Abacus-tournament Models of Hall-Littlewood Polynomials

Andrew J. Wills

Dissertation submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
in
Mathematics

Nicholas A. Loehr, Chair
Ezra A. Brown
William J. Floyd
Peter A. Linnell

December 7, 2015
Blacksburg, Virginia

Keywords: Symmetric polynomials, Hall-Littlewood polynomials, abacus-tournaments,
Pieri rules.

Abacus-tournament Models of Hall-Littlewood Polynomials

Andrew J. Wills

(ABSTRACT)

In this dissertation, we introduce combinatorial interpretations for three types of Hall-Littlewood polynomials (denoted R_λ, P_λ, and Q_λ) by using weighted combinatorial objects called abacus-tournaments. We then apply these models to give combinatorial proofs of properties of Hall-Littlewood polynomials. For example, we show why various specializations of Hall-Littlewood polynomials produce the Schur symmetric polynomials, the elementary symmetric polynomials, or the t-analogue of factorials. With the abacus-tournament model, we give a bijective proof of a Pieri rule for Hall-Littlewood polynomials that gives the P_λ-expansion of the product of a Hall-Littlewood polynomial P_μ with an elementary symmetric polynomial e_k. We also give a bijective proof of certain cases of a second Pieri rule that gives the P_λ-expansion of the product of a Hall-Littlewood polynomial P_μ with another Hall-Littlewood polynomial $Q_{(r)}$. In general, proofs using abacus-tournaments focus on canceling abacus-tournaments and then finding weight-preserving bijections between the sets of uncanceled abacus-tournaments.

Contents

1 Introduction **1**

2 Background **5**

2.1 Partitions 5

2.2 Permutations 8

2.3 Quantum Factorials and Binomial Coefficients 9

2.4 Symmetric Polynomials 10

2.5 Monomial Symmetric Polynomials 11

2.6 Elementary Symmetric Polynomials 12

2.7 Complete Symmetric Polynomials 12

2.8 Schur Polynomials 12

2.9 Monomial Antisymmetric Polynomials 15

2.10 Pieri Rules for Schur Polynomials 18

2.11 Hall-Littlewood Polynomials 19

3 Combinatorial Interpretations of Hall-Littlewood Polynomials **21**

3.1 Abacus-Tournaments 21

3.2 A Combinatorial Interpretation of $a_{\delta(N)}R_\lambda$ 23

3.3 Blocks and Involutions 25

3.4 Global and Local Exponent Collisions 27

3.5 Cancellation Theorem for Local Exponent Collisions 30

3.6 Leading Abacus-Tournaments 32

3.7 A Combinatorial Explanation of Division by t-Factorials 37
3.8 A Combinatorial Interpretation for $a_{\delta(N)}P_\lambda$ 38
3.9 A Combinatorial Interpretation for $a_{\delta(N)}Q_\lambda$ 40

4 Local Exponent Collision Lemmas 43

4.1 The Single Gap Collision Lemma 43
4.2 The Single Bead Collision Lemma 44

5 Specializations of Hall-Littlewood Polynomials 49

5.1 Specialization at $t = 0$. 49
5.2 Specialization at $t = 1$. 51
5.3 Specialization at $x_{N+1} = 0$. 51
5.4 Specializations at Partitions with One Column. 53
5.5 Specializations at Partitions with One Row. 55

6 A Pieri Rule for Hall-Littlewood Polynomials 58

6.1 A Combinatorial Model for the Right Side of the Pieri Rule 59
6.2 Combinatorial Models for $a_{\delta(N)}P_\mu e_k$ 62
6.3 A Bijection Between the Two Models 65

7 A Second Pieri Rule for Hall-Littlewood Polynomials 68

7.1 A Combinatorial Model for $Q_{(r,0^{N-1})}$ 69
7.2 Subproblem P1 71
7.3 Subproblem P2 77
7.3.1 A Combinatorial Model for $a_{\delta(N)}P_\mu \cdot Q_{(r,0^{N-1})}$ 80
7.3.2 A Combinatorial Model for the Right Side of the Pieri Rule 83
7.3.3 A Bijection Between the Two Models 86
7.4 Subproblem P3 89
7.4.1 A Combinatorial Model for $a_{\delta(N)}P_\mu \cdot Q_{(r,0^{N-1})}$ 91
7.4.2 A Combinatorial Model for the Right Side of the Pieri Rule 91

7.4.3 A Bijection Between the Two Models 95
7.5 Subproblem P4 97

List of Figures

1.1 An example of an abacus-tournament. 2

3.1 An example of an abacus-tournament. 22

3.2 The square of block $\{1, 2, 3, 4, 5\}$ in an abacus-tournament. 26

3.3 The squares of the λ-blocks in an abacus-tournament. 27

3.4 An abacus-tournament to pair with the one in Figure 3.2. 29

3.5 An abacus-tournament to pair with the one in Figure 3.3. 30

3.6 An abacus-tournament for $\lambda = (3^3, 1^4, 0)$ that is leading in $\mathcal{B}_\lambda$. 35

3.7 An abacus-tournament for $\lambda = (3^3, 1^4, 0)$ with a permuted abacus. 36

3.8 An abacus-tournament for $\lambda = (3^3, 1^4, 0)$ with local outdegrees in $\mathcal{B}_\lambda$ out of order. 36

3.9 An abacus-tournament for $\lambda = (3^3, 1^4, 0)$ with local outdegrees in $\mathcal{B}_\lambda$ in order. 37

3.10 An abacus-tournament that is leading in $\mathrm{Pos}_\lambda(0)$. 40

3.11 An example of a shaded abacus-tournament. 41

4.1 x_2 and x_8 have equal local exponents in blocks D and $B \cup C$. 45

4.2 An abacus-tournament A such that upset(A) is not right-justified. 46

4.3 An abacus-tournament A' such that upset(A') is right-justified and $|\text{upset}(A')| \geq 3$. 47

4.4 An abacus-tournament A'' such that upset(A'') is right-justified and $|\text{upset}(A'')| < 3$. 47

5.1 An abacus-tournament without upsets. 50

5.2 An abacus-tournament in the set X. 54

5.3 Bead 9 and the edges in row 1 are removed from Figure 5.2 by J^{-1}. 54
5.4 An abacus-tournament with shape $\lambda = (1^3, 0^4)$. 56
5.5 The abacus $J(A'')$ with position set $\mu = (1^3, 0^2)$ and word 12453. 57

6.1 An abacus-tournament in $\mathbb{LAT}_\lambda^{\mathcal{B}_{\lambda/\mu}}$. 62
6.2 An abacus-tournament in $\mathbb{LAT}_\lambda^{\mathcal{B}_{\lambda/\mu}}$ and $\mathbb{LAT}_\lambda^{\mathcal{B}_\mu}$. 64
6.3 An abacus-tournament in the domain of I. 66
6.4 An abacus-tournament that cancels with the one in Figure 6.3. 66

7.1 An abacus-tournament in $\mathbb{JAT}_{(7,0^5)}$. 72
7.2 Abacus-tournament $R(A, T^*) \in \mathbb{JAT}_{(7,0^5)}$. 78
7.3 Outlines for μ and λ meeting Subproblem P2 restrictions. 79
7.4 An abacus-tournament $A \in \mathbb{LAT}_\mu^{\mathcal{B}_\mu}$. 83
7.5 An abacus-tournament $A' \in \mathbb{LAT}_\mu^{\mathcal{B}_\mu}$. 84
7.6 An abacus-tournament $A'' \in \mathbb{LAT}_\mu^{\mathcal{B}_\mu}$. 85
7.7 An abacus-tournament in $\mathbb{JAT}_\mu(\lambda)$. 86
7.8 An abacus-tournament in $\mathbb{JAT}_\mu(\lambda)$. 87
7.9 An abacus-tournament $A'_* \in \mathbb{JAT}_\mu(\lambda)$. 88
7.10 The abacus-tournament A where $R(A, T^*) = A'_*$. 90
7.11 An abacus-tournament A'_* such that $A'_* \in X$ but $A'_* \notin \mathbb{LAT}_\lambda^{\mathcal{B}_\lambda}$. 93
7.12 An abacus-tournament $A'_* \in \mathbb{JAT}_\mu(\lambda)$. 94
7.13 An abacus-tournament $A \in \mathbb{LAT}_\mu^{\mathcal{B}_\mu}$. 96
7.14 The abacus-tournament $R(A, T^*)$. 97
7.15 The abacus-tournament $R(A, S^*)$. 98
7.16 The abacus-tournament A such that $R^{-1}(A'_*) = (A, T^*)$. 99
7.17 An abacus-tournament $A \in \mathbb{LAT}_\mu^{\mathcal{B}_\mu}$. 101
7.18 The abacus-tournament $R(A, T^*)$. 102

List of Tables

2.1 Content monomials for $\mathrm{SSYT}_3((2,1,0))$. 14

2.2 Signs and weights for the set $\mathrm{LAbc}((3,2,2))$. 17

3.1 The signed weights of abacus-tournaments for $\lambda = (1,0)$. 25

3.2 The abacus-tournaments for $\lambda = (1^3,0)$ leading in $\mathcal{B}_\lambda$ and their signed weights. 39

3.3 The shaded abacus-tournaments for $\lambda = (1,0)$. 42

Chapter 1

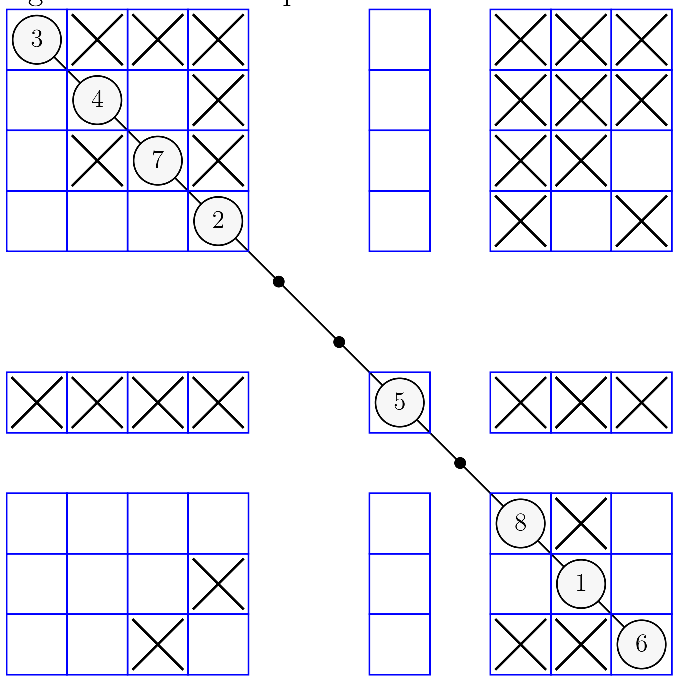
Introduction

Symmetric polynomials are polynomials in multiple variables that are unaffected if any two of the variables are interchanged. For background information on symmetric polynomials, see Sagan [12], Stanley [13], and Loehr [9]. Schur functions are a well-known type of symmetric polynomial that have connections to tableau enumeration and the representation theory of the symmetric group. Many transition matrices between different bases of the vector space of symmetric polynomials have combinatorial interpretations (see [1]). For example, Eğecioğlu and Remmel [4] described the transition matrix between Schur polynomials s_λ and monomial symmetric polynomials m_μ, called the inverse Kostka matrix, with *signed rim hook tabloids*. It is desirable to find combinatorial proofs of Schur function identities. Loehr used objects called *labeled abaci* to give bijective proofs of antisymmetrized versions of many Schur polynomial identities in [8]. Included were Pieri rules and the Littlewood-Richardson rule for Schur polynomials, which describe the Schur expansions of products of Schur polynomials with various other symmetric polynomials.

Hall-Littlewood polynomials (which come in three types, denoted R_λ, P_λ, and Q_λ) are an important basis of symmetric polynomials in N variables. These polynomials have a parameter t and are indexed by partitions λ of precisely N nonnegative parts. A generalization of Schur polynomials and monomial symmetric polynomials, Hall-Littlewood polynomials were first defined in 1961 by Littlewood [7] based on work by Philip Hall [6] studying the lattice structure of finite abelian p-groups. Since then, they have been extensively studied by Macdonald [11, III] and others from a predominantly algebraic standpoint.

As with Schur polynomials, there are combinatorial descriptions of Hall-Littlewood polynomial transition matrices. Loehr, Serrano, and Warrington studied some of these using *starred semistandard tableaux* in [10], and Carbonara used *special tournament matrices* to describe the transition matrix between P_λ and Schur polynomials [2]. There are also many algebraic identities for Hall-Littlewood polynomials; see Macdonald [11] for a thorough algebraic treatment of these identities. Since Schur polynomials are closely related to Hall-Littlewood polynomials, there are identities for Hall-Littlewood polynomials analogous to many of the

Figure 1.1: An example of an abacus-tournament.



identities in Loehr's paper proving Schur function identities with abaci [8]. Consequently, Loehr's paper provides a motivating framework suggesting that certain Hall-Littlewood polynomial identities may have analogous bijective proofs.

In this dissertation, we introduce combinatorial objects called abacus-tournaments. Each abacus-tournament has three associated components: a partition, a labeled abacus, and a tournament. Figure 1.1 shows a visual depiction of one abacus-tournament for the partition $\lambda = (3, 3, 3, 2, 0, 0, 0, 0)$. Each abacus-tournament has an associated monomial in several variables, called its signed weight, that is calculated from various aspects of the abacus-tournament. When the signed weights of a specific set of abacus-tournaments, all sharing the same partition λ, are summed together, the resulting polynomial is exactly $a_{\delta(N)} \cdot R_\lambda$, the product of the Vandermonde determinant $a_{\delta(N)}$ and the Hall-Littlewood polynomial R_λ. Then, giving a bijective explanation for the divisibility of R_λ by products of t-factorials results in combinatorial models for $a_{\delta(N)} \cdot P_\lambda$ and $a_{\delta(N)} \cdot Q_\lambda$. We find that proofs with abacus-tournaments, which incorporate abaci *and* tournaments, have significantly more complicated interactions between objects than proofs with labeled abaci alone.

Our strategy for proving identities for Hall-Littlewood polynomials is to first antisymmetrize (multiply by $a_{\delta(N)}$) and then compare the abacus-tournament models for the two sides of the identity. Such a comparison is formally established by demonstrating a *weight-preserving bijection* between the two sets of abacus-tournaments, and this ensures that the polynomials represented by the two sets are equal. Frequently, such a bijection is impossible to find between an initial pair of sets. In this case, one or both models must be cleared of extraneous objects by pairing off abacus-tournaments that have opposing signed weights using *sign-reversing involutions*. These pairs of objects cancel out in the sum of signed weights, leaving two core sets of abacus-tournaments that can be matched by a bijection.

The identities for Hall-Littlewood polynomials that we prove in this dissertation fall into two categories. The first type of identity relates Hall-Littlewood polynomials to other symmetric polynomials. When evaluated at specific instances of partitions λ or variables $x_1, \ldots, x_N$, and t, Hall-Littlewood polynomials specialize to other symmetric polynomials, or may produce a sum of symmetric polynomials. In proofs of this type of identity, we draw on other established combinatorial models for symmetric polynomials to pair with the abacus-tournament model. A list of some of the identities that we prove combinatorially follows. See Chapter 2 or Macdonald [11] for the definitions of notation used here.

- $R_{(0)}(x_1, \ldots, x_N; t) = [N]!_t$, the t-analogue of N factorial.
- $P_\lambda(x_1, \ldots, x_N; 0) = s_\lambda(x_1, \ldots, x_N)$, where s_λ denotes the Schur symmetric polynomial.
- $P_{(1^r, 0^{N-r})}(x_1, \ldots, x_N; t) = e_r(x_1, \ldots, x_N)$, where e_r denotes the rth elementary symmetric polynomial.
- $P_\lambda(x_1, \ldots, x_N, 0; t) = P_\lambda(x_1, \ldots, x_N; t)$. On the left side, we have set $x_{N+1} = 0$.
- $P_{(r, 0^{N-1})}(x_1, \ldots, x_N; t) = \sum_{k=0}^{r-1} (-t)^k s_{(r-k, 1^k, 0^{N-k-1})}(x_1, \ldots, x_N)$.

The second type of identity found in this dissertation advances our understanding of how Hall-Littlewood polynomials behave in the algebra of symmetric polynomials. It is desirable to prove *Pieri rules* for Hall-Littlewood polynomials that express the product of a Hall-Littlewood polynomial P_μ with some symmetric polynomial f as a sum of Hall-Littlewood polynomials P_λ. To this end, we give a bijective proof of a Pieri rule that describes how to express $P_\mu \cdot e_r$, the product of the Hall-Littlewood polynomial P_μ and the elementary symmetric polynomial e_r, as a sum of Hall-Littlewood polynomials P_λ. We also give a bijective proof of certain cases of a second Pieri rule to describe the product of P_λ and $Q_{(r)}$.

The rest of this dissertation is organized as follows. Chapter 2 provides the necessary definitions and notation involving symmetric polynomials, antisymmetric polynomials, and Hall-Littlewood polynomials. Chapter 2 also introduces abaci and tournaments, which are key ingredients in our combinatorial models for Hall-Littlewood polynomials. Chapter 3 establishes abacus-tournament models for the antisymmetrized Hall-Littlewood polynomials $a_{\delta(N)} R_\lambda$, $a_{\delta(N)} P_\lambda$, and $a_{\delta(N)} Q_\lambda$. We also explain combinatorially why R_λ is divisible by products of t-factorials. Chapter 4 provides two mechanisms for canceling sets of abacus-tournaments: the Single Gap Collision Lemma and the Single Bead Collision Lemma. Chapter 5 is devoted to proving identities relating specialized Hall-Littlewood polynomials to many of the symmetric polynomials defined in Chapter 2. Chapter 6 gives a bijective proof of the first Pieri rule for Hall-Littlewood polynomials. Chapter 7 gives bijective proofs of some special cases of the second Pieri rule for Hall-Littlewood polynomials and discusses future research to prove the general form of this rule.

Chapter 2

Background

We now present preliminary material and establish notation in order to introduce the reader to a number of classic symmetric polynomials, culminating in Hall-Littlewood polynomials. Along the way, we define three important combinatorial objects: tournaments, tableaux, and labeled abaci. We also present combinatorial interpretations for two Pieri rules for Schur polynomials, which inspire Hall-Littlewood polynomial versions of these rules discussed in Chapters 6 and 7 of this dissertation.

2.1 Partitions

Definition 1. A *partition of an integer* k is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ of N nonnegative integers with

$$\lambda_1 + \lambda_2 + \cdots + \lambda_N = k.$$

Informally, a partition breaks up an integer k into integer *parts* $\lambda_1, \lambda_2, \ldots, \lambda_N$. In general, we may be interested in partitions with a particular number of parts N, or partitions of a particular integer k.

Definition 2. Define $\ell(\lambda)$ to be the number of nonzero parts of a partition λ. The *size* of a partition λ is $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_N$. Define $\mathrm{Par}(k)$ to be the set of partitions of size k, define Par_N to be the set of partitions with exactly N nonnegative parts, and define $\mathrm{Par}_N(k)$ to be the set of partitions of k with exactly N nonnegative parts. For $\lambda \in \mathrm{Par}_N$ and $i \geq 0$, let $m_i(\lambda)$ denote the number of parts of λ of size i.

We can append $N - \ell(\lambda)$ parts of size zero to a partition with fewer than N nonzero parts to obtain a partition in Par_N; then $\ell(\lambda) + m_0(\lambda) = N$ for such λ.

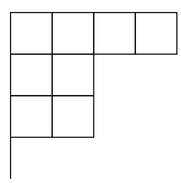
Example 3. The partition $\lambda = (4,2,2,0) \in \mathrm{Par}_4(8)$ is a partition of 8 with 4 parts. An alternative notation for λ combines parts of equal size into a single term with an exponent. In this case, we can write $\lambda = (4,2^2,0)$. This partition has one part of size 4, two parts of size 2, and one part of size 0.

Definition 4. For $\lambda \in \mathrm{Par}_N(k)$, the *Ferrers diagram* of λ is

$$\mathrm{dg}(\lambda) = \{(i,j) \in \mathbb{N} \times \mathbb{N} : 1 \leq i \leq N, 1 \leq j \leq \lambda_i\}.$$

We can visually represent $\mathrm{dg}(\lambda)$ as an array of N left-justified rows having k cells such that row i contains precisely λ_i boxes for $1 \leq i \leq N$. Rows with no boxes are marked with a single vertical line.

Example 5. We can depict $\lambda = (4,2^2,0)$ by left-justifying four rows of boxes where the first row has 4 boxes, the second and third rows have 2 boxes, and the fourth row has no boxes:

$\mathrm{dg}(\lambda) =$ 

Definition 6. For $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathrm{Par}_N$ and all $i \geq 0$, define

$$\lambda'_i = |\{j : \lambda_j \geq i\}|.$$

Note that λ'_i is the number of cells in the ith column of the diagram of λ. Also, $\lambda'_0 = N$, $\lambda'_i = 0$ for all $i > \lambda_1$, and $m_i(\lambda) = \lambda'_i - \lambda'_{i+1}$ for all $i \geq 0$.

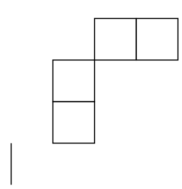
Example 7. For $\lambda = (4,2^2,0)$, $\lambda'_0 = 4$, $\lambda'_1 = 3 = \lambda'_2$, and $\lambda'_3 = 1 = \lambda'_4$.

Definition 8. If $\lambda, \mu \in \mathrm{Par}_N$ are partitions such that $\lambda_i \geq \mu_i$ for all $i \geq 1$, define the *skew shape*

$$\lambda/\mu = \{(i,j) : 1 \leq i \leq N, \mu_i < j \leq \lambda_i\}.$$

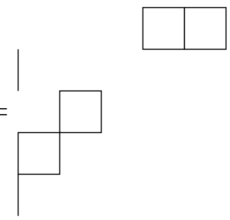
The skew shape λ/μ can be obtained visually from the diagram of λ by overlaying $\mathrm{dg}(\mu)$ and erasing any overlapping squares. If $\mu = (0^N)$ is the zero partition, then $\lambda/\mu = \mathrm{dg}(\lambda)$.

Example 9. If $\lambda = (4,2^2,0)$ and $\mu = (2,1^2,0)$, then

$\lambda/\mu =$ 

On the other hand,

$$(5, 2^2, 1, 0)/(3, 2, 1, 0^2) =$$



Definition 10. A skew shape λ/μ, where $\lambda, \mu \in \mathrm{Par}_N$, is a *vertical r-strip* if λ/μ has precisely r cells and each row has at most one cell. Similarly, λ/μ is a *horizontal r-strip* if λ/μ has precisely r cells and each column has at most one cell. If $\mu \in \mathrm{Par}_N$, let

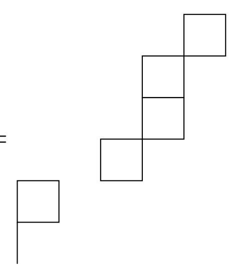
$$\mathrm{V}(\mu, r) = \{\lambda \in \mathrm{Par}_N : \lambda/\mu \text{ is a vertical } r\text{-strip}\},$$

and

$$\mathrm{H}(\mu, r) = \{\lambda \in \mathrm{Par}_N : \lambda/\mu \text{ is a horizontal } r\text{-strip}\}.$$

Example 11. The skew shape $(5, 2^2, 1, 0)/(3, 2, 1, 0^2)$, displayed in the previous example, is a horizontal 4-strip and is not a vertical strip of any size. The skew shape λ/μ for $\lambda = (4, 2^2, 0)$ and $\mu = (2, 1^2, 0)$, also displayed in the previous example, is neither a vertical 4-strip nor a horizontal 4-strip. The following skew shape is a vertical 5-strip but not a horizontal strip.

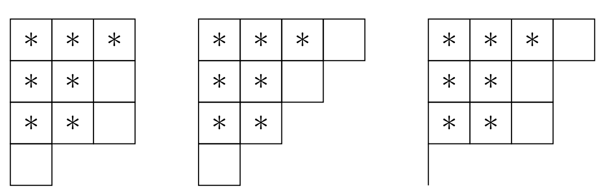
$$(5, 4^2, 3, 1, 0)/(4, 3^2, 2, 0^2) =$$



Example 12. If $\mu = (3, 2^2, 0) \in \mathrm{Par}_4$, then

$$\mathrm{V}(\mu, 3) = \{(3^3, 1), (4, 3, 2, 1), (4, 3^2, 0)\}.$$

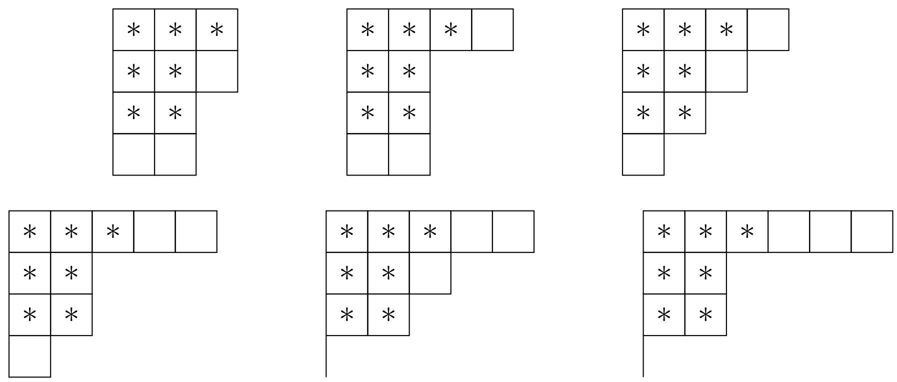
We display the partitions $\lambda \in \mathrm{V}(\mu, 3)$ below with $*$'s in cells which overlap with μ. The cells without $*$'s form the skew shape λ/μ.



Similarly,

$$\mathrm{H}(\mu, 3) = \{(3^2, 2^2), (4, 2^3), (4, 3, 2, 1), (5, 2^2, 1), (5, 3, 2, 0), (6, 2^2, 0)\},$$

which we display below.



Note that $(4, 3, 2, 1)$ is an partition of both $\mathrm{V}(\mu, 3)$ and $\mathrm{H}(\mu, 3)$.

2.2 Permutations

Definition 13. A *word* of length N over $\{1, 2, \ldots, N\}$ is a sequence $w = w_1 w_2 \cdots w_N$ where each $w_i \in \{1, \ldots, N\}$. Such a word is also a *permutation* of $\{1, \ldots, N\}$ provided each element of $\{1, \ldots, N\}$ appears in w exactly once.

Equivalently, we can define a word $w_1 w_2 \cdots w_N$ of length N over $\{1, \ldots, N\}$ as a function $w : \{1, \ldots, N\} \to \{1, \ldots, N\}$ such that $w(i) = w_i$ for all i. In this case, we say w is a permutation of $\{1, \ldots, N\}$ iff w is a bijective function.

Let S_N denote the set of permutations of $\{1, \ldots, N\}$.

Example 14. Both $u = 612354$ and $v = 622352$ are words of length 6. The word u is in S_6, while v is not. As a function, u maps 1 to 6, 2 to 1, 3 to 2, and so on.

Example 15. The set of permutations S_3 is

$$S_3 = \{123, 132, 213, 231, 312, 321\}.$$

Definition 16. Let $w = w_1 w_2 \cdots w_N \in S_N$ be a permutation. An *inversion* of w is a pair (i, j) such that $i < j$ and $w_i > w_j$. Let $\mathrm{inv}(w)$ denote the number of inversions of w and define the *sign* of w to be $\mathrm{sgn}(w) = (-1)^{\mathrm{inv}(w)}$.

Example 17. For $N = 6$, the permutation $u = 612354$ has 6 total inversions:

$$(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (5, 4).$$

For more details on permutations (e.g., group structure, inverses, and compositions) see [9, Ch. 9].

2.3 Quantum Factorials and Binomial Coefficients

Definition 18. For a variable t and positive integer k, define

$$[k]_t = 1 + t + t^2 + \cdots + t^{k-1} = \frac{1-t^k}{1-t}.$$

Furthermore, define $[0]_t = 0$, $[0]!_t = 1$, and

$$[k]!_t = [1]_t[2]_t \cdots [k]_t.$$

We call $[k]!_t$ the *quantum factorial of* k.

Example 19. To find the quantum factorial of 3, we compute

$$[3]!_t = [1]_t[2]_t[3]_t = 1 \cdot (1+t) \cdot (1+t+t^2) = 1 + 2t + 2t^2 + t^3.$$

The next theorem shows that the coefficients of monomials of the form t^b in $[N]_t$ count objects: namely, the number of permutations in S_N with precisely b inversions.

Theorem 20. *For* $N \geq 1$,

$$\sum_{w \in S_N} t^{inv(w)} = [N]!_t.$$

A proof of this theorem can be found in [9, Thm. 6.26].

Definition 21. For a variable t and integers k, n with $0 \leq k \leq n$, define

$$\begin{bmatrix} n \\ k \end{bmatrix}_t = \begin{bmatrix} n \\ k, n-k \end{bmatrix}_t = \frac{[n]!_t}{[k]!_t[n-k]!_t}.$$

We call $\begin{bmatrix} n \\ k \end{bmatrix}_t$ a *quantum binomial coefficient.*

Example 22. The quantum binomial coefficient for $n = 4$ and $k = 2$ is

$$\begin{aligned} \begin{bmatrix} 4 \\ 2 \end{bmatrix}_t &= \frac{[4]!_t}{[2]!_t[2]!_t} \\ &= \frac{(1+t)(1+t+t^2)(1+t+t^2+t^3)}{(1+t)(1+t)} \\ &= 1 + t + 2t^2 + t^3 + t^4 \end{aligned}$$

2.4 Symmetric Polynomials

We now define what it means for a multivariable polynomial to be symmetric, and we define a few examples of symmetric polynomials. The set of symmetric polynomials in a fixed number of variables form a vector space, and some of the symmetric polynomials provide specific bases for this vector space. See [3] for algebra notation not defined here.

Definition 23. For $\beta \in \mathbb{N}^N$, let $x^\beta = x_1^{\beta_1} x_2^{\beta_2} \cdots x_N^{\beta_N}$. We say x^β is a *monomial* of *degree* $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_N$.

Definition 24. For a permutation $w \in S_N$ and a polynomial $f \in K[x_1, x_2, \ldots, x_N]$, define the action $w \cdot f$ by

$$w \cdot f(x_1, x_2, \ldots, x_N) = f(x_{w(1)}, x_{w(2)}, \ldots, x_{w(N)}).$$

Furthermore, if $h = f/g$ is the quotient of two polynomials $f, g \in K[x_1, x_2, \ldots, x_N]$, define

$$w \cdot h(x_1, x_2, \ldots, x_N) = \frac{w \cdot f(x_1, x_2, \ldots, x_N)}{w \cdot g(x_1, x_2, \ldots, x_N)}.$$

Definition 25. For a field K, a polynomial f in $K[x_1, x_2, \ldots, x_N]$ is *symmetric* iff

$$w \cdot f(x_1, x_2, \ldots, x_N) = f(x_1, x_2, \ldots, x_N)$$

for all w in the symmetric group S_N. A polynomial f is *homogeneous of degree* k if every monomial x^β appearing in f with nonzero coefficient has degree k. The 0 polynomial is considered to be homogeneous of every degree.

Example 26. Let $N = 3$ and $K = \mathbb{Q}$. The polynomial

$$f(x_1, x_2, x_3) = x_1^2 x_2^1 x_3^0 + x_1^2 x_2^0 x_3^1 + x_1^1 x_2^2 x_3^0 + x_1^0 x_2^2 x_3^1 + x_1^1 x_2^0 x_3^2 + x_1^0 x_2^1 x_3^2$$

is symmetric. For example, if $w = 231$, then $w \cdot f(x_1, x_2, x_3) = f(x_1, x_2, x_3)$. In particular, the term $x_1^2 x_2^0 x_3^1$ maps to $x_1^1 x_2^2 x_3^0$ in $w \cdot f(x_1, x_2, x_3)$. On the other hand, the polynomial

$$g(x_1, x_2, x_3) = x_1^3 x_2^1 x_3^1 + x_1^1 x_2^3 x_3^1$$

is not symmetric because, for example,

$$w \cdot g(x_1, x_2, x_3) = x_1^1 x_2^3 x_3^1 + x_1^1 x_2^1 x_3^3 \neq g(x_1, x_2, x_3).$$

Note that f is homogeneous of degree 3, and g is homogeneous of degree 5.

Definition 27. Let Λ_N be the set of all symmetric polynomials in $K[x_1, \ldots, x_N]$. For all $k \geq 0$, let Λ_N^k be the set of polynomials in Λ_N that are homogeneous of degree k.

Here and below, we will fix an integer N that determines both the number of variables $x_1, \dots, x_N$ for a multivariable polynomial and the number of parts in a partition $\lambda \in \mathrm{Par}_N$ including parts of size zero. If a polynomial is displayed without a list of variables, the reader may assume such a polynomial involves variables $x_1, \dots, x_N$ unless otherwise specified.

Theorem 28. *Λ_N and Λ_N^k are K-vector spaces and*

$$\Lambda_N = \bigoplus_{k \geq 0} \Lambda_N^k.$$

See [9, p. 386] for a proof of this theorem.

2.5 Monomial Symmetric Polynomials

Definition 29. Given a sequence $\beta \in \mathbb{N}^N$, define $\mathrm{sort}(\beta) \in \mathbb{N}^N$ by sorting the entries of β into weakly decreasing order. For $\lambda \in \mathrm{Par}_N$, let $M(\lambda) = \{\beta \in \mathbb{N}^N : \mathrm{sort}(\beta) = \lambda\}$ denote the set of sequences, or *exponent vectors*, that sort to the particular partition λ.

A symmetric polynomial in N variables indexed by a partition λ can be created by summing every monomial x^β such that its exponent sequence β sorts to λ.

Definition 30. For a partition $\lambda \in \mathrm{Par}_N$ and variables $x_1, \dots, x_N$, the *monomial symmetric polynomial indexed by* λ is

$$m_\lambda(x_1, x_2, \dots, x_N) = \sum_{\beta \in M(\lambda)} x^\beta = \sum_{\beta \in M(\lambda)} x_1^{\beta_1} x_2^{\beta_2} \cdots x_N^{\beta_N}.$$

Example 31. For $N = 3$ and the partition $\lambda = (2, 1, 0)$,

$$m_{(2,1,0)}(x_1, x_2, x_3) = x_1^2 x_2^1 x_3^0 + x_1^2 x_2^0 x_3^1 + x_1^1 x_2^2 x_3^0 + x_1^0 x_2^2 x_3^1 + x_1^1 x_2^0 x_3^2 + x_1^0 x_2^1 x_3^2.$$

For $N = 3$ and the partition $\lambda = (3, 1, 1)$,

$$m_{(3,1,1)}(x_1, x_2, x_3) = x_1^3 x_2^1 x_3^1 + x_1^1 x_2^3 x_3^1 + x_1^1 x_2^1 x_3^3.$$

Note that $m_{(2,1,0)}$ and $m_{(3,1,1)}$ are symmetric polynomials that are homogeneous of degree 3 and 4, respectively.

Theorem 32. *For all $\lambda \in \mathrm{Par}_N$, the monomial symmetric polynomial m_λ is a homogeneous symmetric polynomial of degree $|\lambda|$. For every $k \geq 0$ and $N \geq 1$, $\{m_\lambda : \lambda \in \mathrm{Par}_N(k)\}$ is a basis for Λ_N^k.*

See [9, Thm. 10.29] for a proof of this theorem.

2.6 Elementary Symmetric Polynomials

The next polynomials are another example of homogeneous symmetric polynomials.

Definition 33. For an integer r such that $1 \leq r \leq N$, the *rth elementary symmetric polynomial in N variables* is

$$e_r(x_1, \ldots, x_N) = \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq N} x_{i_1} x_{i_2} \cdots x_{i_r} = \sum_{\substack{S \subseteq \{1, \ldots, N\}: \\ |S| = r}} \left(\prod_{j \in S} x_j \right).$$

Example 34. For $N = 4$ and $r = 3$,

$$e_3(x_1, x_2, x_3, x_4) = x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4.$$

Theorem 35. *For all r such that $1 \leq r \leq N$, the rth elementary symmetric polynomial is a homogeneous symmetric polynomial of degree r.*

See [9, Sec. 10.4] to prove this theorem.

2.7 Complete Symmetric Polynomials

Definition 36. For an integer $r \geq 1$, the *rth complete symmetric polynomial in N variables* is

$$h_r(x_1, \ldots, x_N) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_r \leq N} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

Example 37. For $N = 4$ and $r = 2$,

$$h_2(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1 x_2 + x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4.$$

Theorem 38. *For all $r \geq 1$, the rth complete symmetric polynomial is a homogeneous symmetric polynomial of degree r.*

See [9, Sec. 10.4] to prove this theorem.

2.8 Schur Polynomials

Definition 39. For a partition λ, a *tableau of shape* λ is a function $T : \mathrm{dg}(\lambda) \to \mathbb{N}^+$.

A tableau of shape λ can be displayed by filling in boxes of the Ferrers diagram of λ by placing the value $T(i,j)$ in the box in row i and column j. Each box, corresponding to an ordered pair $(i,j) \in \mathrm{dg}(\lambda)$, is called a *cell of* T. For example,

$$R = \begin{array}{|c|c|c|c|}\hline 1 & 9 & 4 & 4 \\ \hline 4 & 4 \\ \cline{1-2} 1 & 2 \\ \cline{1-2} \\ \end{array} \qquad S = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 3 & 8 \\ \hline 4 & 4 \\ \cline{1-2} 8 & 9 \\ \cline{1-2} \\ \end{array} \qquad T = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 6 \\ \hline 2 & 7 \\ \cline{1-2} 4 & 8 \\ \cline{1-2} \\ \end{array}$$

are tableaux of shape $(4,2,2,0)$.

Definition 40. A tableau T of shape λ is *semistandard* if the values in each row weakly increase from left to right and the values in each column strictly increase from top to bottom. For a partition $\lambda \in \mathrm{Par}_N$, let $\mathrm{SSYT}_N(\lambda)$ denote the set of all semistandard tableaux of shape λ taking values in $\{1,\ldots,N\}$. A semistandard tableau T of shape λ is *standard* if T is a bijection from $\mathrm{dg}(\lambda)$ to $\{1,\ldots,|\lambda|\}$, i.e., each number from 1 to $|\lambda|$ appears exactly once in T.

In the example above, S is semistandard, T is standard, and R is neither.

Example 41. The set of semistandard tableaux for the partition $\lambda = (2,1) \in \mathrm{Par}_2$ is

$$\mathrm{SSYT}_2((2,1)) = \left\{ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array} \right\}.$$

Neither of these are standard tableaux. On the other hand, the set of semistandard tableaux for the partition $\lambda = (2,1,0) \in \mathrm{Par}_3$ is

$$\mathrm{SSYT}_3((2,1,0)) = \left\{ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} \\ \end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1} \\ \end{array}, \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1} \\ \end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1} \\ \end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \\ \end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \\ \end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1} \\ \end{array}, \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1} \\ \end{array} \right\}.$$

The fifth and sixth tableaux are also standard tableaux.

Each monomial symmetric polynomial m_λ is a sum of monomials x^β arising from objects $\beta \in M(\lambda)$. Similarly, we can build symmetric polynomials called Schur polynomials by adding up certain monomials indexed by semistandard tableaux.

Definition 42. The *content* of a tableau T of shape λ is $c(T) = (c_1, c_2, \ldots)$ where

$$c_k = |\{(i,j) \in \mathrm{dg}(\lambda) : T((i,j)) = k\}|.$$

Informally, c_k is the number of times k appears in T. Given variables $x_1, x_2, \ldots$, the *content monomial* of T is

$$x^{c(T)} = x_1^{c_1} x_2^{c_2} \cdots x_k^{c_k} \cdots .$$

Example 43. The tableau S displayed below has one 1, zero 2's, two 3's, and so on, so $c(S) = (1, 0, 2, 2, 0, 0, 0, 2, 1, 0, 0, \dots)$. Therefore, $x^{c(T)} = x_1^1 x_3^2 x_4^2 x_8^2 x_9^1$.

$$S = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 3 & 8 \\ \hline 4 & 4 \\ \cline{1-2} 8 & 9 \\ \cline{1-2} \end{array}$$

Example 44. Table 2.1 shows the content monomials for each semistandard tableau in $\mathrm{SSYT}_3((2,1,0))$. When summed together, these monomials form the polynomial

$$x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + 2x_1 x_2 x_3 + x_1 x_3^2 + x_2 x_3^2,$$

which is a symmetric polynomial in 3 variables. The next definition generalizes this example.

Table 2.1: Content monomials for $\mathrm{SSYT}_3((2,1,0))$.

$T \in \mathrm{SSYT}_3((2,1,0))$	$x^{c(T)}$	$T \in \mathrm{SSYT}_3((2,1,0))$	$x^{c(T)}$
$\begin{array}{\|c\|c\|} \hline 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array}$	$x_1^2 x_2$	$\begin{array}{\|c\|c\|} \hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}$	$x_1 x_2 x_3$
$\begin{array}{\|c\|c\|} \hline 1 & 1 \\ \hline 3 \\ \cline{1-1} \end{array}$	$x_1^2 x_3$	$\begin{array}{\|c\|c\|} \hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$	$x_1 x_2 x_3$
$\begin{array}{\|c\|c\|} \hline 2 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}$	$x_2^2 x_3$	$\begin{array}{\|c\|c\|} \hline 1 & 3 \\ \hline 3 \\ \cline{1-1} \end{array}$	$x_1 x_3^2$
$\begin{array}{\|c\|c\|} \hline 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array}$	$x_1 x_2^2$	$\begin{array}{\|c\|c\|} \hline 2 & 3 \\ \hline 3 \\ \cline{1-1} \end{array}$	$x_2 x_3^2$

Definition 45. For a partition $\lambda \in \mathrm{Par}_N$, the *Schur polynomial in N variables indexed by λ* is

$$s_\lambda(x_1, \dots, x_N) = \sum_{T \in \mathrm{SSYT}_N(\lambda)} x^{c(T)}.$$

Example 46. We calculated in the previous example that

$$s_{(2,1,0)} = x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + x_1x_2^2 + 2x_1x_2x_3 + x_1x_3^2 + x_2x_3^2.$$

From Example 41, we see that $s_{(2,1)} = x_1^2x_2 + x_1x_2^2$.

