Direct and Converse Lyapunov Theorems for Discrete Dynamical Systems

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Abstract

This paper derives the necessary and sufficient conditions for the Lyapunov function such that the equilibrium point of a dynamical system is stable or Globally Asymptotic Stable (GAS). The paper shows that continuity of Lyapunov function at the equilibrium point is the only necessary and sufficient condition for stability. We shows that a certain type of converse theorem can not be proved with continuous Lyapunov function. The *Tower of Babel* 'TOB' is given as an example of stable dynamical system in which no continuous Lyapunov function exists for this system.

I. INTRODUCTION

One of the main topic in the theory of dynamical systems is stability analysis. Stability analysis of dynamical system is important, because without stability a system will behave erratically. Theoretically, stability of discrete dynamical systems refers to stability of solutions of difference equations at the steady state (equilibrium point). A system is called stable if all the equilibrium points are stable and it is unstable otherwise. There are several theorems to prove stability of dynamical systems. The most famous theorem introduced by the russian mathematician *A.M. Lyapunov* in 1892 is called *Lyapunov Theorem*. One may consider both direct and converse theorems. A direct Lyapunov theorem gives sufficient conditions for stability of a system, and a converse theorem gives necessary conditions. These conditions will be developed in detail in this paper.

The main motivation for this study originated with the engineering paper [2], which uses the converse theorem to prove another important theorem. The authors use the converse theorem

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without giving a proof. We develop a proof in this paper. Both Lyapunov theorems (direct and converse) have a great number of applications in stability analysis of dynamical systems, so we will investigate different aspect of these theorems in this paper.

The organization of this paper is as follow: Section II of this paper devoted to the definitions and some examples, which will be used though the paper. In the Section V, we construct a dynamical system, *Tower of Babel* (‘TOB’), which shows that a certain type of converse theorem can not be proved without relaxing some constraints on the Lyapunov function. Specifically, the stability of the equilibrium point for this system cannot be proved using a continuous Lyapunov function. This will be shown in Theorem 3.2. In the Section III, we will prove the direct Lyapunov theorem for stable systems (Theorem 3.1) and its converse (Theorem 3.3). In this section, we also prove that there is no continuous Lyapunov function for ‘TOB’ system. Finally, the paper will be concluded by Section IV, which proves that if there exists V which satisfies certain conditions for a system, then the equilibrium point of that system is Globally Asymptotically Stable (GAS) Theorem 4.1. The converse of this theorem will be proven in Theorem 4.3. GAS is a strong notion in the stability analysis theory, because for a system with GAS equilibrium point all the trajectories starting from arbitrary initial conditions converge to the same equilibrium point. This is the main objective in design of almost all control systems.

This paper should be accessible to anyone familiar with analysis on $\mathbb{R}^n$ at the level of *Advanced Calculus*.

II. DEFINITIONS

Definition 2.1: Let $X \subset \mathbb{R}^n$. A discrete dynamical system on X is a continuous function f which is defined $f : X \to X$.

In this paper, we only consider discrete dynamical systems and so the term ”discrete” will usually be dropped. Although a dynamical system is just another name for a continuous function, the terminology is supposed to suggest a way of thinking about the situation. Namely, the points of X are thought of as describing the state of a physical system of some kind. This systems is imagined to evolve in time and the function f gives the state at time $k + 1$ in terms of the state at time k. We often refer to dynamical system by the notation $\mathbf{x}_{k+1} = f(\mathbf{x}_k)$ when some $\mathbf{x}_0$ has been chosen. The dynamical system is *discrete* because the time is restricted to a discrete set of values. Also, although this term was not mentioned in the definition, the system is *autonomous* because the new state depends only on the old state, not on the time at which that state appeared.

It will be convenient to use the notation $f^{[k]}$ to denote the k^{th} symbolic power of f. We define $f^{[0]} : X \to X$ to be the identity map of X and then recursively set

$$f^{[k]} = f \circ f^{[k-1]}$$

for $k \geq 1$. If $f : X \to X$ happens to be invertible then we may also define negative symbolic powers by letting $f^{[-1]}$ be the inverse function and recursively defining

$$f^{[k-1]} = f^{[-1]} \circ f^{[k]}$$

for $k \leq -1$. By above notation we will define a dynamical system trajectory as follow:

Definition 2.2: Let $\mathbf{x} \in X$. The trajectory of $\mathbf{x}$ is the sequence $\{f^{[k]}(\mathbf{x}) : k \geq 0\}$.

One main topic of interest in studying dynamical systems is the behavior of the trajectories, since this describes the long-term behavior of the physical system that is being represented.

Definition 2.3: A point $\mathbf{x}^*$ is said to be an equilibrium point of the dynamical system $f : X \to X$ if $\mathbf{x}^* = f(\mathbf{x}^*)$.

Since the main objective of this work is to investigate stability analysis of dynamical systems so we will introduce some technical terms as follow.

Definition 2.4: The equilibrium point $\mathbf{x}^*$ is *stable* if $\forall \epsilon > 0, \exists \delta > 0$ such that $\|\mathbf{x}_0 - \mathbf{x}^*\| < \delta$ implies $\|f^{[k]}(\mathbf{x}_0) - \mathbf{x}^*\| < \epsilon$ for all $k \geq 0$.

Intuitively stability at the equilibrium point means that a trajectory starts in the δ neighborhood of the equilibrium point remains in the ϵ neighborhood of the equilibrium point. Since the definition (2.4) for $k = 1$ become the definition of continuity, so stability is a strengthening of continuity at $\mathbf{x}^*$.

Definition 2.5: The equilibrium point $\mathbf{x}^*$ is *attracting* if $\exists \eta > 0$ such that $\|\mathbf{x}_0 - \mathbf{x}^*\| < \eta$ implies $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{x}^*$ for all $k \geq 0$. If $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{x}^*$ for all $\mathbf{x}_0$ and for all $k \geq 0$ then $\mathbf{x}^*$ is called *globally attracting*.

Definition 2.6: The equilibrium point $\mathbf{x}^*$ is *asymptotically stable* if it is stable and attracting. The equilibrium point $\mathbf{x}^*$ is *globally asymptotically stable* if it is stable and globally attracting.

By the definition (2.6) asymptotic stability implies stability whereas the opposite direction is not necessary true. The following example illustrates an example of a system which is stable but not asymptotically stable.

Example 2.1:

Consider $X = \mathbb{R}$ and $f : X \to X$ to be defined as

$$f(x) = -x \tag{II.1}$$

The origin which is the equilibrium point of system (II.1) is stable but it is not asymptotically stable. By the definition (**??**) the origin of the system (II.1) is stable because for any x_0 in the δ neighborhood of the origin the trajectory remains in the ϵ neighborhood of the origin. However the origin is not asymptotically stable because the origin is not an attracting point.

A physical example of system (II.1) is a ball rolling inside a cup without any friction. A rolling ball which is a continuous system can be represented by (II.1) if only the x-coordinate be sampled with the correct sampling time. The correct sampling time samples the movement of the rolling ball at its highest point on either side. The equilibrium point of this system is stable but not asymptotically stable.

