LATERAL VIBRATION OF A BEAM CARRYING A CONCENTRATED MASS AT THE MID-POINT, INCLUDING THE EFFECT OF ROTATORY INERTIA

by

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TABLE OF CONTENTS

I. SUMMARY

A general solution has been obtained for the lateral vibrations of a simply supported beam with constant cross-section carrying a concentrated mass at the mid-point of span. The elementary beam theory including the rotatory inertia of the beam is utilized. The frequency equation involves the product of two terms, roots arising from one of these being associated with symmetric vibration modes and from the other, antisymmetric modes. The effect of the rotatory inertia of the beam on the roots of the frequency equations and on the normal mode shapes is investigated. The roots of the frequency equations are determined for the first ten modes for wide ranges of values of the significant parameters. These numerical results are depicted in tables and graphs. A study is made of the limiting values of the frequency roots for extreme values of the parameters.

II. NOMENCLATURE

A = cross-sectional area of the beam

E = Young's modulus of elasticity

I_b = moment of inertia of the cross-section area of the beam with respect to its neutral axis

I_m = moment of inertia of the mounted mass with respect to an axis parallel to the y-axis lying the neutral plane of the beam

K = $\dfrac{1}{2R(\frac{5}{8})^2}$

K_s = spring constant of the beam in resistance to the change of slope at the point considered. (Torque per unit radian).

M = bending moment

My = moment about y-axis

R = $\frac{m}{m_b}$ = mass ratio of the concentrated mass to the mass of the beam

T_n = time function or amplitude function for n^{th} mode.

V = shearing force

$W_n = X_n \cdot T_n$ = deflection function for the z - direction

X_n = the n^{th} normal function

a = $\sqrt{\dfrac{EI_b}{\rho A}}$

a_z = linear acceleration in z - direction

$k_n = \dfrac{2\beta_n}{\ell}$

l = span length of the beam

m = the concentrated mass mounted on the beam

$m_b = \rho A \ell$ = mass of the beam

p_n = the circular frequency of the n^{th} mode

$r = \sqrt{\frac{I_m}{m}}$ = radius of gyration of the concentrated mass with respect to an axis defined for I_m

$s = \sqrt{\frac{I_b}{A}}$ = radius of gyration of the beam cross-section with respect to an axis defined for I_b

t = time

w = lateral deflection of the beam in z-direction, function of x and t

x, y, z = Cartesian coordinates. xy-plane coincides with the neutral plane of the undeflected beam

α_n = parameter of the frequency equation characterizing the effect of the rotatory inertia of the beam for n^{th} mode.

α_y = angular acceleration about y-axis

β_n = root of the frequency equation for the n^{th} mode

θ = angle of rotation or slope of the deflection curve of the beam

λ_n = the root of the auxililary equation of differential equation for n^{th} mode

ρ = density of the beam

III. INTRODUCTION

The lateral vibration problem of a simply supported beam or shaft with constant cross-section carrying a concentrated mass at the mid-point is of practical importance in design considerations since such configurations frequently occur in structures and machines.

This problem has been studied in two manners. The first way was to treat the problem as for a single degree of freedom system by neglecting the mass of the beam.[1*] The second method included the mass of the beam and treated the configuration as a continuous elastic system.[2] & [3] In both treatments only the case of symmetrical vibrations of the beam and mass were analyzed.

Restricting this system to vibrate in symmetric modes simplified this problem and made unnecessary any consideration of the rotatory inertia of the mounted mass. But if this system is excited by a force applied anywhere on the system other than at a mid-span, or by a couple, antisymmetric vibration modes will be produced.

In this thesis, this problem is solved by starting from the general case, allowing the concentrated mass to rotate about an axis parallel to the y-axis, Figure (1), and to deflect in the z-direction. A general solution including both the rotatory inertia of the beam and of the concentrated mass is obtained by applying elementary beam theory[4] in which the effect of transverse shear deformation is neglected. The latter effect may be significant but was omitted both for simplicity and because it was desired to isolate the effect of rotatory inertia.

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* Numbers in the square parentheses refer to the references in Bibliography

The solutions which apply when the rotatory inertia of the beam is neglected are deduced from the general solutions. A simplified relation is derived for the lowest antisymmetric mode frequency. The antisymmetric mode frequencies for two extreme cases of mounted mass are discussed. Roots of the frequency equations are worked out and plotted as curves for certain ranges of parameters involving the ratio of the concentrated mass to the mass of the beam and certain moments of inertia.

IV. ANALYSIS OF FREE VIBRATIONS

The reference coordinates for this system are oriented as shown in Figure (1). The xy-plane coincides with the neutral plane of the undeflected beam. Consider the whole beam as two spans each of length $\frac{\ell}{2}$.

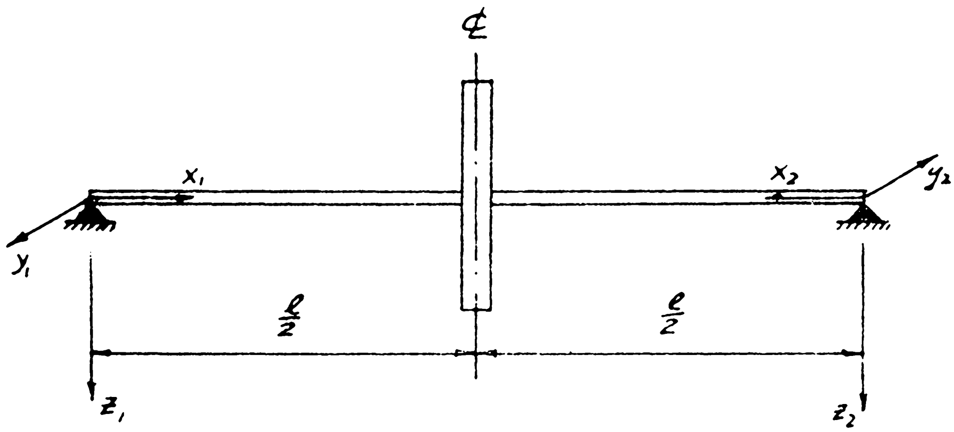

Fig. (1)

The following beam sign convention is adopted in the analysis. The positive values of bending moment M, shear force V, deflection w and rotation θ of the beam are shown in Figure (2).

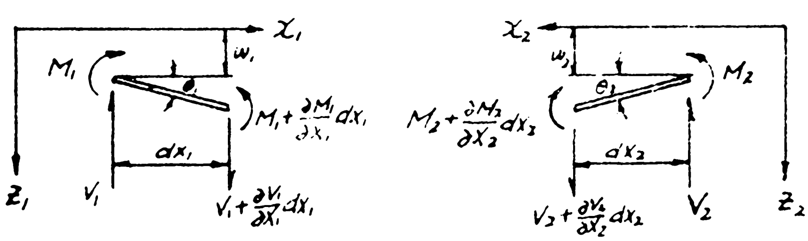

Fig. (2)

The differential equation for the lateral vibration of this system including the rotatory inertia of the beam and neglecting the effect of shear deformation is as follows[4]:

$$a^2 \frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} - s^2 \frac{\partial^4 w}{\partial x^2 \partial t^2} = 0 \qquad 0 \leq x \leq \frac{\ell}{2} \tag{1}$$

where $a^2 = \frac{EI_b}{\rho A}$ (2)

and $s = \sqrt{\frac{I_b}{A}}$ is the radius of gyration of the beam cross section with respect to its neutral axis. The use of the symbols w and x without subscripts denotes that the relations in which they appear apply to both x_1 - and x_2 - coordinate systems.