Theorem 47. *For a partition $\lambda \in \mathrm{Par}_N$, $s_\lambda(x_1, \ldots, x_N)$ is a symmetric polynomial that is homogeneous of degree $|\lambda|$. Furthermore, for fixed $k \geq 0$ and $N \geq 1$, $\{s_\lambda(x_1, \ldots, x_N) : \lambda \in \mathrm{Par}_N(k)\}$ is a basis for Λ_N^k.*

See [9, Thm. 10.49] for a proof.

2.9 Monomial Antisymmetric Polynomials

The next type of polynomial we need is called the *monomial antisymmetric polynomial.* These polynomials have both an algebraic definition and a combinatorial interpretation. As the name suggests, monomial antisymmetric polynomials are *antisymmetric* rather than symmetric.

Definition 48. A polynomial f in N variables $x_1, \ldots, x_N$ is *antisymmetric* iff

$$f(x_{w(1)}, \ldots, x_{w(N)}) = \mathrm{sgn}(w) f(x_1, \ldots, x_N)$$

for all $w \in S_N$.

Definition 49. For $N \geq 1$, define $\delta(N) = (N-1, N-2, \ldots, 2, 1, 0) \in \mathrm{Par}_N$.

We are about to define monomial antisymmetric polynomials $a_\mu(x_1, \ldots, x_N)$ where μ is a partition with N *distinct* parts. There is a one-to-one correspondence between partitions $\lambda \in \mathrm{Par}_N$ and partitions $\mu \in \mathrm{Par}_N$ such that μ has distinct parts, given by $\lambda \mapsto \mu = \lambda + \delta(N)$. Consequently, we often index a monomial antisymmetric polynomial for μ as $a_{\lambda+\delta(N)}$ where $\mu = \lambda + \delta(N)$.

Definition 50. For partitions $\lambda, \mu \in \mathrm{Par}_N$ such that $\mu = \lambda + \delta(N)$ and variables $x_1, \ldots, x_N$, define

$$a_\mu(x_1, \ldots, x_N) = a_{\lambda+\delta(N)}(x_1, \ldots, x_N) = \sum_{w \in S_N} \mathrm{sgn}(w) \prod_{i=1}^{N} x_{w(i)}^{\mu_i}.$$

Example 51. Let $N = 3$ and $\lambda = (3, 2, 2)$. Then $\lambda + \delta(3) = (5, 3, 2)$ and

$$a_{(5,3,2)}(x_1, x_2, x_3) = x_1^5x_2^3x_3^2 + x_1^3x_2^2x_3^5 + x_1^2x_2^5x_3^3 - x_1^3x_2^5x_3^2 - x_1^5x_2^2x_3^3 - x_1^2x_2^3x_3^5.$$

Theorem 52. *The monomial antisymmetric polynomials $a_{\lambda+\delta(N)}$ are antisymmetric polynomials.*

See [9, Def. 10.28] for details of a proof.

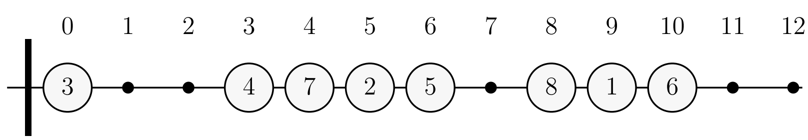
As with the previously studied polynomials, individual monomials in the expression for a monomial antisymmetric polynomial can be obtained by calculating an exponent vector from a combinatorial object. In this case, the objects under consideration are called *abaci* (see [9, Ch. 11], [8]).

Definition 53. A *labeled abacus with N beads* is an ordered pair (λ, v) such that $\lambda \in \mathrm{Par}_N$ and $v \in S_N$. Let LAbc denote the set of all labeled abaci and, for $\lambda \in \mathrm{Par}_N$, define

$$\mathrm{LAbc}(\lambda) = \{(\lambda, v) : v \in S_N\}.$$

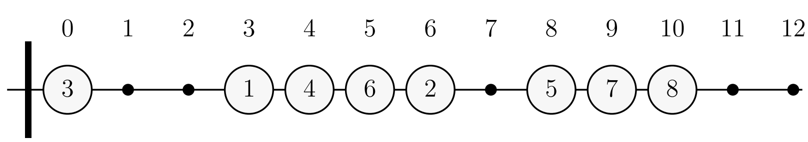
To display an abacus (λ, v), place *beads* on a horizontal line, called the *bead runner*, in positions given by $\mathrm{pos}(\lambda, v) = \lambda + \delta(N)$. By Theorem **??**, $\mu = \mathrm{pos}(\lambda, v) = (\mu_1 > \mu_2 > \cdots > \mu_N) \in \mathrm{Par}_N$ is a set of *distinct* positions. Positions on the bead runner not found in $\mathrm{pos}(\lambda, v)$ are marked on the runner with *bead gaps*. Bead positions start at 0 and are listed *from left to right*. The beads are then labeled with integers given by the *word* $v = v_1 v_2 \cdots v_N$ *from right to left*. If $\mathrm{pos}(\lambda, v) = (\mu_1 > \mu_2 > \cdots > \mu_N)$ and $w(\lambda, v) = v_1 v_2 \cdots v_N$, the ith bead from the right in (λ, v) is located in position μ_i and is labeled v_i.

Example 54. Let $N = 8$, $\lambda = (3^3, 2^4, 0)$, and $v = 61852743$. Then $\mathrm{pos}(\lambda, v) = (10, 9, 8, 6, 5, 4, 3, 0)$. The labeled abacus (λ, v) is drawn below.



In proofs involving abaci, it is common to produce a new abacus by permuting the positions $\{1, \ldots, N\}$ of the labels in $w(\lambda, v) = v_1 v_2 \cdots v_N$. We typically write permutations of positions in cycle notation and abacus words in one-line form (see [3]).

Example 55. As in the previous example, let $N = 8$, $\lambda = (3^3, 2^4, 0)$, and $v = 61852743$. Let $a = (1, 3, 4, 5)(2, 6, 7)(8)$ permute positions in $w(\lambda, v)$. The new abacus (λ, u) with position set $\mathrm{pos}(\lambda, u) = \mathrm{pos}(\lambda, v)$ and word $u = v \circ a = 87526413$ is drawn below.



We can assign a *sign* and a *weight monomial* to each labeled abacus (λ, v) as follows.

Definition 56. Given a labeled abacus (λ, v) with N beads and $\mu = \mathrm{pos}(\lambda, v) = \lambda + \delta(N)$, define the *weight* of (λ, v) to be

$$\mathrm{wt}(\lambda, v) = \prod_{i=1}^{N} x_{v_i}^{\mu_i}.$$

Define the *sign* of (λ, v) to be

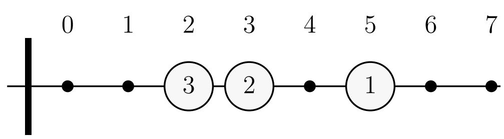
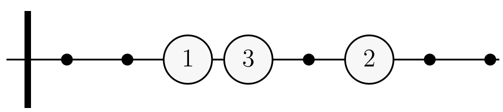
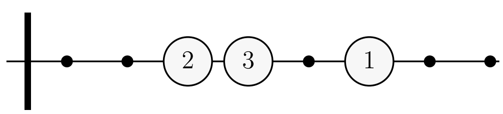
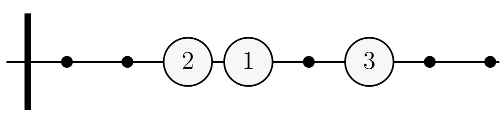
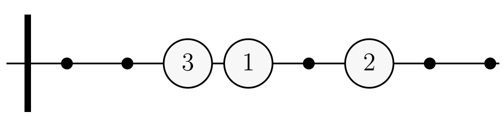
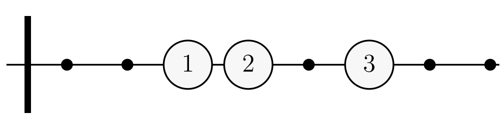
$$\operatorname{sgn}(\lambda, v) = \operatorname{sgn}(v) = (-1)^{\operatorname{inv}(v)}.$$

Informally, a bead labeled j in position k on the abacus contributes x_j^k to the weight of the abacus.

The abacus (λ, v) in the Example 54 has $\operatorname{inv}(v) = 16$, $\operatorname{sgn}(\lambda, v) = (-1)^{16} = 1$, and $\operatorname{wt}(\lambda, v) = x_1^9 x_2^5 x_3^0 x_4^3 x_5^6 x_6^{10} x_7^4 x_8^8$.

Example 57. Let $\lambda = (3, 2, 2)$. Table 2.2 shows the product of the sign and weight of each abacus with position set $\mu = \lambda + \delta(3) = (5, 3, 2)$.

Table 2.2: Signs and weights for the set LAbc((3, 2, 2)).

$(\lambda, v) \in \operatorname{LAbc}((3,2,2))$	$\operatorname{sgn}(\lambda, v)\operatorname{wt}(\lambda, v)$
	$x_1^5 x_2^3 x_3^2$
	$x_1^2 x_2^5 x_3^3$
	$-x_1^5 x_2^2 x_3^3$
	$x_1^3 x_2^2 x_3^5$
	$-x_1^3 x_2^5 x_3^2$
	$-x_1^2 x_2^3 x_3^5$

Summing the monomials from Table 2.2 gives the polynomial

$$\sum_{(\lambda,v)\in\operatorname{LAbc}((3,2,2))} \operatorname{sgn}(\lambda, v)\operatorname{wt}(\lambda, v) = x_1^5x_2^3x_3^2 + x_1^3x_2^2x_3^5 + x_1^2x_2^5x_3^3 - x_1^3x_2^5x_3^2 - x_1^5x_2^2x_3^3 - x_1^2x_2^3x_3^5$$
$$= a_{(5,3,2)}(x_1, x_2, x_3).$$

The next theorem states that this happens in general.

Theorem 58. *For all $\lambda \in \mathrm{Par}_N$,*

$$a_{\lambda+\delta(N)}(x_1, \ldots, x_N) = \sum_{(\lambda,v)\in\mathrm{LAbc}(\lambda)} \mathrm{sgn}(\lambda, v)\,\mathrm{wt}(\lambda, v).$$

See [9, p. 461] for a proof. It can be shown [9, p. 458] that

$$a_{\delta(N)}(x_1, \ldots, x_N) = \det \left\| x_j^{N-i} \right\|_{1\le i,j\le N} = \prod_{1\le i<j\le N} (x_i - x_j), \tag{2.1}$$

which is known as the *Vandermonde determinant*.

2.10 Pieri Rules for Schur Polynomials

The monomial antisymmetric polynomials provide an alternate algebraic definition of Schur polynomials.

Theorem 59. *For all $\lambda \in \mathrm{Par}_N$,*

$$s_\lambda(x_1, \ldots, x_N) = \frac{a_{\lambda+\delta(N)}(x_1, \ldots, x_N)}{a_{\delta(N)}(x_1, \ldots, x_N)}.$$

[8, Thm. 3.1] and [9, Sec. 11.12] give purely bijective proofs of this theorem using abaci, where previous proofs were algebraic (see [11, II.5.17] and [9, Thm. 11.45]) or based on the RSK algorithm on tableaux (see [5]). The next theorems, Theorems 60 and 62, are called "Pieri rules" for Schur polynomials because they describe how to express the product $a_{\mu+\delta(N)} \cdot f$ (or $a_{\delta(N)} s_\mu \cdot f$) in terms of the a_λ's (or $a_{\delta(N)} s_\lambda$'s), where f is h_r or e_r. The abacus proofs of these Pieri rules and Theorem 59 in [9] all make use of a similar mechanism: beads on an abacus are moved according to certain rules to create a new abacus with a new partition. However, if beads collide while moving, the abacus instead cancels with another abacus with a similar collision.

This next theorem describes the product of a monomial antisymmetric polynomial with an elementary symmetric polynomial.

Theorem 60. *For all $\mu \in \mathrm{Par}_N$ and all $r \geq 1$,*

$$a_{\mu+\delta(N)}(x_1, \ldots, x_N) \cdot e_r(x_1, \ldots, x_N) = \sum_{\substack{\lambda\in\mathrm{Par}_N: \\ \lambda\in\mathrm{V}(\mu,r)}} a_{\lambda+\delta(N)}(x_1, \ldots, x_N).$$

See [9, Thm. 11.42] for a bijective proof.

Example 61. Let $N = 4$ and $r = 2$. Then

$$e_2 \cdot a_{(1^2,0^2)+\delta(4)} = a_{(1^4)+\delta(4)} + a_{(2,1^2,0)+\delta(4)} + a_{(2^2,0^2)+\delta(4)}.$$

Similarly, we can describe the product of a monomial antisymmetric polynomial with a complete symmetric polynomial.

Theorem 62. *For all $\mu \in \mathrm{Par}_N$ and all $r \geq 1$,*

$$a_{\mu+\delta(N)}(x_1, \dots, x_N) \cdot h_r(x_1, \dots, x_N) = \sum_{\substack{\lambda \in \mathrm{Par}_N: \\ \lambda \in \mathrm{H}(\mu,r)}} a_{\lambda+\delta(N)}(x_1, \dots, x_N).$$

See [9, Thm. 11.44] for a bijective proof.

Example 63. Let $N = 4$ and $r = 2$. Then

$$h_2 \cdot a_{(1^2,0^2)+\delta(4)} = a_{(2,1^2,0)+\delta(4)} + a_{(3,1,0^2)+\delta(4)}.$$

2.11 Hall-Littlewood Polynomials

We now give algebraic definitions for three versions of the Hall-Littlewood polynomials, which are the central objects studied in this work. These definitions can be found in [11, Ch. III].

Definition 64. For a partition $\lambda \in \mathrm{Par}_N$, variables $x_1, \dots, x_N$, and an indeterminate t, define

$$R_\lambda(x_1, \dots, x_N; t) = \sum_{w \in S_N} w \cdot \left(x_1^{\lambda_1} \cdots x_N^{\lambda_N} \prod_{i<j} \frac{x_i - tx_j}{x_i - x_j} \right). \tag{2.2}$$

We will call R_λ the (unstable) *Hall-Littlewood polynomial indexed by* λ.

Example 65. For the partition $\lambda = (2, 1, 0) \in \mathrm{Par}_3$, a calculation leads to

$$R_{(2,1,0)}(x_1, x_2, x_3) = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + (2 - t - t^2)x_1x_2x_3 + x_2^2x_3 + x_1x_3^2 + x_2x_3^2.$$

Theorem 66. *R_λ is a symmetric polynomial in $x_1, \dots, x_N$ with coefficients in $\mathbb{Z}[t]$ and is homogeneous of degree $|\lambda|$.*

See [11, III.1.5] for a proof. For $w \in S_N$, we can use equation 2.1 to write

$$w \cdot \prod_{i<j} (x_i - x_j) = w \cdot a_{\delta(N)} = \mathrm{sgn}(w) a_{\delta(N)},$$

because $a_{\delta(N)}$ is antisymmetric. Using this identity, we can rewrite equation 2.2 to have the form

$$a_{\delta(N)}R_\lambda(x_1,\ldots,x_N) = \sum_{w\in S_N} \operatorname{sgn}(w)w\cdot\left(x_1^{\lambda_1}\ldots x_N^{\lambda_N}\prod_{i<j}(x_i - tx_j)\right). \tag{2.3}$$

This expression for $a_{\delta(N)}R_\lambda$ has a number of familiar components that motivate the combinatorial interpretation given in Chapter 3.

Theorem 67. *For all $\lambda \in \mathrm{Par}_N$, the coefficients of R_λ are divisible in $\mathbb{Z}[t]$ by $\prod_{i\geq 0}[m_i(\lambda)]!_t$.*

See [11] for a an algebraic proof. We provide a combinatorial proof in Theorem 110 below.

Definition 68. For a partition $\lambda \in \mathrm{Par}_N$, variables $x_1,\ldots,x_N$, and indeterminate t, define

$$P_\lambda(x_1,\ldots,x_N;t) = \frac{1}{\prod_{i\geq 0}[m_i(\lambda)]!_t}\cdot R_\lambda(x_1,\ldots,x_N).$$

We call P_λ the (stable) *Hall-Littlewood polynomial indexed by* λ.

Thanks to Theorem 67, the stable Hall-Littlewood polynomial P_λ is also a symmetric polynomial with coefficients in $\mathbb{Z}[t]$ and is homogeneous of degree $|\lambda|$. See Theorem 130 for the motivation for the term "stable".

Definition 69. For a partition $\lambda \in \mathrm{Par}_N$, variables $x_1,\ldots,x_N$, and indeterminate t, define

$$\begin{aligned} Q_\lambda(x_1,\ldots,x_N;t) &= \left(\prod_{i\geq 1}(1-t)(1-t^2)\cdots(1-t^{m_i(\lambda)})\right)P_\lambda(x_1,\ldots,x_N;t) && (2.4)\\ &= (1-t)^{\ell(\lambda)}\prod_{i\geq 1}[m_i(\lambda)]!_t\ P_\lambda(x_1,\ldots,x_N;t)\\ &= \frac{(1-t)^{\ell(\lambda)}}{[m_0(\lambda)]!_t}\ R_\lambda(x_1,\ldots,x_N;t). && (2.5)\end{aligned}$$

We call Q_λ the (variant) *Hall-Littlewood polynomial indexed by* λ.

Example 70. For the partition $\lambda = (2,2,0) \in \mathrm{Par}_3$,

$$R_{(2,2,0)} = (1+t)x_1^2x_2^2+(1-t^2)x_1^2x_2x_3+(1-t^2)x_1x_2^2x_3+(1+t)x_1^2x_3^2+(1-t^2)x_1x_2x_3^2+(1+t)x_2^2x_3^2,$$

$$P_{(2,2,0)} = x_1^2x_2^2 + (1-t)x_1^2x_2x_3 + (1-t)x_1x_2^2x_3 + x_1^2x_3^2 + (1-t)x_1x_2x_3^2 + x_2^2x_3^2,$$

and

$$\begin{aligned} Q_{(2,2,0)} = {}&(1-t)^2(1+t)x_1^2x_2^2+(1-t)^2(1-t^2)x_1^2x_2x_3+(1-t)^2(1-t^2)x_1x_2^2x_3+(1-t)^2(1+t)x_1^2x_3^2\\ &+(1-t)^2(1-t^2)x_1x_2x_3^2 + (1-t)^2(1+t)x_2^2x_3^2.\end{aligned}$$

Chapter 3

Combinatorial Interpretations of Hall-Littlewood Polynomials

We now introduce the objects used to provide a new combinatorial interpretation for $a_{\delta(N)}R_\lambda$, $a_{\delta(N)}P_\lambda$, and $a_{\delta(N)}Q_\lambda$, which will be our focus for the rest of this paper. The objects in question are ordered pairs of abaci and tournaments, called *abacus-tournaments*.

3.1 Abacus-Tournaments

Definition 71. A *tournament* τ on vertex set $[N] = \{1, 2, \ldots, N\}$ is a subset of $[N] \times [N]$ such that for all $i \neq j$ in $[N]$, exactly one of (i, j) and (j, i) appears in τ, and no pair (i, i) appears in τ. Let $\mathbb{T}_N$ denote the set of all tournaments with vertex set $[N]$.

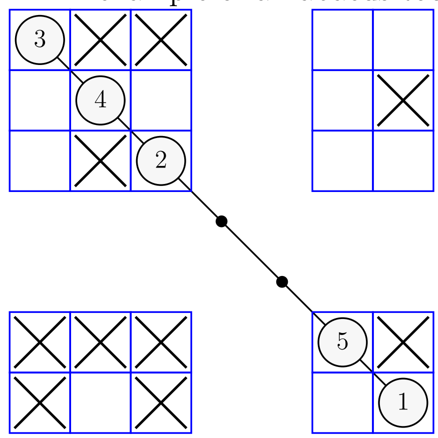
Example 72. Let $\tau = \{(1,5), (1,4), (2,1), (2,5), (2,3), (4,5), (4,2), (4,3), (3,1), (3,5)\}$. Then τ is a tournament in $\mathbb{T}_5$.

Definition 73. For a partition $\lambda \in \text{Par}_N$, an *abacus-tournament for* λ is an ordered triple $A = (\lambda, v, \tau)$ where (λ, v) is a labeled abacus (see Definition 53) and $\tau \in \mathbb{T}_N$ is a tournament. Define the *word* of A to be $w(A) = v \in S_N$, and the *tournament* of A to be $\tau(A) = \tau \in \mathbb{T}_N$. Let $\mathbb{AT}_\lambda = \{\lambda\} \times S_N \times \mathbb{T}_N$ denote the set of abacus-tournaments for the partition λ.

An abacus-tournament $A = (\lambda, v, \tau)$ can be displayed by forming a grid and placing the abacus runner for (λ, v) on the diagonal of the grid. Each column and row of the grid contains either precisely one bead of the abacus or precisely one bead gap of the abacus. Bead position 0 is at the top left corner. Fill in X's in the grid to encode the ordered pairs in τ: if $(v_i, v_j) \in \tau$, place an X in the column of the bead labeled v_i and in the row of the bead labeled v_j.

Example 74. Let $\lambda = (2^2, 0^3) \in \text{Par}_5(4)$, let $v = 15243 \in S_5$, and let $\tau \in T_5$ be as in Example 72. Then $A = (\lambda, v, \tau) \in \mathbb{AT}_\lambda$, and A is displayed in Figure 3.1.

Figure 3.1: An example of an abacus-tournament.



Each abacus-tournament A has a *signed weight*, denoted $\mathrm{swt}(A)$, that is calculated from its components via the following definitions.

Definition 75. For $A = (\lambda, v, \tau) \in \mathbb{AT}_\lambda$, define

- the (global) *outdegree* of a bead $v_i = k$ to be $\mathrm{out}_A(k) = |\{l : (k,l) \in \tau\}|$,
- the (global) *gap count* of a bead $v_i = k$ to be $\mathrm{gap}_A(k) = \lambda_i$, and
- $\mathrm{upset}(A) = \{(v_j, v_i) \in \tau : i < j\}$.

When the abacus-tournament under consideration is understood from context, we often drop the subscript A, writing $\mathrm{out}(k)$ and $\mathrm{gap}(k)$.

Definition 76. If $A = (\lambda, v, \tau)$ is an abacus-tournament for λ, define the *signed weight* of A to be

$$\mathrm{swt}(A) = \mathrm{sgn}(v)(-t)^{|\mathrm{upset}(A)|}\left(\prod_{k=1}^{N} x_k^{\mathrm{out}(k)+\mathrm{gap}(k)}\right). \tag{3.1}$$

Note that the signed weight of an abacus-tournament is a monomial in variables $x_1, \ldots, x_N$ with a coefficient in $\mathbb{Z}[t]$. The signed weight of A can be calculated from the diagram of A as follows. To calculate the exponent of x_k in $\mathrm{swt}(A)$, add the number of X's in the column of the bead labeled k, namely $\mathrm{out}(k)$, to the number of bead gaps on the diagonal northwest of k, namely $\mathrm{gap}(k)$. When $(v_j, v_i) \in \tau$ with $i < j$, the corresponding X appears below the main diagonal in the diagram. Consequently, to calculate the signed coefficient of $\mathrm{swt}(A)$, raise $(-t)$ to the number of X's below the diagonal in the diagram and multiply by the sign of $w(A)$.

Example 77. Let $\lambda = (2^2, 0^3) \in \text{Par}_4(5)$ and let $A = (\lambda, v, \tau)$ be as in Example 74. The bead $v_1 = 1$ has $\text{out}(1) = 2$ and $\text{gap}(1) = 2$, the bead $v_2 = 5$ has $\text{out}(5) = 0$ and $\text{gap}(5) = 2$, the bead $v_3 = 2$ has $\text{out}(2) = 3$ and $\text{gap}(2) = 0$, and so on. Also, there are six X's below the main diagonal corresponding to edges in

$$\text{upset}(A) = \{(3,1), (3,5), (4,5), (4,2), (2,1), (2,5)\},$$

so $|\,\text{upset}(A)| = 6$. Finally, $\text{inv}(v) = 4$, so

$$\text{swt}(A) = (-1)^4(-t)^6 x_1^4 x_2^3 x_3^2 x_4^3 x_5^2.$$

3.2 A Combinatorial Interpretation of $a_{\delta(N)}R_\lambda$

We now show that abacus-tournaments for λ give a combinatorial model for $a_{\delta(N)}R_\lambda$.

Lemma 78. *Let* $\lambda \in \text{Par}_N$ *and* $A = (\lambda, v, \tau) \in \mathbb{AT}_\lambda$. *Then*

$$\text{swt}(A) = \text{sgn}(v) x_{v_1}^{\lambda_1} \cdots x_{v_N}^{\lambda_N} \prod_{\substack{(v_i,v_j)\in\tau:\\ i<j}} x_{v_i} \prod_{\substack{(v_i,v_j)\in\tau:\\ i>j}} (-t x_{v_i}).$$

Proof. For a given bead v_i, edges of the form $(v_i, v_j) \in \tau$ where $i > j$ are counted by $\text{out}(v_i)$ and $\text{upset}(A)$, each such edge contributing a factor of $(-t)x_{v_i}$ to $\text{swt}(A)$. Edges $(v_i, v_j) \in \tau$ such that $i < j$ are only counted by $\text{out}(v_i)$, contributing a factor of x_{v_i} to $\text{swt}(A)$. Therefore

$$\begin{aligned} \text{swt}(A) &= \text{sgn}(v)(-t)^{|\,\text{upset}(A)|} \left(\prod_{k=1}^{N} x_k^{\text{out}(k)+\text{gap}(k)} \right) \\ &= \text{sgn}(v) x_{v_1}^{\text{gap}(v_1)} \cdots x_{v_N}^{\text{gap}(v_N)} (-t)^{|\,\text{upset}(A)|} \left(\prod_{k=1}^{N} x_k^{\text{out}(k)} \right) \\ &= \text{sgn}(v) x_{v_1}^{\lambda_1} \cdots x_{v_N}^{\lambda_N} \prod_{\substack{(v_i,v_j)\in\tau:\\ i<j}} x_{v_i} \prod_{\substack{(v_i,v_j)\in\tau:\\ i>j}} (-t x_{v_i}). \end{aligned} \tag{3.2}$$

□

Theorem 79. *For a partition* $\lambda \in \text{Par}_N$, *the abacus-tournaments for* λ *form a combinatorial model for* $a_{\delta(N)}R_\lambda(x_1, \ldots, x_N; t)$:

$$a_{\delta(N)}R_\lambda = \sum_{A \in \mathbb{AT}_\lambda} \text{swt}(A).$$

Proof. We first show

$$\prod_{i<j}(x_i - tx_j) = \sum_{\tau\in T_N} \prod_{\substack{(i,j)\in\tau: \\ i<j}} x_i \prod_{\substack{(i,j)\in\tau: \\ i>j}} (-tx_i).$$

In general, to multiply the set of factors of the form $(x_i - tx_j)$ together, choose either x_i or $-tx_j$ for each factor with $i < j$ and multiply these together. Then add together each of the possible products formed in the previous step. In the first step, each selection of x_i or $-tx_j$ helps to determine a tournament $\tau \in \mathbb{T}_N$: if x_i is chosen, then $(i, j) \in \tau$, and if $-tx_j$ is chosen, then $(j, i) \in \tau$. The product of the chosen factors encoded by τ is

$$\prod_{\substack{(i,j)\in\tau: \\ i<j}} x_i \prod_{\substack{(j,i)\in\tau: \\ i<j}} (-tx_j).$$

Summing these together and reindexing gives

$$\prod_{i<j}(x_i - tx_j) = \sum_{\tau\in T_N} \prod_{\substack{(i,j)\in\tau: \\ i<j}} x_i \prod_{\substack{(i,j)\in\tau: \\ i>j}} (-tx_i).$$

Now, consider the polynomial obtained by summing the signed weights of all abacus-tournaments for a partition $\lambda \in \mathrm{Par}_N$. By Lemma 78,

$$\begin{aligned}
\sum_{A\in\mathbb{AT}_\lambda} \mathrm{swt}(A) &= \sum_{(\lambda,v,\tau)\in\mathbb{AT}_\lambda} \mathrm{sgn}(v) x_{v_1}^{\lambda_1} \cdots x_{v_N}^{\lambda_N} \prod_{\substack{(v_i,v_j)\in\tau \\ :i<j}} x_{v_i} \prod_{\substack{(v_i,v_j)\in\tau: \\ i>j}} (-tx_{v_i}) \\
&= \sum_{(\lambda,v)\in\mathrm{LAbc}(\lambda)} \mathrm{sgn}(v) x_{v_1}^{\lambda_1} \cdots x_{v_N}^{\lambda_N} \sum_{\tau\in\mathbb{T}_N} \prod_{\substack{(v_i,v_j)\in\tau: \\ i<j}} x_{v_i} \prod_{\substack{(v_i,v_j)\in\tau \\ :i>j}} (-tx_{v_i}) \\
&= \sum_{v\in S_N} \mathrm{sgn}(v) v \cdot \left(x_1^{\lambda_1} \cdots x_N^{\lambda_N} \prod_{i<j}(x_i - tx_j) \right).
\end{aligned}$$

The last line of the above equation matches $a_{\delta(N)} R_\lambda$ by (2.3). □

Example 80. Table 3.1 lists the abacus-tournaments for $\lambda = (1, 0)$ and their signed weights. Observe that

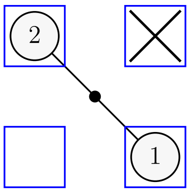
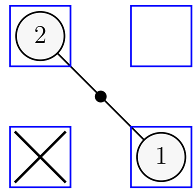
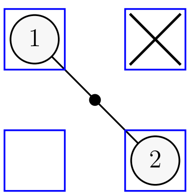
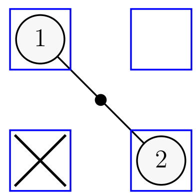
$$\sum_{A\in\mathbb{AT}_\lambda} \mathrm{swt}(A) = x_1^2 - x_2^2 = a_{\delta(2)} R_\lambda(x_1, x_2),$$

since

$$a_{\delta(2)} R_{(1,0)}(x_1, x_2) = x_1(x_1 - tx_2) - x_2(x_2 - tx_1) = x_1^2 - x_2^2$$

by (2.3).

Table 3.1: The signed weights of abacus-tournaments for $\lambda = (1, 0)$.

$A \in \mathbb{AT}_\lambda$	$w(A)$	$\tau(A)$	$\mathrm{swt}(A)$	$A \in \mathbb{AT}_\lambda$	$w(A)$	$\tau(A)$	$\mathrm{swt}(A)$
	12	$\{(1,2)\}$	x_1^2		12	$\{(2,1)\}$	$-tx_1x_2$
	21	$\{(2,1)\}$	$-x_2^2$		21	$\{(1,2)\}$	tx_1x_2

3.3 Blocks and Involutions

Our next goal is to develop combinatorial models for $a_{\delta(N)}P_\lambda$ and $a_{\delta(N)}Q_\lambda$ from the abacus-tournament model for $a_{\delta(N)}R_\lambda$. First, we need some technical constructions to help cancel objects in $\mathbb{AT}_\lambda$.

Definition 81. If $j \geq 1$ and $k \geq 0$, we call the set of word positions $\{j, j+1, \ldots, j+k\}$ a *block*.

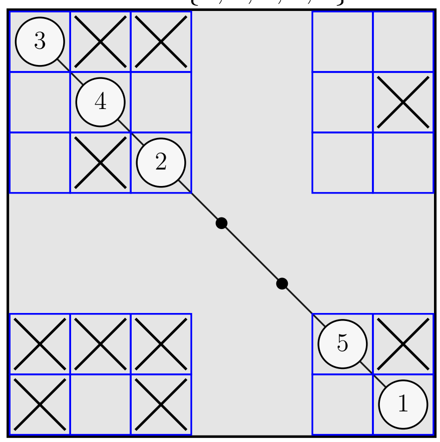
A block B can be visualized on the diagram of an abacus-tournament A by drawing a square box with corners on the main diagonal that captures the beads $v_j, v_{j+1}, \ldots, v_{j+k}$ where $w(A) = v_1 v_2 \cdots v_N$. We call this box the *square of block B*.

Definition 82. Given $\lambda \in \mathrm{Par}_N$, an abacus-tournament $A = (\lambda, v, \tau) \in \mathbb{AT}_\lambda$, and a block $B = \{j, j+1, \ldots, j+k\}$, let the *abacus-tournament restricted to block B*, denoted $A|_B$, refer to the set of beads $\{v_j, v_{j+1}, \ldots, v_{j+k}\}$ together with tournament edges $\{(v_i, v_l) \in \tau : i, l \in B\}$. Furthermore, define for $i \in B$

- the *local outdegree* of bead v_i in block B to be $\mathrm{out}_A^B(v_i) = |\{(v_i, v_l) \in \tau : l \in B\}|$,
- the *local gap count* of bead v_i in block B to be $\mathrm{gap}_A^B(v_i) = \lambda_i - \lambda_{j+k}$, and
- $\mathrm{upset}(A|_B) = \{(v_l, v_i) \in \tau : i, l \in B \text{ and } i < l\}$.

Visually, the local outdegree of a bead v_i can be determined by counting X's in the column of v_i that are also contained in the square delimiting the given block. The local gap count

Figure 3.2: The square of block $\{1, 2, 3, 4, 5\}$ in an abacus-tournament.



can be obtained by counting the bead gaps to the left of bead v_i that are contained in the square for the given block. If the abacus-tournament A under consideration is understood from context, then we may write $\text{out}^B(v_i)$ and $\text{gap}^B(v_i)$, dropping the subscript "A".

Example 83. The single block $B = \{1, \ldots, N\}$ contains every word position. See Figure 3.2. In this and later figures, the portion of an abacus-tournament restricted to a block is outlined in red.

Definition 84. Given $\lambda \in \text{Par}_N$, define $\text{Pos}_\lambda(i)$ be the set of all positions j for which $\lambda_j = i$. Define the collection of *λ-blocks* to be $\mathcal{B}_\lambda = \{\text{Pos}_\lambda(i) : i \geq 0, \text{Pos}_\lambda(i) \neq \emptyset\}$.

The square for $\text{Pos}_\lambda(i)$ encloses the consecutive beads on any abacus for λ that have exactly i gaps above them. Note that $|\text{Pos}_\lambda(i)| = m_i(\lambda)$, the number of parts of λ equal to i. Specifically, for i with $m_i(\lambda) > 0$, $\text{Pos}_\lambda(i)$ contains positions $N - (\sum_{j=0}^{i} m_j(\lambda)) + 1, \ldots, N - (\sum_{j=0}^{i-1} m_j(\lambda))$ of v. Likewise, the beads in the $(\lambda, v, \tau)|_{\text{Pos}_\lambda(i)}$ are located in columns $(\sum_{j=0}^{i-1} m_j(\lambda)) + i, \ldots, (\sum_{j=0}^{i} m_j(\lambda)) + i - 1$ of the abacus-tournament diagram.

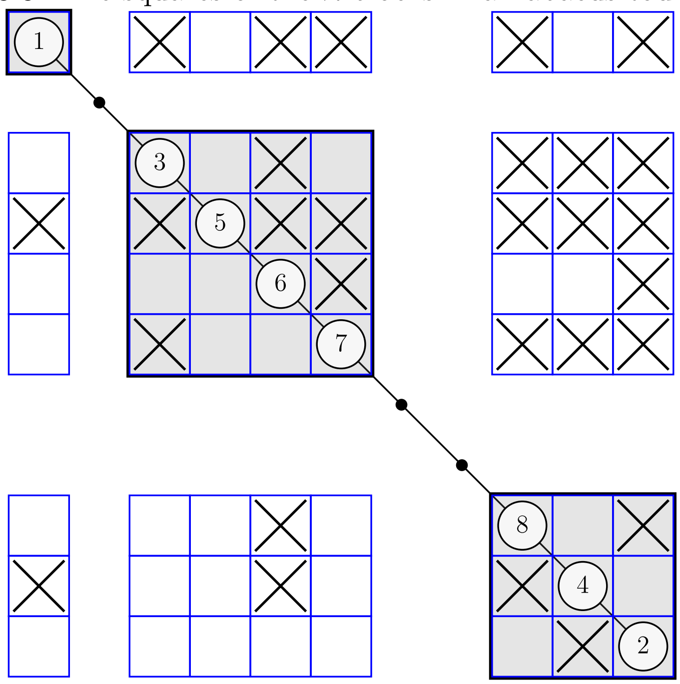
Example 85. Let $N = 8$ and $\lambda = (3^3, 1^4, 0)$. Then $\text{Pos}_\lambda(0) = \{8\}$, $\text{Pos}_\lambda(1) = \{4, 5, 6, 7\}$ and $\text{Pos}_\lambda(3) = \{1, 2, 3\}$. All other λ-blocks for this λ are empty. Figure 3.3 colors the squares of the λ-blocks in an abacus-tournament for λ.

Definition 86. A set of blocks $\mathcal{B}$ is *non-overlapping* iff there does not exist a bead position that is contained in more than one block of $\mathcal{B}$.

For example, the set of λ-blocks in Example 85 is non-overlapping. The λ/μ-blocks from Definition 139 and Example 140 below give a more complicated example of non-overlapping blocks.

Definition 87. Given $\lambda \in \text{Par}_N$, we say that two λ-blocks $\text{Pos}_\lambda(i) \neq \emptyset$ and $\text{Pos}_\lambda(j) \neq \emptyset$ with $i < j$ are *adjacent* if $\text{Pos}_\lambda(k) = \emptyset$ for all $i < k < j$. In general, if $B = \{b_1, \cdots, b_s\}, C = \{c_1, \ldots, c_r\}$ are blocks, we say B and C are *adjacent for* λ iff $b_1 = c_r + 1$ and $\lambda_{c_r} - \lambda_{b_1} > 0$.

Figure 3.3: The squares of the λ-blocks in an abacus-tournament.



Example 88. In Figure 3.3, $\mathrm{Pos}_\lambda(0)$ and $\mathrm{Pos}_\lambda(1)$ are adjacent and $\mathrm{Pos}_\lambda(1)$ and $\mathrm{Pos}_\lambda(3)$ are adjacent. There are no other pairs of adjacent λ-blocks in Figure 3.3.

3.4 Global and Local Exponent Collisions

Definition 89. An abacus-tournament $A = (\lambda, v, \tau)$ is said to have a *(global) exponent collision* if there exist l, m such that

$$\mathrm{out}(v_l) + \mathrm{gap}(v_l) = \mathrm{out}(v_m) + \mathrm{gap}(v_m).$$

If B is a block, then we say A has a *local exponent collision in* B if there exist $l, m \in B$ such that

$$\mathrm{out}^B(v_l) + \mathrm{gap}^B(v_l) = \mathrm{out}^B(v_m) + \mathrm{gap}^B(v_m).$$

Example 90. Let $N = 5$ and $\lambda = (2^2, 0^3)$. The abacus-tournament in Figure 3.2 has a (global) exponent collision between beads $v_2 = 5$ and $v_5 = 3$ because

$$\mathrm{out}(5) + \mathrm{gap}(5) = 0 + 2 = 2 + 0 = \mathrm{out}(3) + \mathrm{gap}(3).$$

Example 91. Let $N = 8$ and $\lambda = (3^3, 1^4, 0)$. The abacus-tournament in Figure 3.3 has a local exponent collision in block $B = \mathrm{Pos}_\lambda(1)$ between beads $v_5 = 6$ and $v_7 = 3$ because

$$\mathrm{out}^B(6) + \mathrm{gap}^B(6) = 2 + 0 = \mathrm{out}^B(3) + \mathrm{gap}^B(3).$$

Definition 92. If λ is a partition and $\mathcal{B}$ is a set of non-overlapping blocks, let $\mathbb{AT}_\lambda^{\mathcal{B}}$ denote the set of abacus-tournaments in $\mathbb{AT}_\lambda$ with no local exponent collisions in any of the blocks of $\mathcal{B}$.

Given a partition λ and a set of non-overlapping blocks $\mathcal{B}$, we can pair together abacus-tournaments with local exponent collisions in $\mathcal{B}$ in such a way that the two abacus-tournaments in each pair have the same weight but opposing signs. As pairs, matched abacus-tournaments with local exponent collisions then contribute a combined signed weight of zero to the total sum of signed weights. With this pairing, the set of all abacus-tournaments with local exponent collisions make no net contribution to the sum of signed weights. So, a sum over only abacus-tournaments *without local exponent collisions* gives the same sum of signed weights as a sum over *all abacus-tournaments*. Consider the following examples to see how such a pairing can work.

Example 93. Let $N = 5$, let $\lambda = (2^2, 0^3)$, let $B = \{1, 2, 3, 4, 5\}$, and let $A = (\lambda, v, \tau)$ be the abacus-tournament in Figure 3.2. Recall from Example 74 that the abacus (λ, v) has position set $\mathrm{pos}(A) = \lambda + \delta(N) = (6, 5, 2, 1, 0)$ and $w(A) = v = 15243$. The tournament τ is given by $\tau = \{(1,5),(1,4),(2,3),(2,5),(2,1),(4,3),(4,2),(4,5),(3,5),(3,1)\}$. The abacus-tournament A has signed weight $(-1)^4(-t)^6 x_1^4 x_2^3 x_3^2 x_4^3 x_5^2$. Since the block B includes every position, local outdegree and gap counts relative to B will be equal to its global outdegree and gap counts of A.

