The Structure of the 2-Sylow Subgroups of the Ideal Class Groups of Imaginary Bicyclic Biquadratic Fields

Ramona Renee Ranalli

Dissertation submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
in
Mathematics

Charles Parry, Chair
Ezra Brown
Robert McCoy
Monte Boisen
John Rossi

December, 1997
Blacksburg, Virginia

Keywords: Biquadratic Fields, Bicyclic Fields, Class Groups
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(ABSTRACT)

In this dissertation class groups of imaginary bicyclic biquadratic fields are considered. In chapter 1 we develop a method for determining the structure of the 2-class group of K. In chapters 3, 4 and 5 this method is applied determine all imaginary bicyclic biquadratic extensions of $\mathbb{Q}$ with class number 4, 8 and 16, as well as determine the specific structure of each.

Dedication

In memory of my father, who never doubted this day would come. I only wish he were here to celebrate with me.

Acknowledgments

Whenever one manages to reach a milestone in ones life there is a myriad of people whose contributions made the achievement possible. It is impossible to list them all in these few words. The greatest of my thanks goes to God who set me on this path and provided me with the necessary support and ability needed to complete it. Without Him I would not be. He blessed me with parents who have believed in me and supported me when I did not believe in myself. My thanks to both my mother and father as well as to the rest of my family for their support. In addition I would like to thank Dr. Parry. Asking him to be my advisor was one of the best choices I ever made. I thank Frank Taylor for his friendship, for influencing me towards a computational problem and, afterwards, his willingness to help me learn Mathematica. My questions must have tried his patience at times and I am grateful for his forbearance. Finally, I thank David Alger who helped me find and keep my sanity for this last year so that I might finish.

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Chapter 1

Introduction

Given a bicyclic biquadratic field K, we wish to be able to determine the structure of the class group of K whenever the structure of the class groups of the three quadratic subfields is known. It is easy to show that the odd part of the class group of K is the direct product of the odd parts of the class groups of the quadratic subfields. However, the structure of the 2-Sylow subgroup, H, of the class group of K is much more difficult to determine.

In the first chapter we will develop an algorithm for determining H when the structure of the 2-class groups of the quadratic subfields is known. This process will include computing the rank of H by computing the rank of a matrix in $\mathbb{Z}_2$ as well as the ranks of $\frac{H^{2^{i+1}}}{H^{2^{i+2}}}$ for certain values of i. In the second chapter we will look at specific examples and discuss some of the twists and turns which can occur. The third chapter develops theory to classify all K with 2-class group $\mathbb{Z}_2 \times \mathbb{Z}_2$. In the final chapter we discuss we go through the computations required to develop complete lists of all imaginary bicyclic biquadratic fields with class numbers 4, 8 and 16, respectively. These lists are given in the various tables.

Chapter 2

Notation

The following notation will be used for the remainder of this dissertation.

K : An imaginary bicyclic biquadratic extension of $\mathbb{Q}$.

k_1, k_2, k_3 : The quadratic subfields of K with k_2 real.

d_1, d_2, d_3 : Square free integers with $k_i = \mathbb{Q}(\sqrt{d_i})$ for $i = 1, 2, 3$.

f : The conductor of K.

H, H_1, H_2, H_3 : The ideal class groups of K, k_1, k_2 and k_3, respectively.

h, h_1, h_2, h_3 : The class numbers of K, k_1, k_2 and k_3, respectively.

$\widehat{H}_i$: The group of quadratic character values on the group H_i. Each element of $\widehat{H}_i$ corresponds to a genus in H_i for $i = 1, 2, 3$.

$\widetilde{A}$: The ideal class determined by the ideal A.

$\widehat{S}$: The subgroup of $\widehat{H}_1 \times \widehat{H}_2 \times \widehat{H}_3$ consisting of those character values which are consistent on each pair of H_1, H_2 and H_3.

S : The subgroup of $H_1 \times H_2 \times H_3$ with character group $\widehat{S}$.

Amb : The subgroup of $\widehat{H}_1 \times \widehat{H}_2 \times \widehat{H}_3$ consisting of the ambiguous classes.

θ : The homomorphism $H_1 \times H_2 \times H_3 \to H$ defined by $\theta(C_1, C_2, C_3) = C_1C_2C_3$.

ker : The kernel of θ.

H_0 : The image of θ.

t : The positive integer determined such that 2^t is the product of the ramification indices of all primes for the extension $K/\mathbb{Q}$. The number of rational primes which ramify in K is $t-1$ or t according as 2 is totally ramified in K or not. Hence t will be referred to as *the total number of discrimintal divisors*.

t_i : The number of rational primes ramified in k_i for $i = 1, 2, 3$.

r_a : The rank of $H_1 \times H_2 \times H_3$.

r_H : The rank of H.

l, q, r, s : Distinct prime numbers.

(l, q, r) : An element of $H_1 \times H_2 \times H_3$ determined by the ideal classes of prime divisors of l, q and r in k_1, k_2 and k_3, respectively.

ψ : The isomorphism from the multiplicative group $\{\pm 1\}$ to the additive group Z_2.

$\left(\frac{a}{b}\right)$: The Kronecker symbol using the convention $\left(\frac{b}{2}\right) = \left(\frac{2}{b}\right)$ for all odd positive integers.

u, v, w, x, y, z : $\psi(\frac{q}{s})$, $\psi(\frac{r}{s})$, $\psi(\frac{q}{r})$, $\psi(\frac{l}{q})$, $\psi(\frac{l}{r})$, $\psi(\frac{l}{s})$ respectively.

M : A Z_2-matrix determined by $\widehat{S} \cdot \widehat{\text{ker}}$.

$G^{(2)}$: The 2-Sylow subgroup of the group G.

Chapter 3

Class Group Structure of K

The structure of the odd part of the class group of K is easily shown to be the direct product of the class groups of its subfields. While the structure of the 2-Sylow subgroup of H depends on the structures of the 2-Sylow subgroups of H_1, H_2 and H_3, the relation is more complicated. In this chapter we describe a method for determining the 2-class group of H.

Lemma 1. *Let $(C_1, C_2, C_3) \in S$. Then there is a prime p of $\mathbb{Q}$ which has a prime divisor P_0 in K such that $\mathfrak{P}_i = P_0 \cap k_i = P_0 P_i$ with $\mathfrak{P}i$ and C_i in the same genus and where $(p) = P_0 P_1 P_2 P_3$ in K.*

Proof Since the characters on C_i in $\widehat{H}i$ are consistent with one another for $i = 1, 2, 3$, there is a prime p of $\mathbb{Q}$ which satisfies these character values. Now p splits completely in K and so $(p) = P_0 P_1 P_2 P_3$ in K. Since the Galois group of $K|\mathbb{Q}$ is transitive on the primes P_0, P_1, P_2, and P_3 there is a σ_i in $G(K|\mathbb{Q})$ such that $\sigma_i(P_0) = P_i$. Number the primes P_1, P_2, and P_3 so that σ_i fixes the subfield k_i of K. Let $\mathfrak{P}_i = P_0 \cap k_i$ for $i = 1, 2, 3$. Then $\mathfrak{P}_i = P_0 P_j \cap k_i$ for some j. Since the Galois group of $K|k_i$ is transitive on the factors of $\mathfrak{P}_i$ and $G(K|k_i) = \{1, \sigma_i\}$ we have $\sigma_i(P_0) = P_i$ and so, by the definition of σ_i, $\mathfrak{P}_i = P_0 \cap k_i = P_0 P_i \cap k_i$. Since $\widetilde{P_i}$ and C_i have the same character values they are in the same genus.

Theorem 1. *The homomorphism θ induces an isomorphism $\frac{S^{2^i}}{S^{2^i}\cap\mathrm{Ker}} \simeq H^{2^{i+1}}$ for any integer $i \geq 0$.*

Proof Let $(C_1^{2^i}, C_2^{2^i}, C_3^{2^i}) \in S^{2^i}$ with $(C_1, C_2, C_3) \in S$. By Lemma 1 we have a prime p in $\mathbb{Q}$ which splits completely in K and has a prime divisor P_0 such that $\mathfrak{P}_i = P_0 \cap k_i = P_0P_i \cap k_i$ where $(p) = P_0P_1P_2P_3$ in K with $\mathfrak{P}i$ and C_i in the same genus. Note that $(\widetilde{\mathfrak{P}}_1^{2^i}, \widetilde{\mathfrak{P}}_2^{2^i}, \widetilde{\mathfrak{P}}_3^{2^i}) \in S^{2^i}$ with $\widetilde{\mathfrak{P}}_i$ and C_i being in the same genus of k_i. Now

$$\theta(\widetilde{\mathfrak{P}}_1^{2^i}, \widetilde{\mathfrak{P}}_2^{2^i}, \widetilde{\mathfrak{P}}_3^{2^i}) = \widetilde{\mathfrak{P}}_1^{2^i}\widetilde{\mathfrak{P}}_2^{2^i}\widetilde{\mathfrak{P}}_3^{2^i} = (\widetilde{\mathfrak{P}}_1\widetilde{\mathfrak{P}}_2\widetilde{\mathfrak{P}}_3)^{2^i} = (\widetilde{P}_0^2\widetilde{p})^{2^i} = \widetilde{P}_0^{2^{i+1}} \in H^{2^{i+1}}.$$

Since $\widetilde{\mathfrak{P}}_iC_i^{-1}$ is in the principal genus of k_i, $\widetilde{\mathfrak{P}}_iC_i^{-1} = B_i^2$ for some class B_i of k_i. Hence

$$(\widetilde{\mathfrak{P}}_1C_1^{-1}, \widetilde{\mathfrak{P}}_2C_2^{-1}, \widetilde{\mathfrak{P}}_3C_3^{-1}) = (B_1^2, B_2^2, B_3^2),$$

so

$$B_1^2B_2^2B_3^2 = (\widetilde{\mathfrak{P}}_1\widetilde{\mathfrak{P}}_2\widetilde{\mathfrak{P}}_3)(C_1C_2C_3)^{-1} = \widetilde{P}_0^2(C_1C_2C_3)^{-1}.$$

Therefore

$$(B_1B_2B_3)^{2^{i+1}} = \widetilde{P}_0^{2^{i+1}}(C_1^{2^i}C_2^{2^i}C_3^{2^i})^{-1}$$

and $C_1^{2^i}C_2^{2^i}C_3^{2^i} \in H^{2^{i+1}}$.

Conversely, let $C^{2^{i+1}} \in H^{2^{i+1}}$ and $P_0 \in C$ be a prime ideal of degree 1 and index 1 over $\mathbb{Q}$. Let $\mathfrak{P}_i = P_0 \cap k_i$ for $i = 1, 2, 3$. Then $\mathfrak{P}_1 = P_0P_1, \mathfrak{P}_2 = P_0P_2$ and $\mathfrak{P}_3 = P_0P_3$ where $P_0 \cap \mathbb{Q} = (p) = P_0P_1P_2P_3$. Now $(\widetilde{\mathfrak{P}}_1, \widetilde{\mathfrak{P}}_2, \widetilde{\mathfrak{P}}_3) \in S$ and $\widetilde{\mathfrak{P}}_1\widetilde{\mathfrak{P}}_2\widetilde{\mathfrak{P}}_3 = \widetilde{P}_0^2 = C^2$. Thus $\widetilde{\mathfrak{P}}_1^{2^i}\widetilde{\mathfrak{P}}_2^{2^i}\widetilde{\mathfrak{P}}_3^{2^i} = \widetilde{P}_0^{2^{i+1}} = C^{2^{i+1}}$. Therefore $\frac{S^{2^i}}{S^{2^i}\cap\mathrm{Ker}} \simeq H^{2^{i+1}}$.

Theorem 2. $\frac{H^{2^{i+1}}}{H^{2^{i+2}}} \simeq \frac{S^{2^i}\cap\mathrm{Amb}}{S^{2^i}\cap\mathrm{Ker}} \cdot (S^{2^{i+1}} \cap \mathrm{Ker})$.

Proof Let $S = \langle C_1, C_2, C_3, \ldots, C_k\rangle$ and $T = S^{(2)}$. Arrange the C_i so that $o(C_1) \geq o(C_2) \geq o(C_3) \geq \ldots \geq o(C_k)$. Then, for any i where $2^i < o(C_1)$, there exists an m_i such that $o(C_{m_i}) > 2^i$ and $o(C_{m_{i+1}}) \leq 2^i$. Thus $T^{2^i} = \langle C_1^{2^i}, C_2^{2^i}, C_3^{2^i}, \ldots, C_{m_i}^{2^i}\rangle$. Then we see that $\frac{S^{2^i}}{S^{2^{i+1}}} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ where the rank of the right-hand side is m_i and

$T^{2^i} \cap Amb \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ with the rank of the right side again being m_i. Now, by Theorem 1,

$$\frac{H^{2^{i+1}}}{H^{2^{i+2}}} \simeq \frac{\frac{S^{2^i}}{S^{2^i}\cap \mathrm{Ker}}}{\frac{S^{2^{i+1}}}{S^{2^{i+1}}\cap \mathrm{Ker}}}.$$

Using a simple order argument we now see that

$$\frac{|S^{2^i}|}{|S^{2^i} \cap \mathrm{Ker}|} \cdot \frac{|S^{2^{i+1}} \cap \mathrm{Ker}|}{|S^{2^{i+1}}|}$$

$$= \frac{|\frac{S^{2^i}}{S^{2^{i+1}}}|}{|S^{2^i} \cap \mathrm{Ker}|} \cdot |S^{2^{i+1}} \cap \mathrm{Ker}|$$

$$= |\frac{S^{2^i} \cap \mathrm{Amb}}{S^{2^i} \cap \mathrm{Ker}}| \cdot |S^{2^{i+1}} \cap \mathrm{Ker}|.$$

The isomorphism follows since both groups are elementary.

Lemma 2. *The order of $\widehat{S}$ is 2^{t-2}.*

Proof Each odd prime which ramifies in K determines a character. If 2 ramifies in K then it determines either 3 or 1 characters depending on whether 2 totally ramifies or not. In the former case only two of these characters are independent. These t characters and their products are only restricted by the conditions $\prod_{\chi \in \widehat{H}i} \chi = +1$, for $i = 1, 2, 3$. However, any two product conditions determine the third. Therefore $\widehat{S}$ has 2^{t-2} elements.

Lemma 3. *The number t is determined by $t_1 + t_2 + t_3 = 2t - \tau$ where $\tau = 1$ if 2 is totally ramified and $\tau = 0$ otherwise. Moreover,*

$$r_a = \begin{cases} t_1 + t_2 + t_3 - 4 & \text{if a prime } q \equiv 3 \pmod 4 \text{ divides } d_2. \\ t_1 + t_2 + t_3 - 3 & \text{otherwise.} \end{cases} \tag{3.1}$$

Consequently,

$$r_a = \begin{cases} 2t-5 & \text{if } 2 \text{ is totally ramified and } q|d_2 \text{ for some prime } q \equiv 3 \pmod 4. \\ 2t-3 & \text{if } 2 \text{ is not totally ramified and no prime } q \equiv 3 \pmod 4 \text{ divides } d_2. \\ 2t-4 & \text{otherwise.} \end{cases} \tag{3.2}$$

Proof Each odd prime which ramifies in K, ramifies in exactly two of the subfields k_i for $i = 1, 2, 3$. The same is true for 2 unless it is totally ramified in K. The 2-rank of H_i is $t_i - 1$ unless $i = 2$ and some prime $q \equiv 3$ (mod4) divides d_2. In that case the 2-rank of H_2 is $t_2 - 2$. The final expressions for r_a are immediate.