The boundary conditions at both ends of the beam are that the deflections and moments vanish.

i.e. $$w\Big]_{x=0} = 0 \tag{3}$$

$$-EI \frac{\partial^2 w}{\partial x^2}\Big]_{x=0} = 0 \tag{4}$$

The conditions of continuity, considering the whole beam as two spans each of length $\frac{\ell}{2}$, are that the deflections are equal and slopes are equal in magnitudes and opposite in sign at the mid-point.

i.e. $$w_1\Big]_{x_1=\frac{\ell}{2}} = w_2\Big]_{x_2=\frac{\ell}{2}} \tag{5}$$

$$\frac{\partial w_1}{\partial x_1}\Big]_{x_1=\frac{\ell}{2}} = -\frac{\partial w_2}{\partial x_2}\Big]_{x_2=\frac{\ell}{2}} \tag{6}$$

By taking the free body of the concentrated mass at the mid-point as shown in Figure (3), the equation of motion for the z-direction is found to be:

$$\sum F_z = ma_z$$

i.e.
$$\left[EI_b \frac{\partial^3 w_1}{\partial x_1^3} + EI_b \frac{\partial^3 w_2}{\partial x_2^3}\right]_{x=\frac{\ell}{2}} = \left[m \frac{\partial^2 w_1}{\partial t^2} + \rho I_b \frac{\partial^3 w_1}{\partial x_1 \partial t^2} + \rho I_b \frac{\partial^3 w_2}{\partial x_2 \partial t^2}\right]_{x=\frac{\ell}{2}}$$

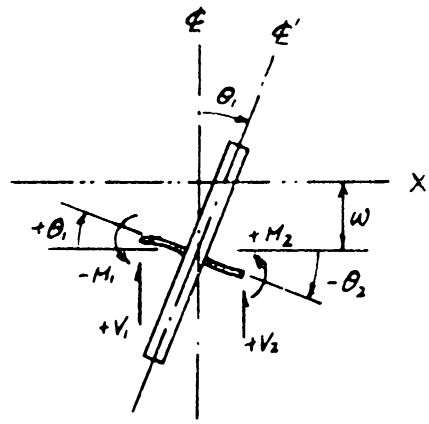

Fig. (3)

But by the continuity of slopes at this point, equation (6), one has

$$\left[\frac{\partial w_1}{\partial x_1} + \frac{\partial w_2}{\partial x_2}\right]_{x=\frac{\ell}{2}} = 0$$

This relation reduces the previous equation to

$$\left[EI_b \frac{\partial^3 w_1}{\partial x_1^3} + EI_b \frac{\partial^3 w_1}{\partial x_2^3}\right]_{x=\frac{\ell}{2}} = m \frac{\partial^2 w_1}{\partial t^2}\Big]_{x_1=\frac{\ell}{2}} \qquad (7)$$

Similarly, the equation for rotatory motion of the concentrated mass about an axis parallel to y-axis lying the neutral plane of the beam is:

$$\sum M_y = I_m \alpha_y$$

i.e. $$\left[EI_b \frac{\partial^2 W_1}{\partial X_1^2} - EI_b \frac{\partial^2 W_2}{\partial X_2^2}\right]_{X=\frac{\ell}{2}} = - m r^2 \frac{\partial^3 W_1}{\partial t^2 \partial X_1}\Bigg]_{X_1=\frac{\ell}{2}} \tag{8}$$

where m is the concentrated mass and r is its radius of gyration with respect to the y-axis defined above.

Assuming a separation of variables of the form

$$W_n = X_n(x) \cdot T_n(t),$$

one obtains

$$X_n^{IV} + k_n^4 s^2 X_n'' + k_n^4 X_n = 0 \tag{9}$$

$$\ddot{T}_n + T_n p_n^2 = 0 \tag{10}$$

where $$k_n^2 = \frac{p_n}{a}. \tag{11}$$

The roots of the auxiliary equation of equation (9) are

$$\lambda_n = \pm \sqrt{\pm\sqrt{\frac{k_n^4 s^4}{4} + 1} - \frac{k_n^2 s^2}{2}}.$$

Let $$\alpha_n = \sqrt{\sqrt{\frac{k_n^4 s^4}{4} + 1} - \frac{k_n^2 s^2}{2}}, \tag{12}$$

then $$i\sqrt{\sqrt{\frac{k_n^4 S^4}{4}+1}+\frac{k_n^2 S^2}{2}} = i\frac{1}{\alpha_n}.$$

Hence $$\lambda_n = \pm \alpha_n k_n \quad \text{and} \quad \pm i\frac{k_n}{\alpha_n}.$$

Considering this α_n as a parameter, one obtains the general solution of equations (9) and (10) to be:

$$X_n = C_n \cosh \alpha_n k_n x + D_n \sinh \alpha_n k_n x + F_n \cos \frac{k_n}{\alpha_n} x + H_n \sin \frac{k_n}{\alpha_n} x \tag{13}$$

$$T_n = A_n \cos p_n t + B_n \sin p_n t \tag{14}$$

where A_n, B_n, C_n, D_n, F_n and H_n are arbitrary constants.

By the boundary conditions of equations (3) and (4), it is necessary that $C_n = F_n = 0$. The general solution (13) is thus reduced to

$$X_n = D_n \sinh \alpha_n k_n x + H_n \sin \frac{k_n}{\alpha_n} x$$

which represents the form of the following two characteristic functions for span 1 and 2, respectively.

$$X_{1n} = D_{1n} \sinh \alpha_n k_n x_1 + H_{1n} \sin \frac{k_n}{\alpha_n} x_1$$

$$X_{2n} = D_{2n} \sinh \alpha_n k_n x_2 + H_{2n} \sin \frac{k_n}{\alpha_n} x_2 \tag{15}$$

Using these functions in conjunction with the continuity equations (5) and (6), and letting

$$\frac{k_n \ell}{2} = \beta_n \tag{16}$$

$$\frac{m}{\rho A \ell} = \frac{m}{m_b} = R \tag{17}$$

one obtains the following relations

$$(D_{1n} - D_{2n}) \sinh \alpha_n \beta_n + (H_{1n} - H_{2n}) \sin \frac{\beta_n}{\alpha_n} = 0 \tag{18}$$

$$\alpha_n (D_{1n} + D_{2n}) \cosh \alpha_n \beta_n + \frac{1}{\alpha_n} (H_{1n} + H_{2n}) \cos \frac{\beta_n}{\alpha_n} = 0 \tag{19}$$

By the equations of motions (7) and (8), the following relations may be obtained:

$$\alpha_n^3 (D_{1n} + D_{2n}) \cosh \alpha_n \beta_n - \frac{1}{\alpha_n^3} (H_{1n} + H_{2n}) \cos \frac{\beta_n}{\alpha_n}$$
$$= -2R\beta_n (D_{1n} \sinh \alpha_n \beta_n + D_{1n} \sin \frac{\beta_n}{\alpha_n}) \tag{20}$$

$$\alpha_n^2 (D_{1n} - D_{2n}) \sinh \alpha_n \beta_n - \frac{1}{\alpha_n^2} (H_{1n} - H_{2n}) \sin \frac{\beta_n}{\alpha_n}$$
$$= 8R(\frac{r}{\ell})^2 \beta_n^3 (\alpha_n D_{1n} \cosh \alpha_n \beta_n + \frac{1}{\alpha_n} H_{1n} \cos \frac{\beta_n}{\alpha_n}) \tag{21}$$

