independent of t and $\hbar$ such that*

$$\|U(t)\phi_0(A(0),B(0),\hbar,a(0),\eta(0),\cdot)-e^{iS(t)/\hbar}\phi_0(A(t),B(t),\hbar,a(t),\eta(t),\cdot)\|_2\leqslant C\hbar^{\lambda}$$

for all $t\in(-\infty,\infty)$, $\hbar\in(0,1)$, any $A(0)$, $B(0)$ satisfying Eqs. (2) and (3) and almost all $a(0)$, $\eta(0)$.

Remark: The theorem is an analogous statement to that of lemma 2 for the semiclassical wave packets.

The proof given in Ref. 1 that is restricted to $n \geqslant 3$ uses the fact that the wave packet decays as $t^{-n/2}$, and thus the wave packet is itself in L^1 when $n \geqslant 3$. For $n=1$ and $n=2$ we remove the portion of the state that has small asymptotic momentum. This portion of the wave packet is $O(\hbar^{1/2})$. The remaining portion of the wave packet decays fast enough in t to prove the estimates we need. Our idea is to write the wave packet as

$$\phi_0(A(t),B(t),\hbar,a(t),\eta(t),x) = \frac{p}{\eta}\phi_0(A(t),B(t),\hbar,a(t),\eta(t),x) + \frac{\eta - p}{\eta}\phi_0(A(t),B(t),\hbar,a(t),\eta(t),x)$$

and then drop the second term at time 0 in order to get the asymptotics to cancel out correctly. The intuition is that the second term above is on the order of $\sqrt{\hbar}$ at time zero and can be disregarded in the semiclassical limit. The idea to write the wave packet in this way was inspired by Ref. 6 and many ideas from this paper can be seen in the proof. We need the portion of the wave packet that is not disregarded to be propogated exactly by the semiclassics given in Sec. II, therefore we write the wave packet as

$$\begin{aligned}\phi_0(A(\tau),B(\tau),\hbar,a(\tau),\eta(\tau),x) &= \left\{1 + \frac{(x-a(\tau))iB_+}{A(\tau)\eta_+}\right\}\phi_0(A(\tau),B(\tau),\hbar,a(\tau),\eta(\tau),x) \\ &\quad - \frac{(x-a(\tau))iB_+}{A(\tau)\eta_+}\phi_0(A(\tau),B(\tau),\hbar,a(\tau),\eta(\tau),x) \\ &= \left\{1 + \frac{(x-a(\tau))iB_+}{A(\tau)\eta_+}\right\}\phi_0(A(\tau),B(\tau),\hbar,a(\tau),\eta(\tau),x) \\ &\quad - \sqrt{\frac{\hbar}{2}}\frac{iB_+}{\eta_+}\phi_1(A(\tau),B(\tau),\hbar,a(\tau),\eta(\tau),x) \\ &= \tilde{\phi}_0(A(\tau),B(\tau),\hbar,a(\tau),\eta(\tau),x) \\ &\quad - \sqrt{\frac{\hbar}{2}}\frac{iB_+}{\eta_+}\phi_1(A(\tau),B(\tau),\hbar,a(\tau),\eta(\tau),x). \end{aligned} \tag{8}$$

We have used the fact that in one dimension

$$\phi_1(A(t),B(t),\hbar,a(t),\eta(t),x) = \sqrt{\frac{2}{\hbar}}\frac{(x-a(t))}{A(t)}\phi_0(A(t),B(t),\hbar,a(t),\eta(t),x). \tag{9}$$

The argument of the theorem almost exactly follows the argument in Ref. 1. Besides the introduction of the modified wave packet the changes that we have made are imbedded in technical lemmas 3 and 4.