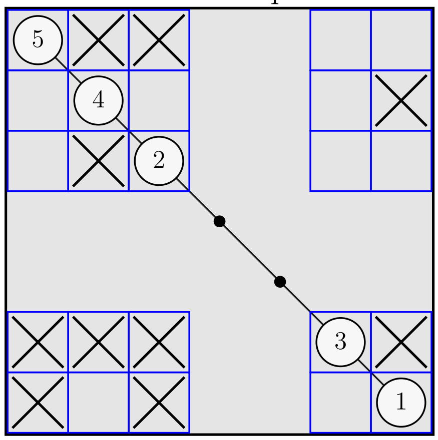
The abacus-tournament A has exponent collisions: for example, x_3 and x_5 have equal exponents in $\mathrm{swt}(A)$. Consider what happens to the abacus-tournament A if the bead labeled 3 in position 5 and the bead labeled 5 in position 2 switch positions, and the X's in the abacus-tournament diagram *do not move locations* to reflect this change. This produces the new abacus-tournament $A' = (\lambda, v', \tau')$ in Figure 3.4, where $\mathrm{pos}(A') = \lambda + \delta(N)$, $w(A') = v' = 13245$, and $\tau' = \{(1,4),(1,3),(2,5),(2,3),(2,1),(4,5),(4,2),(4,3),(5,3),(5,1)\}$. In the new tournament τ', every instance of a 3 in an ordered pair in τ has been replaced by a 5 and vice versa.

The new abacus-tournament has signed weight $(-1)^1(-t)^6 x_1^4 x_2^3 x_3^2 x_4^3 x_5^2$, so the sum of $\mathrm{swt}(A)$ and $\mathrm{swt}(A')$ is zero. Notice that A' also has an exponent collision between the bead labeled 3, now in position 2, and the bead labeled 5, now in position 5. Switching these beads without moving any X's in the diagram of A' restores the original abacus-tournament A.

Example 94. Let $N = 8$ and $\lambda = (3^3, 1^4, 0)$, and consider the set of λ-blocks $\mathcal{B}_\lambda$ and the abacus-tournament $A = (\lambda, v, \tau)$ in Figure 3.3. A has signed weight $(-1)^{18}(-t)^8 x_1^2 x_2^9 x_3^4 x_4^7 x_5^1 x_6^6 x_7^4 x_8^8$.

The previous example suggests switching beads labeled 3 and 7 because x_3 and x_7 have matching exponent values of 4 in $\mathrm{swt}(A)$. However, for the choice of blocks in our current

Figure 3.4: An abacus-tournament to pair with the one in Figure 3.2.

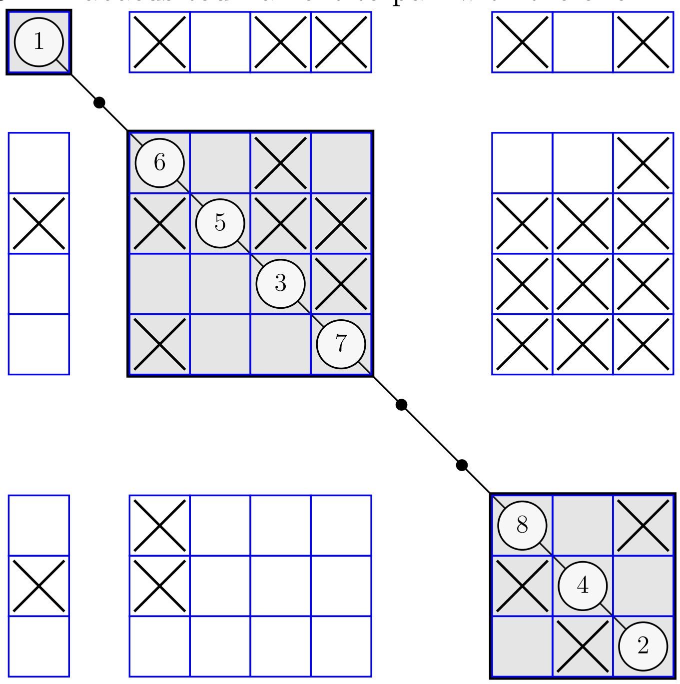


example, global values of outdegree and gap do not coincide with *local* values of outdegree and gap in various blocks. Block $\mathrm{Pos}_\lambda(1)$ has local exponent collisions in positions 4, 5, and 7, and block $\mathrm{Pos}_\lambda(3)$ has local exponent collisions in positions 1, 2, and 3. We will select a local exponent collision from $\mathrm{Pos}_\lambda(1)$, the leftmost λ-block with local exponent collisions. If, like in this example, there is more than one local exponent collision in the leftmost block, first maximize the position of the bead later in the word, then maximize the position of the bead earlier in the word. Beads $v_7 = 3$ and $v_5 = 6$ have the local exponent collision chosen by these rules in this example.

To produce a new abacus-tournament A', begin by switching beads v_5 and v_7. Any X's *within* the square for $\mathrm{Pos}_\lambda(1)$ again *do not move* to reflect the change. However, X's *outside* the square for $\mathrm{Pos}_\lambda(1)$ *do update* to reflect the new positions of v_5 and v_7. In the diagram, the parts of row 2 and row 4 outside of the square for $\mathrm{Pos}_\lambda(1)$ switch locations, and the parts of column 2 and column 4 outside the square for $\mathrm{Pos}_\lambda(1)$ switch. The abacus-tournament $A' = (\lambda, v', \tau')$ is displayed in Figure 3.5. Here, the set of ordered pairs that defines the new tournament τ' can be obtained by selectively switching instances of bead labels 3 and 6: if a 3 (resp. 6) appears in an ordered pair with another bead v_s, then change it to a 6 (resp. 3) if and only if $s \in \mathrm{Pos}_\lambda(1)$. For example, edge $(3, 7)$ in Figure 3.3 becomes $(6, 7)$ in Figure 3.5, whereas edge $(6, 8)$ in Figure 3.3 remains $(6, 8)$ in Figure 3.5.

The signed weight of A' is $(-1)^{15}(-t)^8 x_1^2 x_2^9 x_3^4 x_4^7 x_5^1 x_6^6 x_7^4 x_8^8$ so $\mathrm{swt}(A') = -\mathrm{swt}(A)$. Unlike in the first example, $\mathrm{swt}(A')$ is not obtained from $\mathrm{swt}(A)$ by simply permuting the subscripts of x_3 and x_6. The new abacus-tournament A' also has local exponent collisions in $\mathcal{B}_\lambda$. In fact, the local exponent collision selected by the rules above also involves beads labeled 3 and 6, now in positions 5 and 7 (respectively), so applying the same construction to A' restores the original A.

Figure 3.5: An abacus-tournament to pair with the one in Figure 3.3.



3.5 Cancellation Theorem for Local Exponent Collisions

The following theorem formalizes the pairing process for abacus-tournaments with local exponent collisions demonstrated in the last two examples. This leads to new combinatorial interpretations for $a_{\delta(N)}R_\lambda$ involving fewer signed objects.

Theorem 95. *If $\mathcal{B} = \{B_1, \ldots, B_r\}$ is a set of non-overlapping blocks, then for all $\lambda \in \mathrm{Par}_N$,*

$$\sum_{A \in \mathbb{AT}_\lambda} \mathrm{swt}(A) = \sum_{A \in \mathbb{AT}_\lambda^{\mathcal{B}}} \mathrm{swt}(A).$$

Proof. Define a sign-reversing, weight-preserving involution $I^{\mathcal{B}} : \mathbb{AT}_\lambda \to \mathbb{AT}_\lambda$ with fixed point set $\mathbb{AT}_\lambda^{\mathcal{B}}$ as follows. For $A = (\lambda, v, \tau) \in \mathbb{AT}_\lambda$, assign $I^{\mathcal{B}}(A) = A$ if $A \in \mathbb{AT}_\lambda^{\mathcal{B}}$. Otherwise, $A \notin \mathbb{AT}_\lambda^{\mathcal{B}}$ must have some block $B_i \in \mathcal{B}$ with a local exponent collision between beads v_l and v_m, so

$$\mathrm{out}^{B_i}(v_l) + \mathrm{gap}^{B_i}(v_l) = \mathrm{out}^{B_i}(v_m) + \mathrm{gap}^{B_i}(v_m).$$

If A has more than one block B_i with a local exponent collision, choose one where i is minimal. If block B_i has more than one local exponent collision, choose l and then m to be maximal in B_i. Construct the diagram for a new abacus-tournament $A' = (\lambda, v', \tau')$ from A's diagram by switching beads v_l and v_m and moving X's according to the following rules:

1. Do not move X's not involving v_l or v_m. According to this rule, if $j, k \notin \{l, m\}$ then $(v_j, v_k) \in \tau$ if and only if $(v_j, v_k) \in \tau'$.

2. If an X is contained in the square for block B_i it does not move. This rule says that $(v_l, v_m) \in \tau'$ if and only if $(v_m, v_l) \in \tau$, and for $k \in B_i$ such that $k \notin \{l, m\}$, $(v_m, v_k) \in \tau'$ if and only if $(v_l, v_k) \in \tau$, and $(v_l, v_k) \in \tau'$ if and only if $(v_m, v_k) \in \tau$.

3. Say v_l is in row and column r and v_m is in row and column s of the diagram. Switch the X's in rows r and s outside the square for B_i and switch the X's in columns r and s outside the square for B_i. This rule says that for $k \notin B_i$, $(v_l, v_k) \in \tau'$ if and only if $(v_l, v_k) \in \tau$, and $(v_m, v_k) \in \tau'$ if and only if $(v_m, v_k) \in \tau$, and similarly for (v_k, v_l) and (v_k, v_m).

Define $I^{\mathcal{B}}(A) = A'$.

First we show that $I^{\mathcal{B}}$ is sign-changing and weight-preserving. For all beads v_k except v_l and v_m, every X in the column of v_k stays in that column, and v_k has the same gap count in A and A'. Under rule 2 of the construction, outdegree and gap counts for v_l and v_m outside of the square for B_i do not change. Under rule 1, v_l and v_m interchange local exponent values. However, by assumption, v_l and v_m have equal local exponents, so the local exponents of v_l and v_m are unaffected. Therefore, $\operatorname{swt}(A)$ and $\operatorname{swt}(A')$ have the same exponents for each x-variable. Furthermore, the rules prohibit any X's from moving across the main diagonal. This ensures that $|\operatorname{upset}(A)| = |\operatorname{upset}(A')|$. The two words v and v' have opposing signs since switching v_l and v_m negates $\operatorname{sgn}(v)$. Thus $\operatorname{swt}(A') = -\operatorname{swt}(A)$.

Second, we show that $I^{\mathcal{B}}$ is an involution. If $A \notin \mathbb{AT}_\lambda^{\mathcal{B}}$, then $I^{\mathcal{B}}(A) = A'$ also has a local exponent collision. In fact, one such collision occurs in block B_i between beads labeled $v'_l = v_m$ and $v'_m = v_l$, and this exponent collision is the one used to compute $I(A')$. Note that since there is no local exponent collision in block B_j of A where $j < i$, the same will be true for A', since the rules don't change local outdegrees and local gap counts in B_j. Thus, a second application of $I^{\mathcal{B}}$ reverses the first application, and $I^{\mathcal{B}}(I^{\mathcal{B}}(A)) = I^{\mathcal{B}}(A') = A$. □

Corollary 96. *For a partition $\lambda \in \operatorname{Par}_N$ and $\mathcal{B}$ any set of nonoverlapping blocks,*

$$a_{\delta(N)} R_\lambda(x_1, \ldots, x_N) = \sum_{A \in \mathbb{AT}_\lambda^{\mathcal{B}}} \operatorname{swt}(A).$$

Proof. This follows directly from Theorems 79 and 95. □

The above proof sets a pattern for many later abacus-tournament proofs. If some attribute local to blocks guarantees that some of the abacus-tournaments have local exponent collisions, then we will apply a sign-reversing, weight-preserving involution like $I^{\mathcal{B}}$ to cancel abacus-tournaments. The next "cancellation" lemma abstracts the details of this process and will be referenced whenever we need to excise abacus-tournaments with local exponent collisions from a set T of abacus-tournaments.

Lemma 97. *Let $\mathcal{B}$ be a set of non-overlapping blocks, let $I^{\mathcal{B}}$ be the involution from the proof of Theorem 95, and let T and U be subsets of $\mathbb{AT}_\lambda$. Assume T and U have the following properties:*

1. $I^{\mathcal{B}}(T) \subseteq T$ *(i.e., T is* closed under $I^{\mathcal{B}}$*), and*
2. $U = T \cap \mathbb{AT}_\lambda^{\mathcal{B}}$.

Then

$$\sum_{A \in T} \mathrm{swt}(A) = \sum_{A \in U} \mathrm{swt}(A).$$

Proof. Consider the mapping $I^{\mathcal{B}}|_T$ obtained by restricting $I^{\mathcal{B}}$ to the set T. T is closed under $I^{\mathcal{B}}$, so $I^{\mathcal{B}}|_T$ is a function from T to T. Note that $I^{\mathcal{B}}|_T$ is a sign-reversing, weight-preserving involution because $I^{\mathcal{B}}$ is. Since the fixed points of $I^{\mathcal{B}}$ are $\mathbb{AT}_\lambda^{\mathcal{B}}$, the fixed points of $I^{\mathcal{B}}|_T$ are $T \cap \mathbb{AT}_\lambda^{\mathcal{B}}$. The lemma follows by using $I^{\mathcal{B}}|_T$ to cancel all summands indexed by objects in $T \setminus U$. □

3.6 Leading Abacus-Tournaments

We will next focus on properties of abacus-tournaments in blocks which involve some (not necessarily all) consecutive beads on an abacus.

Definition 98. Given $\lambda \in \mathrm{Par}_N$ and a block B, we say B *contains no gaps for* λ if for all $j, k \in B$, $\lambda_j = \lambda_k$.

Note that this definition is equivalent to requiring that B must be a subset of some λ-block in $\mathcal{B}_\lambda$.

Definition 99. Let $\lambda \in \mathrm{Par}_N$ and $A \in \mathbb{AT}_\lambda$. We say A is *leading* in a block $B = \{j, \ldots, j+k\}$ that contains no gaps for λ if all the X's in the square for B are above the main diagonal. If $\mathcal{B}$ is a set of non-overlapping blocks that all contain no bead gaps, let $\mathbb{LAT}_\lambda^{\mathcal{B}}$ denote the set of abacus-tournaments $A \in \mathbb{AT}_\lambda$ such that A is leading in all blocks $B \in \mathcal{B}$.

An abacus-tournament $A = (\lambda, v, \tau)$ is leading in block B if and only if for all $a, b \in B$ with $a < b$, $(v_a, v_b) \in \tau$, or equivalently $\mathrm{upset}(A|_B) = \emptyset$.

Example 100. Let $N = 8$ and $\lambda = (3^3, 1^4, 0)$. Let A be the abacus-tournament in Figure 3.6. The X's within each λ-block square, also displayed in Figure 3.6, are all found above the diagonal, so $A \in \mathbb{LAT}_\lambda^{\mathcal{B}_\lambda}$. This abacus-tournament has $w(A) = 61852743$ and signed weight $(-1)^{16}(-t)^8 x_1^8 x_2^5 x_3^3 x_4^3 x_5^6 x_6^8 x_7^4 x_8^4$.

Theorem 101. *Given $\lambda \in \mathrm{Par}_N$, let $\mathcal{B}$ be a collection of non-overlapping blocks that contain no gaps for λ. Then $\mathbb{LAT}_\lambda^{\mathcal{B}} \subseteq \mathbb{AT}_\lambda^{\mathcal{B}}$.*

Proof. Let $A = (\lambda, v, \tau) \in \mathbb{LAT}_\lambda^{\mathcal{B}}$ and consider $B = \{j, \ldots, j+k\} \in \mathcal{B}$. Since $\mathcal{B}$ contains no gaps for λ, $\mathrm{gap}^B(v_i) = 0$ for all $i \in B$. Furthermore, A leading in B forces $\mathrm{out}^B(v_i) = j+k-i$ for all $i \in B$. This forms a strictly decreasing sequence of local outdegrees, so A cannot have local exponent collisions in B. □

In particular, it is true that $\mathbb{LAT}_\lambda^{\mathcal{B}_\lambda} \subseteq \mathbb{AT}_\lambda^{\mathcal{B}_\lambda}$ because $\mathcal{B}_\lambda$ is a set of non-overlapping blocks that contains no bead gaps.

Example 102. Let $N = 8$, $\lambda = (3^3, 1^4, 0)$, and A be the abacus-tournament in Figure 3.6. In order, the beads $v_4 = 5, v_5 = 2, v_6 = 7$, and $v_8 = 4$ of the square for A for $\mathrm{Pos}_\lambda(1)$ have outdegrees $3, 2, 1, 0$.

Definition 103. Let $B = \{j, \ldots, j+k\}$ be a block. Let S_N^B denote the set of permutations in S_N that only permute positions contained in block B. In detail, $w \in S_N$ lies in S_N^B iff for all $j \in [N] \setminus B$, $w(j) = j$. For a set of non-overlapping blocks $\mathcal{B}$, let $S_N^{\mathcal{B}}$ denote the set of permutations in S_N that only permute positions within blocks of $\mathcal{B}$. This means $w(B) \subseteq B$ for all $B \in \mathcal{B}$ and $w(i) = i$ for i not in any B.

For $\mathcal{B} = \{B_1, \ldots, B_r\}$ non-overlapping, one can check that $S_N^{\mathcal{B}} \cong S_N^{B_1} \times \cdots \times S_N^{B_r}$.

Theorem 104. *Let $\mathcal{B} = \{B_1, \ldots, B_r\}$ be a set of non-overlapping blocks and let $k_i = |B_i|$. Then*

$$\prod_{i=1}^{r} [k_i]!_t = \sum_{a \in S_N^{\mathcal{B}}} t^{inv(a)}.$$

Proof. Write each $a \in S_N^{\mathcal{B}}$ as $a = a_1 \circ a_2 \circ \cdots \circ a_r$ with each $a_i \in S_N^{B_i}$. Since $\mathcal{B}$ is non-overlapping, if $j \in B_i$, then $a(j) \in B_i$ because $a_i(j) \in B_i$ and $a_m(j) = j$ for all $m \neq i$. Therefore, every inversion (j, k) of a has $j, k \in B_i$ for some i and there is a one-to-one correspondence between the inversions (j, k) of a where $j, k \in B_i$ and the inversions of a_i. Then

$$\mathrm{inv}(a) = \mathrm{inv}(a_1) + \mathrm{inv}(a_2) + \cdots + \mathrm{inv}(a_r),$$

and so

$$\sum_{a \in S_N^{\mathcal{B}}} t^{\mathrm{inv}(a)} = \sum_{a_1 \in S_N^{B_1}} \cdots \sum_{a_r \in S_N^{B_r}} t^{\mathrm{inv}(a_1) + \cdots + \mathrm{inv}(a_r)}.$$

Since the block B_i has size k_i for all $1 \leq i \leq r$, the result follows from Theorem 20. □

Example 105. Let $a = (2,3)(4,7,5)$ be a permutation on positions in v in cycle notation. This permutation has one-line form 13274658, and has 5 inversions: $\{(2,3),(4,5),(4,6),(4,7),(6,7)\}$. The permutation a only permutes word positions within λ-blocks so $a \in S_N^{\mathcal{B}_\lambda}$. In particular, $(2,3) \in S_N^{\mathrm{Pos}_\lambda(3)}$ and $(4,7,5) \in S_N^{\mathrm{Pos}_\lambda(1)}$.

Part 1 of the following lemma says that the beads in a block with no bead gaps and no local exponent collisions can be strictly ordered by local outdegree. According to part 2, the number of X's below the main diagonal in a block B in $(\lambda, v \circ a, \tau)$ is equal to the number of inversions of a when (λ, v, τ) is leading in B.

Lemma 106. *Given $\lambda \in \mathrm{Par}_N$, let $A = (\lambda, v, \tau)$ be an abacus-tournament, and let $B = \{j, \dots, j+k\}$ be a block that contains no bead gaps for λ. If A has no local exponent collisions in B then:*

1. *For $i \in B$, $\mathrm{out}_A^B(v_i) \in \{0,1,\dots,k\}$, and every value in $\{0,1,\dots,k\}$ is used exactly once.*
2. *If A is leading in B and $a \in S_N^B$, then the inversions of a are in 1-1 correspondence with the X's below the diagonal in the square of B in the abacus-tournament $(\lambda, v \circ a, \tau)$.*

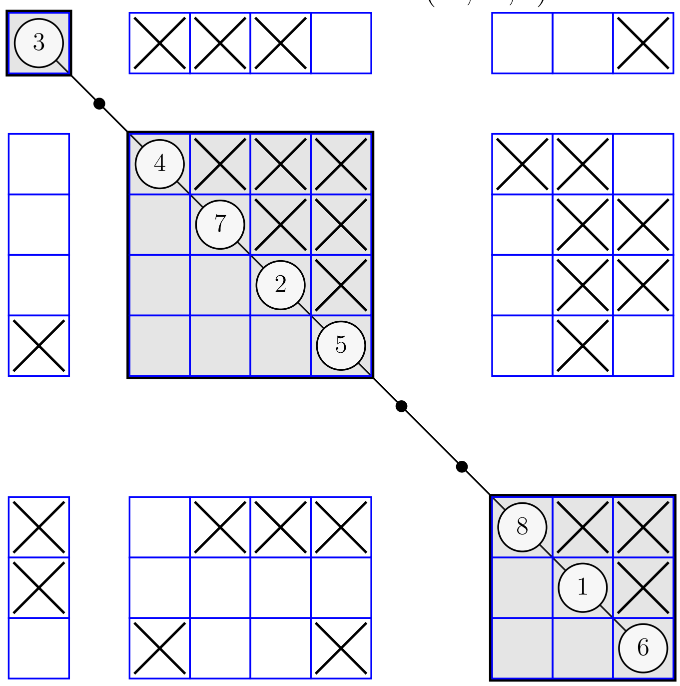
Proof. 1. There are $k+1$ beads in $A|_B$, so $0 \le \mathrm{out}_A^B(v_i) \le k$ for all i. Since B contains no bead gaps for λ, the local gap count of any bead in B is zero. Therefore, if two beads were to have the same local outdegree, they would have the same local exponent, contradicting the assumption that A has no local exponent collisions in B. Thus, each value in $\{0,1,\dots,k\}$ appears exactly once as a local outdegree in B.

2. Now say A is leading in B, let $a \in S_N^B$, let $u = v \circ a$, and let $t > s$ for $s, t \in B$. Then, because A is leading in B, $(v_s, v_t) \in \tau$. Write $a_q = t$ and $a_r = s$. If $q < r$, then t precedes s in the one-line form of a, so v_t precedes v_s in the one-line form of $u = v \circ a$. In this case, (q, r) is an inversion of a and $(v_s, v_t) \in \tau$ has its X below the diagonal in the diagram for (λ, u, τ). Conversely, if $q > r$, then s precedes t in the one-line form of a so v_s precedes v_t in u. In this case, (q, r) is not an inversion of a and $(v_s, v_t) \in \tau$ has its X above the diagonal in the diagram for (λ, u, τ). These remarks set up a bijection between inversions of a and X's below the main diagonal in the square for B of the diagram of (λ, v, τ). Specifically, this bijection sends inversion (q, r) to X for $(v_{a_r}, v_{a_q}) = (v_s, v_t)$ in (λ, v, τ). □

Example 107. Let $N = 8$ and $\lambda = (3^3, 1^4, 0)$. Let A be the abacus-tournament in Figure 3.6. We determined in Example 100 that $A \in \mathbb{LAT}_\lambda^{\mathcal{B}_\lambda}$ and A has $w(A) = 61852743$ and signed weight $(-1)^{16}(-t)^8 x_1^8 x_2^5 x_3^3 x_4^3 x_5^6 x_6^8 x_7^4 x_8^4$. Let $a = (2,3)(4,7,5) \in S_N^{\mathcal{B}_\lambda}$ be the permutation discussed in Example 105.

Now compare $A = (\lambda, v, \tau)$ to the abacus-tournament (λ, u, τ), where $u = v \circ a$ is obtained from v by permuting the positions of beads in the abacus v according to the position permutation a. The new abacus-tournament has word $u = 68145723$, and (λ, u, τ) is displayed

Figure 3.6: An abacus-tournament for $\lambda = (3^3, 1^4, 0)$ that is leading in $\mathcal{B}_\lambda$.



in Figure 3.7. Unlike in the bijection $I^{\mathcal{B}_\lambda}$ from the proof of Theorem 95, A and (λ, u, τ) have the same tournament component τ. This does change the position of X's in the new abacus-tournament diagram: the X's in a row or column of a bead v_i "follow" that bead to a new row or column as v_i changes position to become $u_{a^{-1}(i)}$. Observe that there are precisely 5 X's contained in λ-blocks that are below the diagonal in (λ, u, τ). The set of ordered pairs in τ that correspond to these X's is $\{(5,4),(7,4),(2,4),(2,7),(1,8)\}$. These bead labels correspond to the set of inversions of a.

Example 108. Let $N = 8$ and $\lambda = (3^3, 1^4, 0)$. Let (λ, u, τ) be the abacus-tournament in Figure 3.8. We will show that there is a unique pair $(a, (\lambda, v, \tau))$ such that $u = v \circ a$ and (λ, v, τ) is leading in $\mathcal{B}_\lambda$, which can be obtained from (λ, u, τ) by taking advantage of the fact that (λ, u, τ) has no local exponent collisions in blocks of $\mathcal{B}_\lambda$. First consider the block $\mathrm{Pos}_\lambda(1)$ of (λ, u, τ). Beads in $\mathrm{Pos}_\lambda(1)$ have local exponent contributions of $1, 0, 2, 3$, in order from right to left. We ask what position permutation would arrange the beads of $\mathrm{Pos}_\lambda(1)$ such that the local exponent collisions appear in strictly decreasing order from right to left. The position permutation $b = (4,7,5,6) \in S_N^{\mathrm{Pos}_\lambda(1)}$, in cycle notation, arranges the beads of $\mathrm{Pos}_\lambda(1)$ in the desired order, and $b' = (1,3) \in S_N^{\mathrm{Pos}_\lambda(3)}$ arranges the beads of $\mathrm{Pos}_\lambda(3)$ in the desired order. Set $c = b' \circ b = (1,3)(4,7,5,6)$. Then $v = u \circ c$ and $a = c^{-1}$. Figure 3.9 shows (λ, v, τ).

Figure 3.7: An abacus-tournament for $\lambda = (3^3, 1^4, 0)$ with a permuted abacus.

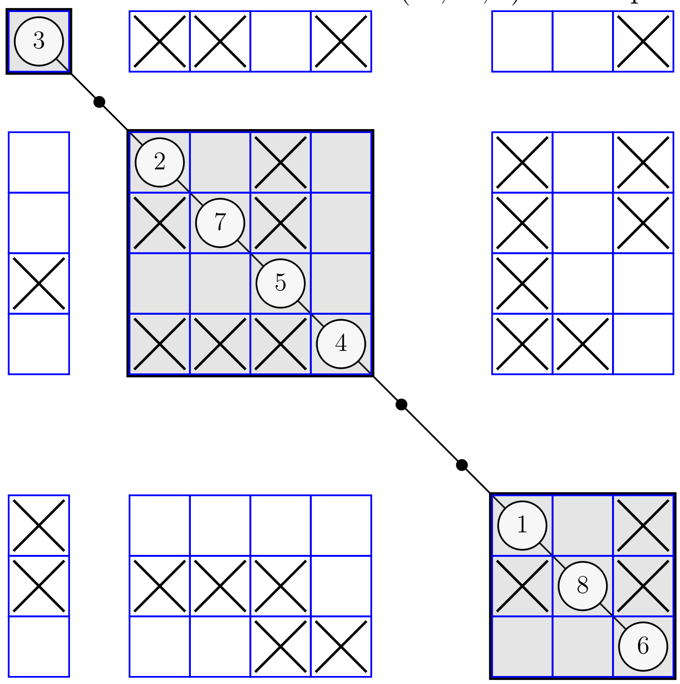


Figure 3.8: An abacus-tournament for $\lambda = (3^3, 1^4, 0)$ with local outdegrees in $\mathcal{B}_\lambda$ out of order.

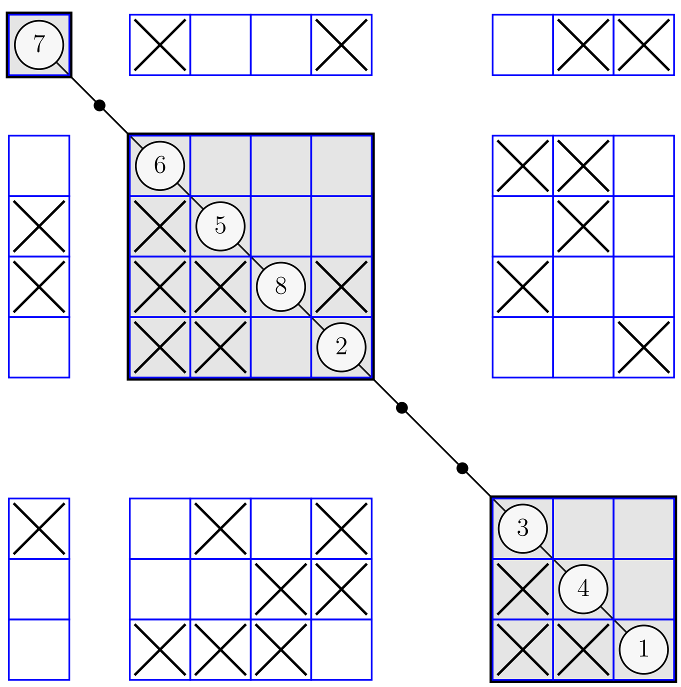
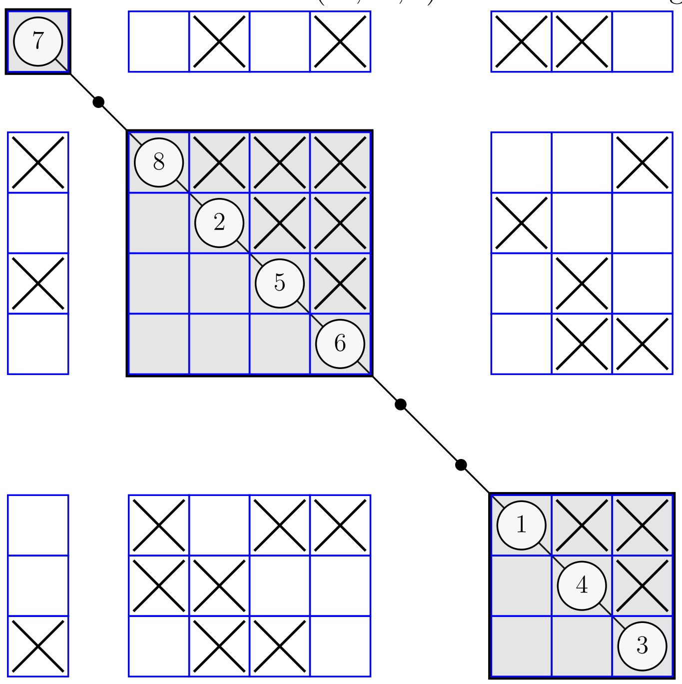


Figure 3.9: An abacus-tournament for $\lambda = (3^3, 1^4, 0)$ with local outdegrees in $\mathcal{B}_\lambda$ in order.



3.7 A Combinatorial Explanation of Division by t-Factorials

Theorem 109 formally describes the relationship illustrated in Example 108 between permutations of $S_N^{\mathcal{B}_\lambda}$, the abacus-tournaments of $\mathbb{AT}_\lambda^{\mathcal{B}_\lambda}$, and the leading abacus-tournaments of $\mathbb{LAT}_\lambda^{\mathcal{B}_\lambda}$.

Theorem 109. *Given $\lambda \in \mathrm{Par}_N$, let $\mathcal{B} = \{B_1, \ldots, B_r\}$ be a set of non-overlapping blocks that contains no bead gap for λ. Let $k_i = |B_i|$. Then $\sum_{A \in \mathbb{AT}_\lambda} \mathrm{swt}(A)$ is divisible by $\prod_{i=1}^r [k_i]!_t$ and*

$$\frac{1}{\prod_{i=1}^r [k_i]!_t} \cdot \sum_{A \in \mathbb{AT}_\lambda} \mathrm{swt}(A) = \sum_{A \in \mathbb{LAT}_\lambda^{\mathcal{B}}} \mathrm{swt}(A).$$

Proof. By Theorem 95,

$$\sum_{A \in \mathbb{AT}_\lambda} \mathrm{swt}(A) = \sum_{A \in \mathbb{AT}_\lambda^{\mathcal{B}}} \mathrm{swt}(A).$$

Define a signed weight-preserving bijection $J : S_N^{\mathcal{B}} \times \mathbb{LAT}_\lambda^{\mathcal{B}} \to \mathbb{AT}_\lambda^{\mathcal{B}}$ such that $(a, (\lambda, v, \tau)) \mapsto (\lambda, v \circ a, \tau)$. The function J maps into the codomain $\mathbb{AT}_\lambda^{\mathcal{B}}$ because $\mathbb{LAT}_\lambda^{\mathcal{B}} \subseteq \mathbb{AT}_\lambda^{\mathcal{B}}$ and J preserves local outdegrees and local gap counts within blocks of $\mathcal{B}$.

We show J is one-to-one and onto by constructing the inverse map J^{-1}. Let $(\lambda, v, \tau) \in \mathbb{AT}_\lambda^{\mathcal{B}}$. The local outdegrees of the beads with positions in $B_i \in \mathcal{B}$ are $0, 1, \ldots, k$ (in some order) by the first result of Lemma 106. For each i, there *exists* a *unique* position permutation $b^{(i)} \in S_N^{B_i}$ such that the beads in $v \circ b_i$ with positions in B_i are strictly decreasing (from right to left in the diagram) by local outdegree. Let $b = b^{(1)} \circ \cdots \circ b^{(r)}$ and $a = b^{-1}$. Then $a, b \in S_N^{\mathcal{B}}$, $(\lambda, v \circ b, \tau) \in \mathbb{LAT}_\lambda^{\mathcal{B}}$, and $J(a, (\lambda, v \circ b, \tau)) = (\lambda, v, \tau)$.

We now show J preserves signed weights, where we define the signed weight of a pair $(a, (\lambda, v, \tau))$ to be $t^{\mathrm{inv}(a)} \operatorname{swt}(\lambda, v, \tau)$. Fix $a \in S_N^{\mathcal{B}}$ and $(\lambda, v, \tau) \in \mathbb{LAT}_\lambda^{\mathcal{B}}$. J preserves the x-variable exponents because $(\lambda, v \circ a, \tau)$ has the same outdegrees in τ as (λ, v, τ) does. Also, beads of $(\lambda, v \circ a, \tau)$ all have positions in the same blocks as beads of (λ, v, τ); therefore, the gap count of each bead is the same in $(\lambda, v \circ a, \tau)$ and (λ, v, τ). Thus, for all bead values i,

$$\operatorname{gap}_{(\lambda, v\circ a, \tau)}(i) + \operatorname{out}_{(\lambda, v\circ a, \tau)}(i) = \operatorname{gap}_{(\lambda, v, \tau)}(i) + \operatorname{out}_{(\lambda, v, \tau)}(i).$$

The bijection J also preserves the t-coefficient. Any X that is below the main diagonal in the diagram of (λ, v, τ) and not in some $(\lambda, v, \tau)|_B$ for $B \in \mathcal{B}$ will still be below the main diagonal and not in $(\lambda, v, \tau)|_B$ in the diagram of $(\lambda, v \circ a, \tau)$. For $B \in \mathcal{B}$, some X's that are above the main diagonal in $(\lambda, v, \tau)|_B$ may be below the diagonal in the diagram of $(\lambda, v \circ a, \tau)$. By the second result of Lemma 106, there are precisely $\mathrm{inv}(a)$ total such X's. So

$$\operatorname{swt}(\lambda, v \circ a, \tau) = \operatorname{sgn}(a)(-t)^{\mathrm{inv}(a)} \operatorname{swt}(\lambda, v, \tau) = t^{\mathrm{inv}(a)} \operatorname{swt}(\lambda, v, \tau),$$

since $\operatorname{sgn}(v \circ a) = \operatorname{sgn}(v) \operatorname{sgn}(a)$ and $(-1)^{\mathrm{inv}(a)} = \operatorname{sgn}(a)$. □

3.8 A Combinatorial Interpretation for $a_{\delta(N)} P_\lambda$

Theorem 110. *For a partition λ with N parts, $a_{\delta(N)} R_\lambda$ is divisible in $\mathbb{Z}[t][x_1, \ldots, x_N]$ by $\prod_{i\geq 0}[m_i(\lambda)]!_t$, and*

$$a_{\delta(N)} P_\lambda(x_1, \ldots, x_N; t) = \frac{1}{\prod_{i\geq 0}[m_i(\lambda)]!_t} \cdot a_{\delta(N)} R_\lambda = \sum_{A \in \mathbb{LAT}_\lambda^{\mathcal{B}_\lambda}} \operatorname{swt}(A).$$

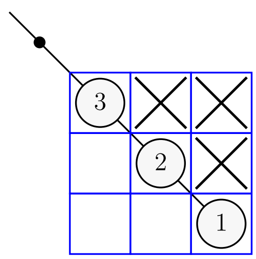
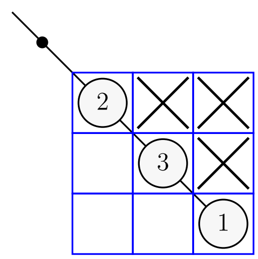
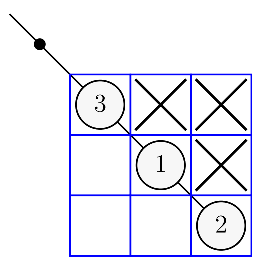
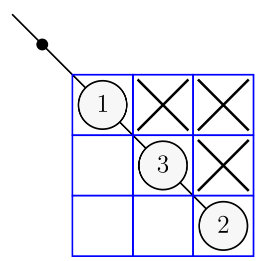
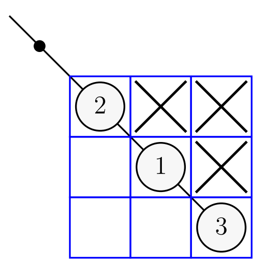
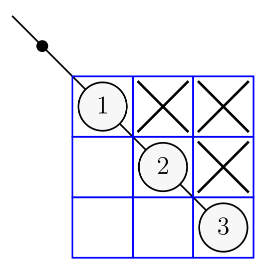
Proof. The first equality comes from Definition 68. The second equality follows directly from Theorem 109, taking $\mathcal{B}$ to be the set of λ-blocks from Definition 84, and using the combinatorial model for $a_{\delta(N)} R_\lambda$ from Theorem 79. Since the sum over $A \in \mathbb{LAT}_\lambda^{\mathcal{B}_\lambda}$ is visibly in $\mathbb{Z}[t][x_1, \ldots, x_N]$, the stated divisibility result follows. □

Example 111. Table 3.2 lists the abacus-tournaments for $\lambda = (1^3, 0)$ that are leading in the set of λ-blocks $\mathcal{B}_\lambda = \{\mathrm{Pos}_\lambda(1)\}$ and their signed weights.

Observe that

$$\sum_{A \in \mathbb{LAT}_\lambda^{\mathcal{B}_\lambda}} \operatorname{swt}(A) = x_1^2 x_2 - x_1^2 x_3 - x_1 x_2^2 + x_2^2 x_3 + x_1 x_3^2 - x_2 x_3^2 = a_{\delta(3)} P_\lambda(x_1, x_2, x_3).$$

Table 3.2: The abacus-tournaments for $\lambda = (1^3, 0)$ leading in $\mathcal{B}_\lambda$ and their signed weights.

$A \in \mathbb{LAT}_\lambda^{\mathcal{B}_\lambda}$	$\mathrm{swt}(A)$	$A \in \mathbb{LAT}_\lambda^{\mathcal{B}_\lambda}$	$\mathrm{swt}(A)$
	$x_1^2 x_2$		$-x_1^2 x_3$
	$-x_1 x_2^2$		$x_2^2 x_3$
	$x_1 x_3^2$		$-x_2 x_3^2$

3.9 A Combinatorial Interpretation for $a_{\delta(N)}Q_\lambda$

Recall from Definition 69 that

$$Q_\lambda(x_1,\ldots,x_N;t) = \frac{(1-t)^{\ell(\lambda)}}{[m_0(\lambda)]!_t}\, R_\lambda(x_1,\ldots,x_N;t).$$

Combinatorially, dividing R_λ by $[m_0(\lambda)]!_t$ corresponds to making only the 0th λ-block leading, whereas dividing by $\prod_{i\geq 0}[m_i(\lambda)]!_t$ makes all λ-blocks leading.