The distinction between asymptotically stable and globally asymptotically stable is also important to mention. There are many systems which are asymptotically stable only inside a specific regions. These specific regions are called *region of attraction* and widely used in stability analysis. In the following example, we will show an example which is asymptotically stable but not globally asymptotically stable.

Example 2.2:

Consider $X = \mathbb{R}$ and a dynamical system defined by $f : X \to X$ as

$$f(x) = x^2 \tag{II.2}$$

The system (II.2) is an example of asymptotically stable system which is not globally asymptotically stable. By the definition (2.6) the origin of system (II.2) is not globally asymptotically stable because it is not globally attracting. However the origin is stable and attracting and thus asymptotically stable.

There are many systems which are globally attracting but they are not stable. In the following example we will show a simple example of globally attracting system which is not stable.

Example 2.3: [4] Consider a dynamical system defined on the unit circle as

$$\theta_{k+1} = \sqrt{2\pi\theta_k} \qquad \forall \theta_k \in [0, 2\pi) \tag{II.3}$$

The solution of above difference equation is as follow

$$\theta_k = (2\pi)^{1-2^{-k}} \theta_0^{2^{-k}} \tag{II.4}$$

$\forall k \geq 0$. The equilibrium point of system (II.3) can be derived by finding the limit of (II.4) as $k \to \infty$. It turns out that the point $\theta^* = 0$ is the equilibrium point of difference equation (II.3). Since $\theta_{k+1} > \theta_k$ hence any trajectories started on the unit circle (excluding the equilibrium point) converge to the equilibrium point in the counter clockwise direction. The point $\theta^* = 0$ is not stable because $\exists \epsilon > 0$ such that $\forall \delta > 0$ there is some θ_0 in the δ neighborhood of θ^* such that θ_k leaves the ϵ neighborhood. This is the negation of definition of stability. However by the definition (2.5) the equilibrium point is globally attracting because all trajectory started on the unit circle converges to the equilibrium point.

III. FORWARD AND CONVERSE STABILITY THEOREMS

The main theorem in stability analysis of dynamical systems is called *Lyapunov theorem*. In this section, we will derive the necessary and sufficient conditions for the Lyapunov function such that the equilibrium point of a system is stable. The necessary and sufficient conditions for GAS of the equilibrium point will be investigated in the next section.

We will begin the section by introducing some definitions which will be used in the rest of the paper.

Definition 3.1: Let $\mathbf{x}^* \in X \subset \mathbb{R}^n$ and V be a function on X with values in $\mathbb{R}$. Then we will say that V satisfies the *annulus condition* with respect to $\mathbf{x}^*$ if

1) $V(\mathbf{x}^*) = 0$,
2) There is a number $\alpha > 0$ such that if $0 < \delta < \alpha$ then

$$\inf\{V(\mathbf{x}) : \mathbf{x} \in X, \delta \leq \ \|\mathbf{x} - \mathbf{x}^*\| \leq \alpha\} > 0. \tag{III.1}$$

Definition 3.2: We say that the real-valued function V is positive definite at $\mathbf{x}^*$ if

1) $V(\mathbf{x}^*) = 0$,
2) $V(\mathbf{x}) > 0$ for all $\mathbf{x} \in B(\mathbf{x}^*, \alpha), \mathbf{x} \neq \mathbf{x}^*$, for some $\alpha > 0$.

Note that annulus condition says that V has a positive lower bound at $\mathbf{x} \in X$ lying on the annulus. In other word, V satisfying the annulus condition with respect to $\mathbf{x}^*$ implies that V is positive definite. The opposite direction is not necessary true, because V needs to be continuous and X be closed.

Definition 3.3: Let X be a non-empty set of $\mathbb{R}^n$ and define dynamical system by $f : X \to X$. Suppose $\mathbf{x}^*$ is an equilibrium point of f. We will define a *weak Lyapunov function* for $\mathbf{x}^*$to be a $V : X \to [0, \infty)$ such that

1) V is non-increasing ($V(f(\mathbf{x})) \leq V(\mathbf{x})$ for all $\mathbf{x} \in X$),
2) V is continuous at $\mathbf{x}^*$,
3) V satisfies the annulus condition with respect to $\mathbf{x}^*$.

Definition 3.4: Let X be a non-empty set of $\mathbb{R}^n$ and define dynamical system by $f : X \to X$. Suppose $\mathbf{x}^*$ is an equilibrium point of f. We will define a *ordinary Lyapunov function* for $\mathbf{x}^*$ to be a $V : X \to [0, \infty)$ such that

1) V is continuous on X,
2) $V(f(\mathbf{x})) \leq V(\mathbf{x})$ for all $\mathbf{x} \in X$.

If X is a closed set then ordinary Lyapunov function will be equal to weak Lyapunov function, however the opposite direction is not true. In the case when X is not a closed set then neither directions can be deducted.

In the following theorem, we will prove that the existence of weak Lyapunov function is enough for stability of the equilibrium point.

Theorem 3.1: Let $f : X \to X$ be a dynamical system and $\mathbf{x}^* \in X$ be an equilibrium point. Suppose that $V : X \to [0, \infty)$ is a weak Lyapunov function for $\mathbf{x}^*$. Then $\mathbf{x}^*$ is stable.

Proof 3.1: By the annulus condition there exists α_1 such that

$$\psi(\epsilon) = \inf\{V(\mathbf{z}) : \mathbf{z} \in X, \epsilon \leq \|\mathbf{z} - \mathbf{x}^*\| \leq \alpha_1\} > 0 \tag{III.2}$$

for all $0 < \epsilon < \alpha_1$. Since f is continuous, there exists some $0 < \alpha_2 < \alpha_1$ such that

$$\|\mathbf{z} - \mathbf{x}^*\| < \alpha_2 \Rightarrow \|f(\mathbf{z}) - \mathbf{x}^*\| < \alpha_1 \tag{III.3}$$

for all $\mathbf{z} \in X$. Choose $0 < \epsilon < \alpha_2$. By the continuity of V at $\mathbf{x}^*$ and the fact that $V(\mathbf{x}^*) = 0$ there exists $0 < \delta < \epsilon$ such that $\|\mathbf{z} - \mathbf{x}^*\| < \delta$ implies $V(\mathbf{z}) < \psi(\epsilon)$ for all $\mathbf{z} \in X$. We claim that if $\|\mathbf{x}_0 - \mathbf{x}^*\| < \delta$ then $\|\mathbf{x}_n - \mathbf{x}^*\| < \epsilon$ for all $n \geq 0$. Suppose not. Then there is some $\mathbf{x}_0$ with $\|\mathbf{x}_0 - \mathbf{x}^*\| < \delta$ but $\|\mathbf{x}_n - \mathbf{x}^*\| \geq \epsilon$ for some n. We let $n = m + 1$ be the least such value, so that $\|\mathbf{x}_r - \mathbf{x}^*\| < \epsilon$ for $0 \leq r \leq m$ but $\|\mathbf{x}_{m+1} - \mathbf{x}^*\| \geq \epsilon$. Since $\|\mathbf{x}_m - \mathbf{x}^*\| < \epsilon < \alpha_2$ then by the property of α_1 and α_2 we can say that $\|\mathbf{x}_{m+1} - \mathbf{x}^*\| < \alpha_1$. Knowing that $\|\mathbf{x}_{m+1} - \mathbf{x}^*\| \geq \epsilon$ and $\|\mathbf{x}_{m+1} - \mathbf{x}^*\| < \alpha_1$ then by the annulus condition

$$V(\mathbf{x}_{m+1}) \geq \psi(\epsilon) \tag{III.4}$$

V is a non-increasing function hence,

$$V(\mathbf{x}_{m+1}) \leq V(\mathbf{x}_m) \leq \ldots \leq V(\mathbf{x}_0) \tag{III.5}$$