Lemma 4. *The order of S is given by $|S| = \frac{h_1 h_2 h_3}{2^{r_a}} |\widehat{S}| = \frac{h_1 h_2 h_3}{2^{s_o}}$ where*

$$s_o = \begin{cases} t-3 & \text{if } r_a = 2t-5. \\ t-2 & \text{if } r_a = 2t-4. \\ t-1 & \text{if } r_a = 2t-3. \end{cases} \tag{3.3}$$

Proof The order of $\widehat{H}_1 \times \widehat{H}_2 \times \widehat{H}_3$ is 2^{r_a}. Now the same number of classes of $H_1 \times H_2 \times H_3$ belong to each character value of $\widehat{H}_1 \times \widehat{H}_2 \times \widehat{H}_3$. Since there are 2^{t-2} character values in $\widehat{S}$, the first result follows. Since $s_o = r_a - t + 2$, the final result is immediate from the previous lemma.

In the remainder of this chapter we will show that the rank of H is given by the rank of a $\mathbb{Z}_2$-matrix.

Theorem 3. *The 2-rank of H is given by*

$$r_H = t - 2 + \log_2[H_1 \times H_2 \times H_3 : S \cdot \text{Ker}]$$

Proof Now

$$\begin{aligned} r_H &= \log_2[H : H^2] \\ &= \log_2[H : H_0] + \log_2[H_0 : H^2] \\ &= \log_2[H_0 : H^2] + t - 2 \end{aligned}$$

where the last equality follows from Kubota [9]. But $H_0/H^2 \simeq \frac{H_1\times H_2\times H_3/\mathrm{Ker}}{S/S\cap \mathrm{Ker}}$ and $S/S\cap\mathrm{Ker} \simeq S\cdot\mathrm{Ker}/\mathrm{Ker}$ so $[H_0 : H^2] = [H_1\times H_2\times H_3 : S\cdot\mathrm{Ker}]$. The result follows.

Corollary 1. *If the unit index is* 2 *then* $s_o \le r_H \le r_a$*. In any case* $t-2 \le r_H \le r_a$*.*

Proof From Kubota [9], $|\mathrm{Ker}| = 2^{t-2}$ or 2^{t-1} according as the unit index of K is 2 or 1. Let $2^e = |\mathrm{Ker}|$. Then

$$\begin{aligned}[H_1\times H_2\times H_3 : S\cdot\mathrm{Ker}] &= \frac{|H_1\times H_2\times H_3|}{|S||\mathrm{Ker}|}|S\cap\mathrm{Ker}| \\ &= 2^{s_o-e}|S\cap\mathrm{Ker}|.\end{aligned}$$

The results now follow from Lemma 4 and Theorem 3.

Corollary 2. *In Theorem 3, the term* $[H_1\times H_2\times H_3 : S\cdot\mathrm{Ker}]$ *can be replaced with* $[(H_1\times H_2\times H_3)^{(2)} : S^{(2)}\cdot\mathrm{Ker}]$.

Proof Note that all elements of odd order in $H_1\times H_2\times H_3$ are in S since they belong to the principal genus of each k_i. Thus $[(H_1\times H_2\times H_3)^{(2)}\cdot S\cdot\mathrm{Ker}] = H_1\times H_2\times H_3$. Moreover, since Ker is a 2-group, $(H_1\times H_2\times H_3)^{(2)}\cap S\cdot\mathrm{Ker} = S^{(2)}\cdot\mathrm{Ker}$. Hence $H_1\times H_2\times H_3/S\cdot\mathrm{Ker} \simeq (H_1\times H_2\times H_3)^{(2)}/S^{(2)}\cdot\mathrm{Ker}$.

Theorem 4. *Let* m *denote the rank of* $\widehat{S}\cdot\widehat{\mathrm{Ker}}$*. Then*

$$r_H = r_a + t - 2 - m = \begin{cases} 3t-7-m & \textit{if } r_a = 2t-5. \\ 3t-6-m & \textit{if } r_a = 2t-4. \\ 3t-5-m & \textit{if } r_a = 2t-3. \end{cases} \tag{3.4}$$

Proof Let $\phi : (H_1\times H_2\times H_3)^{(2)} \to \widehat{H}_1\times\widehat{H}_2\times\widehat{H}_3$ be the mapping determined by taking a class C_i of H_i to its character system in $\widehat{H}i$. Then $\frac{(H_1\times H_2\times H_3)^{(2)}}{(\mathrm{Ker}\phi(S\cdot\mathrm{Ker}))^{(2)}} \simeq \frac{\widehat{H}_1\times\widehat{H}_2\times\widehat{H}_3}{\phi(S^{(2)}\cdot\mathrm{Ker})}$. But $\phi(S^{(2)}\cdot\mathrm{Ker}) = \phi(S^{(2)})\cdot\phi(\mathrm{Ker}) = \widehat{S}\cdot\widehat{\mathrm{Ker}}$. Moreover, Ker$\phi$ is the direct product of the 2-Sylow subgroups of the principal genera of k_1, k_2 and k_3 which is clearly contained in $S^{(2)}$. Thus $\frac{(H_1\times H_2\times H_3)^{(2)}}{S^{(2)}\cdot\mathrm{Ker}} \simeq \frac{\widehat{H}_1\times\widehat{H}_2\times\widehat{H}_3}{\widehat{S}\cdot\widehat{\mathrm{Ker}}}$. The result now follows from Theorem 3 and Corollary 2.

In order to determine r_H all that remains to be done is to compute $m = \mathrm{rank}\ (\widehat{S}\cdot\widehat{\mathrm{Ker}})$. This will be discussed and illustrated in the next chapter.

Chapter 4

Examples

In this section we show how the formula of the last Theorem can be used to commute r_H. Recall that rank $\widehat{S} = t - 2$ and rank $\widehat{\mathrm{Ker}} = t - 1$ or $t - 2$ according as the unit index of K is 1 or 2. Let $n = \text{rank } \widehat{S} + \text{rank } \widehat{\mathrm{Ker}}$ and M be a $n \times r_a$ $\mathbb{Z}_2$-matrix whose rows correspond to generators of $\widehat{S} \cdot \widehat{\mathrm{Ker}}$ by means of the isomorphism ψ. Then m is the rank of M.

Note that the characters of the real field will often have to be normalized in order to ensure that -1 lies in the principal genus. This is done by multiplying one column where -1 has a $-$ character by each of the other columns where -1 also has a $-$ character. The initial choice of column does not matter. The following example illustrates this as well as the technique for finding H.

Example Let $k_1 = \mathbb{Q}(\sqrt{-lqrs})$, $k_2 = \mathbb{Q}(\sqrt{lq})$ and $k_3 = \mathbb{Q}(\sqrt{-rs})$ with $l \equiv q \equiv r \equiv 3 \pmod 4$ and $s \equiv 1 \pmod 4$. Here the unit index is 1, $t = 4$ and $r_a = 5$. Normalizing the character of the real field, the table of consistent character systems is:

l	q	r	s	lq	r	s
$+$	$+$	$+$	$+$	$+$	$+$	$+$
$-$	$-$	$+$	$+$	$+$	$+$	$+$
$+$	$+$	$-$	$-$	$+$	$-$	$-$
$-$	$-$	$-$	$-$	$+$	$-$	$-$

.

Here $\widehat{S}$ is generated by $(1,1,0,0,0,0,0)$ and by $(0,0,1,1,0,1,1)$ representing a + by a 0 and a $-$ by a 1. Moreover, Ker is generated by $\{(l,1,1),\ (q,1,1),\ (r,1,r)\}$. Since in k_2 lq is always + it will just generate a column of zeroes in our matrix and so can be deleted. Thus our columns will correspond to l, q, and r in k_1 and r in k_3. So our matrix is

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \psi(\left(\frac{l}{q}\right)\left(\frac{l}{r}\right)\left(\frac{l}{s}\right)) & \psi\left(\frac{l}{q}\right) & \psi\left(\frac{l}{r}\right) & 0 \\ 1+\psi\left(\frac{l}{q}\right) & \psi(\left(\frac{q}{l}\right)\left(\frac{q}{s}\right)\left(\frac{q}{r}\right)) & \psi\left(\frac{q}{r}\right) & 0 \\ 1+\psi\left(\frac{l}{r}\right) & 1+\psi\left(\frac{q}{r}\right) & \psi(\left(\frac{l}{r}\right)\left(\frac{r}{s}\right)\left(\frac{q}{r}\right) & \psi\left(\frac{r}{s}\right) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ x+y+z & x & y & 0 \\ 1+x & 1+x+w+u & w & 0 \\ 1+y & 1+w & y+v+w & v \end{pmatrix}$$

where the first two rows corresponds to the generators of $\widehat{S}$ and the last three rows correspond to generators of Ker. We have deleted one character from each subfield since the product of the characters for a quadratic field is +1. The first three columns correspond to characters for k_1, determined by l, q and r, and the last column to a character for k_3, determined by r. Performing elementary row operations we find:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & z+y & y & 0 \\ 0 & w+u & w & 0 \\ 0 & w+y & w+y+v & v \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & y & y+z \\ 0 & 0 & w & w+u \\ 0 & v & y+v+w & y+w \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & y & y+z \\ 0 & 0 & w & w+u \\ 0 & 0 & w+y & w+y \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & y & z \\ 0 & 0 & w & u \\ 0 & 0 & y+w & 0 \end{pmatrix}$$

So we see that

$$m = \begin{cases} 2 & \text{if } y = w = u = z = 0. \\ 3 & \text{if } y = z = w = u = 1. \\ & \text{if exactly one of } u,\ w,\ y,\ z \text{ is non-zero.} \\ 4 & \text{if } y = z \text{ and } w = u. \\ & \text{if } y = u \text{ and } w = z. \\ & \text{if exactly one of } u,\ w,\ y,\ z \text{ is zero.} \end{cases} \tag{4.1}$$

Now $r_a = 4$ and $t = 4$ so $r_H = 6 - m$.

Now we are ready to compute the structure of H for any K. The following proposition will be helpful in that it states a basic fact which will be used many times in our computations.

Proposition 1. *Let $b = 2^j$ for some $j > 1$ where $b|h_i$ for some $i = 1, 2, 3$. Also, let $T = S^{(2)}$. If $(C_1^{b/2}, C_2^{b/2}, C_3^{b/2})$ is in $T^{b/2} \cap \mathrm{Amb}$ then $(C_1^{b/2}, 1, 1)$, $(1, C_2^{b/2}, 1)$ and $(1, 1, C_3^{b/2})$ are in $T^{b/4} \cap \mathrm{Amb}$.*

Proof Let $(C_1, C_2, C_3)^{b/2} = (C_1^{b/2}, C_2^{b/2}, C_3^{b/2})$ be in $T^{b/2} \cap \mathrm{Amb}$. Then $C_i^{b/2}$ is ambiguous for each i. Since C_1^2 is in the principal genus of k_1, $(C_1^2, 1, 1)$ is in T, so $(C_1^{b/2}, 1, 1) = (C_1^2, 1, 1)^{b/4}$ is in $T^{b/4} \cap \mathrm{Amb}$. A similar argument is true for $(1, C_2^{b/2}, 1)$ and $(1, 1, C_3^{b/2})$.

Now we will compute H for a variety of cases in order to get a feel for the algorithm. In order to find h we use the following formula due to Kubota [9]:

$$h = \frac{1}{2}[\text{unit index}]h_1 h_2 h_3.$$

Example For our first example let $d_1 = -402$, $d_2 = 3953$ and $d_3 = -354$. Here ${H_1}^{(2)} \simeq {H_3}^{(2)} \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$ and H_2 has odd class number. First note that the Ker is $\{(1, 1, 1)$, $(1, 1, 6)$,

$(2,1,2)$, $(2,1,3)$, $(3,1,2)$, $(3,1,3)$, $(6,1,1)$, $(6,1,6)\}$. The table of consistent characters is

−2	3	67	59	67	−2	3	59
+	+	+	+	+	+	+	+
+	−	−	−	−	+	−	−
−	−	+	+	+	−	−	+
−	+	−	−	−	−	+	−

Normalizing the characters in the real field we get

−2	3	67	$59 \cdot 67$	−2	3	59
+	+	+	+	+	+	+
+	−	−	+	+	−	−
−	−	+	+	−	−	+
−	+	−	+	−	+	−

Now our goal here is to use Theorem 2 to find H. To do this we need to determine the structure of $S^{2^i} \cap \text{Amb}$ and $S^{2^i} \cap \text{Ker}$ for various i. As a first step we will determine which of these lines of characters belong to ambiguous classes. This is equivalent to determining which genera can be generated by the Jacobi symbol of a product of ramified primes of a field over the respective prime divisors. In k_1 we find that 2 and 3 have the character lines $+--$ and that 6 is in the principal genus . In k_3 it is 2 which corresponds to $+--$ and 3 which is in the principal genus. Thus, in both k_1 and k_3, the character line $+--$ represents a genus with elements of order two and the other non-principal genera contain only classes with elements of order eight. In these latter genera, the fourth power of any element is in the principal genus and is ambiguous and so can be represented by 6 in k_1 and by 3 in k_3 respectively. The following table reflects this information.

$-2 \cdot 3 \cdot 67$	$59 \cdot 67$	$-2 \cdot 3 \cdot 59$
$(1,1)$	$(1,1)$	$(1,1)$
$(1,1)$	$(1,1)$	$(1,1)$
$(4,6)$	$(1,1)$	$(4,3)$
$(4,6)$	$(1,1)$	$(4,3)$

In this table each (a,b) means that in that genus there is an element of order $2a$ whose ath power is in the class represented by b. Thus, in this example, we see that

$$S^4 \cap \text{Amb} = \langle (6,1,3) \rangle \text{ and that } S^4 \cap \text{Ker} = \langle (1,1,1) \rangle.$$

Thus

$$\frac{S^4 \cap \text{Amb}}{S^4 \cap \text{Ker}} \simeq \mathbb{Z}_2$$

and

$$\frac{H^8}{H^{16}} \simeq \mathbb{Z}_2.$$

Hence there is a factor of Z_{16} in H. Now, by the Proposition,

$$S^2 \cap \text{Amb} = \langle (6,1,3), (6,1,1), (1,1,3) \rangle$$

and

$$S^2 \cap \text{Ker} = \langle (6,1,1) \rangle.$$

And so,

$$\frac{S^2 \cap \text{Amb}}{S^2 \cap \text{Ker}} \simeq \mathbb{Z}_2$$

making

$$\frac{H^4}{H^8} \simeq \mathbb{Z}_2.$$

This is satisfied by the factor of $\mathbb{Z}_{16}$ in $H^{(2)}$ that we already have.

Using the methods already described, we can determine that the rank of H is three and the order is 2^7. Thus we have two more factors of combined order 2^3. Clearly then,

$$H^{(2)} \simeq \mathbb{Z}_{16} \times \mathbb{Z}_4 \times \mathbb{Z}_2.$$

Example For our next example let $d_1 = -406$, $d_2 = 182$ and $d_3 = -377$. Here ${H_1}^{(2)} \simeq {H_3}^{(2)} \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$ and ${H_2}^{(2)} \simeq \mathbb{Z}_2$. First note that the Ker is $\{(1,1,1)$, $(1,1,13)$, $(1,2,2)$, $(1,2,26)$, $(2,1,2)$, $(2,1,26)$, $(2,2,1)$, $(2,2,13)$, $(7,1,2)$, $(7,1,26)$, $(7,2,1)$, $(7,2,13)$, $(14,1,1)$, $(14,1,13)$, $(14,2,2)$, $(14,2,26)\}$. The table of consistent characters is

2	7	29	$-2 \cdot 7$	13	-1	13	29
+	+	+	+	+	+	+	+
−	−	+	+	+	+	+	+
+	−	−	+	+	−	+	−
−	+	−	+	+	−	+	−
+	−	−	−	−	+	−	−
−	+	−	−	−	+	−	−
+	+	+	−	−	−	−	+
−	−	+	−	−	−	−	+

Note that we have normalized the characters of the real field and that in k_3 -1 denotes the character determined by the ramified prime 2. In k_1 we find that 2 has the character line $- + -$ and that 7 is in the principal genus . Thus the character line $- + -$ represents a genus with elements of order two and the other non-principal genera lines contain only classes with elements of order eight. In these latter genera, the fourth power of any element in the principal genus and is ambiguous and so can be represented by 7. In k_3 2 has the character line $+ - -$ and 13 is in the principal genus. Thus $+ - -$ represents a genus with elements of order two and the other non-principal genera lines contain only classes with elements of order eight. In these latter genera, the fourth power of any element in the principal genus and is ambiguous and so can be represented by 13. The following table reflects this information.