Proof of Theorem 1 for $n=1$: Let $\mu < 1$, $\epsilon \in \left(0, \frac{1}{6}\right)$, and define

$$\chi_1(\hbar,a(t),x) = \begin{cases} 1 & \text{if } |x-a(t)| \leqslant (1+|a(t)|)\mu\hbar^{1/2-\epsilon}, \\ 0 & \text{otherwise}. \end{cases}$$

Define $\chi_2(\hbar,a(t),x) = 1 - \chi_1(\hbar,a(t),x)$. Now define $\tilde{\phi}_0(A(\tau),B(\tau),\hbar,a(\tau),\eta(\tau),x)$ as above and proceed to calculate. By (8) and since $\{U(t)-U_1(t,0)\}$ is bounded by lemma 2 it is clear that

$$\begin{aligned}&\|\{U(t)-U_1(t,0)\}\phi_0(A(0),B(0),\hbar,a(0),\eta(0),\cdot)\|_2 \\ &\quad \leqslant \|\{U(t)-U_1(t,0)\}\tilde{\phi}_0(A(0),B(0),\hbar,a(0),\eta(0),\cdot)\|_2 + k\sqrt{\hbar}, \end{aligned} \tag{10}$$

where

$$k=\frac{|B_+|}{\sqrt{2}|\eta_+|}.$$

By the fundamental theorem of calculus,

$$\begin{aligned}&\|\{U(t)-U_1(t,0)\}\tilde{\phi}_0(A(0),B(0),\hbar,a(0),\eta(0),\cdot)\|_2\\&=\left\|\int_0^t\frac{d}{ds}\{U(s)-U_1(s,0)\}\tilde{\phi}_0(A(0),B(0),\hbar,a(0),\eta(0),\cdot)ds\right\|_2\\&\leqslant\hbar^{-1}\int_0^t\|\{V(\cdot)-W_{a(s)}(\cdot)\}\tilde{\phi}_0(A(s),B(s),\hbar,a(s),\eta(s),\cdot)\|_2\,ds.\end{aligned}\tag{11}$$

Analyzing the integrand in the last expression,

$$\begin{aligned}&\|\{V(x)-W_{a(s)}(x)\}\tilde{\phi}_0(A(s),B(s),\hbar,a(s),\eta(s),x)\|_2\\&\leqslant\|\{V(x)-W_{a(s)}(x)\}\chi_1(\hbar,a(s),x)\tilde{\phi}_0(A(s),B(s),\hbar,a(s),\eta(s),x)\|_2\\&\quad+\|V(x)\chi_2(\hbar,a(s),x)\tilde{\phi}_0(A(s),B(s),\hbar,a(s),\eta(s),x)\|_2\\&\quad+\|W_{a(s)}(x)\chi_2(\hbar,a(s),x)\tilde{\phi}_0(A(s),B(s),\hbar,a(s),\eta(s),x)\|_2=\mathrm{I}(s)+\mathrm{II}(s)+\mathrm{III}(s).\end{aligned}\tag{12}$$

If $|x-a(s)|\leqslant(1+|a(s)|)\mu\hbar^{1/2-\epsilon}$ then following the analysis from Ref. 1 we let $z_*\in\mathcal{Z}=\{z=rx+(1-r)y\}$ such that $|z_*|\leqslant|z|$ for all $z\in\mathcal{Z}$. By the fundamental theorem of calculus and the triangle inequality it can be seen that

$$|V_2(x)-V_2(y)|\leqslant C_3(1+|z|)^{-4-\nu}|x-y|\leqslant C_3(1+|y|-|y-z|)^{-4-\nu}\leqslant C_3[(1-\mu)(1+|y|)^{-4-\nu}|x-y|\ ,\tag{13}$$

where C_3 is taken from the short-range assumption. From here it follows that

$$\|\chi_1(\hbar,a(s),x)(V(x)-W_{a(s)}(x))\|_\infty\leqslant C_3(1+|a(s)|)^{-1-\nu}\hbar^{3/2-3\epsilon}.\tag{14}$$

Hence

$$\mathrm{I}(s)\leqslant C_3(1+|a(s)|)^{-1-\nu}\hbar^{3/2-3\epsilon}\left(1+k\sqrt{\frac{\hbar}{2}}\right).$$

