Output Regulation of Systems Governed by Delay Differential Equations: Approximations and Robustness

Sai Tej Paruchuri

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John A. Burns, Chair

Andrew J. Kurdila

Eugene M. Cliff

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(ABSTRACT)

This thesis considers the problem of robust geometric regulation for tracking and disturbance rejection of systems governed by delay differential equations. It is well known that geometric regulation can be highly sensitive to system parameters and hence such designs are not always robust. In particular, when employing numerical approximations to delay systems, the resulting finite dimensional models inherit natural approximation errors that can impact robustness. This demonstrates this lack of robustness and then addresses robustness by employing versions of robust regulation that have been developed for infinite dimensional systems. Numerical examples are given to illustrate the ideas and to test the robustness of the regulator.

Output Regulation of Systems Governed by Delay Differential Equations: Approximations and Robustness

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(GENERAL AUDIENCE ABSTRACT)

Recent years have seen a surge in the everyday application of complex mechanical and electrical systems. These systems can perform complex tasks; however, the increased complexity makes it harder to control them. An example of such a system is a semi-autonomous car designed to stay within a designated lane. One of the most commonly used approaches for controlling such systems is called output regulation. In the above example, the output regulator regulates the output of the car (position of the car) to follow the reference output (the road lane). Traditionally, the design of output regulators assumes complete knowledge of the system. However, it is impossible to derive equations that govern complex systems like a car. This thesis analyzes the robustness of output regulators in the presence of errors in the system. In particular, the focus is on analyzing output regulators implemented to delay-differential equations. These are differential equations where the rate of change of states at the current time depends on the states at previous times. Furthermore, this thesis addresses this problem by employing the robust versions of the output regulators.

Dedication

Dedicated to my mother who let me pursue my dreams.

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Chapter 1

Introduction

Researchers have extensively used delay-differential equations to model population dynamics, delays arising from sensors and actuators, neural networks, and systems with a hysteresis. Kyrychko and Hogan provide an extensive review of the use of delay differential equations in engineering in [1]. The delay differential equation can be posed as an infinite-dimensional system. The output regulation of delay systems requires the implementation of infinite-dimensional controllers. The practical implementation of the infinite-dimensional controllers requires the approximation of either the controller or the infinite-dimensional system.

The design of output regulators assumes the exact knowledge of the plant. Such controllers are highly sensitive to any errors in system parameters. The approximation of the delay systems or the corresponding controller introduces error into the actual system. Furthermore, any perturbation to the delay system's parameters (for example, the time delay) can deteriorate the effectiveness of the controller.

The objectives of this thesis are

1. analyze the effect of the delay system's parameter error on the controller's performance,
2. study the effect of approximation schemes on the controller gain, and
3. illustrate the effectiveness of robust controllers.

Chapter 2

The Problem of Output Regulation

In the current study, we focus on applying geometric regulation to systems governed by delay differential equations of the form

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-r) + B_0 u(t) + w_{dist}(t) \in \mathbb{R}^n \tag{2.1}$$

with initial data

$$x(0) = \eta \in \mathbb{R}^n,\ x(s) = \ \varphi(s),\ \ \varphi(\cdot) \in L^2(-r,0;\mathbb{R}^n). \tag{2.2}$$

The controlled output of the system is defined by

$$y_c(t) = C_0 x(t) \in \mathbb{R}^p. \tag{2.3}$$

Here, $r > 0$ is a time delay, A_0, A_1 are $n \times n$ matrices, B_0 is $n \times m$ and C_0 is $p \times n$. The goal of output regulation is to track a given reference signal $y_{ref}(t) \in \mathbb{R}^p$ while rejecting the disturbance $w_{dist}(t)$.

We assume that the reference signal $y_{ref}(t)$ and the disturbance $w_{dist}(t)$ are the outputs to a finite dimensional exogenous system. In particular, $w(t) \in \mathbb{R}^q$ satisfies the initial value problem

$$\dot{w}(t) = Sw(t),\ \ w(0) = w_0 \in \mathbb{R}^q, \tag{2.4}$$

where S is an $q \times q$ matrix. The disturbance and reference signals are assumed to have the form

$$w_{dist}(t) = P_0 w(t) \text{ and } y_{ref}(t) = -Q_0 w(t), \tag{2.5}$$

where P_0 is $n \times q$ and Q_0 is $p \times q$. The error is defined by

$$e(t) := y_c(t) - y_{ref}(t), \tag{2.6}$$

which implies

$$e(t) = C_0 x(t) + Q_0 w(t). \tag{2.7}$$

It is well known ([2, 3, 4, 5]) that the initial value problem for the retarded delay equation (2.1) - (2.2) is equivalent to the infinite dimensional system on the state space $Z = \mathbb{R}^n \times L^2(-r, 0; \mathbb{R}^n)$ of the form

$$\dot{z}(t) = \mathscr{A} z(t) + \mathscr{B} u(t) + \mathscr{P} w(t) \tag{2.8}$$

with initial condition

$$z(0) = [\eta \;\; \varphi(\cdot)]^\top \in \mathbb{R}^n \times L^2(-r, 0; \mathbb{R}^n). \tag{2.9}$$

The system operator $\mathscr{A}$ is defined on the domain

$$\mathscr{D}(\mathscr{A}) = \left\{ \begin{bmatrix} \eta \\ \varphi(\cdot) \end{bmatrix} : \varphi(\cdot) \in H^1(-r, 0; \mathbb{R}^n), \; \varphi(0) = \eta \right\} \tag{2.10}$$

by

$$\mathscr{A} \begin{bmatrix} \eta \\ \varphi(\cdot) \end{bmatrix} = \begin{bmatrix} A_0 \eta + A_1 \varphi(-r) \\ \varphi'(\cdot) \end{bmatrix}, \tag{2.11}$$

where $\varphi'(\cdot)$ is the derivative of $\varphi(\cdot)$.

The operators $\mathscr{B}:\mathbb{R}^m \to Z$ and $\mathscr{P}:\mathbb{R}^q \to Z$ are defined by

$$\mathscr{B}u = \begin{bmatrix} B_0 u \\ 0 \end{bmatrix}$$

and

$$\mathscr{P}w = \begin{bmatrix} P_0 w \\ 0 \end{bmatrix},$$

respectively and $\mathscr{C}: Z \to \mathbb{R}^p$ is given by

$$\mathscr{C}\begin{bmatrix} \eta \\ \varphi(\cdot) \end{bmatrix} = C_0\eta. \tag{2.12}$$

The reference signal is assumed to have the form $y_{ref} = -\mathscr{Q}w(t)$. Note that the operator $\mathscr{Q}:\mathbb{R}^q \to \mathbb{R}^m$ and is defined by $\mathscr{Q} = Q_0$ so that the error for the distributed parameter system (2.8) - (2.9) is given by

$$e(t) = y_c(t) - y_{ref}(t) = \mathscr{C}w(t) + \mathscr{Q}w(t) = \mathscr{C}z(t) + Q_0 w(t). \tag{2.13}$$

In [2] it is shown that $\mathscr{A}$ generates the C_0-semigroup $S(t): Z \to Z$ such that for all $[\eta \;\; \varphi(\cdot)]^\top \in Z$

$$S(t)\begin{bmatrix} \eta \\ \varphi(\cdot) \end{bmatrix} = \begin{bmatrix} x(t) \\ x_t(\cdot) \end{bmatrix}, \tag{2.14}$$

where $x(t)$ is the solution to (2.1) - (2.2) with $u(t) = \omega(t) = 0$ and $x_t(\cdot) \in H^1(-r,0;\mathbb{R}^n)$ is the past history function defined by $x_t(s) = x(s+t)$ for all $s \in [-r,0]$. Moreover, it is straightforward to show that the operators $\mathscr{B}$, $\mathscr{P}$, $\mathscr{C}$ and Q_0 are all bounded.