$2\cdot 7\cdot 29$	$-2\cdot 7\cdot 13$	$-1\cdot 13\cdot 29$
$(1,1)$	$(1,1)$	$(1,1)$
$(4,7)$	$(1,1)$	$(1,1)$
$(4,7)$	$(1,1)$	$(4,13)$
$(1,1)$	$(1,1)$	$(4,13)$
$(4,7)$	$(1,1)$	$(1,1)$
$(1,1)$	$(1,1)$	$(1,1)$
$(1,1)$	$(1,1)$	$(4,13)$
$(4,7)$	$(1,1)$	$(4,13)$

Again, in this table each (a,b) means that in that class there is an element of order $2a$ whose ath power is in the class represented by b. Thus, in this example, we see that

$$S^4 \cap \text{Amb} = \langle (7,1,1),(1,1,13)\rangle \text{ and that } S^4 \cap \text{Ker} = \langle (1,1,13)\rangle.$$

Thus

$$\frac{S^4 \cap \text{Amb}}{S^4 \cap \text{Ker}} \simeq \mathbb{Z}_2$$

and

$$\frac{H^8}{H^{16}} \simeq \mathbb{Z}_2.$$

Hence there is a factor of Z_{16} in H. Now we see that for the second power everything remains the same. So

$$S^2 \cap \text{Amb} = \langle (7,1,1),(1,1,13)\rangle$$

and

$$S^2 \cap \text{Ker} = \langle (1,1,13)\rangle.$$

And so, since

$$\frac{S^2 \cap \text{Amb}}{S^2 \cap \text{Ker}} \simeq \mathbb{Z}_2$$

, Theorem 2 shows that

$$\frac{H^4}{H^8} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2.$$

The one $\mathbb{Z}_2$ is satisfied by the factor of $\mathbb{Z}_{16}$ that we already have, so the other one must indicate a factor of $\mathbb{Z}_8$ in $H^{(2)}$.

Using the methods already described, we can determine that the rank of $H^{(2)}$ is three and the order is 2^8. Thus we have one more factor of order 2. Clearly then,

$$H^{(2)} \simeq \mathbb{Z}_{16} \times \mathbb{Z}_8 \times \mathbb{Z}_2.$$

Now, things worked out in a very straight-forward manner in these two examples because each H_i had at most one non-elementary factor. This meant that, once we identified the genera that contained the ambiguous classes, we knew the order of the classes of the other genera. If there are two or more non-elementary factors of unequal order in some H_i then this will not be the case. To determine $S^n \cap \text{Amb}$ for any $n = 2^j$ where j is a whole number, we must know not only the minimum order of a class in each genus but also the ambiguous class in the principal genus which it generates.

Let q be an unramified prime whose character values place its prime divisors in a genus that does not contain ambiguous classes. Say $\mathfrak{q}$, a prime divisor of q in k_i, belongs to a class of order m. Then $\mathfrak{q}^{m/2}$ belongs to an ambiguous class in the principal genus. Assume that $\mathfrak{p}$ is an ideal in this class (this is always the case when k_i is imaginary). Then $\mathfrak{q}^{m/2}\mathfrak{p} \sim (1)$ is a principal ideal of k_i. If $\mathfrak{q}^{m/2}\mathfrak{p} = (x + \sqrt{d_i}y)$ where x and y are both integers or half of integers then $x^2 - d_iy^2 = \mathfrak{q}^{m/2}p$. Conversely, assume $x^2 - d_iy^2 = 2^l q^{m/2}p$, where $l = 0$, unless $d_i \equiv 1 \pmod 4$ and then $l = 0$ or 2 and $\gcd(x, \mathfrak{q}p) = 1$. If p ramifies in the field k_i and if the prime divisor $\mathfrak{p}$ in k_i is not principal then the prime divisors of $\mathfrak{q}$ in k_i belong to classes of order m.

So the key to solving such problems will be to find an x and y in $\mathbb{Z}$ and a prime $\mathfrak{q}$ in a non-principal genus such that $x^2 - d_iy^2 = p\mathfrak{q}^{m/2}$ for each ramified (p) in the principal genus. The genus which contains the divisors of $\mathfrak{q}$ contains the classes of order m.

A slight complication occurs if the H_i contains one or more odd factors. In this case the right-hand side of the equation may be $\mathfrak{q}^{gm}$ where g is a divisor of the product of the orders of the odd factors.

Example For our next problem let $d_1 = -18761$, $d_2 = 2482$ and $d_3 = -8738$. Here ${H_1}^{(2)} \simeq \mathbb{Z}_{32} \times \mathbb{Z}_4$, ${H_2}^{(2)} \simeq \mathbb{Z}_4 \times \mathbb{Z}_2$ and ${H_3}^{(2)} \simeq \mathbb{Z}_{16} \times \mathbb{Z}_4$. First note that the Ker is $((1,1,1),\ (1,2,2),\ (1,17,17),\ (1,34,34),\ (2,1,2),\ (2,2,1),\ (2,17,34),\ (2,34,17),\ (73,1,34),$

(73, 2, 17), (73, 17, 2), (73, 34, 1), (146, 1, 17), (146, 2, 34), (146, 17, 1), (146, 34, 2)). The table of consistent characters is

−1	73	257	2	17	73	−2	17	257
+	+	+	+	+	+	+	+	+
−	+	−	+	+	+	−	+	−
+	−	−	+	−	−	+	−	−
−	−	+	+	−	−	−	−	+
+	−	−	−	+	−	−	+	−
−	−	+	−	+	−	+	+	+
+	+	+	−	−	+	−	−	+
−	+	−	−	−	+	+	−	−

Note that in this case the real field did not have to be normalized. Finding the ambiguous classes in the principal genus, we see that in k_1 the divisors of 2 and of 73 are both in the principal genus, in k_2 it is the divisors of 2 and, in k_3, the divisors of 2 and of 17.

It is necessary in this case, for both of the imaginary fields, to find the quadratic representations of a prime in the non-principal genera. It will be sufficient to find two such primes (perhaps even in the same genus) where some power of each is in the same ambiguous class as the divisors of 2, 73 or 146 in k_1 and as 2, 17 or 34 in k_3. For k_1 we get:

$$(73 \cdot 4)^2 + 18761 \cdot 3^2 = 73 \cdot 59^2$$

$$141^2 + 18761 = 2 \cdot 139^2.$$

Since we find that both 59 and 139 have characters - + -, we know this genus contains classes of order four which square to an ambiguous class containing divisors of 2. The other non-ambiguous genera must have classes of order thirty-two whose sixteenth power is in an ambiguous class containing divisors of 146 (since 73 was also seen to be a square).

For k_3 we get:

$$108^2 + 8738 = 2 \cdot 101^2$$

$$51^2 + 8738 \cdot 2^2 = 17 \cdot 47^2.$$

Here 101 and 47 have characters - + - and so we know this genus contains classes of order four which square to an ambiguous class containing divisors of 2 (and 17). So the other non-principal genera must have classes of order sixteen whose eighth power is in an ambiguous class containing divisors of 34. Thus, using the quadratic representations, we get the following table.

$-1 \cdot 73 \cdot 257$	$2 \cdot 17 \cdot 73$	$-2 \cdot 17 \cdot 257$
$(1,1)$	$(1,1)$	$(1,1)$
$(16,146)$	$(1,1)$	$(2,2)$
$(16,146)$	$(1,1)$	$(8,34)$
$(2,2)$	$(1,1)$	$(8,34)$
$(16,146)$	$(2,2)$	$(2,2)$
$(2,2)$	$(2,2)$	$(1,1)$
$(1,1)$	$(2,2)$	$(8,34)$
$(16,146)$	$(2,2)$	$(8,34)$

Again, in this table each (a,b) means that in that class there is an element of order $2a$ whose ath power is in the class represented by b. Thus, in this example, we see that

$$S^{16} \cap \mathrm{Amb} = \langle (146,1,1) \rangle \text{ and that } S^{16} \cap \mathrm{Ker} = \langle (1,1,1) \rangle.$$

Thus

$$\frac{S^{16} \cap \mathrm{Amb}}{S^{16} \cap \mathrm{Ker}} \simeq \mathbb{Z}_2$$

and

$$\frac{H^{32}}{H^{64}} \simeq \mathbb{Z}_2.$$

Hence there is a factor of Z_{64} in H. Now we see that for the eighth power we have

$$S^8 \cap \mathrm{Amb} = \langle (145,1,1), (1,1,34) \rangle$$

and

$$S^8 \cap \mathrm{Ker} = \langle (1,1,1) \rangle.$$

And so,

$$\frac{S^8 \cap \text{Amb}}{S^8 \cap \text{Ker}} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$$

making

$$\frac{H^{16}}{H^{32}} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2.$$

The one $\mathbb{Z}_2$ is satisfied by the factor of $\mathbb{Z}_{64}$ that we already have, so the other one must indicate a factor of $\mathbb{Z}_{32}$ in H. Clearly, since there are no terms (a, b) in the above table with $a = 4$,

$$S^4 \cap \text{Amb} \simeq S^8 \cap \text{Amb}$$

so we move on to S^2. Since

$$S^2 \cap \text{Amb} = \langle (146, 1, 1), (1, 1, 34), (2, 2, 1) \rangle$$

and

$$S^2 \cap \text{Ker} = \langle (2, 2, 1) \rangle$$

we have

$$\frac{S^2 \cap \text{Amb}}{S^2 \cap \text{Ker}} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$$

and

$$\frac{H^4}{H^8} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Both of these are accounted for by the factors of H already obtained. So all other factors of H are of order 2 or 4. Since it can be determined by previous methods that H has rank 5 and $h = 2^{15}$ we see that

$$H^{(2)} \simeq \mathbb{Z}_{64} \times \mathbb{Z}_{32} \times \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$

In our final example we will examine a special case for the real field: weak ambiguous classes. While these do not happen very often, they do occur and so must be dealt with. If in the real field the characters of -1 are all positive and the norm of the fundamental unit is $+1$ then there exists at least one ambiguous class which does not contain an ambiguous ideal. Such a class is referred to as a weak ambiguous class. The number of such classes occurring is the same as the number of classes generated by taking the product of all strong ambiguous classes with any one of the weak ambiguous classes. In other words, half of the ambiguous classes will be weak. When this occurs we just know that for some w_i in H_2 the divisors of w_i^2 are in the principal genus. It is not necessary to know exactly what w_i is, only that it exists.

Example For our next problem let $d_1 = -1326$, $d_2 = 14722$ and $d_3 = -16887$. Here $H_1 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$, $H_2 \simeq \mathbb{Z}_4 \times \mathbb{Z}_4$ and $H_3 \simeq \mathbb{Z}_4 \times \mathbb{Z}_8 \times \mathbb{Z}_3$. Recall that the odd factor H_3 puts a slight twist on the calculations for the quadratic representations. Next note that the Ker is $\{(1,1,1)$, $(1,17,39)$, $(1,34,39)$, $(2,1,1)$, $(2,17,39)$, $(2,34,39)$, $(3,1,3)$, $(3,17,13)$, $(3,34,13)$, $(6,1,3)$, $(6,17,13)$, $(6,34,13)$, $(13,1,13)$, $(13,17,3)$, $(13,34,3)$, $(17,1,39)$, $(17,17,1)$, $(17,34,1)$, $(26,1,13)$, $(26,17,3)$, $(26,34,3)$, $(34,1,39)$, $(34,17,1)$, $(34,34,1)\}$.

The table of consistent characters is

2	3	13	17	2	17	433	3	13	433
+	+	+	+	+	+	+	+	+	+
+	−	−	+	+	+	+	−	−	+
+	+	−	−	+	−	−	+	−	−
+	−	+	−	+	−	−	−	+	−
−	+	−	+	−	+	−	+	−	−
−	−	+	+	−	+	−	−	+	−
−	+	+	−	−	−	+	+	+	+
−	−	−	−	−	−	+	−	−	+

Note that again in this case the real field did not have to be normalized. Finding the ambiguous classes in the principal genus, we see that in k_1 all genera contain ambiguous classes, and in k_2 the divisors of 17 are in the principal genus. Since the norm of the fundamental unit here is $+1$ and the characters of -1 are all positive, there must be at least one weak ambiguous class. We find that $1517^2 - 14722 \cdot 2^2 = 17 \cdot 19^2$. The characters for 19 in k_2 are -, +, - . Thus the square of any class in this genus is the ambiguous class containing a divisor of 17. So there must be two weak ambiguous classes. Let w_1 and w_2 be their representatives whose divisors are in the principal genus. Then $17w_1 \sim w_2$. Now, in k_3 the divisors of 3 and of 13 are in the principal genus. Using the quadratic representations we get the following table.

$2 \cdot 3 \cdot 13 \cdot 17$	$2 \cdot 17 \cdot 433$	$3 \cdot 13 \cdot 433$
$(1,1)$	$(1,1)$	$(1,1)$
$(1,1)$	$(1,1)$	$(1,1)$
$(1,1)$	$(2,w_2)$	$(4,13)$
$(1,1)$	$(2,w_2)$	$(2,3)$
$(1,1)$	$(2,17)$	$(4,13)$
$(1,1)$	$(2,17)$	$(2,3)$
$(1,1)$	$(2,w_1)$	$(1,1)$
$(1,1)$	$(2,w_1)$	$(4,13)$

Again, in this table each (a,b) means that in that class there is an element of order $2a$ whose ath power is in the class represented by b. Thus, in this example, we see that $S^4 \cap \mathrm{Amb}$ is given by $\langle(1,1,13)\rangle$ and that $S^4 \cap \mathrm{Ker}$ is $\langle(1,1,1)\rangle$. Thus $\frac{S^4\cap\mathrm{Amb}}{S^4\cap\mathrm{Ker}} \simeq \mathbb{Z}_2$ and $\frac{H^8}{H^{16}} \simeq \mathbb{Z}_2$. Hence there is a factor of Z_{16} in H. Now we see that for the second power we have $S^2 \cap \mathrm{Amb} = \langle(1,1,13),(1,w_2,3),(1,17,3)\rangle$ and $S^2 \cap \mathrm{Ker} = \langle(1,17,3)\rangle$. And so, $\frac{S^2\cap\mathrm{Amb}}{S^2\cap\mathrm{Ker}} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ making $\frac{H^4}{H^8} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. The one $\mathbb{Z}_2$ is satisfied by the factor of $\mathbb{Z}_{16}$ that we already have, so the other two must indicate two factors of $\mathbb{Z}_8$ in H. So all other factors of H are of order 2 or 4. Since it can be determined by previous methods that H has rank 4 and $r_a = 2^{11}$, we see that $H \simeq \mathbb{Z}_{16} \times \mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_4$.

Chapter 5

Fields with 2-Class Group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

In this chapter we determine all imaginary bicyclic biquadratic fields which have 2-class group $\mathbb{Z}_2 \times \mathbb{Z}_2$. We have previously determined all such fields with 2-class group Z_4 by different techniques [10]. For the first case, Corollary 1 of Theorem 3 shows that the total number of discrimintal divisors, t, is at most 4.