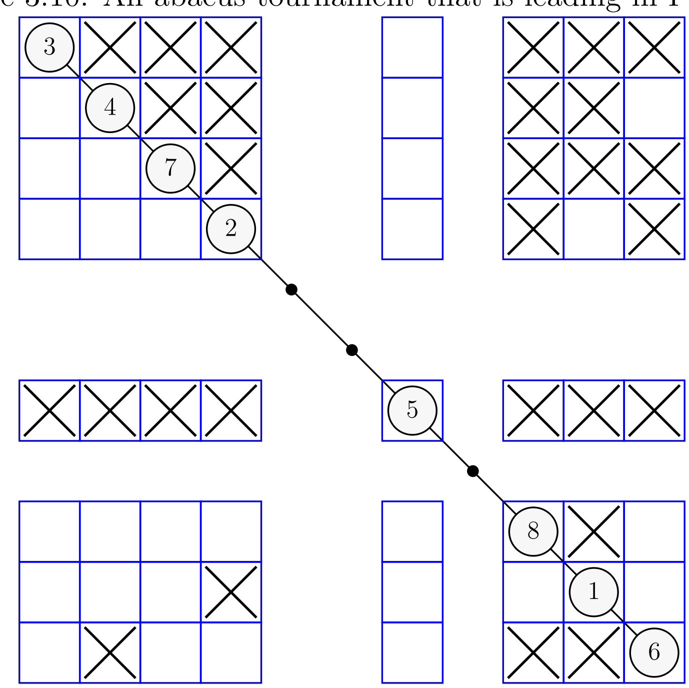
Definition 112. Given $\lambda \in \mathrm{Par}_N$, let $\mathcal{B}_{\lambda,0} = \{\mathrm{Pos}_\lambda(0)\}$ be the set consisting of the single block $\mathrm{Pos}_\lambda(0)$.

By Theorem 109 applied to the set of blocks $\{\mathrm{Pos}_\lambda(0)\}$,

$$R_\lambda/[m_0(\lambda)]!_t = \sum_{A\in\mathbb{LAT}_\lambda^{\mathcal{B}_{\lambda,0}}} \mathrm{swt}(A). \tag{3.3}$$

Example 113. The abacus-tournaments for the partition $\lambda = (3^3, 2, 0^4)$ in Figure 3.10 is leading in $\mathrm{Pos}_\lambda(0)$.

Figure 3.10: An abacus-tournament that is leading in $\mathrm{Pos}_\lambda(0)$.



We adjust the combinatorial model to account for the factors of $(1-t)$ in Q_λ as follows.

Definition 114. A *shaded abacus-tournament for* λ is an abacus-tournament $A_* \in \mathbb{LAT}_\lambda^{\mathcal{B}_{\lambda,0}}$ where each bead of A_* that is not in $\mathrm{Pos}_\lambda(0)$ is either *shaded* or *unshaded*. The *signed weight* of a shaded abacus-tournament A_*, denoted $\mathrm{swt}(A_*)$, is defined by

$$\mathrm{swt}(A_*) = (-t)^{n(A_*)} \cdot \mathrm{swt}(A),$$

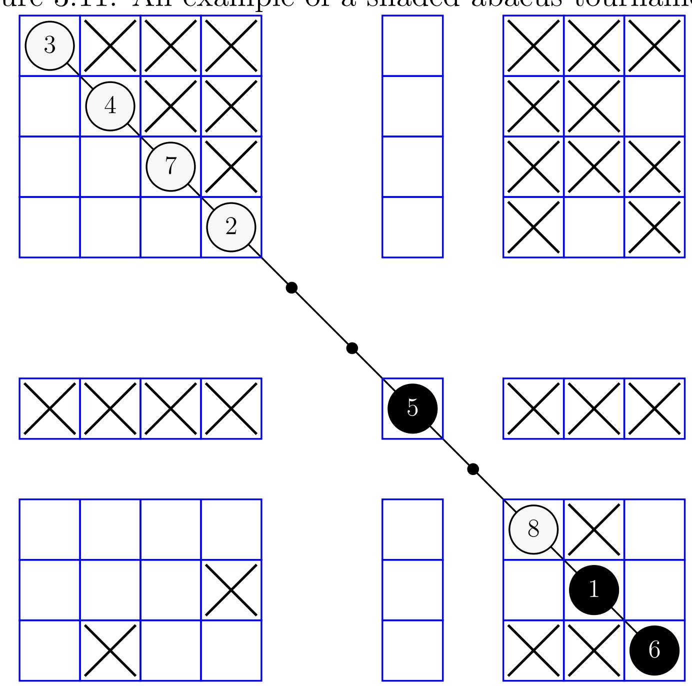
where $n(A_*)$ is the number of shaded beads in A_*, and A is A_* with all shading removed. Let $\mathbb{AT}_\lambda^*$ denote the set of shaded abacus-tournaments for λ.

Example 115. The shaded abacus-tournament for $\lambda = (3^3, 2, 0^4)$ displayed in Figure 3.11 has

$$\mathrm{swt}(A_*) = (-1)^{16}(-t)^8(-t)^3 x_1^9 x_2^5 x_3^1 x_4^3 x_5^2 x_6^7 x_7^3 x_8^9.$$

In particular, the three shaded beads are colored black and contribute an additional factor of $(-t)^3$.

Figure 3.11: An example of a shaded abacus-tournament.

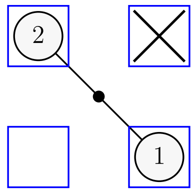
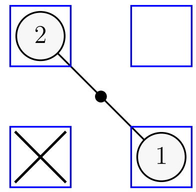
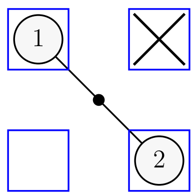
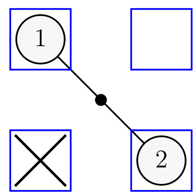
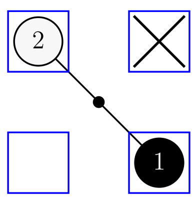
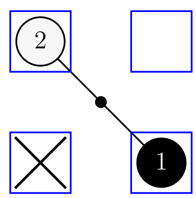
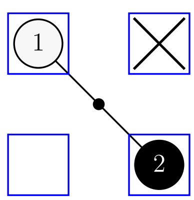
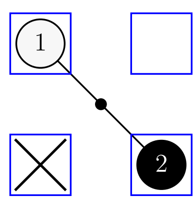


Theorem 116. *Given a partition* $\lambda \in \mathrm{Par}_N$,

$$a_{\delta(N)} \cdot Q_\lambda(x_1, \ldots, x_N; t) = \sum_{A_* \in \mathbb{AT}_\lambda^*} \mathrm{swt}(A_*).$$

Proof. We apply a factor of $(1-t)^{\ell(\lambda)}$ to equation 3.3 using a general distributive law where each of power of $-t$ corresponds to one of the $\ell(\lambda)$ beads in A_* being shaded. $\square$

Table 3.3: The shaded abacus-tournaments for $\lambda = (1, 0)$.

$A_* \in \mathbb{AT}^*_\lambda$	$\mathrm{swt}(A_*)$	$A_* \in \mathbb{AT}^*_\lambda$	$\mathrm{swt}(A_*)$
	x_1^2		$-tx_1x_2$
	$-x_2^2$		tx_1x_2
	$-tx_1^2$		$t^2x_1x_2$
	tx_2^2		$-t^2x_1x_2$

Example 117. Table 3.3 lists the shaded abacus-tournaments for $\lambda = (1, 0)$ and their signed weights. Observe that the four objects on the right cancel, so

$$\sum_{A_* \in \mathbb{AT}^*_\lambda} \mathrm{swt}(A_*) = x_1^2 - x_2^2 - tx_1^2 + tx_2^2 = (1-t)(x_1^2 - x_2^2) = a_{\delta(2)} Q_\lambda(x_1, x_2).$$

Chapter 4

Local Exponent Collision Lemmas

We have seen that abacus-tournaments with local exponent collisions can often be canceled using Lemma 97. We now describe new situations in which local exponent collisions are derived from arrangements of edges *between* two adjacent λ-blocks. If an abacus-tournament has too many upset edges between two λ-blocks or the upset edges do not appear in a specific configuration, then the abacus-tournament is guaranteed to have local exponent collisions. The following two lemmas will be referenced frequently in later chapters.

4.1 The Single Gap Collision Lemma

In the first lemma, we consider when an abacus-tournament has a single bead gap between the squares of two adjacent blocks.

Lemma 118 (The Single Gap Collision Lemma)**.** *Fix $\lambda \in \mathrm{Par}_N$, and fix blocks B, C such that $B \subseteq \mathrm{Pos}_\lambda(i)$, $C \subseteq \mathrm{Pos}_\lambda(i+1)$, and $B \cup C$ is a block. If $A = (\lambda, v, \tau) \in \mathbb{AT}_\lambda$ is leading in B and C, and there exist $j \in B$ and $k \in C$ such that $(v_j, v_k) \in \tau$, then A has a local exponent collision in block $B \cup C$.*

Proof. We will show that abacus-tournament A has a local exponent collision in $B \cup C$ by finding a local exponent collision in a weakly smaller block $D \subseteq B \cup C$. By assumption $\mathrm{upset}(A|_{B\cup C}) \neq \emptyset$. Let q be the smallest word position in C such that $(v_s, v_q) \in \tau$ for some $s \in B$. Let t be the largest word position in B such that $(v_t, v_r) \in \tau$ for some $r \in C$. Then let $D = \{q, \ldots, t\} \subseteq B \cup C$.

Fix $a, c \in B \cup C$. By our construction of D, if $a \in D$ and $c \notin D$, then either $(v_a, v_c) \in \tau$ if $a < c$ or $(v_c, v_a) \in \tau$ if $a > c$. Consequently, given $a \in D$,

$$\mathrm{out}^{B\cup C}(v_a) = \mathrm{out}^{D}(v_a) + |\{c \in B : c > t\}|.$$

If v_a and v_b cause a local exponent collision in D, then

$$\begin{aligned} \text{out}^{B\cup C}(v_a) + \text{gap}^{B\cup C}(v_a) &= \text{out}^D(v_a) + |\{c \in B : c > t\}| + \text{gap}^D(v_a) && (4.1)\\ &= \text{out}^D(v_b) + |\{c \in B : c > t\}| + \text{gap}^D(v_b) && (4.2)\\ &= \text{out}^{B\cup C}(v_b) + \text{gap}^{B\cup C}(v_b). && (4.3) \end{aligned}$$

Therefore, if A has a local exponent collision in D, then A has a local exponent collision in $B \cup C$.

To complete the proof, we must show at least one exponent collision occurs in D. In general, for $a \in D$, values of $\text{out}^D(v_a) + \text{gap}^D(v_a)$ must lie in the range 0 to $(|D| - 1) + 1$. We first show that no bead can have value 0. If $a \in C \cap D$, then $\text{gap}^D(v_a) = 1$. Since A is leading in B and C, A is also leading in $B \cap D$ and $C \cap D$. If $a \in B \cap D$ and $a \neq t$, then $\text{out}^D(v_a) \geq 1$ because A is leading in $B \cap D$. And since $(v_t, v_r) \in \tau$ for some $r \in C \cap D$ (by choice of t and q), $\text{out}^D(v_t) \geq 1$. We conclude that

$$\text{out}^D(v_a) + \text{gap}^D(v_a) \geq 1$$

for all $a \in D$.

Similarly, we show no bead v_a can have $\text{out}^D(v_a) + \text{gap}^D(v_a) = (|D| - 1) + 1$. Observe that $\text{gap}^D(v_a) = 0$ for all $a \in B \cap D$. And if $a \in C \cap D$ such that $a \neq q$, then $(v_q, v_a) \in \tau$ since A is leading in $C \cap D$, so $\text{out}^D(v_a) < |D| - 1$. Also, for some $s \in B \cap D$, $(v_s, v_q) \in \tau$ so $\text{out}^D(v_q) < |D| - 1$. So for all $a \in D$,

$$\text{out}^D(v_a) + \text{gap}^D(v_a) < (|D| - 1) + 1.$$

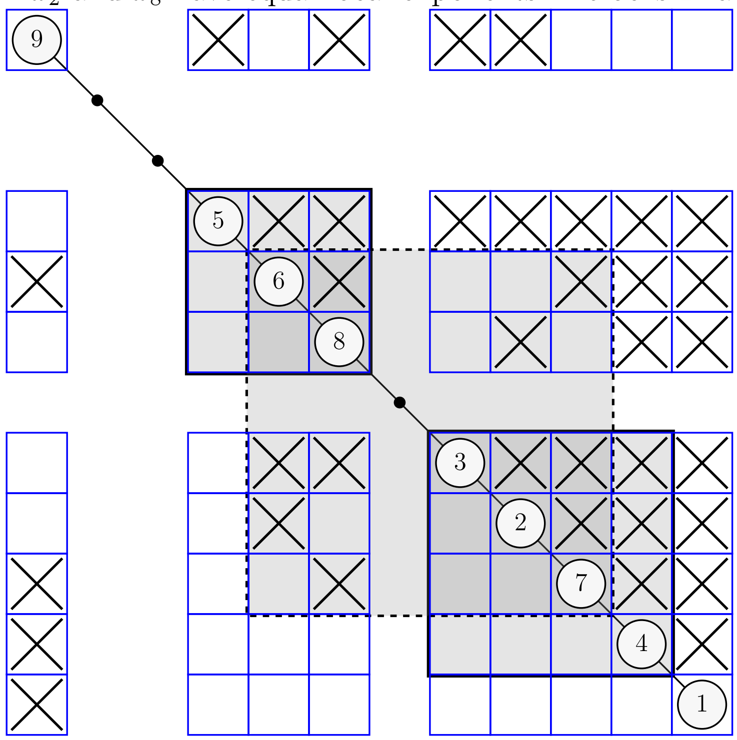
In conclusion, we have $|D|$ beads with values of $\text{out}^D(v_a) + \text{gap}^D(v_a)$ in the restricted range from 1 to $|D|-1$. At least two beads in D must have equal local exponents by the pigeonhole principle. □

Example 119. Let $N = 9$, let $\lambda = (3^5, 2^3, 0)$, and let A be the abacus-tournament displayed in Figure 4.1. The blocks $B = \{6, 7, 8\} = \text{Pos}_\lambda(2)$ and $C = \{2, 3, 4, 5\} \subset \text{Pos}_\lambda(3)$ are separated by a single gap. According to the proof of Lemma 118, we set block $D = \{3, 4, 5, 6, 7\}$. The square for D is displayed with a dashed outline in Figure 4.1. Observe that the local exponents of $x_{v_6} = x_8$ and $x_{v_4} = x_2$ are both 3 in D and both 4 in $B \cup C$.

4.2 The Single Bead Collision Lemma

For the next lemma, consider two adjacent blocks B and C, where $B \subset \text{Pos}_\lambda(i)$ for some i, $C = \{a\}$ has size one, and $D = B \cup C$ is a block. In contrast to the setup for the Single Gap Collision Lemma, the squares of B and C in an abacus-tournament diagram can be separated by any number of bead gaps. In this case, an abacus-tournament $A = (\lambda, v, \tau) \in \mathbb{AT}_\lambda$ has

Figure 4.1: x_2 and x_8 have equal local exponents in blocks D and $B \cup C$.



precisely $|B|$ edges (v_m, v_n) such that exactly one of m, n is in $C = \{a\}$ and exactly one of m, n is in B. In the diagram for A, these edges correspond to X's either in the lowest row in the square of $B \cup C$ below the main diagonal or in the rightmost column in the square of D above the main diagonal. If we further require that $A \in \mathbb{LAT}_\lambda^{\mathcal{B}}$ where $B, C \in \mathcal{B}$, then A must be leading in block B. Consequently,

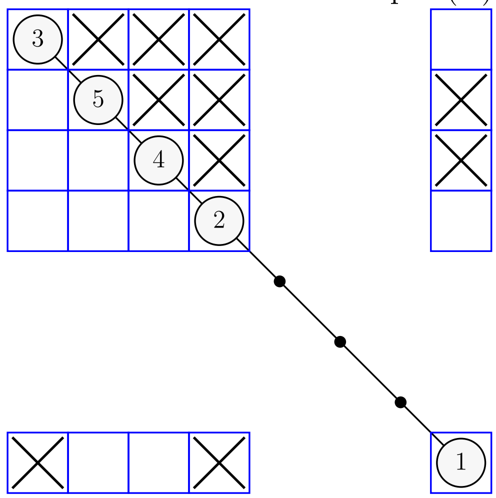
$$\text{upset}(A|_D) \subseteq \{(v_m, v_n) : |\{m,n\} \cap \{a\}| = 1 \text{ and } |\{m,n\} \cap B| = 1\}.$$

Definition 120. Let $\lambda \in \text{Par}_N$, and fix two adjacent blocks B, C such that $B \subseteq \text{Pos}_\lambda(i)$ for some i, $C = \{a\}$ for some $a \in \{1, \ldots, N\}$, and $D = B \cup C$ is a block. Let $A = (\lambda, v, \tau) \in \mathbb{LAT}_\lambda^{\mathcal{B}}$ where $B, C \in \mathcal{B}$. We say that $\text{upset}(A|_D)$ is *right-justified* iff, for all $k \in B$ with $k \neq a+1$, $(v_k, v_a) \in \tau$ implies $(v_{k-1}, v_a) \in \tau$.

Informally, $\text{upset}(A|_D)$ is right-justified iff the X's in its row below the main diagonal in the square of D all appear to the right of any edge gaps. Equivalently, the X's in its column in D above the main diagonal must be top-justified.

Example 121. Let $N = 5$ and $\lambda = (3, 0^4) \in \text{Par}_5$. Consider the abacus-tournaments A,A',A'' $\in \mathbb{LAT}_\lambda^{\mathcal{B}_\lambda}$ displayed in Figures 4.2, 4.3, and 4.4 (respectively). Here, λ has two non-empty λ-blocks which together give every position in $\{1, \ldots, 5\}$: $C = \text{Pos}_\lambda(3) = \{1\}$ and $B = \text{Pos}_\lambda(0) = \{2, 3, 4, 5\}$. Consequently, attributes local to $\text{Pos}_\lambda(1) \cup \text{Pos}_\lambda(3)$ are

Figure 4.2: An abacus-tournament A such that upset(A) is not right-justified.



also global attributes for these abacus-tournaments. There are three gaps between the two blocks.

The set upset(A) is not right-justified because $(3,1) \in \tau$ but $(5,1) \notin \tau$, where $A = (\lambda, v, \tau)$. The set upset(A') is right-justified and

$$|\text{upset}(A')| = |\{(2,1),(4,1),(5,1)\}| = 3.$$

The set upset(A'') is right-justified and $|\text{upset}(A'')| = 2$.

Lemma 122 (The Single Bead Collision Lemma)**.** *Let $\lambda \in \text{Par}_N$ and fix adjacent blocks $B \subseteq \text{Pos}_\lambda(j)$ and $C \subseteq \text{Pos}_\lambda(i)$, where $j < i$, separated by $c = i - j$ gaps, such that $|C| = 1$. Let $D = B \cup C$ be a block (so B must be the smallest $|B|$ members of $\text{Pos}_\lambda(j)$ and C must be the largest $|C|$ members of $\text{Pos}_\lambda(i)$). If $A \in \mathbb{LAT}_\lambda^{\mathcal{B}}$ where $B, C \in \mathcal{B}$ has $|\text{upset}(A|_D)| \geq c$ or $\text{upset}(A|_D)$ is not right-justified, then A has a local exponent collision in D.*

Proof. Let $C = \{a\}$ for some $a \in \{1, \dots, N\}$, and fix $A = (\lambda, v, \tau) \in \mathbb{LAT}_\lambda^{\mathcal{B}}$. First assume $\text{upset}(A|_D)$ is not right-justified. Then there exists $k \in B$ such that $k \neq a+1$, $(v_k, v_a) \in \tau$, but $(v_{k-1}, v_a) \notin \tau$. In other words, the diagram of A has an X in the lower left row of the square of D and no X in the location directly to the right. We observe that

$$\text{gap}^D(v_{k-1}) = 0 = \text{gap}^D(v_k).$$

Also, $(v_k, v_a) \in \tau$, so $\text{out}^D(v_k) = \text{out}^B(v_k) + 1$. On the other hand, $(v_{k-1}, v_a) \notin \tau$ so $\text{out}^D(v_{k-1}) = \text{out}^B(v_{k-1})$. Since A is leading in B, $\text{out}^B(v_{k-1}) = \text{out}^B(v_k) + 1$. Therefore

$$\begin{aligned} \text{out}^D(v_{k-1}) &= \text{out}^B(v_{k-1}) \\ &= \text{out}^B(v_k) + 1 \\ &= \text{out}^D(v_k). \end{aligned}$$

Figure 4.3: An abacus-tournament A' such that upset(A') is right-justified and $|\text{upset}(A')| \geq 3$.

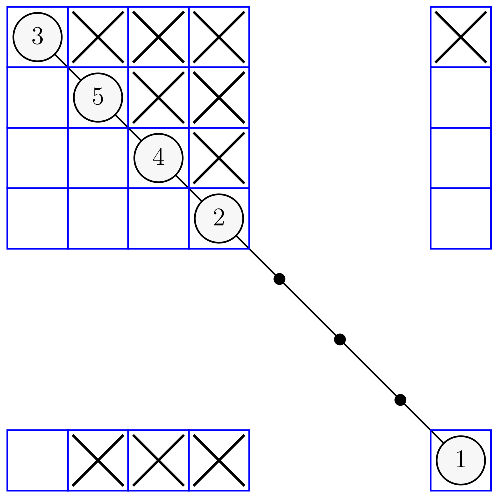
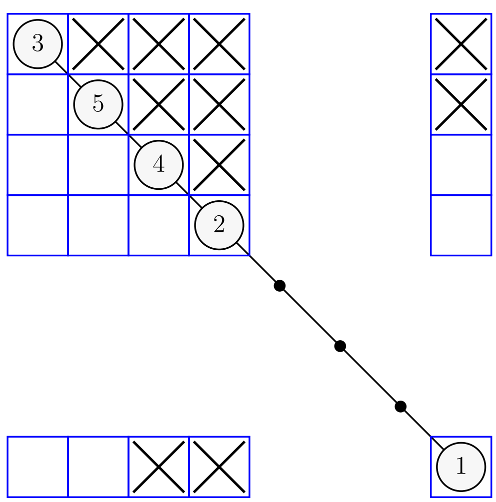


Figure 4.4: An abacus-tournament A'' such that upset(A'') is right-justified and $|\text{upset}(A'')| < 3$.



We conclude that A has a local exponent collision between v_k and v_{k-1} in D.

Now assume $\text{upset}(A|_D)$ is right-justified and $p = |\text{upset}(A|_D)| \geq c$. We will show that v_a must have a local exponent collision with some $v_k \in B$ such that $(v_k, v_a) \in \tau$. Indeed, $\text{gap}^D(v_a) = c$ and $\text{out}^D(v_a) = |B| - p$. Consequently,

$$|B| - p + 1 \leq \text{gap}^D(v_a) + \text{out}^D(v_a) \leq |B|,$$

since $1 \leq c \leq p$. If $a < k \leq a+p$, then $(v_k, v_a) \in \tau$, $\text{gap}^D(v_k) = 0$, and $\text{out}^D(v_k) = |B| - (k - a) + 1$. Therefore, the sequence of values $\text{out}^D(v_k) + \text{gap}^D(v_k)$, for $a < k \leq p + a$, accounts for every integer between $|B| - p + 1$ and $|B|$, including the value of $\text{out}^D(v_a) + \text{gap}^D(v_a)$. We conclude A has a local exponent collision in D. □

Example 123. Let $N = 5$ and $\lambda = (3, 0^4) \in \text{Par}_5$. Consider again the abacus-tournaments A and A' displayed in Figures 4.2 and 4.3. As we observed in the previous example, exponent collisions local to $\text{Pos}_\lambda(3) \cup \text{Pos}_\lambda(0)$ are equivalent to global exponent collisions. Here, $c = 3$. The set $\text{upset}(A)$ is not right-justified, and x_3 and x_5 have equal exponents of 1 in $\text{swt}(A)$, as predicted by Lemma 122. Similarly, according to Lemma 122, A' must have an exponent collision involving x_1 in $\text{swt}(A')$ because $|\text{upset}(A')| \geq 3$. In this case, x_1 and x_2 have equal exponents of 4.

Chapter 5

Specializations of Hall-Littlewood Polynomials

Specializations of Hall-Littlewood polynomials obtained by substituting specific values for $x_1, \ldots, x_N, t$, or λ often match symmetric polynomials introduced in Chapter 2. A few of these relationships are described below and proved with our abacus-tournament models.

5.1 Specialization at $t = 0$.

Theorem 124. *Given a partition $\lambda \in \mathrm{Par}_N$,*

$$P_\lambda(x_1, \ldots, x_N; 0) = s_\lambda(x_1, \ldots, x_N).$$

Proof. We show that

$$a_{\delta(N)} \cdot P_\lambda(x_1, \ldots, x_N; 0) = a_{\delta(N)} \cdot s_\lambda(x_1, \ldots, x_N),$$

where

$$a_{\delta(N)} \cdot s_\lambda(x_1, \ldots, x_N) = a_{\lambda+\delta(N)}(x_1, \ldots, x_N)$$

by Theorem 59.

If $A \in \mathbb{LAT}_\lambda^{\mathcal{B}_\lambda}$ then $\mathrm{swt}(A) = 0$ when $t = 0$ if and only if $\mathrm{swt}(A)$ has nonzero power of t, which is equivalent to $\mathrm{upset}(A) \neq \emptyset$.

In general, given any abacus (λ, v), there is a unique tournament τ_v such that $A_v = (\lambda, v, \tau_v)$ has $\mathrm{upset}(A_v) = \emptyset$, namely $\tau_v = \{(v_i, v_j) : 1 \leq i < j \leq N\}$. This abacus-tournament A_v is leading in all blocks of $\mathcal{B}_\lambda$, and for all i, $\mathrm{gap}(v_i) = \lambda_i$ and $\mathrm{out}(v_i) = N - i$. Thus, by Definition 56,

$$\mathrm{swt}(A_v) = \mathrm{sgn}(v) \cdot x_{v_1}^{\lambda_1} \cdots x_{v_N}^{\lambda_N} \cdot x_{v_1}^{N-1} \cdots x_{v_N}^{0} = \mathrm{sgn}(v)\, \mathrm{wt}(\lambda, v).$$

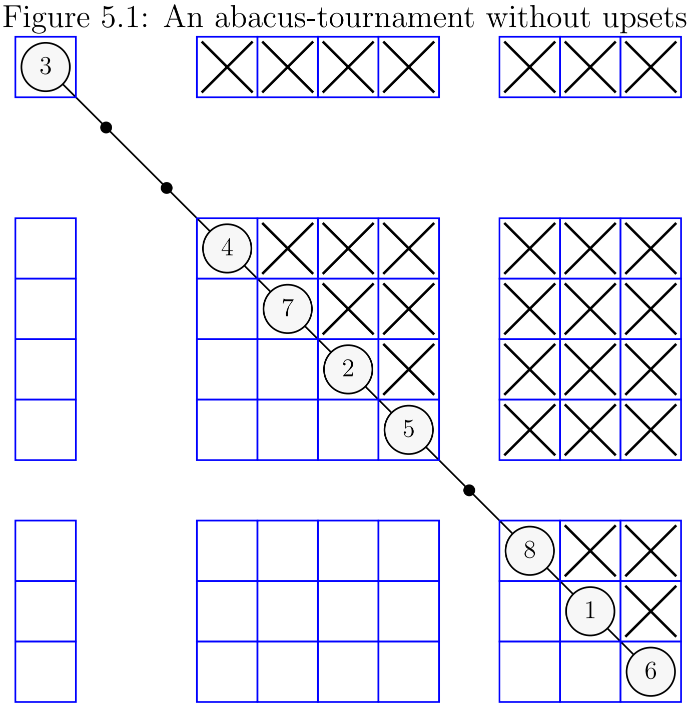
In summary, there exists a signed weight-preserving bijection between abaci in $\mathrm{LAbc}(\lambda)$ and abacus-tournaments A in $\mathbb{LAT}_\lambda^{\mathcal{B}_\lambda}$ such that $\mathrm{upset}(A) = \emptyset$. Therefore,

$$a_{\delta(N)} \cdot P_\lambda(x_1, \ldots, x_N; 0) = \sum_{\substack{A \in \mathbb{LAT}_\lambda^{\mathcal{B}_\lambda}: \\ \mathrm{upset}(A)=\emptyset}} \mathrm{swt}(A) = \sum_{(\lambda,v)\in \mathrm{LAbc}(\lambda)} \mathrm{sgn}(v)\,\mathrm{wt}(\lambda, v).$$

By Theorem 58, the final sum is the combinatorial model for $a_{\lambda+\delta(N)}$, which completes the proof. □

Example 125. Let $N = 8$, $\lambda = (3^3, 1^4, 0)$, and $\mu = (3^3, 2^4, 0)$. The abacus-tournament A of shape λ in Figure 3.6 has eight X's below the main diagonal, so $|\,\mathrm{upset}(A)| = 8$. In Example 107, we calculated the signed weight of A to be $(-1)^{16}(-t)^8 x_1^8 x_2^5 x_3^3 x_4^3 x_5^6 x_6^8 x_7^4 x_8^4$. Evaluating $\mathrm{swt}(A)$ when $t = 0$, we find $\mathrm{swt}(A)|_{t=0} = 0$. On the other hand, if A' is the abacus-tournament of shape μ in Figure 5.1 then $\mathrm{upset}(A') = \emptyset$. Note that $\mathrm{swt}(A') = (-1)^{16}(-t)^0 x_1^9 x_2^5 x_3^0 x_4^3 x_5^6 x_6^{10} x_7^4 x_8^8$ is nonzero even when $t = 0$ (using the convention that $0^0 = 1$). In fact, $A' = (\mu, v', \tau_{v'})$ is the only abacus-tournament with abacus (μ, v') and $\mathrm{upset}(A') = \emptyset$.

Figure 5.1: An abacus-tournament without upsets.



Example 126. The abacus (μ, v') from the second abacus-tournament in Example 125 was drawn in Example 54. The sign and weight of that abacus were calculated to be $\mathrm{sgn}(v')\,\mathrm{wt}(\mu, v') = (-1)^{16} x_1^9 x_2^5 x_3^0 x_4^3 x_5^6 x_6^{10} x_7^4 x_8^8$, which matches the signed weight we calculated for $A' = (\mu, v', \tau_{v'})$.

5.2 Specialization at $t = 1$.

Theorem 127. *[11, III.2.4] Given a partition $\lambda \in \mathrm{Par}_N$,*

$$P_\lambda(x_1, \ldots, x_N; 1) = m_\lambda(x_1, \ldots, x_N).$$

Note that [2] gives a detailed combinatorial treatment of this result based on special tournament matrices; so we will not give an abacus-tournament proof.

Restating Theorems 124 and 127, we can say that as t goes from 0 to 1, $P_\lambda(x_1, \ldots, x_N; t)$ interpolates between $s_\lambda(x_1, \ldots, x_N)$ and $m_\lambda(x_1, \ldots, x_N)$.

5.3 Specialization at $x_{N+1} = 0$.

Up to this point, the number of variables N has been held fixed. To motivate the next discussion, compare what happens to R_λ and P_λ when N changes but the nonzero parts of the partition stay the same. Because P_λ is unaffected by this change, while R_λ is affected, we call P_λ *stable.*

Example 128. For the partitions $\lambda = (2, 0^2)$ and $\mu = (2, 0)$,

$$R_{(2,0^2)}(x_1, x_2, x_3; t) = (1+t)x_1^2 + (1-t^2)x_1x_2 + (1+t)x_2^2 + (1-t^2)x_1x_3 + (1-t^2)x_2x_3 + (1+t)x_3^2,$$
$$R_{(2,0)}(x_1, x_2; t) = x_1^2 + (1-t)x_1x_2 + x_2^2.$$

Example 129. For the partitions $\lambda = (2, 0^2) \in \mathrm{Par}_3$ and $\mu = (2, 0) \in \mathrm{Par}_2$,

$$P_{(2,0^2)}(x_1, x_2, x_3; t) = x_1^2 + (1-t)x_1x_2 + x_2^2 + (1-t)x_1x_3 + (1-t)x_2x_3 + x_3^2,$$
$$P_{(2,0)}(x_1, x_2; t) = x_1^2 + (1-t)x_1x_2 + x_2^2.$$

When we set $x_3 = 0$, $R_{(2,0^2)}(x_1, x_2, 0; t) \neq R_{(2,0)}(x_1, x_2; t)$ while $P_{(2,0^2)}(x_1, x_2, 0; t) = P_{(2,0)}(x_1, x_2; t)$. The following theorem states that this is true in general for the polynomials P_λ. This *stability* property also holds true for Q_λ and has a similar proof to the one for P_λ. As demonstrated by Example 128 the property does not hold true for the polynomials R_λ.

Theorem 130. *Given a partition $\lambda \in \mathrm{Par}_{N+1}$ such that $\lambda_{N+1} = 0$, let $\bar{\lambda} \in \mathrm{Par}_N$ be obtained from λ by deleting the last part of λ. Then*

$$P_\lambda(x_1, \ldots, x_N, 0; t) = P_{\bar{\lambda}}(x_1, \ldots, x_N; t).$$

Proof. We will show that

$$a_{\delta(N+1)}(x_1, \ldots, x_N, 0) \cdot P_\lambda(x_1, \ldots, x_N, 0; t) = a_{\delta(N+1)}(x_1, \ldots, x_N, 0) \cdot P_{\bar{\lambda}}(x_1, \ldots, x_N; t).$$

First, recall from (2.1) that

$$a_{\delta(N+1)}(x_1,\ldots,x_N,x_{N+1}) = \prod_{1\le j<k\le N+1}(x_j-x_k) = \prod_{1\le j<k\le N}(x_j-x_k)\prod_{1\le j\le N}(x_j-x_{N+1}).$$

So, setting $x_{N+1}=0$,

$$a_{\delta(N+1)}(x_1,\ldots,x_N,0) = \left(\prod_{1\le j\le N} x_j\right)\cdot a_{\delta(N)}(x_1,\ldots,x_N).$$

Now consider the abacus-tournament model $\mathbb{LAT}_\lambda^{\mathcal{B}_\lambda}$ for $a_{\delta(N+1)}\cdot P_\lambda(x_1,\ldots,x_{N+1};t)$. When x_{N+1} is set to zero, the signed weights of all abacus-tournaments are eliminated except for those abacus-tournaments having signed weights with exponent of x_{N+1} equal to zero. Let X be the set of abacus-tournaments in $\mathbb{LAT}_\lambda^{\mathcal{B}_\lambda}$ that are not eliminated when $x_{N+1}=0$.

Define a bijection $J:\mathbb{LAT}_{\bar\lambda}^{\mathcal{B}_{\bar\lambda}}\to X$ such that $J(A)=A'$ as follows. Given $A=(\bar\lambda,v,\tau)\in\mathbb{LAT}_{\bar\lambda}^{\mathcal{B}_{\bar\lambda}}$, let (λ,v') be the abacus obtained from $(\bar\lambda,v)$ by shifting every bead $v_1,\ldots,v_N$ to the right one position in the abacus and inserting a new bead labeled $N+1$ in column 0. Form τ' by copying the edges of τ and including the set of edges of the form $(j,N+1)$ for all $j\in\{1,\ldots,N\}$. The new abacus-tournament $A'=(\lambda,v',\tau')$ is contained in $\mathbb{LAT}_\lambda^{\mathcal{B}_\lambda}$, has the same sign as A, and $\mathrm{upset}(A)=\mathrm{upset}(A')$. The exponent of x_{N+1} is zero in $\mathrm{swt}(A')$; so A' is not eliminated when x_{N+1} is set to 0. Thus $A'\in X$. While every bead $v_j\neq N+1$ has the same gap value in A' as in A, $\mathrm{out}_{A'}(v'_j)=\mathrm{out}_A(v_j)+1$. So

$$x_1\cdots x_N\cdot\mathrm{swt}(A)=\mathrm{swt}(A').$$

To show that J is a bijection, we will describe the inverse mapping $J^{-1}:X\to\mathbb{LAT}_{\bar\lambda}^{\mathcal{B}_{\bar\lambda}}$. Before doing so, we describe the abacus-tournaments in the set X. Let $A'=(\lambda,v',\tau')\in\mathbb{LAT}_\lambda^{\mathcal{B}_\lambda}$ be one such abacus-tournament in X where A' is not eliminated when setting $x_{N+1}=0$. The exponent of x_{N+1} in $\mathrm{swt}(A')$, equal to the sum $\mathrm{gap}(N+1)+\mathrm{out}(N+1)$, must be zero. Therefore, $\mathrm{gap}(N+1)=0=\mathrm{out}(N+1)$ in A'. Then, for all $i\neq N+1$, $(v'_i,N+1)\in\tau'$ because $\mathrm{out}(N+1)=0$. Since $\mathrm{gap}(N+1)=0$, the bead labeled $N+1$ must be positioned in $\mathrm{Pos}_\lambda(0)$. In fact, $N+1$ must be the label of the bead v'_{N+1} in word position $N+1$. For, if $v'_{N+1}\neq N+1$, then $(N+1,v'_{N+1})\in\tau'$ because A' is leading in $\mathrm{Pos}_\lambda(0)$; this forces $\mathrm{out}(N+1)>0$, which is impossible. In summary, the only abacus-tournaments in $\mathbb{LAT}_\lambda^{\mathcal{B}_\lambda}$ that are not eliminated when $x_{N+1}=0$ are those with $v'_{N+1}=N+1$ and $(v'_i,N+1)\in\tau'$ for all $i\neq N+1$. See Example 131 below.

Define $J^{-1}(A')=A$ by reversing the steps in J as follows. If $A'=(\lambda,v',\tau')$, let $(\bar\lambda,v)$ be the abacus obtained from (λ,v') by removing the bead labeled $N+1$ in abacus position 0 and shifting the remaining beads to the left in the abacus one position. Let τ be the tournament obtained from τ' by removing all edges of the form $(j,N+1)$ where $j\neq N+1$. The new abacus-tournament $A = (\bar{\lambda}, v, \tau)$ is in $\mathbb{LAT}_{\bar{\lambda}}^{\mathcal{B}_{\bar{\lambda}}}$ and $J(J^{-1}(A')) = A'$. Consequently,

$$\sum_{A \in \mathbb{LAT}_{\bar{\lambda}}^{\mathcal{B}_{\bar{\lambda}}}} x_1 \cdots x_N \cdot \mathrm{swt}(A) = \sum_{A' \in X} \mathrm{swt}(A').$$

This means

$$a_{\delta(N+1)}(x_1, \dots, x_N, 0) \cdot P_{\bar{\lambda}}(x_1, \dots, x_N; t) = x_1 \cdots x_N \cdot a_{\delta(N)}(x_1, \dots, x_N) \cdot P_{\bar{\lambda}}(x_1, \dots, x_N; t) \tag{5.1}$$

$$= \sum_{A \in \mathbb{LAT}_{\bar{\lambda}}^{\mathcal{B}_{\bar{\lambda}}}} x_1 \cdots x_N \cdot \mathrm{swt}(A) \tag{5.2}$$

$$= \sum_{A' \in X} \mathrm{swt}(A') \tag{5.3}$$

$$= a_{\delta(N+1)}(x_1, \dots, x_N, 0) \cdot P_{\lambda}(x_1, \dots, x_N, 0; t). \tag{5.4}$$

□

Example 131. Let $N = 9$ and $\lambda = (3^3, 1^2, 0^4)$. The abacus-tournament A' displayed in Figure 5.2 has signed weight $\mathrm{swt}(A') = (-1)^8(-t)^{15} x_1^7 x_2^8 x_3^5 x_4^7 x_5^5 x_6^6 x_7^5 x_8^4 x_9^0$, which is nonzero even when $x_9 = 0$. We form $A = J^{-1}(A')$ by removing the bead labeled 9 along with its edges to obtain the abacus-tournament A displayed in Figure 5.3. J maps A to A' as described above. Note that

$$\mathrm{swt}(A) = (-1)^8(-t)^{15} x_1^6 x_2^7 x_3^4 x_4^6 x_5^4 x_6^5 x_7^4 x_8^3 = \frac{\mathrm{swt}(A')}{x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8}.$$

5.4 Specializations at Partitions with One Column.

Theorem 132. *For the partition* $\lambda = (1^r, 0^{N-r}) \in \mathrm{Par}_N$,

$$P_{(1^r, 0^{N-r})}(x_1, \dots, x_N; t) = e_r(x_1, \dots, x_N).$$

Proof. We will prove that

$$a_{\delta(N)} \cdot P_{\lambda} = a_{\delta(N)} \cdot e_r.$$

Theorem 60 states that for a general partition $\mu \in \mathrm{Par}_N$,

$$a_{\mu+\delta(N)} \cdot e_r = \sum_{\substack{\beta \in \mathrm{Par}_N: \\ \beta \in \mathrm{V}(\mu, r)}} a_{\beta+\delta(N)},$$

Figure 5.2: An abacus-tournament in the set X.