From the above four equations, the frequency equation is obtained and has the following form:

$$\left[R\beta_n(\alpha_n \tanh \alpha_n \beta_n - \frac{1}{\alpha_n} \tan \frac{\beta_n}{\alpha_n}) - (\frac{1}{\alpha_n^2} + \alpha_n^2)\right]$$

$$\left[4R(\frac{r}{\ell})^2 \beta_n^3 (\alpha_n \tan \frac{\beta_n}{\alpha_n} - \frac{1}{\alpha_n} \tanh \alpha_n \beta_n) - (\frac{1}{\alpha_n^2} + \alpha_n^2) \tan \frac{\beta_n}{\alpha_n} \tanh \alpha_n \beta_n\right] = 0 \tag{22}$$

It is possible to separate this equation into two parts, each of which may be equated to zero. The first part is

$$R\beta_n(\alpha_n \tanh \alpha_n \beta_n - \frac{1}{\alpha_n} \tan \frac{\beta_n}{\alpha_n}) - (\frac{1}{\alpha_n^2} + \alpha_n^2) = 0 \tag{23}$$

If the second part is divided by tanh $\alpha_n \beta_n \cdot \tan \frac{\beta_n}{\alpha_n}$, one obtains

$$4R\left(\frac{r}{\ell}\right)^2 \beta_n^3 \left(\alpha_n \coth \alpha_n \beta_n - \frac{1}{\alpha_n} \cot \frac{\beta_n}{\alpha_n}\right) - \left(\frac{1}{\alpha_n^2} + \alpha_n^2\right) = 0 \tag{24}$$

The characteristic functions X and X may be expressed as:

$$X_{1n} = D_{1n}\left(\sinh \alpha_n \beta_n x_1 + \frac{H_{1n}}{D_{1n}} \sin \frac{\beta_n}{\alpha_n} x_1\right)$$

$$X_{2n} = D_{2n}\left(\sinh \alpha_n \beta_n x_2 + \frac{H_{2n}}{D_{2n}} \sin \frac{\beta_n}{\alpha_n} x_2\right) \tag{25}$$

From equations (18) to (21), one obtains the following ratios

$$\frac{H_{1n}}{D_{1n}} = \frac{R\beta_n \sin \frac{\beta_n}{\alpha_n} \sinh \alpha_n \beta_n - 4R\left(\frac{r}{\ell}\right)^2 \beta_n^3 \cosh \alpha_n \beta_n \cos \frac{\beta_n}{\alpha_n}}{\frac{1}{\alpha_n}\left(\frac{1}{\alpha_n^2} + \alpha_n^2\right) \sin \frac{\beta_n}{\alpha_n} \cos \frac{\beta_n}{\alpha_n} + 4R\left(\frac{r}{\ell}\right)^2 \beta_n^3 \frac{1}{\alpha_n^2} \cos^2 \frac{\beta_n}{\alpha_n} - R\beta_n \sin^2 \frac{\beta_n}{\alpha_n}} \tag{26}$$

$$\frac{D_{2n}}{D_{1n}} = 1 - 8R\left(\frac{r}{\ell}\right)^2 \beta_n^3 \left[\frac{\alpha_n^3}{1+\alpha_n^4} \coth \alpha_n \beta_n - \frac{H_{1n}}{D_{1n}} \frac{\alpha_n}{1+\alpha_n^4} \frac{\cos \frac{\beta_n}{\alpha_n}}{\sinh \alpha_n \beta_n}\right] \tag{27}$$

Equation (26) may be simplified by use of the frequency equation (23); the result is:

$$\frac{H_{1n}}{D_{1n}} = -\alpha_n^2 \frac{\cosh \alpha_n \beta_n}{\cos \frac{\beta_n}{\alpha_n}} . \tag{28}$$

The same result for $\frac{H_{2n}}{D_{2n}}$ may be obtained in a similar manner.

i.e.

$$\frac{H_{2n}}{D_{2n}} = -\alpha_n^2 \frac{\cosh \alpha_n \beta_n}{\cos \frac{\beta_n}{\alpha_n}} .$$

Substituting the ratio of equation (28) into equation (27), one obtains

$$\frac{D_{2n}}{D_{1n}} = 1$$

Substituting these ratios in equation (25), one finds

$$X_{1n} = D_{1n}\left(\sinh \frac{2\alpha_n\beta_n}{\ell} x_1 - \alpha_n^2 \frac{\cosh \alpha_n\beta_n}{\cos \frac{\beta_n}{\alpha_n}} \sin \frac{2\beta_n}{\ell\alpha_n} x_1\right) \qquad 0 \leq x_1 \leq \frac{\ell}{2}$$

$$X_{2n} = D_{1n}\left(\sinh \frac{2\alpha_n\beta_n}{\ell} x_2 - \alpha_n^2 \frac{\cosh \alpha_n\beta_n}{\cos \frac{\beta_n}{\alpha_n}} \sin \frac{2\beta_n}{\ell\alpha_n} x_2\right) \qquad 0 \leq x_2 \leq \frac{\ell}{2} \tag{29}$$

Thus the characteristic functions resulting from the use of frequency equation (23) are symmetric with respect to the plane $x = \frac{\ell}{2}$; this equation will therefore be termed the frequency equation for symmetric modes.

If instead, one uses frequency equation (24) to reduce equation (26), one obtains the ratios

$$\frac{H_{1n}}{D_{1n}} = \frac{H_{2n}}{D_{2n}} = - \frac{\sinh \alpha_n\beta_n}{\sin \frac{\beta_n}{\alpha_n}}$$

$$\frac{D_{2n}}{D_{1n}} = -1 .$$

From these ratios, one finds the characteristic functions to be

$$X_{1n} = D_{1n}\left(\sinh \frac{2\alpha_n\beta_n}{\ell} x_1 - \frac{\sinh \alpha_n\beta_n}{\sin \frac{\beta_n}{\alpha_n}} \sin \frac{2\beta_n}{\ell\alpha_n} x_1\right) \qquad 0 \leq x_1 \leq \frac{\ell}{2}$$

$$X_{2n} = -D_{1n}\left(\sinh \frac{2\alpha_n\beta_n}{\ell} x_2 - \frac{\sinh \alpha_n\beta_n}{\sin \frac{\beta_n}{\alpha_n}} \sin \frac{2\beta_n}{\ell\alpha_n} x_2\right) \qquad 0 \leq x_2 \leq \frac{\ell}{2} \tag{30}$$

These eigen-functions all have a nodal point at $x = \frac{\ell}{2}$ and are symmetric with respect to this point; these modes, which are associated with frequency equation (24), will henceforth be referred to as antisymmetric modes.

The determination of the roots of the frequency equations will be discussed in details in the next chapter. However, it should be noted at this point that the roots of equations (23) and (24) when arranged in ascending order arise alternately from equation (23) and (24) with the lowest, or fundamental root being derived from equation (23). Thus, the symmetric modes will be associated with n = 1,3,5,... and the antisymmetric modes with n = 2,4,6, ...

Finally, the solution for the deflection is found to be:

$$W_1 = \sum_{1,3,5}^{\infty} (A_n \cos p_n t + B_n \sin p_n t)\left(\sinh \frac{2\alpha_n \beta_n}{\ell} x_1 - \alpha_n^2 \frac{\cosh \alpha_n \beta_n}{\cos \frac{\beta_n}{\alpha_n}} \sin \frac{2\beta_n}{\ell \alpha_n} x_1\right)$$

$$+ \sum_{2,4,6}^{\infty} (A_n \cos p_n t + B_n \sin p_n t)\left(\sinh \frac{2\alpha_n \beta_n}{\ell} x_1 - \frac{\sinh \alpha_n \beta_n}{\sin \frac{\beta_n}{\alpha_n}} \sin \frac{2\beta_n}{\ell \alpha_n} x_1\right) \tag{31}$$

where A_n and B_n are arbitrary constants to be determined from the initial conditions. The expression for W_2 will be similar to that obtained for W_1 except for the sign of the even terms which will be negative.