When $t = 2$, $r_a \leq 1$ and the 2-Sylow subgroup of H is cyclic. In this chapter we determine r_H when $t = 3$ and when $t = 4$.

Theorem 5. *Let 2 be totally ramified in K, $t = 3$, $r_a = 2$ and l be the unique odd prime which ramifies in K. Then $r_H = 1$ or 2 according as $l \equiv 5$ (mod 8) or $l \equiv 1$ (mod 8).*

Proof Since 2 is totally ramified in K, the only possible values for d_1, d_2 and d_3 are $-l$, $2l$, -2; $-2l$, $2l$, -1 and $-2l$, 2, $-l$ with $l \equiv 1 \pmod 4$. In each case $\widehat{S}$ has the form $\pm\pm$. Moreover $\text{Ker} = \langle\ (2,1,1), (1,2,1), (1,1,2)\rangle$.

Thus $\widehat{S} \cdot \widehat{\text{Ker}}$ is generated by character values corresponding to the matrix $M = \begin{pmatrix} 1 & 1 \\ x & 0 \\ 0 & x \end{pmatrix}$

over $\mathbb{Z}_2$ which reduces to $\begin{pmatrix} 1 & 0 \\ 0 & x \\ 0 & 0 \end{pmatrix}$. Now the rank $\widehat{S} \cdot \widehat{\text{Ker}} = \text{rank } M$, so Theorem 4 shows

$r_H = 1$ or 2 according as $x = 1$ or 0; i.e. according as $l \equiv 5$ or 1 (mod 8).

Theorem 6. *Let $t = 3$, $r_a = 2$ and 2 not be totally ramified in K. Assume the 2-Sylow subgroup of each H_i is cyclic. If the unit index of K is 1 then $r_H = 1$ or 2 according as $h_i \equiv 2$ (mod 4) for some i or not. If the unit index of K is 2 then $r_H = 2$.*

Proof Since $r_a = 2$, $(H_1 \times H_2 \times H_3)^{(2)} \simeq \mathbb{Z}_{2^a} \times \mathbb{Z}_{2^b}$. When the unit index is 1, $|\text{Ker}| = 4$ and $\text{Ker} \simeq \langle\ (2^{a-1}, 0),\ (0, 2^{b-1})\rangle$. Here $S^{(2)} \simeq \langle(2i, 2j), (2i+1, 2j+1)\rangle$. Thus if either $a = 1$ or $b = 1$ then $S^{(2)} \cdot \text{Ker} \simeq \mathbb{Z}_{2^a} \times \mathbb{Z}_{2^b}$, Theorem 3 and Corollary 2 show $r_H = 1$. If $a > 1, b > 1$ then $S^{(2)} \cdot \text{Ker} = S^{(2)}$, Theorem 3 and Corollary 2 show $r_H = 2$. If the unit index is 2 then $S^{(2)}$ is the same as above, but $|\text{Ker}| = 2$. In order for the unit index to be 2, either d_1 and d_3 are principal divisors of k_2 or $d_3 = -1$ and 2 is a principal divisor of k_2. In the first case $\text{Ker} \simeq \langle(2^{a-1}, 2^{b-1})\rangle$. In the second case, since 2 is not totally ramified in K, it must be unramified in k_1. Hence Ker is the same as above. Theorem 3 and Corollary 2 show $r_H = 2$ if and only if $\text{Ker} \subseteq S^{(2)}$ if and only if $a = b = 1$ or $a > 1, b > 1$. To complete the proof we show $a = 1$ if and only if $b = 1$. Since the unit index is 2, the discriminant of k_2 is divisible by all three primes which ramify in K. Using the symmetry of k_1 and k_3 we may assume $H_1^{(2)} \simeq \mathbb{Z}_{2^a}$ and $H_2^{(2)} \simeq \mathbb{Z}_{2^b}$. Thus the discriminant of k_1 is divisible by exactly two primes l and r with $l \not\equiv r$ (mod 4), where we choose l to be odd. Now $a > 1$ if and only if $(\frac{r}{l}) = +1$. From above we see r is not a principal divisor of k_2, so $b > 1$ if and only if $(\frac{r}{l}) = +1$.

Corollary 3. *For each of the following values of d_1, d_2 and d_3, $r_H = 2$. Moreover, if $r_H = 2$ and K satisfies the hypothesis of Theorem 5 or 6 then K is listed below. Here l, q, and r denote primes with $l \equiv 1$, $q \equiv r \equiv 3$ (mod 4).*

d_1	d_2	d_3	conditions
$-2l$	2	$-l$	$l \equiv 1 \pmod 8$
$-2l$	$2l$	-1	$l \equiv 1 \pmod 8$
$-l$	$2l$	-2	$l \equiv 1 \pmod 8$
$-lq$	qr	$-lr$	$(\frac{l}{q}) = (\frac{l}{r}) = +1$
$-lq$	lqr	$-r$	$(\frac{l}{r}) = +1$
$-2q$	qr	$-2r$	$(\frac{2}{q}) = (\frac{2}{r}) = +1$
$-lq$	q	$-l$	$l \equiv 1 \pmod 8$, $(\frac{l}{q}) = +1$
$-lq$	lq	-1	$l \equiv 1 \pmod 8$
$-l$	lq	$-q$	$(\frac{l}{q}) = +1$
$-2l$	$2q$	$-lq$	$l \equiv 1 \pmod 8$, $(\frac{l}{q}) = +1$
$-2l$	$2lq$	$-q$	$(\frac{l}{q}) = +1$
$-lq$	$2lq$	-2	$(\frac{2}{l}) = +1$
$-2q$	$2qr$	$-r$	$(\frac{2}{r}) = +1$

Proof The results follow from Theorems 5 and 6 and conditions for $h_i \equiv 0 \pmod 4$, see Brown [3, 4, 2].

Theorem 7. *If 2 is not totally ramified in K, $t = 3$, $r_a = 2$, and exactly one quadratic subfield of K has even class number then $r_H = 2$ for precisely the values listed below where l, q and r are primes with $l \equiv q \equiv r \equiv 3 \pmod 4$:*

d_1	d_2	d_3	*conditions*
$-lqr$	qr	$-l$	$(\frac{l}{q}) = (\frac{l}{r}) = -1$
$-2lq$	lq	-2	$(\frac{2}{l}) = (\frac{l}{q}) = -1$
$-2lq$	$2q$	$-l$	$(\frac{2}{l}) = +1$, $(\frac{l}{q}) = -1$
$-lq$	q	$-l$	$(\frac{2}{l}) = +1$, $(\frac{l}{q}) = -1$

When these conditions are not satisfied $r_H = 1$.

Proof Since $r_a = 2$ and 2 is not totally ramified in K, $q|d_2$ for some prime $q \equiv 3 \pmod 4$.

Since $t = 3$, 2-rank $H_2 \leq 1$, so 2-rank $H_2 = 0$. The only possible values for d_1, d_2 and d_3 other than those listed above are $-lq$, lq, -1. In all cases $\text{Ker} = \langle (l, 1, 1), (r, 1, 1) \rangle$ where $r = 2$ when it is not explicitly defined. In the first two lines the generating matrix for $\widehat{S} \cdot \widehat{\text{Ker}}$ is $M = \begin{pmatrix} 0 & 1 \\ x+y & x \\ y+1 & w+1 \end{pmatrix}$, where the columns correspond to l and q respectively. The matrix reduces to $\begin{pmatrix} 0 & 1 \\ y+1 & 0 \\ x+1 & 0 \end{pmatrix}$. In the last two lines the generating matrices are $\begin{pmatrix} 0 & 1 \\ x+y+1 & x \\ y & w \end{pmatrix}$, and $\begin{pmatrix} 0 & 1 \\ x+1 & x \\ y & w \end{pmatrix}$, where the columns correspond to l and q. Both of these reduce to $\begin{pmatrix} 0 & 1 \\ x+1 & 0 \\ y & 0 \end{pmatrix}$. The results follow. In the case $-lq$, lq, -1, $\text{Ker} = \langle (l, 1, 1), (2, 1, 1) \rangle$ and the generating matrix is $\begin{pmatrix} 0 & 1 \\ 1 & x+1 \\ y+w & y \end{pmatrix}$, with columns corresponding to -1 and l. The matrix reduces to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ y+w & 0 \end{pmatrix}$. Thus $r_H = 1$ in this case.

Next we turn to the case where $t = 3$ and $r_a = 3$. Here 2 is not totally ramified in K and d_2 is divisible by no prime congruent to 3 modulo 4. We consider separately the cases where each H_i is cyclic and where some H_i is noncyclic for $i = 1, 2, 3$. In the following theorems k_1, k_2 and k_3 will be defined by two ordered n-tuples where the first n-tuple defines d_1, d_2 and d_3 and the second n-tuple specifies congruence conditions on l, q, r etc. modulo 4.

Theorem 8. *If $t = 3$ and $r_a = 3$ with each H_i cyclic then K and r_H are determined as follows:*

a $(-lq,\ lr,\ -qr)$, (1, 2 or 3, 1 or 2), $r_H = 3 - w - x - y + wxy$.

b $(-l,\ lr,\ -r)$, (1, 2, 1), $r_H = 3 - w - x - y + wx + wxy$.

Proof Since each H_i is cyclic and d_2 is divisible by no prime congruent to 3 modulo 4, the discriminant of each k_i is divisible by exactly two primes. Since 2 is not totally ramified the only possible values are those listed above. In case (a) Ker is generated by $\langle (l,l,1),(q,1,q)\rangle$ and the matrix M reduces to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & y & w \\ 0 & x & x+w \end{pmatrix}$. In case (b) Ker $= \langle (1,l,1),(2,1,2)\rangle$ and M reduces $\begin{pmatrix} 1 & 0 & 0 \\ 0 & y & 0 \\ 0 & x & x+w \end{pmatrix}$. The results follow.

Theorem 9. *If $t = 3$ and $r_a = 3$ with some H_i noncyclic then K and r_H are determined as follows:*

a $(-lqr,\ lr,\ -q)$, (1, 2 or 3, 1 or 2), $r_H = 3 - x - w$.

b $(-lr,\ lr,\ -1)$, (1, 2, 1), $r_H = 3 - x - w + xw$.

c $(-lr,\ l,\ -r)$, (1, 2, 1) or $(-lqr,\ l,\ -qr)$, (1 or 2, 2 or 3, 1 or 2), $r_H = 3 - x - y$.

Proof Since 2 is not totally ramified in K and no prime congruent to 3 (modulo 4) divides d_2, it is seen that H_2 is cyclic and we may assume H_1 is noncyclic. It follows that only the three cases listed above can occur. In each case we list generators for Ker and a matrix which M reduces to. The results follow immediately.

a $\langle (l,l,1),(q,1,1)\rangle, \begin{pmatrix} 1 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & w \end{pmatrix}$

b $\langle (l,l,1),(2,1,1)\rangle, \begin{pmatrix} 1 & 0 & 0 \\ 0 & x & w \\ 0 & 0 & 0 \end{pmatrix}$

c $\langle (l,1,1),(q,1,q)\rangle, \begin{pmatrix} 1 & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & x \end{pmatrix}$

When $t = 4$, $3 \leq r_a \leq 5$ and $2 \leq r_H \leq r_a$. We now determine r_H for all fields K with $t = 4$. The notation here is similar to that used in Theorems 8 and 9.

Theorem 10. *When $r_a = 3$, the fields K and the values of r_H are listed below:*

a $(-2lq,\ q,\ -2l),\ (1,3),\ r_H = 3 - x - y + xy.$

b $(-2lq,\ q,\ -2l),\ (3,3),\ r_H = 3 - \bar{x} - y + \bar{x}y.$

c $(-2lq,\ 2q,\ -l),\ (1,3),\ r_H = 3 - x - y + xy.$

d $(-lq,\ 2q,\ -2l),\ (3,3),\ r_H = 3 - \bar{x} - y + \bar{x}y.$

e $(-2lq,\ lq,\ -2),\ (1,3),\ r_H = 3 - y.$

f $(-2q,\ lq,\ -2l),\ (1,3),\ r_H = 3 - x - y + xy.$

g $(-2lq, 2lq, -1), (1,3), r_H = 3 - y.$

h $(-2lq, 2lq, -1), (3,3), r_H = 2.$

i $(-lq, 2lq, -2), (3,3), r_H = 3 - \bar{w} - \bar{y} + \bar{w}\bar{y}.$

j $(-2q, 2lq, -l), (1,3), r_H = 3 - x - y + xy$.

Proof In order that $r_a = 3$, 2 must be totally ramified in K and there exists a prime $q \equiv 3 \pmod 4$ which divides d_2. Hence there is exactly one other odd prime l which ramifies in K. The only possible fields are those listed above. In part (a), Ker $= \langle (2,1,1), (l,1,1), (1,1,2) \rangle$ and the matrix M reduces to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & y \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}$. Thus $m = 2 + x + y - xy$ and the formula for r_H follows from Theorem 4. The remaining parts are done similarly. Also, each case has been computer checked.

We now will determine all fields K with 2-class group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Theorem 11. *The fields with 2-class group $\mathbb{Z}_2 \times \mathbb{Z}_2$ are exactly those listed below:*

a $(-2lq, q, -2l), (1,3), (\frac{2}{l}) = -1$ and either $(\frac{l}{q}) = -1$ or $(\frac{2}{q}) = -1$.

b $(-2lq, q, -2l), (3,3), (\frac{2}{l}) = (\frac{l}{q}) = -1, (\frac{2}{q}) = +1$.

c $(-2lq, 2q, -l), (1,3), (\frac{2}{l}) = -1$ and either $(\frac{l}{q}) = -1$ or $(\frac{2}{q}) = -1$.

d $(-lq, 2q, -2l), (3,3), (\frac{2}{l}) = -1$ and either $(\frac{l}{q}) = -1$ or $(\frac{2}{q}) = -1$.

e $(-2lq, lq, -2), (1,3), (\frac{2}{l}) = -1$ and either $(\frac{l}{q}) = -1$ or $(\frac{2}{q}) = -1$.

f $(-2q, lq, -2l), (1,3), (\frac{l}{q}) = +1, (\frac{2}{l}) = (\frac{2}{q}) = -1$.

g $(-2lq, 2lq, -1), (1,3), (\frac{2}{l}) = -1$ and either $(\frac{l}{q}) = -1$ or $(\frac{2}{q}) = -1$.

h $(-2lq, 2lq, -1), (3,3), (\frac{2}{l}) = (\frac{l}{q}) = -(\frac{2}{q})$.

i $(-lq, 2lq, -2), (3, 3), (\frac{2}{l}) = (\frac{l}{q}) = -(\frac{2}{q})$.

j $(-2q, 2lq, -l), (1, 3), (\frac{2}{l}) = (\frac{2}{q}) = -1, (\frac{l}{q}) = +1$.

k $(-lqr, qr, -l), (3, 3, 3), (\frac{l}{q}) = (\frac{l}{r}) = -1, h_1 \equiv 8 \pmod{16}$.

l $(-2lq, 2q, -l), (3, 3), (\frac{2}{l}) = +1, (\frac{l}{q}) = -1, h_1 \equiv 8 \pmod{16}$.

m $(-2lq, lq, -2), (3, 3), (\frac{2}{l}) = (\frac{2}{q}) = -1, h_1 \equiv 8 \pmod{16}$.

n $(-lq, q, -l), (3, 3), (\frac{2}{l}) = +1, (\frac{l}{q}) = -1, h_1 \equiv 8 \pmod{16}$.

o $(-l, lr, -r), (1, 1), (\frac{l}{r}) = (\frac{2}{l}) = (\frac{2}{r}) = -1$.

p $(-lqr, lr, -q), (1, 2 \text{ or } 3, 1 \text{ or } 2), (\frac{l}{r}) = -1, (\frac{q}{r}) \neq (\frac{q}{l})$.

q $(-lr, lr, -1), (1, 1), (\frac{l}{r}) = -1, (\frac{2}{l}) \neq (\frac{2}{r})$.

r $(-lr, l, -r), (1, 1), (\frac{2}{r}) = (\frac{l}{r}) = -1, (\frac{2}{l}) = +1$.

s $(-lqr, l, -qr), (1, 2 \text{ or } 3, 1 \text{ or } 2), (\frac{q}{r}) = -1, (\frac{l}{q}) \neq (\frac{l}{r})$.

t $(-2qr, 2, -qr), (1, 3), (\frac{q}{r}) = -1, (\frac{2}{q}) \neq (\frac{2}{r})$.

u $(-lq, lqr, -r), (1, 3, 3), (\frac{l}{q}) = -1, (\frac{l}{r}) = +1$.

v $(-lq, lq, -1), (1, 3), (\frac{2}{l}) = +1, (\frac{l}{q}) = -1$.

w $(-l, lq, -q), (1, 3), (\frac{2}{l}) = -1, (\frac{l}{q}) = +1$.

x $(-2l, 2lq, -q), (1, 3), (\frac{2}{l}) = -1, (\frac{l}{q}) = +1$.

y $(-lq, 2lq, -2), (1, 3), (\frac{2}{l}) = +1, (\frac{l}{q}) = -1$.

z $(-2l, 2lq, -q), (3, 3), (\frac{2}{l}) = -1, (\frac{2}{q}) = +1$.