Since we assumed that $\|\mathbf{x}_0 - \mathbf{x}^*\| < \delta$ then $V(\mathbf{x}_0) < \psi(\epsilon)$ by the construction of δ. Combining (III.5) we can conclude that

$$V(\mathbf{x}_{m+1}) \leq V(\mathbf{x}_m) \leq \ldots \leq V(\mathbf{x}_0) < \psi(\epsilon) \tag{III.6}$$

which is not possible because it contradicts (III.4). Hence the hypothesis was wrong and $\|\mathbf{x}_n - \mathbf{x}^*\| < \epsilon$ for all n which proves that $\mathbf{x}^*$ is stable.

In the following theorem, we will prove that the stability of the equilibrium point of 'TOB' system can not be proved with continuous Lyapunov function.

Theorem 3.2: If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is the dynamical system defined in the Section V then $\mathbf{x}^* = (0, 0)$ is a stable equilibrium point but there is no ordinary Lyapunov function for $\mathbf{x}^*$.

Proof 3.2: Suppose that $V : \mathbb{R}^2 \to [0, \infty)$ is a continuous weak Lyapunov function and denote set of circles by C_j as follow

$$C_j = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| = 2^{-j}\} \tag{III.7}$$

for all $j > 0$. The set C_j is compact and V is continuous and so by *Extreme Value Theorem* (EVT) V achieve maximum value of C_j. Denote the maximum value of V on C_j as M_j and consider the maximum value occur at $p_j \in C_j$. We claim that $M_{j+1} \geq M_j$.

In order to prove above claim, consider a point q in the neighborhood of the the point p_j and inside the region between C_j and C_{j+1}. Due to the continuity of V for all $\epsilon > 0$ there exists $\delta > 0$ such that $\|p_j - q\| < \delta$ implies $\|M_j - V(q)\| < \epsilon$. Since M_j is the maximum value of V and a real number then

$$V(q) > M_j - \epsilon \tag{III.8}$$

By the assumption V is non-increasing ($V(f(\mathbf{x})) \leq V(\mathbf{x})$). Since V is non-increasing and one-to-one function then the backward trajectory from q yields a sequence $\{f^{[-k]}(q) : k \geq 0\}$ and $V(f^{[-k]}(q)) \geq V(q)$. The sequence $\{f^{[-k]}(q) : k \geq 0\}$ is a bounded sequence so by Bolzano-Weierstrass theorem it has a convergent subsequence. Let the convergent subsequence to be $\{f^{[-k]}(q) : k \in J\}$, which converges to s. Since $s = \lim\limits_{k\to\infty\ ,\ k\in J} f^{[-k]}(q)$ then by the continuity of V,

$$\begin{aligned} V(s) &= \lim_{k\to\infty\ ,\ k\in J} V(f^{[-k]}(q)) \\ &\geq V(q) \end{aligned} \tag{III.9}$$

Since $\|f^{[-k]}(q)\| \to 2^{-(j+1)}$ as $k \to \infty$ then $s \in C_{j+1}$. By (III.8), (III.9) and the fact that $s \in C_{j+1}$ we can conclude that $M_{j+1} \geq V(s) \geq V(q) > M_j - \epsilon$. Since $\epsilon > 0$ was arbitrary positive number then $M_{j+1} \geq M_j$. Also as $j \to \infty$, $p_j \to 0$, hence $M_j = V(p_j) \to V(0) = 0$ and for large enough j, $M_j = V(p_j) > 0$. Since $p_j \neq 0$ so there exist $\alpha > 0$ such that if $0 < \delta < \alpha$ then

$$\inf\{V(\mathbf{z}) : \mathbf{z} \in X \quad \delta < \|\mathbf{z} - \mathbf{x}^*\| < \alpha\} > 0 \tag{III.10}$$

Once j is large enough $0 < \|p_j\| < \alpha$ and it follows that $V(p_j) > 0$. In summary we derived the following facts:

- $M_{j+1} \geq M_j$
- $p_j \to 0$, hence $M_j = V(p_j) \to V(0) = 0$
- For large enough j, $M_j = V(p_j) > 0$

These three properties of M_j are inconsistent, hence the hypothesis is wrong and there is no ordinary Lyapunov function for the function f.

In the following theorem the converse of theorem 3.1 will be proved.

Theorem 3.3: Let $f : X \to X$ be a dynamical system and $\mathbf{x}^* \in X$ be a stable equilibrium point. Then there is a weak Lyapunov function for $\mathbf{x}^*$.

Proof 3.3: Let $g : [0, \infty) \to [0, 1]$ defined as

$$g(t) = \begin{cases} t, & t \in [0, 1] \\ 1, & t \in (1, \infty) \end{cases}$$

which is continuous at 0, non-decreasing and bounded above by 1. For $\mathbf{x} \in X$, define

$$V(\mathbf{x}) = \sup\{g(\|f^{[k]}(\mathbf{x}) - \mathbf{x}^*\|) : k \geq 0\} \tag{III.11}$$

Given $\mathbf{x} \in X$, the set $S = \{g(\|f^{[k]}(\mathbf{x}) - \mathbf{x}^*\|) : k \geq 0\}$ is non-empty since it contains $g(\|\mathbf{x} - \mathbf{x}^*\|)$; Because g is bounded above by 1 and so S is bounded above by the same number. Hence $\sup(S)$ exists and so $V(\mathbf{x})$ is well-defined. In addition, note that this argument implies that $V(\mathbf{x}) \leq 1$ for all $\mathbf{x} \in X$.

