PROCESS CONTROL:
A DYNAMIC PROGRAMMING APPROACH

by

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Chapter I

INTRODUCTION

Traditionally, the design of quality control systems has been based on statistical criteria. Specifically, control charts have been used extensively since their initial introduction by Walter A. Shewhart in 1924. Although tools such as these are statistically appealing, their economic aspects are often undesirable. In light of this fact, much research and effort has been afforded to the economic design of quality control systems. For the most part, this research has been divided between the areas of process control and acceptance sampling.

With regard to process control, it is normally a manager's goal to in some way minimize the cost of operating his process. Upon this premise, this thesis will endeavor to develop an economic model describing a typical process control situation. A Bayesian decision theory approach will be undertaken and the model will be formulated using the techniques of dynamic programming.

Process Quality Control

The function of a quality control procedure is to determine the condition of the process. In most general terms, the process is either in-control or out-of-control. But more specifically, it is often desirable to know the degree, if any, of maladjustment. That is to say, how far out-of-control is the process? Because of the inherent variation of all processes, no machine can be expected to manufacture all items at the same level of quality. A machine may operate automatically, but, in practice, the quality of its performance will deteriorate unless it is examined from time to time and repaired when it is deemed necessary.

In general, any operating process is subject to two types of variation. There are variations due to randomness or chance in the process operation. Such deviations are uncontrollable, but are normally tolerable since their impact is small. But secondly, variations due to assignable causes are of great concern to a process operator. Often such variations plague a process and are responsible for the production of both undesirable and unacceptable quality. In view of this latter cause, a quality control procedure should specify regular inspections of the process output to determine whether or not to stop and overhaul the machine.

The use of control charts, first introduced by Walter A. Shewhart [29], has been a valuable statistical tool in process control. Undoubtedly the most widely used chart is for controlling the average operating level of a process, $\overline{X}$. Traditionally accepted control limits have been positioned three standard deviations on either side of the center line while sample sizes of either four or five items have been inspected. Additionally, samples of the process output have been collected at convenient intervals.

Because of the increased emphasis on economical operation in industry over the past twenty-five years and the mushrooming of competition, the expensive use of purely statistical quality control procedures has, in some process control applications, become less and less desirable. Conditions such as these initiated the interest in the design of economically based quality control plans.

Survey of Previous Research

There are numerous authors in the area of designing cost-based or economic quality control systems. One such author is Duncan [10] [11] who proposed a procedure for determining the sample size, the interval between samples, and the control limits for an $\overline{X}$ control chart which maximize the average net income when a single assignable cause or out-of-control state exists. His assumptions were that the process could shift from the in-control state to a single out-of-control state at anytime during the day and that the process possessed the Markov property. Goel et al. [15] then extended Duncan's model by developing an algorithm for computing the optimal test parameters. Lave [23] considered the optimal choice of process control plans based on a two-state Markov model and in addition examined means in selecting the optimum parameters for the respective plans.

More recent research by Duncan [12] and Knappenberger and Grandage [20] addressed the problem of determining the optimal parameters for an $\overline{X}$ control chart when there are several assignable causes. The latter authors assumed that the process parameter, μ, was a continuous random variable but that it could be satisfactorily approximated by a discrete random variable. One value, μ_0, of the discrete random variable was associated with the single in-control state, and the remaining values μ_1, μ_2, ..., μ_s were associated with out-of-control values of the process parameter. Their model was then formulated based on three pertinent costs: sampling, investigating and correcting the process, and producing defective units.

An even more recent article was published by Gabria [13] who again considered the economic design of an $\overline{X}$ control chart to optimally determine its parameters. This author introduced a new element into the cost analysis defined as the worst cycle quality level (W.C.Q.L.). This W.C.Q.L. offered the chart user the option of placing a constraint on the permissible mean expected number of defectives produced within a quality cycle time. Chiu and Wetherill [8] proposed a semi-economic scheme for the design of a control plan using an $\overline{X}$ chart. These researchers allow the Q.C. engineer to select a consumer's risk point on the OC-curve and then proceed to outline a step-by-step procedure for determining the optimal control chart design parameters.

In all cases, the previously discussed authors only addressed their research for a process where a single measurable quality characteristic was of interest. Contrary to this doctrine, Montgomery and Klatt [24] formulated an economic process control model based on dependent multiple characteristics and utilized the Hotelling T^2 control chart. Latimer et al. [22] have proposed a multicharacteristic model with the dual purpose of controlling the lot mean and determining the lot disposition.

The problem of establishing control policies has been undertaken by only a few researchers using the Bayesian decision theory approach. Such authors as Girshick and Rubin [14] inspected the problem of minimizing the expected running and repair costs while operating a process in an equilibrium condition. They assumed that the process level could occur in four possible states: two operating states and two repair states. Bather [2] formulated a model using dynamic programming and avoided the difficulties in assuming the existence of steady state conditions. Bather developed optimal stopping rules which indicated when the process should be stopped and overhauled and when it should be allowed to continue production. His work was later extended by Carter [7] who included the determination of optimal sample sizes to collect at each inspection and the optimal interval between sampling. Both Bather and Carter addressed their work to a process which has a single quality characteristic of interest. Further, they assumed that the quality characteristic was a continuous random variable and was normally distributed. The process mean was also considered to be a continuous random variable whose successive increments behaved as a random walk. Additionally, Carter proved that the recursive equations of the dynamic programming formulation converged to a limit as the time index on operating the process grew without bound. Hence, a single set of operating policies was shown to exist in the steady state environment.

Another application of dynamic programming was effected by Pruzan [27] who considered a system of consecutive operations and determined points of either no inspection or 100% inspection. Taylor [30] has shown that inspecting a fixed number of items at a fixed interval of time is non-optimal. Instead, decisions with respect to sampling should be executed in a sequential manner based on current posterior probabilities. Using this approach, Taylor [31] developed an optimal control procedure assuming that the process exists in only two states, in-control and out-of-control.

The previously discussed background in the design of cost-based process control plans is by no means intended to exhaust the body of research. But rather, it is offered only as a general description of the primary areas of concentration.

Purpose

The review of previous research has revealed a variety of approaches to the economical design of quality control systems. Generally, most authors have assumed that the Markov property exists in the operation of the process model and also that a steady state or equilibrium condition has been achieved. Several authors have employed a Bayesian decision theory approach to incorporate historical data with current sample observations in making present operating decisions.

Still the influence of the $\overline{X}$ control chart has persuaded many researchers to attempt to increase its salability. But one of the shortcomings of the $\overline{X}$ chart is that explicit consideration of historical decisions is not incorporated while making present operating decisions. In order to overcome this malady, this thesis will employ the techniques of Bayesian decision theory. It is next proposed that not all processes operate in the steady state environment. For example, consider a job shop operation which may require a production run of several days. In following the operating policies of an $\overline{X}$ control chart, the situation may arise where the policy indicates an out-of-control condition and that an overhaul is warranted. But suppose only five to six more hours of production may complete the run. Query--is it better, economically, to initiate an expensive overhaul now, rather than turning out a higher percent of defective products for the remainder of the production run? Such a question is often not easily answered, but here is where a dynamic programming approach can supply the necessary guidelines. Accordingly, this thesis will first formulate the model in a time dependent environment and employ the technique of dynamic programming. The cost factors to be included in the model formulation are for sampling, investigating and overhauling the process, and producing defective items. The pertinent design parameters to be determined are the sample size at each inspection, the interval between successive inspections, and finally the values of the sample outcome which will dictate the proper operating decisions.

The model will only accommodate process operations which have a single quality characteristic of interest. It will be assumed that this characteristic is a continuous random variable as is the process parameter μ, the mean operating level. From this point, this thesis will follow the approach of Knappenberger and Grandage [20] in assuming that the parameter μ can be satisfactorily approximated by a discrete random variable and that the time between successive shifts in the operating level is dictated by the exponential distribution.

Finally, following the guidelines of Bather [2] and Carter [7] the recursive equations will be shown to converge to a limit as the time index grows without bound. Hence an optimal set of design parameters will be obtained to illustrate the steady state optimal control policies. Upon accomplishing this, a comparison of the results obtained will be made with the optimally designed $\overline{X}$ control chart proposed by

Knappenberger and Grandage [20]. The evaluation will be based upon how well each approach attains its premise of minimizing the operating cost.

Outline of Succeeding Chapters

In Chapter II, a general development of the dynamic programming formulation of the process control model is presented. A cursory description of the techniques of Bayesian decision theory is provided. After developing the recursive equation of the dynamic model, a steady state optimal policy is shown to exist via an inductive proof of convergence.

In Chapter III, the dynamic model is presented in detail. Initially, the Markovian assumption is used to derive the probabilities of shifts in the state of the process mean. The transition probability matrix is derived along with an explicit form of the dynamic recursive equation. In addition, the steady state $\overline{X}$ control chart model developed by Knappenberger and Grandage [20] is presented to illustrate the similarities in its formulation.

In Chapter IV, the numerical results for five example problems are presented. For each example, the optimal steady state operating policies are derived from both the dynamic model and the $\overline{X}$ control chart model. These policies are interpreted and a comparison of the optimal costs obtained is effected.

Chapter V provides a summary of, and conclusions about, the findings of the research. Also, areas for further research are discussed.

Chapter II

DEVELOPMENT OF THE DYNAMIC APPROACH

In this chapter a general description of the process model is provided along with assumptions concerning the pertinent costs incurred while operating the process. A cursory discussion of the Bayesian decision theory approach is included and the recursive equation of the dynamic programming formulation is derived. Then employing some results due to Carter [7], the optimal operating policies are shown to converge to a steady state optimal policy as the time index on operating the process grows without bound.

General Description of the Model

Consider a typical manufacturing process which fabricates units that have a quality characteristic of interest which can be described in terms of a single continuous random variable. This random variable, x, is assumed to be dependent upon the process mean, μ, which itself is a continuous random variable. Adopting the approach of Knappenberger and Grandage [20], it is assumed that the process mean, μ, can be satisfactorily approximated by a discrete random variable. One value, $\mu(0)$, of the discrete random variable is associated with a single in-control state of the process mean, and the remaining values $\mu(-s)$, $\mu(-s+1)$, ..., $\mu(-1)$ and $\mu(1)$, $\mu(2)$, ..., $\mu(s)$ are all associated with the out-of-control states of the process mean.

At any time t during the process operation, the quality characteristic, x_t, is assumed to be normally distributed with mean $\mu_t(i)$ and variance σ^2, where σ^2 is constant for any time t. For the purpose of the analysis, the time index will run backwards starting with T, then T-1, T-2, ..., t+1, t, ..., 2, 1, 0. Samples of size n_t are collected from the process output at each period t and the sample mean, $\bar{x}_t$, is calculated. It is assumed that $\bar{x}_t$ is normally distributed with mean $\mu_t(i)$ and variance σ^2/n_t.

In order to further describe the stochastic nature of the process, it is assumed that the time between successive shifts in the state of the process mean is exponentially distributed with parameter λ. Any shift from the in-control state, $\mu(0)$, to an out-of-control state $\mu(i)$ ($i = -s, -s+1, \ldots, -1, 1, 2, \ldots, s$), is considered to be a deterioration of the process; and once a shift does occur, the process can only be returned to the in-control state by an adjustment or an overhaul.

General Bayesian Decision Theory Approach

The main objective of a process operator is to monitor and control his process so as to minimize the total future costs of its operation over some finite or possibly infinite planning horizon. Since the processing operation is inherently random, an operator cannot know with certainty what costs he will incur. Accordingly, he must make inspections of the process from time to time and based on the results obtained, effect operating decisions with the purpose of minimizing total future expected costs.

Prior and Posterior Probabilities

At any time t, the operating decision, either to overhaul or not to overhaul, should be based on the value of the process mean. But the process mean is never known with certainty. Accordingly, the operator must assess a prior probability for the state of the process (i.e., prior to sampling). These prior beliefs can be based on past experience in the form of historical data, or even upon subjective judgement.

Define the ith element, $\hat{\alpha}_t'(i)$, of the probability state vector $\underline{\hat{\alpha}}_t'$, to be the prior probability that the process mean is in state i at time t (the prime denotes a prior belief). After observing a sample of size n_t at time t, the operator must make an operating decision. At this point, he has two sources of information describing the process: the prior state vector $\underline{\hat{\alpha}}_t'$, and the sample result $\bar{x}_t$. These two sources may be combined to depict the posterior condition of the process. Now define the element $\hat{\alpha}_t''(i)$ of the probability state vector $\underline{\hat{\alpha}}_t''$, to be the posterior probability that process mean is in state i at time t (the double prime denotes a posterior belief). Having the posterior information about the state of the process, the operator can effectively choose to either overhaul the process or allow it to continue production at its present level.

In order to derive the posterior state vector, Bayes formula must be employed. In general terms the formula is given by

$$P(A/B) = \frac{P(B/A) \cdot P(A)}{P(B)} \quad . \qquad (2.01)$$

The posterior probabilities are dependent upon the prior probability state vector $\hat{\underline{\alpha}}_t'$ and the likelihood function of the sample mean, denoted by $\ell[\bar{x}_t|\mu_t(i),n_t]$. Hence the elements of the posterior state vector are derived in the following manner:

$$\hat{\alpha}_t''(i) = \frac{\ell[\bar{x}_t|\mu_t(i),n_t]\cdot\hat{\alpha}_t'(i)}{\sum_{\forall i}\ell[\bar{x}_t|\mu_t(i),n_t]\cdot\hat{\alpha}_t'(i)} \qquad (2.02)$$

where i = -s, -s+1, ..., 1, 0, 1, ..., s.

The prior beliefs about the state of the process must be explicitly defined for each period t, t-1, and so on. Since the stochastic nature of the process mean is described by a Markov process, the transition probabilities of the process, that is, the probabilities of a shift in the process mean from its present state to another state, must be defined. These transition probabilities are depicted in matrix form in Figure 2-1.

The row index of the matrix indicates the present state of the process mean at time t, whereas the column index describes the state of the process mean at the beginning of the following period, t-1. Each element, $\hat{b}_{ij}$, of the transition probability matrix $\hat{\underline{B}}$, represents the probability of a shift in the process mean from state i to state j in a single transition. It is also assumed that the transition probabilities remain stationary from one period to the next.

With the transition probability matrix defined in general, the procedure for obtaining the prior state vector at each period can be defined. Since the posterior beliefs about the process at time t incorporate the greatest amount of information, they will be used in deriving the prior beliefs for period t-1. Hence, the prior state

From	To: $\mu(-s)$	$\mu(-s+1)$	...	$\mu(-1)$	$\mu(0)$	$\mu(1)$	...	$\mu(s-1)$	$\mu(s)$
$\mu(-s)$	$\hat{b}_{-s,-s}$	$\hat{b}_{-s,-s+1}$	...	$\hat{b}_{-s,-1}$	$\hat{b}_{-s,0}$	$\hat{b}_{-s,1}$	...	$\hat{b}_{-s,s-1}$	$\hat{b}_{-s,s}$
$\mu(-s+1)$	$\hat{b}_{-s+1,-s}$	$\hat{b}_{-s+1,-s+1}$	...	$\hat{b}_{-s+1,-1}$	$\hat{b}_{-s+1,0}$	$\hat{b}_{-s+1,1}$	...	$\hat{b}_{-s+1,s-1}$	$\hat{b}_{-s+1,s}$
⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮
$\mu(-1)$	$\hat{b}_{-1,-s}$	$\hat{b}_{-1,-s+1}$	...	$\hat{b}_{-1,-1}$	$\hat{b}_{-1,0}$	$\hat{b}_{-1,1}$	...	$\hat{b}_{-1,s-1}$	$\hat{b}_{-1,s}$
$\mu(0)$	$\hat{b}_{0,-s}$	$\hat{b}_{0,-s+1}$	...	$\hat{b}_{0,-1}$	$\hat{b}_{0,0}$	$\hat{b}_{0,1}$	...	$\hat{b}_{0,s-1}$	$\hat{b}_{0,s}$
$\mu(1)$	$\hat{b}_{1,-s}$	$\hat{b}_{1,-s+1}$	...	$\hat{b}_{1,-1}$	$\hat{b}_{1,0}$	$\hat{b}_{1,1}$	...	$\hat{b}_{1,s-1}$	$\hat{b}_{1,s}$
⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮
$\mu(s-1)$	$\hat{b}_{s-1,-s}$	$\hat{b}_{s-1,-s+1}$	...	$\hat{b}_{s-1,-1}$	$\hat{b}_{s-1,0}$	$\hat{b}_{s-1,1}$	...	$\hat{b}_{s-1,s-1}$	$\hat{b}_{s-1,s}$
$\mu(s)$	$\hat{b}_{s,-s}$	$\hat{b}_{s,-s+1}$	...	$\hat{b}_{s,-1}$	$\hat{b}_{s,0}$	$\hat{b}_{s,1}$	...	$\hat{b}_{s,s-1}$	$\hat{b}_{s,s}$

Figure 2-1 The Generalized Transition Probability Matrix

vector, $\hat{\underline{\alpha}}_{t-1}'$, is obtained by the matrix multiplication:

$$\hat{\underline{\alpha}}_{t-1}' = \hat{\underline{\alpha}}_t'' \cdot \hat{\underline{B}} \; . \tag{2.03}$$

Each individual element of the prior state vector at t-1 is then

$$\hat{\alpha}_{t-1}'(j) = \sum_{\forall i} \hat{\alpha}_t''(i) \cdot \hat{b}_{i,j} \tag{2.04}$$

where i, j = -s, -s+1, ..., 1, 0, 1, ..., s.

A detailed derivation of the prior and posterior state vectors as well as the elements of the transition probability matrix is provided in Chapter III.

In order to ease the notational burden in subsequent sections let u_t' and u_t'' denote the general random variables for the prior and posterior means of the process, respectively, and let u^0 denote the state of the process mean immediately after an overhaul is completed.

Cost Assumptions

The costs incurred in operating and maintaining the repetitive process generally arise from three sources.

1. Overhauling cost - This cost is incurred whenever the process is halted to be adjusted or to be examined for assignable causes. In general, this cost can be assumed to be a function of the state of the process mean, but for most actual processes a fixed average cost for maintenance is probably most realistic. Let $k[\mu_t(i)]$ be the general cost expression for the overhauling cost where $\mu_t(i)$ is the mean of the process at time t. After sampling at time t, the state of the process

mean is still not known with certainty; thus, the overhaul cost must be replaced by an expected cost. Let $K(u_t'')$ be the posterior expected cost of an overhaul where

$$K(u_t'') = \underset{\mu_t(i)/u_t''}{E} k[\mu_t(i)] \cdot \qquad (2.05)$$

It is assumed that $K(0) > 0$. That is to say, a cost will be incurred regardless of the state of the process mean whenever an overhaul is initiated.

2. Operating cost - This cost is incurred during the productive cycle of the process. The cost will be solely attributed to the manufacture of defective units when the process is in a particular state $\mu_t(i)$. The cost of operating for one period will be a function of the state of process mean, the production rate, the length of the period, and the cost per defective unit produced. Let this cost be denoted by:

$$c[\mu_t(i)] = R \cdot h \cdot a_4 \cdot \hat{f}(i) \qquad (2.06)$$

where

R is the production rate,

h is the length of the period,

a_4 is the cost of a defective unit, and

$\hat{f}(i)$ is the conditional probability of producing a defective unit given that the process mean is $\mu_t(i)$.