Proof Since $h \equiv 4 \pmod 8$ and $h = h_1h_2h_3$ or $\frac{1}{2}\ h_1h_2h_3$ according as the unit index of K is 2 or 1, it follows that $r_a = 2$ or 3. The only possible fields come from Corollary 3 to Theorem 6 and Theorems 7, 8, 9 and 10 where $r_H = 2$. The results follow by using conditions from Brown [3, 4, 2, 5] to determine when $h \equiv 4 \pmod 8$.

Chapter 6

Fields with Class Numbers 4, 8 and 16.

Complete lists of all imaginary bicyclic biquadratic fields with class number 1 and 2 have already been obtained [5, 7]. In this chapter we calculate all imaginary bicyclic biquadratic fields with class groups of order four, eight and sixteen as well as compute the actual structure of each class group and give the results accordingly.

In order to determine all fields of class number 2^n, we use the well known class number formula

$$h = \frac{q_o}{2} h_1 h_2 h_3$$

where $q_o = 1$ or 2 is the index in the unit group of K of the subgroup of units contained in the quadratic subfields. Hence, if $h = 2^n$, then $h_1 h_3 \leq 2^{n+1}$. Since we wish to find all fields K with $h = 2^n$ where $n \leq 4$, we need to know complete lists of imaginary quadratic fields of class number 2^n for $n \leq 5$. Stark [12, 11] has shown that there are 9 such fields of class number 1,

$$\mathbb{Q}(\sqrt{-d}) : d = 1, 2, 3, 7, 11, 19, 43, 67, 163$$

and 18 fields of class number 2,

$$\mathbb{Q}(\sqrt{-d}) : d = 5, 6, 10, 13, 15, 22, 35, 37, 51, 58, 91, 115, 123, 187, 235, 267, 403, 427.$$

Moreover, Arno [1] has shown that

$$\mathbb{Q}(\sqrt{-d}) : d = 14, 17, 21, 30, 33, 34, 39, 42, 46, 55, 57, 70, 73,$$

$$78, 82, 85, 93, 97, 102, 130, 133, 142, 155, 177, 190, 193, 195, 203, 219,$$

$$253, 259, 291, 323, 355, 435, 483, 555, 595, 627, 667, 715, 723, 763, 795,$$

$$955, 1003, 1027, 1227, 1243, 1387, 1411, 1435, 1507, 1555.$$

are the only imaginary quadratic fields of class number 4. While complete lists of imaginary quadratic fields of class number 8, 16 and 32 have not been determined, Buell [6] and again in more recent calculations has computed the class numbers of all imaginary quadratic fields with discriminant 2.2 billion. In fact, Hoffstein's [8] bounds on the L-series show there can exist at most one imaginary field of class number 8, 16 or 32 with discriminant greater than 2.2 million. Therefore, Buell's calculations almost certainly provide us with complete lists of necessary imaginary quadratic fields.

Now the unit index is 2 if and only if $\sqrt{\omega\varepsilon} \in K$ where ω is a root of unity in a quadratic subfield of K and ε is the fundamental unit of the real quadratic subfield. Since K is an imaginary field $\sqrt{\omega\varepsilon}$ can not be real so we can assume $\omega = -1$ or i. If ε has norm -1 then the unit index is always 1. If ε has norm 1 then

$$\sqrt{\varepsilon} = \frac{(\sqrt{N(\varepsilon + 1)} + \sqrt{-N(\varepsilon - 1)}}{2}$$

where N is the norm function from k_2 to $\mathbb{Q}$. Now

$$-N(\varepsilon+1)N(\varepsilon-1) = -N(\varepsilon^2-1) = u^2 d_2$$

for some rational number u. The square free kernels of $N(\varepsilon+1)$ and $-N(\varepsilon-1)$ are called the *principal divisors* of k_2. Thus the unit index is 2 only when $-d_1$ and $-d_3$ are the principal divisors or $d_1 = -1$ and 2 is a principal divisor. Moreover, the product of the two principal divisors is d_2 or $4d_2$ where the latter value occurs only when $d_2 \equiv 3(\text{mod} 4)$. Hence if some prime divides the discriminant of K but not that of k_2 the unit index is always 1.

Since we may assume $h_1 \leq h_3$, we first let k_1 and k_3 run through all fields of class number 1, 2 and 4. If $h_3 = 8$, 16 or 32 we only need to let k_1 run through those fields with $h_1 h_3 \leq 32$. We first estimate h_2 using genus theory since this gives a lower bound for h_2, if the class formula shows $h > 16$ the field is discarded. Otherwise, h_2 is determined to give the exact value of h.

When $h = 4$ or 8, a simple computation of the rank determines the structure of the class group. For fields with class number 16 the basic procedure is the same except that the structure of the class group is not always completely determined by the rank. If the class group has rank 2 there are two possible group structures. The final determination requires that we use Theorem 2. These computations were done on computer. Assuming that Buell's lists [6] of imaginary quadratic fields of class number 8, 16 and 32 are complete, all fields K with class number 16 are listed in Table's 6.6, 6.7, 6.8 , 6.9 and 6.10. Specifically, fields with class group $\mathbb{Z}_{16}$ are listed in Table 6.6, with class group $\mathbb{Z}_8 \times \mathbb{Z}_2$ in Table 6.7, with class group $\mathbb{Z}_4 \times \mathbb{Z}_4$ in Table 6.8, with class group $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ in Table 6.9, and with class group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ in Table 6.10. In all but Table 6.1 the fields are listed by their conductor, f.

Table 6.1: Fields where $H \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$

	d_1	d_2	d_3		d_1	d_2	d_3
1,a	−30	3	−10	36	−1947	177	−11
2	−70	7	−10	37	−2163	721	−3
3	−190	19	−10	38	−2667	889	−3
4,b	−42	7	−6	39	−5467	781	−7
5,c	−30	6	−5	40,l	−154	22	−7
6	−70	14	−5	41	−322	46	−7
7	−78	6	−13	42	−66	33	−2
8	−190	38	−5	43,m	−114	57	−2
9,d	−21	14	−6	44	−258	129	−2
10	−33	22	−6	45	−418	209	−2
11	−33	6	−22	46	−498	249	−2
12	−57	38	−6	47	−77	11	−7
13	−93	62	−6	48,n	−301	43	−7
14	−177	118	−6	49	−5	65	−13
15	−253	46	−22	50,o	−5	185	−37
16,e	−30	15	−2	51	−13	481	−37
17	−70	35	−2	52	−195	65	−3
18	−78	39	−2	53,p	−555	185	−3
19	−190	95	−2	54	−715	65	−11
20,f	−22	55	−10	55	−70	10	−7
21,g	−30	30	−1	56	−78	26	−3
22	−70	70	−1	57	−190	10	−19
23	−78	78	−1	58	−85	85	−1
24	−190	190	−1	59	−85	17	−5
25	−42	42	−1	60	−195	13	−15
26,h	−21	42	−2	61	−435	29	−15
27,i	−93	186	−2	62	−555	37	−15
28	−133	266	−2	63	−1435	41	−35
29	−22	110	−5	64	−78	13	−6
30	−6	78	−13	65	−102	17	−6
31	−6	222	−37	66	−70	2	−35
32	−22	814	−37	67,t	−102	2	−51
33	−651	217	−3	68	−15	165	−11
34,k	−1659	553	−3	69,u	−15	285	−19
35	−1771	253	−7	70	−35	385	−11

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Fields where $H \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$

continued from previous page

	d_1	d_2	d_3
71	−35	665	−19
72	−51	969	−19
73	−51	2193	−43
74	−51	3417	−67
75	−91	273	−3
76	−91	3913	−43
77	−115	1265	−11
78	−115	2185	−19
79	−123	5289	−43
80	−123	20049	−163
81	−187	3553	−19
82	−187	8041	−43
83	−187	12529	−67
84	−235	2585	−11
85	−235	4465	−19
86	−267	2937	−11
87	−403	1209	−3
88	−403	17329	−43
89	−427	1281	−3
90	−427	69601	−163
91	−51	51	−1
92,v	−123	123	−1
93	−187	187	−1
94	−267	267	−1
95	−5	55	−11
96,w	−5	95	−19
97	−13	39	−3
98	−13	559	−43
99	−37	111	−3
100	−37	259	−7
101	−37	407	−11
102	−37	2479	−67
103	−10	110	−11
104,x	−10	190	−19
105	−58	406	−7
106	−58	3886	−67
107	−51	102	−2
108,y	−123	246	−2
109	−187	374	−2
110	−267	534	−2
111	−6	42	−7
112	−22	154	−7

Table 6.2: Fields where $H \simeq \mathbb{Z}_4$

f	d_1	d_2	d_3	f	d_1	d_2	d_3
56	−2	7	−14	385	−35	77	−55
56	−1	14	−14	385	−7	385	−55
95	−19	5	−95	408	−34	6	−51
111	−3	37	−111	408	−3	34	−102
120	−15	10	−6	408	−3	102	−34
120	−15	2	−30	429	−11	429	−39
120	−6	5	30	440	−10	22	−55
120	−3	10	−30	440	−2	110	−55
156	−13	3	−39	452	−1	113	−113
156	−1	39	−39	455	−35	65	−91
164	−1	41	−41	456	−19	6	−114
165	−15	33	−55	456	−3	38	−114
165	−3	165	−55	465	−15	93	−155
168	−6	21	−14	465	−3	465	−155
168	−3	42	−14	476	−7	119	−17
183	−3	61	−183	520	−10	13	−130
184	−2	23	−46	520	−2	65	−130
184	−1	46	−46	548	−1	137	−137
195	−39	5	−195	552	−3	46	−138
204	−17	3	−51	552	−6	69	−46
204	−3	51	−17	552	−3	138	−46
220	−5	11	−55	552	−2	69	−138
220	−1	55	−55	564	−1	141	−141
255	−15	85	−51	564	−3	47	−141
264	−11	6	−66	568	−2	71	−142
264	−3	22	−66	579	−3	193	−579
273	−91	21	−39	583	−11	53	−583
273	−7	273	−39	595	−35	17	−595
276	−1	69	−69	595	−7	85	−595
276	−3	23	−69	609	−3	609	−203
280	−7	10	−70	615	−15	205	−123
308	−11	7	−77	616	−11	14	−154
308	−1	77	−77	616	−22	77	−14
312	−2	78	−39	616	−11	154	−14
371	−7	53	−371	616	−2	77	−154

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Fields where $H \simeq \mathbb{Z}_4$

continued from previous page

f	d_1	d_2	d_3
620	−5	31	−155
620	−1	155	−155
651	−7	93	−651
712	−2	89	−178
741	−19	741	−39
748	−17	11	−187
748	−11	187	−17
795	−15	53	−795
795	−3	265	−795
812	−1	203	−203
868	−1	217	−217
816	−51	34	−6
852	−3	71	−213
868	−7	31	−217
904	−2	113	−226
939	−3	313	−939
952	−7	238	−34
969	−51	57	−323
969	−3	969	−323
979	−11	89	−979
984	−82	6	−123
984	−3	246	−82
987	−7	141	−987
987	−3	329	−987
1032	−43	6	−258
1032	−3	86	−258
1036	−37	7	−259
1036	−1	259	−259
1043	−7	149	−1043
1064	−19	266	−14
1065	−15	213	−355
1065	−3	1065	−355
1085	−35	217	−155
1085	−7	1085	−155
1128	−3	94	−282
1128	−2	141	−282
1204	−1	301	−301
1204	−43	7	−301
1209	−403	93	−39
1240	−10	62	−155
1240	−2	310	−155
1252	−1	313	−313
1265	−115	253	−55
1288	−2	161	−322
1299	−3	433	−1299
1348	−1	337	−337
1420	−5	71	−355
1420	−1	355	−355
1435	−7	205	−1435
1496	−187	34	−22
1496	−34	22	−187
1496	−11	374	−34
1533	−7	1533	−219
1624	−58	14	−203
1624	−2	406	−203
1659	−7	237	−1659
1672	−19	22	−418
1672	−11	38	−418
1704	−6	213	−142
1704	−3	426	−142
1771	−11	161	−1771
1803	−3	601	−1803
1828	−1	457	−457
1864	−2	233	−466
1939	−7	277	−1939
1947	−3	649	−1947
1992	−3	166	−498
2001	−3	2001	−667
2024	−22	253	−46
2037	−7	2037	−291