In order to prove that V is a weak Lyapunov function we need to show that V satisfies the following conditions as follow

(a) $V(\mathbf{x}^*) = 0$

(b) V is continuous at $\mathbf{x}^*$

(c) V is non-increasing

(d) There is a number $\alpha > 0$ such that if $0 < \delta < \alpha$ then

$$\inf\{V(\mathbf{x}) \;\mid\; \mathbf{x} \in X \quad : \delta \leq \|\mathbf{x} - \mathbf{x}^*\| \leq \alpha\} > 0 \tag{III.12}$$

The condition in part (a) is true because

$$\begin{aligned} V(\mathbf{x}^*) &= \sup\{g(\|f^{[k]}(\mathbf{x}^*) - \mathbf{x}^*\|) : k \geq 0\} \\ &= \sup\{g(\|\mathbf{x}^* - \mathbf{x}^*\|) : k \geq 0\} \\ &= \sup\{g(0) : k \geq 0\} \\ &= 0 \qquad \text{since } g(0) = 0 \end{aligned}$$

To prove the condition in part (b) we need to show that $\forall\epsilon > 0$, $\exists\delta > 0$ such that $\|\mathbf{x} - \mathbf{x}^*\| < \delta$ implies $V(\mathbf{x}) < \epsilon$ for all $k \geq 0$. If $\epsilon > 1$ then any $\delta > 0$ will suffices because we have already observed that $V(\mathbf{x}) \leq 1$ for all $\mathbf{x} \in X$. Now suppose that $\epsilon \leq 1$. Since f is stable at $\mathbf{x}^*$ we may find $\delta > 0$ such that if $\mathbf{x} \in X$ and $\|\mathbf{x} - \mathbf{x}^*\| < \delta$ then $\|f^{[k]}(\mathbf{x}) - \mathbf{x}^*\| < \epsilon/2$ for all $k \geq 0$. Suppose that $\|\mathbf{x} - \mathbf{x}^*\| < \delta$. Then

$$\begin{aligned} V(\mathbf{x}) &= \sup\{g(\|f^{[k]}(\mathbf{x}) - \mathbf{x}^*\|) : k \geq 0\} \\ &\leq \sup\{g(\epsilon/2) : k \geq 0\} \quad \text{since } g(\epsilon/2) = \epsilon/2 \\ &< \epsilon \end{aligned}$$

Thus V is continuous at $\mathbf{x}^*$ and this proves the condition in part (b).

For the proof of the condition in part (c) we need to show $V(f(\mathbf{x})) \leq V(\mathbf{x})$. Define the set A and B as follow

$$A \triangleq \{g(\|f^{[k]}(\mathbf{x}) - \mathbf{x}^*\|) : k \geq 0\}$$
$$B \triangleq \{g(\|f^{[k+1]}(\mathbf{x}) - \mathbf{x}^*\|) : k \geq 0\}$$

By inspection every elements of B is element of A and so $\sup B \leq \sup A$. By definition $V(f(\mathbf{x})) = \sup B$ and $V(\mathbf{x}) = \sup A$. This proves the condition in part (c).

We now prove condition (d) with $\alpha = 1$. Suppose that $0 < \epsilon < 1$ and that $\|\mathbf{x} - \mathbf{x}^*\| \leq 1$. Then

$$\|\mathbf{x} - \mathbf{x}^*\| \in \{g(\|f^{[k]}(\mathbf{x}) - \mathbf{x}^*\|) : k \geq 0\}$$
$$\epsilon \leq \|\mathbf{x} - \mathbf{x}^*\| \leq \sup\{g(\|f^{[k]}(\mathbf{x}) - \mathbf{x}^*\|) : k \geq 0\} = V(\mathbf{x})$$

Thus $\inf\{V(\mathbf{x}) : \epsilon \leq \|\mathbf{x} - \mathbf{x}^*\| \leq 1\} \geq \epsilon > 0$, as required. This proves the condition in part (d). Since (a), (b), (c) and (d) are true hence V is a weak Lyapunov function.

IV. FORWARD AND CONVERSE GLOBAL ASYMPTOTIC STABILITY THEOREMS

The main contribution of this section is to derive the necessary and sufficient conditions for global asymptotic stability of the equilibrium point. We will prove that if there exists V (satisfying additional constraints compared to the previous section) then the equilibrium point not only is stable but also is asymptotically stable. We will also prove the converse of this theorem.

The next two propositions will be quoted, since they will be used later in the proof of the theorems.

Proposition 4.1: Let $\{x_k\}$ be a sequence in $\mathbb{R}^n$. Let $\mathbf{p} \in \mathbb{R}^n$. Suppose that every subsequence of $\{x_k\}$ has a further subsequence that converges to $\mathbf{p}$. Then $\{x_k\}$ converges to $\mathbf{p}$.

Proof 4.1: See [1] Section 2.5 Theorem $2.5.2$, page 55.

Proposition 4.2: If $\{x_k\}$ is a non-increasing sequence of real numbers bounded bellow then $\{x_k\}$ is convergent.

Proof 4.2: See [3] Section 1.6, page 47.

In order to show that the equilibrium point is GAS, we need to prove that the it is stable and globally attracting. In the previous section, we found the necessary and sufficient conditions for

V which guarantee stability of the equilibrium point. In the following section, we will prove if V satisfies additional constraints then the equilibrium point will be globally attracting.

Theorem 4.1: Let $X \subset \mathbb{R}^n$ be a closed set and $f : X \to X$ a dynamical system with an equilibrium point $\mathbf{x}^* \in X$. Suppose that there is a function $V : X \to [0, \infty)$ such that

(a) $V(\mathbf{x}^*) = 0$,

(b) For all $\mathbf{x} \in X - \{\mathbf{x}^*\}, V(f(\mathbf{x})) < V(\mathbf{x})$,

(c) V is continuous at every point of $X - \{\mathbf{x}^*\}$,

(d) For all $C \in [0, \infty)$, the set $\{\mathbf{z} \in X : V(\mathbf{z}) \leq C\}$ is bounded.

Then $\mathbf{x}^*$ is globally attracting.

Proof 4.3: Let $\mathbf{x} \in X$. We are required to show that the sequence $\{f^{[k]}(\mathbf{x})\}$ converges to $\mathbf{x}^*$. If $\mathbf{x} = \mathbf{x}^*$ then this is obvious, so assume that $\mathbf{x} \in X - \{\mathbf{x}^*\}$.

Let $\{f^{[k]}(\mathbf{x}) : k \in J\}$ be a subsequence of $\{f^{[k]}(\mathbf{x}) : k \in \mathbb{N}\}$. By Proposition 4.1, it will suffice to show that this subsequence has a further subsequence that converges to $\mathbf{x}^*$.

Considering the assumptions (a) and (b) and the fact that $f(\mathbf{x}^*) = \mathbf{x}^*$ yields

$$V(f(\mathbf{x})) \leq V(\mathbf{x}) \qquad \forall \mathbf{x} \in X. \tag{IV.1}$$

Let $C = V(\mathbf{x})$. Then $V(f^{[k]}(\mathbf{x})) \leq V(f^{[k-1]}(\mathbf{x})) \leq \ldots \leq V(\mathbf{x}) = C$ for all $k \geq 0$. Thus

$$\{f^{[k]}(\mathbf{x}) : k \geq 0\} \subset \{\mathbf{z} \in X : V(\mathbf{z}) \leq C\}. \tag{IV.2}$$

By Assumption (d), the set $\{\mathbf{z} \in X : V(\mathbf{z}) \leq C\}$ is bounded and hence the sequence $\{f^{[k]}(\mathbf{x}) : k \geq 0\}$ is also bounded. In particular, the subsequence $\{f^{[k]}(\mathbf{x}) : k \in J\}$ is bounded and so has a subsequence $\{f^{[k]}(\mathbf{x}) : k \in J'\}(J' \subset J)$ that converges to some point $\mathbf{p}$ of $\mathbb{R}^n$. Since X is closed $\mathbf{p} \in X$. If $\mathbf{p} = \mathbf{x}^*$ then the theorem proved, so assume that $\mathbf{p} \neq \mathbf{x}^*$.