2.1 The Problem of Output Regulation: Full Information

The goal is to find bounded linear operators $\mathscr{K} : Z \to \mathbb{R}^m$ and $\mathscr{L} : \mathbb{R}^q \to \mathbb{R}^m$ such that if

$$u(t) = -\mathscr{K} z(t) + \mathscr{L} w(t), \tag{2.15}$$

then the system

$$\dot{z}_{cl}(t) = [\mathscr{A} - \mathscr{B}\mathscr{K}] z_{cl}(t) \tag{2.16}$$

is stable and

$$\lim_{t\to\infty} ||e(t)|| = 0, \tag{2.17}$$

for all $z_0 \in Z$ and $w_0 \in \mathbb{R}^q$. Observe that if the system is stable or there is a known feedback operator $\mathscr{K}$ that stabilizes the system, then the problem reduces to finding the gain $\mathscr{L}$.

2.2 The Problem of Output Regulation: Error Feedback

In this case one assumes that only the error is available for measurement and consequently the controller is dynamic. Now the goal is to find a (well-posed) dynamical system on a Hilbert space V of the form

$$\dot{\xi}(t) = \mathscr{G}_1 \xi(t) + \mathscr{G}_2 e(t) \tag{2.18}$$

where $\mathscr{G}_2 : \mathbb{R}^p \to V$ is bounded and a bounded operator $\mathscr{F} : V \to \mathbb{R}^m$ which defines a feedback controller by

$$u(t) = -\mathscr{F} \xi(t). \tag{2.19}$$

Observe that the well-posedness conditions implies that the operator $\mathscr{G}_1$ generates a C_0-semigroup on V .

The resulting closed-loop system is a dynamical system on $Z_e = Z \times V \times \mathbb{R}^q$ defined by

$$\dot{z}(t) = \mathscr{A}z(t) - \mathscr{B}\mathscr{F}\xi(t) + \mathscr{P}w(t), \tag{2.20}$$

$$\dot{\xi}(t) = \mathscr{G}_2\mathscr{C}z(t) + \mathscr{G}_1\xi(t) + \mathscr{G}_2Q_0w(t) \tag{2.21}$$

$$\dot{w}(t) = Sw(t). \tag{2.22}$$

Lemma 2.1. *The closed-loop system* (2.20)- (2.22) *is well-posed on* Z_e*,*

Proof: Let $\mathscr{A}_{dia}$ and $\mathscr{A}_{pert}$ denote the operators

$$\mathscr{A}_{dia} = \begin{bmatrix} \mathscr{A} & 0 & 0 \\ 0 & \mathscr{G}_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathscr{A}_{pert} = \begin{bmatrix} 0 & -\mathscr{B}\mathscr{F} & \mathscr{P} \\ \mathscr{G}_2\mathscr{C} & 0 & \mathscr{G}_2Q_0 \\ 0 & 0 & S \end{bmatrix},$$

respectively. The diagonal operator $\mathscr{A}_{dia}$ generates a C_0-semigroup on Z_e and since $\mathscr{A}_{pert}$ is a bounded operator, the operator

$$\mathscr{A}_e = \mathscr{A}_{dia} + \mathscr{A}_{pert} = \begin{bmatrix} \mathscr{A} & -\mathscr{B}\mathscr{F} & \mathscr{P} \\ \mathscr{G}_2\mathscr{C} & \mathscr{G}_1 & \mathscr{G}_2Q_0 \\ 0 & 0 & S \end{bmatrix}$$

also generates a C_0-semigroup on Z_e. This completes the proof.

The goal of output regulation with error feedback is to find operators $\mathscr{G}_1$, $\mathscr{G}_2$ and $\mathscr{F}$ such that the system

$$\dot{z}(t) = \mathscr{A}z(t) - \mathscr{B}\mathscr{F}\xi(t), \tag{2.23}$$

$$\dot{\xi}(t) = \mathscr{G}_2\mathscr{C}z(t) + \mathscr{G}_1\xi(t), \tag{2.24}$$

is asymptotically (exponentially) stable and the error defined by (2.13) satisfies

$$\lim_{t\to\infty} ||e(t)|| = 0, \tag{2.25}$$

for all $z_0 \in Z$, $\xi_0 \in V$ and $w_0 \in \mathbb{R}^q$.

In this setting, we can apply the approaches in [6, 7, 8, 9] and [10] to formulate regulator and robust regulator problems that extend the finite dimensional problems as defined in the classic book [11]. In fact, in [7] the authors used a simple delay equation as an example to illustrate their theory. Also, in certain cases the dynamic regulator approach developed in the book [12] can be applied to this system without the need to use the distributed parameter formulation (see [13]).

One of the important practical issues is the problem of computing the geometric regulator. As always, there are two basic approaches to dealing with computation. The design then approximate approach (DTA) solves the infinite dimensional regulator problem to produce an infinite dimensional controller and then approximates the controller. In the approximate then design (ATD) approach one employs numerical methods to approximate the distributed parameter system, producing finite dimensional system and then solves the regulator problem for these approximate systems. Roughly speaking, the approach in the recent paper [13] is a DTA method and the approach in [10] is a ATD method.

There is a very nice analysis of the convergence of the approximate controllers (and corresponding reduced order controllers) in [10]. As noted in that paper, dual convergence is important and, for the parabolic and hyperbolic PDE problems in that paper, standard Galerkin methods yield convergence to the dual system. Dual convergence is obtained since these types of systems are defined by normal dynamic operators $\mathscr{A}$. However, the system operator $\mathscr{A}$ defined by (2.10)-(2.11) for the delay equation is highly non-normal and dual convergence can fail for certain Galerkin methods (see [14]).

2.3 Approximating Systems

In this section, we consider the (ATD) approach applied to problems governed by the delay system (2.1)-(2.3) by introducing approximations. Results are presented for three approximating schemes:

- (AVE) a finite volume (averaging) method [2],
- (BK) a finite element method [15] and
- (IK) a spline based scheme [16].

The resulting approximating models are ODE systems of the form

$$\dot{z}_N(t) = A_N z_N(t) + B_N u(t) + P_N w(t) \tag{2.26}$$

$$y_N(t) = C_N z_N(t) \tag{2.27}$$

with error

$$e_N(t) = C_N z_N(t) + Q_N w(t), \tag{2.28}$$

where $Q_N = Q_0$ for all $N \geq 1$. Since we are only considering finite dimensional exogenous systems, the exogenous system is still defined by

$$\dot{w}(t) = Sw(t). \tag{2.29}$$

All the approximations above are obtained by first selecting a finite dimensional subspace Z_N of the state space $Z = \mathbb{R}^n \times L^2(-r,0;\mathbb{R}^n)$ and then projecting the system onto Z_N. A general framework for convergence of controllers is given in [17]. Following [17] we consider a sequence (finite dimensional) approximating problems defined by $(Z_N, A_N, B_N, P_N, C_N, Q_N)$, where $Z_N \subset Z$ and π_N is the orthogonal projection onto Z_N. Let $S(t)$, $S^*(t)$, $S_N(t)$ and $S_N^*(t)$ denote the C_0-semigroups generated by the operators $\mathscr{A}$, $\mathscr{A}^*$, A_N and A_N^*, respectively.

Definition 2.2. We say this approximating system is *convergent* if

HC: For each $z \in Z$, $u \in \mathbb{R}^m$ and $w \in \mathbb{R}^q$,

(HCa) : $S_N(t)\pi_N z \to S(t)z$,

(HCb) : $B_N u \to \mathscr{B}u$,

(HCc) : $C_N \pi_N z \to \mathscr{C}z$,

(HCp) : $P_N w \to \mathscr{P}w$,

where the convergence in (HCa) is uniform on compact time intervals.

Definition 2.3. We say this approximating system is *dual convergent* if

HD: For each $z \in Z$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$,

(HDa) : $S_N^*(t)\pi_N z \to S^*(t)z$,

(HDb) : $B_N^* \pi_N z \to \mathscr{B}^* z$,

(HDc) : $C_N^* y \to \mathscr{C}^* y$,

(HDp) : $P_N^* \pi_N z \to \mathscr{P}^* z$,

where the convergence in (HDa) is uniform on compact time intervals.