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Fields where $H \simeq \mathbb{Z}_4$

continued from previous page

f	d_1	d_2	d_3
2044	−7	511	−73
2072	−2	518	−259
2135	−35	305	−427
2136	−267	178	−6
2139	−3	713	−2139
2163	−7	309	−2163
2212	−1	553	−553
2248	−2	281	−562
2261	−7	2261	−323
2307	−3	769	−2307
2365	−43	2365	−55
2408	−43	602	−14
2409	−11	2409	−219
2451	−43	57	−2451
2451	−19	129	−2451
2485	−35	497	−355
2485	−7	2485	−355
2585	−235	517	−55
2611	−7	373	−2611
2613	−67	2613	−39
2632	−7	94	−658
2632	−2	329	−658
2667	−7	381	−2667
2668	−1	667	−667
2716	−7	679	−97
2840	−2	710	−355
2865	−15	573	−955
2865	−3	2865	−955
2947	−7	421	−2947
3009	−51	177	−1003
3009	−3	3009	−1003
3052	−1	763	−763
3212	−11	803	−73
3243	−3	1081	−3243
3507	−7	501	−3507
3507	−3	1169	−2507
3553	−187	209	−323
3553	−11	3553	−323
3565	−115	713	−155
3608	−11	902	−82
3685	−67	3685	−55
3729	−3	3729	−1243
3752	−67	938	−14
3787	−7	541	−3787
3820	−5	191	−955
3820	−1	955	−955
3857	−19	3857	−203
3883	−11	353	−3883
3963	−3	1321	−3963
4108	−1	1027	−1027
4123	−19	217	−4123
4123	−7	589	−4123
4233	−51	249	−1411
4233	−3	4233	−1411
4323	−11	393	−4323
4323	−3	1441	−4323
4521	−3	4521	−1507
4539	−51	1513	−267
4665	−15	933	−1555
4665	−3	4665	−1555
4921	−19	4921	−259
5061	−7	5061	−723
5336	−58	46	−667
5336	−2	1334	−667
5467	−11	497	−5467
5947	−19	313	−5947
6104	−2	1526	−763
6220	−5	311	−1555
6222	−1	1555	−1555
6232	−19	1558	−82

continued on following page

Fields where $H \simeq \mathbb{Z}_4$

continued from previous page

f	d_1	d_2	d_3
6248	−22	781	−142
6248	−11	1562	−142
6357	−163	6357	−39
6665	−43	6665	−155
6685	−35	1337	−955
7021	−7	7021	−1003
7189	−91	553	−1027
7189	−7	7189	−1027
7285	−235	1457	−155
7337	−11	7337	−667
7372	−19	1843	−97
7640	−10	382	−955
7640	−2	1910	−955
7912	−43	1978	−46
7953	−11	7953	−723
8165	−115	1633	−355
8216	−2	2054	−1027
8393	−11	8393	−763
8492	−11	2123	−193
8589	−7	8589	−1227
8729	−43	8729	−203
8965	−163	8965	−55
9128	−163	2282	−14
9417	−43	9417	−219
9709	−7	9709	−1387
9877	−7	9877	−1411
10385	−67	10385	−155
10792	−19	2698	−142
10947	−123	3649	−267
11033	−187	649	−1003
11033	−11	11033	−1003
11084	−163	2771	−17
11297	−11	11297	−1027
12328	−67	3082	−46
12440	−10	622	−1555

f	d_1	d_2	d_3
12440	−2	3110	−1555
12556	−43	3139	−73
12673	−19	12673	−667
13497	−11	13497	−1227
13737	−19	13737	−723
14497	−19	14497	−763
15521	−187	913	−1411
16685	−235	3337	−355
19497	−67	19497	−291
19513	−19	19513	−1027
21965	−115	4393	−955
21976	−67	5494	−82
23313	−19	23313	−1227
23785	−67	23785	−355
24424	−43	6106	−142
25265	−163	25265	−155
28681	−43	28681	−667
29992	−163	7498	−46
31089	−43	31089	−723
31837	−403	2449	−1027
33089	−163	33089	−203
35697	−163	35697	−219
35765	−115	7153	−1555
38056	−67	9514	−142
41065	−43	41065	−955
42217	−163	42217	−259
44885	−235	8977	−955
47596	−163	11899	−73
51121	−67	51121	−763
52649	−163	52649	−323
52761	−43	52761	−1227
53449	−43	53449	−1243
57865	−163	57865	−355
59641	−43	59641	−1387
64801	−43	64801	−1507

continued on following page

Fields where $H \simeq \mathbb{Z}_4$

continued from previous page

f	d_1	d_2	d_3
66865	−43	66865	−1555
68809	−67	68809	−1027
73085	−235	14617	−1555
82209	−67	82209	−1227
83281	−67	83281	−1243
100969	−67	100969	−1507
104185	−67	104185	−1555
108721	−163	108721	−667

f	d_1	d_2	d_3
124369	−163	124369	−763
155665	−163	155665	−955
163489	−163	163489	−1003
167401	−163	167401	−1027
226081	−163	226081	−1387
229993	−163	229993	−1411
245641	−163	245641	−1507
253465	−163	253465	−1555

Table 6.3: Fields where $H \simeq \mathbb{Z}_8$

f	d_1	d_2	d_3
195	−15	65	−39
248	−2	31	−62
248	−1	62	−62
260	−13	5	−65
260	−5	13	−65
280	−35	10	−14
280	−14	5	−70
285	−15	57	−95
285	−3	285	−95
312	−39	2	−78
312	−6	26	−39
340	−17	5	−85
340	−5	85	−17
376	−2	47	−94
376	−1	94	−94
380	−5	19	−95
380	−1	95	−95
399	−19	21	−399
399	−7	57	−399
407	−11	37	−407
435	−3	145	−435
440	−22	10	−55
444	−37	3	−111
444	−1	111	−111
471	−3	157	−471
492	−123	3	−41
492	−3	123	−41
559	−43	13	−559
584	−2	73	−146
632	−1	158	−158
644	−1	161	−161
665	−35	133	−95
665	−7	665	−95
680	−10	85	−34
715	−55	13	−715
728	−91	26	−14
732	−1	183	−183
744	−6	93	−62
744	−3	186	−62
760	−2	190	−95
885	−15	177	−295
885	−3	885	−295
888	−2	222	−111
903	−43	21	−903
903	−7	129	−903
920	−115	10	−46
935	−187	85	−55
984	−6	82	−123
1016	−2	127	−254
1023	−3	341	−1023
1047	−3	349	−1047
1060	−5	53	−265
1095	−15	365	−219
1113	−3	1113	−371
1128	−3	282	−94
1139	−67	17	−1139
1148	−7	287	−41
1159	−19	61	−1159
1160	−10	145	−58
1180	−1	295	−295
1185	−15	237	−395
1185	−3	1185	−395
1196	−13	23	−299
1196	−1	299	−299
1281	−427	21	−183
1281	−7	1281	−183
1295	−35	185	−259
1335	−15	445	−267
1356	−3	339	−113
1379	−7	197	−1379

continued on following page

Fields where $H \simeq \mathbb{Z}_8$

continued from previous page

f	d_1	d_2	d_3	f	d_1	d_2	d_3
1412	−1	353	−353	2211	−67	33	−2211
1416	−3	118	−354	2211	−3	737	−2211
1455	−15	485	−291	2292	−3	191	−573
1460	−5	365	−73	2292	−1	573	−573
1464	−2	366	−183	2332	−1	583	−583
1484	−1	371	−371	2356	−19	31	−589
1528	−2	191	−382	2356	−1	589	−589
1528	−1	382	−382	2360	−10	118	−295
1580	−1	395	−395	2360	−2	590	−295
1599	−123	533	−39	2392	−2	598	−299
1608	−67	6	−402	2516	−37	629	−17
1608	−3	134	−402	2568	−3	214	−642
1636	−1	409	−409	2712	−3	678	−226
1644	−3	411	−137	2728	−22	341	−62
1749	−3	1749	−583	2728	−11	682	−62
1804	−11	451	−41	2739	−11	249	−2739
1876	−67	7	−469	2739	−3	913	−2739
1896	−6	237	−158	2765	−7	2765	−395
1896	−3	474	−158	2811	−3	937	−2811
1912	−2	239	−478	2868	−3	239	−717
1912	−1	478	−478	2868	−1	717	−717
1940	−5	485	−97	2884	−7	103	−721
1983	−3	661	−1983	2884	−1	721	−721
2004	−3	167	−501	2937	−267	33	−979
2004	−1	501	−501	2937	−3	2937	−979
2013	−11	2013	−183	2968	−2	742	−371
2015	−403	65	−155	2985	−15	597	−995
2019	−3	673	−2019	2985	−3	2985	−995
2065	−35	413	−295	3063	−3	1021	−3063
2065	−7	2065	−295	3116	−19	779	−41
2093	−91	161	−299	3144	−3	262	−786
2093	−7	2093	−299	3160	−10	158	−395
2109	−19	2109	−111	3160	−2	790	−395
2136	−267	6	−178	3171	−7	453	−3171
2136	−3	534	−178	3289	−11	3289	−299

continued on following page

Fields where $H \simeq \mathbb{Z}_8$

continued from previous page

f	d_1	d_2	d_3
3336	−3	278	−834
3363	−19	177	−3363
3363	−3	1121	−3363
3367	−91	481	−259
3416	−427	122	−14
3443	−11	313	−3443
3448	−2	431	−862
3448	−1	862	−862
3471	−267	1157	−39
3531	−3	1177	−3531
3556	−1	889	−889
3585	−15	717	−1195
3603	−3	1201	−3603
3723	−51	1241	−219
3732	−3	311	−933
3732	−1	933	−933
3752	−67	14	−938
3784	−43	22	−946
3784	−11	86	−946
3796	−13	949	−73
3819	−67	57	−3819
3819	−19	201	−3819
3832	−2	479	−958
3832	−1	958	−958
3860	−5	965	−193
3944	−58	493	−34
3976	−2	497	−994
3980	−5	199	−995
3980	−1	995	−995
4008	−3	334	−1002
4008	−2	501	−1002
4085	−43	4085	−95
4136	−22	517	−94
4136	−11	1034	−94
4171	−43	97	−4171

f	d_1	d_2	d_3
4172	−1	1043	−1043
4179	−7	597	−4179
4228	−1	1057	−1057
4324	−1	1081	−1081
4427	−19	233	−4427
4584	−6	573	−382
4584	−3	1146	−382
4596	−3	383	−1149
4596	−1	1149	−1149
4612	−1	1153	−1153
4664	−2	1166	−583
4683	−7	669	−4683
4687	−43	109	−4687
4712	−19	62	−1178
4712	−19	1178	−62
4712	−2	589	−1178
4773	−43	4773	−111
4780	−5	239	−1195
4780	−1	1195	−1195
4804	−1	1201	−1201
4899	−3	1633	−4899
4939	−11	449	−4939
4947	−51	1649	−291
4971	−3	1657	−4971
4984	−7	1246	−178
5044	−13	1261	−97
5356	−13	103	−1339
5385	−3	5385	−1795
5403	−3	1801	−5403
5428	−1	1357	−1357
5551	−91	793	−427
5572	−7	199	−1393
5592	−3	1398	−466
5681	−19	5681	−299
5736	−6	717	−478

continued on following page

Fields where $H \simeq \mathbb{Z}_8$

continued from previous page

f	d_1	d_2	d_3
5736	−3	1434	−478
5752	−2	719	−1438
5752	−1	1438	−1438
5896	−67	22	−1474
5896	−11	134	−1474
5907	−3	1969	−5907
5908	−1	1477	−1477
5992	−2	749	−1498
6099	−3	2033	−6099
6177	−3	6177	−2059
6312	−3	526	−1578
6312	−2	789	−1578
6365	−67	6365	−95
6369	−11	6369	−579
6532	−1	1633	−1633
6573	−7	6573	−939
6604	−13	127	−1651
6604	−1	1651	−1651
6628	−1	1657	−1657
6744	−3	1686	−562
6785	−115	1357	−295
6792	−3	566	−1698
6853	−7	6853	−979
6923	−43	161	−6923
6952	−22	869	−158
6952	−11	158	−1738
6952	−11	1738	−158
6952	−2	869	−1738
6965	−7	6965	−995
6969	−3	6969	−2323
7051	−11	641	−7051
7144	−19	1786	−94
7180	−1	1795	−1795
7185	−3	7185	−2395
7257	−123	177	−2419
7257	−3	7257	−2419
7288	−2	911	−1822
7288	−1	1822	−1822
7368	−3	614	−1842
7491	−11	681	−7491
7491	−3	2497	−7491
7588	−7	271	−1897
7756	−1	1939	−1939
7819	−7	1117	−7819
7828	−19	103	−1957
7869	−43	7869	−183
7923	−19	417	−7923
7960	−2	1990	−995
8043	−7	1149	−8043
8043	−3	2681	−8043
8236	−1	2059	−2059
8283	−3	2761	−8283
8299	−43	193	−8299
8308	−67	31	−2077
8308	−1	2077	−2077
8344	−2	2086	−1043
8365	−35	1673	−1195
8365	−7	8365	−1195
8452	−1	2113	−2113
8481	−3	8481	−2827
8548	−1	2137	−2137
8588	−19	2147	−113
8643	−67	129	−8643
8643	−43	201	−8643
8683	−19	457	−8683
8764	−7	2191	−313
8859	−3	2953	−8859
8968	−19	118	−2242
8985	−3	8985	−2995
9085	−115	1817	−395

continued on following page

Fields where $H \simeq \mathbb{Z}_8$

continued from previous page

f	d_1	d_2	d_3
9093	−7	9093	−1299
9208	−2	1151	−2302
9219	−7	1317	−9219
9269	−403	713	−299
9292	−1	2323	−2323
9373	−91	721	−1339
9373	−7	9373	−1339
9507	−3	3169	−9507
9508	−1	2377	−2377
9512	−58	1189	−82
9560	−2	2390	−1195
9580	−5	479	−2395
9580	−1	2395	−2395
9919	−91	1417	−763
10036	−13	2509	−193
10203	−19	537	−10203
10203	−3	3401	−10203
10209	−123	249	−3403
10209	−3	10209	−3403
10227	−7	1461	−10227
10312	−2	1289	−2578
10344	−3	2586	−862
10348	−13	199	−2587
10348	−1	2587	−2587
10444	−1	2611	−2611
10563	−3	3521	−10563
10587	−3	3529	−10587
10785	−3	10785	−3595
10843	−7	1549	−10843
10988	−67	2747	−41
11001	−19	11001	−579
11077	−19	11077	−583
11092	−1	2773	−2773
11179	−7	1597	−11179
11473	−11	11473	−1043

f	d_1	d_2	d_3
11512	−2	1439	−2878
11523	−3	3841	−11523
11557	−91	889	−1651
11557	−7	11557	−1651
11572	−11	263	−2893
11572	−1	2893	−2893
11649	−3	11649	−3883
11788	−1	2947	−2947
11848	−2	1481	−2962
11931	−123	3977	−291
11980	−5	599	−2995
11980	−1	2995	−2995
12008	−19	3002	−158
12027	−19	633	−12027
12117	−7	12117	−1731
12261	−67	12261	−183
12383	−427	1769	−203
12387	−3	4129	−12387
12523	−7	1789	−12523
12565	−7	12565	−1795
12585	−3	12585	−4195
12621	−7	12621	−1803
12747	−7	1821	−12747
12772	−1	3193	−3193
12801	−51	753	−4267
12801	−3	12801	−4267
12859	−11	1169	−12859
12859	−7	1837	−12859
12868	−1	3217	−3217
13123	−11	1193	−13123
13161	−3	13161	−4387
13208	−2	3302	−1651
13288	−11	302	−3322
13288	−2	1661	−3322
13363	−7	1909	−13363

continued on following page

Fields where $H \simeq \mathbb{Z}_8$

continued from previous page

f	d_1	d_2	d_3
13528	−19	3382	−178
13827	−3	4609	−13827
13865	−235	2773	−295
14241	−3	14241	−4747
14356	−37	3589	−97
14360	−10	718	−1795
14371	−7	2053	−14371
14380	−5	719	−3595
14403	−3	4801	−14403
14707	−11	1337	−14707
14729	−11	14729	−1339
14828	−11	3707	−337
15148	−1	3787	−3787
15387	−3	5129	−15387
15485	−163	15485	−95
15512	−2	3878	−1939
15547	−7	2221	−15547
15799	−427	2257	−259
16149	−7	16149	−2307
16156	−7	4039	−577
16168	−43	4042	−94
16261	−7	16261	−2323
16347	−3	5449	−16347
16472	−2	4118	−2059
16616	−67	4154	−62
16723	−7	2389	−16723
16765	−7	16765	−2395
16780	−1	4195	−4195
16808	−11	4202	−382
16933	−7	16933	−2419
16985	−43	16985	−395
17347	−11	1577	−17347
18093	−163	18093	−111
18161	−11	18161	−1651
18403	−11	1673	−18403

f	d_1	d_2	d_3
18584	−2	4646	−2323
18601	−19	18601	−979
18988	−1	4747	−4747
19160	−10	958	−2395
19160	−2	4790	−2395
19436	−43	4859	−113
19765	−67	19765	−295
19789	−7	19789	−2827
19833	−11	19833	−1803
19987	−11	1817	−19987
20108	−11	5027	−457
20504	−11	5126	−466
20696	−2	5174	−2587
20888	−2	5222	−2611
20965	−7	20965	−2995
21032	−22	2629	−478
21032	−11	5258	−478
21131	−187	1921	−1243
21329	−11	21329	−1939
22348	−37	151	−5587
22348	−1	5587	−5587
22649	−11	22649	−2059
22828	−1	5707	−5707
23564	−43	5891	−137
23576	−2	5894	−2947
23821	−7	23821	−3403
23960	−10	1198	−2995
23960	−2	5990	−2995
24681	−19	24681	−1299
24728	−11	6182	−562
24857	−67	24857	−371
25165	−7	25165	−3595
25192	−67	6298	−94
25267	−11	2297	−25267
25441	−19	25441	−1339