We may assume without loss of generality that $0 \notin J'$. Consider the sequence $\{f^{[k-1]}(\mathbf{x}) : k \in J'\}$. This sequence is bounded, so we may extract a convergent subsequence $\{f^{[k-1]}(\mathbf{x}) : k \in J''\}$ with $J'' \subset J'$. Say that this sequence converges to $\mathbf{q} \in X$. Since $\{f^{[k]}(\mathbf{x}) : k \in J''\}$ is a subsequence of $\{f^{[k]}(\mathbf{x}) : k \in J'\}$, $\{f^{[k]}(\mathbf{x}) : k \in J''\}$ converges to $\mathbf{p}$. Since f is continuous and the terms in $\{f^{[k-1]}(\mathbf{x}) : k \in J''\}$ are each one step before the terms in $\{f^{[k]}(\mathbf{x}) : k \in J''\}$

$$f(\mathbf{q}) = \mathbf{p}. \tag{IV.3}$$

In particular, $\mathbf{q} \neq \mathbf{x}^*$ and $f(\mathbf{q}) \neq \mathbf{x}^*$. By Assumption (c) V is continuous at $\mathbf{q}$ and so

$$V(\mathbf{q}) = \lim_{k \in J''} V(f^{[k-1]}(\mathbf{x})) \tag{IV.4}$$

Since $\{V(f^{[k]}(\mathbf{x})) : k \geq 0\}$ is a decreasing sequence, it follows that

$$V(\mathbf{q}) = \lim_{k \geq 0} V(f^{[k]}(\mathbf{x})) \tag{IV.5}$$

The sequence $\{f^{[k]}(\mathbf{x}) : k \in J''\}$ converges to $f(\mathbf{q}) = \mathbf{p}$. By the continuity of V at $\mathbf{q}$

$$V(f(\mathbf{q})) = V(\mathbf{p}) = \lim_{k \in J''} V(f^{[k]}(\mathbf{x})) = V(\mathbf{q}) \tag{IV.6}$$

This is a contradiction to $V(f(\mathbf{q})) < V(\mathbf{q})$ and the theorem proved.

Consider again the system given in example (2.3). Let X be all the points on the circle and take $V : X \to [0, \infty)$ to be the function $V(\mathbf{x}) = 2\pi - \theta$ where $\theta \in (0, 2\pi]$ describes $\mathbf{x}$ in polar coordinates. The equilibrium $\mathbf{x}^*$ has $\theta = 2\pi$. All the conditions given in Theorem 4.1 are satisfied, hence the equilibrium point is globally attracting. Figure IV.1 shows that the function V is continuous everywhere except at the the equilibrium point.

Fig. IV.1: $V(\mathbf{x})$ versus θ for Example 2.3

Theorem 4.2: Let $X \subset \mathbb{R}^n$ be a closed set and $f : X \to X$ a dynamical system with an equilibrium point $\mathbf{x}^*$. Suppose that there is a function $V : X \to [0, \infty)$ such that

(a) V satisfies the annulus condition with respect to $\mathbf{x}^*$,

(b) For all $\mathbf{x} \in X - \{\mathbf{x}^*\}, V(f(\mathbf{x})) < V(\mathbf{x})$,

(c) V is continuous at every point of $X - \{\mathbf{x}^*\}$,

(d) For all $C \in [0, \infty)$, the set $\{\mathbf{z} \in X : V(\mathbf{z}) \leq C\}$ is bounded.

Then $\mathbf{x}^*$ is *globally asymptotically stable*.

Proof 4.4: This theorem follows combining of Theorem 3.1 and Theorem 4.1.

In the following theorem, we will prove the converse of Theorem 4.1.

Theorem 4.3: Let $X \subset \mathbb{R}^n$ and $f : X \to X$ be a dynamical system with the globally asymptotically stable equilibrium point $\mathbf{x}^*$. Then there is a function $V : X \to [0, \infty)$ such that

(a) V satisfies the annulus condition with respect to $\mathbf{x}^*$,

(b) For all $\mathbf{x} \in X - \{\mathbf{x}^*\}, V(f(\mathbf{x})) < V(\mathbf{x})$,

(c) V is continuous,

(d) For all $C \in [0, \infty)$, the set $\{\mathbf{z} \in X : V(\mathbf{z}) \leq C\}$ is bounded.

Proof 4.5: For each $k \geq 0$, we define $\rho_k : X \to [0, \infty)$ by

$$\rho_k(\mathbf{x}) = \sup\{\|f^{[j]}(\mathbf{x}) - \mathbf{x}^*\| : j \geq k\} \tag{IV.7}$$

The set $\{\|f^{[j]}(\mathbf{x}) - \mathbf{x}^*\| : j \geq k\}$ is non-empty. Since $\{f^{[j]}(\mathbf{x})\} \to \mathbf{x}^*$. $\|f^{[j]}(\mathbf{x}) - \mathbf{x}^*\|$ is bounded and so $\rho_k(\mathbf{x})$ is well-defined.

The sequence $\{\rho_k(\mathbf{x})\}$ is a non-increasing sequence. This can be shown as follow. Define

$$\begin{aligned} A &\triangleq \{\|f^{[j]}(\mathbf{x}) - \mathbf{x}^*\| : j \geq k\} \\ B &\triangleq \{\|f^{[j]}(\mathbf{x}) - \mathbf{x}^*\| : j \geq k+1\} \end{aligned} \tag{IV.8}$$

By inspection $B \subseteq A$ so $\sup B \leq \sup A$. That is $\rho_{k+1}(\mathbf{x}) \leq \rho_k(\mathbf{x})$. In other words,

$$\rho_{k+1}(\mathbf{x}) \leq \rho_k(\mathbf{x}) \leq \ldots \leq \rho_0(\mathbf{x}) \tag{IV.9}$$

Note moreover that $\rho_{k+1}(\mathbf{x}) = \rho_k(f(\mathbf{x}))$, because

$$\begin{aligned} \rho_k(f(\mathbf{x})) &= \sup\{\|f^{[j]}(f(\mathbf{x})) - \mathbf{x}^*\| : j \geq k\} \\ &= \sup\{\|f^{[j+1]}(\mathbf{x}) - \mathbf{x}^*\| : j \geq k\}. \end{aligned} \tag{IV.10}$$

Let $j' = j + 1$ then

$$\begin{aligned} \rho_k(f(\mathbf{x})) &= \sup\{\|f^{[j']}(\mathbf{x}) - \mathbf{x}^*\| : j' - 1 \geq k\} \\ &= \sup\{\|f^{[j']}(\mathbf{x}) - \mathbf{x}^*\| : j' \geq k+1\} \\ &= \rho_{k+1}(\mathbf{x}) \end{aligned} \tag{IV.11}$$

We claim that $\{\rho_k(\mathbf{x})\} \to 0$. Since $\mathbf{x}^*$ is globally attracting equilibrium point $\{f^{[j]}(\mathbf{x})\} \to \mathbf{x}^*$. Let $\epsilon > 0$. There is a K such that for $j \geq K$, $\|f^{[j]}(\mathbf{x}) - \mathbf{x}^*\| \leq \epsilon/2$. If $k \geq K$ then $\rho_k(\mathbf{x}) \leq \epsilon/2 < \epsilon$, this establishes the claim.