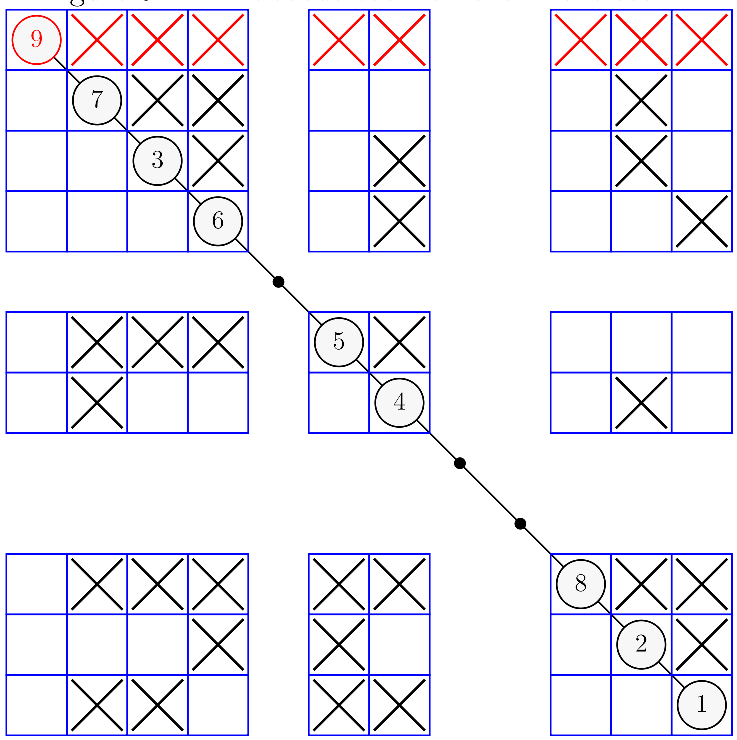
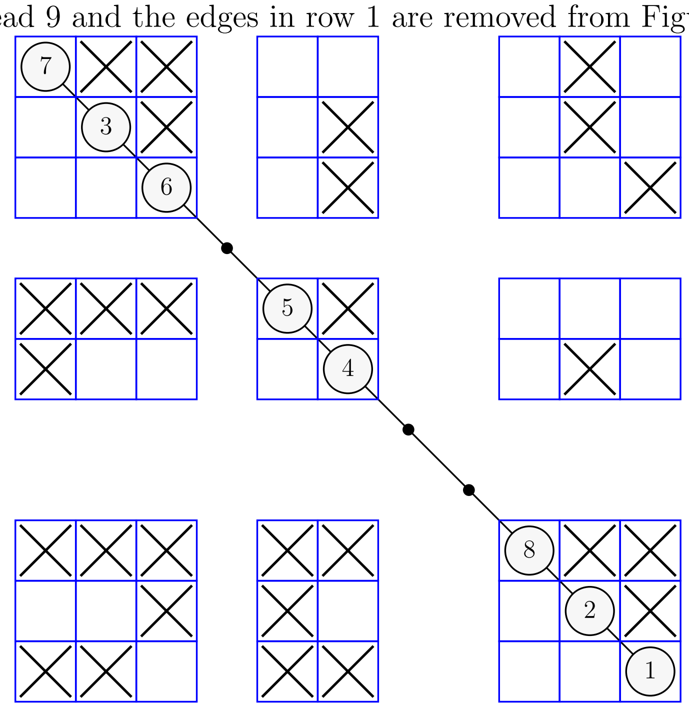


Figure 5.3: Bead 9 and the edges in row 1 are removed from Figure 5.2 by J^{-1}.



where the sum is taken over all β such that β/μ is a vertical r-strip. When $\mu = (0^N)$, $\lambda = (1^r, 0^{N-r})$ is the only partition in $\mathrm{V}(\mu, r)$. Therefore, by Theorem 58, we have the following combinatorial interpretation:

$$a_{\delta(N)} \cdot e_r = a_{(1^r,0^{N-r})+\delta(N)} = \sum_{(\lambda,v)\in \mathrm{LAbc}(\lambda)} \mathrm{sgn}(\lambda, v)\,\mathrm{wt}(\lambda, v).$$

In the proof of Theorem 124, we proved that there was a bijection between the set $\mathrm{LAbc}(\mu)$ and the set $\{A \in \mathbb{LAT}^{\mathcal{B}_\mu}_\mu : \mathrm{upset}(A) = \emptyset\}$ where $(\mu, v) \mapsto (\mu, v, \tau_v)$ and $\mathrm{sgn}(v)\,\mathrm{wt}(\mu, v) = \mathrm{swt}(\mu, v, \tau_v)$. In the case where $r = N$, $\mathcal{B}_{(1^N)} = \{\mathrm{Pos}_{(1^N)}(1)\}$ consists of a single block containing all word positions $1, \ldots, N$. Therefore, every $A \in \mathbb{LAT}^{\mathcal{B}_{(1^N)}}_{(1^N)}$ automatically has $\mathrm{upset}(A) = \emptyset$. This explains the equality in (5.5). This proves the result for $r = N$, since

$$a_{\delta(N)} \cdot P_{(1^N)}(x_1, \ldots, x_N; t) = a_{\delta(N)} \cdot P_{(1^N)}(x_1, \ldots, x_N; 0) \tag{5.5}$$
$$= a_{(1^N)+\delta(N)}(x_1, \ldots, x_N) \tag{5.6}$$
$$= a_{\delta(N)} \cdot e_N. \tag{5.7}$$

Consider the case when $r < N$. Now, for $\lambda = (1^r, 0^{N-r})$, $\mathbb{LAT}^{\mathcal{B}_\lambda}_\lambda$ contains abacus-tournaments that are not in the image of the bijection from $\mathrm{LAbc}(\lambda)$. We show that these "extra" abacus-tournaments $\{A \in \mathbb{LAT}^{\mathcal{B}_\lambda}_\lambda : \mathrm{upset}(A) \neq \emptyset\}$ are guaranteed to have exponent collisions.

Because $r < N$, $\lambda = (1^r, 0^{N-r})$ has precisely two nonempty blocks $\mathrm{Pos}_\lambda(0)$ and $\mathrm{Pos}_\lambda(1)$ and, together, they contain every bead position. Let $(\lambda, v, \tau) \in \{A \in \mathbb{LAT}^{\mathcal{B}_\lambda}_\lambda : \mathrm{upset}(A) \neq \emptyset\}$. If $(v_j, v_i) \in \mathrm{upset}(\lambda, v, \tau)$, then it must be true that $j \in \mathrm{Pos}_\lambda(0)$ and $i \in \mathrm{Pos}_\lambda(1)$ since (λ, v, τ) is leading in each block. Applying Lemma 118 to $\mathrm{Pos}_\lambda(0) \cup \mathrm{Pos}_\lambda(1)$, we conclude that (λ, v, τ) has an exponent collision. Lemma 97 shows that $\{A \in \mathbb{LAT}^{\mathcal{B}_\lambda}_\lambda : \mathrm{upset}(A) = \emptyset\}$ and $\mathbb{LAT}^{\mathcal{B}_\lambda}_\lambda$ have the same sum of signed weights. The result follows as in the $r = N$ case. □

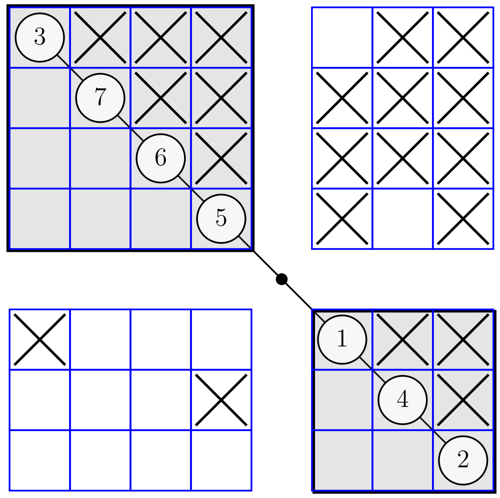
Example 133. Let $N = 7$, $r = 3$, and $\lambda = (1^3, 0^4)$. The abacus-tournament $A \in \mathbb{LAT}^{\mathcal{B}_\lambda}_\lambda$ in Figure 5.4 has $|\mathrm{upset}(A)| = 2$ and signed weight $\mathrm{swt}(A) = (-1)^6(-t)^2 x_1^4 x_2^7 x_3^1 x_4^5 x_5^4 x_6^2 x_7^1$. The squares for blocks $\mathrm{Pos}_\lambda(0)$ and $\mathrm{Pos}_\lambda(1)$ are displayed to emphasize that X's in these squares are not eligible to be in $\mathrm{upset}(A)$. This abacus-tournament has (global) exponent collisions involving beads $v_3 = 1$ and $v_4 = 5$ and involving beads $v_6 = 7$ and $v_7 = 3$.

5.5 Specializations at Partitions with One Row.

Theorem 134. *For the partition* $\lambda = (r, 0^{N-1}) \in \mathrm{Par}_N$*,*

$$P_{(r,0^{N-1})}(x_1, \ldots, x_N; t) = \sum_{k=0}^{\min(r-1,N-1)} (-t)^k s_{(r-k,1^k,0^{N-k-1})}(x_1, \ldots, x_N).$$

Figure 5.4: An abacus-tournament with shape $\lambda = (1^3, 0^4)$.



Proof. For this proof, we name $\mu^{(k)}$ to be the partition $(r-k, 1^k, 0^{N-k-1})$ for $0 \le k \le \min(r-1, N-1)$. We will show that

$$a_{\delta(N)} P_{(r,0^{N-1})}(x_1, \ldots, x_N; t) = \sum_{k=0}^{\min(r-1,N-1)} (-t)^k a_{\delta(N)} s_{\mu^{(k)}}(x_1, \ldots, x_N)$$
$$= \sum_{k=0}^{\min(r-1,N-1)} (-t)^k \sum_{(\mu^{(k)},v) \in \mathrm{LAbc}(\mu^{(k)})} \mathrm{sgn}(v)\, \mathrm{wt}(\mu^{(k)}, v).$$

If $N = 1$, then

$$a_{\delta(1)} P_{(1)} = x_1^r = a_{\delta(1)} s_{(r)}.$$

For $r > 1$, note that $\lambda = (r, 0^{N-1})$ has exactly two nonempty blocks, namely $\mathrm{Pos}_\lambda(r) = \{1\}$ and $\mathrm{Pos}_\lambda(0) = \{2, \ldots, N\}$, separated by r gap positions. Therefore $\mathrm{Pos}_\lambda(r) \cup \mathrm{Pos}_\lambda(0) = \{1, \ldots, N\}$ accounts for every bead position. By Lemma 122, any abacus-tournament $A \in \mathbb{LAT}_\lambda^{\mathcal{B}_\lambda}$ such that $|\mathrm{upset}(A)| \ge r$ or $\mathrm{upset}(A)$ is not right-justified must have an exponent collision. Furthermore, the involution from Lemma 97, applied to the set consisting of the single block $\{1, \ldots, N\}$, fixes all X locations in the diagram of A. We conclude that $a_{\delta(N)} P_{(r,0^{N-1})}$ is given by summing the signed weights of abacus-tournaments $A \in \mathbb{LAT}_\lambda^{\mathcal{B}_\lambda}$ such that $|\mathrm{upset}(A)| < r$ and $\mathrm{upset}(A)$ is right-justified. Let X denote this set of abacus-tournaments for λ.

We define a mapping J from X to $\bigcup_{k=0}^{\min(r-1,N-1)} \mathrm{LAbc}(\mu^{(k)})$ as follows. Given $A = (\lambda, v, \tau) \in X$, let $p = |\mathrm{upset}(A)| \le \min(r-1, N-1)$. Since $r > p$, $r - p \ge 1$ and we can define $\mu = \mu^{(p)}$. Define (μ, v') such that $v' = v$. Then bead v_1 has $\mathrm{out}(v_1) + \mathrm{gap}(v_1) = (N-1-p) + r$ in A, and bead $v_1' = v_1$ is in abacus position $\mu_1 + \delta(N)_1 = (r-p) + (N-1)$ of v'. Similarly, for $1 < j \le p+1$,

$$\mathrm{out}(v_j) + \mathrm{gap}(v_j) = [(N-j)+1] + 0 = 1 + (N-j) = \mu_j + \delta(N)_j,$$

and for $p+1 < j \leq N$,

$$\mathrm{out}(v_j) + \mathrm{gap}(v_j) = (N-j) + 0 = \mu_j + \delta(N)_j.$$

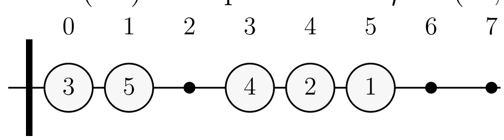
We conclude that $\mathrm{swt}(\lambda, v, \tau) = (-t)^p \,\mathrm{sgn}(v') \,\mathrm{wt}(\mu, v')$.

Now we show J is a bijection by describing J^{-1}. If $v' \in \mathrm{LAbc}(\mu^{(k)})$ for some $0 \leq k \leq \min(r-1, N-1)$, then define $J^{-1}(v) = A = (\lambda, v, \tau)$ to be the abacus-tournament of shape $\lambda = (r, 0^{N-1})$ with $v = v'$ and $(v'_j, v'_i) \in \tau$ if and only if $i < j$ or $i = 1$ and $1 < j \leq k+1$. By construction, $|\mathrm{upset}(A)| = k < r$, $\mathrm{upset}(A)$ is right-justified, and $A \in \mathbb{LAT}_{\lambda}^{\mathcal{B}_\lambda}$, so $A \in X$. Furthermore, $J(A) = v'$ so $J \circ J^{-1} = \mathrm{id}$. Similarly, $J^{-1} \circ J = \mathrm{id}$ so J is a bijection. □

Example 135. Let $N = 5$, $r = 3$, and $\lambda = (3, 0^4) \in \mathrm{Par}_5$. Consider the abacus-tournament A'' displayed in Figure 4.4. There are $p = 2$ upset edges in $\mathrm{upset}(A'')$ and they are right-justified. To calculate $J(A'')$, we construct an abacus with position set $\mu = (1, 1^2, 0^2) + \delta(5) = (5, 4, 3, 1, 0)$ and word $w(A'') = 12453$. The abacus $J(A'')$ is displayed in Figure 5.5. Note that

$$(-t)^2 \,\mathrm{sgn}(v) \,\mathrm{wt}(\mu, v) = t^2 x_1^5 x_2^4 x_3^0 x_4^3 x_5^1 = \mathrm{swt}(A'').$$

Figure 5.5: The abacus $J(A'')$ with position set $\mu = (1^3, 0^2)$ and word 12453.



Chapter 6

A Pieri Rule for Hall-Littlewood Polynomials

We will bijectively prove a "Pieri rule" for Hall-Littlewood polynomials that shows how to write the product of a Hall-Littlewood polynomial P_μ and the kth elementary symmetric polynomial e_k as a linear combination of Hall-Littlewood polynomials. This is analogous to the Pieri rules for Schur polynomials (see Theorems 60, 62 and examples). As discussed in Chapter 2.10, [8, Thm. 3.1] and [9, Sec. 11.12] give bijective proofs of Theorem 59 using abaci, where previous proofs were algebraic (see [11, II.5.17]) or based on the RSK algorithm on tableaux (see [5]). Each different proof reveals new facets of the underlying identity.

In the proof of the Schur Pieri rule using abaci, the multiplication of the signed weight monomial of an abacus v with position set $\mu+\delta(N)$ by a term from e_k is modeled by a bead movement on v. If no collisions occur, the resulting abacus has position set $\lambda+\delta(N)$, where λ/μ is a vertical k-strip. If bead collisions do occur, the abacus and term from e_k cancel with some other pair of objects. We will define a similar bead movement for abacus-tournaments, but the edges of the tournament will necessitate more intricate cancellation mechanisms.

The Pieri rule for multiplying P_μ by e_k is stated as follows.

Theorem 136. *For $\mu \in \mathrm{Par}_N$ and $k \in [N]$,*

$$P_\mu(x_1,\ldots,x_N;t)\cdot e_k(x_1,\ldots,x_N) = \sum_{\lambda\in \mathrm{V}(\mu,k)} \left(\prod_{i\geq 1} \begin{bmatrix} \lambda'_i - \lambda'_{i+1} \\ \lambda'_i - \mu'_i \end{bmatrix}_t P_\lambda(x_1,\ldots,x_N;t) \right).$$

See [11, III.3.2] for an algebraic proof of this theorem.

Example 137. Let $N=3$, $\mu=(1,0^2)$, and $k=2$,. On one hand, we obtain

$$\begin{aligned} P_{(1,0^2)}(x_1,x_2,x_3;t)\cdot e_2(x_1,x_2,x_3) &= (x_1+x_2+x_3)\cdot(x_1x_2+x_1x_3+x_2x_3) && (6.1)\\ &= x_1^2x_2+x_1^2x_3+x_1x_2^2+x_1x_3^2+3x_1x_2x_3+x_2x_3^2+x_2^2x_3. && (6.2)\end{aligned}$$

On the other hand, $V(\mu,2)=\{(2,1,0),(1^3)\}$, so we compute (using Example 65)

$$\begin{aligned}\begin{bmatrix}3\\2\end{bmatrix}_t P_{(1^3)}(x_1,x_2,x_3;t)+P_{(2,1,0)}(x_1,x_2,x_3;t) &= (1+t+t^2)(x_1x_2x_3)\\ &\quad+\left((2-t-t^2)x_1x_2x_3+x_1^2x_2+x_1^2x_3+x_1x_2^2+x_1x_3^2+x_2x_3^2+x_2^2x_3\right)\\ &= x_1^2x_2+x_1^2x_3+x_1x_2^2+x_1x_3^2+3x_1x_2x_3+x_2x_3^2+x_2^2x_3.\end{aligned}$$

Example 138. Let $N=4$, $\mu=(2,1^2,0)\in \mathrm{Par}_4$, and $k=2$. The Pieri rule states that

$$\begin{aligned}P_{(2,1^2,0)}(x_1,x_2,x_3,x_4;t)\cdot e_2(x_1,x_2,x_3,x_4) &= P_{(3,2,1,0)}(x_1,x_2,x_3,x_4;t)+\begin{bmatrix}3\\1\end{bmatrix}_t P_{(3,1^3)}(x_1,x_2,x_3,x_4;t)\\ &\quad+\begin{bmatrix}3\\2\end{bmatrix}_t P_{(2^3,0)}(x_1,x_2,x_3,x_4;t)+\left(\begin{bmatrix}2\\1\end{bmatrix}_t\right)^2 P_{(2^2,1^2)}(x_1,x_2,x_3,x_4;t).\end{aligned}$$

Note that even though $(2,1^4)/(2,1^2,0^2)$ is a vertical 2-strip, $(2,1^4)\notin \mathrm{Par}_4$. This is consistent with the stability property of P_λ (see Theorem 130): $P_{(2,1^2,0^2)}e_2$ would have the extra term $\begin{bmatrix}4\\2\end{bmatrix}_t P_{(2,1^4)}$ but it goes away when $x_5=0$.

After multiplying both sides of the Pieri rule equation by $a_{\delta(N)}$ to obtain

$$a_{\delta(N)}P_\mu\cdot e_k=\sum_{\lambda\in \mathrm{V}(\mu,k)}\left(\prod_{i\geq 1}\begin{bmatrix}\lambda_i'-\lambda_{i+1}'\\ \lambda_i'-\mu_i'\end{bmatrix}_t a_{\delta(N)}P_\lambda\right), \tag{6.3}$$

we find sign-canceling involutions on both sides, establish combinatorial models for both sides, and then find a weight-preserving bijection between the two models.

6.1 A Combinatorial Model for the Right Side of the Pieri Rule

Fix $\mu\in \mathrm{Par}_N$, $k\in[N]$, and $\lambda\in \mathrm{V}(\mu,k)$. For all $i\geq 0$, $m_i(\lambda)=\lambda_i'-\lambda_{i+1}'$ and $\left(\lambda_i'-\lambda_{i+1}'\right)-(\lambda_i'-\mu_i')=\mu_i'-\lambda_{i+1}'$. Recall from Definition 68 that $R_\lambda=\left(\prod_{i\geq 0}[m_i(\lambda)]!_t\right)\cdot P_\lambda$. Then

$$\begin{aligned}\prod_{i\geq 1}\begin{bmatrix}\lambda_i'-\lambda_{i+1}'\\ \lambda_i'-\mu_i'\end{bmatrix}_t P_\lambda &= \prod_{i\geq 1}\frac{[\lambda_i'-\lambda_{i+1}']!_t}{[\lambda_i'-\mu_i']!_t\cdot[\mu_i'-\lambda_{i+1}']!_t}\cdot P_\lambda & (6.4)\\ &= \left(\prod_{i\geq 1}\frac{1}{[\lambda_i'-\mu_i']!_t\cdot[\mu_i'-\lambda_{i+1}']!_t}\right)\cdot\frac{1}{[m_0(\lambda)]!_t}\cdot R_\lambda & (6.5)\\ &= \left(\prod_{i\geq 0}\frac{1}{[\lambda_i'-\mu_i']!_t\cdot[\mu_i'-\lambda_{i+1}']!_t}\right)\cdot R_\lambda. & (6.6)\end{aligned}$$

Since $\lambda'_0 = \mu'_0 = N$, $m_0(\lambda) = \mu'_0 - \lambda'_1$. Thus we can incorporate the $1/[m_0(\lambda)]!_t$ term in (6.5) into the product in (6.6).

Recall from Definition 84 that the λ-blocks $\mathcal{B}_\lambda = \{\mathrm{Pos}_\lambda(0), \dots, \mathrm{Pos}_\lambda(i), \dots\}$ are a collection of blocks for λ such that the ith block $\mathrm{Pos}_\lambda(i)$ is the set of $m_i(\lambda)$ word positions that correspond to the parts of λ of size i. The proof of the Pieri rule will involve a bead movement on an abacus-tournament with position set $\mu + \delta(N)$ to create a new abacus-tournament A with position set $\lambda + \delta(N)$. It will be important to keep track of which beads moved in the bead movement. Consequently, we further divide the λ-blocks into *λ/μ-blocks* such that each λ-block is composed of a block of *new positions* and a block of *original positions*, as defined below. Beads of A found in the new positions of a block are beads that moved. Beads of A found in original positions are beads unaffected by the bead movement.

Definition 139. For $\mu \in \mathrm{Par}_N$, $k \in [N]$, and $\lambda \in \mathrm{V}(\mu, k)$, let the ith block of *new positions* be

$$\mathrm{nPos}_{\lambda/\mu}(i) = \{\text{word positions } j : \lambda_j = i \text{ and } \mu_j = i - 1\},$$

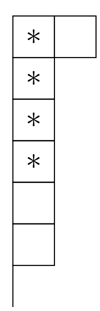
and let the ith block of *original positions* be

$$\mathrm{oPos}_{\lambda/\mu}(i) = \{\text{word positions } j : \lambda_j = \mu_j = i\}.$$

Let $\mathcal{B}_{\lambda/\mu}$ denote the set of all λ/μ blocks of both types.

Note that $\mathcal{B}_{\lambda/\mu}$ is a set partition of $\{1, \dots, N\}$ (i.e. all word positions belong to exactly one block).

Example 140. Let $N = 7$, $\mu = (1^4, 0^3)$, $k = 3$, and $\lambda = (2^1, 1^5, 0^1)$:

$\lambda =$  .

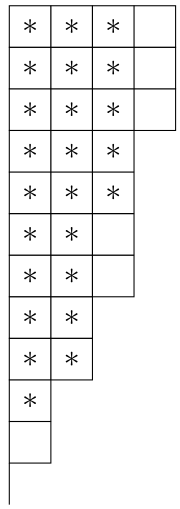
Recall Example 12 for the star notation used in this diagram of λ. Note that $\mathrm{Pos}_\lambda(0) = \{7\} = \mathrm{oPos}_{\lambda/\mu}(0)$ while $\mathrm{nPos}_{\lambda/\mu}(0) = \emptyset$. The 1st λ-block contains both new positions and original positions: $\mathrm{nPos}_{\lambda/\mu}(1) = \{5, 6\}$, $\mathrm{oPos}_{\lambda/\mu}(1) = \{2, 3, 4\}$, and

$$\mathrm{oPos}_{\lambda/\mu}(1) \cup \mathrm{nPos}_{\lambda/\mu}(1) = \{2, 3, 4\} \cup \{5, 6\} = \mathrm{Pos}_\lambda(1).$$

Last, $\mathrm{Pos}_\lambda(2) = \{1\} = \mathrm{nPos}_{\lambda/\mu}(2)$ while $\mathrm{oPos}_{\lambda/\mu}(2) = \emptyset$.

Example 141. Let $N = 12$, $\mu = (3^5, 2^4, 1, 0^2)$, $k = 6$, and $\lambda = (4^3, 3^4, 2^2, 1^2, 0)$:

$\lambda =$



.

Here, $\mathrm{nPos}_{\lambda/\mu}(2) = \emptyset$ because there are no starred cells in the rows of size 2. Similarly, $\mathrm{oPos}_{\lambda/\mu}(4)$ is empty because μ has no parts of size 4 and so every part of λ of size 4 has a starred cell. Also, $\mathrm{nPos}_{\lambda/\mu}(3) = \{6, 7\}$ and $\mathrm{oPos}_{\lambda/\mu}(3) = \{4, 5\}$.

Definition 142. Let $\mu \in \mathrm{Par}_N$, $k \in [N]$, and $\lambda \in \mathrm{V}(\mu, k)$. Let $\mathbb{LAT}_\lambda^{\mathcal{B}_{\lambda/\mu}}$ denote the set of abacus-tournaments for λ that are leading in the λ/μ-blocks $\mathcal{B}_{\lambda/\mu}$.

One can visually verify that an abacus-tournament is in $\mathbb{LAT}_\lambda^{\mathcal{B}_{\lambda/\mu}}$ by checking, for all i, that all X's in the squares of the λ/μ-blocks $\mathrm{nPos}_{\lambda/\mu}(i)$ and $\mathrm{oPos}_{\lambda/\mu}(i)$ are above the main diagonal. By Theorem 101, $\mathbb{LAT}_\lambda^{\mathcal{B}_{\lambda/\mu}} \subseteq \mathbb{AT}_\lambda^{\mathcal{B}_{\lambda/\mu}}$.

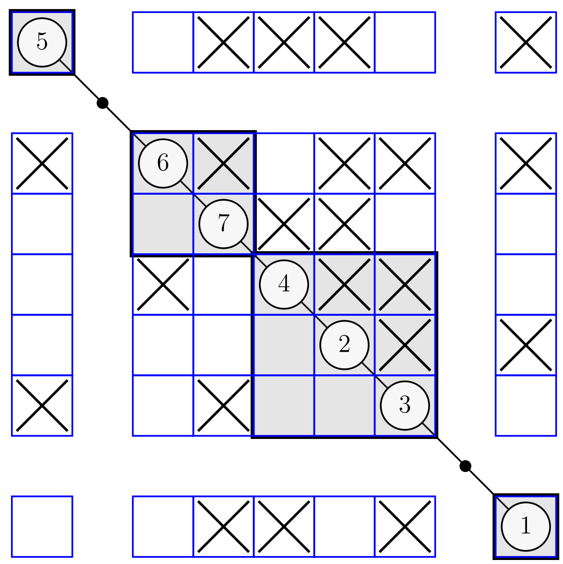
Example 143. As in Example 140, let $N = 7$, $k = 3$, $\mu = (1^4, 0^3)$, and $\lambda = (2^1, 1^5, 0^1)$. Let A be the abacus-tournament in Figure 6.1. Then $A \in \mathbb{LAT}_\lambda^{\mathcal{B}_{\lambda/\mu}}$. Indeed, any abacus-tournament with position set $\lambda + \delta(7)$ will be leading in $\mathcal{B}_{\lambda/\mu}$ provided the X's in the gray squares in Figure 6.1 are present. Abacus-tournaments in $\mathbb{LAT}_\lambda^{\mathcal{B}_{\lambda/\mu}}$ need not be in $\mathbb{LAT}_\lambda^{\mathcal{B}_\lambda}$. For example, A is not in $\mathbb{LAT}_\lambda^{\mathcal{B}_\lambda}$.

$\mathcal{B}_{\lambda/\mu}$ is a set of non-overlapping blocks that do not contain bead gaps for λ. Consequently, we can use Theorem 109 to form a new combinatorial model for the expression in (6.6) multiplied by $a_{\delta(N)}$.

Theorem 144. *For $\mu \in \mathrm{Par}_N$, $k \in [N]$, and $\lambda \in \mathrm{V}(\mu, k)$, R_λ is divisible in $\mathbb{Z}[t][x_1, \ldots, x_N]$ by $\prod_{i \geq 0} [\lambda'_i - \mu'_i]!_t \cdot [\mu'_i - \lambda'_{i+1}]!_t$, and*

$$\left(\prod_{i \geq 0} \frac{1}{[\lambda'_i - \mu'_i]!_t \cdot [\mu'_i - \lambda'_{i+1}]!_t} \right) \cdot a_{\delta(N)} R_\lambda (x_1, \ldots, x_N; t) = \sum_{A \in \mathbb{LAT}_\lambda^{\mathcal{B}_{\lambda/\mu}}} \mathrm{swt}(A).$$

Figure 6.1: An abacus-tournament in $\mathbb{LAT}_\lambda^{\mathcal{B}_{\lambda/\mu}}$.



Proof. The λ/μ-blocks $\mathrm{oPos}_{\lambda/\mu}(i)$ and $\mathrm{nPos}_{\lambda/\mu}(i)$ contain $\mu'_i - \lambda'_{i+1}$ and $\lambda'_i - \mu'_i$ bead positions, respectively. Since these blocks are non-overlapping and do not contain bead gaps for λ, the result follows from Theorem 109 and the combinatorial model for $a_{\delta(N)}R_\lambda$ (see Thm. 79). $\square$

We now have a combinatorial model for (6.6) multiplied by $a_{\delta(N)}$. A union of these models over all $\lambda \in \mathrm{V}(\mu, k)$ provides a combinatorial model for the right hand side of the Pieri rule (6.3). This is stated in the lemma below.

Lemma 145. $\mu \in \mathrm{Par}_N$ *and* $k \in [N]$

$$\sum_{\lambda \in \mathrm{V}(\mu,k)} \left(\prod_{i \geq 1} \begin{bmatrix} \lambda'_i - \lambda'_{i+1} \\ \lambda'_i - \mu'_i \end{bmatrix}_t a_{\delta(N)} P_\lambda \right) = \sum_{\lambda \in \mathrm{V}(\mu,k)} \sum_{A \in \mathbb{LAT}_\lambda^{\mathcal{B}_{\lambda/\mu}}} \mathrm{swt}(A).$$

6.2 Combinatorial Models for $a_{\delta(N)}P_\mu e_k$

To provide an initial combinatorial model for $a_{\delta(N)}P_\mu e_k$, we combine known models for $a_{\delta(N)}P_\mu$ and e_k.

Lemma 146. *Given* $\mu \in \mathrm{Par}_N$ *and* $k \in [N]$,

$$a_{\delta(N)} P_\mu e_k = \sum_{(S,A) \in X} \left(\prod_{j \in S} x_j \right) \mathrm{swt}(A),$$

where $X = \{S \subseteq [N] : |S| = k\} \times \mathbb{LAT}_{\mu}^{\mathcal{B}_\mu}$.

Proof. This follows directly from Definition 33 and Theorem 110. □

Using this model, we will describe a bead movement on abacus-tournaments $A \in \mathbb{LAT}_{\mu}^{\mathcal{B}_\mu}$ where beads in A are moved based on the members of a set $S \subseteq [N]$ with $|S| = k$. Many such abacus-tournaments will cancel with others due to bead collisions. Abacus-tournament A' resulting from successful bead movements will have partition λ, where $\lambda \in \mathrm{V}(\mu, k)$. This bead movement is a mapping from the uncanceled objects in $X = \{S \subseteq [N] : |S| = k\} \times \mathbb{LAT}_{\mu}^{\mathcal{B}_\mu}$ to $\bigcup_{\lambda \in \mathrm{V}(\mu,k)} \mathbb{LAT}_{\lambda}^{\mathcal{B}_{\lambda/\mu}}$, which is the set of objects that form the combinatorial model in Lemma 145. However, we will see that the mapping *is not onto*. Consequently, our immediate goal is to describe the image of this mapping, which will form the objects in a revised combinatorial model for $a_{\delta(N)} P_\mu e_k$.

Intuitively, when a k-strip is added to μ to obtain some $\lambda \in \mathrm{V}(\mu, k)$, certain beads positioned in the ith μ-block are moved up to form the $\lambda'_{i+1} - \mu'_{i+1}$ beads in the new positions of the $(i+1)$th λ-block, leaving the $\mu'_i - \lambda'_{i+1}$ beads behind in the original positions of the ith λ-block. When the ith μ-block is leading, we should expect X's between beads in new positions of the $(i+1)$th λ-block and beads in original positions of the ith λ-block to be above the diagonal. These are precisely the abacus-tournaments in $\mathbb{LAT}_{\lambda}^{\mathcal{B}_\mu}$, the set of abacus-tournaments with partition λ that are leading in μ-blocks. Given $i \geq 0$, $\mathrm{Pos}_\mu(i) = \mathrm{nPos}_{\lambda/\mu}(i+1) \cup \mathrm{oPos}_{\lambda/\mu}(i)$, while $\mathrm{Pos}_\lambda(i) = \mathrm{nPos}_{\lambda/\mu}(i) \cup \mathrm{oPos}_{\lambda/\mu}(i)$. Consequently, $\mathbb{LAT}_{\lambda}^{\mathcal{B}_\mu} \subseteq \mathbb{LAT}_{\lambda}^{\mathcal{B}_{\lambda/\mu}}$. We will see that $\bigcup_{\lambda \in \mathrm{V}(\mu,k)} \mathbb{LAT}_{\lambda}^{\mathcal{B}_\mu}$ is the image of the mapping mentioned above.

Example 147. Let $N = 7$, $\mu = (1^4, 0^3)$, $k = 3$, and $\lambda = (2^1, 1^5, 0^1)$. While both the abacus-tournament from Figure 6.1 and the abacus-tournament from Figure 6.2 are contained in $\mathbb{LAT}_{\lambda}^{\mathcal{B}_{\lambda/\mu}}$, only the abacus-tournament from Figure 6.2 is also contained in $\mathbb{LAT}_{\lambda}^{\mathcal{B}_\mu}$. In Figure 6.2, the X's in the squares with dashed outlines show X's in the abacus-tournament diagram that must be above the diagonal for the abacus-tournament to be in $\mathbb{LAT}_{\lambda}^{\mathcal{B}_{\lambda/\mu}}$. When the X's located in squares with solid outlines in Figure 6.2 are above the diagonal, the abacus-tournament is an element of $\mathbb{LAT}_{\lambda}^{\mathcal{B}_\mu}$.

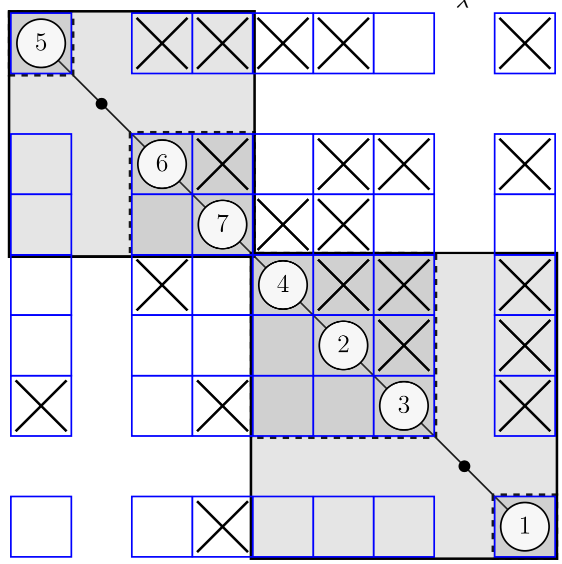
Theorem 148. *Let* $\mu \in \mathrm{Par}_N$ *and* $k \in [N]$. *Then*

$$a_{\delta(N)} P_\mu \cdot e_k = \sum_{\lambda \in \mathrm{V}(\mu,k)} \sum_{A \in \mathbb{LAT}_{\lambda}^{\mathcal{B}_\mu}} \mathrm{swt}(A).$$

Proof. By Lemma 146,

$$a_{\delta(N)} P_\mu e_k = \sum_{(S,A) \in X} \left(\prod_{j \in S} x_j \right) \mathrm{swt}(A),$$

Figure 6.2: An abacus-tournament in $\mathbb{LAT}_{\lambda}^{\mathcal{B}_{\lambda/\mu}}$ and $\mathbb{LAT}_{\lambda}^{\mathcal{B}_{\mu}}$.



where $X = \{S \subseteq [N] : |S| = k\} \times \mathbb{LAT}_{\mu}^{\mathcal{B}_{\mu}}$. Define an involution $I : X \to X$ as follows. Given $(S, A) \in X$ where $A = (\mu, v, \tau)$, scan the beads in the diagram of A from right to left. If a bead's label belongs to S, move that bead one step diagonally to the right in A along the abacus.

If this ever causes two beads to occupy the same position, we say that a *bead collision* has occurred. In this case, $I(S, A)$ will be another object that cancels with (S, A). Say the first bead collision occur when bead v_i moves to occupy the same position as bead v_{i-1}. Since v is scanned from right to left and bead v_i moves only one position, $v_{i-1} \notin S$ and word positions i and $i-1$ are in the same μ-block. Since $A \in \mathbb{LAT}_{\mu}^{\mathcal{B}_{\mu}}$, $(v_{i-1}, v_i) \in \tau$. Define $I(S, A) = (S', A')$, where S' is obtained from S by removing v_i and adding v_{i-1}, and A' is obtained from A by switching beads v_i and v_{i-1} and switching edge (v_{i-1}, v_i) to (v_i, v_{i-1}). This changes the sign of A but leaves the weight of (S, A) unchanged: in particular, $\mathrm{swt}(A') = -\frac{x_{v_i}}{x_{v_{i-1}}}\mathrm{swt}(A)$ and $\prod_{j\in S'} x_j = \frac{x_{v_{i-1}}}{x_{v_i}}\left(\prod_{j\in S'} x_j\right)$. Furthermore, S' is still a k-element subset of $[N]$ and A' is still leading in all μ-blocks. If $I(S, A)$ is scanned as directed in I, the same two beads are the first to collide and so $I(I(S, A)) = (S, A)$. So I cancels all objects in X with bead collisions. See Example 149 below.

Now suppose $(S, A) \in X$ is an object where all bead motions can be completed with no bead collisions. Define $I(S, A) = (S, A)$. We will now define a weight-preserving bijection J from the set of fixed points of I to the set $\bigcup_{\lambda \in \mathrm{V}(\mu,k)} \mathbb{LAT}_{\lambda}^{\mathcal{B}_{\mu}}$. By making all the bead motions encoded by S, A turns into a new abacus-tournament A'' with partition λ where (as one readily checks) $\lambda \in \mathrm{V}(\mu, k)$. Define $J(S, A) = A''$. Note that $\mathrm{swt}(A'') = \left(\prod_{j\in S} x_j\right)\mathrm{swt}(A)$ because each bead v_i in A that moves exactly one position increases $\mathrm{swt}(A)$ by a multiplicative factor of x_{v_i} due to the increased gap count of v_i; and these are exactly the beads identified by S.

In the bead movement changing (S, A) to A'', the beads of A are not rearranged: $v(A) = v(A'')$. Because for all i, A is leading in the entire block $\mathrm{Pos}_\mu(i)$ and $\tau(A) = \tau(A'')$, we can conclude that A'' is leading in $\mathrm{Pos}_\mu(i)$, for all i. Thus $A'' \in \mathbb{LAT}_\lambda^{\mathcal{B}_\mu}$.

To show that J is a bijection, we fix any $A^\dagger \in \mathbb{LAT}_\lambda^{\mathcal{B}_\mu}$ for some $\lambda \in \mathrm{V}(\mu, k)$. Let S be the set of labels for the k beads in positions $\bigcup_{i\geq 1} \mathrm{nPos}_{\lambda/\mu}(i)$, and let A be the abacus-tournament obtained by moving each of those beads in $A^\dagger$ diagonally to the left exactly one position. This is possible because $\lambda \in \mathrm{V}(\mu, k)$, and A must have partition μ. Because $A^\dagger$ is leading in the ith μ-block for all i, so too is A. Thus $A \in \mathbb{LAT}_\mu^{\mathcal{B}_\mu}$, and we define $J^{-1}(A^\dagger) = (S, A)$. It is routine to check that $J \circ J^{-1}$ and $J^{-1} \circ J$ are identity maps.

Applying I to cancel objects in X and then J to change fixed points of I to elements of $\bigcup_{\lambda\in \mathrm{V}(\mu,k)} \mathbb{LAT}_\lambda^{\mathcal{B}_\mu}$, the theorem follows. □

Example 149. To illustrate the previous proof, let $N = 7$, $\mu = (1^4, 0^3)$, $k = 3$, and $\lambda = (2^1, 1^5, 0^1)$. Consider the abacus-tournament A displayed in Figure 6.3. This abacus-tournament has signed weight $(-1)^4(-t)^4 x_1^6 x_2^5 x_3^4 x_4^3 x_5^1 x_6^2 x_7^4$ and is contained in $\mathbb{LAT}_\mu^{\mathcal{B}_\mu}$.

Let $S = \{1, 5, 6\}$ and note $(S, A) \in X$. When we try to move the bead labeled 6 one position diagonally to the right in A to position 2, a bead collision occurs with the bead labeled 7. Consequently, I cancels (S, A) with (S', A'), where $S' = \{1, 5, 7\}$ and where A' is the abacus-tournament obtained from A by switching beads labeled 6 and 7 and changing the ordered pair $(7, 6)$ to $(6, 7)$. All other edges in τ do not change but the X's in the diagram move around with beads 6 and 7. The new abacus-tournament A' is shown in Figure 6.4, has signed weight $(-1)^3(-t)^4 x_1^6 x_2^5 x_3^4 x_4^3 x_5^1 x_6^3 x_7^3$, and is contained in $\mathbb{LAT}_\mu^{\mathcal{B}_\mu}$. Then

$$\mathrm{wt}(S)\,\mathrm{swt}(A) = t^4 x_1^7 x_2^5 x_3^4 x_4^3 x_5^2 x_6^3 x_7^4 = -\,\mathrm{wt}(S')\,\mathrm{swt}(A').$$

Consequently, the terms for the pairs (S, A) and (S', A') cancel. Furthermore, moving the bead labeled 7 in A' diagonally one position to the right causes a bead collision with the bead labeled 6. So $I(S', A') = (S, A)$, as expected.

Now, let $T = \{1, 6, 7\}$, and consider $(T, A) \in X$. The beads labeled 1, then 7, then 6 move diagonally to the right one position each. The change in positions increases the exponents of x_1, x_6, and x_7 by one in the signed weight. The new abacus-tournament A'' has signed weight $(-1)^4(-t)^4 x_1^7 x_2^5 x_3^4 x_4^3 x_5^1 x_6^3 x_7^5 = \left(\prod_{j\in T} x_j\right) \mathrm{swt}(A)$ and is displayed in Figure 6.2. The partition of A'' is $\lambda = (2, 1^5, 0)$, and λ/μ is a vertical 3-strip.