continued on following page

Fields where $H \simeq \mathbb{Z}_8$

continued from previous page

f	d_1	d_2	d_3	f	d_1	d_2	d_3
25553	−11	25553	−2323	41629	−7	41629	−5947
25612	−19	6403	−337	41657	−11	41657	−3787
26465	−67	26465	−395	42097	−43	42097	−979
26609	−11	26609	−2419	42344	−67	10586	−158
27176	−43	6794	−158	42785	−43	42785	−995
27307	−7	3901	−27307	44849	−43	44849	−1043
27741	−7	27741	−3963	45961	−19	45961	−2419
28457	−11	28457	−2587	46937	−187	2761	−4267
28564	−37	7141	−193	46937	−11	46937	−4267
28721	−11	28721	−2611	48085	−163	48085	−295
28760	−10	1438	−3595	48257	−11	48257	−4387
28760	−2	7190	−3595	48737	−163	48737	−299
28963	−11	2633	−28963	49153	−19	49153	−2587
29032	−19	7258	−382	51385	−43	51385	−1195
29365	−7	29365	−4195	52217	−11	52217	−4747
29869	−7	29869	−4267	53273	−11	53273	−4843
30296	−2	7574	−3787	53713	−19	53713	−2827
30616	−43	7654	−178	53836	−43	13459	−313
30709	−7	30709	−4387	55993	−19	55993	−2947
33229	−7	33229	−4747	60568	−67	15142	−226
33560	−2	8390	−4195	61288	−163	15322	−94
34069	−7	34069	−4867	62777	−11	62777	−5707
36328	−19	9082	−478	64347	−267	21449	−723
36716	−67	9179	−137	64385	−163	64385	−395
37433	−11	37433	−3403	64657	−19	64657	−3403
37928	−11	9482	−862	65512	−19	16378	−862
37976	−2	9494	−4747	66665	−67	66665	−995
38744	−58	334	−4843	75297	−19	75297	−3963
38744	−2	9686	−4843	77185	−43	77185	−1795
38936	−2	9734	−4867	78604	−43	19651	−457
39121	−19	39121	−2059	80065	−67	80065	−1195
39949	−91	3073	−5707	83353	−19	83353	−4387
39949	−7	39949	−5707	83884	−67	20971	−313
40377	−43	40377	−939	87033	−67	87033	−1299
41509	−403	3193	−1339	89324	−163	22331	−137

continued on following page

Fields where $H \simeq \mathbb{Z}_8$

continued from previous page

f	d_1	d_2	d_3	f	d_1	d_2	d_3
89713	−67	89713	−1339	218257	−163	218257	−1339
90316	−67	22579	−337	231016	−67	57754	−862
92017	−19	92017	−4843	240865	−67	240865	−3595
94377	−163	94377	−579	249064	−163	62266	−382
99244	−43	24811	−577	253729	−67	253729	−3787
102376	−67	25594	−382	255721	−43	255721	−5947
102985	−43	102985	−2395	260161	−67	260161	−3883
103016	−163	25754	−158	265521	−67	265521	−3963
106153	−19	106153	−5587	269113	−163	269113	−1651
110617	−67	110617	−1651	292585	−163	292585	−1795
112273	−43	112273	−2611	293889	−163	293889	−1803
115977	−67	115977	−1731	303832	−163	75958	−466
116056	−163	29014	−178	316057	−163	316057	−1939
120265	−67	120265	−1795	335617	−163	335617	−2059
121561	−43	121561	−2827	376041	−163	376041	−2307
124888	−67	31222	−466	378649	−163	378649	−2323
126721	−43	126721	−2947	382369	−67	382369	−5707
128104	−67	32026	−478	390385	−163	390385	−2395
128785	−43	128785	−2995	398449	−67	398449	−5947
148264	−43	37066	−862	421681	−163	421681	−2587
150616	−67	37654	−562	488185	−163	488185	−2995
154569	−67	154569	−2307	562024	−163	140506	−862
154585	−43	154585	−3595	585985	−163	585985	−3595
154636	−67	38659	−577	617281	−163	617281	−3787
155641	−67	155641	−2323	632929	−163	632929	−3883
159577	−163	159577	−979	645969	−163	645969	−3963
162073	−67	162073	−2419	683785	−163	683785	−4195
162185	−163	162185	−995	695521	−163	695521	−4267
170009	−163	170009	−1043	789409	−163	789409	−4843
173329	−67	173329	−2587	793321	−163	793321	−4867
174937	−67	174937	−2611	910681	−163	910681	−5587
194785	−163	194785	−1195	930241	−163	930241	−5707

Table 6.4: Fields where $H \simeq \mathbb{Z}_4 \times \mathbb{Z}_2$

f	d_1	d_2	d_3	f	d_1	d_2	d_3
120	−6	15	−10	616	−22	14	−77
120	−5	30	−6	616	−2	154	−77
136	−17	2	−34	616	−1	154	−154
136	−2	34	−17	644	−7	23	−161
136	−1	34	−34	660	−15	11	−165
168	−14	3	−42	660	−15	55	−33
168	−14	6	−21	660	−5	33	−165
260	−1	65	−65	660	−5	165	−33
264	−22	3	−66	660	−3	55	−165
264	−6	11	−66	660	−1	165	−165
328	−1	82	−82	663	−51	221	−39
399	−3	133	−399	680	−10	34	−85
408	−34	3	−102	696	−6	87	−58
408	−17	6	−102	776	−2	194	−97
408	−6	51	−34	777	−3	777	−259
408	−6	102	−17	780	−13	195	−15
420	−35	3	−105	780	−5	39	−195
420	−35	15	−21	820	−1	205	−205
420	−15	7	−105	840	−35	6	−210
420	−15	35	−21	840	−35	30	−42
420	−7	15	−105	840	−35	42	−30
420	−5	21	−105	840	−30	21	−70
420	−3	35	−105	840	−15	14	−210
420	−1	105	−105	840	−15	42	−70
456	−6	19	−114	840	−15	70	−42
520	−13	10	−130	840	−10	21	−210
520	−10	130	−13	840	−10	105	−42
520	−5	26	−130	840	−7	30	−210
552	−6	23	−138	840	−6	105	−70
552	−6	46	−69	840	−6	210	−35
552	−2	138	−69	840	−3	70	−210
552	−1	138	−138	840	−3	210	−70
580	−5	29	−145	840	−2	105	−210
584	−2	146	−73	876	−73	3	−219
616	−22	7	−154	876	−3	219	−73

continued on following page

Fields where $H \simeq \mathbb{Z}_4 \times \mathbb{Z}_2$

continued from previous page

f	d_1	d_2	d_3
876	−1	219	−219
903	−3	301	−903
915	−3	305	−915
924	−21	77	−33
924	−11	231	−21
952	−7	34	−238
984	−6	123	−82
1020	−51	15	−85
1020	−15	51	−85
1020	−5	255	−51
1020	−3	255	−85
1023	−11	93	−1023
1032	−6	43	−258
1045	−19	1045	−55
1060	−1	265	−265
1064	−14	38	−133
1092	−91	3	−273
1092	−91	39	−21
1092	−13	21	−273
1092	−13	273	−21
1092	−7	39	−273
1092	−1	273	−273
1128	−6	47	−282
1128	−2	282	−141
1128	−1	282	−282
1131	−3	377	−1131
1140	−5	285	−57
1144	−13	286	−22
1155	−35	33	−1155
1155	−15	77	−1155
1155	−7	165	−1155
1155	−3	385	−1155
1160	−5	290	−58
1164	−97	3	−291
1164	−3	291	−97

f	d_1	d_2	d_3
1240	−10	31	−310
1240	−5	62	−310
1240	−2	155	−310
1240	−1	310	−310
1288	−14	161	−46
1288	−7	322	−46
1292	−19	323	−17
1292	−17	19	−323
1292	−1	323	−323
1320	−22	165	−30
1320	−15	22	−330
1320	−15	330	−22
1320	−3	110	−330
1365	−91	105	−195
1365	−35	273	−195
1365	−15	1365	−91
1365	−7	1365	−195
1380	−15	23	−345
1380	−3	115	−345
1380	−1	345	−345
1416	−2	177	−354
1428	−51	7	−357
1428	−51	119	−21
1428	−7	51	−357
1428	−3	119	−357
1480	−10	370	−37
1496	−22	187	−34
1496	−22	374	−17
1540	−5	77	−385
1540	−1	385	−385
1544	−2	386	−193
1560	−130	6	−195
1560	−15	78	−130
1560	−10	78	−195
1560	−2	390	−195

continued on following page

Fields where $H \simeq \mathbb{Z}_4 \times \mathbb{Z}_2$

continued from previous page

f	d_1	d_2	d_3
1596	−57	21	−133
1608	−2	201	−402
1624	−58	203	−14
1635	−3	545	−1635
1640	−10	205	−82
1672	−22	19	−418
1704	−2	426	−213
1705	−55	341	−155
1716	−13	429	−33
1740	−5	87	−435
1752	−6	146	−219
1752	−3	146	−438
1752	−2	438	−219
1768	−2	221	−442
1785	−51	105	−595
1785	−15	357	−595
1820	−13	455	−35
1820	−5	455	−91
1848	−22	21	−462
1848	−11	42	−462
1848	−11	462	−42
1848	−6	77	−462
1848	−3	154	−462
1860	−5	465	−93
1876	−7	67	−469
1932	−1	483	−483
1992	−6	83	−498
1995	−35	57	−1995
1995	−7	285	−1995
2020	−5	101	−505
2024	−46	22	−253
2040	−10	510	−51
2072	−37	518	−14
2145	−195	33	−715
2145	−15	429	−715

f	d_1	d_2	d_3
2184	−91	42	−78
2184	−91	78	−42
2211	−11	201	−2211
2220	−15	555	−37
2220	−5	111	−555
2244	−187	51	−33
2244	−51	187	−33
2280	−30	57	−190
2280	−15	114	−190
2280	−6	285	−190
2280	−3	570	−190
2289	−3	2289	−763
2328	−6	194	−291
2328	−2	582	−291
2380	−85	7	−595
2380	−35	119	−85
2380	−7	595	−85
2380	−5	119	−595
2392	−13	46	−598
2392	−13	598	−46
2392	−2	299	−598
2392	−1	598	−598
2408	−2	602	−301
2415	−115	105	−483
2415	−35	345	−483
2415	−15	805	−483
2508	−57	11	−627
2508	−33	19	−627
2508	−33	209	−57
2508	−19	627	−33
2508	−11	627	−57
2552	−22	319	−58
2584	−34	38	−323
2604	−7	651	−93
2639	−91	377	−203

continued on following page

Fields where $H \simeq \mathbb{Z}_4 \times \mathbb{Z}_2$

continued from previous page

f	d_1	d_2	d_3
2660	−35	95	−133
2660	−5	665	−133
2760	−115	138	−30
2760	−6	690	−115
2805	−15	2805	−187
2840	−10	355	−142
2840	−5	710	−142
2849	−11	2849	−259
2856	−51	238	−42
2860	−13	55	−715
2892	−1	723	−723
2924	−43	731	−17
2945	−19	2945	−155
2964	−13	741	−57
2968	−2	371	−742
2968	−1	742	−742
3003	−91	33	−3003
3003	−11	273	−3003
3003	−7	429	−3003
3016	−13	754	−58
3036	−3	759	−253
3080	−22	385	−70
3081	−39	237	−1027
3144	−2	393	−786
3171	−3	1057	−3171
3172	−13	61	−793
3180	−5	159	−795
3192	−19	798	−42
3336	−2	417	−834
3355	−11	305	−3355
3432	−22	429	−78
3432	−11	858	−78
3444	−123	287	−21
3480	−58	30	−435
3480	−10	174	−435
3540	−15	295	−177
3540	−5	885	−177
3556	−7	127	−889
3608	−22	451	−82
3612	−43	903	−21
3615	−15	1205	−723
3640	−91	70	−130
3640	−35	182	−130
3740	−187	55	−85
3752	−7	134	−938
3815	−35	545	−763
3819	−3	1273	−3819
3864	−2	966	−483
3905	−55	781	−355
3905	−11	3905	−355
3976	−14	497	−142
4012	−17	59	−1003
4092	−33	341	−93
4161	−219	57	−1387
4228	−7	151	−1057
4268	−11	1067	−97
4305	−123	105	−1435
4305	−15	861	−1435
4389	−7	4389	−627
4440	−10	222	−555
4440	−2	1110	−555
4488	−187	66	−102
4488	−22	561	−102
4669	−203	161	−667
4683	−3	1561	−4683
4760	−10	238	−595
4836	−403	39	−93
4836	−13	1209	−93
4872	−58	609	−42
4956	−21	413	−177

continued on following page

Fields where $H \simeq \mathbb{Z}_4 \times \mathbb{Z}_2$

continued from previous page

f	d_1	d_2	d_3
4972	−1	1243	−1243
5005	−91	385	−715
5005	−35	1001	−715
5005	−7	5005	−715
5016	−22	114	−627
5016	−6	418	−627
5060	−115	55	−253
5060	−5	1265	−253
5124	−427	183	−21
5180	−35	1295	−37
5320	−70	133	−190
5320	−35	266	−190
5336	−58	667	−46
5412	−123	451	−33
5548	−73	19	−1387
5548	−19	1387	−73
5565	−35	1113	−795
5628	−67	1407	−21
5640	−235	282	−30
5640	−6	1410	−235
5644	−17	83	−1411
5676	−43	1419	−33
5720	−130	22	−715
5720	−10	286	−715
5720	−2	1430	−715
5784	−6	482	−723
5848	−43	1462	−34
5852	−11	1463	−133
5896	−2	737	−1474
5907	−11	537	−5907
5908	−7	211	−1477
5928	−19	1482	−78
5992	−7	214	−1498
6028	−1	1507	−1507
6045	−403	465	−195
6135	−15	2045	−1227
6324	−51	527	−93
6360	−10	318	−795
6424	−22	1606	−73
6545	−187	385	−595
6545	−35	6545	−187
6745	−19	6745	−355
6792	−2	849	−1698
6916	−91	247	−133
6923	−7	989	−6923
7068	−19	1767	−93
7347	−3	2449	−7347
7368	−2	921	−1842
7476	−267	623	−21
7707	−3	2569	−7707
7820	−115	391	−85
7843	−11	713	−7843
7923	−3	2641	−7923
8008	−22	2002	−91
8024	−34	118	−1003
8024	−2	2006	−1003
8060	−5	2015	−403
8211	−51	2737	−483
8283	−11	753	−8283
8436	−37	2109	−57
8540	−5	2135	−427
8584	−37	2146	−58
8643	−3	2881	−8643
8968	−2	1121	−2242
9112	−67	2278	−34
9165	−235	1833	−195
9204	−13	2301	−177
9219	−3	3073	−9219
9348	−123	779	−57
9672	−403	186	−78