Since $\mu_t(i)$ is a random variable, the general cost expression, $c[\mu_t(i)]$, must be replaced by an expected cost. Let $C(u_t'')$ be the posterior expected cost of operating the process for one period where

$$C(u_t'') = \underset{\mu_t(i)/u_t''}{E} c[\mu_t(i)] \, . \tag{2.07}$$

3. Sampling cost - This cost is incurred at each period when the process output is inspected. In general, this cost will involve two components. Let a_1 be the fixed cost of sampling for each period and let a_2 be the variable cost per unit sampled. Let the general cost expression be given by

$$S(n_t) = a_1 + a_2 \cdot n_t \, . \tag{2.08}$$

It will be assumed that the fixed portion of the sampling cost, a_1, is incurred at each sampling point, even if the optimal size is zero. This seems to be a reasonable assumption for many real processes.

<u>List of Variables</u>

t - time (stage) index; where t runs backwards from $t = T$, $t = T-1, \ldots, t = 1, t = 0$.

x_t - continuous random variable representing the quality characteristic of the product at time t.

$\mu_t(i)$ - the ith state of the process mean at time t; assumed to be a discrete random variable where $i = -s, -s-1, \ldots, -1, 0, 1, \ldots, s-1. s$.

σ^2 - the process variance of the quality characteristic x_t; assumed to be constant for all t.

$N[\mu_t(i), \sigma^2]$ - the distribution of x_t at time t.

$\bar{x}_t$ - the sample mean observed at time t.

n_t - the sample size at time t.

$N[\mu_t(i),\sigma^2/n_t]$ - the distribution of the sample mean, $\bar{x}_t$, at time t.

u_t' - the general expression for the prior mean of the process at time t.

$\hat{\underline{\alpha}}_t'$ - the prior probability mass function of the state of the process at time t with elements $\hat{\alpha}_t'(i)$.

u_t'' - the general expression for the posterior mean of the process at time t.

$\hat{\underline{\alpha}}_t''$ - the posterior probability mass function of the state of the process at time t.

u^0 - the state of the process mean immediately after an overhaul; known with certainty to be $\mu_t(0)$, the in-control state.

a_1,a_2 - the fixed and variable costs for sampling.

a_4 - the unit cost of a defect.

R - the production rate of the process.

h - the length of a time period from t to t-1.

λ^{-1} - the mean time between successive shifts in the process mean; assumed to be constant for all t.

$\hat{\underline{f}}$ - the conditional probability mass function of producing a defective unit given that the process is in state $\mu_t(i)$; a row vector with elements $\hat{f}(i)$.

Development of the Recursive Equation

After sampling at time t and computing the posterior state vector, $\hat{\underline{\alpha}}_t''$, for the process mean, the process operator must choose to either overhaul or not overhaul the process. At each period, this decision should not be based entirely on the present economic state of the system, but should also include the minimization of the future expected costs of operating the process. If an overhaul is initiated, the cost of the overhaul plus the cost of operating the process for one period at the adjusted level will be incurred. Otherwise, if no overhaul is performed, the cost of operating the process at the current state of the process mean will be contracted. Additionally, the decision now will effect the costs of operating in future periods since the state of the process mean is highly dependent upon when the process was last overhauled.

Consequently, after sampling at time t, the process operator should choose,

$$\min \begin{bmatrix} K(u_t'') + C(u^0) + V_{t-1}(u_{t-1}') \\ \\ C(u_t'') + V_{t-1}(u_{t-1}') \end{bmatrix} \tag{2.09}$$

where $V_{t-1}(u_{t-1}')$ represents the minimum future expected cost of having prior beliefs, u_{t-1}', with t-1 periods left to operate. It should be reemphasized that the values of $V_{t-1}(u_{t-1}')$ included in each equation are dependent upon the present operating decision, and consequently will not be identical.

Prior to sampling at time t, the process operator cannot know what sample result he will see. But he does have a predictive distribution for the sample mean, $\bar{x}_t$, which is dependent upon the prior state of the process and the sample size. This density function may be derived in the following fashion:

$$g(\bar{x}_t | u_t', n_t) = \sum_{\forall i} \ell[\bar{x}_t | \mu_t(i), n_t] \cdot \hat{\alpha}_t'(i) \ . \tag{2.10}$$

Taking the expected value of equation (2.09) for all values of $\bar{x}_t$, the process operator may find the future expected costs before sampling, conditioned upon the prior state at time t and n_t. Hence, the prior future expected cost is given by

$$\underset{\bar{x}_t | u_t', n_t}{E} \quad \min \begin{bmatrix} K(u_t'') + C(u^0) + V_{t-1}(u_{t-1}') \\ C(u_t'') + V_{t-1}(u_{t-1}') \end{bmatrix} + S(n_t) \ . \tag{2.11}$$

But at this point, the process operator should choose the sample size n_t, which will minimize his prior expected costs. Thus, the minimum future expected cost given prior beliefs, u_t', with t periods left to operate is:

$$V_t(u_t') = \min_{n_t} \left\{ \underset{\bar{x}_t | u_t', n_t}{E} \quad \min \begin{bmatrix} K(u_t'') + C(u^0) + V_{t-1}(u_{t-1}') \\ C(u_t'') + V_{t-1}(u_{t-1}') \end{bmatrix} + S(n_t) \right\} . \tag{2.12}$$

This expression is the general form of the recursive equation for a dynamic programming problem. It can be solved recursively each period (t = 1, 2, 3, ...) for the optimal policy [7]. The policies will depict, for each set of prior probabilities (prior state vector $\underline{\hat{\alpha}}_t'$), what sample size to collect and, based on the sample result, what operating decision to make.

Convergence of the Optimal Policy to a Steady State Condition

The model formulation, up to this point, has been in a time dependent environment. Consequently, a process operator must always be cognizant of the number of periods left to run whenever he chooses a sample size and makes an operating decision. In many process control applications the number of periods of operation is very large and consequently the use of such an approach may tend to be impractical. But in such a situation it is plausible to consider the existence of a steady state optimal policy. That is to say, as the number of periods of operation becomes large, do the optimal policies at each period converge to a steady state policy? If such a policy exists, it would greatly simplify the use of the control procedure since an operator would not have to be cognizant of time while making operating decisions.

In order to show that a steady state optimal policy does exist, the recursive equation of the dynamic programming formulation must be shown to converge to a limit as the time index grows without bound. If such a limit can be shown to exist, then the so-called infinite horizon equation will be obtained.

In attempting to prove the existence of the infinite horizon equation it is necessary to include a discount factor, a, in the recursive equation to restrict the costs from becoming infinite as the time index approaches infinity. Including this discount factor, the recursive equation becomes

$$V_t(u_t') =$$

$$\min_{n_t} \left\{ \underset{\bar{x}_t | u_t', n_t}{E} \min \begin{bmatrix} K(u_t'') + C(u^0) + aV_{t-1}(u_{t-1}') \\ C(u_t'') + a\, V_{t-1}(u_{t-1}') \end{bmatrix} + S(n_t) \right\} \tag{2.13}$$

In order to derive the infinite horizon equation, the sequence $\{V_t(u_t')\}$ must initially be shown to be uniformly bounded, and secondly be shown to be uniformly convergent. The following theorem is a slightly modified version of the derivation outlined by Carter [7]:

Theorem I:

Assume,

i. $k[\mu_t(i)]$ is bounded for all possible states of the process mean (i.e., $k[\mu_t(i)] \leq K_0 < \infty$ for all $\mu_t(i)$).

ii. The cost of operating the process for one period after an overhaul is bounded (i.e., $C(u_0) = C_0 < \infty$).

iii. The cost of maintaining an inspection procedure each period is a constant positive value when the optimal sample size is zero (i.e., $S(0) = S_0 < \infty$).

Then, $V_t(u_t')$ is uniformly bounded for all states of the process mean and for all values of u_t' and t.

Proof:

At each period t, the process operator has the opportunity of not sampling, overhauling the process, and running for one period. The cost of this policy when t = 1 is

$$K(u_1') + C(u^0) + S(0) \leq K_0 + C_0 + S_0 \ . \tag{2.14}$$

Thus,

$$V_1(u_1') \leq K_0 + C_0 + S_0 \ . \tag{2.15}$$

For t = 2 the same policy is again an alternative, so

$$V_2(u_2') \leq K(u_2') + C(u^0) + S(0) + aV_1(u_1') \tag{2.16}$$

$$\leq K_0 + C_0 + S_0 + a(K_0+C_0+S_0) \ . \tag{2.17}$$

Then by induction,

$$V_t(u_t') \leq [K_0+C_0+S_0] \ (1+a+a^2+\ldots+a^{t-1}) \tag{2.18}$$

$$V_t(u_t') \leq \frac{K_0 + C_0 + S_0}{1 - a} \ . \tag{2.19}$$

The inequality holds for all t and all states of the prior mean u_t'; hence, $V_t(u_t')$ is uniformly bounded.

Q.E.D.

Now, if it can be shown that the sequence $\{V_t(u_t')\}$ is monotonically nondecreasing in t, then the sequence must be uniformly convergent since it is uniformly bounded by Theorem I. Before uniform convergence is proven, an important assumption must first be outlined. Because the process mean is assumed to occur in a finite number of discrete states, the possibility of considering too few states could arise in a practical analysis. Suppose, for example, the cost of performing an overhaul is extremely large and a single out-of-control state of the process mean is chosen relatively close to the in-control state. In such a case, it may be cheaper to let the process run even if it is out-of-control, rather than incur the cost of an overhaul. Consequently, to overcome this malady, it will always be assumed that the number of out-of-control states are chosen and positioned such that there exists one or more states of the process mean which make the cost of running the process for one period sufficiently greater than the overhaul cost. A convenient way to state this assumption is that as $\mu_t(s) \to \infty$, $C(u_t'') \to \infty$, where $\mu_t(s)$ is the state of the process mean which is farthest out-of-control; hence, a decision to overhaul will minimize the operating costs.

The following proof of uniform convergence is again a slightly modified version of a theorem proposed by Carter [7].

Theorem II:

Assume,

i. $C(u_t'') \to \infty$ as $u_t'' \to \infty$ ($u_t'' \to \infty$ implies $\mu_t(s) \to \infty$).

Then, the sequence $\{V_t(u_t')\}$ is uniformly convergent as $t \to \infty$.

Proof:

Let $\bar{n}_t$ be the value of n_t which minimizes

$$\underset{\bar{x}_t|u_t',n_t}{E} \min \begin{bmatrix} K(u_t'') + C(u^0) + aV_{t-1}(u_{t-1}') \\ C(u_t'') + aV_{t-1}(u_{t-1}') \end{bmatrix} + S(n_t) \tag{2.20}$$

Define

$$G_t(u'') = \min \begin{bmatrix} K(u_t'') + C(u^0) + aV_{t-1}(u_{t-1}') \\ C(u_t'') + aV_{t-1}(u_{t-1}') \end{bmatrix} \tag{2.21}$$

Substituting $\bar{n}_{t+1}$ and equation (2.21) into equation (2.13), one obtains

$$V_{t+1}(u_{t+1}') = \underset{\bar{x}_{t+1}|u_{t+1}',\bar{n}_{t+1}}{E} [G_{t+1}(u'')] + S(\bar{n}_{t+1}) \qquad (2.22)$$

$$V_t(u_t') = \underset{\bar{x}_t|u_t',\bar{n}_t}{E} [G_t(u'')] + S(\bar{n}_t) \qquad (2.23)$$

$$\leq \underset{\bar{x}_t|u_t',\bar{n}_{t+1}}{E} [G_t(u'')] + S(\bar{n}_{t+1}) \qquad (2.24)$$

Observe that this inequality must hold since $\bar{n}_t$ was defined to be the minimizing value of n_t. Hence, the value of $V_t(u_t')$ evaluated at $\bar{n}_{t+1}$ must at least be equal to or greater than the value of $V_t(u_t')$ evaluated at $\bar{n}_t$. Subtracting equation (2.24) from equation (2.22) yields

$$V_{t+1}(u') - V_t(u') \geq \underset{\bar{x}|u',\bar{n}_{t+1}}{E} [G_{t+1}(u'')-G_t(u'')] \qquad (2.25)$$

Observe that the time index has been omitted from the expressions u', u" and the parameter $\bar{x}$. This change was effected since the expressions and parameter take on the same set of values regardless of the time index. Additionally, the time index would only be a source of confusion in the remainder of the analysis.

If $[G_{t+1}(u'')-G_t(u'')] \geq 0$, for all possible values of u", then the value of $[V_{t+1}(u')-V_t(u')]$ must also be greater than or equal to zero by equation (2.25). The following inductive proof shows that $[G_{t+1}(u'')-G_t(u'')] \geq 0$ for all t and u".

It is first necessary to show that the inequality holds for t = 1. But first, it is desirable to distinguish between the values of u' when an overhaul was performed in the preceding period and when no overhaul was undertaken. If an overhaul did occur, the following notation will be used:

$$u' = u^* . \qquad (2.26)$$

From equation (2.21) one obtains

$$G_1(u'') = \min \begin{bmatrix} K(u'') + C(u^0) \\ \\ C(u'') \end{bmatrix} \qquad (2.27)$$

and

$$G_2(u'') = \min \begin{bmatrix} K(u'') + C(u^0) + aV_1(u^*) \\ \\ C(u'') + aV_1(u') \end{bmatrix} \qquad (2.28)$$

For simplicity, let OH denote a decision to overhaul and let R denote a decision to run without an overhaul; then there are four permutations of decisions that can be effected at periods one and two:

1) R-R 2) R-OH 3) OH-R 4) OH-OH .

The value of $[G_2(u'')-G_1(u'')]$ will now be examined for each case.

Case 1:

$$G_1(u'') = C(u'') \qquad (R)\ (2.29)$$

$$G_2(u'') = C(u'') + aV_1(u') \qquad (R)\ (2.30)$$

$$G_2(u'') - G_1(u'') = aV_1(u'') \geq 0 . \qquad (2.31)$$

The last inequality must hold since all costs are positive and consequently $V_1(u')$ must be positive.

Case 2:

$$G_1(u'') = C(u'') \qquad \text{(R)} \quad (2.32)$$

$$G_2(u'') = K(u'') + C(u^0) + aV_1(u^*) \qquad \text{(OH)} \quad (2.33)$$

$$G_2(u'') - G_1(u'') = K(u'') + C(u^0) - C(u'') + aV_1(u^*) \qquad (2.34)$$

Note that at period 1 the decision was made to run without an overhaul, which implies that this decision had the least expected cost, or that

$$K(u'') + C(u^0) - C(u'') = \Delta \geq 0 \qquad (2.35)$$

Hence,

$$G_2(u'') - G_1(u'') = \Delta + aV_1(u^*) \geq 0 \ . \qquad (2.36)$$

Case 3:

$$G_1(u'') = K(u'') + C(u^0) \qquad \text{(OH)} \quad (2.37)$$

$$G_2(u'') = C(u'') + aV_1(u') \qquad \text{(R)} \quad (2.38)$$

$$G_2(u'') - G_1(u'') = C(u'') - K(u'') - C(u^0) + aV_1(u') \qquad (2.39)$$

For $t = 1$ the process was overhauled which implies that

$$C(u'') - [K(u'')+C(u^0)] = \Delta \geq 0 \ . \qquad (2.40)$$

Hence,

$$G_2(u'') - G_1(u'') = \Delta + aV_1(u') \geq 0 \ . \qquad (2.41)$$

Case 4:

$$G_1(u'') = K(u'') + C(u^0) \qquad \text{(OH)} \quad (2.42)$$

$$G_2(u'') = K(u'') + C(u^0) + aV_1(u^*) \qquad \text{(OH)} \quad (2.43)$$

$$G_2(u'') - G_1(u'') = aV_1(u^*) \geq 0 \ . \qquad (2.44)$$

The preceding portion of the proof shows that for $t = 1$, $[G_{t+1}(u'')-G_t(u'')] \geq 0$ for all values of u". Consequently, $[V_2(u')-V_1(u')] \geq 0$.

In effecting a proof by induction, the next step is to assume that the inequality $[V_t(u')-V_{t-1}(u')] \geq 0$ holds for period t and then demonstrate that it holds for period t+1.

Assume that $V_t(u') \geq V_{t-1}(u')$. Again from equation (2.21) one obtains

$$G_{t+1}(u'') = \min \begin{bmatrix} K(u'') + C(u^0) + aV_t(u^*) \\ C(u'') + aV_t(u') \end{bmatrix} \tag{2.45}$$

and

$$G_t(u'') = \min \begin{bmatrix} K(u'') + C(u^0) + aV_{t-1}(u^*) \\ C(u'') + aV_{t-1}(u') \end{bmatrix} \tag{2.46}$$

Once again consider the four permutations of decisions that can be effected at periods t and t+1.

Case 1:

$$G_t(u'') = C(u'') + aV_{t-1}(u') \qquad \text{(R)} \tag{2.47}$$

$$G_{t+1}(u'') = C(u'') + aV_t(u') \qquad \text{(R)} \tag{2.48}$$

$$G_{t+1}(u'') - G_t(u'') = a[V_t(u')-V_{t-1}(u')] \geq 0 \tag{2.49}$$

This follows from the assumption that $V_t(u') \geq V_{t-1}(u')$.

Case 2:

$$G_t(u'') = C(u'') + aV_{t-1}(u') \qquad (R) \quad (2.50)$$

$$G_{t+1}(u'') = K(u'') + C(u^0) + aV_t(u^*) \qquad (OH) \quad (2.51)$$

$$G_{t+1}(u'') - G_t(u'') = K(u'') + C(u^0) - C(u'')$$

$$+ aV_t(u^*) - aV_{t-1}(u') \qquad (2.52)$$

$$= K(u'') + C(u^0) - C(u'') + aV_t(u^*)$$

$$- aV_{t-1}(u') + [aV_{t-1}(u^*) - aV_{t-1}(u^*)] \qquad (2.53)$$

$$= [K(u'') + C(u^0) + aV_{t-1}(u^*)]$$

$$- [C(u'') + aV_{t-1}(u')] + a[V_t(u^*) - V_{t-1}(u^*)] \qquad (2.54)$$

Observe that since the decision at period t was to run without an overhaul, the sum of the terms in the first bracket must be greater than the sum of the terms in the second bracket. Also, by assumption, the sum of the terms in the third bracket is greater than or equal to zero. Hence,

$$G_{t+1}(u'') - G_t(u'') \geq 0 \ . \qquad (2.55)$$

Case 3:

$$G_t(u'') = K(u'') + C(u^0) + aV_{t-1}(u^*) \qquad (OH) \quad (2.56)$$

$$G_{t+1}(u'') = C(u'') + aV_t(u') \qquad (R) \quad (2.57)$$

$$G_{t+1}(u'') - G_t(u'') = C(u'') - K(u'') - C(u^0)$$

$$+ aV_t(u') - aV_{t-1}(u^*) \qquad (2.58)$$

$$= C(u'') - K(u'') - C(u^0) + aV_t(u')$$

$$- aV_{t-1}(u^*) + [aV_{t-1}(u') - aV_{t-1}(u')] \qquad (2.59)$$

$$= [C(u'') + aV_{t-1}(u')] - [K(u'') + C(u^0) + aV_{t-1}(u^*)]$$

$$+ a[V_t(u') - V_{t-1}(u')] \qquad (2.60)$$

By the same argument as before, since the decision at period t was to overhaul, the sum of the terms in the first bracket must be greater than the sum of the terms in the second bracket. Also, by assumption, the sum of the terms in the third bracket is greater than or equal to zero. Hence,

$$G_{t+1}(u'') - G_t(u'') \geq 0. \tag{2.61}$$

Case 4:

$$G_t(u'') = K(u'') + C(u^0) + aV_{t-1}(u^*) \qquad \text{(OH)} \quad (2\ 62)$$

$$G_{t+1}(u'') = K(u'') + C(u^0) + aV_t(u^*) \qquad \text{(OH)} \quad (2.63)$$

$$G_{t+1}(u'') - G_t(u'') = a[V_t(u^*)-V_{t-1}(u^*)] \geq 0 \tag{2.64}$$

This follows by assumption.