6.3 A Bijection Between the Two Models

Fix $\mu \in \mathrm{Par}_N$, $k \in [N]$, and $\lambda \in \mathrm{V}(\mu, k)$. To complete the proof of the Pieri rule, we must show that the sum of the signed weights of objects in $\mathbb{LAT}_\lambda^{\mathcal{B}_{\lambda/\mu}}$ is equal to the sum of the

Figure 6.3: An abacus-tournament in the domain of I.

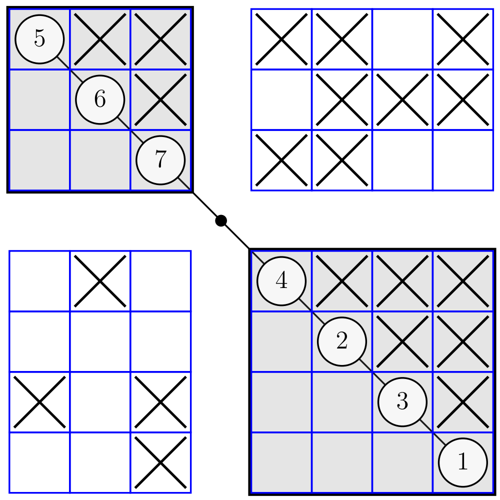
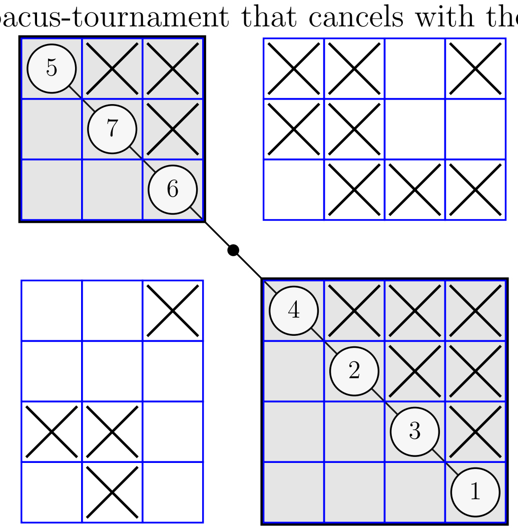


Figure 6.4: An abacus-tournament that cancels with the one in Figure 6.3.



signed weights of objects in $\mathbb{LAT}_\lambda^{\mathcal{B}_\mu}$. To prove this, we use Lemma 118, which shows that we are guaranteed local exponent collisions in any object in $\mathbb{LAT}_\lambda^{\mathcal{B}_{\lambda/\mu}}$ that is not in $\mathbb{LAT}_\lambda^{\mathcal{B}_\mu}$. The result will then follow from Lemma 97.

Theorem 150. *Let $\mu \in \mathrm{Par}_N$ and $k \in [N]$. Then*

$$a_{\delta(N)} P_\mu \cdot e_k = \sum_{\lambda \in \mathrm{V}(\mu,k)} \left(\prod_{i \geq 1} \begin{bmatrix} \lambda_i' - \lambda_{i+1}' \\ \lambda_i' - \mu_i' \end{bmatrix}_t \right) a_{\delta(N)} P_\lambda.$$

Proof. We have combinatorial models for both sides of the equation by Lemma 145 and Theorem 148:

$$\sum_{\lambda \in \mathrm{V}(\mu,k)} \left(\prod_{i \geq 1} \begin{bmatrix} \lambda_i' - \lambda_{i+1}' \\ \lambda_i' - \mu_i' \end{bmatrix}_t a_{\delta(N)} P_\lambda \right) = \sum_{\lambda \in \mathrm{V}(\mu,k)} \sum_{A \in \mathbb{LAT}_\lambda^{\mathcal{B}_{\lambda/\mu}}} \mathrm{swt}(A)$$

and

$$a_{\delta(N)} P_\mu \cdot e_k = \sum_{\lambda \in \mathrm{V}(\mu,k)} \sum_{A \in \mathbb{LAT}_\lambda^{\mathcal{B}_\mu}} \mathrm{swt}(A).$$

For each fixed $\lambda \in \mathrm{V}(\mu, k)$, apply Lemma 97 to the sets $\mathbb{LAT}_\lambda^{\mathcal{B}_{\lambda/\mu}}$ and $\mathbb{LAT}_\lambda^{\mathcal{B}_\mu}$. First, the set $\mathbb{LAT}_\lambda^{\mathcal{B}_{\lambda/\mu}}$ is closed under $I^{\mathcal{B}_\mu}$ because each of $\mathrm{nPos}_{\lambda/\mu}(i)$ and $\mathrm{oPos}_{\lambda/\mu}(i)$ is a subset of $\mathrm{Pos}_\mu(j)$ for some j: $\mathrm{nPos}_{\lambda/\mu}(i) \subseteq \mathrm{Pos}_\mu(i-1)$ and $\mathrm{oPos}_{\lambda/\mu}(i) \subseteq \mathrm{Pos}_\mu(i)$. This means that the configurations of X's inside the λ/μ-blocks are not changed in $I^{\mathcal{B}_\mu}$. Second, we claim

$$\mathbb{LAT}_\lambda^{\mathcal{B}_\mu} = \mathbb{LAT}_\lambda^{\mathcal{B}_{\lambda/\mu}} \cap \mathbb{AT}_\lambda^{\mathcal{B}_\mu}.$$

Since elements of $\mathbb{LAT}_\lambda^{\mathcal{B}_\mu}$ have no exponent collisions in μ-blocks, $\mathbb{LAT}_\lambda^{\mathcal{B}_\mu} \subseteq \mathbb{AT}_\lambda^{\mathcal{B}_\mu}$, and $\mathbb{LAT}_\lambda^{\mathcal{B}_\mu} \subseteq \mathbb{LAT}_\lambda^{\mathcal{B}_{\lambda/\mu}}$ by definition. By Lemma 118, any object in $\mathbb{LAT}_\lambda^{\mathcal{B}_{\lambda/\mu}}$ that is not in $\mathbb{LAT}_\lambda^{\mathcal{B}_\mu}$ has local exponent collisions, so $\mathbb{LAT}_\lambda^{\mathcal{B}_{\lambda/\mu}} \cap \mathbb{AT}_\lambda^{\mathcal{B}_\mu} \subseteq \mathbb{LAT}_\lambda^{\mathcal{B}_\mu}$. We conclude that

$$\sum_{A \in \mathbb{LAT}_\lambda^{\mathcal{B}_{\lambda/\mu}}} \mathrm{swt}(A) = \sum_{A \in \mathbb{LAT}_\lambda^{\mathcal{B}_\mu}} \mathrm{swt}(A)$$

as desired. Since this is true for all $\lambda \in \mathrm{V}(\mu, k)$, we have completed the proof. □

Chapter 7

A Second Pieri Rule for Hall-Littlewood Polynomials

We now consider a second Pieri rule for the product of a Hall-Littlewood polynomial P_μ with a Hall-Littlewood polynomial $Q_{(r,0^{N-1})}$.

Theorem 151. *For $\mu \in \mathrm{Par}_N$ and $r \in \mathbb{N}$,*

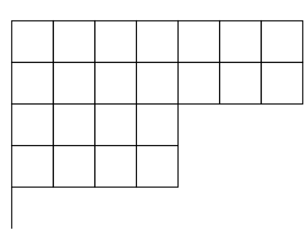
$$P_\mu(x_1, \ldots, x_N; t) \cdot Q_{(r,0^{N-1})}(x_1, \ldots, x_N; t) = \sum_{\lambda \in \mathrm{H}(\mu,r)} \left(\prod_{\substack{i:\lambda'_i=\mu'_i+1,\\ \lambda'_{i+1}=\mu'_{i+1}}} (1 - t^{m_i(\lambda)}) \right) \cdot P_\lambda(x_1, \ldots, x_N; t),$$

See [11, III.3.10] for an algebraic proof of this theorem.

We will give a bijective proof when r is "not too big" compared to μ by tackling three smaller subproblems, each handling successively more general partitions $\mu \in \mathrm{Par}_N$. In the first subproblem (P1), we will prove the result for the special case where $\mu = (0^N)$. In the second subproblem (P2), $\mu \in \mathrm{Par}_N$ is any partition such that for all $i < N$, if $c_i = \mu_i - \mu_{i+1} \neq 0$, then $c_i > r$. In other words, horizontal segments on the "frontier" of μ must have length strictly greater than r. In the third subproblem (P3), μ must be such that $c_i \geq r$ for all $c_i \neq 0$. In an unsolved fourth subproblem (P4), μ has no restrictions; this will be the subject of future work.

Example 152. Let $N = 5$ and $r = 3$. Let $\mu = (7^2, 4^2, 0)$:

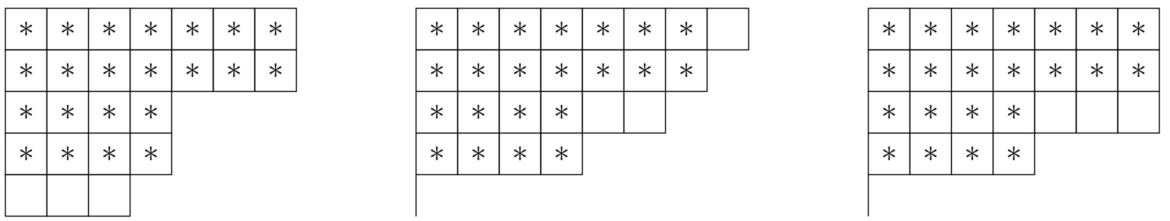
$\mu =$



.

Note that μ satisfies the requirements for problem (P3) because $c_2 = 3$ and $c_4 = 4$ which are both greater than or equal to $r = 3$. All other $c_i = 0$. Because $c_2 = 3$, μ is not covered by (P2).

Consider the partitions $(7^2, 4^2, 3), (8, 7, 6, 4, 0), (7^3, 4, 0) \in \mathrm{H}(\mu, 3)$:



According to this Pieri rule, in the expansion of $P_{(7^2,4^2,0)} \cdot Q_{(3,0^4)}$, $P_{(7^2,4^2,3)}$ appears with coefficient $(1-t)$. Similarly, $P_{(8,7,6,4,0)}$ will have coefficient $(1-t)^2$, and $P_{(7^3,4,0)}$ will have coefficient $(1-t^3)$.

In the previous chapter, we found new models for $a_{\delta(N)} P_\mu \cdot e_r$ by moving beads on abacus-tournaments according to r-element subsets of $\{1, \ldots, N\}$ to obtain abacus-tournaments with new position sets. *Multiple beads* in a given μ-block could each move *at most one position* southeast. Now, computing $a_{\delta(N)} P_\mu \cdot Q_{(r,0^{N-1})}$ will involve moving *at most one bead* in each μ-block *multiple positions* southeast.

7.1 A Combinatorial Model for $Q_{(r,0^{N-1})}$

To get a tableau-based combinatorial model for $Q_{(r,0^{N-1})}$, we specialize the model for $Q_{\lambda/\mu}$ given in [10, p. 2010] to the case $\lambda = (r, 0^{N-1})$ and $\mu = (0^N)$. This model was derived from Macdonald's algebraic formula for $Q_{\lambda/\mu}$ in [11, III.5, p. 299]. Objects in this model consist of a semistandard tableau $T \in \mathrm{SSYT}_N(\lambda/\mu)$ paired with a subset E of the cells of T meeting certain conditions. These pairs (T, E) are called *Q-starred semistandard tableaux* and the set E is denoted visually on the Ferrers diagram of T by starring the entries in cells in T that belong to E. In our case, when $\lambda = (r, 0^{N-1})$ and $\mu = (0^N)$, these objects simplify greatly. Any $T \in \mathrm{SSYT}_N((r, 0^{N-1}))$ consists of a single weakly increasing row of r values, and the only cells in T eligible to have starred entries are those cells c such that the cell immediately to the right of c, if there is one, has an entry with strictly greater value than that of c. Consequently, we use the following definition.

Definition 153. Given N and r, a *starred semi-standard tableau* T^* *of shape* $(r, 0^{N-1})$ *over an* N*-letter alphabet* consists of a weakly increasing row of r cells, relative to the standard ordering $1 < 2 < \cdots < N$, where the last copy of each symbol is optionally starred. Let $\mathrm{SSYT}^*_N(r)$ denote the set of such starred semi-standard tableaux.

In general, we say a value is starred in T^* to mean that the last copy of that value is starred. As the next example shows, there are different ways of displaying starred semi-standard tableaux, and we alternate between them as convenient. We will often wish to display T^* in a manner adapted to the word v of an abacus-tournament. We write T^*_v for T^* displayed with the contents arranged according to the total ordering $v_1 < \cdots < v_N$ determined by v.

Example 154. Let $N = 6$ and $r = 7$. Then

$$T^* = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1^* & 2^* & 3 & 4^* \\ \hline \end{array} = 1^{4*}2^*34^*$$

is a starred semi-standard tableau of shape r. Note that T^* can be displayed with cells as a semi-standard tableau or without cells where exponents denote multiple copies of an entry. T^* is displayed

$$T^*_v = \begin{array}{|c|c|c|c|c|c|c|}\hline 2^* & 1 & 1 & 1 & 1^* & 3 & 4^* \\ \hline \end{array} = 2^*1^{4*}34^*$$

with the ordering $v = 213456$.

Definition 155. Given $r \in \mathbb{N}$ and $T^* \in \mathrm{SSYT}^*_N(r)$, define the *content* of T^* to be $c(T^*) = (c_1, c_2, \dots)$ where c_k is the number of copies of k in T^*. Given variables $x_1, x_2, \dots,$ the *content monomial* of T^* is

$$x^{T^*} = x^{c(T^*)} = x_1^{c_1} x_2^{c_2} \cdots x_k^{c_k} \cdots .$$

Let $e(T^*)$ be the number of starred entries in T^*.

Example 156. Let $N = 6$, $r = 7$, and

$$T^* = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1^* & 2^* & 3 & 4^* \\ \hline \end{array} .$$

The content monomial of T^* is

$$x^{T^*} = x_1^4 x_2 x_3 x_4$$

and $e(T^*) = 3$.

Theorem 157. *Given* $r \in \mathbb{N}$,

$$Q_{(r,0^{N-1})}(x_1, \dots, x_N; t) = \sum_{T^* \in \mathrm{SSYT}^*_N(r)} (-t)^{e(T^*)} x^{T^*}.$$

We refer readers to [10, §4.2] for details of a proof.

Multiplying both sides of the second Pieri rule by $a_{\delta(N)}$ gives the antisymmetric version

$$\left(a_{\delta(N)} P_\mu\right) \cdot Q_{(r,0^{N-1})} = \sum_{\lambda \in \mathrm{H}(\mu,r)} \prod_{\substack{i : \lambda'_i = \mu'_i + 1, \\ \lambda'_{i+1} = \mu'_{i+1}}} (1 - t^{m_i(\lambda)}) \cdot a_{\delta(N)} P_\lambda, \tag{7.1}$$

which will be the subject of our bijective proof. Combining individual combinatorial models, we obtain the following model for $a_{\delta(N)} P_\mu \cdot Q_{(r,0^{N-1})}$.

Theorem 158. *Given* $\mu \in \mathrm{Par}_N$ *and* $r \in \mathbb{N}$,

$$a_{\delta(N)} P_\mu \cdot Q_{(r,0^{N-1})} = \sum_{(A,T^*) \in \mathbb{LAT}_\mu^{\mathcal{B}_\mu} \times \mathrm{SSYT}^*_N(r)} (-t)^{e(T^*)} x^{T^*} \operatorname{swt}(A).$$

Proof. This is a direct consequence of Theorem 110 and Theorem 157. ∎

7.2 Subproblem P1

In this subproblem, we prove the second Pieri rule for the partition $\mu = (0^N)$. Given r, the only partition $\lambda \in \mathrm{H}((0^N), r)$ is $\lambda = (r, 0^{N-1})$. Therefore, we must show

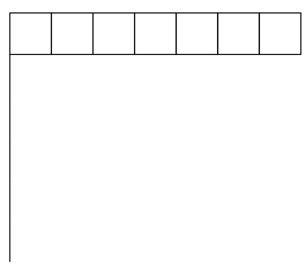
$$\left(a_{\delta(N)} P_{(0^N)}\right) \cdot Q_{(r,0^{N-1})} = (1-t) \cdot a_{\delta(N)} P_{(r,0^{N-1})}.$$

This is an immediate algebraic consequence of Definition 69 but is combinatorially subtle because the left-hand side is modeled by pairs of abacus-tournaments and starred semistandard tableaux while the right-hand side is modeled by shaded abacus-tournaments with moved beads.

Definition 159. Given $r \in \mathbb{N}$, let $\mathbb{JAT}_{(r,0^{N-1})}$ denote the set of all shaded abacus-tournaments A_* in $\mathbb{AT}^*_{(r,0^{N-1})}$ such that A_* is leading in blocks $\mathcal{B}_{(r,0^{N-1})}$, $\mathrm{upset}(A_*) < r$, $\mathrm{upset}(A_*)$ is right-justified (see Definition 120), and v_1 is either shaded or unshaded, where $v = v(A_*)$ is the word of A_*. We say that abacus-tournaments in $\mathbb{JAT}_{(r,0^{N-1})}$ are *justified for r*.

Recall that shading v_1 multiplies the signed weight of A_* by $-t$.

Example 160. Let $N = 6$ and $r = 7$. Consider $(7, 0^5) \in \mathrm{H}((0^6), 7)$:



.

The abacus-tournament A_* found in Figure 7.1 has

$$\mathrm{upset}(A_*) = \{(3,2), (6,2)\}.$$

Since the X's for these upsets are right-justified, $A_* \in \mathbb{JAT}_{(7,0^5)}$.

Definition 161. Given $T^* \in \mathrm{SSYT}^*_N(r)$, $v \in S_N$, and $k > 1$, we say v_k is a *lone star* in T^* iff v_k occurs exactly once in T^*, v_k is starred in T^*, and v_{k-1} occurs in T^*.

Example 162. Let $N = 6$ and $r = 7$. Let $v = 123456$ and let

$$T^* = \boxed{1^*}\boxed{2^*}\boxed{3}\boxed{3^*}\boxed{3^*}\boxed{4^*}\boxed{5}\,.$$

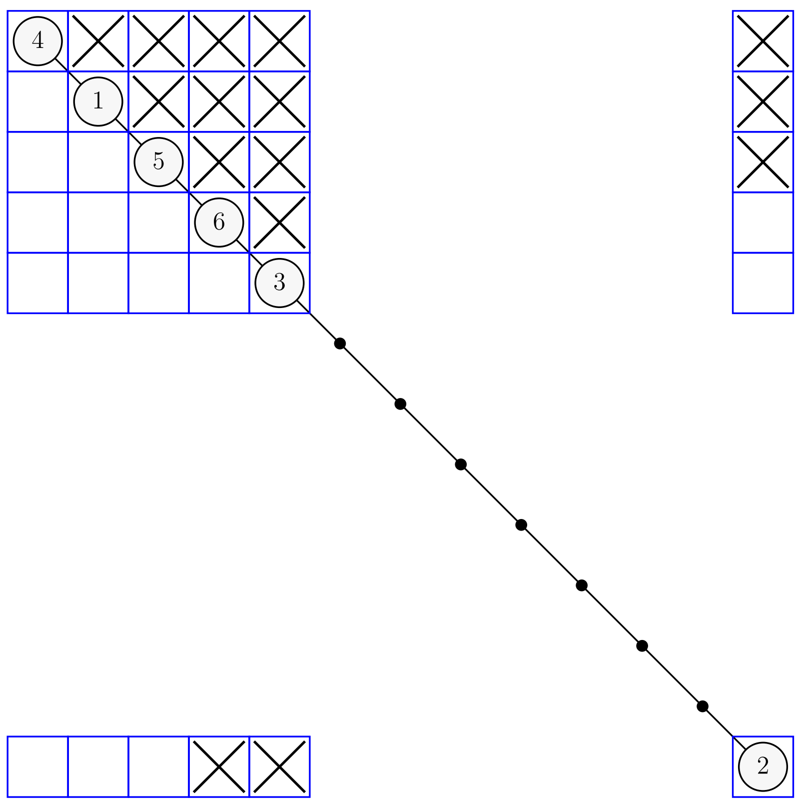
The single copies of $v_2 = 2$ and $v_4 = 4$ are the lone stars in T^*.

Definition 163. Given $T^* \in \mathrm{SSYT}^*_N(r)$, $\mu = (0^N)$, and $A = ((0^N), v, \tau) \in \mathbb{LAT}^{\mathcal{B}_\mu}_\mu$, we say (A, T^*) is a *lone star pair for $\mu = (0^N)$* iff T^* has the form

$$T^*_v = v_1^b v_2^* v_3^* \dots v_{p+1}^*,$$

where the last copy of v_1 could be starred, $r = b + p$, $b > 0$, and $p \geq 0$. Let $\mathbb{LSP}_{(0^N)}$ denote the set of lone star pairs for (0^N). Call the sequence of consecutive lone stars $v_2^* v_3^* \dots v_{p+1}^*$ the *lone star suffix of T^**.

Figure 7.1: An abacus-tournament in $\mathbb{JAT}_{(7,0^5)}$.



Example 164. Let $N = 6$ and $r = 7$. Let $v = 123456$, let $A = ((0^6), v, \tau_v)$, and let

$$T^* = \boxed{1}\boxed{1}\boxed{1}\boxed{1^*}\boxed{2^*}\boxed{3^*}\boxed{4^*}\ .$$

Then $(A, T^*) \in \mathbb{LSP}_{(0^6)}$. The single copies of 2, 3 and 4 form the lone star suffix of T^*. On the other hand, if T^* is paired with $A' = ((0^6), v', \tau_{v'})$ where $v' = 213456$, then

$$T^*_{v'} = \boxed{2^*}\boxed{1}\boxed{1}\boxed{1}\boxed{1^*}\boxed{3^*}\boxed{4^*}$$

and $(A', T^*) \notin \mathbb{LSP}_{(0^6)}$.

The next lemma states that it suffices to consider only lone star pairs when forming a combinatorial interpretation for $a_{\delta(N)} P_{(0^N)} \cdot Q_{(r,0^{N-1})}$.

Lemma 165. *For all $r \in \mathbb{N}$,*

$$a_{\delta(N)} P_{(0^N)} \cdot Q_{(r,0^{N-1})} = \sum_{(A,T^*) \in \mathbb{LSP}_{(0^N)}} (-t)^{e(T^*)} x^{T^*} \operatorname{swt}(A).$$

Proof. By Theorem 158, it is enough to show that there exists a weight-preserving, sign-reversing involution I on $\mathbb{LAT}^{\mathcal{B}_{(0^N)}}_{(0^N)} \times \mathrm{SSYT}^*_N(r)$ with fixed point set $\mathbb{LSP}_{(0^N)}$.

Given (A, T^*) in the domain of I with word $v = v(A)$, scan T^*_v from right to left, ignoring any lone stars. Say we reach v_i, the rightmost non-lone star in T^*. If $i = 1$, then (A, T^*) is a lone star pair and we define $I(A, T^*) = (A, T^*)$, a fixed point of I. If $i > 1$, then $(A, T^*) \notin \mathbb{LSP}_{(0^N)}$ and T^* has the form

$$T^*_v = \cdots v_{i-1}^a v_i^b v_{i+1}^* v_{i+2}^* \cdots v_{i+p}^*,$$

where the last copies of v_{i-1} and v_i could be starred, $b \geq 1$, $a \geq 0$, and $p \geq 0$.

Writing $A = ((0^N), v, \tau)$, we use the following rules to define $I(A, T^*) = (A', S^*)$ where $A' = ((0^N), v', \tau')$:

1. Switch the order of v_{i-1} and v_i in v to get v'. This says that $v' = v \circ (i, i-1) = v_1 \cdots v_i v_{i-1} \cdots v_N$.

2. Switch edge $(v_{i-1}, v_i) \in \tau$ to (v_i, v_{i-1}) to obtain τ'.

3. Increase the number of copies of v_{i-1} in T^* by one and decrease the number of copies of v_i in T^* by one to obtain S^*. Assign stars to v_{i-1} or v_i in S^* according to cases (A)-(D) below and assign to all other v_k, where $k \neq i, i-1$, the same status (starred or not) in S^* as in T^*.

Steps 1 and 2 guarantee that $A' \in \mathbb{LAT}_{(0^N)}^{\mathcal{B}_{(0^N)}}$ and

$$\mathrm{swt}(A') = \left(-\frac{x_{v_i}}{x_{v_{i-1}}}\right)\mathrm{swt}(A).$$

According to step 3, the new starred tableau S^* has the form

$$S^*_{v'} = \cdots v_i^{b-1} v_{i-1}^{a+1} v_{i+1}^* v_{i+2}^* \ldots v_{i+p}^*,$$

where v_{i-1} and v_i are starred according cases (A)-(D). The content monomial x^{S^*} is equal to $\frac{x_{v_{i-1}}}{x_{v_i}} \cdot x^{T^*}$. Consequently, provided I preserves the number of starred entries in T^*, I will be weight-preserving and sign-reversing when acting on pairs $(A, T^*) \notin \mathbb{LSP}_{(0^N)}$.

Maintaining the number of starred entries in T^* requires that step 3 have specific cases. See Example 166 for examples of each case.

Case A: $b > 1$ and $a > 0$. In this case, at least one copy of v_{i-1} and v_i occur in both T^* and S^*. Consequently, let v_i be starred in S^* if and only if v_i is starred in T^*, and let v_{i-1} be starred in S^* if and only if v_{i-1} is starred in T^*. Note that if T^* falls under Case A, then S^* also meets the conditions for Case A.

Case B: $b > 1$ and $a = 0$. Here, there is no copy of v_{i-1} in T^* so v_{i-1} will not be starred in S^*. As before, let v_i be starred in S^* if and only if v_i is starred in T^*. If T^* falls under Case B, then S^* meets the conditions for Case C below.

Case C: $b = 1$ and $a > 0$. Since $b = 1$ in this case, we are guaranteed that v_i is not starred in T^*. Otherwise, it would be a lone star and would be contained in the lone star suffix. Consequently, I erases the last copy of v_i and no star is erased. Let v_{i-1} be starred in S^* if and only if v_{i-1} is starred in T^*. Note that if T^* falls under Case C, then S^* satisfies the conditions for Case B.

Case D: $b = 1$ and $a = 0$. In this case, let v_{i-1} be starred in S^* if and only if v_i is starred in T^*. (Intuitively, T^* has no copies of v_{i-1} and S^* has no copies of v_i, so we let v_{i-1} in S^* inherit v_i's "star status" in T^*.) Note that if T^* falls under Case D, then S^* meets the conditions for Case D.

In each case, the number of starred entries in S^* is equal to the number of starred entries in T^*, so I is weight preserving. Furthermore, in all four cases, v_{i-1} is the first non-lone star from the right in S^* and so, based on the cases above, $I(I(A, T^*)) = (A, T^*)$. □

Example 166. Let $N = 6$ and $r = 7$. Let $v = 123456$, $v' = 132456$, $A = ((0^6), v, \tau_v)$ (where $\tau_v = \{(v_i, v_j) : i < j\}$), and $A' = ((0^6), v', \tau_{v'})$.

- Let
$$T^*_v = \begin{array}{|c|c|c|c|c|c|c|}\hline 1^* & 2 & 3 & 3 & 3^* & 4^* & 5^* \\ \hline\end{array}\ .$$
Here, $i = 3$, $a = 1$, and $b = 3$. Then (A, T^*) falls under Case A. Consequently, $I(A, T^*) = (A', S^*)$ where
$$S^*_{v'} = \begin{array}{|c|c|c|c|c|c|c|}\hline 1^* & 3 & 3^* & 2 & 2 & 4^* & 5^* \\ \hline\end{array}\ .$$
Now, $i = 3$ (note $v'_3 = 2$), $a = 2$, $b = 2$, and $I(A', S^*) = (A, T^*)$ following Case A.

- Let
$$T^*_v = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 1^* & 3 & 3 & 3^* & 4^* & 5^* \\ \hline\end{array}\ .$$
Here, $i = 3$, $a = 0$ and $b = 3$. Under Case B, I produces
$$S^*_{v'} = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 1^* & 3 & 3^* & 2 & 4^* & 5^* \\ \hline\end{array}\ .$$
Now, (A', S^*) falls under Case C because with $i = 3$, $a = 2$ and $b = 1$. So $I(A', S^*) = (A, T^*)$.

- Let
$$T^*_v = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 1 & 1^* & 3^* & 4^* & 5^* & 6^* \\ \hline\end{array}\ .$$
Here, $i = 3$, $a = 0$, and $b = 1$. Then, under Case D, $I(A, T^*) = (A', S^*)$, where
$$S^*_{v'} = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 1 & 1^* & 2^* & 4^* & 5^* & 6^* \\ \hline\end{array}\ .$$
Now, $i = 3$, $a = 0$, and $b = 1$, so following Case D, $I(A', S^*) = (A, T^*)$.

Theorem 167. *For $r \in \mathbb{N}$,*
$$a_{\delta(N)} P_\mu \cdot Q_\lambda = (1 - t) \cdot a_{\delta(N)} P_\lambda. \tag{7.2}$$

Proof. Define $\mu = (0^N)$ and $\lambda = (r, 0^{N-1})$. By Lemma 165, the left hand side of (7.2) has combinatorial model
$$a_{\delta(N)} P_\mu \cdot Q_\lambda = \sum_{(A,T^*) \in \mathbb{LSP}_\mu} (-t)^{e(T^*)} x^{T^*} \operatorname{swt}(A).$$
On the right hand side of (7.2), $a_{\delta(N)} P_\lambda$ has combinatorial model
$$a_{\delta(N)} P_\lambda = \sum_{A \in \mathbb{LAT}_\lambda^{\mathcal{B}_\lambda}} \operatorname{swt}(A),$$
by Theorem 110. By (Single Bead Collision) Lemma 122 applied to blocks $\mathrm{Pos}_\lambda(0)$ and $\mathrm{Pos}_\lambda(1)$, $A \in \mathbb{LAT}_\lambda^{\mathcal{B}_\lambda}$ has a global exponent collision if $\operatorname{upset}(A) \geq r$ or $\operatorname{upset}(A)$ is not right-justified. Let X be the set of abacus-tournaments $A \in \mathbb{LAT}_\lambda^{\mathcal{B}_\lambda}$ such that $\operatorname{upset}(A) < r$

and upset(A) is right-justified. By Lemma 97, applied to $\mathbb{LAT}_\lambda^{\mathcal{B}_\lambda}$ and X, we have a new combinatorial model

$$a_{\delta(N)} P_\lambda = \sum_{A \in X} \text{swt}(A).$$

Consequently,

$$(1-t) a_{\delta(N)} P_\lambda = \sum_{A_* \in \mathbb{JAT}_\lambda} \text{swt}(A_*)$$

because the property that the southeastern-most bead in any A_* may be shaded accounts for the factor of $(1-t)$.

We define an edge-modifying bead movement R on $(A, T^*) \in \mathbb{LSP}_\mu$ that produces a new abacus-tournament $A'_* \in \mathbb{JAT}_\lambda$ from A using motions encoded by T^*. Fix $(A, T^*) \in \mathbb{LSP}_\mu$, where $A = (\mu, v, \tau)$ and T^* has the form

$$T^*_v = v_1^b v_2^* v_3^* \dots v_{p+1}^*,$$

where the last copy of v_1 could be starred, $b > 0$, $p \geq 0$, and $r = b + p$. Move v_1 precisely $r = b + p$ abacus positions to the right, switch all edges of the form $(v_1, v_j) \in \tau$ such that $1 < j \leq p+1$ to (v_j, v_1), and make v_1 a shaded bead if and only if the last copy of v_1 is starred in T^*. Define $A'_* = (\lambda, v, \tau')$ to be the resulting abacus-tournament. Note that $A'_* \in \mathbb{LAT}_\lambda^{\mathcal{B}_\lambda}$ because A is leading in $\text{Pos}_\mu(0) = \{1, 2, \dots, N\}$, $\text{Pos}_\lambda(0) = \{2, \dots, N\}$, and $\text{Pos}_\lambda(1) = \{1\}$. Furthermore, upset$(A'_*) = p < r$ and upset(A'_*) is right-justified, so $A'_* \in \mathbb{JAT}_\lambda$. R serves as a bijection between $\mathbb{LSP}_\mu$ and $\mathbb{JAT}_\lambda$. Indeed, given $A'_* = (\lambda, v, \tau') \in \mathbb{JAT}_\lambda$ where v_1 is either shaded or unshaded, we set $A = (\mu, v, \tau_v)$ and $T^*_v = v_1^{r-p} v_2^* v_3^* \dots v_{p+1}^*$ where $p = |\text{upset}(A'_*)|$ and the last copy of v_1 in T^* is starred if and only if v_1 is shaded in A'_*. Then $(A, T^*) \in \mathbb{LSP}_\mu$ and $R((A, T^*)) = A'_*$.

To complete the proof, we must show that R preserves signed weights. Let $(A, T^*) \in \mathbb{LSP}_\mu$ with the notation above, and $R(A, T^*) = A'_*$. The exponent of x_{v_i} in $(-t)^{e(T^*)} x^{T^*} \text{swt}(A)$ is equal to the sum

$$\text{out}_A(v_i) + \text{gap}_A(v_i) + c_{v_i},$$

where $c(T^*)$ is the content of T^*. On the other hand, the exponent of x_{v_i} in swt(A'_*) is equal to

$$\text{out}_{A'_*}(v_i) + \text{gap}_{A'_*}(v_i).$$

If $i = 1$, then $\text{out}_A(v_1) = N - 1$ and $\text{gap}_A(v_1) = 0$. In A'_*, v_1 is located precisely $r = b + p = c_{v_1} + p$ gaps to the right of its original position in A, so $\text{gap}_{A'_*}(v_1) = c_{v_1} + p$. Also $\text{out}_{A'_*}(v_1) = N - 1 - p$, so the two exponents of x_{v_1} are equal. If $1 < i \leq p + 1$, then v_i has equal gap counts in A and A'_*, namely zero, and v_i's global outdegree in A'_* is one greater than in A. Since $c_{v_i} = 1$ for such i, the exponents of x_{v_i} match. All other i have $p + 1 < i \leq N$. In this case, v_i has equal gap counts and global outdegrees in A and A'_*. Since $c_{v_i} = 0$ for these values of i, the exponents of x_{v_i} match. The weights of (A, T^*) and A'_* also have the same coefficient of $(-t)$: the exponent of $(-t)$ for (A, T^*) is $e(T^*)$, the number of starred entries in T^*, because $|\text{upset}(A)| = 0$. On the other hand, the exponent of $(-t)$ for A'_* is the sum of $|\text{upset}(A'_*)|$ and the number of shaded beads in A'_*. If v_1 is starred in T^*, then v_1 is shaded in A'_*. Every other starred entry v_i in T^* corresponds uniquely to an edge $(v_i, v_1) \in \text{upset}(A'_*)$. A and A'_* have the same word v, so both have the same factor of $(-1)^{\text{inv}(v)}$ in their signed weights. We conclude that

$$(-t)^{e(T^*)} x^{T^*} \text{swt}(A) = \text{swt}(A'_*).$$

□

Example 168. Let $N = 6$ and $r = 7$. Let $v = 123456$, $A = ((0^6), v, \tau_v)$, and

$$T_v^* = \boxed{1}\boxed{1}\boxed{1}\boxed{1^*}\boxed{2^*}\boxed{3^*}\boxed{4^*}\ .$$

Since $(A, T^*) \in \mathbb{LSP}_{(0^6)}$, we can apply the bead movement given by R. In this case, to obtain $R(A, T^*)$, the bead $v_1 = 1$ is moved $r = 7$ abacus positions diagonally to the right and the south-most three X's in the column of v_1 are moved below the diagonal. The bead v_1 is shaded because the final copy of 1 is starred in T^*. The resulting abacus-tournament is displayed in Figure 7.2 and has signed weight $(-t)^4 x_1^9 x_2^5 x_3^4 x_4^3 x_5^1 x_6^0$. On the other hand, consider the abacus-tournament A' in Figure 7.1. We have already determined that $A'_* \in \mathbb{JAT}_{(7,0^5)}$. We compute $R^{-1}(A'_*) = (A'', S^*)$, where $A'' = ((0^6), u = 236514, \tau_u)$ and S^* is reconstructed from A'_* as follows. Notice that A' has two X's below the main diagonal, corresponding to edges $(3, 2)$ and $(6, 2)$. Since $r = 7$, we conclude that S^* has 5 copies of $u_1 = 2$, a starred $u_2 = 3$, and a starred $u_3 = 6$: when ordered according to $u = 236514$,

$$S_u^* = \boxed{2}\boxed{2}\boxed{2}\boxed{2}\boxed{2}\boxed{3^*}\boxed{6^*}\ .$$

Note that the last copy of 2 is not starred because the bead labeled 2 in A' is not shaded.

7.3 Subproblem P2

We now assume for Subproblem P2 that $r \in \mathbb{N}$ and $\mu \in \text{Par}_N$ is any partition such that for all $i \geq 1$, if $\mu_i - \mu_{i+1} > 0$, then $\mu_i - \mu_{i+1} > r$. To see how this affects the Pieri rule, let $\lambda \in \text{H}(\mu, r)$ and consider any i such that $\lambda'_i = \mu'_i + 1$ and $\lambda'_{i+1} = \mu'_{i+1}$. Visually (see Figure 7.3), this means column i of $\text{dg}(\lambda)$ is composed of μ'_i cells that are also part of $\text{dg}(\mu)$, then exactly one additional cell not part of $\text{dg}(\mu)$. Let $j = \lambda'_i$ (the row of the last cell in column i). Then $\lambda_j > \mu_j$. The number of new cells in row j of λ must be less than or equal to r, which is strictly less than the "rim" $\mu_{j-1} - \mu_j$. Therefore, we see that $\lambda_j < \lambda_{j-1}$, so $m_i(\lambda) = 1$ here.

Consequently, our goal is to prove the Pieri rule that

$$a_{\delta(N)} P_\mu \cdot Q_{(r,0^{N-1})} = \sum_{\lambda \in \text{H}(\mu,r)} \left(\prod_{\substack{i:\lambda'_i = \mu'_i + 1, \\ \lambda'_{i+1} = \mu'_{i+1}}} (1-t) \right) a_{\delta(N)} P_\lambda.$$

Figure 7.2: Abacus-tournament $R(A, T^*) \in \mathbb{JAT}_{(7,0^5)}$.

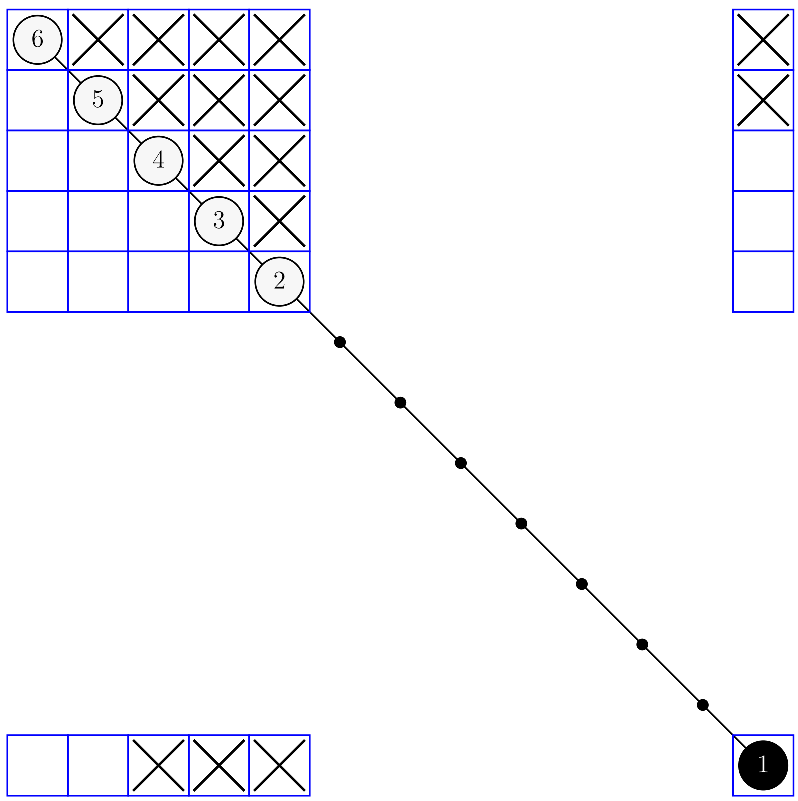
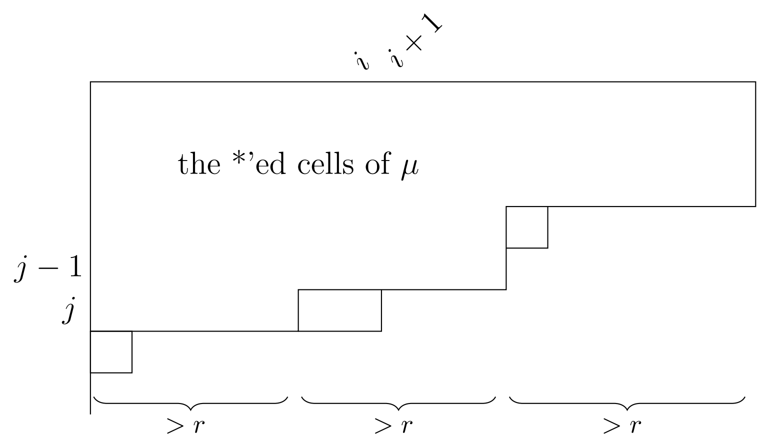


Figure 7.3: Outlines for μ and λ meeting Subproblem P2 restrictions.