Again we follow the argument in Ref. 1. Due to continuity and asymptotics of the classical quantities $a(s)$, $A(s)$ that

$$\begin{aligned}\mathrm{II}(s)&\leqslant\left\|\chi_2(\hbar,a(s),x)\exp\left\{\frac{-(x-a(s))^2}{4|A(s)|^2\hbar}\right\}\right\|_\infty\left\|\chi_2(\hbar,a(s),x)V(x)\left(1+\frac{(x-a(s))iB_+}{A(s)\eta_+}\right)\right.\\&\quad\left.\times(\pi\hbar)^{-1/4}(A(s))^{-1/2}\exp\left\{\frac{-(x-a(s))^2}{4|A(s)|^2\hbar}\right\}\right\|_2\\&\leqslant\exp\{-C'\hbar^{-2\epsilon}\}\left\|\chi_2(\hbar,a(s),x)V(x)\left(1+\frac{(x-a(s))iB_+}{A(s)\eta_+}\right)\right.\\&\quad\left.\times(\pi\hbar)^{-1/4}(A(s))^{-1/2}\exp\left\{\frac{-(x-a(s))^2}{4|A(s)|^2\hbar}\right\}\right\|_2,\end{aligned}\tag{15}$$

where C' is some constant independent of s and $\hbar$. By lemma 5.1 and dividing by $A(s)\eta_+$ there exists C_V, T_1 such that for $s>T_1$,

$$\mathrm{II}(s) \leqslant C_V \hbar^{-1/2-\nu/2} \exp\{-C' \hbar^{-2\epsilon}\} |s|^{-1-\nu/2}.$$

We can do the same thing with $\mathrm{III}(s)$ as well. By lemma 5.2 there exists T_2, C_W such that for $s > T_2$,

$$\mathrm{III}(s) \leqslant C_W \hbar^{-1} \exp\{-C' \hbar^{-2\epsilon}\} |s|^{-1-\nu}.$$

The theorem is now proven by taking $T = \max\{T_1, T_2\}$ and writing for $t > T$,

$$\hbar^{-1} \int_0^t (\mathrm{I}(s) + \mathrm{II}(s) + \mathrm{III}(s)) \mathrm{d}s = \hbar^{-1} \left\{ \int_0^T (\mathrm{I}(s) + \mathrm{II}(s) + \mathrm{III}(s)) \mathrm{d}s + \int_0^t (\mathrm{I}(s) + \mathrm{II}(s) + \mathrm{III}(s)) \mathrm{d}s \right\}.$$

The first term is bounded by some $C_T \hbar^{1/2}$ by lemma 1. The second term is bounded by some $C\hbar^{-2} \exp\{-C' \hbar^{-2\epsilon}\} + C_3 \hbar^{1/2-3\epsilon}$ by the work shown here. In order to propagate to large negative times we write the modified wave packet with η_-, B_- in place of η_+, B_+ and the details are the same. □

V. TECHNICAL LEMMAS

Lemma 3: In space dimension one if $V(x)$ satisfies the short-range assumption (D), then there exists some constant C such that for t sufficiently large, $\hbar \in (0,1)$,

$$\left\| \chi_2(\hbar, a(t), x) V(x) \{ \eta_+ A(t) + (x - a(t)) i B_+ \} (\pi\hbar)^{-1/4} (A(t))^{-1/2} \exp\left\{ \frac{-(x-a(t))^2}{4|A(t)|^2 \hbar} \right\} \right\|_2 \leqslant C \hbar^{-1/2-\nu/2} t^{-\nu/2},$$

where $\chi_2(\hbar, a(t), x)$ is as defined in the proof of Theorem 1.