The preceding manipulations show that if it is assumed that $V_t(u') \geq V_{t-1}(u')$, then $G_{t+1}(u'') \geq G_t(u'')$, for all values of t and u". It was previously shown by equation (2.25) that,

$$V_{t+1}(u') - V_t(u') \geq \underset{\bar{x}|u',\bar{n}_{t+1}}{E} [G_{t+1}(u'')-G_t(u'')] \tag{2.65}$$

thus,

$$V_{t+1}(u') - V_t(u') \geq 0 \tag{2.66}$$

for all values of u' and t.

Hence, $V_t(u')$ is monotonically nondecreasing in t and is uniformly bounded. Thus, the sequence $\{V_t(u')\}$ is uniformly convergent as t approaches infinity.

Q.E.D.

Now, because the sequence $\{V_t(u')\}$ has been shown to converge to a limit, it can be stated with certainty that, as the number of periods of operation becomes large, the optimal control policies at each period converge to a steady state optimal control policy. Consequently, a process operator will make his decisions based on the prior beliefs about the state of the process mean, and, assuming an equilibrium condition has been achieved, will not have to be cognizant of time.

Summary

In this chapter a generalized process control model has been formulated using the techniques of dynamic programming. A number of assumptions have been outlined and the pertinent costs have been discussed. A cursory explanation of the Bayesian decision theory approach was provided to serve as a preface to a more detailed development which follows in Chapter III. And finally, the optimal control policies were shown to converge to steady state policies which are independent of time.

Chapter III

DEVELOPMENT OF THE GENERAL MULTI-STATE MODELS

In this chapter, two multi-state models are presented. The dynamic model employs the technique described in Chapter II. All relations essential to the utilization of the dynamic model are developed and the procedure for optimizing the sampling interval is outlined. The steady state $\bar{X}$ control chart model, proposed by Knappenberger and Grandage [20], is presented to illustrate its similar formulation and as a preface to its utilization in Chapter IV.

The Dynamic Model

Recall from Chapter II that the process being modeled fabricates units which possess a single quality characteristic of interest. This characteristic, x_t, is assumed to be a continuous random variable which is normally distributed with mean $\mu_t(i)$ and variance σ^2. Also, the process mean, $\mu_t(i)$ is assumed to be a discrete random variable which can occur in 2s + 1 different states. These states are denoted by $\mu_t(-s)$, $\mu_t(-s+1)$, ..., $\mu_t(-1)$, $\mu_t(0)$, $\mu_t(1)$, ..., $\mu_t(s-1)$, $\mu_t(s)$ where $\mu_t(0)$ represents a single in-control state and the remaining states are all considered to represent an out-of-control condition of the process. Finally, the process is assumed to possess the Markov property.

The Markovian Assumption

Since shifts in the state of the process mean are described by a Markov process, the time between successive shifts is exponentially distributed. It will be assumed that the average amount of time the process stays in the in-control state before a shift occurs, given by λ^{-1}, remains constant for all periods of operation. The probability density function is then

$$f(t) = \lambda e^{-\lambda t} \tag{3.01}$$

for $t > 0$ and $\lambda > 0$.

If the length of a period of operation is h time units, then the probability that the process mean remains in the in-control state for a single period is

$$\hat{P}_{00} = 1 - \int_0^h \lambda e^{-\lambda t}\, dt = e^{-\lambda h} . \tag{3.02}$$

Given that the process has a production rate R, $\hat{P}_{00}$ can be expressed as

$$\hat{P}_{00} = e^{-\lambda k/R} \tag{3.03}$$

where k is the number of units produced during a period of length h. Now, letting $\lambda^* = \lambda/R$ one obtains,

$$\hat{P}_{00} = e^{-\lambda^* k} . \tag{3.04}$$

Observe that λ^{*-1} represents the average number of units produced before a shift occurs.

The probability that the process shifts out of the in-control state, $\mu_t(0)$, during the production is k units is then $1 - \hat{P}_{00}$ or $1 - e^{-\lambda^* k}$. Since there are 2s states other than $\mu_t(0)$, this probability must be allocated such that

$$\sum_{\substack{j=-s \\ j \neq 0}}^{s} \hat{P}_{0j} = 1 - e^{-\lambda^* k} \tag{3.05}$$

where $\hat{P}_{0j}$ represents the probability of a shift from state $\mu_t(0)$ to state $\mu_t(j)$. One method for allocating this probability, proposed by Knappenberger and Grandage [20] and later modified by Latimer et al. [22], will be employed. If it is assumed that $\hat{P}_{0,-j} = \hat{P}_{0j}$, then equation (3.05) may be expressed as

$$\sum_{j=1}^{s} \hat{P}_{0j} = \frac{1}{2} (1-e^{-\lambda^* k}) \ . \tag{3.06}$$

This probability may be distributed over each of the s states by utilizing the binomial mass function which represents the probability of j successes in s trials and is given by

$$Q(j) = \binom{s}{j} p^j (1-p)^{s-j} \tag{3.07}$$

where $j = 0, 1, \ldots, s$ and $0 < p < 1$. To allocate this probability, some fraction c, where $0 < c < 1$, is required such that

$$\sum_{j=1}^{s} \hat{P}_{0j} = c \sum_{j=1}^{s} Q(j) \ . \tag{3.08}$$

Summing equation (3.07) over the range of j gives

$$\sum_{j=0}^{s} Q(j) = 1 \tag{3.09}$$

which may be written as

$$\sum_{j=1}^{s} Q(j) = 1 - Q(0) \tag{3.10}$$

$$= 1 - (1-p)^s \ . \tag{3.11}$$

Substituting equations (3.06) and (3.11) into (3.08) and solving for c yields

$$c = \frac{\frac{1}{2}\,(1-e^{-\lambda^* k})}{1 - (1-p)^s} \,. \tag{3.12}$$

Consequently, the probability of a shift from state $\mu_t(0)$ to state $\mu_t(j)$ is given by

$$\hat{P}_{0,-j} = \hat{P}_{0j} = \frac{(1-e^{-\lambda^* k})}{2[1-(1-p)^s]} \binom{s}{j} p^j (1-p)^{s-j} \tag{3.13}$$

for j = 1, 2, ..., s [22].

Before deriving the remaining probabilities, $\hat{P}_{ij}$ and $\hat{P}_{ii}$, $i \neq 0$, it is necessary to outline several assumptions about the process. Initially, any shift from state $\mu_t(0)$ is considered as a deterioration of the process; and once the process goes out-of-control it stays out-of-control until an overhaul is performed. Thus, the process cannot correct itself. Secondly, once the direction of deterioration away from $\mu_t(0)$ is established, the process cannot improve but may get worse. That is to say, if the process shifts from $\mu_t(i)$ to $\mu_t(j)$ (or $\mu_t(-i)$ to $\mu_t(-j)$) where $|j| > |i|$, then subsequent shifts can deteriorate only to $\mu_t(j+1)$, ..., $\mu_t(s)$ (or $\mu_t(-j-1)$, ..., $\mu_t(-s)$). Finally, it is assumed that only one shift can occur during a single period of operation.

Now, the remaining probabilities $\hat{P}_{ij}$ and $\hat{P}_{ii}$, $i \neq 0$, can be derived. It seems plausible to assume that the probability of a deterioration from state $\mu_t(i)$ to state $\mu_t(j)$, $\hat{P}_{ij}$, is proportional to the probability of a deterioration from state $\mu_t(0)$ to state $\mu_t(j)$ [22]. Let

$$\hat{P}_{ij} = \frac{\hat{P}_{0j}}{\sum_{j=1}^{s} \hat{P}_{0j}} \tag{3.14}$$

$$= \frac{2\hat{P}_{0j}}{1 - \hat{P}_{00}} . \tag{3.15}$$

Then, $\hat{P}_{ii}$ is given by

$$\hat{P}_{ii} = 1 - \sum_{j=i+1}^{s} \hat{P}_{ij} \tag{3.16}$$

$$= \sum_{j=1}^{i} \hat{P}_{ij} \tag{3.17}$$

$$= \frac{2}{1 - \hat{P}_{00}} \sum_{j=1}^{i} \hat{P}_{0j} . \tag{3.18}$$

<u>The Transition Probability Matrix</u>

Recall the brief description of the transition probability matrix, $\hat{\underline{B}}$, in Chapter II. Each element of $\hat{\underline{B}}$, $\hat{b}_{ij}$, represents the probability of the process mean being in state j at period t-1, given that it was in state i at period t. In order to define each element, it is necessary to consider the two possible actions at time t. Namely, the process can either be overhauled or it can be allowed to run without an overhaul.

First, suppose no overhaul was performed at period t. Then the element $\hat{b}_{00}$, the probability of the process mean being in state 0 at period t-1, given that it was in state 0 at period t, is simply the probability of no shift. Hence, $\hat{b}_{00} = \hat{P}_{00}$. Likewise, $\hat{b}_{0j}$, the probability of the process mean being in state j at period t-1, given that it was in state 0 at period t, is the probability of a shift to state j. Hence, $\hat{b}_{0j} = \hat{P}_{0j}$ $(\hat{b}_{0,-j} = \hat{P}_{0,-j})$. By the same logic, the elements $\hat{b}_{ii}$, $i \neq 0$, and $\hat{b}_{ij}$, $j > i$ and $-j < -i$, must be given by $\hat{b}_{ii} = \hat{P}_{ii}$ and $\hat{b}_{ij} = \hat{P}_{ij}$. Since the process can only be corrected by an overhaul, the remaining elements of $\hat{\underline{B}}$ must all be equal to zero since each represents an improvement in, or a change in the direction of deterioration of, the process mean.

Now suppose that the process was overhauled at period t. In Chapter II it was stated that whenever the process is adjusted, it is always returned to the in-control state, $\mu_t(0)$. Thus, after an overhaul at period t, the value of the element $\hat{\alpha}_t''(0)$, of posterior state vector $\hat{\underline{\alpha}}_t''$, will always be one; the remaining elements $\hat{\alpha}_t''(i)$, $i \neq 0$, will be equal to zero. Also recall from Chapter II that the prior state vector at the next period, t-1, is obtained by the matrix multiplication:

$$\hat{\underline{\alpha}}_{t-1}' = \hat{\underline{\alpha}}_t'' \cdot \hat{\underline{B}} \, . \tag{3.19}$$

It can then be observed, after performing this operation, that

$$\hat{\alpha}_{t-1}'(-s) = \hat{b}_{0,-s} \tag{3.20}$$

$$\hat{\alpha}_{t-1}'(-s+1) = \hat{b}_{0,-s+1} \tag{3.21}$$

$$\vdots \qquad\qquad \vdots$$

$$\hat{\alpha}_{t-1}'(0) = \hat{b}_{00} \tag{3.22}$$

$$\vdots \qquad\qquad \vdots$$

$$\hat{\alpha}_{t-1}'(s) = \hat{b}_{0s} \quad . \tag{3.23}$$

Each element, $\hat{b}_{0j}$, is simply the probability of a shift in the process mean from state 0 to state j during a single period of operation. Consequently, $\hat{b}_{0j} = \hat{P}_{0j}$. All the remaining elements of $\underline{\hat{B}}$ need not be considered. Note that the elements $\hat{b}_{0j}$ are identical for the two cases considered. Thus, a single transition probability matrix is now explicitly defined in Figure 3-1.

The Probability State Vector $\underline{\hat{\gamma}}$

The posterior state vector at period t, $\underline{\hat{\alpha}}_t''$, represents the probability of the process mean being in each state immediately after sampling. In order to determine the expected cost of operating the process for a single period until the next sample is taken, the probability mass function for the state of the process at any point in time during the t'th period is needed. Accordingly, define the i'th element $\hat{\gamma}_t(i)$, of the probability state vector $\underline{\hat{\gamma}}_t$, to be the probability that the process mean is in state $\mu_t(i)$ at any point in time during the t'th period.

Duncan [11] has shown that, given a shift between the samples at t and t-1, the average fraction of the interval that elapses before the shift occurs is

$$\Delta = \frac{1 - (1+\lambda^* k)\ e^{-\lambda^* k}}{(1-e^{-\lambda^* k})\ \lambda\ k} \tag{3.24}$$

where k is the number of units produced between samples.

The following derivation of $\underline{\hat{\gamma}}_t$ parallels that given by Knappenberger and Grandage [20]. Since the process is assumed to be unable to correct itself, $\hat{\gamma}_t(0)$ depends on the probability that the process mean is in

From	$\mu_{t-1}(-s)$	$\mu_{t-1}(-s+1)$	$\mu_{t-1}(-s+2)$	...	$\mu_{t-1}(-1)$	$\mu_{t-1}(0)$	$\mu_{t-1}(1)$	...	$\mu_{t-1}(s-2)$	$\mu_{t-1}(s-1)$	$\mu_{t-1}(s)$
$\mu_t(-s)$	1	0	0	...	0	0	0	...	0	0	0
$\mu_t(-s+1)$	$\hat{P}_{-s+1,-s}$	$\hat{P}_{-s+1,-s+1}$	0	...	0	0	0	...	0	0	0
$\mu_t(-s+2)$	$\hat{P}_{-s+2,-s}$	$\hat{P}_{-s+2,-s+1}$	$\hat{P}_{-s+2,-s+2}$	...	0	0	0	...	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
$\mu_t(-1)$	$\hat{P}_{-1,-s}$	$\hat{P}_{-1,-s+1}$	$\hat{P}_{-1,-s+2}$	...	$\hat{P}_{-1,-1}$	0	0	...	0	0	0
$\mu_t(0)$	$\hat{P}_{0,-s}$	$\hat{P}_{0,-s+1}$	$\hat{P}_{0,-s+2}$	...	$\hat{P}_{0,-1}$	$\hat{P}_{00}$	$\hat{P}_{01}$	...	$\hat{P}_{0,s-2}$	$\hat{P}_{0,s-1}$	$\hat{P}_{0,s}$
$\mu_t(1)$	0	0	0	...	0	0	$\hat{P}_{11}$	...	$\hat{P}_{1,s-2}$	$\hat{P}_{1,s-1}$	$\hat{P}_{1,s}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
$\mu_t(s-2)$	0	0	0	...	0	0	0	...	$\hat{P}_{s-2,s-2}$	$\hat{P}_{s-2,s-1}$	$\hat{P}_{s-2,s}$
$\mu_t(s-1)$	0	0	0	...	0	0	0	...	0	$\hat{P}_{s-1,s-1}$	$\hat{P}_{s-1,s}$
$\mu_t(s)$	0	0	0	...	0	0	0	...	0	0	1

Figure 3-1 The Transition Probability Matrix for the Multi-state Dynamic Model

state $\mu_t(0)$ at the time the sample is taken (at time t) and remains there until the next sample is taken, and the probability that the process mean is in state $\mu_t(0)$ at the time of sampling and then shifts to a different state during the production of the next k units. Hence,

$$\hat{\gamma}_t(0) = \hat{\alpha}_t''(0)\ \hat{P}_{00} + \hat{\alpha}_t''(0)\ (1-\hat{P}_{00})\ \Delta\ . \tag{3.25}$$

The remaining probabilities $\hat{\gamma}_t(i)$, $i \neq 0$, depend upon the probability that the process mean is in state $\mu_t(i)$ at the time a sample is inspected and remains there during the production of the next k units, the probability that the process mean is in state $\mu_t(0)$ after sampling (at time t) and shifts to state $\mu_{t-1}(i)$ during the production of the next k units, and the probability that the process mean is in some lower state (say, $\mu_t(m)$ where $m < i$) at the time a sample is taken and shifts to state $\mu_{t-1}(i)$ during the production of the next k units. Each $\hat{\gamma}_t(i)$ also depends on the probability that the process mean is in state $\mu_t(i)$ after sampling (at time t) and deteriorates to some higher state (say, $\mu_{t-1}(n)$ where $n > i$) during the production of the next k units [20].

It is assumed that the fraction of time the process mean spends in a lower state before it deteriorates to a higher state is on the average, the same fraction, Δ, of the time the process mean spends in state $\mu_t(0)$, given that a shift to some other state has occurred [20]. Hence,

$$\begin{aligned}\hat{\gamma}_t(i) = {} & \hat{\alpha}_t''(i)\ \hat{P}_{ii} + \hat{\alpha}_t''(0)\ \hat{P}_{0i}\ (1-\Delta) \\ & + \sum_{m=1}^{i-1} \hat{\alpha}_t''(m)\ \hat{P}_{mi}\ (1-\Delta) \\ & + \hat{\alpha}_t''(i)\ \Delta \sum_{n=i+1}^{s} \hat{P}_{in}\end{aligned} \tag{3.26}$$

and

$$\hat{\gamma}_t(-i) = \hat{\alpha}_t''(-i)\,\hat{P}_{-i,-i} + \hat{\alpha}_t''(0)\,\hat{P}_{0,-i}\,(1-\Delta)$$
$$+ \sum_{m=-1}^{-i+1} \hat{\alpha}_t''(m)\,\hat{P}_{m,-i}\,(1-\Delta)$$
$$+ \hat{\alpha}_t''(-i)\,\Delta \sum_{n=-i-1}^{-s} \hat{P}_{-i,n} \qquad (3.27)$$

for $i = 1, 2, \ldots, s$. Observe that the third term in each of the equations (3.26) and (3.27) is zero when $i = 1$ and the last term is zero when $i = s$. That is to say, it is impossible to shift to state $\mu_t(1)$ or $\mu_t(-1)$ from any other state except $\mu_t(0)$, and it is impossible to shift from state $\mu_t(s)$ or $\mu_t(-s)$ to some higher state.