Define $V : X \to [0, \infty)$ by

$$V(\mathbf{x}) = \sum_{k=0}^{\infty} 2^{-k} \rho_k(\mathbf{x}) \tag{IV.12}$$

The infinite sum given in (IV.12) converges, because

$$0 \leq 2^{-k} \rho_k(\mathbf{x}) \leq 2^{-k} \rho_0(\mathbf{x}), \qquad \forall k$$

and $\sum_{k=0}^{\infty} 2^{-k} \rho_0(\mathbf{x})$ is a convergent geometric series. Hence, by comparison test, $V(\mathbf{x})$ also converges.

Now we will prove (b). Since $\mathbf{x} \neq \mathbf{x}^*$, $\rho_0(\mathbf{x}) > 0$. We know that the sequence $\{\rho_k(\mathbf{x})\}$ is non-increasing and converges to 0. Thus there must exists some l such that $\rho_l(\mathbf{x}) > \rho_{l+1}(\mathbf{x}) = \rho_l(f(\mathbf{x}))$. By (IV.12) we have

$$V(f(\mathbf{x})) \;\; = \;\; \sum_{k=0}^{\infty} 2^{-k} \rho_k(f(\mathbf{x})) \tag{IV.13}$$

All the terms in (IV.13) are less than or equal to the corresponding terms in (IV.12), and at least one of these inequalities is strict. Thus $V(f(\mathbf{x})) < V(\mathbf{x})$, as required.

Now we will prove (d). Let $C \geq 0$ and suppose that $\mathbf{z} \in X$ with $V(\mathbf{z}) \leq C$. Since all the terms in (IV.12) are positive with $\rho_0(\mathbf{x})$ as the first term we have

$$V(\mathbf{z}) \geq \rho_0(\mathbf{z}) = \sup\{\|f^{[j]}(\mathbf{z}) - \mathbf{x}^*\| : j \geq 0\} \geq \|\mathbf{z} - \mathbf{x}^*\| \tag{IV.14}$$

and so $\|\mathbf{z} - \mathbf{x}^*\| \leq C$. That is, $\mathbf{z}$ lies in the ball $\bar{B}(\mathbf{x}^*, C)$. It follows that $\{\mathbf{z} \in X : V(\mathbf{z}) \leq C\} \subset \bar{B}(\mathbf{x}^*, C)$ and so the former set is bounded.

Next we will prove (a), the annulus condition, with any $\alpha > 0$. First we need to show $V(\mathbf{x}^*) = 0$. This is true because

$$\begin{aligned} V(\mathbf{x}^*) \;\; &= \;\; \sum_{k=0}^{\infty} 2^{-k} \rho_k(\mathbf{x}^*) \\ &= \;\; 0 \qquad \text{since } \rho_k(\mathbf{x}^*) = 0 \end{aligned} \tag{IV.15}$$

Let $\alpha > 0$ and choose δ such that $0 < \delta < \alpha$. Suppose that $\mathbf{z} \in X$ and $\alpha \geq \|\mathbf{z} - \mathbf{x}^*\| \geq \delta$. Then, by (IV.14), $V(\mathbf{z}) \geq \delta$. Thus

$$\inf\{V(\mathbf{z}) : \delta \leq \|\mathbf{z} - \mathbf{x}^*\| \leq \alpha\} \geq \delta > 0 \tag{IV.16}$$

as required.

For the part (c), we divide the proof into two sections. First, we will show that V is continuous at $\mathbf{x}^*$ and secondly, we will show that V is continuous at $\mathbf{x} \in X - \{\mathbf{x}^*\}$. Let $\epsilon > 0$. Since $\mathbf{x}^*$ is stable, there is a $\delta > 0$ such that if $\|\mathbf{x} - \mathbf{x}^*\| < \delta$ then $\|f^{[k]}(\mathbf{x}) - \mathbf{x}^*\| < \epsilon/3$, for all $k \geq 0$. Thus if $\mathbf{x} \in X$ and $\|\mathbf{x} - \mathbf{x}^*\| < \delta$ then

$$\rho_k(\mathbf{x}) = \sup\{\|f^{[j]}(\mathbf{x}) - \mathbf{x}^*\| : j \geq k\} \leq \epsilon/3.$$

It follows that

$$\begin{aligned} V(\mathbf{x}) &= \sum_{k=0}^{\infty} 2^{-k}\rho_k(\mathbf{x}) \\ &\leq \sum_{k=0}^{\infty} 2^{-k}(\epsilon/3) \\ &= 2\epsilon/3 \\ &< \epsilon \end{aligned}$$

Hence, for all $\epsilon > 0$ there exist $\delta > 0$ such that $\|\mathbf{x} - \mathbf{x}^*\| < \delta$ implies $V(\mathbf{x}) < \epsilon$. Since $V(\mathbf{x}^*) = 0$, this shows that V is continuous at $\mathbf{x}^*$.

For the second part of the proof, assume $\mathbf{x} \in X - \{\mathbf{x}^*\}$. Let $\eta > 0$. Since $\mathbf{x}^*$ is stable, we can choose $\delta_1 > 0$ such that if $\mathbf{z} \in X$ and $\|\mathbf{z} - \mathbf{x}^*\| < \delta_1$ then $\|f^{[k]}(\mathbf{z}) - \mathbf{x}^*\| < \eta$ for all $k \geq 0$. Since $\mathbf{x}^*$ is globally attracting, so that $f^{[k]}(\mathbf{x}) \to \mathbf{x}^*$, we may find m such that

$$\|f^{[m]}(\mathbf{x}) - \mathbf{x}^*\| < \delta_1/2 \tag{IV.17}$$

By the continuity of $f^{[m]}$, we may then find $\delta_2 > 0$ such that if $\mathbf{y} \in X$ and $\|\mathbf{y} - \mathbf{x}\| < \delta_2$ then

$$\|f^{[m]}(\mathbf{y}) - f^{[m]}(\mathbf{x})\| < \delta_1/2 \tag{IV.18}$$

Considering (IV.17) and (IV.18) and if $\mathbf{y} \in X$ and $\|\mathbf{y} - \mathbf{x}\| < \delta_2$ then

$$\begin{aligned} \|f^{[m]}(\mathbf{y}) - \mathbf{x}^*\| &\leq \|f^{[m]}(\mathbf{y}) - f^{[m]}(\mathbf{x})\| + \|f^{[m]}(\mathbf{x}) - \mathbf{x}^*\| \\ &< \delta_1/2 + \delta_1/2 \\ &= \delta_1 \end{aligned}$$