Remark 2.4. The (BK) scheme is not "dual convergent" which causes problems for optimization based controller designs (see [14] and [18] for details). This is not an issue for the regulator problem, but can be a problem for model reduction and optimization based methods that require dual convergence.

The family of pairs (A_N, B_N) is said to be *uniformly stabilizable* if there is a sequence of bounded operators $K_N : Z_N \to \mathbb{R}^m$ such that $\sup_N ||K_N|| < +\infty$ and the closed-loop system satisfies

$$||e^{(A_N - B_N K_N)t}|| \leq M_1 e^{-\omega_1 t} \tag{2.30}$$

for fixed $M_1 \geq 1$ and $\omega_1 > 0$. Likewise, the family of pairs (A_N, C_N) is said to be *uniformly detectable* if there is a sequence of bounded operators $F_N : Z_N \to \mathbb{R}^p$ such that $\sup_N ||F_N|| < +\infty$ and the closed-loop system satisfies

$$||e^{(A_N - F_N C_N)t}|| \leq M_2 e^{-\omega_2 t} \tag{2.31}$$

for fixed $M_2 \geq 1$ and $\omega_2 > 0$.

Remark 2.5. Observe that if the semigroups $S(t)$ and $S_N(t)$ are exponentially stable, then the family of pairs are both uniformly stabilizable and uniformly detectable since we can take $K_N = F_N = 0$.

We assume that the following two conditions hold in the analysis going forward.

Assumption 2.1. The semigroup $S(t)$ is stable and family $S_N(t)$ is *uniformly stable* in the sense that there exist $M \geq 1$ and $\omega > 0$ such that

$$||S(t)|| \leq Me^{-\omega t} \text{ and } ||e^{(A_N)t}|| \leq Me^{-\omega t} \tag{2.32}$$

for all $N \geq 1$.

Assumption 2.2. The exogenous system (2.29) is neutrally stable.

In this case we take the feedback gain operators $\mathscr{K}$ and K_N to be zero. This is not essential, but it simplifies the analysis and retains the emphasis of this thesis on geometric regulation. Also, we note that if condition (HCa) holds, then the Trotter-Kato Theorem implies that for N sufficiently large (2.32) holds.

The three numerical schemes above are detailed in [18]. For example, for $N \geq 1$ the finite volume

(AVE) scheme in [2] produces the following approximating system of size $N+1$:

$$A_N = \begin{bmatrix} A_0 & 0 & 0 & \cdots & A_1 \\ \frac{N}{r}I_n & -\frac{N}{r}I_n & 0 & \cdots & 0 \\ 0 & \frac{N}{r}I_n & -\frac{N}{r}I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{N}{r}I_n & -\frac{N}{r}I_n \end{bmatrix}, \quad B_N = \begin{bmatrix} B_0 & 0 & \cdots & 0 \end{bmatrix}^\top, \tag{2.33}$$

$$B_N = \begin{bmatrix} B_0 & 0 & \cdots & 0 \end{bmatrix}^\top, \quad P_N = \begin{bmatrix} P_0 & 0 & \cdots & 0 \end{bmatrix}^\top, \tag{2.34}$$

$$C_N = \begin{bmatrix} C_0 & 0 & \cdots & 0 \end{bmatrix} \text{ and } Q_N = Q_0. \tag{2.35}$$

The form of the approximating systems for the (BK) finite element method and the (IK) spline based Galerkin scheme can be found in [15] and [16], respectively. In all cases we note that $A_N = A_N(A_0, A_1, r)$, so that perturbations of the system parameters A_0, A_1 and the delay r could impact robustness. Indeed, we will show this is the case in Section 2.5 and address this issue by applying the methods in [6, 8, 9, 10] in Chapter 3.

2.4 Existence and Convergence

In this section, let us take a look at when the problem with full state information has a solution. The discussion for the problem with error feedback is similar (see [8, 10]). In light of Remark 2.5, we set $\mathscr{K} = 0$ and $K_N = 0$ so that the solution to the output regulation problem with full information reduces to the computation of the operator $\mathscr{L} : \mathbb{R}^q \to \mathbb{R}^m$ and the feedback controller becomes

$$u(t) = \mathscr{L}w(t).$$

Also, the solutions to the approximate problems defined by (2.26)-(2.28) with exogenous system (2.29) are given by

$$u_N(t) = L_N w(t), \tag{2.36}$$

where $L_N : \mathbb{R}^q \to \mathbb{R}^m$. Noting that $Q_N = Q_0$ for all $N \geq 1$, then the next two results follow from Theorem 1.1 in [12].

Theorem 2.6. *If Assumptions 2.1 and 2.2 hold, then the regulator problem with full information is solvable if and only if there exist bounded linear operators $\Pi : \mathbb{R}^q \to Z$ with Range$(\Pi) \subset \mathscr{D}(\mathscr{A})$ and $\mathscr{L} : \mathbb{R}^q \to \mathbb{R}^m$ satisfying the regulator equations*

$$\Pi S = \mathscr{A}\Pi + \mathscr{B}\mathscr{L} + \mathscr{P}, \tag{2.37}$$

$$0 = \mathscr{C}\Pi + Q_0. \tag{2.38}$$

Here, $u(t) = \mathscr{L}w(t)$ solves the infinite dimensional problem.

The proof of this theorem can be found in [12].

Theorem 2.7. *If Assumptions 2.1 and 2.2 hold, then the approximate regulator problems with full information are solvable if and only if there exist bounded linear operators $\Pi_N : \mathbb{R}^q \to Z_N$ and $L_N : \mathbb{R}^q \to \mathbb{R}^m$ satisfying the regulator equations*

$$\Pi_N S = A_N \Pi_N + B_N L_N + P_N, \tag{2.39}$$

$$0 = C_N \Pi_N + Q_0. \tag{2.40}$$

Here, $u(t) = L_N w(t)$ solves the finite dimensional problem.

The proof of this theorem can be found in [11].

Before we look at the theorem about convergence of the gain operators, let us take go over a Lemma that is essential for the convergence proof.

Lemma 2.8. *Assume T_N are bounded linear operators, T_N converges strongly to the bounded linear operator T (i.e., $\|T_N z - Tz\| \to 0$ for all $z \in Z$). If there is a bound c so that $\|T_N\| \leq c$ and if B is a compact operator, then $T_N B$ converges in norm to TB. (i.e., $\|T_N B - TB\| \to 0$).*

Theorem 2.9. *If Assumptions 2.1 and 2.2 hold and the approximating system is convergent (i.e., satisfies condition* **HC** *given in Definition 2.2 above), then the approximate gain operators L_N converge in norm to $\mathscr{L}$.*

Proof. We prove this theorem for the SISO system case with $\mathscr{P} = P_N = 0$ and $S = 0$. This proof easily extends to more general cases. Under these assumptions, the regulator equations simplify to

$$0 = A_N \Pi_N + B_N L_N,$$
$$0 = C_N \Pi_N + Q_0.$$

We can rewrite the first equation above to get $\Pi_N = -A_N^{-1} B_N L_N$. Substituting the expression for Π_N into the second regulator equation, we get

$$0 = C_N (-A_N)^{-1} B_N L_N + Q_0,$$

which implies

$$L_N = -\left(C_N (-A_N)^{-1} B_N\right)^{-1} Q_0.$$

In the above equation, Q_0 is a scalar that does not depend on N. By Trotter-Kato's theorem [19, 20], we note that if the approximating system is convergent, then $\mathscr{R}(\lambda_0, A_N)\pi_N z \to \mathscr{R}(\lambda_0, \mathscr{A})z$ for all λ_0

such that $\text{Real}(\lambda_0) > -\omega$ and for all $z \in Z$. The notation $\mathscr{R}(\lambda_0, A)$ denotes the resolvent of A and is defined as $\mathscr{R}(\lambda_0, A) = (\lambda_0 I - A)^{-1}$. Recall that ω satisfies $||S(t)|| \leq Me^{-\omega t}$ and $||e^{(A_N)t}|| \leq Me^{-\omega t}$. Choose $\lambda_0 = 0 > -\omega$. Thus, Trotter-Kato theorem gives us $(-A_N)^{-1}\pi_N z \to (-\mathscr{A})^{-1}z$ for all $z \in Z$. We note that the norm $\|(-A_N)^{-1}\|$ is bounded above by a constant c for all N. We can show this using strong convergence of $(-A_N)^{-1}$ to $(-\mathscr{A})^{-1}$, that is $(-A_N)^{-1}\pi_N z \to (-\mathscr{A})^{-1}z$ for all $z \in Z$, and uniform boundedness principle. We also note that $B_N u = \mathscr{B}u$ and $C_N \pi_N z = \mathscr{C}z$ for all $u \in \mathbb{R}^m$ and $z \in Z$. Thus, using Lemma 2.8, we conclude that $(-A_N)^{-1}B_N$ converges in norm to $(-\mathscr{A})^{-1}\mathscr{B}$, which in turn implies that L_N converges in norm to $\mathscr{L}$. □