The Probability Vector $\hat{\underline{f}}$

In Chapter II it was assumed that the cost of operating the process for each period is solely attributed to the production of defective units. Consequently, define the i'th element $\hat{f}(i)$, of the probability vector $\hat{\underline{f}}$, to be the conditional probability of producing a defect given that the process mean is in state $\mu_t(i)$. Specifically, assume that S_L and S_U represent the lower and upper specification limits, respectively, for the process quality characteristic x_t. Whenever x_t falls outside the specification limits the unit is considered to be defective. Since x_t is assumed to be normally distributed with mean $\mu_t(i)$ and variance σ^2, the i'th element of $\hat{\underline{f}}$ is given by

$$\hat{f}(i) = 1 - \int_{S_L}^{S_U} \frac{1}{\sqrt{2\Pi}}\,\frac{1}{\sigma}\, e^{\frac{-[x_t-\mu_t(i)]^2}{2\sigma^2}}\, dx_t \,. \qquad (3.28)$$

Derivation of the Posterior State Vector $\hat{\underline{\alpha}}_t''$

The posterior probabilities describing the states of the process mean are derived from the prior probabilities and the sample result. From equation (2.02), the i'th element of $\hat{\underline{\alpha}}_t''$ is given by

$$\hat{\alpha}_t''(i) = \frac{\ell[\bar{x}_t|\mu_t(i),n_t]\ \hat{\alpha}_t'(i)}{\sum_{\forall i} \ell[\bar{x}_t|\mu_t(i),n_t]\ \hat{\alpha}_t'(i)}\ . \tag{3.29}$$

Also, the likelihood function for the sample mean is normally distributed with mean $\mu_t(i)$ and variance σ^2/n_t. Thus, the density function for $\bar{x}_t$ is

$$\ell[\bar{x}_t|\mu_t(i),n_t] = \frac{1}{\sqrt{2\Pi}}\ \frac{\sqrt{n_t}}{\sigma}\ e^{\frac{-n_t[\bar{x}_t-\mu_t(i)]^2}{2\sigma^2}} \tag{3.30}$$

where $-\infty < \bar{x}_t < \infty$ and $i = -s, \ldots, -1, 0, 1, \ldots, s$. Therefore,

$$\hat{\alpha}_t''(i) = \frac{\frac{1}{\sqrt{2\Pi}}\ \frac{\sqrt{n_t}}{\sigma}\ e^{\frac{-n_t[\bar{x}_t-\mu_t(i)]^2}{2\sigma^2}}\ \hat{\alpha}_t'(i)}{\sum_{\forall i}\left\{\frac{1}{\sqrt{2\Pi}}\ \frac{\sqrt{n_t}}{\sigma}\ e^{\frac{-n_t[\bar{x}_t-\mu_t(i)]^2}{2\sigma^2}}\ \hat{\alpha}_t'(i)\right\}}\ . \tag{3.31}$$

This expression can be reduced to

$$\hat{\alpha}_t''(i) = \frac{e^{\frac{-n_t[\bar{x}_t-\mu_t(i)]^2}{2\sigma^2}}\ \hat{\alpha}_t'(i)}{\sum_{\forall i}\left\{e^{\frac{-n_t[\bar{x}_t-\mu_t(i)]^2}{2\sigma^2}}\ \hat{\alpha}_t'(i)\right\}}\ . \tag{3.32}$$

It can be observed that

$$\sum_{\forall i} \hat{\alpha}_t''(i) = 1 \ . \tag{3.33}$$

Derivation of the Predictive Distribution

As stated previously, the sampling distribution is normally distributed with parameters $\mu_t(i)$ and σ^2/n_t. The joint distribution of $\bar{x}_t$ and $\mu_t(i)$, given n_t, is given by the product

$$\ell[\bar{x}_t|\mu_t(i),n_t] \ \hat{\alpha}_t'(i) \ . \tag{3.34}$$

Then summing over all possible states of the process mean, one obtains the marginal or predictive distribution for the sample mean conditioned upon the prior beliefs about the process and the sample size. In equation (2.10), the predictive distribution is expressed as

$$g(\bar{x}_t|u_t',n_t) = \sum_{\forall i} \ell[\bar{x}_t|\mu_t(i),n_t] \ \hat{\alpha}_t'(i) \tag{3.35}$$

which can be explicitly defined as

$$g(\bar{\alpha}_t|u_t',n_t) = \sum_{\forall i} \frac{\sqrt{n_t}}{\sqrt{2\Pi}} \frac{1}{\sigma} e^{\frac{-n_t[\bar{x}_t-\mu_t(i)]^2}{2\sigma^2}} \hat{\alpha}_t'(i) \ . \tag{3.36}$$

The Explicit Form of the Recursive Equation

Recall from Chapter II the general form of the recursive equation given by

$$V_t(u_t') =$$

$$\min_{n_t} \left\{ \underset{\overline{x}_t | u_t', n_t}{E} \min \begin{bmatrix} K(u_t'') + C(u^0) + V_{t-1}(u_{t-1}') \\ C(u_t'') + V_{t-1}(u_{t-1}') \end{bmatrix} + S(n_t) \right\} \tag{3.37}$$

Now, in order to explicitly define each of the cost terms, let the posterior expected cost of an overhaul, $K(u_t'')$, be equal to a constant average cost a_3. Hence,

$$K(u_t'') = a_3 \tag{3.38}$$

for all u_t''.

The posterior expected running cost is dependent upon the state of the process mean throughout a single period of operation. The probability that the process mean is in a particular state, $\mu_t(i)$, at any point in time during the t'th period, given by $\hat{\gamma}(i)$, is needed. Then, utilizing equation (2.07) in Chapter II, the posterior expected running cost, $C(u_t'')$, when no overhaul is performed, is given by

$$C(u_t'') = R\ h\ a_4 \sum_{\forall i} \hat{f}(i)\ \hat{\gamma}(i) \tag{3.39}$$

$$= R\ h\ a_4\ \underline{\hat{f}}\ \underline{\hat{\gamma}}^t \tag{3.40}$$

where $\underline{\hat{\gamma}}^t$ is $\underline{\hat{\gamma}}$ transpose.

The posterior expected running cost, $C(u^0)$, given an overhaul was just completed, is given by

$$C(u^0) = R\ h\ a_4\ \underline{\hat{f}}\ \ \underline{\hat{\gamma}}_*^t \tag{3.41}$$

where $\hat{\underline{\gamma}}_*$ is derived from the posterior state vector, $\hat{\underline{\alpha}}_t''$, whose 0'th element $\hat{\alpha}_t''(0) = 1$ ($\hat{\underline{\gamma}}_*$ is used to signify an overhaul was just completed).

The cost of sampling, $S(n_t)$, is simply a restatement of equation (2.08):

$$S(n_t) = a_1 + a_2\, n_t \;. \tag{3.42}$$

Then to summarize, the explicit recursive equation is given by

$$V_t(u_t') = \min_{n_t} \left\{ \underset{\overline{x}_t | u_t', n_t}{E} \; \min \begin{bmatrix} a_3 + R\, h\, a_4\, \hat{\underline{f}}\, \hat{\underline{\gamma}}_*{}^t + V_{t-1}(u_{t-1}') \\ R\, h\, a_4\, \hat{\underline{f}}\, \hat{\underline{\gamma}}^t + V_{t-1}(u_{t-1}') \end{bmatrix} + a_1 + a_2\, n_t \right\} \tag{3.43}$$

where

u_{t-1}' implies $\left\{ \begin{array}{l} \hat{\alpha}_{t-1}' \text{ is obtained from equations (3.20) through} \\ \text{(3.23) if an overhaul was performed at period t.} \\ \hat{\alpha}_{t-1}' \text{ is obtained in general from equation (3.19)} \\ \text{when no overhaul was performed at period t.} \end{array} \right\}$

Optimization of the Sampling Interval

Two approaches, each proposed by Carter [7], for determining the optimal interval between inspections will be discussed. In considering h, the time between samples, it may be observed for certain kinds of cost functions, the total expected cost per period may decrease as the value of h decreases. But, on the other hand, the total cost for sampling over the planning horizon may increase, especially if there is a fixed component of the sampling cost. Thus, the effect of h during the optimization procedure is not clear.

For a finite horizon problem, it would be possible to examine the total expected cost for a fixed <u>total</u> time using different values for the length of the sampling interval. If T is the total time and h is the interval length, then T/h is the number of periods of operation that would be considered. This procedure, however, has several limitations [7]:

1) There is no guarantee that the optimal policies derived from a fixed interval length will yield a global optimum.
2) A search or enumeration technique would have to be employed and only a local optimum might be found.
3) Each time the total time of operation, T, changes, a new value for h would have to be determined.

The second approach, which will be employed in the optimization procedure, is to consider the "gain" of the process. Recall from Chapter II that $V_t(u_t')$ is the minimum future expected cost with t periods left to operate. The "gain" of the process is then defined as

$$\theta = \lim_{t\to\infty}\left\{V_t(u_t') - V_{t-1}(u_{t-1}')\right\} \qquad (3.44)$$

with the value of a in equation (2.13) set equal to one. The gain is interpreted as the rate per period at which the expected future total costs decrease. For the infinite horizon problem, the steady state optimal policy yields a minimum gain. (Conversely, the optimal policy can be found by minimizing the gain) [7].

The optimization procedure would proceed as follows: a starting value for h would have to be chosen. The recursive equation (3.43) would be solved for each period t = 1, 2, 3, ..., for all sets of prior probabilities and the value of the gain, θ, would be computed. When t increases such that θ converges, then the steady state control policies for the given value of h would be obtained. The value of the gain would then be divided by k, the number of units produced in that period, and an average steady state expected cost per unit would be obtained. The procedure would then be repeated by searching over values of h until the optimal fixed interval is determined such that the average steady state expected cost per unit is minimized.

Utilization of the Model

The application of the dynamic model would proceed as follows. Initially, estimates of the costs for overhauling and sampling the process would have to be determined. The cost of producing defective units as well as the specification limits would have to be given. Secondly, the possible discrete states of the process mean would have to be outlined. An estimate of both the process variance and the average time between shifts from the in-control state of the process mean would have to be obtained. And lastly, the production rate would have to be determined.

Assuming that the process starts out in the in-control state, the process operator would run the process for a single period. He would then derive his prior beliefs about the process using equation (3.19).

Given the new prior probabilities he would check the control policy to determine what optimal sample size to inspect. After sampling he would compute the posterior probabilities and again check the control policies to determine the optimal operating decision. If the optimal sample size is zero, then the operator immediately checks the control policies to determine the optimal operating decision based on his present prior probabilities. The same procedure is repeated after operating the process each period. The operator, by using the steady state control policy, does not have to be cognizant of how many periods the process will be operated.

The Steady State $\bar{X}$ Control Chart Model

One of the main purposes of this thesis is to compare the results obtained for the dynamic model with those of the optimally designed $\bar{X}$ control chart model proposed by Knappenberger and Grandage [20]. Accordingly, a cursory development of the steady state $\bar{X}$ control chart model is included. For a detailed derivation and description see [20].

It is the purpose of this model to derive the test parameters, which include the control limits, the sample size, and the interval between samples, such that the expected total cost per unit manufactured by the process is minimized. It is assumed, likewise for the dynamic model, that there is a single measurable quality characteristic, x, which is a continuous random variable and is normally distributed. The process parameter, μ, is a continuous random variable which can be satisfactorily approximated by a discrete random variable. One value, $\mu(0)$, of the discrete random variable is associated with the in-control value of the process mean and the remaining values $\mu(1)$, $\mu(2)$, ..., $\mu(s)$ are associated with out-of-control values of the process parameter. Here, Knappenberger and Grandage develop their model assuming the process mean only shifts in a positive direction, unlike the formulation of the dynamic model. But since they assume that the control limits are symmetric, the additional burden of developing a model which allows shifts in either direction is avoided.

General Cost Model [20]

The expected total cost E(C), per unit of product, associated with a quality control test procedure can be written as

$$E(C) = E(C_1) + E(C_2) + E(C_3), \tag{3.45}$$

where $E(C_1)$ is the expected cost per unit associated with carrying out the test procedure, $E(C_2)$ is the expected cost per unit associated with investigating and correcting the process when the test indicated the process is out-of-control (that is, when the null hypothesis is rejected), and $E(C_3)$ is the expected cost per unit associated with the production of defective products.

The expected sampling and testing cost per unit is

$$E(C_1) = \frac{a_1}{k} + \frac{a_2 N}{k} \tag{3.46}$$

where

N is the sample size,

a_1 is the fixed cost per sample,

a_2 is the cost per unit of product sampled, and

k is the number of units produced between samples.

After each sample is collected, the null hypothesis, H_0 (the assertion that the process is in-control), can be either accepted or rejected. Let $\underline{q}$ be the row vector of probabilities q(i), where q(i) is the conditional probability of rejecting H_0 given $\mu = \mu(i)$. Let $\underline{\alpha}$ be the row vector of probabilities $\alpha(i)$, where $\alpha(i)$ is the probability that $\mu = \mu(i)$ at the time the test is performed. Then the expected cost of rejecting the null hypothesis is given by

$$E(C_2) = \frac{a_3}{k} \sum_{i=0}^{s} q(i)\ \alpha(i) \tag{3.47}$$

$$= \frac{a_3}{k}\ \underline{q}\ \underline{\alpha}^t \quad , \tag{3.48}$$

where

$\underline{\alpha}^t$ is the transpose of $\underline{\alpha}$, and

a_3 is the average cost of investigating and correcting the process.

If a_4 is the cost associated with producing a defective unit, $\underline{f}$ is the row vector of probabilities f(i), where f(i) is the conditional probability of producing a defective unit given $\mu = \mu(i)$, and $\underline{\gamma}$ is the row vector of probabilities $\gamma(i)$, where $\gamma(i)$ is the probability that the process mean is in state $\mu(i)$, then the expected cost per unit associated with accepting the null hypothesis is

$$E(C_3) = a_4 \sum_{i=0}^{s} f(i)\ \gamma(i) \tag{3.49}$$

$$= a_4\ \underline{f}\ \underline{\gamma}^t \ . \tag{3.50}$$

Combining equations (3.45), (3.46), (3.48), and (3.50), the total expected cost per unit becomes

$$E(C) = \frac{a_1}{k} + \frac{a_2 N}{k} + \frac{a_3}{k} \underline{q}\, \underline{\alpha}^t + a_4\, \underline{f}\, \underline{\gamma}^t \, . \tag{3.51}$$

In the above function, the a's are cost coefficients which are independent of the test parameters. The vector $\underline{f}$ depends only on the nature of the process parameter, but the vectors $\underline{q}$, $\underline{\alpha}$, and $\underline{\gamma}$ are all functionally dependent upon the test parameter. The form of this dependency is outlined in later sections.

Thus far, only two (that is, N and k) of the three test parameters have been defined. In order to express the third test parameter (that is, the critical region) as a single parameter, some restrictions must be placed on the nature of the test. It will be assumed that the test statistic, T, is normally distributed with mean μ and variance σ^2/N, where σ^2 is constant, and that the critical region is symmetric and defined by the critical region parameter L such that the null hypothesis is rejected if

$$T > \mu(0) + \frac{L\sigma}{\sqrt{N}} \qquad \text{or if} \tag{3.52}$$

$$T < \mu(0) - \frac{L\sigma}{\sqrt{N}} \qquad . \tag{3.53}$$

<u>Probability Vector $\underline{q}$ [20]</u>

Based on the assumption that T is normally distributed with mean μ and variance σ^2/N, the probability of rejecting H_0 when $\mu = \mu(i)$ can be written as

$$q(i) = P(T > \mu(0) + \frac{L\sigma}{\sqrt{N}}) + P(T < \mu(0) - \frac{L\sigma}{\sqrt{N}})$$

$$= \phi(\frac{\mu(0)-\mu(i)}{\sigma} \sqrt{N} + L) + 1$$

$$- \phi(\frac{\mu(0)-\mu(i)}{\sigma} \sqrt{N} - L) , \qquad (3.54)$$

where

$$\phi(d) = \int_{d}^{\infty} \frac{1}{\sqrt{2\Pi}} e^{-\frac{z^2}{2}} dz . \qquad (3.55)$$

When i = 0, the above reduces to

$$q(0) = 2\,\phi(L) . \qquad (3.56)$$

Probability Vector $\underline{\alpha}$ [20]

The elements, $\alpha(i)$, of the probability state vector $\underline{\alpha}$ represent the steady state probability that the process mean is in state i (that is, $\mu = \mu(i)$) at the time a sample is inspected. To obtain these steady state probabilities, the transition probability matrix, $\underline{B}$, is required. The elements, b_{ij} of the matrix $\underline{B}$. represent the probability of the process mean shifting from state $\mu(i)$ to state $\mu(j)$ during the production of k units between samples.

It is assumed that the time between successive shifts in the state of the process mean is exponentially distributed with parameter λ and whose probability density function is given by equation (3.01). At this point, the development proceeds in the same manner as for the dynamic model, hence only the results will be presented. The probability that the process mean remains in the in-control state during the production of k units is

$$P_{00} = e^{-\lambda^* k} \ . \tag{3.57}$$

The probability of a shift from state $\mu(0)$ to some other state $\mu(j)$ during the production of k units is

$$P_{0j} = \frac{(1-e^{-\lambda^* k})}{[1-p(1-p)^s]} \binom{s}{j} p^j (1-p)^{s-j} \tag{3.58}$$

for j = 1, 2, ..., s. Note that equation (3.58) is different from equation (3.13) derived for the dynamic model. The factor $(\frac{1}{2})$ is omitted from equation (3.58) since shifts are only permitted in one direction.

It is assumed, likewise for the dynamic model, that when the process goes out-of-control it stays out-of-control until detected. Once an overhaul has been performed the process mean is always returned to the in-control state $\mu(0)$. Also, shifts in the state of the process mean are only allowed to proceed from one state $\mu(i)$ to a higher state $\mu(j)$, where $j > i$. And only one shift is allowed in each testing period.

On the basis of these assumptions, the elements of $\underline{B}$ can now be defined. When j is less than i, the probability, b_{ij}, that the process mean is in state $\mu(i)$ at some time t and in state $\mu(j)$ at time t + k/R (recall R is the production rate) is simply the probability of rejecting H_0 (that is, q(i)) at time t multiplied by the probability P_{0j} of shifting to $\mu(j)$ during the production of the next k units. Thus, for $j < i$,

$$b_{ij} = q(i)\ P_{0j} \ . \tag{3.59}$$

For the case $j > i$, the probability of failing to reject H_0 at time t multiplied by the probability, P_{ij}, of shifting directly from $\mu(i)$ to $\mu(j)$ in the time required to produce k units must be added to the above equation. Again, the development of P_{ij} proceeds in the same fashion as for the dynamic model, thus the results are as follows:

$$P_{ii} = \frac{\sum_{i=1}^{j} P_{0j}}{1 - P_{00}} \tag{3.60}$$

where $i \neq 0$, and

$$P_{ij} = \frac{P_{0j}}{1 - P_{00}} \tag{3.61}$$

where $j > i$.

Hence, for $j > i$, b_{ij} becomes

$$b_{ij} = q(i)\ P_{0j} + \frac{[1-q(i)]\ P_{0j}}{1 - P_{00}}\ . \tag{3.62}$$

For $i \neq 0$, the probability, b_{ii}, of being in state $\mu(i)$ at time t and also at time $t + (k/R)$ is equal to the probability of rejecting H_0 at time t, multiplied by the probability of returning to $\mu(i)$ during the production of the next k units, plus the probability of failing to reject H_0 at time t, multiplied by the probability of remaining in state $\mu(i)$ during the production of the next k units. Thus, for $i \neq 0$, b_{ii} is

$$b_{ii} = q(i)\ P_{0i} + [1-q(i)]\ P_{ii}, \tag{3.63}$$

and, hence,

$$b_{ii} = q(i)\ P_{0i} + \frac{[1-q(i)]\ \sum_{i=1}^{j} P_{0j}}{1 - P_{00}}\ . \tag{3.64}$$

Finally, the probability of shifting from state $\mu(0)$ at time t to state $\mu(j)$ at time $t + (k/R)$ is simply P_{0j}. Hence,

$$b_{0j} = P_{0j}\ . \tag{3.65}$$

It is shown in many texts on stochastic processes [26], [5] that the above conditions define a transition matrix of an irreducible aperiodic positive recurrent Markov chain. Thus, there exists a vector $\underline{\alpha}$ such that

$$\underline{\alpha}\ \underline{B} = \underline{\alpha}\ , \tag{3.66}$$

where

$$\underline{\alpha} = [\alpha(0), \alpha(1), \ldots, \alpha(s)] \text{ and} \tag{3.67}$$

$$\sum_{i=0}^{s} \alpha(i) = 1. \tag{3.68}$$

Furthermore, $\alpha(i)$ is the long-run (or steady state) unconditional probability that the process mean is in state $\mu(i)$ regardless of the initial state of the process mean. The solution of equation (3.66) for the i'th element of $\underline{\alpha}$ is

$$\alpha(i) = b^{*-1}_{s,i+1} \tag{3.69}$$

where b^{*-1}_{ij}'s are the elements of B^{*-1}; and

$$b^{*}_{ij} = b_{ij} \tag{3.70}$$

for $i \neq j-1$ and $j \neq s+1$,

$$b^{*}_{i,s+1} = 1 \tag{3.71}$$

for $j = s+1$, and

$$b^{*}_{j-1,j} = b_{jj} - 1 \tag{3.72}$$

for $j \neq s+1$. For further details see [20].