The choice of δ_1 then implies that

$$\|f^{[j]}(f^{[m]}(\mathbf{y})) - \mathbf{x}^*\| < \eta$$

for all $j \geq 0$ or equivalently

$$\|f^{[k]}(\mathbf{y}) - \mathbf{x}^*\| < \eta \tag{IV.19}$$

for all $k \geq m$. If $k \geq m$ and $\|\mathbf{y} - \mathbf{x}\| < \delta_2$ then

$$0 \leq \|f^{[k]}(\mathbf{x}) - \mathbf{x}^*\| < \eta \tag{IV.20}$$

and

$$0 \leq \|f^{[k]}(\mathbf{y}) - \mathbf{x}^*\| < \eta, \tag{IV.21}$$

and so

$$\big|\|f^{[k]}(\mathbf{x}) - \mathbf{x}^*\| - \|f^{[k]}(\mathbf{y}) - \mathbf{x}^*\|\big| < \eta. \tag{IV.22}$$

Next we will show that, by making δ_2 smaller if necessary, we may arrange that (IV.22) holds for $0 \leq k \leq m$ as well. Define $t_k : X \to \mathbb{R}$ by $t_k(\mathbf{w}) = \|f^{[k]}(\mathbf{w}) - \mathbf{x}^*\|$. For all $k \geq 0$, t_k is continuous. Thus we may find $\lambda_k > 0$ such that if $\mathbf{y} \in X$ and $\|\mathbf{x} - \mathbf{y}\| < \lambda_k$ then $|t_k(\mathbf{y}) - t_k(\mathbf{x})| < \eta$. Let $\delta_3 = \min\{\lambda_0, \lambda_1, ..., \lambda_{m-1}, \delta_2\}$. Then $\delta_3 > 0$ and if $\mathbf{y} \in X$ and $\|\mathbf{y} - \mathbf{x}\| < \delta_3$ then

$$\big|\|f^{[k]}(\mathbf{x}) - \mathbf{x}^*\| - \|f^{[k]}(\mathbf{y}) - \mathbf{x}^*\|\big| < \eta \tag{IV.23}$$

for all $k \geq 0$. This implies that if $\|\mathbf{y} - \mathbf{x}\| < \delta_3$, then $|\rho_k(\mathbf{x}) - \rho_k(\mathbf{y})| \leq \eta$ for all $k \geq 0$. This is because if $l \geq k$, $\mathbf{y} \in X$, $\|\mathbf{y} - \mathbf{x}\| < \delta_3$ then

$$\|f^{[l]}(\mathbf{x}) - \mathbf{x}^*\| < \|f^{[l]}(\mathbf{y}) - \mathbf{x}^*\| + \eta \leq \rho_k(\mathbf{y}) + \eta \tag{IV.24}$$

Thus $\rho_k(\mathbf{y}) + \eta$ is an upper bound for $\{\|f^{[l]}(\mathbf{x}) - \mathbf{x}^*\| : l \geq k\}$ and so $\rho_k(\mathbf{x}) \leq \rho_k(\mathbf{y}) + \eta$. Switching the roles of $\mathbf{x}$ and $\mathbf{y}$ in this argument, we get $\rho_k(\mathbf{y}) \leq \rho_k(\mathbf{x}) + \eta$. Thus if $\|\mathbf{y} - \mathbf{x}\| < \delta_3$ then

$$|\rho_k(\mathbf{x}) - \rho_k(\mathbf{y})| \leq \eta \tag{IV.25}$$

for all $k \geq 0$.

Considering (IV.25) and the definition of $V(\mathbf{x})$, if $\|\mathbf{y}-\mathbf{x}\| < \delta_3$ then we will have

$$\begin{aligned} |V(\mathbf{x})-V(\mathbf{y})| &= |\sum_{k=0}^{\infty} 2^{-k}(\rho_k(\mathbf{x})-\rho_k(\mathbf{y}))| \\ &\leq \sum_{k=0}^{\infty} 2^{-k}|\rho_k(\mathbf{x})-\rho_k(\mathbf{y})| \qquad \text{(IV.26)} \\ &\leq \sum_{k=0}^{\infty} 2^{-k}\eta \\ &= 2\eta \qquad \text{(IV.27)} \end{aligned}$$

Take $\eta = \epsilon/3$ yields,

$$\begin{aligned} |V(\mathbf{x})-V(\mathbf{y})| &< 2\epsilon/3 \\ &< \epsilon \end{aligned}$$

This concludes the proof of the theorem.

V. AN ILLUSTRATIVE EXAMPLE

In this section, we will construct an example which will show certain type of converse theorem can not be proven. We will prove above claim in theorem 3.2.

Consider a dynamical system on $[0,1]$ defined by

$$f(x) = \sqrt{x} \qquad \text{(V.1)}$$

The fact that $\sqrt{x} > x$ for all $x \in (0,1)$ makes the function $f(x)$ lies above the dashed line shown in Figure V.1. This implies that system (V.1) on $[0,1]$ has 0 as an unstable equilibrium point and 1 as an asymptotically stable equilibrium point. More precisely if $x \in (0,1]$ then the trajectory $\{f^{[k]}(x) : k \geq 0\}$ approaches 1.

Fig. V.1: Graph of system defined in $(V.1)$

The dynamical system given in (V.1) can be defined on the interval $[a,b](\forall a < b)$ by defining a new function $g_a^b : [a,b] \to [a,b]$ as follow

$$g_a^b(x) = a + \sqrt{(b-a)(x-a)} \tag{V.2}$$

The dynamical system given in (V.1) and (V.2) has the same behavior with the difference that system (V.1) defined on $[0,1]$ and system (V.2) defined on $[a,b](\forall a < b)$. Hence the dynamical system defined by g_a^b has an unstable equilibrium point a and an asymptotically stable equilibrium point b. The graph of the function g_a^b is shown in Figure V.2.

Fig. V.2: Graph of function defined in (V.2)

At this point we will introduce a dynamical system $g : [0,\infty) \to [0,\infty)$ based on g_a^b. The new system will be used as a counterexample that is, to show that a certain 'theorem' is not, in fact true in section III. This system can be defined by

$$g(r) = \begin{cases} 0 & r = 0 \\ g_{2^{-(j+1)}}^{2^{-j}}(r), & 2^{-(j+1)} \leq r \leq 2^{-j} \end{cases} \tag{V.3}$$

$\forall j \geq 0$. The function g is a continuous function on each interval $[2^{-(j+1)}, 2^{-j}]$ where the left and right limits at the ends of the intervals overlap each other. The graph of the function g is shown in Figure V.3.