Proof: Let $k_1 > 0$. By lemma 2 there exists T such that $t > T$ implies that

$$\begin{aligned} &\left\| \chi_2(\hbar, a(t), x) V(x) \{ \eta_+ A(t) + (x - a(t)) i B_+ \} (\pi\hbar)^{-1/4} (A(t))^{-1/2} \exp\left\{ \frac{-(x-a(t))^2}{4|A(t)|^2 \hbar} \right\} \right\|_2 \\ &\leqslant \Bigg\| \chi_2(\hbar, a(t), x) V(x) \cdot \{ \eta_+ (A_+ + i B_+ t) + (x - a_+ - \eta_+ t) i B_+ \} \\ &\quad \times (\pi\hbar)^{-1/4} (A(t))^{-1/2} \exp\left\{ \frac{-(x-a(t))^2}{4|A(t)|^2 \hbar} \right\} \Bigg\|_2 + k_1 \hbar^{-1/4} |A(t)|^{-1/2}. \end{aligned} \tag{16}$$

Using the triangle inequality we find that

$$\begin{aligned} &\left\| \chi_2(\hbar, a(t), x) V(x) \cdot \{ \eta_+ (A_+ + i B_+ t) + (x - a_+ - \eta_+ t) i B_+ \} (\pi\hbar)^{-1/4} (A(t))^{-1/2} \exp\left\{ \frac{-(x-a(t))^2}{4|A(t)|^2 \hbar} \right\} \right\|_2 \\ &\leqslant \left\| V(x) [\eta_+ A_+] (\pi\hbar)^{-1/4} (A^{-1/2}(t)) \exp\left\{ \frac{-(x-a(t))^2}{4|A(t)|^2 \hbar} \right\} \right\|_2 + \Bigg\| \chi_2(\hbar, a(t), x) V(x) [i B_+ (x - a_+)] \\ &\quad \times (\pi\hbar)^{-1/4} (A^{-1/2}(t)) \exp\left\{ \frac{-(x-a(t))^2}{4|A(t)|^2 \hbar} \right\} \Bigg\|_2. \end{aligned} \tag{17}$$

Since $V(x) \in L^2(\mathbb{R})$ we have some constant k_2 such that for large enough t,

$$\left\| V(x) [\eta_+ A_+] (\pi\hbar)^{-1/4} (A^{-1/2}(t)) \exp\left\{ \frac{-(x-a(t))^2}{4|A(t)|^2 \hbar} \right\} \right\|_2 \leqslant k_2 \hbar^{-1/4} t^{-1/2}. \tag{18}$$

Similarly,

$$\left\| \chi_2(\hbar,a(t),x)V(x)[iB_+(x-a_+)](\pi\hbar)^{-1/4}(A^{-1/2}(t))\exp\left\{\frac{-(x-a(t))^2}{4|A(t)|^2\hbar}\right\}\right\|_2$$
$$\leqslant \|iB_+V(x)(x-a_+)^{1/2+\nu/2}\|_2 \left\| \chi_2(\hbar,a(t),x)(\pi\hbar)^{-1/4}(A(t))^{-1/2}(x-a_+)^{1/2-\nu/2} \times\exp\left\{\frac{-(x-a(t))^2}{4|A(t)|^2\hbar}\right\}\right\|_\infty. \tag{19}$$

The first factor is a constant independent of t and $\hbar$. Evaluating the second term further we see that