Note that dual convergence is not needed to establish convergence of the gain matrices. However, if one were to use an optimization based method to first stabilize the system (say a LQR controller), then dual convergence is also necessary.

Although the theoretical results above provide a basis to claim convergence, there remains the questions of robustness with respect to numerical approximations and to model parameters. In the next section, we see that the solution to the problem with full information need not be robust with respect to errors caused by numerical computation and not robust with respect to model parameters. However, we also see in Chapter 3 that the robust methods developed by Paunonen and co-workers can be used to recover robustness for these delay systems.

2.5 An Example and Numerical Results

Example 2.10. Consider the scalar system

$$\dot{x}(t) = a_0 x(t) + a_1 x(t-r) + u(t) + w_{dist}(t), \tag{2.41}$$

$$y(t) = cx(t), \tag{2.42}$$

Table 2.1: AVE System Gains for Increasing N values

N	l_1	l_2	l_3
8	-1.3618	0.15006	2.0000
16	-1.3549	0.16986	2.0000
32	-1.3520	0.18023	2.0000
64	-1.3506	0.18552	2.0000
128	-1.3500	0.18820	2.0000

with disturbance and reference signals

$$w_{dist}(t) = cos(t) \ \text{ and } \ y_{ref}(t) = 1 - sin(t), \tag{2.43}$$

respectively.

Thus, the exogenous system $\dot{w}(t) = Sw(t)$ is defined by

$$S = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ with } w(0) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top \tag{2.44}$$

and

$$P_0 = \begin{bmatrix} (1/2) & (1/2) & 0 \end{bmatrix}, \ Q_0 = \begin{bmatrix} (1/2) & (-1/2) & -1 \end{bmatrix}. \tag{2.45}$$

Note that the system (2.41) is stable for any a_0, a_1 satisfying $a_0 < 1/r$ and $-\pi/2r < a_1 < 0$ (see page 54 in [21]). For the numerical example we set $a_0 = a_1 = -1$, $r = 1$ and $c = 1$. We set the initial data as $x(0) = 1$ and $\varphi(s) = 1$ for all $s \in [-1, 0)$.

Since $q = 3$, the gains L_N and $\mathscr{L}$ are 3×1 matrices of the form $L = \begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix}$. Tables 2.1, 2.2 and 2.3 demonstrate convergence of the gains for all three approximation schemes. Observe that the (BK) finite element method produces convergent gains even though this scheme is not dual convergent. Also, note that the higher order scheme (IK) has converged to the "infinite dimensional

Table 2.2: BK System Gains for Increasing N values

N	l_1	l_2	l_3
8	-1.3286	0.17416	2.0000
16	-1.3396	0.18254	2.0000
32	-1.3447	0.18671	2.0000
64	-1.3471	0.18888	2.0000
128	-1.3483	0.18984	2.0000

Table 2.3: IK System Gains for Increasing N values

N	l_1	l_2	l_3
8	-1.3503	0.19108	2.0000
16	-1.3496	0.19094	2.0000
32	-1.3495	0.19090	2.0000
64	-1.3494	0.19089	2.0000
128	-1.3494	0.19089	2.0000

gain" $\mathscr{L} = \begin{bmatrix} -1.3494 & 0.19089 & 2.0000 \end{bmatrix}$ when $N = 64$. Although all schemes are convergent (as stated in Theorem 2.9), the (IK) scheme converges much more rapidly.

For the simulations we assume the initial data for the delay system is the constant function $\varphi(s) = 1$ and $\eta = 1$. As one can see in Figure 2.1, the controller does an excellent job of tracking the reference signal. Note that the tracking error is essentially zero for time $t > 20$. Figure 2.2 shows the reference signal y_{ref} and the controlled output $y(t)$.

As per Theorem 2.7, the gain L_N obtained for the above case should work for all initial conditions $w_0 \in \mathbb{R}^q$ of the exosystem. This means, the controller should track for any disturbance $w_{dist}(t)$ and reference signal y_{ref} generated by the exosystem defined the matrix S. We ran the output regulator for $w_0 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^\top$. The regulator tracks the reference output perfectly as seen from Figures 2.3 and 2.4.

Observe that the gain $l_3^N = l_3 = 2$ for all $N \geq 8$ and for all three schemes. This is due to the special structure of the delay system and the corresponding approximation schemes. A natural question is,