Fig. V.3: Graph of function defined in (V.3)

Next we will define a function f and investigate its stability and continuity through a theorem. These properties will be used through the rest of the report.

Theorem 5.1: Define a dynamical system $f : \mathbb{R}^2 \to \mathbb{R}^2$ by saying that f takes the origin to the origin and if $\mathbf{p}$ is not the origin and has polar coordinates (r,θ) with $r > 0$ then $f(\mathbf{p})$ has polar coordinates $(g(r), \theta + \pi/2)$. Then f is continuous and the dynamical system f has a stable equilibrium point at the origin and no other equilibrium points.

Proof 5.1: The proof of this theorem contains several steps. We will itemized the proof steps as follow:

- **Step 1**: In the first step we will show that the origin is the only equilibrium point and it is stable. For any non-zero point **p** there exists j such that the point **p** lies between two circles centered at the origin with the radius $2^{-(j+1)}$ and 2^{-j}. According to the function f definition at every step k the radius follows the function $g(r)$ and the angle shifts $90°$. For instant starting from a point **p** the $f(\mathbf{p})$ shifts $90°$ (see fig(V.4)) and getting closer to the outer circle. The radius of $f(\mathbf{p})$ become larger than the radius of **p** because of the way the function $g(r)$ defined. Hence $f^{[k]}(\mathbf{p})$ gets closer to the outer circle. Since **p** could be any point in the domain and lies between two circles and $f^{[k]}(\mathbf{p})$ always converges to the outer circle. This will assure that the origin is the only equilibrium point of f which is stable

- **Step 2**: In this step we will prove the continuity of f at the equilibrium point. In order to prove that f is continuous at the origin it suffices to show $\forall\epsilon > 0, \exists\delta > 0$ such that

$$\mathbf{p} \in \mathbb{R}^2, \|\mathbf{p}\| < \delta \Longrightarrow \|f(\mathbf{p})\| < \epsilon \tag{V.4}$$

Since $\|\mathbf{p}\| = r$ and $\|f(\mathbf{p})\| = g(r)$ the we need to show given $\epsilon > 0$, $\exists\delta > 0$ such that if $0 \leq r \ < \delta$ then $g(r) < \epsilon$. Let $\epsilon > 0$. Choose j such that $2^{-j} < \epsilon$ and $\delta = 2^{-j}$. If $0 \leq r < 2^{-j}$ then $2^{-(i+1)} \leq r \leq 2^{-i}$ with $i \geq j$, hence

$$\begin{aligned} g(r) &= g_{2^{-(i+1)}}^{2^{-i}}(r) \\ &\leq 2^{-i} \\ &\leq 2^{-j} \\ &< \epsilon \end{aligned} \tag{V.5}$$

which proves continuity of g or equivalently continuity of f at the origin.

- **Step 3** In this step we will prove that g is continuous. Denote C_j as a compact set on $[2^{(j-1)}, 2^j]$ and C_{j+1} as a compact set on $[2^j, 2^{(j+1)}]$. Similar to step 2 define a continuous function $g_j : C_j \to \mathbb{R}^2$ and $g_{j+1} : C_{j+1} \to \mathbb{R}^2$ for all $j > 0$. Since the left and right limits at the end of the intervals overlaps so $g_j(\mathbf{p}) = g_{j+1}(\mathbf{p})$ if $\mathbf{p} \in C_j \cap C_{j+1}$. By the proof given in the previous step g_j and g_{j+1} are continuous on C_j and C_{j+1}, hence we can define another continuous function $g : C_j \cup C_{j+1} \to \mathbb{R}^2$ such that

$$g(\mathbf{p}) = \begin{cases} g_j(\mathbf{p}) \quad if \qquad \mathbf{p} \in C_j \\ g_{j+1}(\mathbf{p}) \quad if \quad \mathbf{p} \in C_{j+1} \end{cases} \tag{V.6}$$

By induction it can be proved that g is a continuous function.

- **Step 4**: Finally we will prove that$f$ is continuous at other points than the equilibrium point. Suppose that $(x, y) \neq (0, 0)$ and that $(X, Y) = f(x, y)$. Let (x, y) have polar coordinates (r, θ). Then

$$X = g(r)\cos(\theta + \pi/2) = -g(r)\sin(\theta) = -\frac{g(r)}{r}y$$
$$Y = g(r)\sin(\theta + \pi/2) = g(r)\cos(\theta) = \frac{g(r)}{r}x$$

and so the map f may be written as

$$f(x, y) = \left(-\frac{yg(\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}}, \frac{xg(\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}} \right)$$

provided that $(x, y) \neq (0, 0)$. evidently a continuous function from $\mathbb{R}^2 - \{(0, 0)\}$ to itself.

The dynamical behavior of the function f defined in theorem (5.1) illustrated in Figure V.4. The figure shows the behavior of this system for three separate initial conditions. The system trajectory for the initial condition **s** located on the outer circle remains on the same circle. The system trajectory for the initial condition **p** lying on the annulus getting closer to the outer circle. The system trajectory for the initial condition **q** located on the inner circle remains on the same circle. The special behavior of function f reminds the author the structure of *Tower of Babel* (TOB).

Fig. V.4: Graphical representation of the 'TOB' function. The gray circles have radii 2^{-j} for various $j \in \mathbb{Z}$. Points on these circles are rotated anticlockwise by $90°$ by f. Points that lie between two of the circles have trajectories that spiral out towards the outer circle.

Next, we will illustrate the proof of theorem 3.3 by using it to construct weak Lyapunov function for the 'TOB' dynamical system. The weak Lyapunov function defined in (III.11) can be written for the 'TOB' system as follow

$$\begin{aligned} V(\mathbf{x}) &= \sup\{h(\|f^{[k]}(\mathbf{x})\|) : k \geq 0\} \\ &= \sup\{h(g^{[k]}(r)) : k \geq 0\} \end{aligned} \tag{V.7}$$

where h defined as

$$h(t) = \begin{cases} t, & t \in [0,1] \\ 1, & t \in (1,\infty) \end{cases}$$

Considering the dynamical behavior of $g(r)$ explained in the previous section, the weak Lyapunov function in (V.7) can be rewritten as

$$V(\mathbf{x}) = \begin{cases} 1, & \|\mathbf{x}\| > 2^{-1} \\ 2^{-j}, & 2^{-(j+1)} < \|\mathbf{x}\| \leq 2^{-j}, \quad \forall j \geq 1 \end{cases} \tag{V.8}$$

Note that the function V is not continuous for the 'TOB' dynamical system. The discontinuity of V verse $\|\mathbf{x}\|$ illustrated in Figure V.5.

Fig. V.5: Weak Lyapunov function for 'TOB' dynamical system

VI. CONCLUSION

The necessary and sufficient conditions for stability and GAS stability of dynamical system is derived. We show that the

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