Probability Vector $\underline{\gamma}$ [20]

In the previous section, the vector $\underline{\alpha}$, where $\alpha(i)$ is the steady state probability that the process mean is in state $\mu(i)$ at the time a sample is taken, was developed. In order to determine the cost of producing defectives, as defined in equation (3.50), the steady state probability, $\gamma(i)$, of the process mean being in state $\mu(i)$ at any point in time is required. Again, at this point, the development of $\underline{\gamma}$ directly parallels the development of $\hat{\underline{\gamma}}$ for the dynamic model; hence, only the results will be reiterated. The steady state probability that the process mean is in state $\mu(0)$ at any point in time, $\gamma(0)$, is

$$\gamma(0) = \alpha(0)\ P_{00} + \alpha(0)\ (1-P_{00})\ \Delta \tag{3.73}$$

where Δ is defined by equation (3.24). The remaining probabilities, $\gamma(i)$, $i \neq 0$, are given by

$$\begin{aligned}\gamma(i) &= \alpha(i)\ P_{00} + \alpha(0)\ P_{0i}\ (1-\Delta) \\ &= + \sum_{m=1}^{i-1} \alpha(m)\ P_{mi}\ (1-\Delta) \\ &\quad + \alpha(i)\ \Delta \sum_{n=i+1}^{s} P_{in}\end{aligned} \tag{3.74}$$

where the third term is zero when $i = 1$ and the last term is zero when $i = s$.

Summary

In this chapter, two multi-state models have been presented. The dynamic model was explicitly formulated along with all the relations which are necessary for its use. Methods for the optimization of the sampling interval are outlined and discussed. The steady state $\overline{X}$ control chart model, formulated by Knappenberger and Grandage [20], was presented briefly to describe the technique of its design.

In Chapter IV, each model will be illustrated by the solution of several example problems. A comparison of the results of each will then be made based on how well each attains its premise of minimizing operating costs.

Chapter IV

COMPUTATIONAL RESULTS

Utilization of the dynamic model is illustrated in this chapter by a number of examples. Three of the example problems describe hypothetical processes which possess a single out-of-control state, one depicts a three state process, and one illustrates a five state process. The results obtained for the dynamic model are then compared to those of the steady state $\overline{X}$ control chart model designed by Knapperberger and Grandage [20].

Design of the Numerical Analysis

Two computer programs were written in FORTRAN IV and processed on an IBM model 370/158 computer to solve the dynamic model and the steady state $\overline{X}$ model.

The possible states of the process mean were determined arbitrarily as follows:

Two state process: $\mu(1) = \mu(0) + 3\sigma$;

Three state process: $\mu(-1) = \mu(0) - 3\sigma$

$\mu(1) = \mu(0) + 3\sigma$;

Five state process: $\mu(-2) = \mu(0) - 3\sigma$

$\mu(-1) = \mu(0) - 1.5\sigma$

$\mu(1) = \mu(0) + 1.5\sigma$

$\mu(2) = \mu(0) + 3\sigma$.

In order to parallel the approach by Knappenberger and Grandage [20], the lower and upper specification limits on the process quality characteristic were positioned at

$$\mu(0) - 3\sigma$$

and

$$\mu(0) + 3\sigma ,$$

respectively.

Optimization of the Sampling Interval

Optimization of the sampling interval for the dynamic model was effected in the manner described in Chapter III. The gain, θ, given by equation (3.44), was evaluated at each period of the dynamic recursion for a given value of the sampling interval, k units (recall $k = h \cdot R$, where h is the time (hours) between sampling and R is the production rate). The gain per unit produced in the sampling interval was then found by the division: θ/k. When the change in the gain per unit from one period to the next was less than 0.01, the criterion for convergence was satisfied. In all problems, convergence was attained after three to five periods.

Since there is no assurance that the steady state cost per unit as a function of k is convex, the computer program initially attempted to establish bounds on the optimal value of k. Once this "interval of uncertainty" was established, the internal points were enumerated.

Grid Values for $\hat{\underline{\alpha}}_t'$ and $\bar{x}_t$

In order to limit the magnitude of the optimization procedure, appropriate grid values for the prior probabilities, $\hat{\alpha}_t'(i)$, and sample mean, $\bar{x}_t$, were chosen. The values of $\hat{\alpha}_t'(i)$ chosen for the two state process example problems were as follows:

$\hat{\underline{\alpha}}_t' = [\hat{\alpha}_t'(0), \hat{\alpha}_t'(1)]$:

[1.0,0.0],

[.9, .1],

[.8, .2],

[.7, .3],

[.6, .4],

[.5, .5],

[.4, .6],

[.3, .7],

[.2, .8],

[.1, .9], and

[0.0,1.0].

A greater number of different prior probabilities could have been utilized, but the affect on the optimal control policies and the resulting optimal cost per unit would probably not have been significant. On the other hand, for an actual application of the dynamic model, more prior probabilities should be considered to insure accuracy.

Since the distribution of $\bar{x}_t$ was assumed to be normal, only sample means in the following interval were considered:

$$\mu(-s) - 3\sigma \leq \overline{x}_t \leq \mu(s) + 3\sigma \ .$$

The incremental values of $\overline{x}_t$ in this interval were evaluated every $\sigma/3$ units.

Example Problems

For each of the examples the following process parameters were utilized:

$\sigma^2 = = .36$

$\mu(0) = 0.0$

$S_L = -1.8$

$S_U = 1.8$

The remaining parameters and cost components are presented in Table 4-1.

Two State Processes

Example #1

From Table 4-2, the optimal cost per unit is shown to be \$.2702. The associated optimal sampling interval, k, is 730 units. In solving the steady state $\overline{X}$ control chart model, the following results were obtained:

$E^*(C) = \$.3998$

$K^* = 220$

$N^* = 1$

$L^* = 2.32$.

Table 4-1

Cost Terms and Process Parameters
for Example Problems

Example #	a_1	a_2	a_3	a_4	λ	R
1	\$ 25.00	\$ 6.00	\$ 200.00	\$ 3.00	.0625	200
2	\$150.00	\$15.00	\$1000.00	\$100.00	.10	100
3	\$ 50.00	\$ 5.00	\$ 125.00	\$ 15.00	.10	100

Table 4-2

Example #1: Sampling Interval Results

Sampling Interval (units)	Expected Cost Per Unit
650	.2714
670	.2708
690	.2704
710	.2703
730*	.2702
750	.2704
770	.2706
790	.2710
810	.2722
830	.2728
850	.2735

* denotes optimum sampling interval

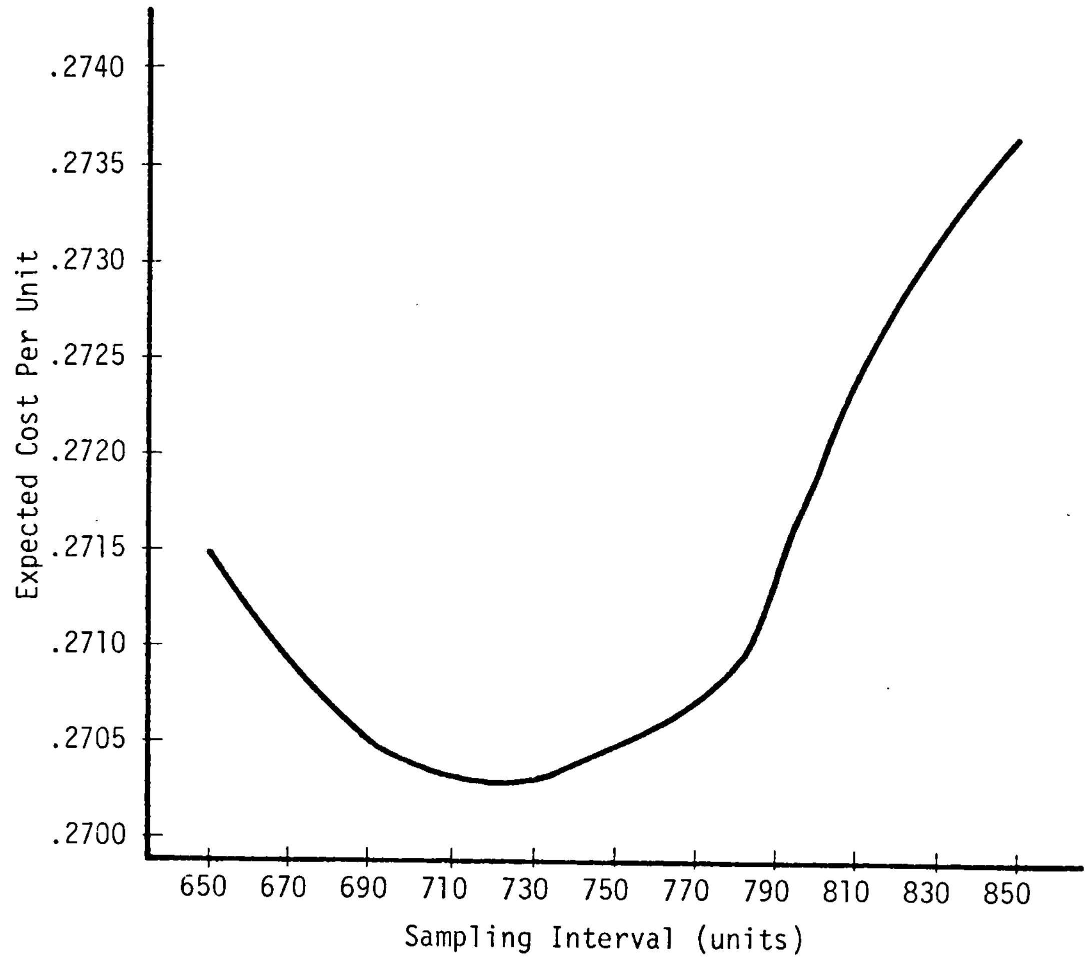

Figure 4-1

Steady State Cost Per Unit for Example #1

Hence, the optimal testing procedure is to sample 1 unit every 220 units produced and to reject H_0 if $\overline{x} > \mu(0) + 2.32\sigma$ or if $\overline{x} < \mu(0) - 2.32\sigma$.

The optimal steady state control policies specified by the dynamic model are listed in Table 4-3. For each prior probability of the process mean being in the in-control state the corresponding optimal sample size to collect is listed. Then based on the sample mean observed, the process operator will select the optimal operating decision.

Consider, $\hat{\alpha}_t'(0) = 1.0$. The process operator may observe that the optimal sample, $\overline{n}$, is zero. Consequently, the optimal operating decision when $\overline{n} = 0$ is to let the process run for another period, (R). The operator may now obtain the prior probabilities for the next period by employing equation (3.19). ($\hat{\underline{\alpha}}_t'' = \hat{\underline{\alpha}}_t'$ since no sample was observed.)

Consider, $\hat{\alpha}_t'(0) = .80$. The optimal sample size to collect is 3. After sampling, if $\overline{x}_t$ is found to be ≤ 1.20, then the optimal operating decision is to let the process run. If $\overline{x}_t > 1.2$, then the process should be overhauled.

After sampling, the posterior probabilities are then derived from equation (3.32). If the process is allowed to run, the prior probabilities for the next period are obtained from equation (3.19). If the process is overhauled $\hat{\underline{\alpha}}_t''$ becomes (1.0,0.0) and then equation (3.19) is used to obtain $\hat{\alpha}_{t-1}''$.

Results obtained in optimizing the sampling interval are listed in Table 4-2 and illustrated in Figure 4-1. Observe that the expected unit cost as a function of k appears to be convex. Also note that expected unit cost is extremely insensitive to large changes in the

Table 4-3

Optimal Steady State Control Policies:
Example #1

Prior Probability of State 0 $\hat{\alpha}_t'(0)$	Optimal Sample Size $\bar{n}$	Range of the sample mean $\bar{x}_t$ for which "Run Without Overhaul" is Optimal Operating Decision	Optimal Operating Decision if $\bar{n}$ = 0
1.00	0	-	R
.90	2	$\leq$ 1.20	
.80	3	$\leq$ .95	
.70	3	$\leq$ .95	
.60	3	$\leq$ .95	
.50	3	$\leq$ .95	
.40	3	$\leq$ .70	
.30	3	$\leq$ .70	
.20	3	$\leq$ .70	
.10	1	$\leq$ -.30	
0.00	0	-	OH

sampling interval. That is to say, the difference in the expected cost per unit for $k = 650$ and $k = 850$ is a mere \$.0021.

Now consider how well each model attains its premise of minimizing the expected cost per unit for operating the respective quality control procedure. Observe that by employing the control policies dictated by the dynamic model, a 32.4% savings in the expected cost per unit may be achieved over the $\overline{X}$ control chart method.

Example #2

The cost terms in Table 4-1 were employed in optimizing both the dynamic model and the $\overline{X}$ steady state model. The results for the dynamic model yielded a minimum steady state expected cost per unit of \$.6493 for a sampling interval of 800 units. These results are listed in Table 4-4. The optimal expected cost per unit using the steady state $\overline{X}$ model was found to be:

$$E^*(C) = \$1.049 \ .$$

The optimal testing procedure is to sample 3 units every 512 units produced and reject H_0 is $\overline{x} < \mu(0) - 3.12\sigma$ or if $\overline{x} > \mu(0) + 3.12\sigma$.

Observe in Figure 4-2 that the expected unit cost as a function of k appears to be convex as in Example 1. But note that the sensitivity of the function is more significant than in Example 1. Still a large change in the sampling interval, k, from 500 to 800 only effects a change of approximately \$.04 in the expected cost per unit.

The optimal operating policies dictated by the dynamic model are illustrated in Table 4-5.

Table 4-4

Example #2: Sampling Interval Results

Sampling Interval (units)	Expected Cost Per Unit
500	.6883
550	.6723
600	.6626
650	.6559
700	.6518
750	.6497
800*	.6493
850	.6507
900	.6532
950	.6567
1000	.6610

* denotes optimal sampling interval

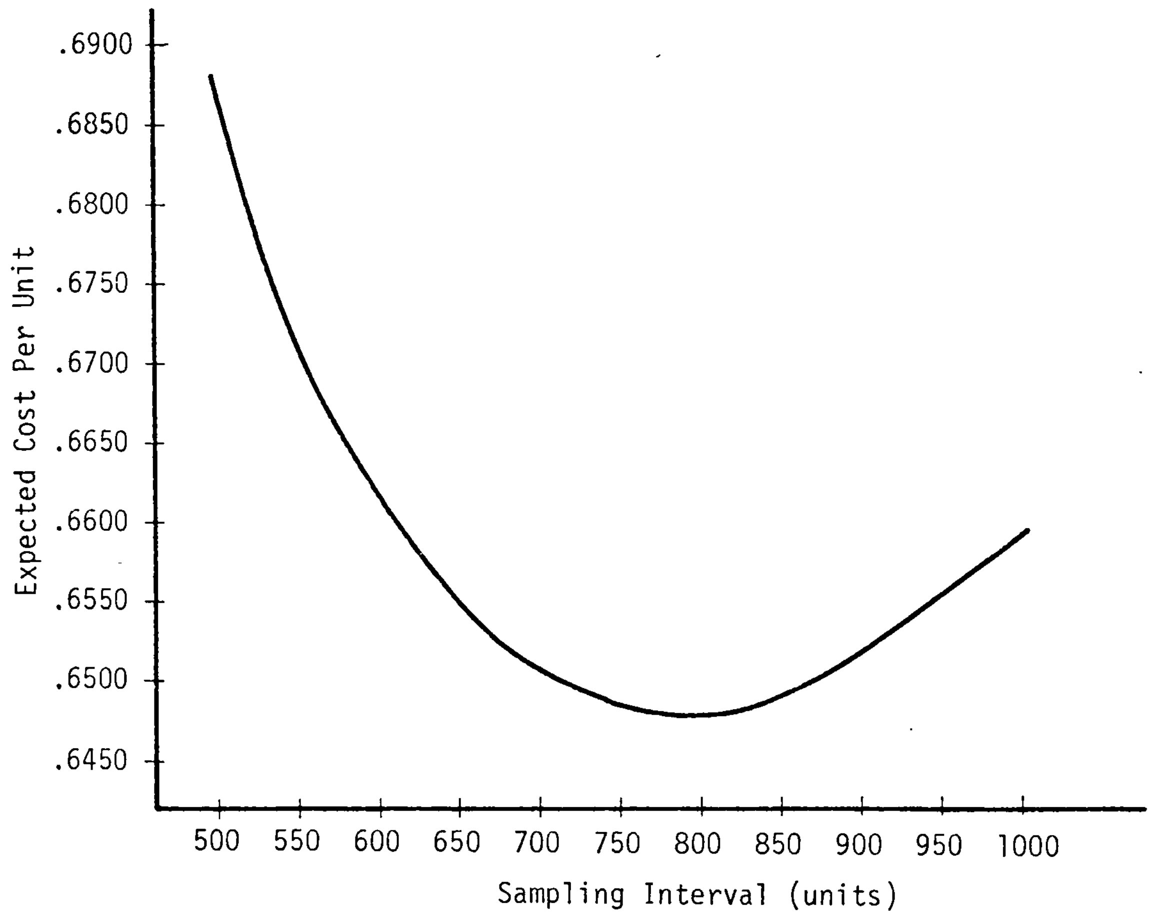

Figure 4-2

Steady State Expected Cost Per Unit for Example #2

Table 4-5

Optimal Steady State Control Policies:
Example #2

Prior Probability of State 0 $\hat{\alpha}_t'(0)$	Optimal Sample Size $\overline{n}$	Range of the sample mean $\overline{x}_t$ for which "Run Without Overhaul" is Optimal Operating Decision	Optimal Operating Decision if $\overline{n}$ = 0
1.00	0	-	R
.90	4	$\leq$ .95	
.80	6	$\leq$ .95	
.70	6	$\leq$ .95	
.60	6	$\leq$ .95	
.50	6	$\leq$ .70	
.40	6	$\leq$ .70	
.30	6	$\leq$ .70	
.20	6	$\leq$ .70	
.10	6	$\leq$ .70	
0.00	0	-	OH

Again consider how well each model attains its premise of minimizing expected unit costs. Again, a significant savings, 38.1%, in the expected cost per unit to operate the quality control procedure can be effected by employing the operating policies of the dynamic model.

Example #3

The resultant optimal expected cost per unit derived from the dynamic model was $1.067 for a sampling interval of 160 units. These results are listed in Table 4-6. The optimal expected cost per unit derived from the $\overline{X}$ control chart model was:

$$E^*(C) = \$1.791 \ .$$

The corresponding test procedure is to sample 2 units every 78 units manufactured, and overhaul the process if $\overline{x} < \mu(0) - 2.51\sigma$ or if $\overline{x} > \mu(0) + 2.51\sigma$.