Intuitively, we will be generalizing the bead movement from Subproblem P1: given a pair (A, T^*) from the model for the left side, the starred tableau T^* will move beads and modify edges on an abacus-tournament A to either produce a shaded abacus-tournament A'_* with partition $\lambda \in \mathrm{H}(\mu, r)$, modeled by the right side, or cancel with another pair (A', S^*). Unlike in Subproblem P1, the abacus-tournament A may contain multiple nonempty blocks of beads, and T^* may move the southeastern-most bead in any number of these blocks. The restrictions of Subproblem P2 on μ prevent a bead from moving from its position in one block $\mathrm{Pos}_\mu(i)$ all the way to a new position in another *nonempty* block $\mathrm{Pos}_\mu(j)$ because the number of gaps between beads in any two nonempty μ-blocks is guaranteed to be larger than r, which will be the combined number of abacus positions that all beads may travel.

We use a weight-preserving bijection in order to formalize this bead movement. Lemma 175 below will provide a refined combinatorial model for $a_{\delta(N)} P_\mu \cdot Q_{(r,0^{N-1})}$ by canceling extra pairs (A, T^*) from $\mathbb{LAT}_\mu^{\mathcal{B}_\mu} \times \mathrm{SSYT}_N^{*r}$ that would have bead collisions with the involution J. Lemma 180 shrinks the number of combinatorial objects in the combinatorial model for

$$\left(\prod_{\substack{\lambda'_i = \mu'_i + 1, \\ \lambda'_{i+1} = \mu'_{i+1}}} (1 - t) \right) a_{\delta(N)} P_\lambda,$$

for each $\lambda \in \mathrm{H}(\mu, r)$; the union of these models will form the image of the map from the fixed points of J to abacus-tournaments. Both lemmas are then used in Theorem 181 to complete the proof of Subproblem P2 which exhibits a weight-preserving bijection between the models.

7.3.1 A Combinatorial Model for $a_{\delta(N)} P_\mu \cdot Q_{(r,0^{N-1})}$

We first require additional definitions to formulate Lemma 175. While the mechanics of Subproblem P1 are based on *global* outdegrees and gap counts of abacus-tournaments, bead movements and cancellations in Subproblem P2 are *local* to blocks. Consequently, given a block B, we introduce the idea of a "zone for B" in a starred semistandard tableaux, which is analogous to the "square for $A|_B$" in an abacus-tournament A. The definitions and results here apply to all $\mu \in \mathrm{Par}_N$, not just μ satisfying the conditions of Subproblem 2.

Definition 169. If B is a block, $T^* \in \mathrm{SSYT}^*_N(r)$, and $v \in S_N$, then the *zone* of T^*_v for B is the set of cells

$$\{(1,k) : T^*_v(1,k) = v_j \text{ for some } j \in B\}.$$

Example 170. Let $N = 8$, $r = 8$, and $\mu = (22^3, 9^4, 0)$. Let $v = 215|8734|6$, and

$$T^*_v = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 2 & 1^* & 5 & 5 & 3^* & 6 & 6 & 6 \\ \hline\end{array}\ .$$

Then, for example, the zone of T^*_v for block $\mathrm{Pos}_\mu(22)$ is $\{(1,1),(1,2),(1,3),(1,4)\}$.

T^*_v is ordered according to $v \in S_N$, so all cells in a zone of T^*_v for a particular block B appear consecutively in T^*_v. When a set of blocks $\mathcal{B}$ accounts for every position in $\{1,\ldots,N\}$, we often add an additional vertical line between two cells in T^*_v to indicate a division between zones for blocks in $\mathcal{B}$. We similarly place vertical lines in a word v to emphasize which symbols belong to various zones. In the above example, $v = 215|8734|6$ and T^*_v looks like

$$\begin{array}{|c|c|c|c||c||c|c|c|}\hline 2 & 1^* & 5 & 5 & 3^* & 6 & 6 & 6 \\ \hline\end{array}\ .$$

Another example is

$$S^*_v = \begin{array}{|c|c|c|c||c|c|c||c|}\hline 2 & 2 & 2^* & 1^* & 8 & 7^* & 3^* & 6^* \\ \hline\end{array}\ .$$

Given $\mu \in \mathrm{Par}_N$, we are particularly interested in the zones of T^*_v for the blocks $\mathcal{B}_\mu$. Consequently, these zones are given a special name.

Definition 171. Given $\mu \in \mathrm{Par}_N$, $T^* \in \mathrm{SSYT}^*_N(r)$, and $v \in S_N$, define the *ith μ-zone of T^*_v* to be the set of cells

$$\{(1,k) : T^*_v(1,k) = v_j \text{ for some } j \in \mathrm{Pos}_\mu(i)\}.$$

We now proceed as in Subproblem P1 with definitions and theorems that now apply to zones of starred semistandard tableaux. As in Subproblem P1, the first step is to refine the combinatorial model for $a_{\delta(N)} P_\mu \cdot Q_{(r,0^{N-1})}$ from Theorem 158. We continue to work under the assumption that $r \in \mathbb{N}$ and $\mu \in \mathrm{Par}_N$ is partition such that for all $i < N$, if $\mu_i - \mu_{i+1} > 0$, then $\mu_i - \mu_{i+1} > r$.

Definition 172. Given $T^* \in \mathrm{SSYT}^*_N(r)$ and $v \in S_N$, we say v_k is a *lone star in the ith μ-zone of T^*_v* iff v_k occurs exactly once in the ith μ-zone of T^*_v, v_k is starred in T^*, and v_{k-1} occurs in the ith μ-zone of T^*_v.

Example 173. Let $N = 8$, $r = 8$, and $\mu = (22^3, 9^4, 0)$. Let $v = 215|8734|6$,

$$T_v^* = \begin{array}{|c|c|c|c||c||c|c|c|}\hline 2 & 1^* & 5 & 5 & 3^* & 6 & 6 & 6 \\ \hline\end{array}$$

and

$$S_v^* = \begin{array}{|c|c|c|c||c|c|c||c|}\hline 2 & 2 & 2^* & 1^* & 8 & 7^* & 3^* & 6^* \\ \hline\end{array}\ .$$

The single copy of 1 in T_v^* is the only lone star in T_v^*. The single copies of 1, 7, and 3 are all lone stars in their respective μ-zones of S_v^*.

Definition 174. Let $T^* \in \mathrm{SSYT}_N^*(r)$, let $A = (\mu, v, \tau) \in \mathbb{LAT}_\mu^{\mathcal{B}_\mu}$, and let d_i be the first position in $\mathrm{Pos}_\mu(i)$ for all i such that $\mathrm{Pos}_\mu(i) \neq \emptyset$. We say (A, T^*) is a *lone star pair for* μ iff each nonempty ith μ-zone of T_v^* has the form

$$v_{d_i}^{b_i} v_{d_i+1}^* v_{d_i+2}^* \cdots v_{d_i+p_i}^*,$$

where the last copy of v_{d_i} may be starred, $r = \sum_{i \geq 0}(b_i + p_i)$, $b_i > 0$, and $p_i \geq 0$. Let $\mathbb{LSP}_\mu(r)$ denote the set of lone star pairs for μ.

As in Subproblem P1, the lone star pairs form a refined combinatorial model for $a_{\delta(N)} P_\mu \cdot Q_{(r,0^{N-1})}$.

Lemma 175. *Given* $r \in \mathbb{N}$ *and* $\mu \in \mathrm{Par}_N$,

$$a_{\delta(N)} P_\mu \cdot Q_{(r,0^{N-1})} = \sum_{(A,T^*) \in \mathbb{LSP}_\mu(r)} (-t)^{e(T^*)} x^{T^*} \operatorname{swt}(A).$$

Proof. Starting from the model for the left side in Theorem 158, we demonstrate a weight-preserving, sign-reversing involution J on $X = \mathbb{LAT}_\mu^{\mathcal{B}_\mu} \times \mathrm{SSYT}_N^*(r)$ with fixed point set $\mathbb{LSP}_\mu(r)$. Let $(A, T^*) \in X$ where $A = (\mu, v, \tau)$. Beginning with $i = 0$, scan each of the ith μ-zones of T_v^* from right to left, ignoring any lone stars for μ in the ith μ-zone. Say we reach entry v_j in the ith μ-zone, where v_j is the rightmost non-lone star entry for μ in the ith μ-zone of T_v^*. If $j = d_i$, then proceed to the $(i+1)$th μ-zone. If, for every i with ith μ-zone nonempty, the first non-lone star entry for μ in T_v^* in the ith μ-zone is d_i, then $(A, T^*) \in \mathbb{LSP}_\mu(r)$ and we define $J(A, T^*) = (A, T^*)$. Otherwise, let i be the least integer such that the ith μ-zone of T_v^* has $j > d_i$, where v_j is the rightmost non-lone star entry for μ in the ith μ-zone of T_v^*. This μ-zone of T_v^* has the form

$$\cdots v_{j-1}^a v_j^b v_{j+1}^* v_{j+2}^* \cdots v_{j+p}^*,$$

where the last copies of v_{j-1} and v_j may be starred, $a \geq 0$, $b \geq 1$, and $p \geq 0$.

In this case, $(A, T^*) \notin \mathbb{LSP}_\mu$, and we define $J(A, T^*) = (A', S^*)$ as follows. Let $v' = v \circ (j, j-1)$, where $(j, j-1)$ is the transposition switching positions j and $j-1$. Let τ'

be the tournament obtained from τ by switching edge $(v_{j-1}, v_j) \in \tau$ to (v_j, v_{j-1}). Define $A' = (\mu, v', \tau')$. Then $A' \in \mathbb{LAT}_{\mu}^{\mathcal{B}_\mu}$ and

$$\operatorname{swt}(A') = \left(-\frac{x_{v_j}}{x_{v_{j-1}}}\right) \operatorname{swt}(A).$$

To obtain S^*, apply the directions in the proof of Lemma 165 to the ith μ-zone of T^* to obtain a new zone. Define S^* to be the starred semistandard tableau that is identical to T^* except with the new zone in place of the ith μ-zone of T^*. In particular, this will increase the number of copies of v_{j-1} by one, decrease the number of copies of v_j by one, and maintain the number of stars. Consequently

$$x^{S^*} = \frac{x_{v_{j-1}}}{x_{v_j}} \cdot x^{T^*}.$$

Therefore, J is weight-preserving and sign-reversing for pairs $(A, T^*) \notin \mathbb{LSP}_\mu(r)$. Note that if we apply J to (A', S^*), the ith μ-zone of S^* will again be the first μ-zone of S^* to have a non-lone star entry not equal to v_{d_i}. Furthermore, this non-lone star entry of S^* must be v_{j-1}. Thus $J(J(A, T^*)) = (A, T^*)$. $\square$

Example 176. Let $N = 6$, $r = 4$, and $\mu = (4^2, 0^4)$. Let $v = 12|6543$ and let $A = (\mu, v, \tau)$ be the abacus-tournament in Figure 7.4.

- Let

  $$T_v^* = \begin{array}{|c|c|c|}\hline 6 & 6 & 5^* \\ \hline \end{array}\ ,$$

  and

  $$S_v^* = \begin{array}{|c|c||c|}\hline 1^* & 2^* & 6^* \\ \hline \end{array}\ .$$

  Then $(A, T^*), (A, S^*) \in \mathbb{LSP}_\mu(r)$.

- Let

  $$T_v^* = \begin{array}{|c|c|c|}\hline 1 & 2 & 2^* \\ \hline \end{array}\ .$$

  Here $i = 4$, $j = 2$ (note $v_2 = 2$ has position in $\operatorname{Pos}_\mu(4)$), $a = 1$, and $b = 2$, so (A, T^*) falls under Case A in the proof of Lemma 165. Then, $J(A, T^*) = (A', S^*)$, where $v' = 21|6543$, $A' = (\mu, v', \tau')$ is the abacus-tournament in Figure 7.5, and

  $$S_{v'} = \begin{array}{|c|c|c|}\hline 2^* & 1 & 1 \\ \hline \end{array}\ .$$

  Again, $i = 4$, $j = 2$, $a = 1$, and $b = 2$, so $J(A', S^*) = (A, T^*)$ following Case A.

- Let

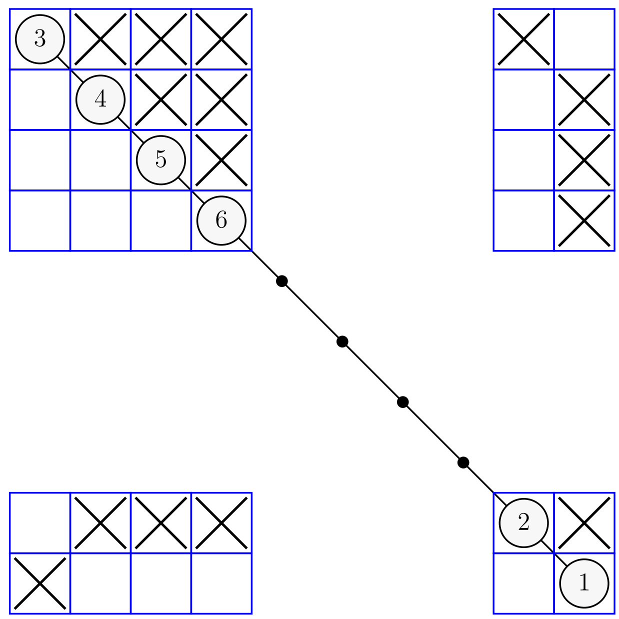
  $$T_v^* = \begin{array}{|c||c|c|}\hline 2 & 6^* & 5 \\ \hline \end{array}\ .$$

  Here $i = 0$, $j = 4$, $a = 1$, and $b = 1$. Under Case B, $J(A, T^*) = (A'', S^*)$, where $v'' = 12|5643$, $A'' = (\mu, v'', \tau'')$ is the abacus-tournament in Figure 7.6, and

  $$S_{v''} = \begin{array}{|c||c|c|}\hline 2 & 6 & 6^* \\ \hline \end{array}\ .$$

  Here, $i = 0$, $j = 4$, $a = 0$, and $b = 2$. So, under Case D, $J(A'', S^*) = (A, T^*)$.

Figure 7.4: An abacus-tournament $A \in \mathbb{LAT}_{\mu}^{\mathcal{B}_{\mu}}$.



7.3.2 A Combinatorial Model for the Right Side of the Pieri Rule

Definition 177. Let $\lambda \in \mathrm{H}(\mu, r)$, and let d_i be the lowest position in $\mathrm{Pos}_{\mu}(i)$ for all i such that $\mathrm{Pos}_{\mu}(i) \neq \emptyset$. Define $\mathbb{JAT}_{\mu}(\lambda)$ to be the set of all shaded abacus-tournaments $A_* \in \mathbb{AT}_{\lambda}^{*}$ that are leading in λ-blocks B_{λ} and for all $i \geq 0$ with $\mathrm{Pos}_{\mu}(i) \neq \mathrm{Pos}_{\lambda}(i)$, $\mathrm{upset}(A_*|_{\mathrm{Pos}_{\mu}(i)}) < \lambda_{d_i} - \mu_{d_i}$, $\mathrm{upset}(A_*|_{\mathrm{Pos}_{\mu}(i)})$ is right-justified, and v_{d_i} is either shaded or unshaded. No other beads v_j may be shaded (where $j \neq d_i$ for any i where $\mathrm{Pos}_{\mu}(i) \neq \mathrm{Pos}_{\lambda}(i)$).

Example 178. Let $N = 6$, $r = 3$, and $\mu = (4^2, 0^4)$. Consider $(4^2, 3, 0^3), (6, 4, 1, 0^3) \in \mathrm{H}(\mu, 3)$:

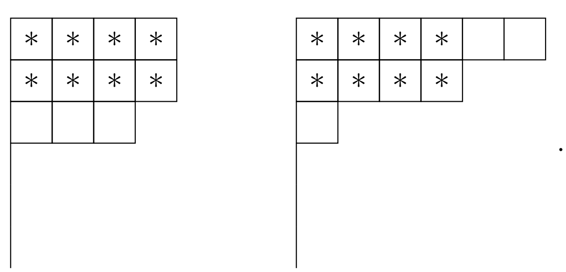
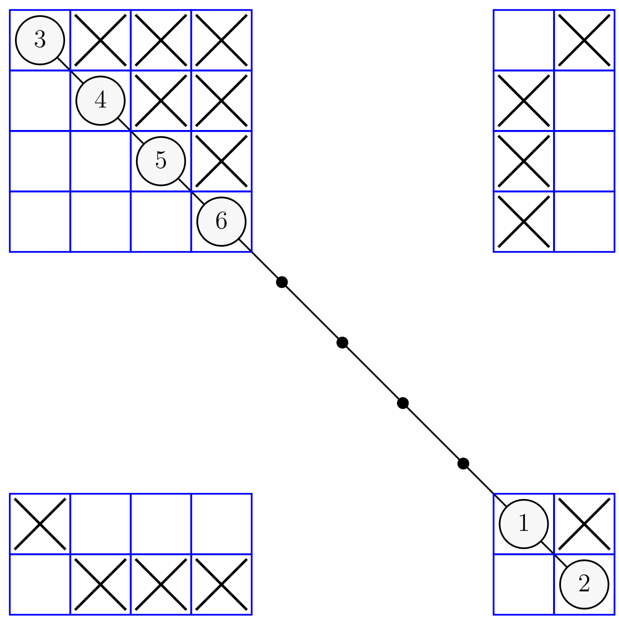


Figure 7.7 shows an abacus-tournament in $\mathbb{JAT}_{\mu}(\lambda)$ where $\lambda = (4^2, 3, 0^3)$. The bead $v_3 = 6$ is the only bead in this abacus-tournament that is eligible to be shaded. Figure 7.8 shows an abacus-tournament in $\mathbb{JAT}_{\mu}(\lambda)$ where $\lambda = (6, 4, 1, 0^3)$. Here, $v_1 = 1$ and $v_3 = 6$ are eligible to be shaded (and are shaded in this example).

Figure 7.5: An abacus-tournament $A' \in \mathbb{LAT}_{\mu}^{\mathcal{B}_\mu}$.



Example 179. Let $N = 6$, $r = 4$, and $\mu = (5^4, 1^2)$. Consider $\lambda = (7, 5^3, 3, 1) \in \mathrm{H}(\mu, 4)$:

$\lambda =$

*	*	*	*	*		
*	*	*	*	*		
*	*	*	*	*		
*	*	*	*	*		
*						
*						

.

The abacus-tournament in Figure 7.9 is in $\mathbb{JAT}_\mu(\lambda)$. Here, $v_1 = 1$ and $v_5 = 3$ are eligible to be shaded (only $v_1 = 1$ is shaded in this example).

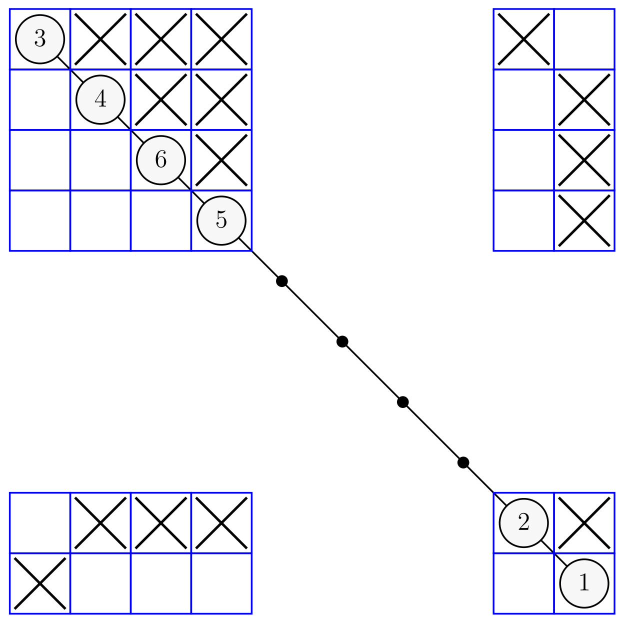
Lemma 180. *Given* $\lambda \in \mathrm{H}(\mu, r)$,

$$\left(\prod_{\substack{i: \lambda'_i = \mu'_i + 1, \\ \lambda'_{i+1} = \mu'_{i+1}}} (1 - t) \right) a_{\delta(N)} P_\lambda = \sum_{A_* \in \mathbb{JAT}_\mu(\lambda)} \mathrm{swt}(A_*).$$

Proof. By Theorem 110,

$$a_{\delta(N)} P_\lambda = \sum_{A \in \mathbb{LAT}_\lambda^{\mathcal{B}_\lambda}} \mathrm{swt}(A).$$

Figure 7.6: An abacus-tournament $A'' \in \mathbb{LAT}_{\mu}^{\mathcal{B}_\mu}$.



By (Single Bead Collision) Lemma 122, $A \in \mathbb{LAT}_{\lambda}^{\mathcal{B}_\lambda}$ has local exponent collisions in block $\mathrm{Pos}_\mu(i)$ if $\mathrm{upset}(A|_{\mathrm{Pos}_\mu(i)}) \geq \lambda_{d_i} - \mu_{d_i}$ or $\mathrm{upset}(A|_{\mathrm{Pos}_\mu(i)})$ is not right-justified. Since both of these conditions can be verified by examining only labels and edges in the square for $\mathrm{Pos}_\mu(i)$, by Lemma 97, we can cancel all abacus-tournaments such that $\mathrm{upset}(A|_{\mathrm{Pos}_\mu(i)}) \geq \lambda_{d_i} - \mu_{d_i}$ or $\mathrm{upset}(A|_{\mathrm{Pos}_\mu(i)})$ is not right-justified for any $i \geq 0$. Also,

$$|\{i : \mathrm{Pos}_\mu(i) \neq \mathrm{Pos}_\lambda(i)\}| = |\{k : \mu_k < \lambda_k\}| = |\{i : \lambda'_i = \mu'_i + 1, \lambda'_{i+1} = \mu'_{i+1}\}|,$$

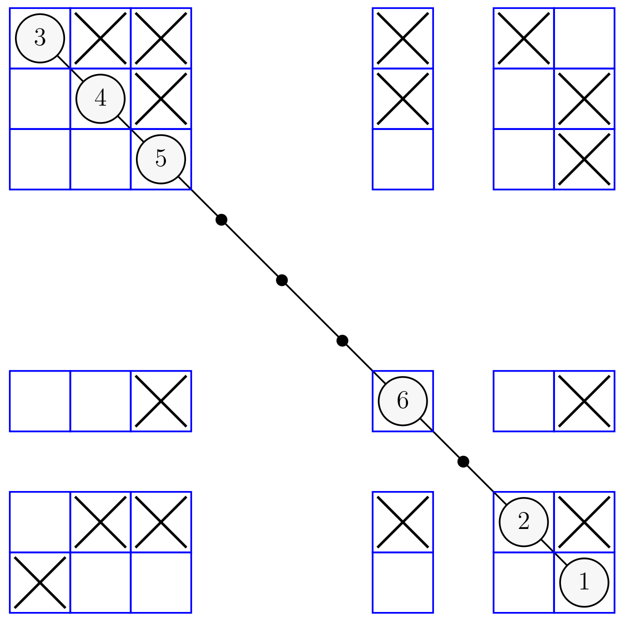
so the choice of shading for beads eligible to be shaded account for a factor of

$$\prod_{\substack{i:\lambda'_i=\mu'_i+1,\\ \lambda'_{i+1}=\mu'_{i+1}}} (1-t).$$

This completes the proof. □

See examples from the proof of Lemma 122 for example cancellations.

Figure 7.7: An abacus-tournament in $\mathbb{JAT}_\mu(\lambda)$.



7.3.3 A Bijection Between the Two Models

Theorem 181. *Let $r \in \mathbb{N}$ and let $\mu \in \mathrm{Par}_N$ be such that for all $i < N$, if $\mu_i - \mu_{i+1} > 0$, then $\mu_i - \mu_{i+1} > r$. Then*

$$a_{\delta(N)} P_\mu \cdot Q_{(r,0^{N-1})} = \sum_{\lambda \in \mathrm{H}(\mu,r)} \left(\prod_{\substack{i:\lambda'_i=\mu'_i+1,\\ \lambda'_{i+1}=\mu'_{i+1}}} (1-t) \right) \cdot a_{\delta(N)} P_\lambda.$$

Proof. By Lemma 175 and Lemma 180, we can complete this proof by providing a sign and weight-preserving bijection R between $\mathbb{LSP}_\mu(r)$ and $\bigcup_{\lambda\in\mathrm{H}(\mu,r)} \mathbb{JAT}_\mu(\lambda)$.

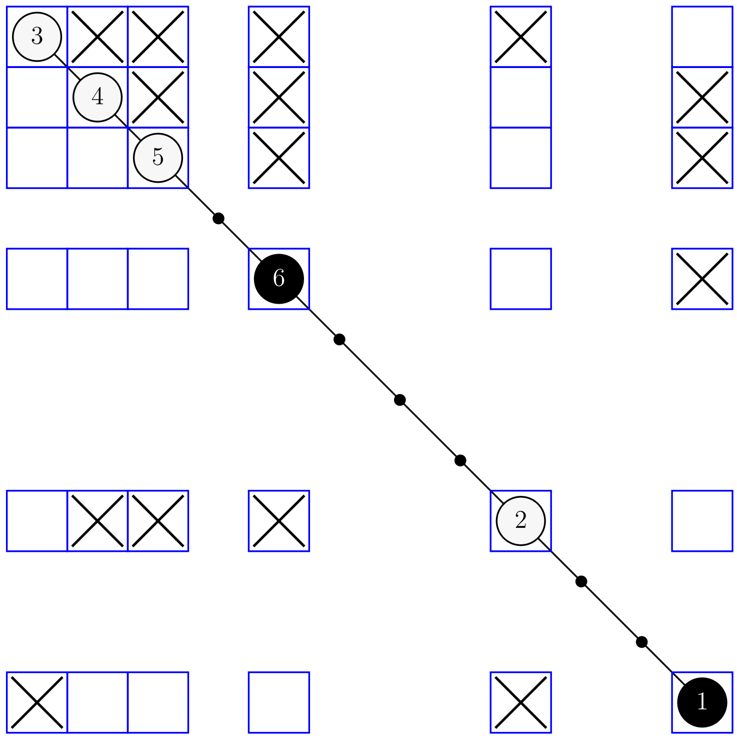
Let $(A, T^*) \in \mathbb{LSP}_\mu(r)$ be a lone star pair for μ, where $A = (\mu, v, \tau)$. Because (A, T^*) is a lone star pair for μ, for each $i \geq 0$, the ith μ-zone of T^*_v is empty or has the form

$$v_{d_i}^{b_i} v^*_{d_i+1} v^*_{d_i+2} \cdots v^*_{d_i+p_i},$$

where d_i is the lowest position in $\mathrm{Pos}_\mu(i)$, the last copy of v_{d_i} may be starred, $b_i > 0$, $p_i \geq 0$, and $r = \sum_i (b_i + p_i)$. Define λ such that for all j, if $j = d_i$ for some i, then $\lambda_j = \mu_j + (b_j + p_j)$, and otherwise $\lambda_j = \mu_j$. For a given i with $d_i > 1$, $\mu_{d_i-1} \neq \mu_{d_i}$ since d_i is the lowest position in $\mathrm{Pos}_\mu(i)$. Then, by the restrictions on μ,

$$\mu_{d_i-1} - \mu_{d_i} > r \geq b_i + p_i.$$

Figure 7.8: An abacus-tournament in $\mathbb{JAT}_\mu(\lambda)$.



Consequently, λ is a partition with N parts and since $\sum_i (b_i + p_i) = r$, $\lambda \in \mathrm{H}(\mu, r)$. We define $R(A, T^*) = A'_*$ where $A'_* = (\lambda, v, \tau')$, for all $i \geq 0$ the beads v_{d_i} of A'_* are shaded if and only if the last copy of v_{d_i} is starred in T^*, and τ' is the tournament obtained from τ by making the following alterations: for all i where d_i exists, change edges of the form (v_{d_i}, v_{d_i+j}) where $0 < j \leq p_i$ to (v_{d_i+j}, v_{d_i}).

To complete the proof, we must show that R^{-1} exits, $A'_* \in \mathbb{JAT}_\mu(\lambda)$, and (A, T^*) and A'_* have equal signed weights. Each v_{d_i} of A'_* is either shaded or unshaded. Given $i \geq 0$, there are two cases for λ. In the first case, $\mathrm{Pos}_\lambda(i) = \emptyset$ or $\mathrm{Pos}_\lambda(i) = \mathrm{Pos}_\mu(i)$, and we can conclude that A'_* is leading in $\mathrm{Pos}_\lambda(i)$. In the second case, $\mathrm{Pos}_\lambda(i) = \mathrm{Pos}_\mu(i) \setminus \{d_i\}$, and all the X's of $A'_*|_{\mathrm{Pos}_\lambda(i)}$ will be above the main diagonal because all the X's of $A|_{\mathrm{Pos}_\mu(i)}$ must be above the main diagonal. Therefore, $A'_* \in \mathbb{LAT}_\lambda^{\mathcal{B}_\lambda}$. By construction

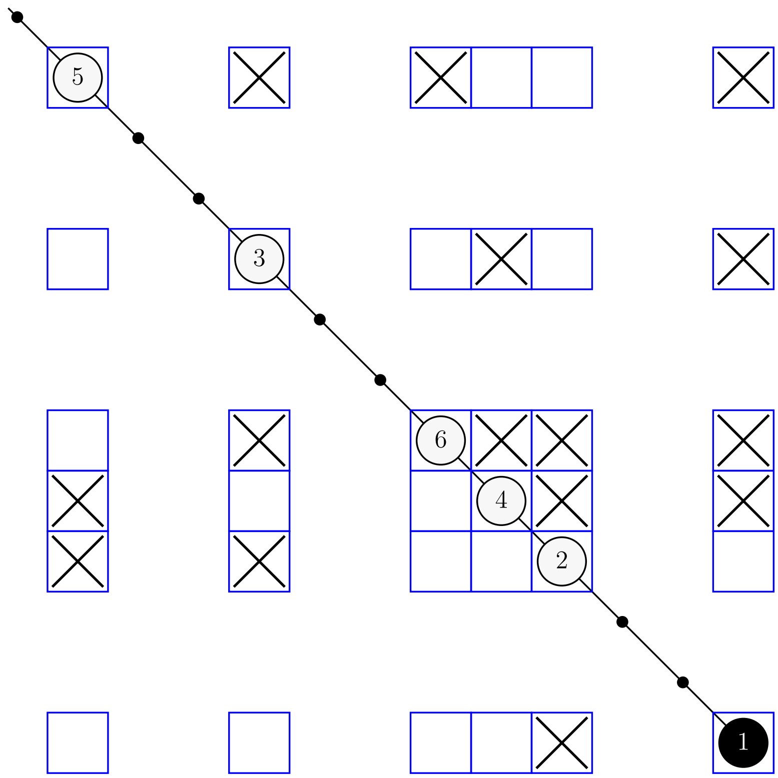
$$|\mathrm{upset}(A'_*|_{\mathrm{Pos}_\mu(i)})| = p_i < p_i + b_i = \lambda_{d_i} - \mu_{d_i},$$

and $\mathrm{upset}(A'_*|_{\mathrm{Pos}_\mu(i)})$ is right-justified. Thus $A'_* \in \mathbb{JAT}_\mu(\lambda)$.

Now consider the exponents of x_{v_k} for $1 \leq k \leq N$ in $\mathrm{swt}(A'_*)$ and $(-t)^{e(T^*)} x^{T^*} \mathrm{swt}(A)$. The exponent x_{v_k} in $(-t)^{e(T^*)} x^{T^*} \mathrm{swt}(A)$ is equal to

$$\mathrm{out}_A(v_k) + \mathrm{gap}_A(v_k) + c_{v_k},$$

Figure 7.9: An abacus-tournament $A'_* \in \mathbb{JAT}_\mu(\lambda)$.



where $c(T^*)$ is the content of T^*. The exponent of x_{v_k} in $\mathrm{swt}(A'_*)$ is equal to

$$\mathrm{out}_{A'_*}(v_k) + \mathrm{gap}_{A'_*}(v_k).$$

If $k = d_i$ for some i, then $\mathrm{out}_{A'_*}(v_{d_i}) = \mathrm{out}_A(v_{d_i}) - p_i$, $\mathrm{gap}_{A'_*}(v_{d_i}) = \mathrm{gap}_A(v_{d_i}) + (b_i + p_i)$, and $c_{d_i}(T^*) = b_i$. If $k = d_i + j$ for some $i \geq 0$ and $0 < j \leq p_i$, then $\mathrm{out}_{A'_*}(v_{d_i+j}) = \mathrm{out}_A(v_{d_i+j}) + 1$, $\mathrm{gap}_{A'_*}(v_{d_i+j}) = \mathrm{gap}_A(v_{d_i+j})$, and $c_{d_i+j}(T^*) = 1$. For all other values of k, $\mathrm{out}_{A'_*}(v_k) = \mathrm{out}_A(v_k)$, $\mathrm{gap}_{A'_*}(v_k) = \mathrm{gap}_A(v_k)$, and $c_k(T^*) = 0$. In every case, the exponents v_k in the two expressions match. Furthermore, $\mathrm{swt}(A'_*)$ and $(-t)^{e(T^*)} x^{T^*} \mathrm{swt}(A)$ have matching t-coefficients because A'_* has precisely $e(T^*) - \sum_{i\geq 0} p_i$ shaded beads and exactly $\sum_{i \geq 0} p_i$ additional X's below the main diagonal compared to A. Because A and A'_* have the same word v, the signed weights of A and A'_* have matching factors of $(-1)^{\mathrm{inv}(v)}$.

Given $A'_* = (\lambda, v, \tau') \in \mathbb{JAT}_\mu(\lambda)$, where $\lambda \in \mathrm{H}(\mu, r)$, define $J^{-1}(A'_*) = (A, T^*)$ as follows. Set $A = (\mu, v, \tau) \in \mathbb{LAT}_\mu^{\mathcal{B}_\mu}$ to be the abacus-tournament such that τ is identical to τ' except that all X's in squares for $\mathrm{Pos}_\mu(i)$ must be above the main diagonal. Define $T^* \in \mathrm{SSYT}^*_N(r)$ by making the following μ-zones: if $i \geq 0$ and $\mathrm{Pos}_\mu(i) \neq \mathrm{Pos}_\lambda(i)$, give T^* an ith μ-zone of the form

$$v_{d_i}^{b_i} v^*_{d_i+1} v^*_{d_i+2} \cdots v^*_{d_i+p_i},$$

where $p_i = |\mathrm{upset}(A'_*|_{\mathrm{Pos}_\mu(i)})|$, $b_i = \lambda_{d_i} - \mu_{d_i} - p_i$, and v_{d_i} is starred in T^* if and only if v_{d_i} is shaded in A'_*. Otherwise let T^* have empty ith μ-zone. Then $R^{-1}(A'_*) \in \mathbb{LSP}_\mu(r)$ and R^{-1} is the two-sided inverse of R, showing R to be a bijection. □

Example 182. Let $N = 6$, $r = 3$, and $\mu = (4^2, 0^4) \in \mathrm{Par}_6$. Let $A = (\mu, v, \tau)$ be the abacus-tournament in Figure 7.4 and let

$$T_v^* = \boxed{6}\,\boxed{6}\,\boxed{5^*} \; .$$

Then $(A, T^*) \in \mathbb{LSP}_\mu(3)$, and we calculate $R(A, T^*)$ to be the abacus-tournament in Figure 7.7. Now let

$$S_v^* = \boxed{1^*}\,\boxed{2^*}\|\boxed{6^*} \; .$$

Note that (A, S^*) is also a lone-star pair for μ, and $R(A, S^*)$ is the abacus-tournament in Figure 7.8.

Example 183. Let $N = 6$, $r = 4$, and $\mu = (5^4, 1^2) \in \mathrm{Par}_6$. The abacus-tournament $A'_* = (\lambda, v, \tau)$ from Figure 7.9 has partition $\lambda = (7, 5^3, 3, 1) \in \mathrm{H}(\mu, 4)$ and $A'_* \in \mathbb{JAT}_\mu(\lambda)$. To calculate $R^{-1}(A'_*) = (A, T^*)$, set A to be the abacus-tournament (μ, v, χ) where χ is obtained from τ by placing all X's in each square for $A|_{\mathrm{Pos}_\mu(i)}$ above the main diagonal. A is displayed in Figure 7.10. To obtain T^*, we look at the first and fifth μ-blocks of A'_*. In the fifth μ-block of A'_*, composed of the fifth and seventh λ-blocks of A'_*, the bead $v_1 = 1$ is separated from the other beads by $\lambda_1 - \mu_1 = 2$ bead gaps. Since $\mathrm{upset}(A'_*|_{\mathrm{Pos}_\mu(i)}) = \{(2, 1)\}$, T^* must contain one copy of 1 and one copy of 2. Consequently, the fifth μ-zone of T_v^* must be $\boxed{1^*}\,\boxed{2^*}$. Note that the copy of 1 in T^* is also starred because the bead 1 is shaded in A'_*. Similarly, the first μ-zone of T^* must be $\boxed{3}\,\boxed{3}$ because copies of $v_5 = 3$ must account for each of the $\lambda_5 - \mu_5 = 2$ gaps between beads $v_5 = 3$ and $v_6 = 5$. We conclude that

$$T_v^* = \boxed{1^*}\,\boxed{2^*}\|\boxed{3}\,\boxed{3}$$

and $R(A, T^*) = A'_*$.

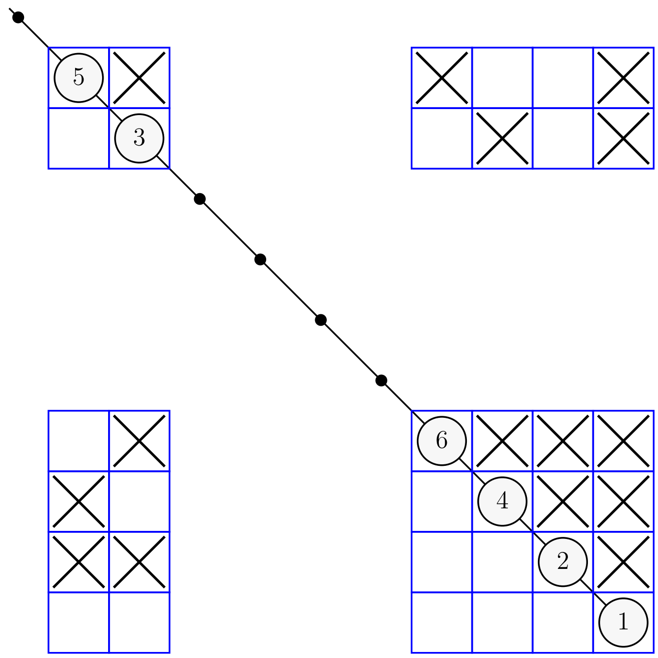
7.4 Subproblem P3

We now let $r \in \mathbb{N}$ and $\mu \in \mathrm{Par}_N$ be any partition such that for all $i < N$, if $\mu_i - \mu_{i+1} > 0$, then $\mu_i - \mu_{i+1} \geq r$. We will show

$$a_{\delta(N)} P_\mu \cdot Q_{(r,0^{N-1})} = \sum_{\lambda \in \mathrm{H}(\mu,r)} \prod_{\substack{i:\lambda'_i = \mu'_i + 1, \\ \lambda'_{i+1} = \mu'_{i+1}}} (1 - t^{m_i(\lambda)}) \cdot a_{\delta(N)} P_\lambda \tag{7.3}$$

for such a partition μ. It is now possible for the right-hand side of (7.3) to contain a factor of $(1 - t^{m_i(\lambda)})$ where $m_i(\lambda) > 1$. For this to happen, since $\mu_j - \mu_{j+1} \geq r$ for all $j < N$ with $\mu_j \neq \mu_{j+1}$, the entire horizontal r-strip must be added to a single row of μ: $\lambda_{j+1} = \mu_{j+1} + r$.

Figure 7.10: The abacus-tournament A where $R(A, T^*) = A'_*$.



This moves the largest position in $\text{Pos}_\mu(\mu_{j+1})$ to become the least position in $\text{Pos}_\lambda(\mu_j)$. A necessary and sufficient condition for a factor other than $1-t$ to appear is that there exists $i > 0$ such that $m_i(\lambda) > m_i(\mu) > 0$. Accounting for partitions λ with this condition is the key difference between Subproblem P2 and Subproblem P3.

The outline of Subproblem P3 is similar to the outline of Subproblem P2. Step 1 is to find a combinatorial model for the left-hand side of (7.3). In fact, Lemma 175 still applies in this setting because its statement and proof are independent of conditions on μ. Step 2 is to find a combinatorial model for the right-hand side of (7.3). For most $\lambda \in \text{H}(\mu, r)$, we will use exactly the same objects $\mathbb{JAT}_\mu(\lambda)$ used in Subproblem P2. However, for any $\lambda \in \text{H}(\mu, r)$ such that there exists i with $m_i(\lambda) > m_i(\mu) > 0$, we will require a slight modification of the objects (discussed in detail below) in which $\text{Pos}_\lambda(i)$ is not required to be completely leading. Step 3 is to define a sign and weight-preserving bijection between the models in Step 1 and Step 2. This bijection is exactly like the one used in Subproblem P2, except that we must check that the image now consists of the expanded set of objects from Step 2.

7.4.1 A Combinatorial Model for $a_{\delta(N)}P_\mu \cdot Q_{(r,0^{N-1})}$

Lemma 184. *Given $r \in \mathbb{N}$ and $\mu \in \mathrm{Par}_N$,*

$$a_{\delta(N)}P_\mu \cdot Q_{(r,0^{N-1})} = \sum_{(A,T^*)\in\mathbb{LSP}_\mu(r)} (-t)^{e(T^*)} x^{T^*} \operatorname{swt}(A).$$

This restates Lemma 175; since Lemma 175 and its proof do not utilize any of the restrictions on μ from Subproblem P2, its proof still applies here.