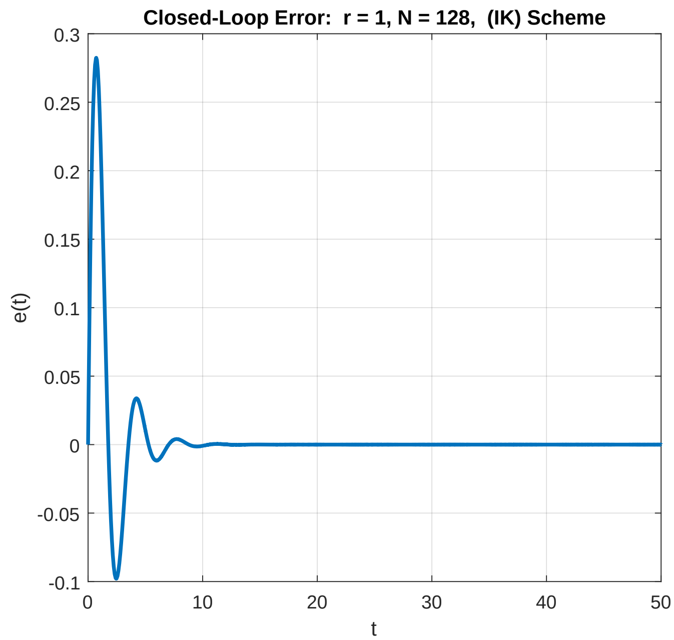

Figure 2.1: Error for $N = 128$ (IK) Model: Output Regulator

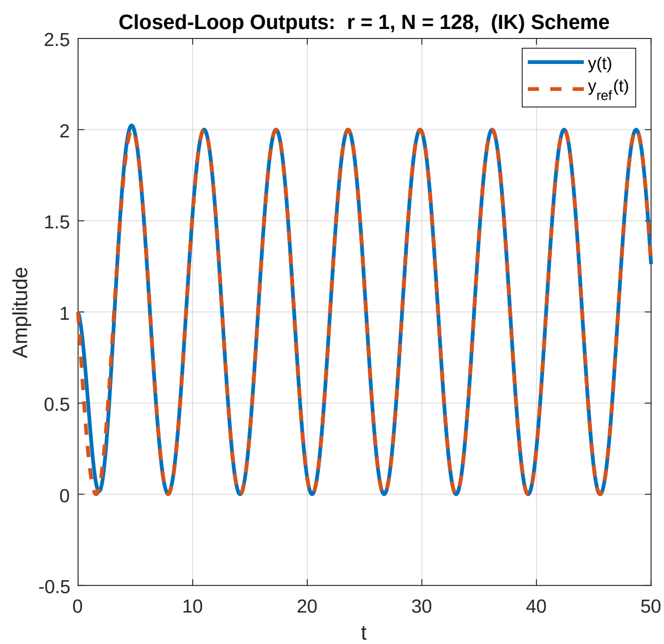

Figure 2.2: Outputs for $N = 128$ (IK) Model: Output Regulator

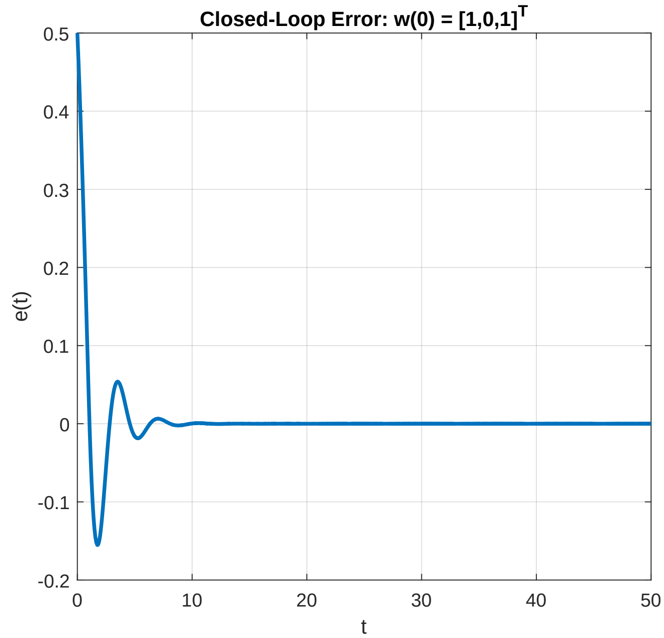

Figure 2.3: Error for $w_0 = [1,0,1]^\top$: Output Regulator

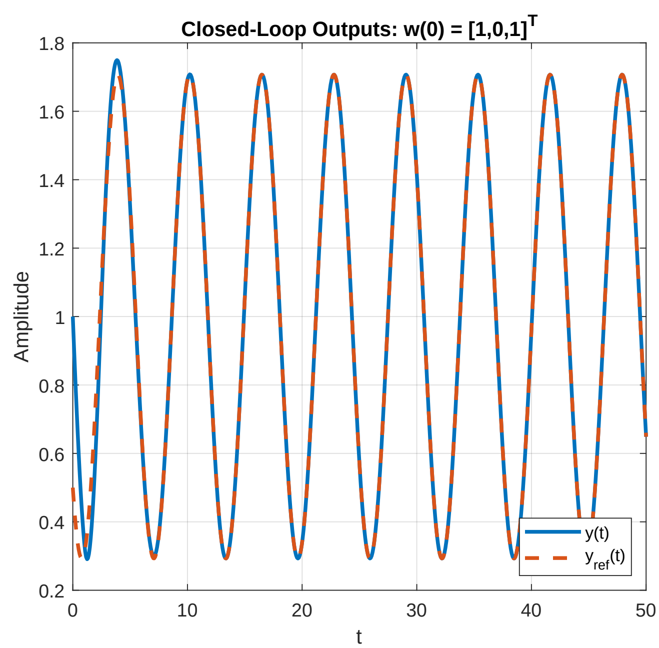

Figure 2.4: Outputs for $w_0 = [1,0,1]^\top$: Output Regulator

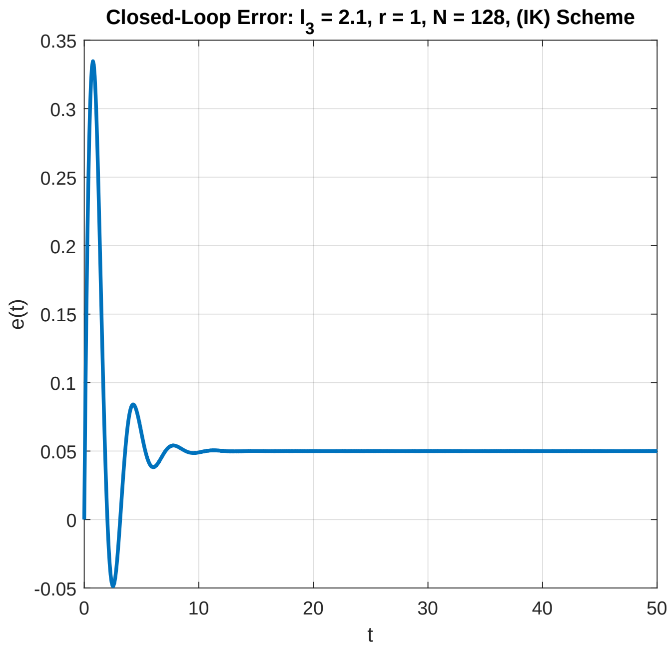

Figure 2.5: Error for $l_3 = 2 + \delta = 2.1$: Output Regulator Fails

"What happens if the numerical gain computed for a low order model such as the (AVE) scheme is applied to delay system (or a sufficiently accurate high order model)"? A more general question is, "What is the impact of perturbations in gains on regulator performance"? Thus, we consider two robustness questions:

Q1: Is the control robust with respect to numerical errors that naturally occur in the computation of the gains?

Q2: Is the control robust with respect to changes in the system parameters?

For the problem here, the answer to both questions is no. To illustrate this lack of robustness for question Q1, we ran several cases with small perturbations in the computed gains and the resulting closed-loop systems do not track the reference signal. The most dramatic example was obtained by perturbing the third gain $l_3 = 2$. For example, if one sets $\delta = 0.1$ and perturb the infinite dimensional gain $\mathscr{L}$ to $\mathscr{L}_{delta} = \begin{bmatrix} -1.3494 & 0.19089 & 2.00+\delta \end{bmatrix}$, then as shown in Figure 2.5 and

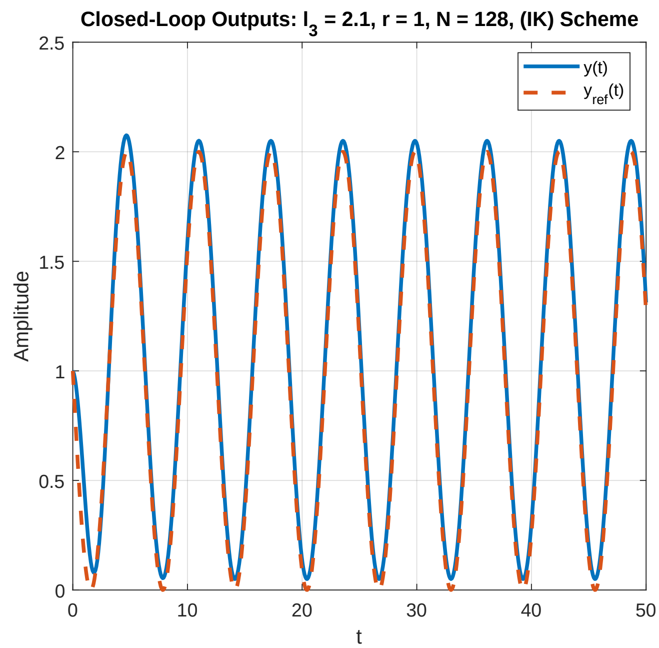

Figure 2.6: Outputs for $l_3 = 2+\delta = 2.1$: Output Regulator Fails

Figure 2.6 below, the error reaches a non-zero steady state constant so that the control fails to track $y_{ref} = 1 - sin(t)$. The case with $l_3 = 2-\delta = 1.9$ produces the same type of results.

We also ran the case where we use the (AVE) scheme gain L_{128} in the (IK) scheme plant with $N = 128$. Calculating the (AVE) scheme gain is more computationally efficient than that of the (IK) scheme. On the other hand, the (IK) scheme converges to the actual system faster, which is evident from the gain convergence in Table 2.3. Thus, to test the actual controller, it might be efficient to compute the gain using a computationally efficient scheme ((AVE) scheme in this case) and implement it in another scheme which is more close to the actual system ((IK) scheme in this case). Before looking at the results, we note from Table 2.1 that the (AVE) scheme gain L_{128} is close but not equal to the infinite-dimensional gain L. Figures 2.7 and 2.8 show that the error does not go to zero demonstrating the lack of robustness for question Q1.