By substituting the k_n value defined by equation (16) into equation (12), the parameter α_n becomes

$$\alpha_n = \sqrt{\sqrt{4\left(\frac{s}{\ell}\right)^4 \beta_n^4 + 1} - 2\left(\frac{s}{\ell}\right)^2 \beta_n^2} \tag{32}$$

If one neglects the effect of the rotatory inertia of the beam by setting s = 0, one finds, from equation (32), that $\alpha_n = 1$.

Substituting this value into equations (23), (24), (29), (30) and (31) respectively, one obtains the following solutions which apply when the rotatory inertia of the beam is neglected (Bernoulli-Euler theory [5]). The frequency equation, for symmetric modes is

$$R\beta_n(\tan\beta_n - \tanh\beta_n) - 2 = 0 \tag{33}$$

and for antisymmetric modes is

$$2R\left(\frac{r}{\ell}\right)^2\beta_n^3(\coth\beta_n - \cot\beta_n) - 1 = 0 \tag{34}$$

The normal mode equations for the symmetric case are

$$X_{1n} = D_{1n}\left(\sin\frac{2\beta_n}{\ell}x_1 - \frac{\cosh\beta_n}{\cos\beta_n}\sin\frac{2\beta_n}{\ell}x_1\right) \qquad 0 \le x_1 \le \frac{\ell}{2}$$

$$X_{2n} = D_{1n}\left(\sin\frac{2\beta_n}{\ell}x_2 - \frac{\cosh\beta_n}{\cos\beta_n}\sin\frac{2\beta_n}{\ell}x_2\right) \qquad 0 \le x_2 \le \frac{\ell}{2} \tag{35}$$

and for the antisymmetric case are

$$X_{1n} = D_{1n}\left(\sinh\frac{2\beta_n}{\ell}x_1 - \frac{\sinh\beta_n}{\sin\beta_n}\sin\frac{2\beta_n}{\ell}x_1\right) \qquad 0 \le x_1 \le \frac{\ell}{2}$$

$$X_{2n} = -D_{1n}\left(\sinh\frac{2\beta_n}{\ell}x_2 - \frac{\sinh\beta_n}{\sin\beta_n}\sin\frac{2\beta_n}{\ell}x_2\right) \qquad 0 \le x_2 \le \frac{\ell}{2} \tag{36}$$

The deflection curve function is

$$W_1 = \sum_{1,3,5}^{\infty}(A_n\cos p_nt + B_n\sin p_nt)\left(\sinh\frac{2\beta_n}{\ell}x_1 - \frac{\cosh\beta_n}{\cos\beta_n}\sin\frac{2\beta_n}{\ell}x_1\right)$$

$$+\sum_{2,4,6}^{\infty}(A_n\cos p_nt + B_n\sin p_nt)\left(\sin\frac{2\beta_n}{\ell}x_1 - \frac{\sinh\beta_n}{\sin\beta_n}\sin\frac{2\beta_n}{\ell}x_1\right) \tag{37}$$

The expression for W will be similar to that for W except for the sign in front of the even terms which will be negative.

Equations (33) and (35) agree with the results that have been derived in the book by Von Karman & Biot [2] and in the paper by Professor Hoppmann [3].

The first six normal mode curves are plotted as shown by Figure (4) for R = 1 and $\frac{1}{2R(\frac{s}{\ell})^2} = K = 500$, based on equations (33) and (35) for odd modes and equations (34) and (36) for even modes.

From equation (32), we see that the influence of the rotatory inertia of the beam is dependent on the $\frac{s}{\ell}$ ratio, the "slenderness ratio" of the beam, and β_n, the roots of the frequency equation. For low frequencies and small $\frac{s}{\ell}$ ratios, this effect may be negligible. For higher frequencies, the effect will be greater. The fifth and sixth normal mode curves for $\frac{s}{\ell} = 0$ and $\frac{s}{\ell} = 0.05$ are plotted as shown in Figure (5) for R = 1 and K = 500. It can be seen from these curves by the change of the position of the nodal points that the effect of the rotatory inertia of the beam is greater for the antisymmetric modes than for the symmetric modes.