$$\left\| \chi_2(\hbar,a(t),x)(\pi\hbar)^{-1/4}(A(t))^{-1/2}(x-a_+)^{1/2-\nu/2}\exp\left\{\frac{-(x-a(t))^2}{4|A(t)|^2\hbar}\right\}\right\|_\infty$$
$$= \left\| \chi_2(\hbar,a(t),x)\frac{(x-a_+)^{1/2-\nu/2}}{(x-a(t))^{1/2-\nu/2}}\frac{(x-a(t))^{1/2-\nu/2}}{(\pi\hbar)^{1/4}(A(t))^{1/2}}\exp\left\{\frac{-(x-a(t))^2}{4|A(t)|^2\hbar}\right\}\right\|_\infty$$
$$\leqslant \left\| \chi_2(\hbar,a(t),x)\frac{(x-a_+)^{1/2-\nu/2}}{(x-a(t))^{1/2-\nu/2}}\right\|_\infty \left\| \chi_2(\hbar,a(t),x)\frac{(x-a(t))^{1/2-\nu/2}}{(\pi\hbar)^{1/4}(A(t))^{1/2}}\exp\left\{\frac{-(x-a(t))^2}{4|A(t)|^2\hbar}\right\}\right\|_\infty$$
$$\leqslant \left\| \chi_2(\hbar,a(t),x)\frac{(x-a_+)^{1/2-\nu/2}}{(x-a(t))^{1/2-\nu/2}}\right\|_\infty \frac{\hbar^{-\nu/4+\epsilon\nu/2}}{(\mu(1+|a(t)|))^{\nu/2}} \left\| \frac{(x-a(t))^{1/2}}{(\pi\hbar)^{1/4}(A(t))^{1/2}}\exp\left\{\frac{-(x-a(t))^2}{4|A(t)|^2\hbar}\right\}\right\|_\infty. \tag{20}$$

The second norm in the last expression is bounded by a constant. For the first norm we see that

$$\left\| \chi_2(\hbar,a(t),x)\frac{(x-a_+)^{1/2-\nu/2}}{(x-a(t))^{1/2-\nu/2}}\right\|_\infty \leqslant \max\left\{1, \left\| \chi_2(\hbar,a(t),x)\frac{(x-a_+)}{(x-a(t))}\right\|_\infty\right\}. \tag{21}$$

Now we see that

$$\left\| \chi_2(\hbar,a(t),x)\frac{(x-a_+)}{(x-a(t))}\right\|_\infty \leqslant \left\| \chi_2(\hbar,a(t),x)\frac{(x-a(t)+a(t)-a_+)}{(x-a(t))}\right\|_\infty 1 + \left\| \chi_2(\hbar,a(t),x)\frac{(a(t)-a_+)}{(x-a(t))}\right\|_\infty$$
$$\leqslant 1 + \left\| \chi_2(,a(t),x)\frac{(a(t)-a_+)}{(1+a(t))\mu\hbar^{1/2-\nu/2}}\right\|_\infty \leqslant 1 + k_3\hbar^{-1/2+\nu/2}, \tag{22}$$

where k_3 is a constant independent of t and $\hbar$. The lemma now follows. □

Lemma 4: If $V(x)$ *satisfies the short-range assumption* (D), *then there exists some constant* C *such that for large enough* t, *and* $\hbar \in (0,1)$,

$$\left\| W_{a(t)}(x)\left[1+\sqrt{\frac{\overline{A(t)}}{A(t)}}\frac{(x-a(t))iB_+}{|A(t)|\eta_+}\right](\pi\hbar)^{-1/4}(A^{-1/2}(t))\exp\left\{\frac{-(x-a(t))^2}{4|A(t)|^2\hbar}\right\}\right\|_2 \leqslant C\hbar^{-1}t^{-1-\nu}.$$

Proof: Since $V(x)$ satisfies the short-range condition there exists C_j, $j=0,1,2$ such that

$$\left\| W_{a(t)}(x)\left[1+\sqrt{\frac{\overline{A(t)}}{A(t)}}\frac{(x-a(t))iB_+}{|A(t)|\eta_+}\right](\pi\hbar)^{-1/4}(A^{-1/2}(t))\exp\left\{\frac{-(x-a(t))^2}{4|A(t)|^2\hbar}\right\}\right\|_2$$
$$\leq \sum_{j=0}^{2} C_j(1+|a(t)|)^{-1-j-\nu}\cdot|A(t)|^j\cdot 2^j\cdot\hbar^{j/2}\left\|\left(\frac{(x-a(t))}{2|A(t)|\hbar^{1/2}}\right)^j\cdot\left\{1+\sqrt{\frac{\overline{A(t)}}{A(t)}}\frac{(x-a(t))iB_+}{|A(t)|\eta_+}\right\}\right.$$
$$\left.\times(\pi\hbar)^{-1/4}(A(t))^{-1/2}\exp\left\{\frac{-(x-a(t))^2}{4|A(t)|^2\hbar}\right\}\right\|_2. \tag{23}$$