7.4.2 A Combinatorial Model for the Right Side of the Pieri Rule

Definition 185. For $\lambda \in \mathrm{Par}_N$ and $i > 0$ such that $m_i(\lambda) > 0$, define $\mathcal{B}_{\lambda,i}$ to be the set of blocks consisting of all blocks $\mathrm{Pos}_\lambda(j)$ with $i \neq j$, together with the block $\mathrm{Pos}_\lambda(i)$ with its largest position removed.

Example 186. Let $N = 7$ and $\lambda = (4^4, 1^3)$. Then

$$\mathcal{B}_\lambda = \{\{1,2,3,4\},\{5,6,7\}\},$$

$$\mathcal{B}_{\lambda,1} = \{\{1,2,3,4\},\{5,6\}\},$$

and

$$\mathcal{B}_{\lambda,4} = \{\{1,2,3\},\{5,6,7\}\}.$$

Lemma 187. *Given $\lambda \in \mathrm{Par}_N$ and i such that $m_i(\lambda) > 0$, let X be the set of shaded abacus-tournaments in $\mathbb{AT}^*_\lambda$ that are leading in the blocks $\mathcal{B}_{\lambda,i}$ and where the bead in the largest position of $\mathrm{Pos}_\lambda(i)$ is optionally shaded. Then*

$$(1 - t^{m_i(\lambda)})a_{\delta(N)}P_\lambda = \sum_{A_*\in X} \operatorname{swt}(A_*).$$

Proof. Theorem 110 proved that $a_{\delta(N)}R_\lambda$ is divisible by $\prod_{j\geq 0}[m_j(\lambda)]!_t$ by applying Theorem 109 to the λ-blocks $\mathcal{B}_\lambda$. We can similarly conclude that $a_{\delta(N)}R_\lambda$ is divisible by a product of factors $[m_j(\lambda)]!_t$ where $j \neq i$, together with the factor $[m_i(\lambda) - 1]!_t$. This results from Theorem 109 applied to the set of blocks $\mathcal{B}_{\lambda,i}$; we conclude that

$$\frac{a_{\delta(N)}R_\lambda}{[m_i(\lambda) - 1]!_t \prod_{j\neq i}[m_j(\lambda)]!_t} = \sum_{A\in\mathbb{LAT}_\lambda^{\mathcal{B}_{\lambda,i}}} \operatorname{swt}(A).$$

Consequently, because

$$1 - t^{m_i(\lambda)} = (1-t)[m_i(\lambda)]_t = \frac{(1-t)[m_i(\lambda)]!_t}{[m_i(\lambda) - 1]!_t},$$

it must be the case that

$$\begin{aligned}(1-t^{m_i(\lambda)})a_{\delta(N)}P_\lambda &= \frac{(1-t)[m_i(\lambda)]!_t}{[m_i(\lambda)-1]!_t}\left(\prod_{j\geq 0}\frac{1}{[m_j(\lambda)]!_t}a_{\delta(N)}R_\lambda\right)\\ &= (1-t)\left(\frac{1}{[m_i(\lambda)-1]!_t}\prod_{j\neq i}\frac{1}{[m_j(\lambda)]!_t}\right)a_{\delta(N)}R_\lambda\\ &= (1-t)\sum_{A\in\mathbb{LAT}_\lambda^{\mathcal{B}_{\lambda,i}}}\operatorname{swt}(A).\end{aligned}$$

Allowing the bead in the largest position of $\operatorname{Pos}_\lambda(i)$ to be shaded accounts for the factor of $1-t$. □

Example 188. Let $N=7$, $\mu=(4^3,1^4)$, $r=3$, and $\lambda=(4^4,1^3)$:

$$\lambda = \begin{array}{|c|c|c|c|}\hline * & * & * & * \\ \hline * & * & * & * \\ \hline * & * & * & * \\ \hline * & & & \\ \hline * \\ \cline{1-1} * \\ \cline{1-1} * \\ \cline{1-1}\end{array}\ .$$

The abacus-tournament in Figure 7.11 is an element of X, where $i=4$, but it is not an element of $\mathbb{LAT}_\lambda^{\mathcal{B}_\lambda}$. In this figure, the outlined squares are for the blocks $\mathcal{B}_{\lambda,4}$.

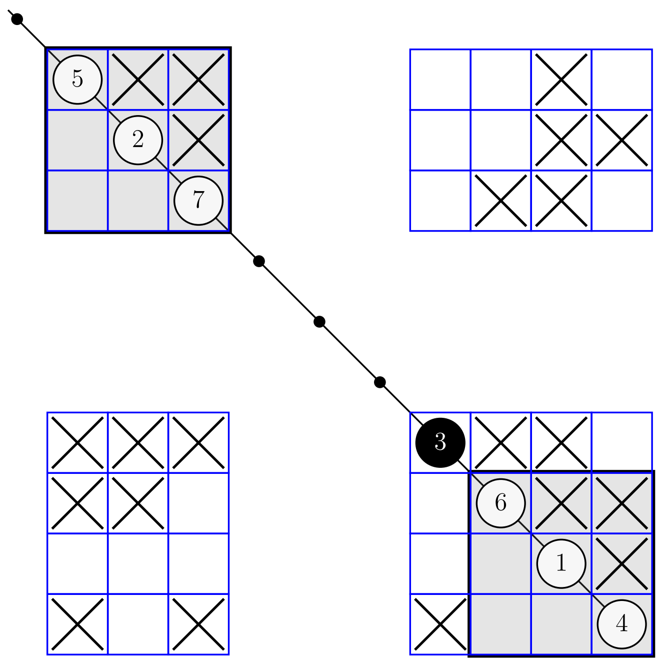
Definition 189. Let $\lambda\in\mathrm{H}(\mu,r)$, and let d_i be the lowest position in $\operatorname{Pos}_\mu(i)$ for all i such that $\operatorname{Pos}_\mu(i)\neq\emptyset$. If there exists $i>0$ such that $m_i(\lambda)>m_i(\mu)>0$, then define $\mathbb{JAT}_\mu(\lambda)$ to be the set of all shaded abacus-tournaments $A_*\in\mathbb{AT}_\lambda^*$ that are leading in blocks $\mathcal{B}_{\lambda,i}$, $|\operatorname{upset}(A_*|_{\operatorname{Pos}_\mu(i-r)})|<r$, $\operatorname{upset}(A_*|_{\operatorname{Pos}_\mu(i-r)})$ is right-justified, and where $v_{d_{i-r}}$ may be shaded. If such an i does not exist, then define $\mathbb{JAT}_\mu(\lambda)$ as in Definition 177.

Example 190. Let $N=7$, $\mu=(4^3,1^4)$, $r=3$, and $\lambda=(4^4,1^3)$. The abacus-tournament in Figure 7.12 is an element of $\mathbb{JAT}_\mu(\lambda)$. In this figure, the outlined square displays $\operatorname{Pos}_\mu(1)$, where there are two right-justified X's below the diagonal. Note that the abacus-tournament in Figure 7.11 is not an element of $\mathbb{JAT}_\mu(\lambda)$ because there are 3 X's below the main diagonal.

Lemma 191. *Given* $\lambda\in\mathrm{H}(\mu,r)$,

$$\prod_{\substack{i:\lambda'_i=\mu'_i+1,\\ \lambda'_{i+1}=\mu'_{i+1}}}(1-t^{m_i(\lambda)})a_{\delta(N)}P_\lambda = \sum_{A_*\in\mathbb{JAT}_\mu(\lambda)}\operatorname{swt}(A_*).$$

Figure 7.11: An abacus-tournament A'_* such that $A'_* \in X$ but $A'_* \notin \mathbb{LAT}_\lambda^{\mathcal{B}_\lambda}$.



Proof. This is done by cases, depending on whether or not there exists $i > 0$ such that $m_i(\lambda) > m_i(\mu) > 0$.

Case 1: Suppose there exists such an $i > 0$. By the restrictions on μ of Subproblem P3,

$$\prod_{\substack{j:\lambda'_j=\mu'_j+1,\\ \lambda'_{j+1}=\mu'_{j+1}}} (1-t^{m_j(\lambda)})a_{\delta(N)}P_\lambda = (1-t^{m_i(\lambda)})a_{\delta(N)}P_\lambda.$$

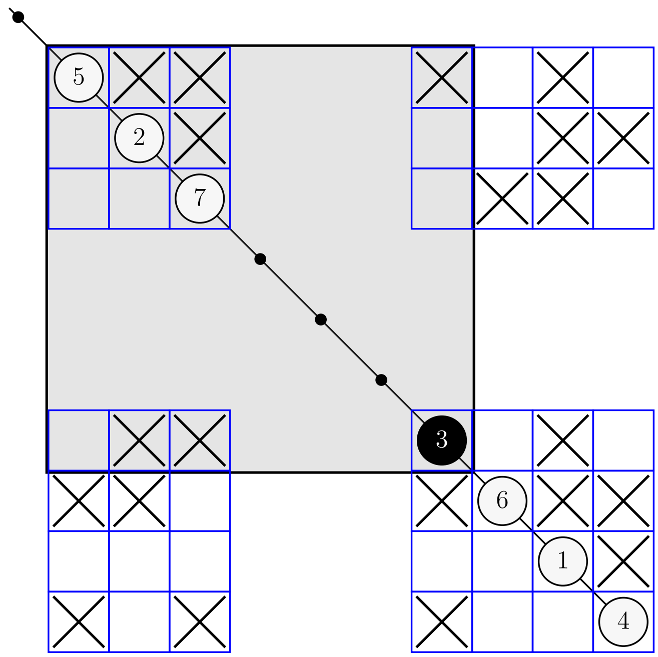
By Lemma 187,

$$\prod_{\substack{j:\lambda'_j=\mu'_j+1,\\ \lambda'_{j+1}=\mu'_{j+1}}} (1-t^{m_j(\lambda)})a_{\delta(N)}P_\lambda = \sum_{A_*\in X} \operatorname{swt}(A_*),$$

where X is the set of shaded abacus-tournaments that are leading in the blocks $\mathcal{B}_{\lambda,i}$ and where the bead in the largest position of $\mathrm{Pos}_\lambda(i)$ may be shaded.

By (Single Bead Collision) Lemma 122, $A_* \in X$ has a local exponent collision in the block $\mathrm{Pos}_\mu(i-r)$ if $|\mathrm{upset}(A|_{\mathrm{Pos}_\mu(i-r)})| \geq r$ or $\mathrm{upset}(A|_{\mathrm{Pos}_\mu(i-r)})$ is not right-justified. Both of these conditions can be verified by examining only labels and edges in the square for $\mathrm{Pos}_\mu(i-r)$. Consequently, X is closed under $I^{\mathcal{B}_\mu}$ (see Theorem 95), which fixes X's in squares for all blocks $\mathrm{Pos}_\mu(j)$. By Lemma 97, we can cancel all abacus-tournaments such that $|\mathrm{upset}(A|_{\mathrm{Pos}_\mu(i-r)})| \geq r$ or $\mathrm{upset}(A|_{\mathrm{Pos}_\mu(i-r)})$ is not right-justified, leaving fixed point set $\mathbb{JAT}_\mu(\lambda)$. Note that the

Figure 7.12: An abacus-tournament $A'_* \in \mathbb{JAT}_\mu(\lambda)$.



results used here (Th. 95, L. 97, L. 122) are not affected by the optional shading of the bead in the largest position of $\mathrm{Pos}_\lambda(i)$.

Case 2: Suppose there does not exist $i > 0$ such that $m_i(\lambda) > m_i(\mu) > 0$. Then

$$\prod_{\substack{j:\lambda'_j=\mu'_j+1,\\ \lambda'_{j+1}=\mu'_{j+1}}} (1 - t^{m_j(\lambda)}) a_{\delta(N)} P_\lambda = \left(\prod_{\substack{j:\lambda'_j=\mu'_j+1,\\ \lambda'_{j+1}=\mu'_{j+1}}} (1-t) \right) a_{\delta(N)} P_\lambda.$$

We proceed as in Subproblem P2: by Theorem 110,

$$a_{\delta(N)} P_\lambda = \sum_{A \in \mathbb{LAT}_\lambda^{\mathcal{B}_\lambda}} \mathrm{swt}(A).$$

By (Single Bead Collision) Lemma 122, $A \in \mathbb{LAT}_\lambda^{\mathcal{B}_\lambda}$ has a local exponent collision in block $\mathrm{Pos}_\mu(j)$ if $|\mathrm{upset}(A|_{\mathrm{Pos}_\mu(j)})| \geq \lambda_{d_j} - \mu_{d_j}$ or $\mathrm{upset}(A|_{\mathrm{Pos}_\mu(j)})$ is not right-justified. By Lemma 97, we can cancel all abacus-tournaments such that $|\mathrm{upset}(A|_{\mathrm{Pos}_\mu(j)})| \geq \lambda_{d_j} - \mu_{d_j}$ or $\mathrm{upset}(A|_{\mathrm{Pos}_\mu(j)})$ is not right-justified for any $j \geq 0$. Optionally shading beads in positions $\bigcup_j (\mathrm{Pos}_\mu(j) \setminus \mathrm{Pos}_\lambda(j))$ of the uncanceled abacus-tournaments gives the set $\mathbb{JAT}_\mu(\lambda)$

and accounts for the factor of

$$\prod_{\substack{j:\lambda'_j=\mu'_j+1,\\ \lambda'_{j+1}=\mu'_{j+1}}} (1-t).$$

□

7.4.3 A Bijection Between the Two Models

Theorem 192. *Let $r \in \mathbb{N}$ and let $\mu \in \mathrm{Par}_N$ be such that for all $i < N$, if $\mu_i - \mu_{i+1} > 0$, then $\mu_i - \mu_{i+1} \geq r$. Then*

$$a_{\delta(N)} P_\mu \cdot Q_{(r,0^{N-1})} = \sum_{\lambda \in \mathrm{H}(\mu,r)} \prod_{\substack{i:\lambda'_i=\mu'_i+1,\\ \lambda'_{i+1}=\mu'_{i+1}}} (1-t^{m_i(\lambda)}) \cdot a_{\delta(N)} P_\lambda.$$

Proof. By Lemma 184 and Lemma 191, we can complete the proof by providing a sign and weight-preserving bijection R between $\mathbb{LSP}_\mu(r)$ and $\bigcup_{\lambda\in \mathrm{H}(\mu,r)} \mathbb{JAT}_\mu(\lambda)$.

To do so, define R (and R^{-1}) as in the proof of Theorem 181. Fix $(A, T^*) \in \mathbb{LSP}_\mu(r)$ where $A = (\mu, v, \tau)$, let variables d_i, b_i, and p_i be as in the proof of Theorem 181, and let $R(A, T^*) = A'_* = (\lambda, v, \tau')$. Because for all i with d_i existing and $d_i > 1$,

$$\mu_{d_i-1} - \mu_{d_i} \geq r \geq b_i + p_i,$$

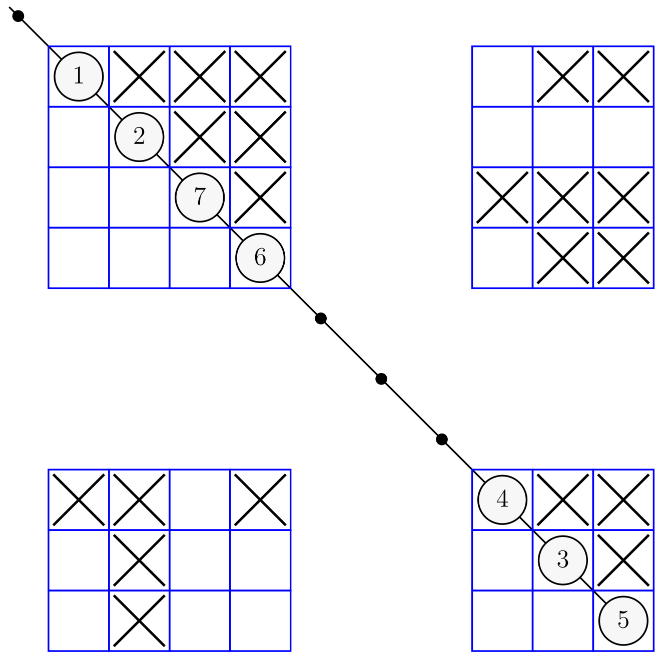
it is still the case that $\lambda \in \mathrm{H}(\mu, r)$. As before, R is sign and weight-preserving:

$$(-t)^{e(T^*)} x^{T^*} \,\mathrm{swt}(A) = \mathrm{swt}(A'_*).$$

The key new point is to confirm that the image of R is precisely $\bigcup_{\lambda\in \mathrm{H}(\mu,r)} \mathbb{JAT}_\mu(\lambda)$.

If there does not exist $i > 0$ such that $m_i(\lambda) > m_i(\mu) > 0$, then it is clear from the proof of Theorem 181 that $R(A, T^*) \in \mathbb{JAT}_\mu(\lambda)$. Suppose that there does exist i such that $m_i(\lambda) > m_i(\mu) > 0$. Then the $(i-r)$th μ-zone of T^* is the only nonempty μ-zone of T^* and $b_{i-r} + p_{i-r} = r$. R moves the bead $v_{d_{i-r}}$ from the lowest position of $\mathrm{Pos}_\mu(i-r)$ in A to the largest position of $\mathrm{Pos}_\lambda(i)$ in A'_*. Also, $v_{d_{i-r}}$ is shaded in A'_* if and only if $v_{d_{i-r}}$ is starred in T^*. In fact, $v_{d_{i-r}}$ is the only bead of A'_* that can be shaded. Since $A \in \mathbb{LAT}_\mu^{\mathcal{B}_\mu}$, A'_* is leading in all blocks $\mathrm{Pos}_\lambda(j) = \mathrm{Pos}_\mu(j)$, where $j \notin \{i-r, i\}$, and A'_* is leading in the block $B = \mathrm{Pos}_\mu(i)$ obtained from $\mathrm{Pos}_\lambda(i)$ by removing the largest position d_{i-r}. The only X's in the square for $\mathrm{Pos}_\mu(i-r)$ in the diagram of A that R alters are the lowest p_{i-r} X's in the column of $v_{d_{i-r}}$. Consequently, A'_* is also leading in $\mathrm{Pos}_\lambda(i-r)$, and so A'_* is leading in all the blocks of $\mathcal{B}_{\lambda,i}$. For the same reason, $|\mathrm{upset}(A'_*|_{\mathrm{Pos}_\mu(i-r)})| = p_{i-r} < r$ (since $b_{i-r} > 0$) and $\mathrm{upset}(A'_*|_{\mathrm{Pos}_\mu(i-r)})$ is right-justified. Therefore, $A'_* \in \mathbb{JAT}_\mu(\lambda)$. Conversely, for any $A'_* \in \mathbb{JAT}_\mu(\lambda)$, one routinely checks that $(A, T^*) = R^{-1}(A'_*)$ is in $\mathbb{LSP}_\mu(r)$ and $R(A, T^*) = A'_*$. □

Figure 7.13: An abacus-tournament $A \in \mathbb{LAT}_\mu^{\mathcal{B}_\mu}$.



Example 193. Let $N = 7$, $\mu = (4^3, 1^4)$, and $r = 3$. Let $A \in \mathbb{LAT}_\mu^{\mathcal{B}_\mu}$ be the abacus-tournament in Figure 7.13, let

$$T_v^* = \begin{array}{|c|c|c|}\hline 6 & 6^* & 7^* \\ \hline\end{array}\ ,$$

and let

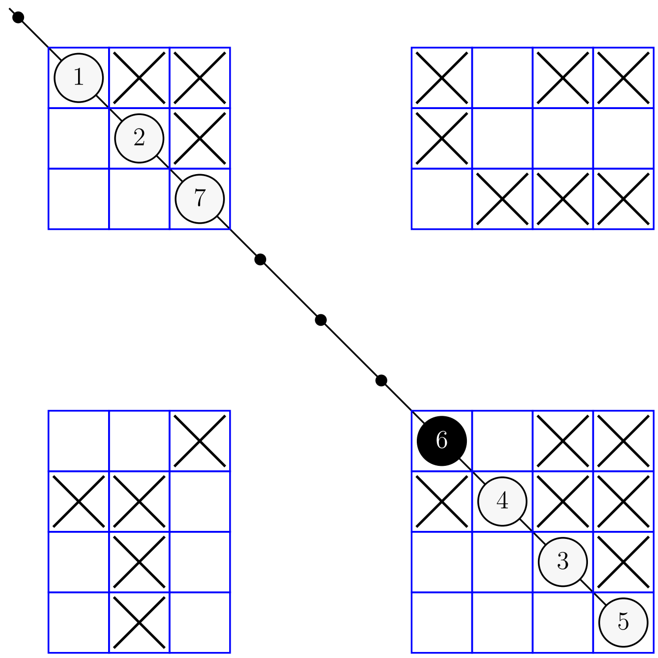
$$S_v^* = \begin{array}{|c||c|c|}\hline 5^* & 6 & 7^* \\ \hline\end{array}\ .$$

Then $(A, T^*), (A, S^*) \in \mathbb{LSP}_\mu(r)$, $R(A, T^*) \in \mathbb{JAT}_\mu(\lambda)$ is the abacus-tournament in Figure 7.14, and $R(A, S^*) \in \mathbb{JAT}_\mu(\chi)$ is the abacus-tournament in Figure 7.15, where $\lambda = (4^4, 1^3)$ (see Example 188) and $\chi = (5, 4^2, 3, 1^3)$:

$$\chi = \begin{array}{|c|c|c|c|c|}\hline * & * & * & * & \\ \hline * & * & * & * \\ \cline{1-4} * & * & * & * \\ \cline{1-4} * & & \\ \cline{1-3} * \\ \cline{1-1} * \\ \cline{1-1} * \\ \cline{1-1}\end{array}\ .$$

Example 194. Let $N = 7$, $\mu = (4^3, 1^4)$, $r = 3$, and $\lambda = (4^4, 1^3)$. Let $A'_* = (\lambda, \upsilon, \tau')$ be the abacus-tournament in Figure 7.12 from Example 190. To compute $R^{-1}(A'_*) = (A, T^*)$, note

Figure 7.14: The abacus-tournament $R(A, T^*)$.



that $|\text{upset}(A'_*|_{\text{Pos}_\mu(1)})| = 2$ and the bead labeled 3 is shaded, so set A to be the abacus-tournament in Figure 7.16 and let

$$T_v^* = \boxed{3^*}\boxed{7^*}\boxed{2^*} \ .$$

7.5 Subproblem P4

We now outline how ingredients from Subproblems P1 through P3 extend to the general case of the second Pieri rule. Here, we let $r \in \mathbb{N}$ and $\mu \in \text{Par}_N$ with no restrictions. We want to show

$$a_{\delta(N)} P_\mu \cdot Q_{(r,0^{N-1})} = \sum_{\lambda \in \text{H}(\mu,r)} \prod_{\substack{i:\lambda'_i=\mu'_i+1,\\ \lambda'_{i+1}=\mu'_{i+1}}} (1 - t^{m_i(\lambda)}) \cdot a_{\delta(N)} P_\lambda. \tag{7.4}$$

We can again reuse Lemma 175 to find a combinatorial model for the left hand side of (7.4). This requires no additional proof since Lemma 175 and its proof do not utilize any restrictions on μ from Subproblem P2.

Figure 7.15: The abacus-tournament $R(A, S^*)$.

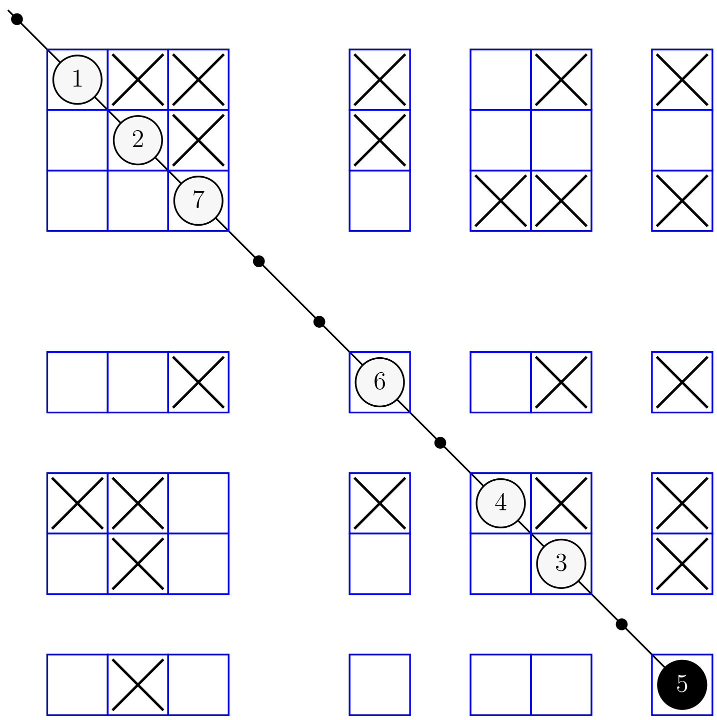
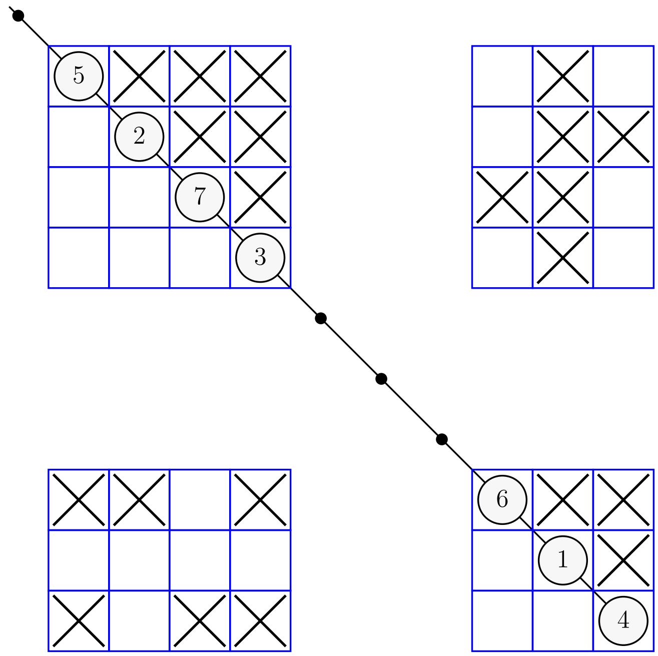


Figure 7.16: The abacus-tournament A such that $R^{-1}(A'_*) = (A, T^*)$.



Lemma 195. *Given $r \in \mathbb{N}$ and $\mu \in \mathrm{Par}_N$,*

$$a_{\delta(N)} P_\mu \cdot Q_{(r,0^{N-1})} = \sum_{(A,T^*) \in \mathbb{LSP}_\mu(r)} (-t)^{e(T^*)} x^{T^*} \operatorname{swt}(A).$$

A bead movement similar to the bijection R from the proof of Theorem 192 suggests that a second involution J must be applied to $\mathbb{LSP}_\mu(r)$ with fixed point set X characterized as follows. If $A = (\mu, v, \tau)$, then (A, T^*) is a fixed point of J if and only if for each $i \geq 0$, the ith μ-zone of T^*_v is empty or has the form given in Definition 174 and where all variables d_i, b_i and p_i are as in Definition 174, *with the added conditions that $b_i + p_i \leq \mu_{d_{i-1}} - \mu_{d_i}$*. This is a natural condition to impose and is motivated by analogous proofs of Schur polynomial Pieri rules. Without this condition for a given $i \geq 0$, the bead v_{d_i} will "collide" with one or more beads in the adjacent nonempty block containing bead $v_{d_{i-1}}$ in the computation of $R(A, T^*)$. This scenario was prevented in the previous subproblems because of restrictions on μ.

Even without knowing how to define J, we can still attempt to define $R(A, T^*) = A'_* = (\lambda, v, \tau')$ as in the proof of Theorem 192 for $(A, T^*) \in X$. Abacus-tournaments in the image set of R can be conjecturally characterized as follows. Fix $i \geq 0$ such that $\mathrm{Pos}_\mu(i) \neq \emptyset$ (compare to Example 196).

1. If $m_i(\lambda) > m_i(\mu) > 0$, then A'_* is leading in the block obtained by deleting the largest

position of $\mathrm{Pos}_\lambda(i)$. The bead in the largest position of $\mathrm{Pos}_\lambda(i)$ may be shaded.

2. If $m_i(\mu) > m_i(\lambda) > 0$, then A'_* is leading in the block $\mathrm{Pos}_\lambda(i)$, $|\mathrm{upset}(A'_*|_{\mathrm{Pos}_\mu(i)})| < \lambda_{d_i} - \mu_{d_i}$, and $\mathrm{upset}(A'_*|_{\mathrm{Pos}_\mu(i)})$ is right-justified.

3. If $m_i(\lambda) = m_i(\mu)$ and $\mathrm{Pos}_\lambda(i) \neq \mathrm{Pos}_\mu(i)$, then a combination of the previous observations is true: A'_* is leading in the block $\mathrm{Pos}_\lambda(i)$ with its largest position removed, the bead in the largest position of $\mathrm{Pos}_\lambda(i)$ may be shaded, $|\mathrm{upset}(A'_*|_{\mathrm{Pos}_\mu(i)})| < \lambda_{d_i} - \mu_{d_i}$, and $\mathrm{upset}(A'_*|_{\mathrm{Pos}_\mu(i)})$ is right-justified.

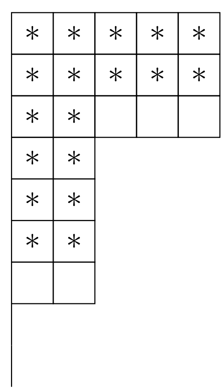
4. If $\mathrm{Pos}_\lambda(i) = \mathrm{Pos}_\mu(i)$, then A'_* is leading in $\mathrm{Pos}_\lambda(i)$.

Example 196. Let $N = 9$, $\mu = (5^2, 2^4, 0^3)$, $r = 5$. Let $A \in \mathbb{LAT}_\mu^{\mathcal{B}_\mu}$ be the abacus-tournament in Figure 7.17 and let

$$T_v^* = \begin{array}{|c|c|c|c|c|}\hline 3 & 3^* & 4^* & 7^* & 8^* \\ \hline\end{array} \ .$$

Then $R(A, T^*) = A'_* = (\lambda, v, \tau)$ is the shaded abacus-tournament in Figure 7.18, where $\lambda = (5^3, 2^4, 0^2)$:

$$\lambda = $$



- For $i = 5$, $m_5(\lambda) = 3 > 2 = m_5(\mu) > 0$. A'_* is leading in the block $\mathrm{Pos}_\lambda(5) \backslash \{3\} = \{1, 2\}$, but A'_* is not leading in $\mathrm{Pos}_\lambda(5)$. In this case, the bead v_3 is shaded.

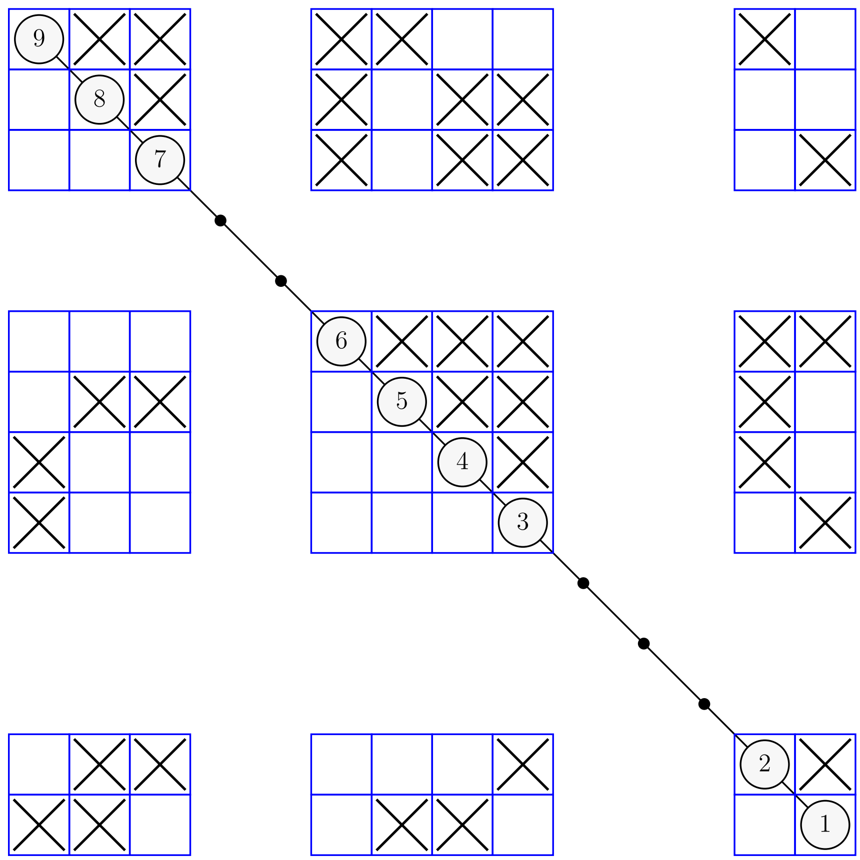
- For $i = 2$, $m_2(\lambda) = 4 = m_2(\mu)$, $\mathrm{Pos}_\lambda(2) = \{4, 5, 6, 7\}$, and $\mathrm{Pos}_\mu(2) = \{3, 4, 5, 6\}$. Note that A'_* is leading in $\{4, 5, 6\}$, the bead v_7 is shaded, and $\mathrm{upset}(A'_*|_{\mathrm{Pos}_\mu(2)}) = \{(4, 3)\}$ has one element and is right-justified.

- For $i = 0$, $m_0(\mu) = 3 > 2 = m_0(\lambda)$¿0. A'_* is leading in the block $\mathrm{Pos}_\lambda(0) = \{8, 9\}$ and $\mathrm{upset}(A'_*|_{\mathrm{Pos}_\mu(0)}) = \{(8, 9)\}$ has one element and is right-justified.

We can generalize the definition of $\mathbb{JAT}_\mu(\lambda)$ from Subproblem P3 as follows.

Definition 197. Let $\lambda \in \mathrm{H}(\mu, r)$, and let d_i be the lowest position in $\mathrm{Pos}_\mu(i)$ for all i such that $\mathrm{Pos}_\mu(i) \neq 0$. Define $\mathbb{JAT}_\mu(\lambda)$ to be the set of all shaded abacus-tournaments $A_* \in \mathbb{AT}_\lambda^*$ such that the following conditions are true:

Figure 7.17: An abacus-tournament $A \in \mathbb{LAT}_{\mu}^{\mathcal{B}_\mu}$.



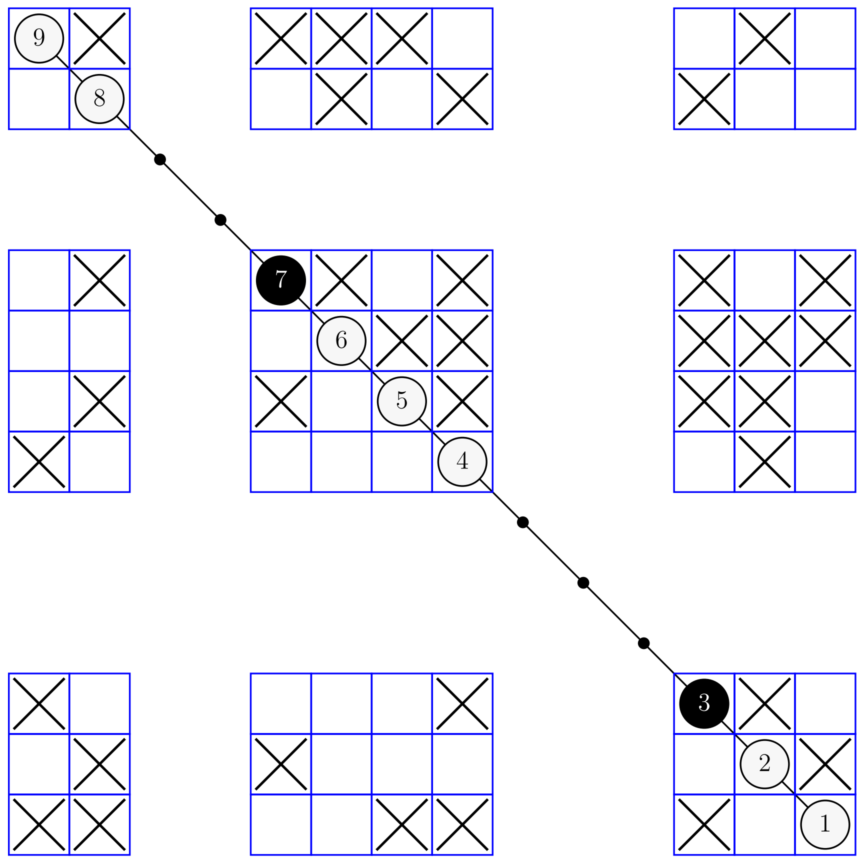
1. If $m_i(\lambda) > m_i(\mu) > 0$, then A'_* is leading in the block obtained by deleting the largest position of $\mathrm{Pos}_\lambda(i)$. The bead in the largest position of $\mathrm{Pos}_\lambda(i)$ may be shaded.
2. If $m_i(\mu) > m_i(\lambda) > 0$, then A'_* is leading in the block $\mathrm{Pos}_\lambda(i)$, $|\mathrm{upset}(A'_*|_{\mathrm{Pos}_\mu(i)})| < \lambda_{d_i} - \mu_{d_i}$, and $\mathrm{upset}(A'_*|_{\mathrm{Pos}_\mu(i)})$ is right-justified.
3. If $m_i(\lambda) = m_i(\mu)$ and $\mathrm{Pos}_\lambda(i) \neq \mathrm{Pos}_\mu(i)$, then A'_* is leading in the block $\mathrm{Pos}_\lambda(i)$, $|\mathrm{upset}(A'_*|_{\mathrm{Pos}_\mu(i)})| < \lambda_{d_i} - \mu_{d_i}$, and $\mathrm{upset}(A'_*|_{\mathrm{Pos}_\mu(i)})$ is right-justified.
4. If $\mathrm{Pos}_\lambda(i) = \mathrm{Pos}_\mu(i)$, then A'_* is leading in $\mathrm{Pos}_\lambda(i)$.

Then we make the following conjecture without proof.

Conjecture 198. *Given* $\lambda \in \mathrm{H}(\mu, r)$,

$$\prod_{\substack{i:\lambda'_i=\mu'_i+1,\\ \lambda'_{i+1}=\mu'_{i+1}}} (1 - t^{m_i(\lambda)}) a_{\delta(N)} P_\lambda = \sum_{A_* \in \mathbb{JAT}_\mu(\lambda)} \mathrm{swt}(A_*).$$

Figure 7.18: The abacus-tournament $R(A, T^*)$.



Note that abacus-tournaments $A'_* \in \text{im}(R)$ and $A''_* \in \mathbb{JAT}_\mu(\lambda)$ can differ in two fundamental ways, both involving requirement (3). First, if i is such that $m_i(\lambda) = \mu_i(\lambda)$ and $\text{Pos}_\lambda(i) \neq \text{Pos}_\mu(i)$, then the bead in the largest position of $\text{Pos}_\lambda(i)$ may be shaded in A'_*, but not in A''_*. Second, while A'_* must be leading in the block obtained by deleting the largest position of $\text{Pos}_\lambda(i)$, A''_* must be leading in the full block $\text{Pos}_\lambda(i)$.

Example 199. Let $N = 9$, $\mu = (5^2, 2^4, 0^3)$, $r = 5$, and $\lambda = (5^3, 2^4, 0^2)$. Let $A'_* = (\lambda, v, \tau) = R(A, T^*) \in \text{im}(R)$ be the shaded abacus-tournament in Figure 7.18, where $A_* \in \mathbb{LAT}_\mu^{\mathcal{B}_\mu}$ is the abacus-tournament in Figure 7.17 and

$$T^*_v = \boxed{3}\boxed{3^*}\boxed{4^*}\boxed{7^*}\boxed{8^*} \ .$$

Note that $A'_* \notin \mathbb{JAT}_\mu(\lambda)$ because it is not leading in $\text{Pos}_\lambda(2) = \{4, 5, 6, 7\}$ and because the bead v_7 is shaded.

Bibliography

[1] D. Beck, J. Remmel, and T. Whitehead, "The combinatorics of transition matrices between the bases of symmetric functions and the B_n analogues," Proceedings of FPSAC, *Discrete Math.* **153** (1996), 3–27.

[2] Joaquin Carbonara, "A combinatorial interpretation of the inverse t-Kostka matrix," *Discrete Math.* **193** (1998), 117–145.

[3] D. Dummit and R. Foote, *Abstract Algebra*, John Wiley & Sons, Inc. (2004).

[4] Ö. Eğecioğlu and J. Remmel, "A combinatorial interpretation of the inverse Kostka matrix," *Linear Multilinear Algebra* **26** (1990), 59–84.

[5] William Fulton, *Young Tableaux*, Cambridge University Press (1997).

[6] Philip Hall, "The algebra of partitions," *Proc. 4th Canadian Math. Congress*, Banff (1959), 147–159.

[7] D. E. Littlewood, "On certain symmetric functions," *Proc. London Math. Soc.* **43** (1961), 485–498.

[8] Nicholas Loehr, "Abacus proofs of Schur function identities," *SIAM J. Discrete Math.* **24** (2010), 1356–1370.

[9] Nicholas Loehr, *Bijective Combinatorics*, Chapman and Hall/CRC Press (2011).

[10] Nicholas Loehr, Luis Serrano, and Gregory Warrington, "Transition matrices for symmetric and quasisymmetric Hall-Littlewood polynomials," *J. Combin. Theory Ser. A* **120** (2013), 1996–2019.

[11] Ian Macdonald, *Symmetric Functions and Hall Polynomials* (second ed.), Oxford University Press (1995).

[12] Bruce Sagan, *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions.* Springer, New York (2001).

[13] Richard Stanley, *Enumerative Combinatorics Volume 2*, Cambridge University Press (1999).