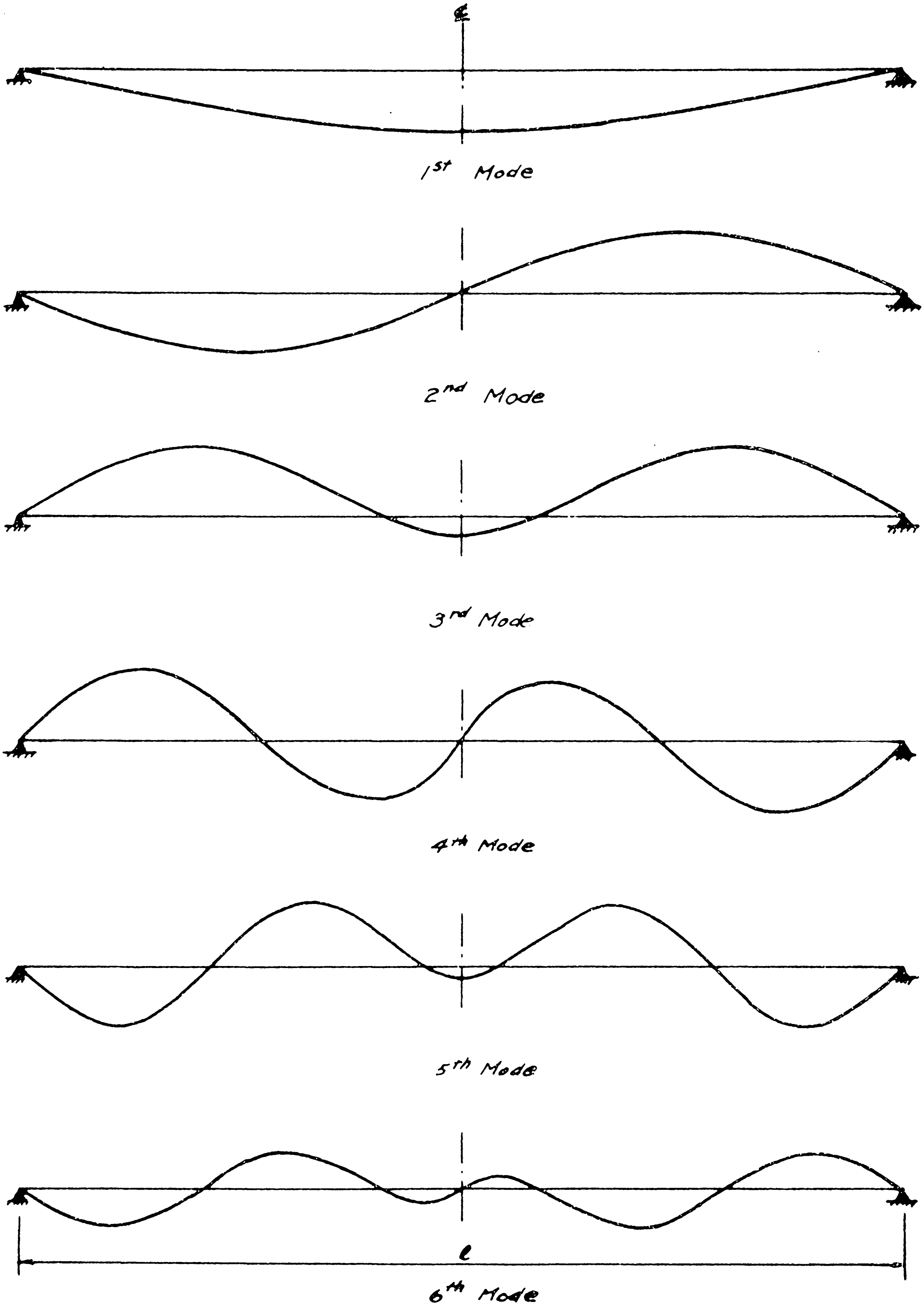

Fig (4) The First 6 Normal Mode Curves of $R=1$, $K=500$, $\frac{a}{l}=0$

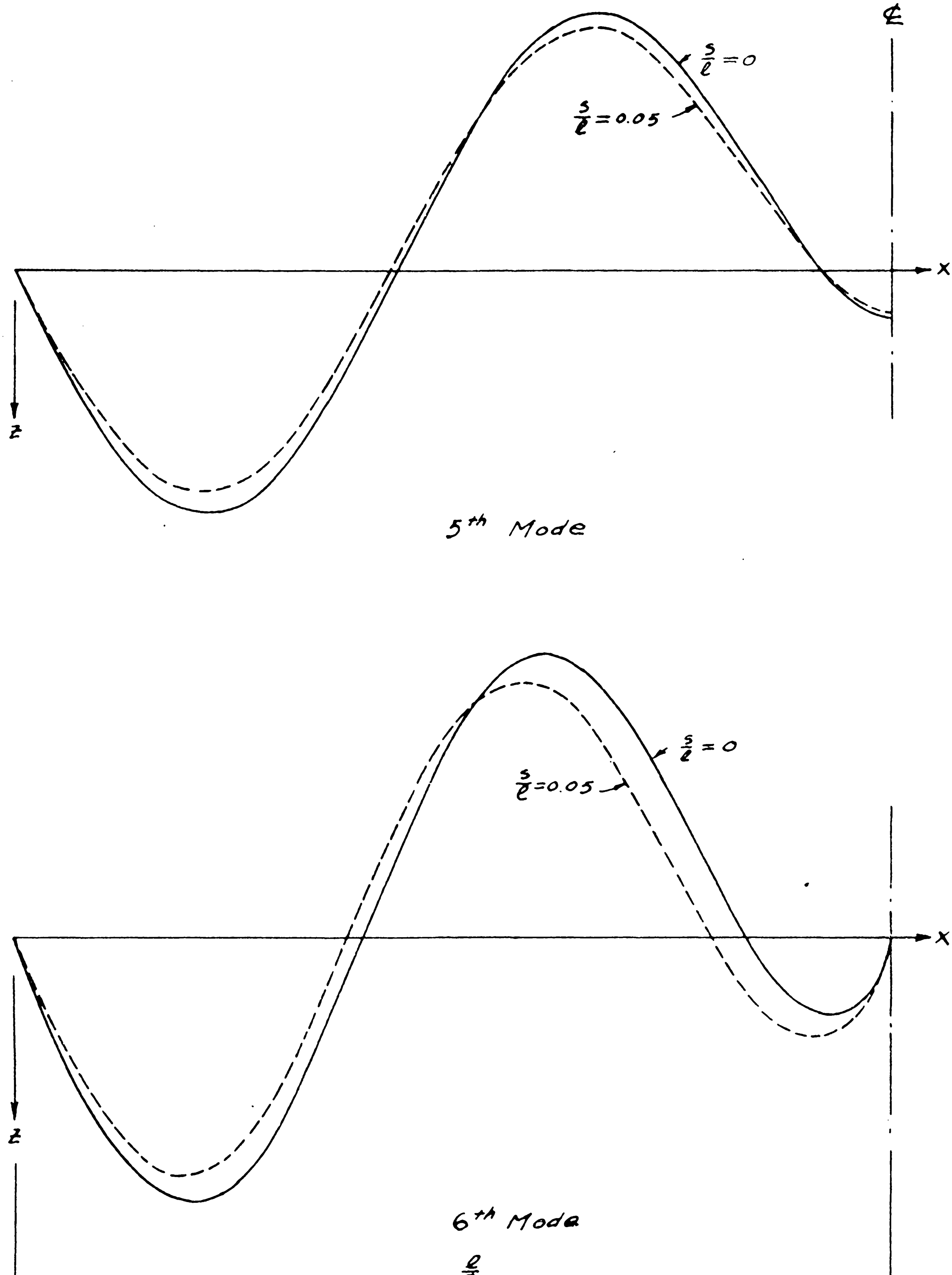

Fig. (5) The Comparison of the 5^{th} and 6^{th} Normal Mode Deflection Curves for $\frac{s}{l} = 0$ and $\frac{s}{l} = 0.05$

V. DISCUSSION OF FREQUENCY EQUATIONS

A discussion of the frequency equation associated with the symmetric mode vibrations has been given in the book by von Karman and Biot [2] . Here, particular attention will be paid to the frequency equation for antisymmetric modes. The rotatory inertia of the beam will be neglected in the following discussion.

1. Approximate Formula for Lowest Antisymmetric Mode Frequency

The frequency equation for antisymmetric modes, equation (34), may be written as

$$K = \beta_n^3 \left(\coth \beta_n - \cot \beta_n \right) \tag{38}$$

where $$K = \frac{1}{2R\left(\frac{r}{\ell}\right)^2} = \frac{m_b \ell^2}{2m\, r^2}$$

In order to obtain β_2 , the frequency number associated with the lowest antisymmetric mode, it will be convenient to replace the transcendental functions occurring in equation (38) by their power series expansions, which are

$$\coth \beta_2 = \frac{1}{\beta_2} + \frac{\beta_2}{3} - \frac{\beta_2^3}{45} + \frac{2\beta_2^6}{945} - \frac{\beta_2^7}{4725} + \frac{2\beta_2^9}{93555} - \cdots, \quad \beta_2^2 < \pi^2$$

$$\cot \beta_2 = \frac{1}{\beta_2} - \frac{\beta_2}{3} - \frac{\beta_2^3}{45} - \frac{2\beta_2^5}{945} - \frac{\beta_2^7}{4725} - \frac{2\beta_2^9}{93555} - \cdots . \quad \beta_2^2 < \pi^2$$

Substituting these series into equation (38), one finds

$$K = \beta_2^3 \left(\frac{2}{3}\beta_2 + \frac{4\beta_2^5}{945} + \frac{4\beta^9}{93555} + \cdots \right). \tag{39}$$

To obtain an explicit expression for , it is necessary to invert this series. To accomplish this, assume that

$$\beta_2^4 = bK + cK^2 + dK^3 + \cdots$$

and substitute this expression into equation (39). Equating the coefficients of the same powers of K, the constants b,c and d are found to be

$$b = \frac{3}{2}$$

$$c = -\frac{3}{2}\left(\frac{4}{945}b^2\right) = -\frac{1}{70}$$

$$d = -\frac{3}{2}\left(\frac{8bc}{945} - \frac{4}{93555}b^3\right) = \frac{13}{23100}$$

Consequently $$\beta_2^4 = \frac{3}{2}K - \frac{1}{70}K^2 + \frac{13}{23100}K^3 - \cdots \qquad (40)$$

If the mass of the beam m_b is small in comparison to the mounted mass m, then K is small and the value of β_2^4 is given with sufficient accuracy by the first term of the series of equation (40); i.e.,

$$\beta_2^4 \approx \frac{3}{2}K \qquad (41)$$

By equations (2), (11) and (16) which are $a^2 = \frac{EI_b}{\rho A}$, $k_n^2 = \frac{p_n}{a}$ and $\beta_n = \frac{k_n \ell}{2}$, respectively, one has

$$p_2 = a\left(\frac{2\beta_n}{\ell}\right)^2 = \sqrt{\frac{EI_b}{\rho A}} \cdot \frac{4}{\ell}\sqrt{\frac{3K}{2}}$$

or $$p_2 = \sqrt{\frac{12EI_b}{I_m \ell}} \qquad (42)$$

since $K = \frac{1}{2}\frac{\rho A \ell}{I_m}\ell^2$.