Now consider question Q2. As noted above $A_N = A_N(A_0, A_1, r)$, so it is natural to consider per-

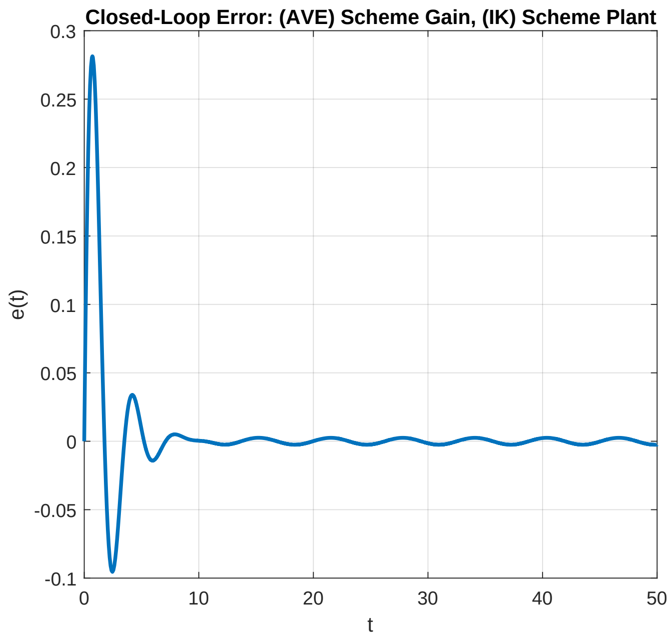

Figure 2.7: Error for (AVE) Gain with (IK) Model: Output Regulator Fails

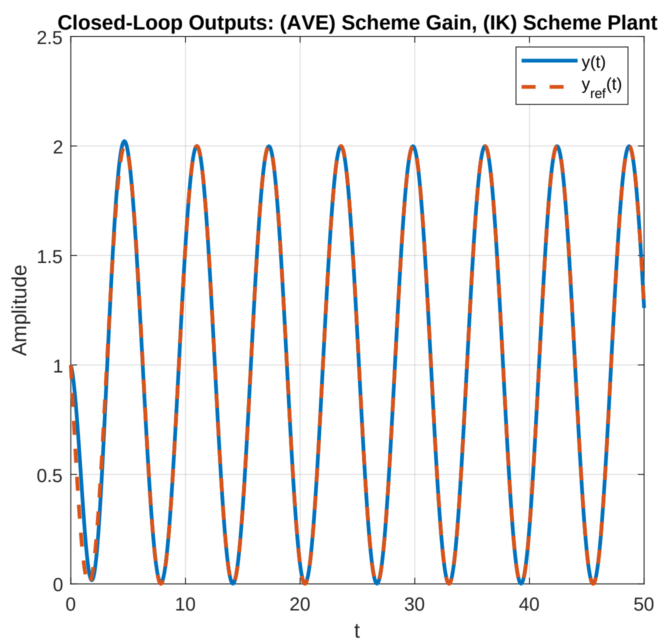

Figure 2.8: Outputs for (AVE) Gain with (IK) Model: Output Regulator Fails

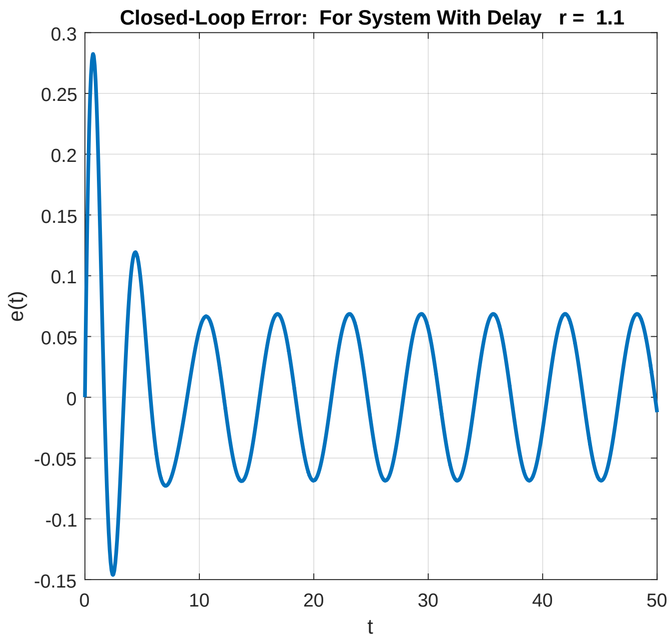

Figure 2.9: Error for $r = 1.1$: Output Regulator Fails

turbations in the delay $r = 1$. In this numerical experiment we apply the infinite dimensional gain $\mathscr{L}$ (computed for the nominal value $r = 1$) to the perturbed systems defined with $r_{pert} = 1.1$. As shown in Figure 2.9, the controller designed for the delay $r = 1$ applied to the perturbed system with $r = 1.1$ fails to drive the error to zero. Thus, there is no robustness with respect to the delay parameter. The case with $r = 1 - 0.1 = 0.9$ is essentially the same.

In view of the results in Example 2.10, it is clear that robustness is an issue with output regulation even with full information. In the next chapter, we apply the method described in [6, 8, 9, 10] to produce a robust regulator.

Chapter 3

The Problem of Robust Regulation

In this chapter, we show that one can recover robustness by using the robust regulator designs discussed in [6, 8, 9, 10]. We consider the same delay system defined in Example 2.10 and illustrate that robust regulators can deal with the type of perturbations discussed there.

The robust controller is a dynamic controller similar in structure to (2.18)-(2.19). Thus, one seeks operators $\mathscr{G}_1, \mathscr{G}_2$ and $\mathscr{F}$ such that

$$\dot{z}_e(t) = \mathscr{G}_1 z_e(t) + \mathscr{G}_2 e(t), \tag{3.1}$$

where the bounded operator $\mathscr{F} : V_e \to \mathbb{R}^m$ defines a feedback controller by

$$u(t) = -\mathscr{F} z_e(t). \tag{3.2}$$

Although the system (3.1)-(3.2) is similar in structure to (2.18)-(2.19), the spaces are different and the construction is more complex. For simplicity, we consider the derivation of the robust controller for the finite-dimensional approximation of the delay system (2.26) and (2.27). The controller construction for the infinite-dimensional case is well documented in [6, 8, 9, 10] and we refer the reader to those papers for details of the method. The finite-dimensional dynamic controller has the form

$$\begin{aligned}\dot{z}_{N,e}(t) &= \mathscr{G}_{N,1} z_{N,e}(t) + \mathscr{G}_{N,2} e(t), \qquad z_{N,e}(0) = z_{N,0} = \pi_N z_0, \text{ for all } z_0 \in Z,\\ u(t) &= F_N z_{N,e}(t)\end{aligned}$$

with $e_N(t) := y_N(t) - y_{ref}(t)$ to achieve robust output regulation.