Again the sensitivity of the expected unit cost as a function of k is depicted in Figure 4-3. Observe that for this example, the expected cost per unit is extremely sensitive to even small changes in the sampling interval.

The optimal steady state operating policies are presented in Table 4-7.

Again, comparing the procedures outlined by each model, a 40.4% savings is attained by following the doctrines of the dynamic model rather than those of the steady state $\overline{X}$ control chart model.

Table 4-6

Example #3: Sampling Interval Results

Sampling Interval (units)	Expected Cost Per Unit
100	1.189
120	1.127
140	1.095
160*	1.067
180	1.179
200	1.181
220	1.912
240	1.208
260	1.230
280	1.196
300	1.285

* denotes optimal sampling interval

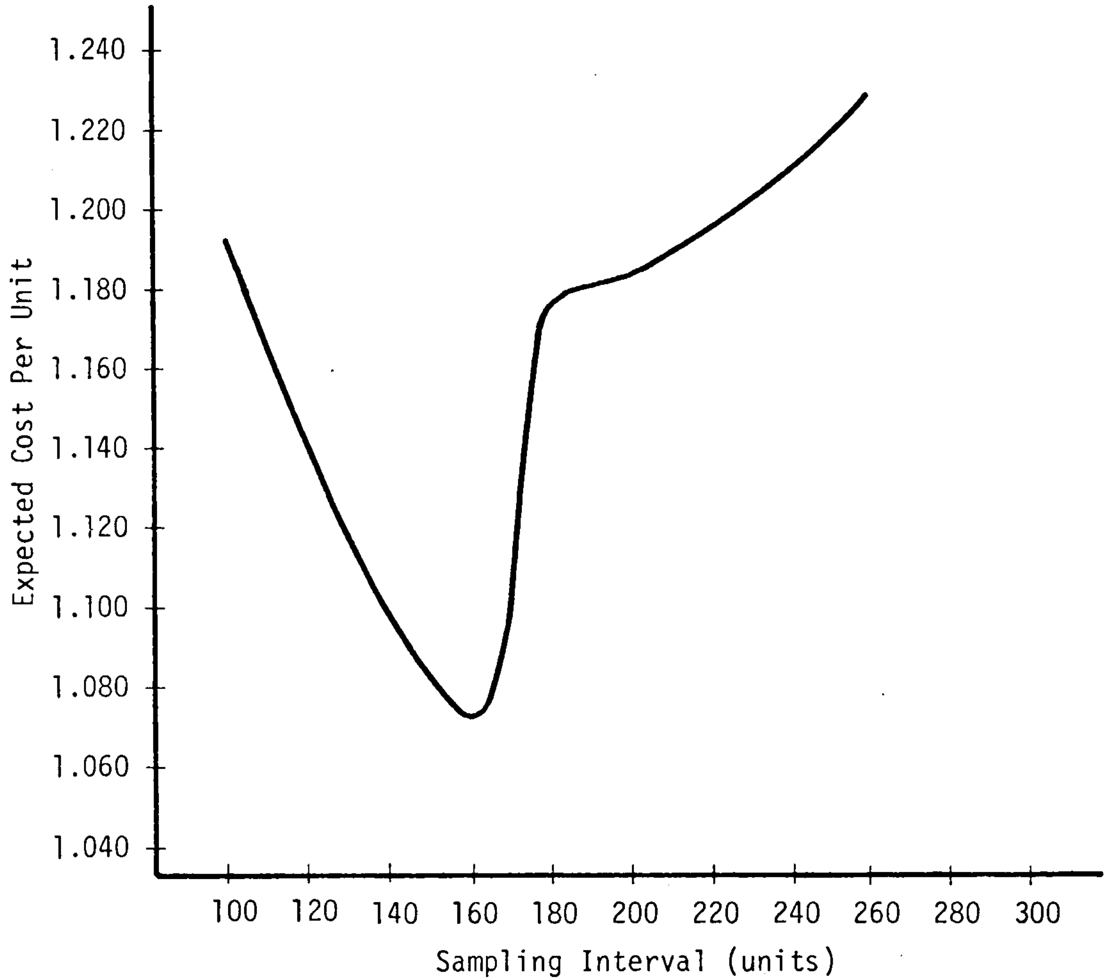

Figure 4-3

Steady State Cost Per Unit for Example #3

Table 4-7

Optimal Steady State Control Policies:
Example #3

Prior Probability of State 0 $\hat{\alpha}_t'(0)$	Optimal Sample Size $\bar{n}$	Range of the sample mean $\bar{x}_t$ for which "Run Without Overhaul" is Optimal Operating Decision	Optimal Operating Decision if $\bar{n} = 0$
1.00	0	-	R
.90	3	$\leq .95$	
.80	3	$\leq .95$	
.70	3	$\leq .95$	
.60	3	$\leq .95$	
.50	6	$\leq .95$	
.40	6	$\leq .95$	
.30	6	$\leq .95$	
.20	6	$\leq .95$	
.10	6	$\leq .70$	
0.00	0	-	OH

Three State Process

The cost components and process parameters used in Example #1 were also used in the analysis of the three state process. This process possesses two out-of-control states which are located symmetrically about the single in-control state. Since the process mean can be in three states, a greater number of combinations of prior probabilities had to be considered in the analysis. These combinations are listed in Table 4-9.

In solving the dynamic model, the minimum expected cost per unit was found to be \$.2826 for a sampling interval of 710 units. The same control procedure as in Example #1 was derived for the $\overline{X}$ control chart model. Recall that $E^*(C)$ = \$.3998. The results of the optimization procedure are listed in Table 4-8 and the sensitivity of the expected unit cost as a function of k is illustrated in Figure 4-4. The optimal steady state operating policies are outlined in Table 4-9.

Note that the results obtained for the three state process differ only slightly from those of the two state:

	Expected Cost Per Unit	Sampling Interval
Two state:	\$.2702	730
Three state:	\$.2826	710

In utilizing the three state process dynamic model a savings of 29.3% can be achieved over the $\overline{X}$ control chart model.

Table 4-8

Three State Process: Sampling Interval Results

Sampling Interval (units)	Expected Cost Per Unit
650	.2833
670	.2829
690	.2827
710*	.2826
730	.2828
750	.2831
770	.2834
790	.2840
810	.2847
830	.2854
850	.2862

* denotes optimal sampling interval

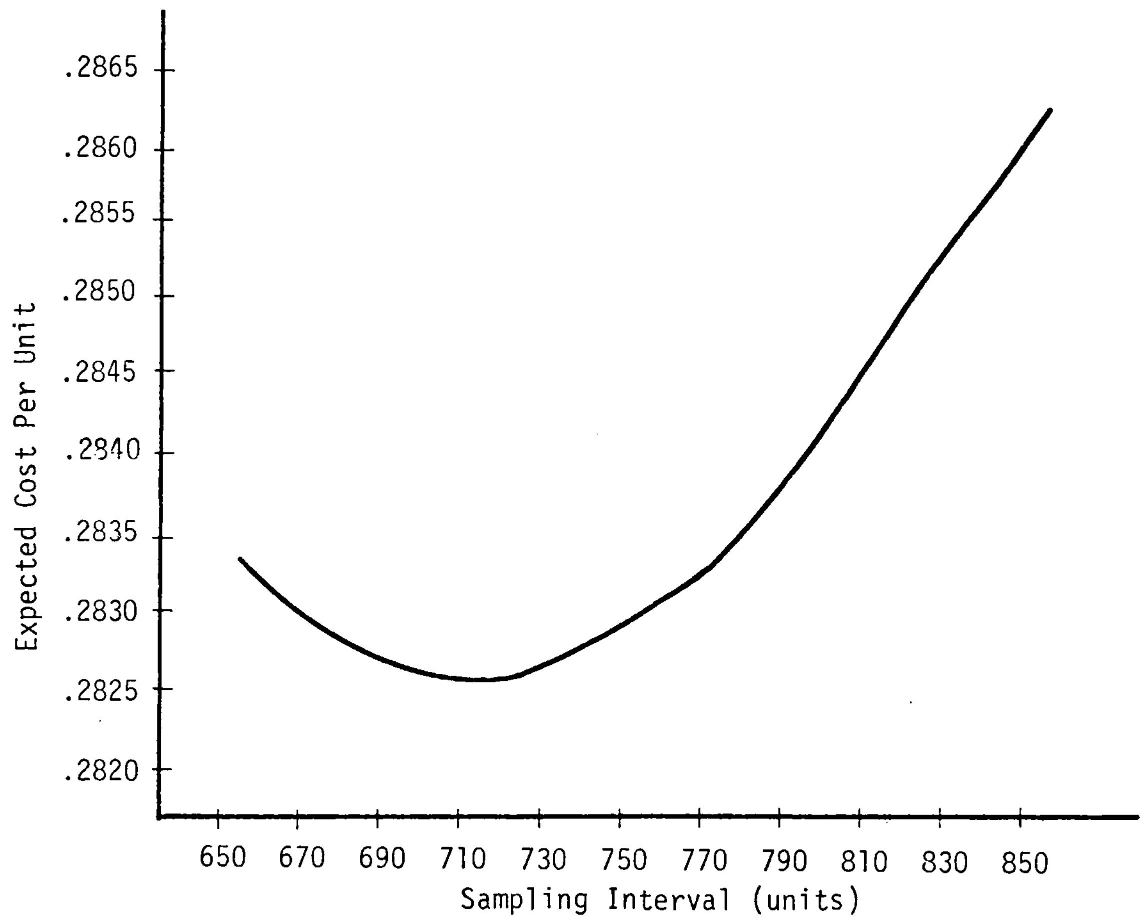

Figure 4-4

Steady State Cost Per Unit for the Three State Process

Table 4-9

Optimal Steady State Control Policies:
Three State Process

Prior State Vector $\hat{\alpha}_t'(i)$			Optimal Sample Size $\bar{n}$	Range of the sample mean $\bar{x}_t$ for which "Run Without Overhaul" is Optimal Operating Decision	Optimal Operating Decision when $\bar{n}$ = 0
-1	0	+1			
0.00	1.00	0.00	0	-	R
.05	.90	.05	2	$-.90 \leq \bar{x}_t \leq 1.2$	
.10	.80	.10	2	$-.90 \leq \bar{x}_t \leq 1.2$	
.15	.70	.15	4	$-.90 \leq \bar{x}_t \leq 1.2$	
.20	.60	.20	4	$-.60 \leq \bar{x}_t \leq .90$	
.25	.50	.25	4	$-.60 \leq \bar{x}_t \leq .90$	
.30	.40	.30	4	$-.60 \leq \bar{x}_t \leq .90$	
.33	.34	.33	4	$-.60 \leq \bar{x}_t \leq .90$	
.70	.2	.10	4	$-.60 \leq \bar{x}_t \leq .90$	
.10	.2	.70	4	$-.60 \leq \bar{x}_t \leq .90$	
0.00	.1	.90	2	$\leq -.30$	
.90	.1	0.00	1	$\geq .60$	
1.00	0.0	0.00	0	-	OH
0.00	0.0	1.00	0	-	OH

Five State Process

Again the cost components of Example #1 are used in the analysis of the five state process. Like the three state process, a greater number of combinations of prior probabilities were needed in the analysis. These combinations are listed in Table 4-11.

The minimum expected cost per unit obtained from the dynamic model was found to be \$.2357 for a sampling interval of 1150 units. The optimal expected cost per unit found for the $\bar{X}$ model was:

$$E^*(C) = \$.3478 \ .$$

The corresponding test procedure is to sample 2 units every 318 units produced and overhaul the process (reject H_0) if $\bar{x} < \mu(0) - 2.25\sigma$ or if $\bar{x} > \mu(0) + 2.25\sigma$.

In performing the analysis, the binomial parameter, p, in equation (3.13) was equal to .597 (value used in [20]). The results of the optimization procedure for the dynamic model are listed in Table 4-10. Figure 4-5 illustrates that when the sampling interval is greater than 1150 the increase in the expected cost per unit is very significant.

Note, however, that the results for the five state process model are significantly different from those of the two and three state process models. The expected cost per unit has decreased approximately \$.035 and the optimal sampling interval has increased from 710 to 1150. Observe in Table 4-11 that in order to compensate for a longer sampling interval, the optimal sample sizes have increased from 3 and 4 to 7.

Table 4-10

Five State Process: Sampling Interval Results

Sampling Interval (units)	Expected Cost Per Unit
1050	.2369
1070	.2366
1090	.2366
1110	.2365
1130	.2361
1150*	.2357
1170	.2440
1190	.2441

* denotes optimal sampling interval

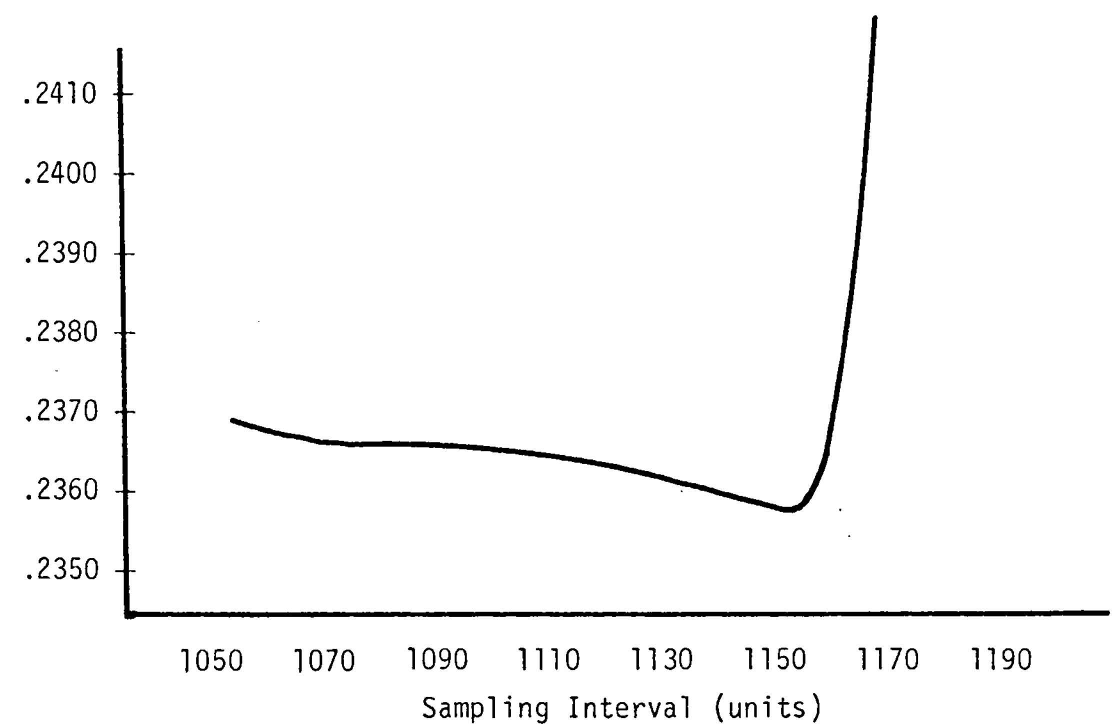

Figure 4-5

Steady State Cost Per Unit for the Five State Process

Table 4-11

Optimal Steady State Control Policies:
Five State Process

Prior State Vector $\hat{\alpha}_t'(i)$ -2	-1	0	+1	+2	Optimal Sample Size $\bar{n}$	Range of the sample mean $\bar{x}_t$ for which "Run Without Overhaul" is Optimal Operating Decision	Optimal Operating Decision if $\bar{n} = 0$
0.00	0.00	1.00	0.00	0.00	0	-	R
0.00	.05	.90	.05	0.00	2	$-1.20 \leq \bar{x}_t \leq 1.50$	
0.00	.10	.80	.10	0.00	7	$-.30 \leq \bar{x}_t \leq .60$	
.05	.10	.70	.10	.05	7	$-.30 \leq \bar{x}_t \leq .60$	
.05	.15	.60	.15	.05	7	$-.30 \leq \bar{x}_t \leq .60$	
.05	.20	.50	.20	.05	7	$-.30 \leq \bar{x}_t \leq .60$	
.05	.25	.40	.25	.05	7	$-.30 \leq \bar{x}_t \leq .60$	
0.00	0.00	.33	.34	.33	7	$\leq .60$	
.33	.34	.33	0.00	0.00	7	$-.30 \leq$	
.05	.30	.30	.30	.05	7	$-.30 \leq \bar{x}_t \leq .60$	
.20	.20	.20	.20	.20	7	$-.30 \leq \bar{x}_t \leq .60$	
-	-	*	-	-	0	-	OH

* For all observed values of $\hat{\alpha}_t'(0) < .20$, the optimal sample size was found to be 0.

Upon comparing the expected unit cost derived from the two models, a 32.2% savings may be achieved by utilizing the steady state optimal operating policies derived from the dynamic model.

Computational Efficiency

The execution time on the IBM model 370/158 computer for optimizing the dynamic model for each of the two state example problems ranged between 4 and 7 minutes, depending on how fast the gain, θ, converged. More specifically, to solve a single period of the dynamic recursive equation required approximately 8 seconds.

In attempting to solve the three state example problem, approximately 9 minutes of execution time was required. In solving the five state process dynamic model approximately 30 minutes of execution time was required.

Hence, as the size of the problem increases, the corresponding execution time required for optimization increases drastically. Also bear in mind that only a coarse grid on the prior probabilities was considered. Consequently, in an actual application of the dynamic model the amount of computer time required would be considerable.

On the other hand, the optimization procedure for the steady state $\overline{X}$ control chart model for the two, three, and five state process example problems only required at most 6 seconds of execution time.

Time Dependent Dynamic Model

Recall that in order to show the existence of steady state optimal policies the dynamic model was initially formulated in a time dependent environment. Then after establishing the convergence of the gain, θ, the steady state policies were obtained.

In observing the numerical results for each of the example problems, the gain has been shown to converge in three to five periods. Therefore, any application of the time dependent model would only be reasonable for operations of only one or two periods. For operations of any greater length of time, the steady state policies would be applicable.

Consequently, it seems reasonable to assume that the use of a time dependent set of operating policies is not practical.

Summary

In this chapter the numerical results of five example problems have been described, tabularized, and illustrated. It has been shown for each example problem that by utilizing the steady state optimal control policies derived from the dynamic model, a savings of from 29% to 40% can be achieved in the expected cost per unit over the use of the control policies derived from the $\overline{X}$ control chart model. Additionally, it was shown that, in most cases, the expected cost per unit was relatively insensitive to substantial changes in the sampling interval.

Chapter V

SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

In this chapter, conclusions are made on the numerical results presented in Chapter IV and areas for future research are discussed.

Summary and Conclusions

The conclusions to be drawn from the results of Chapter IV are primarily in three areas. Initially, it has been shown in all cases considered, that there is a substantial cost savings available by employing the optimal steady state operating policies derived from the dynamic model rather than utilizing the operating policies of an $\overline{X}$ control chart. In process control situations where quality costs are relatively high, the dynamic approach can yield significant results.

However, the simplicity in using this approach in a practical application is not comparable to $\overline{X}$ control chart policies. Since an increased amount of computations are required in the use of the dynamic model (i.e., deriving posterior probabilities and prior probabilities for future periods), the use of a computer terminal or high-speed calculator would most likely be needed. Otherwise, decision tables and/or conversion tables would have to be utilized. And for a process consisting of more than two or three states, this method would probably be extremely cumbersome. Consequently, it is suggested that the dynamic model be utilized in controling a process whose quality costs are high and in a moderately sophisticated environment.