This result could also be obtained by considering a single degree of freedom system as shown in Figure (6).

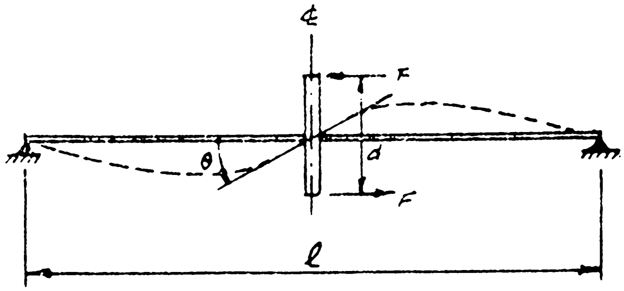

Fig. (6)

A torque Fd is applied on the mounted mass and then released. The differential equation of motion is

$$\frac{d^2\theta}{dt^2} + \frac{K_s}{I_m}\theta = 0$$

where I_m is the moment inertia of the mounted mass with respect to an axis parallel to y-axis intersecting with the elastic line of the beam and K is the spring constant of the beam in resistance to the change of the slope at the point where the mass is attached. If a unit torque is applied, i.e. Fd = 1, then the slope at the mid-point is

$$\theta = \frac{\ell}{12EI_b}$$

Therefore $$K_s = \frac{12EI_b}{\ell}$$

Hence, one has $$p_2 = \sqrt{\frac{K_s}{I_m}} = \sqrt{\frac{12EI_b}{I_m\ell}}$$

By contrast, the single degree of freedom approximation for the fundamental (symmetric) frequency [2] is

$$p_1 = \sqrt{\frac{48 E I_b}{m \ell^3}}$$

2. Antisymmetric Mode Frequencies for Two Extreme Cases of the Mounted Mass

There are two interesting cases associated with limiting values of the constant K. The case K = 0 may arise physically either by $m_b \rightarrow 0$ or by $m \rightarrow \infty$. The first possibility corresponds to the single degree of freedom system discussed in the last article. The second possibility will be investigated as Case A below. The other limiting value, $K = \infty$, may be achieved by either $m_b \rightarrow \infty$ or $m \rightarrow 0$. The first of these possibilities is of no physical interest and the second will be discussed below as Case B.

Case A: $m \rightarrow \infty$ Physically, this means that the mounted mass is extremely large in comparison to the mass of the beam, so that the mid-point of the beam is essentially fixed. Since K = 0, equation (38) becomes

$$\coth \beta_n = \cot \beta_n \quad (\text{for } \beta_n \neq 0)$$

which may also be written as

$$\tanh \beta_n = \tan \beta_n \tag{43}$$

This is the frequency equation for a beam simply supported at one end and built in at other end [5]. The roots of this equation are given with satisfactory accuracy by the expression

$$\beta_n = \left(\frac{n}{2} - \frac{3}{4} \right) \pi \qquad \text{for } n = 4, 6, 8, \ldots$$

The root β_2 is degenerate and approaches zero as $K \rightarrow 0$.

While this value agrees with the limiting value of β_2 given by equation (40), it should be noted that these results are obtained by a physically different method of approaching $K = 0$.

Case B: $m \rightarrow 0$ This means that there is no mass m on the beam. Then $K \rightarrow \infty$. If β_n is not very large, one finds, from equation (38),

$$\cot \beta_n = -\infty$$

Therefore $$\beta_n = \frac{n}{2} \pi \qquad \text{where } n = 2, 4, 6, \ldots$$

These are the roots of the frequency equation for the antisymmetric modes of a simply supported beam [6].

3. Numerical Results

The frequency roots of the first ten modes are worked out for practical reference, based on equations (33) and (38). The effect of the rotatory inertia of the beam is neglected. The mass ratios of R used are $\frac{1}{4}$, 1, 2, 3, 4 and 5. The K values used are 1, 3, 10, 50, 150, 500, 1500 and 5000. These roots are tabulated in Tables (1) and (2) and also are plotted as curves shown in Figure (7) and (8). The variation of β_n with the parameters R and K may be depicted more compactly by introduction of the functions

$$N_s = \beta_n - (n-1)\pi \qquad n = 1, 3, 5, \ldots$$

for symmetric modes, and

$$N_A = \beta_n - (\frac{n}{2} - 1)\pi \qquad n = 2, 4, 6, \ldots$$

for antisymmetric modes. The N_S and N_A values are plotted as Figure (9) and (10), respectively.

The roots, β_n for the 1st, 2nd, 5th, 6th, 9th and 10th modes have been obtained based on equations (23) and (24), which include the effect of the rotatory inertia of the beam. The latter effect was introduced by assuming $\frac{s}{\ell}$ = 0.05, which is considered to be the upper limit of this ratio for practical cases. These results are tabulated in Tables (3) and (4) and shown in Figure (7) and (8) by dotted lines. The roots for the 1st and 2nd modes are quite close to that for the case $\frac{s}{\ell}$ = 0. Hence, these lines coincide with the lines for $\frac{s}{\ell}$ = 0 to the scale used for these figures. It may be seen from these figures that, for a given $\frac{s}{\ell}$ ratio, the effect of rotatory inertia on the frequency increases with the mode number, the magnitude of the increase depending on the $\frac{s}{\ell}$ ratio.

TABLE (1) ROOTS (β_n) OF FREQUENCY EQUATION FOR SYMMETRIC MODES ($\frac{s}{\ell}=0$)

n \ K (β_n)	0	$\frac{1}{4}$	1	2	3	4	5	∞
1	1.571	1.419	1.191	1.048	0.963	0.904	0.860	0.785
3	4.712	4.363	4.120	4.037	4.003	3.981	3.975	3.927
5	7.854	7.406	7.207	7.134	7.113	7.103	7.096	7.069
7	10.996	10.470	10.297	10.256	10.242	10.234	10.229	10.210
9	14.137	13.575	13.421	13.387	13.376	13.370	13.367	13.352

TABLE (2) ROOTS (β_n) OF FREQUENCY EQUATION FOR ANTISYMMETRIC MODES ($\frac{s}{\ell}=0$)

n \ K (β_n)	0	1	3	10	50	150	500	1500	5000	∞
2	0	1.104	1.446	1.921	2.624	2.940	3.060	3.121	3.136	π
4	3.927	3.935	3.952	4.011	4.341	4.947	5.739	6.106	6.213	2π
6	7.069	7.070	7.073	7.083	7.142	7.303	7.877	8.747	9.256	3π
8	10.210	10.210	10.211	10.215	10.234	10.283	10.478	11.094	11.936	4π
10	13.352	13.352	13.352	13.354	13.362	13.384	13.465	13.738	14.666	5π