By explicit evaluation, we see that the norms in the last expression are bounded by constants independent of t and $\hbar$. □

VI. EXTENSION TO TWO DIMENSIONS

The extension of this result to two dimensions has a few complications due to the structure of higher order states in more than one dimension. Here we point out the changes that need to be made in the proof of Theorem 1 in order to extend it to $n=2$. The techniques follow the construction given in Ref. 3. We present this in a less general manner for the sake of clarity. Let $\{e_1,e_2\}$ be the standard basis for $\mathbb{R}^2$. By the polar decomposition theorem for all t there exists a unique unitary matrix $U_A(t)$ such that $A(t)=|A(t)|U_A(t)$. We then define

$$\tilde{H}_1(v,x)=2\langle v,x\rangle$$

and

$$H_{e_j}(A(t);x)=\tilde{H}_1(U_A(t)e_j,x).$$

Now we proceed to define the higher order wave packet,

$$\phi_{e_j}(A(t),B(t),\hbar,a(t),\eta(t),x)=2^{-1/2}H_{e_j}(A(t);\hbar^{-1/2}|A(t)|^{-1}(x-a(t)))\phi_0(A(t),B(t),\hbar,a(t),\eta(t),x) \tag{24}$$

$$=2^{1/2}\langle U_A(t)e_j,\hbar^{-1/2}|A(t)|^{-1}(x-a(t))\rangle\phi_0(A(t),B(t),\hbar,a(t),\eta(t),x). \tag{25}$$

Now define

$$\tilde{\phi}_0(A(t),B(t),\hbar,a(t),\eta(t),x)=\left\{1+\left\langle U_A(t)e_1,\frac{i|A(t)|^{-1}B_+(x-a(t))}{\langle e_1,\eta_+\rangle}\right\rangle\right\}\phi_0(A(t),B(t),\hbar,a(t),\eta(t),x) \tag{26}$$

and the modified wave packet is again propagated as in Theorem 1. Since we have assumed that $\eta_+\neq 0$ we can use e_2 instead of e_1 if $\langle e_1,\eta_+\rangle=0$. Recall

$$U_A(t)=|A(t)|^{-1}A(t),$$

implying

$$U_A^*(t)=A^{-1}(t)|A(t)|,$$

and so similar to the analysis in one dimension we have

$$\tilde{\phi}_0(A(t),B(t),\hbar,a(t),\eta(t),x) = \left\{ \frac{\langle e_1,\eta_+\rangle}{\langle e_1,\eta_+\rangle} + \left\langle e_1, \frac{iA^{-1}(t)B_+(x-a(t))}{\langle e_1,\eta_+\rangle} \right\rangle \right\} \phi_0(A(t),B(t),\hbar,a(t),\eta(t),x)$$
$$= \frac{1}{\langle e_1,\eta_+\rangle}\langle e_1,\eta_+ + A^{-1}(t)(x-a(t))iB_+\rangle \phi_0(A(t),B(t),\hbar,a(t),\eta(t),x)$$
$$= \frac{1}{\langle e_1,\eta_+\rangle}\langle e_1,A^{-1}(t)\{A(t)\eta_+ + (x-a(t))iB_+\}\rangle \phi_0(A(t),B(t),\hbar,a(t),\eta(t),x). \tag{27}$$

Noting that

$$V(x)(1+x)^{\nu/2} \in L^2(\mathbb{R}^2)$$

and thus

$$\|iB_+V(x)(x-a_+)^{\nu/2}\|_2$$

is constant in place of

$$\|iB_+V(x)(x-a_+)^{1/2+\nu/2}\|_2$$

in the one dimensional case, the proof is now analogous to the proof for $n=1$.

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