3.1 The Robust Regulator Problem with Partial Information

The goal of the robust regulator problem is to find matrices $\mathscr{G}_{N,1}, \mathscr{G}_{N,2}$ and F_N such that if

$$\begin{aligned}\dot{z}_{N,e}(t) &= \mathscr{G}_{N,1} z_{N,e}(t) + \mathscr{G}_{N,2} e(t), \qquad z_{N,e}(0) = z_{N,0} = \pi_N z_0, \text{ for all } z_0 \in Z,\\ u(t) &= F_N z_{N,e}(t)\end{aligned}$$

with $e_N(t) := y_N(t) - y_{ref}(t)$, then the closed loop system

$$\begin{Bmatrix} \dot{z}_{N,cl}(t) \\ \dot{z}_{N,cl,e}(t) \end{Bmatrix} = \underbrace{\begin{bmatrix} A_N & B_N F_N \\ \mathscr{G}_{N,2} C_N & \mathscr{G}_{N,1} \end{bmatrix}}_{A_{N,e}} \begin{Bmatrix} z_{N,cl}(t) \\ z_{N,cl,e}(t) \end{Bmatrix}$$

is stable, and

$$\lim_{t\to\infty} ||e_N(t)|| = 0, \tag{3.3}$$

for all $z_{N,0} = \pi_N z_0 \in Z_N$ and $w_0 \in \mathbb{R}^q$. Furthermore, if the matrices $(A_N, B_N, C_N, P_N, Q_0)$ are perturbed to $\left(\tilde{A}_N, \tilde{B}_N, \tilde{C}_N, \tilde{P}_N, \tilde{Q}_0\right)$ such that the closed system remain exponentially stable, then for all initial conditions,

$$\lim_{t\to\infty} ||e_N(t)|| = 0. \tag{3.4}$$

3.2 Robust Regulator Design

In this section, we go over the individual steps involved in the design of robust controller. The controller design given in this section is inspired from the infinite dimensional controller design given in Section 5 in [8]. Refer [8] for the more general controller design discussion. Note, Assumptions 2.1 and 2.2 still hold in the following controller design. Additionally, we assume the following conditions.

Assumption 3.1. The algebraic multiplicity of the eigenvalues of the matrix S is one.

Assumption 3.2. The spectrums of A_N and S do not intersect $(\rho(A_N) \cap \rho(S) = \emptyset)$.

Assumption 3.3. The system defined by A_N, B_N, C_N is SISO (single input single output).

1. Define the matrices $\mathscr{G}_{N,1}, \mathscr{G}_{N,2}$ and F_N as

$$\mathscr{G}_{N,1} = \begin{bmatrix} G_1 & G_2 C_N \\ 0 & A_N + LC_N \end{bmatrix}, \qquad \mathscr{G}_{N,2} = \begin{bmatrix} G_2 \\ L \end{bmatrix},$$
$$F_N = \begin{bmatrix} K_1 & 0 \end{bmatrix}.$$

   In the above definition, the matrix G_1 is defined as

$$G_1 = \begin{bmatrix} i\omega_1 & & \\ & \ddots & \\ & & i\omega_q \end{bmatrix},$$

   where $i\omega_1, \ldots, i\omega_q$ are the eigenvalues of the matrix S. Choose the matrix K_1 as

$$K_1 = \begin{bmatrix} K_1^1 & \ldots & K_1^q \end{bmatrix},$$

where $K_1^k := P(i\omega_k)^{-1}$ with $P(\lambda) := C_N\mathscr{R}(\lambda, A_N)B_N$. Note, the matrix $\mathscr{R}(\lambda, A_N)$ denotes the resolvent of A_N and for finite dimensional cases, it can be expressed as $\mathscr{R}(\lambda, A_N) = (\lambda I - A_N)^{-1}$.

2. Define the matrix

$$H = \begin{bmatrix} H_1 & & \\ & \ddots & \\ & & H_q \end{bmatrix},$$

where $H_k = \mathscr{R}(i\omega_k, A_N)B_N K_1^k$.

Now, define the matrix $C_1 = C_N H$.

3. Choose the matrix G_2 such that $G_1 + G_2 C_1$ is Hurwitz.

Now, calculate the matrix L using the relation $L = HG_2$.

Lemma 3.1. *The matrix H solves the Sylvester equation $HG_1 = A_N H + B_N K_1$.*

Proof. The diagonal structure of matrices G_1 and H allows us to simplify the Sylvester equation to the form

$$H_k i\omega_k = A_N H_k + B_N K_1^k$$

for $k = 1, \ldots, q$. The above equation can be rewritten as

$$(i\omega_k - A_N) H_k = B_N K_1^k.$$

Thus, we get $H_k = (i\omega_k - A_N)^{-1} B_N K_1^k$, which is how we defined H_k during the controller design.

□

Lemma 3.2. *The pair* (C_1, G_1) *is detectable.*

Proof. The pair (C_1, G_1) is detectable if and only if there exists no eigenvector of G_1 that is orthogonal to all the rows of C_1 (Theorem 6.2-5 of [22]). Thus, the pair (C_1, G_1) is detectable if and only if $G_1 p = \lambda p$, $C_1 p = 0 \implies p = 0$. Since G_1 is a diagonal matrix with $i\omega_1, \dots, i\omega_r$ as its diagonal entries, the components of the vector p should satisfy $p^j = 0$ for $j \neq k$ and $p^k \neq 0$. Now consider $C_1 p$. We have

$$\begin{aligned} C_1 p &= C_N H p = C_N H_k p_k = C_N (i\omega_k - A_N)^{-1} B_N K_1^k p_k \\ &= P(i\omega^k) K_1^k p_k = p_k \neq 0. \end{aligned}$$

Thus, the pair C_1, G_1 is detectable. □

Theorem 3.3. *The closed loop system*

$$A_{N,e} = \begin{bmatrix} A_N & B_N F_N \\ \mathscr{G}_{N,2} C_N & \mathscr{G}_{N,1} \end{bmatrix}$$

is stable.

Proof. We substitute for $\mathscr{G}_{N,1}$ and $\mathscr{G}_{N,2}$ to express the matrix A_e in the form

$$A_{N,e} = \begin{bmatrix} A_N & B_N K_1 & 0 \\ G_2 C_N & G_1 & G_2 C_N \\ L C_N & 0 & A_N + L C_N \end{bmatrix}$$

To prove $A_{N,e}$ is stable, it is sufficient to prove the stability of the similarity transform $\hat{A}_{N,e} =$

$Q_e A_{N,e} Q_e^{-1}$, where

$$Q_e = Q_e^{-1} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -I & H & -I \end{bmatrix}.$$

We have

$$\begin{aligned}
\hat{A}_{N,e} &= \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -I & H & -I \end{bmatrix} \begin{bmatrix} A_N & B_N K_1 & 0 \\ G_2 C_N & G_1 & G_2 C_N \\ L C_N & 0 & A_N + L C_N \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -I & H & -I \end{bmatrix} \\
&= \begin{bmatrix} A_N & B_N K_1 & 0 \\ 0 & G_1 + G_2 C_N H & -G_2 C_N \\ 0 & -B_N K_1 + H G_1 + H G_2 C_N H - A_N H - L C_N H & -H G_2 C_N - A_N - L C_N \end{bmatrix} \\
&= \begin{bmatrix} A_N & B_N K_1 & 0 \\ 0 & G_1 + G_2 C_N H & -G_2 C_N \\ 0 & -B_N K_1 + H G_1 - A_N H & A_N \end{bmatrix}
\end{aligned}$$

From Lemma 3.1, it is clear that $-B_N K_1 + H G_1 - A_N H = 0$. Thus, we get

$$\hat{A}_{N,e} = \begin{bmatrix} A_N & B_N K_1 & 0 \\ 0 & G_1 + G_2 C_1 & -G_2 C_N \\ 0 & 0 & A_N \end{bmatrix}.$$

The stability of the block diagonal matrix $\hat{A}_{N,e}$ depends on the stability of A_N and $G_1 + G_2 C_1$. By assumption, the system matrix A_N is stable. Recall that we choose G_2 such that the matrix $G_1 + G_2 C_1$ is stable. Note, the detectability of the pair (C_1, G_1) (shown in Lemma 3.2) allows us to choose the required G_2. This proves that the closed loop system matrix $A_{N,e}$ is stable. □