TABLE (3) FREQUENCY ROOTS FOR SYMMETRIC MODES ($\frac{s}{\ell}=0.05$)

n \ β_n \ R	0	$\frac{1}{4}$	1	2	3	4	5	∞
1	1.561	1.413	1.189	1.047	0.962	0.904	0.860	0.785
5	6.965	6.705	6.555	6.515	6.498	6.490	6.486	6.465
9	11.539	10.567	10.503	10.488	10.482	10.478	10.477	10.468

TABLE (4) FREQUENCY ROOTS FOR ANTISYMMETRIC MODES ($\frac{s}{\ell}=0.05$)

n \ β_n \ K	0	1	3	10	50	150	500	1500	5000	∞
2	0	1.104	1.445	1.917	2.597	2.888	3.013	3.050	3.064	3.069
6	6.465	6.467	6.470	6.482	6.547	6.723	7.247	7.753	8.023	8.040
10	10.469	10.470	10.470	10.472	10.484	10.515	10.633	10.950	11.328	11.511

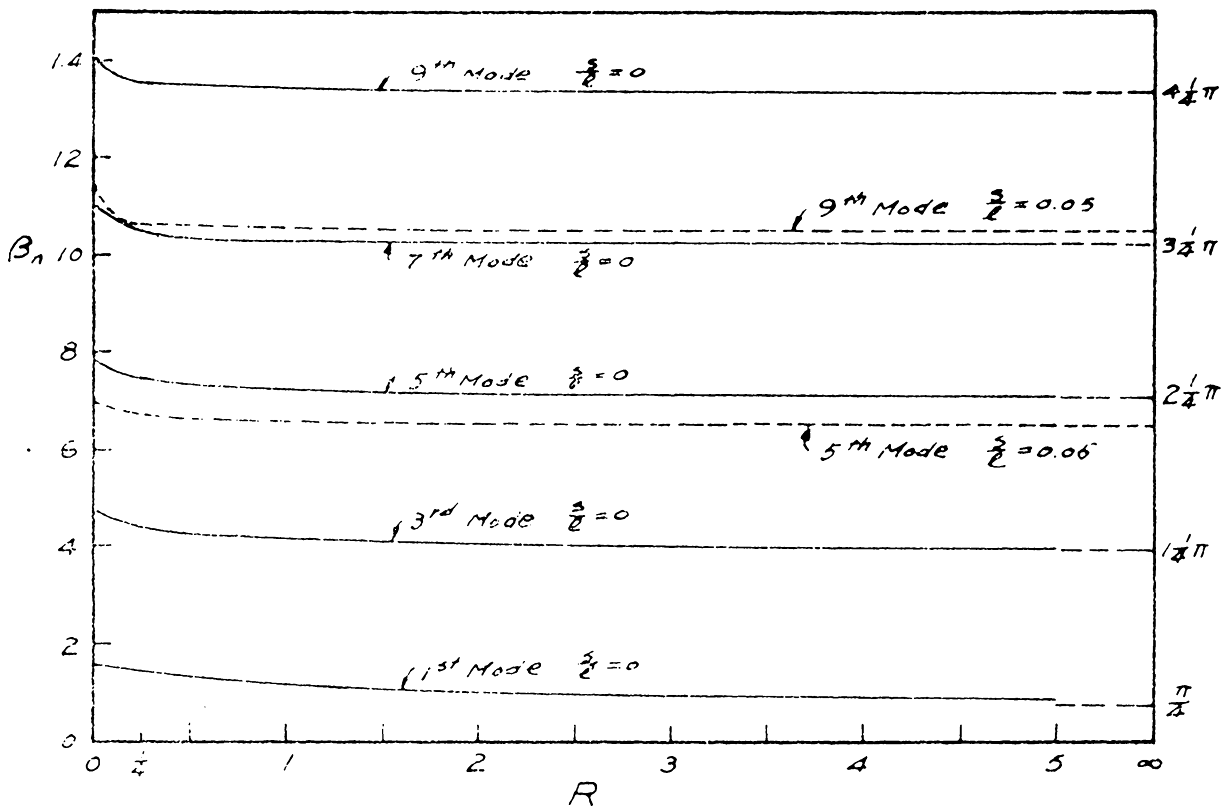

Fig. (7) β_n - R Curves For Symmetric Modes

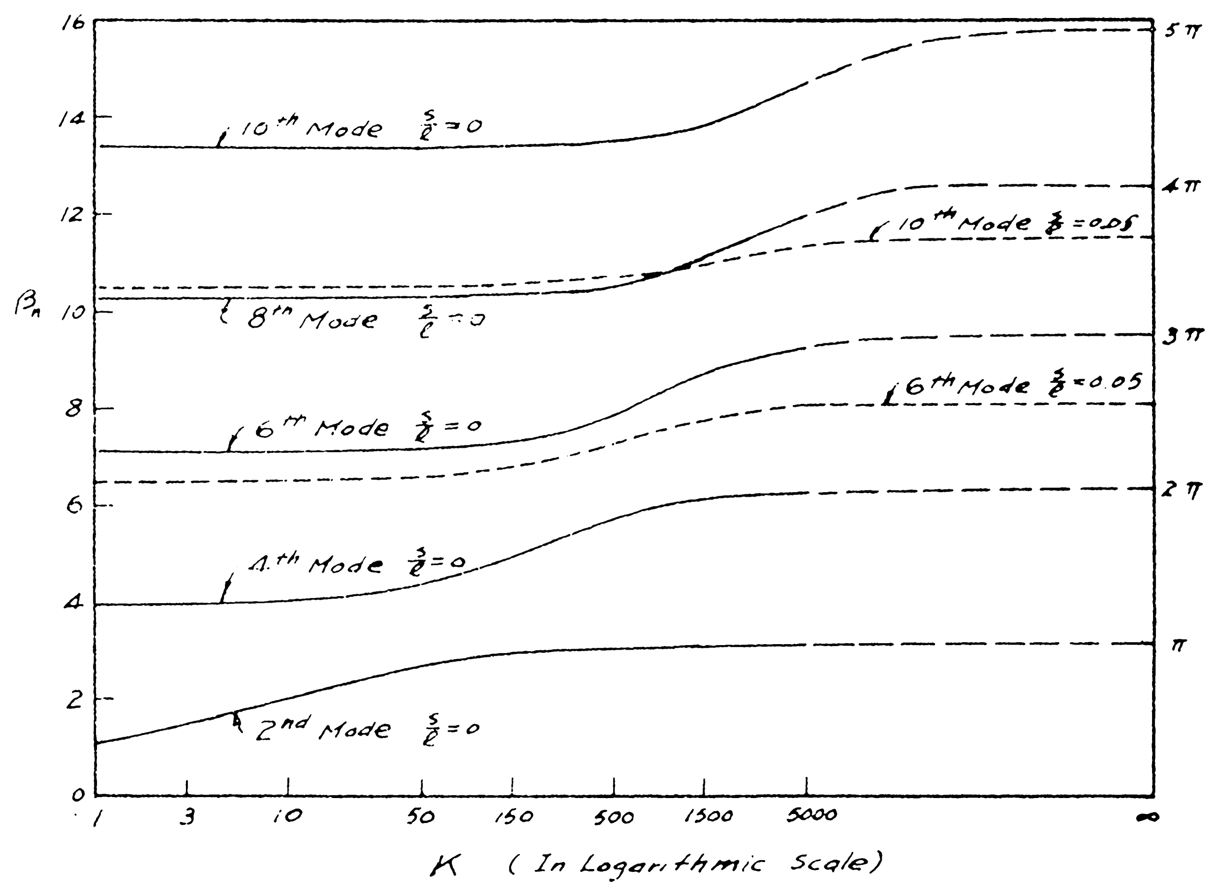

Fig. (8) β_n - K Curves For Antisymmetric Modes

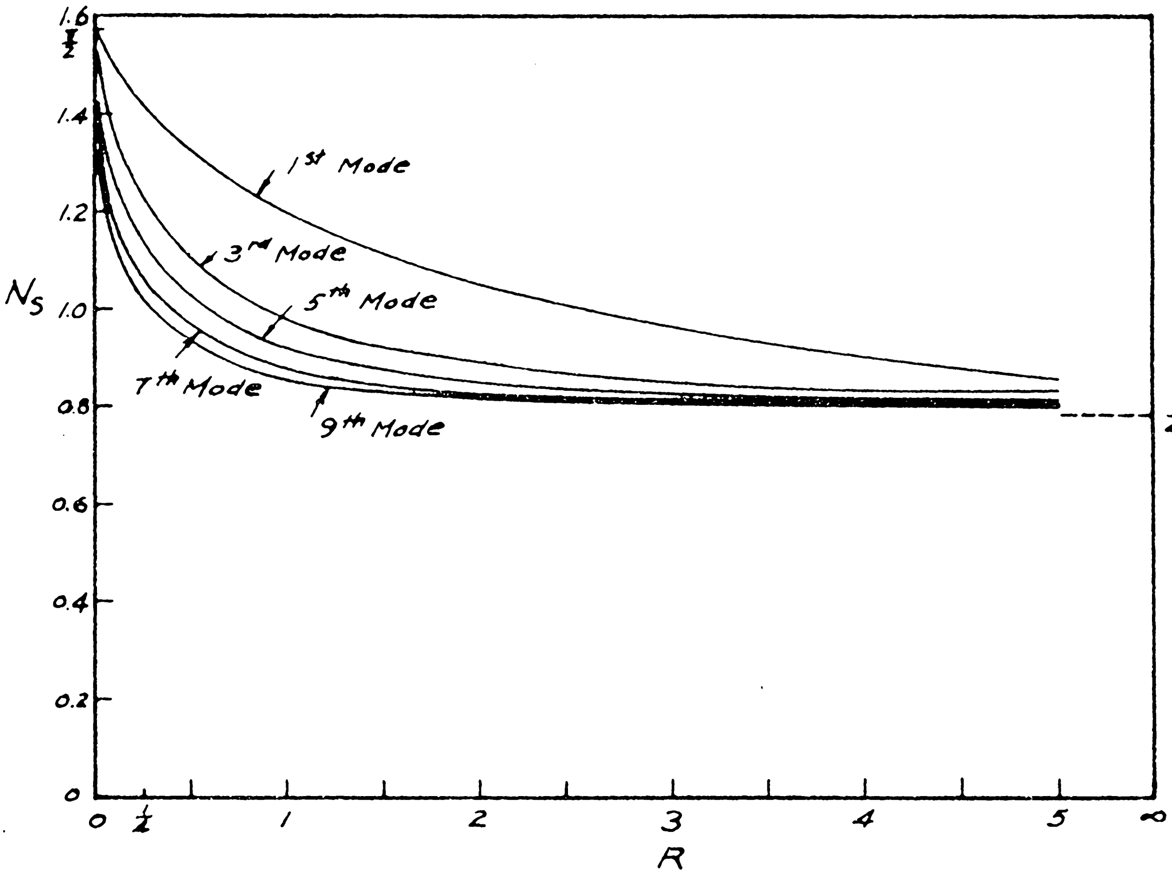

Fig. (9) $N_s - R$ Curves For Symmetric Modes

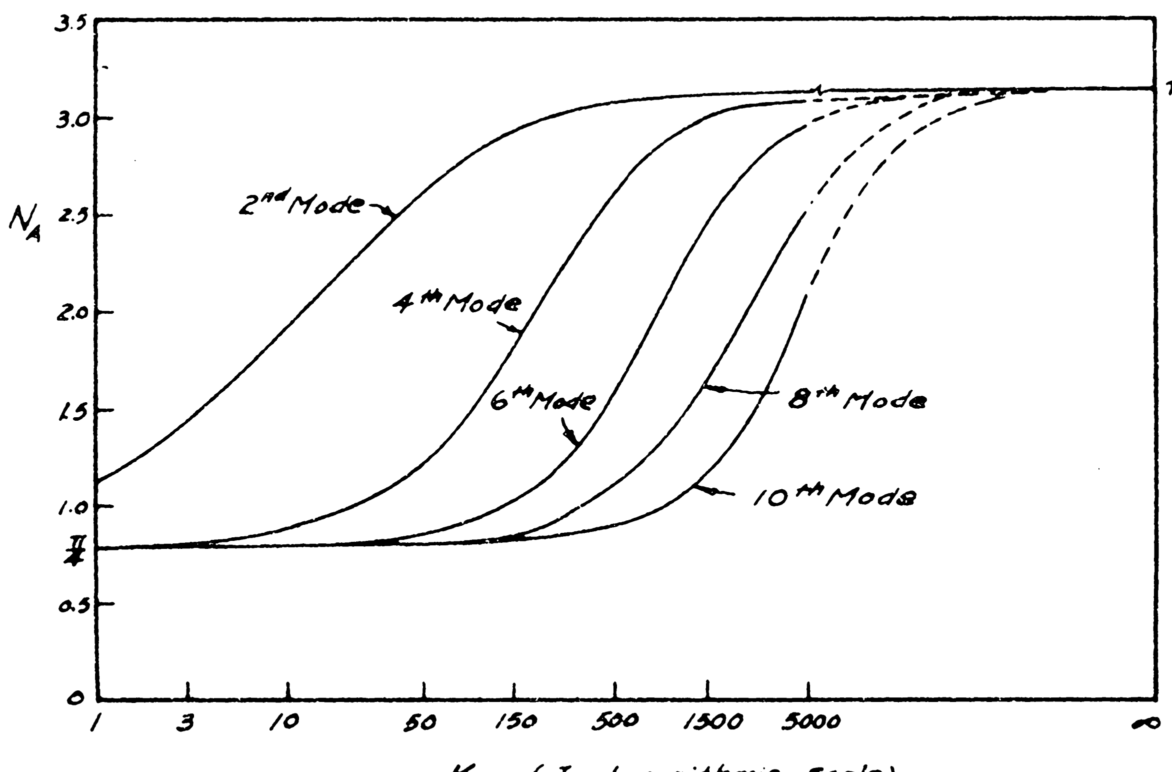

Fig. (10) $N_A - K$ Curves For Antisymmetric Modes

VI. CONCLUSIONS

A general solution has been obtained for the lateral free vibrations of a simply supported beam with constant cross-section carrying a concentrated mass at the mid-point of the span. The effect of the rotatory inertias of the beam and the mounted mass has been included. The frequency equation of the solution consists of two separable parts, one of which leads to the symmetric normal modes, the other leads to the antisymmetric normal modes. The frequencies associated with these two sets of modes when arranged in ascending order occur alternately with the lowest frequency being the fundamental of the (symmetric mode) frequency. In general, the solution will consist of contributions from both symmetric and antisymmetric modes, but for certain initial conditions only one of these types of modes may enter the solution.

The effect of the rotatory inertia of the beam on the roots of the frequency equations and on the normal mode shapes was found to be dependent on the "slenderness ratio" of the beam and mode number, increasing in importance as these factors increase. This influence on the normal mode curves is greater for antisymmetric modes than for symmetric modes. The roots of the frequency equations were determined for the first ten modes for wide ranges of values of the parameters R and K.

VII. ACKNOWLEDGMENT

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