Secondly, a great deal of accuracy in deriving an optimal set of operation policies is not warranted. Judging from most of the example problems, the optimal unit cost appears to be relatively insensitive over a wide range of sampling intervals. Like many quality control cost models, a necessary condition is only to operate in a region near an optimal solution.

Finally, the efficiency of the optimization procedure is considered to be very poor. The application of the dynamic model to a process of any considerable size may require many hours of execution time to derive an optimal set of steady state operating policies. For a process of considerable size, it is suggested that this factor be considered. Perhaps in such a case, the application of the $\overline{X}$ control chart technique would suffice.

Areas for Future Research

Probably the most important follow-up to this research is the examination of the sensitivity of the optimal operating policies to the accuracy designed into the optimization procedure. That is to say, a much finer grid of values for the prior state vector and possible sample means would need to be considered.

Extensions to the dynamic model could include the consideration of an additional random variable in the overhaul cost term. The average cost of an overhaul could be replaced by random variable which describes the cost of lost profit due to machine down time. In addition, the dynamic model could be reformulated in order to describe multiple quality characteristics.

In following the research of Carter [7], a logical extension to his work would be effected through the development of a multi-characteristic model. The process characteristic could be described in terms of a multivariate normal random variable. Then, assuming multivariate normal sampling, the conjugate prior relationship could be utilized to simplify model analysis.

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Appendix A

DOCUMENTATION OF COMPUTER PROGRAM

General Information

This appendix contains a documented listing of the computer program used to optimize the dynamic model. The program is written in FORTRAN IV and was run on an IBM 370/158 system.

Input

In order to utilize the program,a variable amount of input data is necessary. The necessary data cards are described as follows:

Card	Field	Variable	Description
1	2I5	S	Specifies the number of out-of-control states on one side of $\mu(0)$. (e.g., three states, s = 1.) s = 0, if two state process is to be analyzed.
		NPS	Specifies the number of different prior state vectors, $\hat{\underline{\alpha}}_t{}'$, to consider.
2	6F10.0	A1	The fixed cost of sampling.
		A2	The variable cost of sampling.
		A3	The average cost of an overhaul.
		A4	The cost per unit of a defect.
3	6F10.0	LAM	The exponential parameter, λ.
		PI	The binomial parameter, p.
		SIG	The process variance, σ^2.
		SL	Lower specification limit.
		SU	Upper specification limit.
		R	Production rate.
4	6F10.0	MU(I)	The possible states of the process mean; listed as $\mu(-s)$, $\mu(-s+1)$, ..., $\mu(0)$, $\mu(1)$, ..., $\mu(s)$.

Card	Field	Variable	Description
5 to (NPS+4)	6F10.0	PRIOR(I,J)	Prior state vector, $\hat{\alpha}_t'$; listed as $\hat{\alpha}_t'(-s)$, $\hat{\alpha}_t'(-s+1)$, ..., $\hat{\alpha}_t'(\overline{0})$, ..., $\hat{\alpha}_t'(s)$.
NPS+5	6F10.0	H	Starting value of the sampling interval where h is in time units.
		XBARL	Minimum value of $\overline{x}$.
		XBARU	Maximum value of $\overline{x}$.
		XBINT	Incremental steps in the value of $\overline{x}$ to be evaluated between XBARL and XBARU.
		DELTA	Convergence factor for the unit gain, θ/k.
NPS+6	2I5	NSMAX	Maximum sample size to consider.
		ITMAX	Maximum number of iterations to establish gain.
NPS+7	6F10.0	KSTEP	Step size for the sampling interval bounding procedure (in terms of units produced). The "interval of uncertainty" is equal to 2*KSTEP.
		KENUM	Incremental values of k, the sampling interval, to enumerate within the "interval of uncertainty."
NPS+8	2I5	IPNS	= 1 if output is to include a listing of all sample sizes considered for each $\underline{\hat{\alpha}}_t$ at each period of the dynamic recursion also with the corresponding information.
		IPR	= 1 if output is to include the operating policies found in the initial bounding procedure on the sampling interval.

Output

The output includes a listing and description of all input parameters. In addition, for each period of the dynamic recursion for a given value of the sampling interval, each prior state vector is listed along with the corresponding optimal operating policies (i.e., the optimal sample size, the minimum future expected cost, and the description of optimal operating decisions). Also a gain analysis is provided after each period of the dynamic recursion.

Program Restrictions

Due to the dimension of variables within the program, a maximum of five states of the process mean can be analyzed and no more than 30 different prior state vectors, $\hat{\underline{\alpha}}_t'$, can be considered.

```
C***********************************************************************
C                                                                      *
C  MAIN PROGRAM                                                        *
C                                                                      *
C  THIS PROGRAM DRIVES THE DYNAMIC PROGRAM.                            *
C                                                                      *
C  UPPER AND LOWER BOUNDS ON THE OPTIMAL SAMPLING INTERVAL ARE         *
C  ESTABLISHED.  THEN POINTS AT SPECIFIED INCREMENTS ARE               *
C  ENUMERATED WITHIN THE ESTABLISHED INTERVAL.                         *
C                                                                      *
C***********************************************************************
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/BK2/ LAM,LAMS,PI,SIG,SL,SU,R
      COMMON/BK8/ PV(5),FGOH,IPASS
      COMMON/BK9/K,IPERD
      COMMON/BK10/ KSTEP,KENUM
      COMMON/BK11/ IPNS,IPR
      REAL*8 MU,LAM,LAMS,KSTEP,KENUM,K
      INTEGER S
      IPASS=0
      ISTEP=0
      CALL INPUT(H)
      IB=IPNS
      IPNS=0
      I=H*R
      K=I
      CALL DYNPRG(UNTCST)
      AMIN=UNTCST
      K=K+KSTEP
      IST=1
C
C  ATTEMPT TO ESTABLISH UPPER AND LOWER BOUNDS ON THE OPTIMAL
```

```
C   SAMPLING INTERVAL.
C
    1 ISTEP=ISTEP+1
      CALL DYNPRG(UNTCST)
      IF(UNTCST.GE.AMIN) GO TO 2
      IF(ISTEP.GE.5) GO TO 9
      IF(IST.LT.0) K=K-KSTEP
      IF(IST.GT.0) K=K+KSTEP
      AMIN=UNTCST
      IF(K.LE.0.) GO TO 5
      GO TO 1
    2 IF(ISTEP.EQ.1) GO TO 4
      IF(IST.GT.0) GO TO 3
      KMIN=K
      KMAX=K+2.D0*KSTEP
      GO TO 6
    3 KMAX=K
      KMIN=K-2.D0*KSTEP
      GO TO 6
    4 IF(IST.LT.0) GO TO 5
      ISTEP=0
      K=K-2.D0*KSTEP
      IST=-1
      GO TO 1
    5 KMIN=K
      KMAX=KMIN+2.D0*KSTEP
      IF(KMIN.LE.0.) GO TO 8
C
C   ENUMERATE POINTS WITHIN THE INTERVAL OF UNCERTAINTY.
C
    6 K=KMIN+KENUM
      IPNS=IB
```

```
      IPR=1
    7 CALL DYNPRG(UNTCST)
      IF(K.GE.KMAX) GO TO 10
      K=K+KENUM
      GO TO 7
    8 WRITE(6,600)
      KMIN=0.D0
      GO TO 6
    9 WRITE(6,601)
   10 STOP
  600 FORMAT('-',10('*'),'POSSIBLE BOUNDRY CONDITION-- THE SAMPLING INTE
     *RVAL IS BOUNDED BY ZERO.  THUS, 100% INSPECTION MAY BE OPTIMAL.')
  601 FORMAT('-',10('*'),'POSSIBLE BOUNDRY CONDITION-- AN UPPER BOUND ON
     * THE SAMPLING HAS NOT BEEN ESTABLISHED********EXECUTION IS BEING T
     *ERMINATED***********')
      END
```

```
      SUBROUTINE INPUT(H)
C***********************************************************************
C                                                                      *
C  THIS PROGRAM INPUTS ALL THE COST TERMS, PROCESS PARAMETERS,         *
C  AND PROGRAM OPERATING PARAMETERS.                                   *
C                                                                      *
C***********************************************************************
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/BK1/ A1,A2,A3,A4
      COMMON/BK2/ LAM,LAMS,PI,SIG,SL,SU,R
      COMMON/BK3/ S,NPS,NST,NSMAX,ITMAX
      COMMON/BK4/ F(5),P(5),MU(5)
      COMMON/BK5/ PRIOR(30,5),V(30)
      COMMON/BK7/ XBARL,XBARU,XBINT,DELTA
      COMMON/BK10/ KSTEP,KENUM
      COMMON/BK11/ IPNS,IPR
      REAL*8 MU,LAM,LAMS,KSTEP,KENUM
      INTEGER S
      READ(5,500) S,NPS
      NST=2*S+1
      IF(S.EQ.0) NST=2
      READ(5,501) A1,A2,A3,A4
      READ(5,501) LAM,PI,SIG,SL,SU,R
      READ(5,501) (MU(I),I=1,NST)
      DO 1 I=1,NPS
      READ(5,501) (PRIOR(I,J),J=1,NST)
    1 V(I)=0.D0
      READ(5,501) H,XBARL,XBARU,XBINT,DELTA
      READ(5,500) NSMAX,ITMAX
      READ(5,501) KSTEP,KENUM
      READ(5,500) IPNS,IPR
      LAMS=LAM/R
```

```
      CALL PVECF
      WRITE(6,600)
      WRITE(6,601) A1,A2,A3,A4
      DO 2 I=1,NST
      IS=S+1-I
      IT=I-S-1
      IF(I.LT.(S+1)) WRITE(6,602) IS,MU(I)
      IF(I.GE.(S+1)) WRITE(6,603) IT,MU(I)
    2 CONTINUE
      WRITE(6,604) SL,SU
      WRITE(6,605) LAM,R,SIG,PI
      WRITE(6,606)
      WRITE(6,607) XBARL,XBARU
      WRITE(6,608) DELTA
      WRITE(6,609) NSMAX
      WRITE(6,610) ITMAX
      RETURN
  500 FORMAT(2I5)
  501 FORMAT(6F10.0)
  600 FORMAT('1',T15,'THE DYNAMIC PROCESS CONTROL MODEL'/'0','COST TERMS
     * AND PROCESS PARAMETERS:')
  601 FORMAT('0',5X,'FIXED COST OF SAMPLING: $ ',F12.4/'0',5X,'VARIABLE
     *COST OF SAMPLING: $ ',F12.4/'0',5X,'AVERAGE COST OF AN OVERHAUL: $
     * ',F12.4/'0',5X,'COST PER DEFECTIVE UNIT: $ ',F12.4/'0',5X,'DISCRE
     *TE STATES OF THE PROCESS MEAN:')
  602 FORMAT('0',10X,'MU(-',I1,')= ',F12.5)
  603 FORMAT('0',10X,'MU( ',I1,')= ',F12.5)
  604 FORMAT('0',5X,'SPECIFICATION LIMITS:  SL=',F12.5,'   SU=',F12.5)
  605 FORMAT('0',5X,'LAMDA=',F10.7/'0',5X,'PRODUCTION RATE=',F8.0/'0',
     *5X,'PROCESS VARIANCE=',F12.6/'0',5X,'BINOMIAL PROBABILITY PARAMETE
     *R=',F10.7)
  606 FORMAT('-','PROGRAM OPERATING PARAMETERS:')
```

```
607 FORMAT('0',5X,'RANGE OF POSSIBLE VALUES OF THE SAMPLE MEAN:'/'0', DYN 1450
   *10X,'MINIMUM VALUE=',F12.5/'0',10X,'MAXIMUM VALUE=',F12.5)        DYN 1460
608 FORMAT('0',5X,'CONVERGENCE FACTOR ON THE UNIT GAIN:  $ ',F10.7)   DYN 1470
609 FORMAT('0',5X,'LARGEST SAMPLE SIZE CONSIDERED IN THE ANALYSIS: ', DYN 1480
   * I5)                                                              DYN 1490
610 FORMAT('0',5X,'MAXIMUM NUMBER OF ITERATIONS TO ESTABLISH GAIN CONVDYN 1500
   *ERGENCE: ',I5)                                                    DYN 1510
    END                                                               DYN 1520
```

```
      SUBROUTINE DYNPRG(UNTCST)
C***********************************************************************
C                                                                      *
C  THIS PROGRAM SOLVES THE DYNAMIC MODEL FOR A GIVEN VALUE OF THE      *
C  SAMPLING INTERVAL 'K' UNITS.                                        *
C                                                                      *
C  THE COST PER UNIT 'UNTCST', ASSOCIATED WITH 'K' FOR OPERATING       *
C  THE QUALITY CONTROL PROCEDURE IS RETURNED TO THE MAIN PROGRAM.      *
C                                                                      *
C***********************************************************************
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/BK2/ LAM,LAMS,PI,SIG,SL,SU,R
      COMMON/BK3/ S,NPS,NST,NSMAX,ITMAX
      COMMON/BK4/ F(5),P(5),MU(5)
      COMMON/BK5/ PRIOR(30,5),V(30)
      COMMON/BK6/ B(5,5)
      COMMON/BK7/ XBARL,XBARU,XBINT,DELTA
      COMMON/BK9/K,IPERD
      COMMON/BK11/ IPNS,IPR
      DIMENSION VH(30),SAV(16,4),PR(5),XB(2)
      REAL*8 K,LAM,LAMS,MU
      INTEGER S
C
C  EVALUATE THE PROBABILITY VECTOR 'P' AND THE TRANSITION
C  PROBABILITY MATRIX 'B'.
C
      CALL PVECP(K)
      CALL BMTRX
C
C  NOW BEGIN THE DYNAMIC RECURSION.
C
      DO 10 IPERD=1,ITMAX
```

```
      IF(IPR.EQ.1) WRITE(6,600) IPERD
C
C  NOW ITERATE OVER THE PRIOR PROBABILITIES.
C
      DO 5 IJK=1,NPS
      DO 2 I=1,NST
    2 PR(I)=PRIOR(IJK,I)
      IF(IPR.EQ.1) WRITE(6,601) (PR(IJ),IJ=1,NST)
      NMX=NSMAX+1
C
C  NOW ITERATE OVER THE SAMPLE SIZES.
C
      DO 3 NN=1,NMX
      NS=NN-1
C
C  EVALUATE THE MINIMUM FUTURE EXPECTED COST, GIVEN PRIOR
C  PROBABILITIES 'PR(I)', SAMPLE SIZE 'NS',AT PERIOD 'IPERD'.
C
      CALL EXPCST(NS,XB,PR,EC)
      SAV(NN,1)=NS
      SAV(NN,2)=EC
      SAV(NN,3)=XB(1)
      SAV(NN,4)=XB(2)
      IF(IPNS.EQ.1) CALL DECIDE(NN,NS,EC,XB)
    3 CONTINUE
      AMIN=1.D50
C
C  DETERMINE OPTIMAL SAMPLE SIZE FOR PRIOR PROBABILITIES 'PR(I)'.
C
      DO 4 I=1,NMX
      IF(SAV(I,2).GT.AMIN) GO TO 4
      ISV=I
```

```
      AMIN=SAV(I,2)
    4 CONTINUE
      IF(IPR.EQ.1) WRITE(6,602)
      NS=SAV(ISV,1)
      XB(1)=SAV(ISV,3)
      XB(2)=SAV(ISV,4)
      IF(IPR.EQ.1) CALL DECIDE(1,NS,AMIN,XB)
    5 VH(IJK)=AMIN
      IF(IPERD.EQ.1) GO TO 8
      A=0.DO
      DO 6 I=1,NPS
    6 A=A+(VH(I)-V(I))
      AVGAIN=A/NPS
      UNITGN=AVGAIN/K
      IP=IPERD-1
      IF(IPR.EQ.1) WRITE(6,603) IPERD,IP
      IF(IPR.EQ.1) WRITE(6,604) AVGAIN,UNITGN
      IF(IPERD.GT.2) GO TO 7
      UNH=UNITGN
      GO TO 8
C
C   TEST FOR GAIN CONVERGENCE.
C
    7 IF(DABS(UNITGN-UNH).LE.DELTA .OR.IPERD.GE.ITMAX) GO TO 11
      UNH=UNITGN
    8 DO 9 I=1,NPS
    9 V(I)=VH(I)
   10 CONTINUE
   11 UNTCST=UNITGN
      WRITE(6,606)
      WRITE(6,605) IPERD,K,UNTCST
      WRITE(6,606)
```

```
      RETURN
  600 FORMAT('1','DYNAMIC RECURSION: PERIOD ',I2)
  601 FORMAT('0'/' ',5X,'PRIOR STATE VECTOR'/'0',5X,9F8.5)
  602 FORMAT('0',5X,'OPTIMAL SAMPLING RULE AND MINIMUM FUTURE EXPECTED C
     *OST:')
  603 FORMAT('-','GAIN ANALYSIS BETWEEN PERIODS ',I2,' AND ',I2)
  604 FORMAT('0',5X,'AVERAGE GAIN  $ ',F14.7,'  Q C PLAN: UNIT COST  $ '
     *,F10.6)
  605 FORMAT('-',70('$')/'0','STEADY STATE CONTROL POLICY FOUND AT PERIO
     *D ',I2/'0','FOR A SAMPLING INTERVAL OF ',F6.0,' UNITS,'/'0','THE C
     *OST PER UNIT TO OPERATE THE Q C PLAN IS  $',F10.6/'0',70('$'))
  606 FORMAT('-',70('$')/' ',70('$'))
      END
```

```
      SUBROUTINE EXPCST(NS,XBRK,PR,EC)
C**********************************************************************
C                                                                     *
C  THIS PROGRAM DETERMINES THE MINIMUM FUTURE EXPECTED COST ,         *
C  GIVEN A PRIOR STATE VECTOR AND SAMPLE SIZE.                        *
C                                                                     *
C**********************************************************************
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/BK7/ XBARL,XBARU,XBINT,DELTA
      DIMENSION XBRK(2),PR(5),PST(5)
      IF(NS.NE.0) GO TO 1
      CALL EXPC(PR,EO,ER)
      XBRK(1)=1.D50
      IF(EO.LT.ER) XBRK(1)=-1.D50
      XBRK(2)=XBRK(1)
      EC=DMIN1(EO,ER)
      GO TO 5
    1 XBRK(1)=-1.D50
      XBRK(2)=XBARU
      I1=0
      I2=0
      XB=XBARL-XBINT
      EC=0.D0
C
C  ITERATE OVER VALUES OF THE SAMPLE MEAN.
C
      DO 4 I=1,100000
      XB=XB+XBINT
C
C  DETERMINE CORRESPONDING POSTERIOR PROBABILITIES.
C
      CALL POST(NS,PR,XB,PST)
```

```
C
C  EVALUATE MARGINAL PROBABILITY OF THE SAMPLE MEAN.
C
      CALL PRXB(NS,XB,PR,PX)
C
C  EVALUATE THE EXPECTED FUTURE COST OF AN OVERHAUL 'EO' AND TO
C  LET THE PROCESS RUN FOR A SINGLE PERIOD 'ER'.
C
      CALL EXPC(PST,EO,ER)
      EC=EC+DMIN1(EO,ER)*PX
      IF(I1.EQ.1) GO TO 2
C
C  DETERMINE BREAKEVEN POINTS OF THE OPTIMAL OPERATING DECISIONS.
C
      IF((EO-ER).LT.0.DO) GO TO 3
      XBRK(1)=XB
      IF(I.EQ.1) XBRK(1)=XBARL
      I1=1
      GO TO 3
    2 IF(I2.EQ.1) GO TO 3
      IF((ER-EO).LT.0.DO) GO TO 3
      XBRK(2)=XB
      I2=1
    3 IF(XB.GE.XBARU) GO TO 5
    4 CONTINUE
    5 EC=EC+SAMPL(NS)
      RETURN
      END
```

```
      FUNCTION SAMPL(NS)
C******************************************************************
C                                                                 *
C  THIS PROGRAM EVALUATES THE COST OF SAMPLING.                   *
C                                                                 *
C******************************************************************
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/BK1/ A1,A2,A3,A4
      SAMPL=A1+NS*A2
      RETURN
      END
```

```
      SUBROUTINE EXPC(PS,EO,ER)
C**********************************************************************
C                                                                     *
C  THIS PROGRAM DETERMINES THE FUTURE COSTS OF EACH OPERATING         *
C  DECISION.                                                          *
C                                                                     *
C**********************************************************************
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/BK1/ A1,A2,A3,A4
      COMMON/BK3/ S,NPS,NST,NSMAX,ITMAX
      COMMON/BK8/ PV(5),FGOH,IPASS
      COMMON/BK9/K,IPERD
      DIMENSION PS(5),P1(5),PSR(5)
      REAL*8 K
      INTEGER S
      ID=0
      IF(IPASS.EQ.1) GO TO 2
      DO 1 I=1,NST
      P1(I)=0.D0
      IF(I.EQ.(S+1)) P1(I)=1.D0
    1 CONTINUE
C
C  DETERMINE THE PRIOR PROBABILITIES FOR PERIOD 'T-1' GIVEN THE
C  POSTERIOR PROBABILITIES.
C
      CALL TRANS(P1,PV)
      IPASS=1
C
C  EVALUATE THE 'GAMMA' PROBABILITIES.
C
      CALL FGAM(P1,FGOH)
    2 DO 3 I=1,NST
```

```
    3 PSR(I)=PV(I)
C
C   FIND THE MINIMUM FUTURE EXPECTED COSTS AT PERIOD 'T-1', GIVEN
C   THE PRIOR PROBABILITIES FOUND PREVIOUSLY.
C
    4 CALL SERCHV(PSR,VSV)
      IF(ID.EQ.1) GO TO 5
      EO=A3+A4*K*FGOH+VSV
      CALL TRANS(PS,PSR)
      ID=1
      GO TO 4
    5 CALL FGAM(PS,FGRN)
      ER=A4*K*FGRN+VSV
      RETURN
      END
```

```
      SUBROUTINE POST(NS,PR,XB,PST)
C**********************************************************************
C                                                                     *
C  THIS PROGRAM DETERMINES THE POSTERIOR PROBABILITIES GIVEN          *
C  THE SAMPLE SIZE, THE SAMPLE MEAN, AND THE PRIOR PROBABILITIES.     *
C                                                                     *
C**********************************************************************
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/BK2/ LAM,LAMS,PI,SIG,SL,SU,R
      COMMON/BK3/ S,NPS,NST,NSMAX,ITMAX
      COMMON/BK4/ F(5),P(5),MU(5)
      DIMENSION H(5),PR(5),PST(5)
      REAL*8 MU,LAM,LAMS
      INTEGER S
      DN=NS
      AMULT=DN/(2.D0*DSQRT(SIG))
      DO 1 I=1,NST
    1 H(I)=(XB-MU(I))**2*AMULT
      A=0.D0
    2 DO 3 I=1,NST
      IF(H(I).GE.150.D0) GO TO 4
    3 CONTINUE
      GO TO 6
    4 DO 5 I=1,NST
    5 H(I)=H(I)/2.D0
      GO TO 2
    6 DO 7 I=1,NST
      PST(I)=DEXP(-H(I))*PR(I)
    7 A=A+PST(I)
      DO 8 I=1,NST
    8 PST(I)=PST(I)/A
      RETURN
```