3.3 Robust Controller Example

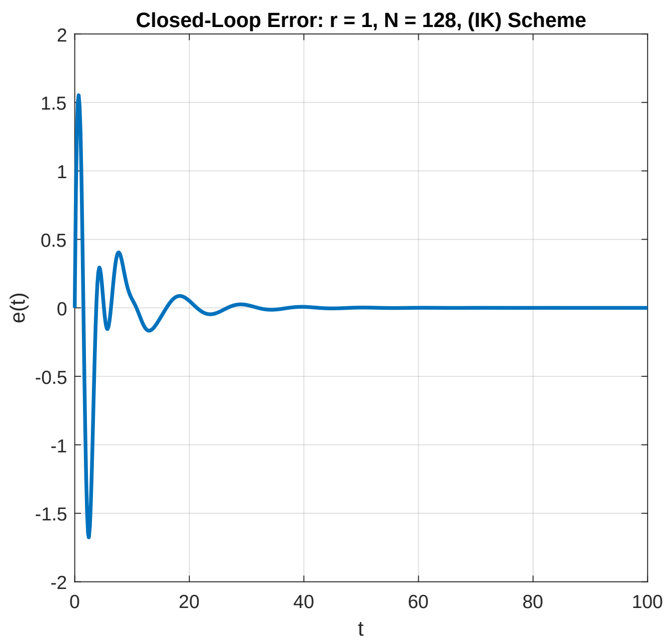

Figure 3.1: Error for $r = 1$: Robust Controller

Applying the robust regulation theory discussed above to the example given in Section 2.5, we generated a robust controller of the form (3.1)-(3.2) for the nominal value of the delay $r = 1$. In our design, we chose $G_2 = \begin{bmatrix} -1 & -1 & -1 \end{bmatrix}^\top$, which made the matrix $G_1 + G_2C_1$ Hurwitz. Figure 3.1 shows the error $e(t)$, and Figure 3.2 shows the outputs that generate this error. Although the robust controller drives the tracking error to zero, good tracking does not occur until $t > 40$ as compared to $t > 20$ for the classical output regulator as shown in Figure 2.1.

From the above analysis, we saw that the implementation of the robust regulator for this control problem reduced the rate at which error converges to zero. However, the advantages of the robust regulator become evident when the actual plant parameters are perturbed. To show this, we perturbed the delay r to 1.1 and 0.9. In Figure 3.3 we plot the tracking error for the controlled system with $r_{pert} = 1.1$. In Figure 3.4 we plot the same for $r_{pert} = 0.9$. In both these cases, the

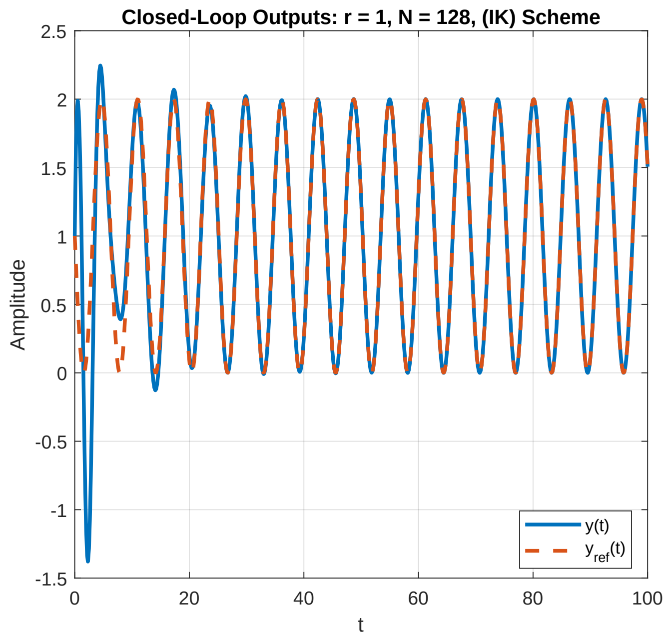

Figure 3.2: Outputs for $r = 1$: Robust Controller

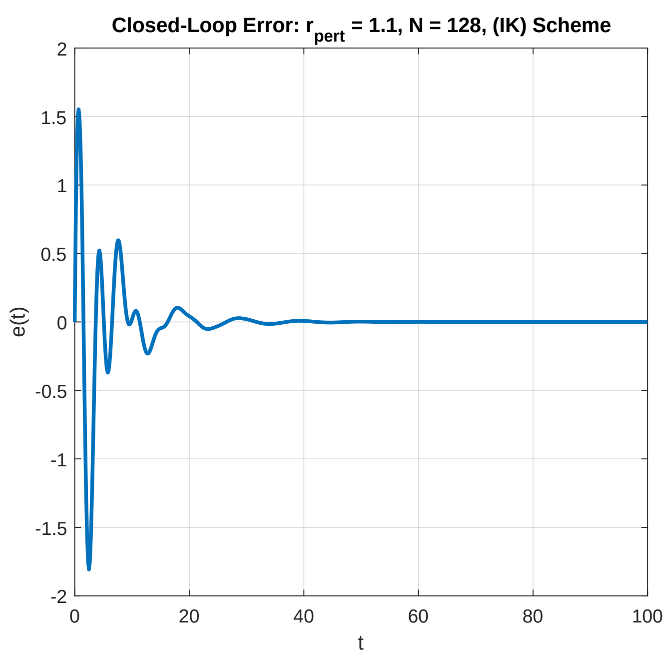

Figure 3.3: Error for $r = 1 + 0.1 = 1.1$: Robust Controller

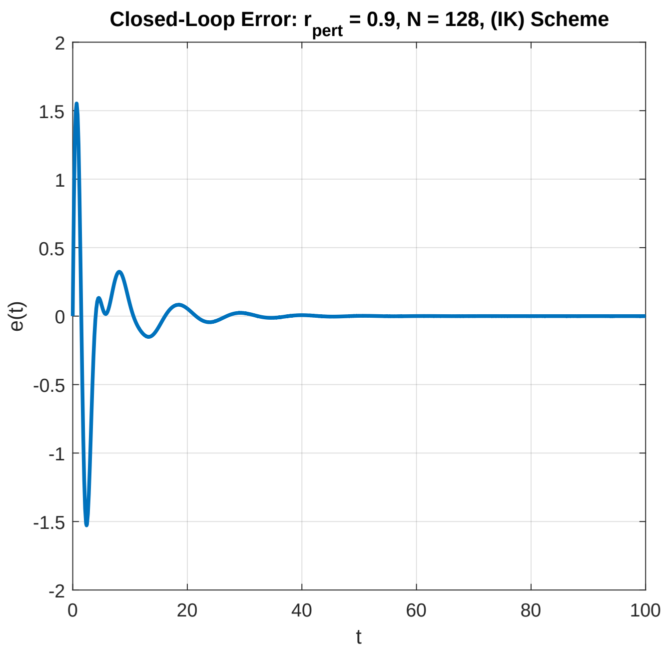

Figure 3.4: Error for $r = 1 - 0.1 = 0.9$: Robust Controller

robust controller drives the tracking error to zero. We also note that reference tracking occurs at approximately same time as $r = 1$ case. This was typical of the many numerical runs we conducted to test the method.

Chapter 4

Conclusion

In this study, we use a system defined by a delay differential equation to set up and formulate the standard distributed parameter regulator problem in order to establish convergence of numerical methods and to investigate the robustness of the output regulator control with respect to both numerical errors and time delays. We showed that forward convergence is sufficient to establish convergence of the gain operators. We also observed that the full information controller is not robust to:

(i) numerical errors and/or perturbations in certain gains, and

(ii) perturbations in the delay.

We showed that by applying the recent results on robust regulation of distributed parameters systems one could deal with both problems.

The delay equation is an infinite dimensional distributed parameter system and as long as the operator $\mathscr{A}$ defined by (2.10)-(2.11) generates an exponentially stable semigroup, the results in this paper hold. In Example 2.10, this is satisfied even for $a_0 = 0$ with $-\pi/2 < -\pi/4 = a_1 < 0$.

Under certain conditions (e.g., the lead matrix A_0 in equation (2.1) is stable) the methods in [12] can be applied directly to the (finite dimensional) delay system (see [13]). Moreover, this can be done without requiring an exogenous system to generate disturbances or the reference signal. However, the approach in [13] cannot be directly applied to Example 2.10 with $a_0 = 0$. We are currently working to modify the approach in [13] to deal with such cases.

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