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END

```
      SUBROUTINE TRANS(PS,PSR)
C***********************************************************************
C                                                                      *
C  THIS PROGRAM DETERMINES THE PRIOR PROBABILITIES AT PERIOD 'T-1' ,  *
C  GIVEN THE POSTERIOR PROBABILITIES AT 'T' AND THE TRANSITION         *
C  PROBABILITY MATRIX 'B'.                                             *
C                                                                      *
C***********************************************************************
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/BK3/ S,NPS,NST,NSMAX,ITMAX
      COMMON/BK6/ B(5,5)
      DIMENSION PS(5),PSR(5)
      INTEGER S
      DO 2 I=1,NST
      A=0.DO
      DO 1 J=1,NST
    1 A=A+PS(J)*B(J,I)
    2 PSR(I)=A
      RETURN
      END
```

```
      SUBROUTINE FGAM(PP,FG)
C**********************************************************************
C                                                                    *
C  THIS PROGRAM DETERMINES THE 'GAMMA' PROBABILITIES.                *
C                                                                    *
C**********************************************************************
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/BK3/ S,NPS,NST,NSMAX,ITMAX
      COMMON/BK4/ F(5),P(5),MU(5)
      COMMON/BK9/K,IPERD
      DIMENSION GP(2,2),PP(5)
      INTEGER S
      REAL*8 MU,K
      IF(S.EQ.0) GO TO 3
      GPO=PP(S+1)
      DO 1 J=1,S
      GP(1,J)=PP(S-J+1)
    1 GP(2,J)=PP(S+J+1)
      CALL PVECGM(GPO,GP,GPO,GP)
      FG=GPO*F(S+1)
      DO 2 J=1,S
    2 FG=FG+F(S-J+1)*GP(1,J)+F(S+J+1)*GP(2,J)
      GO TO 4
    3 DEL=FDEL(K)
      GO=PP(1)*P(1)+DEL*PP(1)*P(2)
      G1=PP(2)+(1.DO-DEL)*PP(1)*P(2)
      FG=F(1)*GO+F(2)*G1
    4 RETURN
      END
```

```
      SUBROUTINE PRXB(N,XB,PR,PX)
C**********************************************************************
C                                                                     *
C  THIS PROGRAM EVALUATES THE MARGINAL PROBABILITY OF THE SAMPLE      *
C  MEAN GIVEN THE SAMPLE SIZE AND THE PRIOR STATE VECTOR.             *
C                                                                     *
C**********************************************************************
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/BK2/ LAM,LAMS,PI,SIG,SL,SU,R
      COMMON/BK3/ S,NPS,NST,NSMAX,ITMAX
      COMMON/BK4/ F(5),P(5),MU(5)
      COMMON/BK7/ XBARL,XBARU,XBINT,DELTA
      DIMENSION PR(5)
      REAL*8 MU,LAM,LAMS
      INTEGER S
      PX=0.D0
      DN=N
      DNM=DSQRT(SIG/DN)
      AINT=XBINT/2.D0
      XL=XB-AINT
      XU=XB+AINT
      DO 1 I=1,NST
      CU=(XU-MU(I))/DNM
      CL=(XL-MU(I))/DNM
      IF(DABS(CU).GT.3.1D0) GO TO 1
      CALL FAPROX(CU,PU)
      CALL FAPROX(CL,PL)
      PX=PX+(PU-PL)*PR(I)
    1 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE PVECP(K)
C***********************************************************************
C                                                                      *
C  THIS PROGRAM EVALUATES THE PROBABILITY VECTOR 'P'.                  *
C                                                                      *
C***********************************************************************
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/BK2/ LAM,LAMS,PI,SIG,SL,SU,R
      COMMON/BK3/ S,NPS,NST,NSMAX,ITMAX
      COMMON/BK4/ F(5),P(5),MU(5)
      REAL*8 K,MU,LAM,LAMS
      INTEGER S
      P(1)=DEXP(-LAMS*K)
      IF(S.EQ.0) GO TO 2
      J=S
      DJ=J
      DA=(1.D0-(1.D0-PI)**J)
      FS=DGAMMA(DJ+1.D0)
      DI=0.D0
      DO 1 I=1,J
      DI=DI+1.D0
      FI=DGAMMA(DI+1.D0)
      FSI=DGAMMA(DJ-DI+1.D0)
    1 P(I+1)=(1.D0-P(1))*FS*(PI**I)*(1.D0-PI)**(J-I)/(DA*FI*FSI)/2.D0
      GO TO 3
    2 P(2)=1.D0-P(1)
    3 RETURN
      END
```

```
      SUBROUTINE BMTRX
C*************************************************************************
C                                                                        *
C  THIS PROGRAM DETERMINES THE TRANSITION PROBABILITY MATRIX 'B'.        *
C                                                                        *
C*************************************************************************
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/BK3/ S,NPS,NST,NSMAX,ITMAX
      COMMON/BK4/ F(5),P(5),MU(5)
      COMMON/BK6/ B(5,5)
      DIMENSION C(3,3)
      REAL*8 MU
      INTEGER S
      IF(S.EQ.0) GO TO 13
      DO 1 I=1,NST
      DO 1 J=1,NST
    1 B(I,J)=0.D0
      II=S+1
      DO 7 I=1,II
      DO 7 J=1,II
      IF(I.GT.1) GO TO 2
      C(I,J)=P(J)
      GO TO 7
    2 IF(J-I) 3,4,6
    3 C(I,J)=0.D0
      GO TO 7
    4 A=0.D0
      DO 5 IJ=2,I
    5 A=A+P(IJ)
      C(I,J)=2.D0*A/(1.D0-P(1))
      GO TO 7
    6 C(I,J)=2.D0*P(J)/(1.D0-P(1))
```

```
    7 CONTINUE
      DO 12 I=1,NST
      DO 11 J=1,NST
      IF(J.GT.I.AND.I.LT.(S+1)) GO TO 12
      IF(I.GE.(S+1)) GO TO 8
      B(I,J)=C((S+2-I),(S+2-J))
      GO TO 11
    8 IF(I.GT.(S+1)) GO TO 10
      DO 9 IJ=1,II
      B(I,IJ)=C(1,(S+2-IJ))
    9 B(I,(IJ+S))=C(1,IJ)
      GO TO 12
   10 IF(J.LT.I) GO TO 11
      B(I,J)=C((I-S),(J-S))
   11 CONTINUE
   12 CONTINUE
      GO TO 14
   13 B(1,1)=P(1)
      B(1,2)=P(2)
      B(2,1)=0.DO
      B(2,2)=1.DO
   14 RETURN
      END
```

```
      SUBROUTINE PVECGM(PSO,PS,GMO,GM)
C***********************************************************************
C                                                                     *
C  THIS PROGRAM EVALUATES THE 'GAMMA' PROBABILITIES.                  *
C                                                                     *
C***********************************************************************
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/BK4/ F(5),P(5),MU(5)
      COMMON/BK3/ S,NPS,NST,NSMAX,ITMAX
      COMMON/BK9/ K,IPERD
      DIMENSION PS(2,2),GM(2,2)
      REAL*8 K,MU
      INTEGER S
      DEL=FDEL(K)
      GMO=PSO*P(1)+PSO*(1.DO-P(1))*DEL
      DO 5 I=1,2
      DO 5 J=1,S
      T3=0.DO
      T4=0.DO
      A=0.DO
      DO 1 II=1,J
    1 A=A+P(II+1)*2.DO
      T1=PS(I,J)*A/(1.DO-P(1))
      T2=PSO*P(J+1)*(1.DO-DEL)
      IF(J.EQ.1) GO TO 3
      A=0.DO
      IN=J-1
      DO 2 II=1,IN
    2 A=A+PS(I,II)
      T3=A*P(J+1)*2.DO/(1.DO-P(1))*(1.DO-DEL)
    3 IF(J.EQ.S) GO TO 5
      A=0.DO
```

```
      IN=J+1                                                            DYN 6080
      DO 4 II=IN,S                                                      DYN 6090
    4 A=A+P(II+1)*2.DO                                                  DYN 6100
      T4=PS(I,J)*DEL*A/(1.DO-P(1))                                      DYN 6110
    5 GM(I,J)=T1+T2+T3+T4                                               DYN 6120
      RETURN                                                            DYN 6130
      END                                                               DYN 6140
```

```
      SUBROUTINE PVECF
C***********************************************************************
C                                                                      *
C  THIS PROGRAM EVALUATES THE PROBABILITY VECTOR 'F' ; THE             *
C  CONDITIONAL PROBABILITY OF PRODUCING A DEFECT.                      *
C                                                                      *
C***********************************************************************
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/BK2/ LAM,LAMS,PI,SIG,SL,SU,R
      COMMON/BK3/ S,NPS,NST,NSMAX,ITMAX
      COMMON/BK4/ F(5),P(5),MU(5)
      REAL*8 MU,LAM,LAMS
      INTEGER S
      DSIG=DSQRT(SIG)
      DO 1 I=1,NST
      CU=(SU-MU(I))/DSIG
      CL=(SL-MU(I))/DSIG
      CALL FAPROX(CU,PU)
      CALL FAPROX(CL,PL)
    1 F(I)=1.D0-(PU-PL)
      RETURN
      END
```

```
      SUBROUTINE FAPROX(X,FK)
C****************************************************************
C                                                               *
C  THIS PROGRAM EVALUATES THE CUMULATIVE NORMAL PROBABILITY.    *
C                                                               *
C****************************************************************
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(6)
      IF(DABS(X).LE.1.D-11) GO TO 2
      B=2.D0
      A(1)=.0705230784
      A(2)=.0422820123
      A(3)=.0092705272
      A(4)=.0001520143
      A(5)=.0002765672
      A(6)=.0000430638
      T=DABS(X)/DSQRT(B)
      SUM=0.
      DO 1 I=1,6
    1 SUM=SUM+A(I)*T**I
      DENOM=(1.+SUM)**16
      GK=1.-1./DENOM
      FK=.5*(1.+X/DABS(X)*GK)
      RETURN
    2 FK=.5D0
      RETURN
      END
```

```
      SUBROUTINE SERCHV(PRS,VSV)
C***********************************************************************
C                                                                      *
C  THIS PROGRAM DETERMINES THE MINIMUM FUTURE EXPECTED COST AT         *
C  PERIOD 'T-1' FOR A SET OF PRIOR PROBABILITIES.                      *
C                                                                      *
C***********************************************************************
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/BK3/ S,NPS,NST,NSMAX,ITMAX
      COMMON/BK5/ PRIOR(30,5),V(30)
      COMMON/BK9/ K,IPERD
      INTEGER S
      REAL*8 K
      DIMENSION PRS(5)
      IF(IPERD.EQ.1) GO TO 8
      PIC=PRS(S+1)
      DO 1 I=1,NPS
      IF(PIC.LT.PRIOR(I,(S+1)).AND.PIC.GE.PRIOR((I+1),(S+1))) GO TO 2
    1 CONTINUE
    2 ISV=I
      DIF1=0.D0
      DIF2=0.D0
      DO 3 J=1,NST
      DIF1=DIF1+DABS(PRIOR(ISV,J)-PRS(J))
    3 DIF2=DIF2+DABS(PRIOR((ISV+1),J)-PRS(J))
      ISAV=ISV
      IF(DIF2.GE.DIF1) GO TO 4
      ISAV=ISV+1
      DIF1=DIF2
    4 I2=ISV+2
      IF(I2.GT.NPS) GO TO 7
      DO 6 I=I2,NPS
```

```
      IF(PIC.LT.PRIOR(I,(S+1))) GO TO 7
      DIF2=0.D0
      DO 5 J=1,NST
    5 DIF2=DIF2+DABS(PRIOR(I,J)-PRS(J))
      IF(DIF2.GE.DIF1) GO TO 6
      ISAV=I
      DIF1=DIF2
    6 CONTINUE
    7 VSV=V(ISAV)
      GO TO 9
    8 VSV=0.D0
    9 RETURN
      END
```

```
      FUNCTION FDEL(K)
C***********************************************************************
C                                                                      *
C  THIS PROGRAM EVALUATES THE FUNCTION DELTA, THE AVERAGE              *
C  PROPORTION OF A SAMPLING INTERVAL THAT ELAPSES WHEN A SHIFT IN      *
C  THE PROCESS MEAN OCCURS.                                            *
C                                                                      *
C***********************************************************************
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/BK2/ LAM,LAMS,PI,SIG,SL,SU,R
      REAL*8 K,LAM,LAMS
      FDEL=( 1.DO-(1.DO+LAMS*K)*DEXP(-LAMS*K) )/( LAMS*K*(1.DO-DEXP(-LAM
     *S*K)) )
      RETURN
      END
```

```
      SUBROUTINE DECIDE(NN,NS,EC,XB)
C*********************************************************************
C                                                                    *
C  THIS PROGRAM OUTPUTS THE SAMPLE SIZE, MINIMUM FUTURE EXPECTED     *
C  COST, AND THE OPTIMAL OPERATING DECISION FOR A SET OF PRIOR       *
C  PROBABILITIES.                                                    *
C                                                                    *
C*********************************************************************
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/BK7/ XBARL,XBARU,XBINT,DELTA
      DIMENSION XB(2)
      IF(NN.NE.1) GO TO 1
      WRITE(6,600)
      WRITE(6,601)
    1 WRITE(6,602) NS,EC
      IF(XB(1).EQ.XBARL.AND.XB(2).EQ.XBARU) XB(1)=1.D50
      IF(DABS(XB(1)).GE.1.D49) GO TO 5
      IF(XB(1).EQ.XBARL) GO TO 2
      WRITE(6,603) XB(1)
      GO TO 3
    2 WRITE(6,604)
    3 IF(XB(2).EQ.XBARU) GO TO 4
      WRITE(6,605) XB(2)
      GO TO 6
    4 WRITE(6,606)
      GO TO 6
    5 IF(XB(1).LT.0.) WRITE(6,607)
      IF(XB(1).GT.0.) WRITE(6,608)
    6 RETURN
  600 FORMAT('0',T53,'OVERHAUL IF SAMPLE MEAN IS')
  601 FORMAT(' ',10X,'SAMPLE SIZE',5X,'EXPECTED FUTURE COST',5X,'LESS TH
     *AN     GREATER THAN')
```

```
602 FORMAT('0',15X,I2,11X,F14.7)                                  DYN 7560
603 FORMAT('+',T53,F10.5)                                         DYN 7570
604 FORMAT('+',T53,'*LET RUN**')                                  DYN 7580
605 FORMAT('+',T69,F10.5)                                         DYN 7590
606 FORMAT('+',T69,'*LET RUN**')                                  DYN 7600
607 FORMAT('+',T53,5('*'),'OVERHAUL ALWAYS',6('*'))               DYN 7610
608 FORMAT('+',T53,6('*'),'LET RUN ALWAYS',6('*'))                DYN 7620
    END                                                           DYN 7630
```

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PROCESS CONTROL:

A DYNAMIC PROGRAMMING APPROACH

by

Wm. Howard Beverly Jr.

(ABSTRACT)

In this thesis, a cost based process control model is formulated. A dynamic programming approach is used and along with the techniques of Bayesian decision theory, an optimal set of steady state control policies are shown to exist which are dependent upon prior beliefs about the condition of the process.

It is the objective of this thesis to compare the results obtained from this approach to those of an X control chart approach. The model proposed by Knappenberger and Grandage [20] is used as a basis for comparison. Numerical examples are used to illustrate each procedure.

The results obtained illustrate that by using the operating policies specified by the dynamic approach, a savings of from 29% to 40% in the optimal cost per unit to operate the quality control procedure can be achieved rather than utilizing the policies of the $\overline{X}$ control chart model.