continued on following page

Fields where $H \simeq \mathbb{Z}_4 \times \mathbb{Z}_2$

continued from previous page

f	d_1	d_2	d_3
9816	−2	2454	−1227
9944	−2	2486	−1243
10227	−3	3409	−10227
10248	−427	366	−42
10465	−91	10465	−115
10659	−187	969	−627
10659	−51	3553	−627
11005	−155	2201	−355
11193	−91	11193	−123
11288	−34	166	−1411
11288	−2	2822	−1411
11832	−51	2958	−58
11960	−115	598	−130
11985	−51	11985	−235
12040	−43	3010	−70
12220	−13	3055	−235
12513	−43	12513	−291
12747	−3	4249	−12747
12765	−115	2553	−555
13064	−46	1633	−142
13065	−67	13065	−195
13156	−13	3289	−253
13452	−57	1121	−177
13468	−37	3367	−91
13601	−67	13601	−203
13685	−115	2737	−595
13692	−163	3423	−21
13764	−37	3441	−93
14105	−35	14105	−403
14673	−67	14673	−219
14952	−267	1246	−42
16027	−11	1457	−16027
16120	−403	310	−130
16120	−10	4030	−403
16445	−115	3289	−715

f	d_1	d_2	d_3
16472	−58	2059	−142
16984	−22	4246	−193
17020	−37	4255	−115
17105	−55	3421	−1555
17105	−11	17105	−1555
17204	−187	391	−253
17227	−7	2461	−17227
17347	−19	913	−17347
18285	−115	3657	−795
18403	−7	2629	−18403
18760	−67	4690	−70
19057	−323	1121	−1003
19564	−67	4891	−73
19684	−37	4921	−133
20292	−267	1691	−57
20445	−235	4089	−435
20769	−43	20769	−483
20904	−67	5226	−78
21016	−37	5254	−142
21777	−51	21777	−427
21948	−93	1829	−177
22204	−13	5551	−427
22876	−43	5719	−133
23865	−43	23865	−555
24440	−235	1222	−130
24924	−67	6231	−93
25619	−187	2329	−1507
25899	−267	8633	−291
26809	−323	1577	−1411
26809	−19	26809	−1411
26961	−43	26961	−627
27965	−235	5593	−595
28536	−58	7134	−123
28633	−19	28633	−1507
29463	−427	4209	−483

continued on following page

Fields where $H \simeq \mathbb{Z}_4 \times \mathbb{Z}_2$

continued from previous page

f	d_1	d_2	d_3
29545	−19	29545	−1555
29605	−155	5921	−955
30444	−43	7611	−177
32361	−67	32361	−483
32452	−427	1159	−133
32680	−43	8170	−190
32809	−43	32809	−763
33108	−267	2759	−93
33605	−235	6721	−715
34780	−37	8695	−235
35464	−22	8866	−403
35644	−67	8911	−133

f	d_1	d_2	d_3
83249	−1003	4897	−1411
86716	−163	21679	−133
94537	−67	94537	−1411
94705	−235	94705	−403
99064	−58	24766	−427
107601	−267	107601	−403
109203	−267	36401	−1227
110405	−355	22081	−1555
114009	−267	114009	−427
116545	−163	116545	−715
164956	−163	41239	−253
297005	−955	59401	−1555

Table 6.5: Fields where $H \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

f	d_1	d_2	d_3
168	−21	2	−42
264	−2	66	−33
408	−2	51	−102
408	−1	102	−102
456	−2	114	−57
520	−1	130	−130
660	−11	15	−165
680	−2	170	−85
780	−13	15	−195
780	−1	195	−195
820	−5	41	−205
840	−7	210	−30
915	−15	61	−915
924	−7	231	−33
1092	−3	91	−273
1140	−15	95	−57
1155	−11	105	−1155
1320	−11	30	−330
1320	−11	330	−30
1320	−10	33	−330
1320	−2	165	−330
1380	−115	3	−345
1416	−2	354	−177
1428	−1	357	−357
1443	−3	481	−1443
1540	−35	11	−385
1540	−11	35	−385
1540	−7	55	−385
1560	−3	390	−130
1596	−21	57	−133
1596	−21	133	−57
1635	−15	109	−1635
1740	−1	435	−435
1752	−6	73	−438
1780	−5	89	−445
1848	−7	66	−462
1860	−15	155	−93
1932	−21	23	−483
1995	−19	105	−1995
1995	−15	133	−1995
1995	−3	665	−1995
2035	−11	185	−2035
2145	−11	2145	−195
2145	−3	2145	−715
2184	−7	546	−78
2184	−6	546	−91
2220	−37	15	−555
2220	−1	555	−555
2280	−19	570	−30
2380	−1	595	−595
2460	−5	615	−123
2604	−21	217	−93
2715	−15	181	−2715
2856	−7	714	−102
2860	−5	143	−715
2860	−1	715	−715
2968	−7	106	−742
3003	−3	1001	−3003
3036	−33	69	−253
3045	−35	609	−435
3045	−7	3045	−435
3080	−22	770	−35
3080	−11	770	−70
3180	−1	795	−795
3705	−19	3705	−195
3740	−11	935	−85
3740	−5	935	−187
3864	−42	46	−483
3885	−7	3885	−555
4760	−2	1190	−595

continued on following page

Fields where $H \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

continued from previous page

f	d_1	d_2	d_3
4785	−11	4785	−435
4920	−10	1230	−123
5313	−11	5313	−483
5320	−19	1330	−70
5320	−7	1330	−190
5340	−5	1335	−267
5565	−7	5565	−795
5720	−11	1430	−130
5740	−5	287	−1435
5740	−1	1435	−1435
6045	−15	6045	−403
6405	−15	6405	−427
6545	−11	6545	−595
7068	−57	589	−93
7480	−10	1870	−187
7548	−37	1887	−51
7752	−19	1938	−102
7788	−33	649	−177
8120	−35	2030	−58
8265	−19	8265	−435
8360	−11	2090	−190
8385	−43	8385	−195
9177	−19	9177	−483
9672	−6	2418	−403
9724	−13	2431	−187
9880	−19	2470	−130
10005	−115	2001	−435

f	d_1	d_2	d_3
14145	−115	14145	−123
14620	−43	3655	−85
15105	−19	15105	−795
15785	−11	15785	−1435
17544	−43	4386	−102
20553	−51	20553	−403
20680	−22	5170	−235
21112	−58	5278	−91
22360	−43	5590	−130
22780	−67	5695	−85
23496	−22	5874	−267
25585	−43	25585	−595
27265	−19	27265	−1435
27676	−37	6919	−187
29145	−67	29145	−435
30745	−43	30745	−715
34185	−43	34185	−795
37185	−67	37185	−555
39516	−37	9879	−267
39865	−67	39865	−595
43945	−187	43945	−235
49569	−123	49569	−403
49929	−187	49929	−267
52521	−123	52521	−427
61705	−43	61705	−1435
100345	−235	100345	−427
233905	−163	233905	−1435

Table 6.6: Fields where $H \simeq \mathbb{Z}_{16}$

f	d_1	d_2	d_3
555	−555	5	−111
555	−15	185	−111
740	−37	5	−185
740	−5	37	−185
760	−190	2	−95
888	−6	74	−111
1464	−6	122	−183
1495	−115	65	−299
1628	−37	11	−407
1628	−1	407	−407
1855	−35	265	−371
1880	−235	10	−94
1884	−1	471	−471
1887	−51	629	−111
1924	−37	13	−481
1924	−13	37	−481
2132	−13	533	−41
2236	−13	43	−559
2236	−1	559	−559
2260	−5	565	−113
2397	−51	141	−799
2397	−3	2397	−799
2685	−15	537	−895
2685	−3	2685	−895
2740	−5	685	−137
2829	−123	69	−943
2829	−3	2829	−943
2895	−15	965	−579
3048	−6	381	−254
3048	−3	762	−254
3084	−3	771	−257
3111	−51	1037	−183
3140	−5	157	−785
3224	−403	26	−62
3256	−2	814	−407
3297	−7	3297	−471
3417	−51	201	−1139
3417	−3	3417	−1139
3460	−5	173	−865
3580	−5	179	−895
3768	−2	942	−471
3832	−1	958	−958
3913	−91	301	−559
3913	−7	3913	−559
4088	−7	1022	−146
4137	−3	4137	−1379
4188	−1	1047	−1047
4236	−3	1059	−353
4472	−2	1118	−559
4520	−10	565	−226
4636	−1	1159	−1159
4664	−22	106	−583
4695	−15	1565	−939
5516	−1	1379	−1379
5593	−7	5593	−799
5668	−13	109	−1417
5752	−1	1438	−1438
5860	−5	293	−1465
6149	−11	6149	−559
6168	−3	1542	−514
6260	−5	1565	−313
6265	−35	1253	−895
6265	−7	6265	−895
6285	−3	6285	−2095
6424	−11	1606	−146
6495	−15	2165	−1299
6585	−15	1317	−2195
6585	−3	6585	−2195
6740	−5	1685	−337
7124	−13	1781	−137

continued on following page

Fields where $H \simeq \mathbb{Z}_{16}$

continued from previous page

f	d_1	d_2	d_3
7160	−10	358	−895
7160	−2	1790	−895
7196	−7	1799	−257
7288	−1	1822	−1822
7329	−7	7329	−1047
7733	−19	7733	−407
7932	−1	1983	−1983
7973	−7	7973	−1139
8113	−427	133	−1159
8376	−2	2094	−1047
8380	−5	419	−2095
8380	−1	2095	−2095
8655	−15	2885	−1731
8780	−1	2195	−2195
8789	−187	517	−799
9140	−5	2285	−457
9272	−2	2318	−1159
9320	−10	1165	−466
9695	−35	1385	−1939
9879	−267	3293	−111
9884	−7	2471	−353
10373	−11	10373	−943
10621	−19	10621	−559
11176	−22	1397	−254
11176	−11	2794	−254
11452	−7	2863	−409
11496	−6	1437	−958
11496	−3	2874	−958
11517	−11	11517	−1047
11540	−5	2885	−577
12252	−1	3063	−3063
12529	−187	737	−1139
12529	−11	12529	−1139
12749	−11	12749	−1159
13055	−35	1865	−2611

f	d_1	d_2	d_3
13137	−3	13137	−4379
13281	−3	13281	−4427
13881	−7	13881	−1983
14665	−35	2933	−2095
14665	−7	14665	−2095
14817	−3	14817	−4939
15169	−11	15169	−1379
15365	−35	3073	−2195
15365	−7	15365	−2195
15585	−15	3117	−5195
15585	−3	15585	−5195
15963	−51	5321	−939
16089	−3	16089	−5363
16172	−13	311	−4043
16172	−1	4043	−4043
16287	−267	5429	−183
16724	−37	4181	−113
16760	−10	838	−2095
16760	−2	4190	−2095
17256	−6	2157	−1438
17256	−3	4314	−1438
17329	−403	1333	−559
17501	−43	17501	−407
17516	−1	4379	−4379
17560	−10	878	−2195
17560	−2	4390	−2195
17917	−19	17917	−943
17996	−11	4499	−409
18748	−1	4687	−4687
19185	−15	3837	−6395
19185	−3	19185	−6395
19304	−19	4826	−254
19532	−19	4883	−257
19677	−7	19677	−2811
19785	−3	19785	−6595

continued on following page

Fields where $H \simeq \mathbb{Z}_{16}$

continued from previous page

f	d_1	d_2	d_3
20253	−43	20253	−471
20585	−115	4117	−895
20648	−58	2581	−178
20780	−5	1039	−5195
20780	−1	5195	−5195
21009	−3	21009	−7003
21153	−3	21153	−7051
21441	−7	21441	−3063
21452	−1	5363	−5363
21585	−15	4317	−7195
21585	−3	21585	−7195
21864	−6	2733	−1822
21864	−3	5466	−1822
22209	−11	22209	−2019
22631	−427	3233	−371
23385	−15	4677	−7795
23385	−3	23385	−7795
23739	−123	7913	−579
23764	−13	5941	−457
24101	−7	24101	−3443
24249	−3	24249	−8083
24504	−2	6126	−3063
25112	−43	6278	−146
25580	−5	1279	−6395
26216	−58	3277	−226
26380	−5	1319	−6595
26380	−1	6595	−6595
27032	−2	6758	−3379
27624	−6	3453	−2302
27624	−3	6906	−2302
28012	−1	7003	−7003
28301	−7	28301	−4043
28780	−5	1439	−7195
28780	−1	7195	−7195
29197	−7	29197	−4171
30004	−13	7501	−577
30585	−15	6117	−10195
30585	−3	30585	−10195
30651	−51	10217	−1803
30921	−11	30921	−2811
30936	−3	7734	−2578
30988	−1	7747	−7747
31084	−19	7771	−409
31180	−5	1559	−7795
31180	−1	7795	−7795
31276	−1	7819	−7819
32284	−7	8071	−1153
32344	−2	8086	−4043
33609	−51	1977	−11203
33609	−3	33609	−11203
34536	−6	4317	−2878
34536	−3	8634	−2878
34797	−7	34797	−4971
35032	−58	302	−4379
35032	−2	8758	−4379
35544	−3	8886	−2962
35985	−15	7197	−11995
35985	−3	35985	−11995
36365	−7	36365	−5195
37496	−2	9374	−4687
37677	−19	37677	−1983
38311	−91	5473	−2947
38361	−19	38361	−2019
38485	−43	38485	−895
38499	−123	12833	−939
39052	−13	751	−9763
39052	−1	9763	−9763
39369	−3	39369	−13123
40780	−5	2039	−10195
40780	−1	10195	−10195

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Fields where $H \simeq \mathbb{Z}_{16}$

continued from previous page

f	d_1	d_2	d_3
41560	−10	2078	−5195
41560	−2	10390	−5195
42065	−235	8413	−895
42904	−2	10726	−5363
43372	−1	10843	−10843
43688	−43	10922	−254
44204	−43	11051	−257
44428	−1	11107	−11107
44473	−11	44473	−4043
44716	−1	11179	−11179
44765	−35	8953	−6395
44765	−7	44765	−6395
44985	−15	8997	−14995
44985	−3	44985	−14995
45021	−43	45021	−1047
46165	−35	9233	−6595
46165	−7	46165	−6595
46324	−37	11581	−313
46396	−7	11599	−1657
47980	−1	11995	−11995
48169	−11	48169	−4379
48185	−115	9637	−2095
48697	−11	48697	−4427
49231	−91	7033	−3787
49676	−11	12419	−1129
49837	−43	49837	−1159
50092	−1	12523	−12523
50365	−35	10073	−7195
50365	−7	50365	−7195
50485	−115	10097	−2195
51160	−10	2558	−6395
51160	−2	12790	−6395
51531	−267	17177	−579
51557	−11	51557	−4687
51969	−51	3057	−17323

f	d_1	d_2	d_3
51969	−3	51969	−17323
52760	−2	13190	−6595
52844	−11	13211	−1201
54028	−13	1039	−13507
54028	−1	13507	−13507
54229	−427	889	−7747
54229	−7	54229	−7747
54565	−35	10913	−7795
54565	−7	54565	−7795
54681	−11	54681	−4971
55276	−13	1063	−13819
56024	−2	14006	−7003
56649	−3	56649	−18883
57441	−123	1401	−19147
57441	−3	57441	−19147
57484	−1	14371	−14371
58197	−19	58197	−3063
58993	−11	58993	−5363
59164	−7	14791	−2113
59433	−11	59433	−5403
59965	−67	59965	−895
59980	−5	2999	−14995
59980	−1	14995	−14995
61185	−15	12237	−20395
61185	−3	61185	−20395
61976	−2	15494	−7747
62360	−10	3118	−7795
62360	−2	15590	−7795
62552	−2	15638	−7819
63181	−67	63181	−943
63272	−22	7909	−1438
63272	−11	15818	−1438
64201	−19	64201	−3379
64209	−51	3777	−21403
64209	−3	64209	−21403

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Fields where $H \simeq \mathbb{Z}_{16}$

continued from previous page

f	d_1	d_2	d_3
66341	−163	66341	−407
66549	−7	66549	−9507
67371	−51	22457	−3963
67636	−37	16909	−457
68072	−67	17018	−254
68341	−91	5257	−9763
68341	−7	68341	−9763
68529	−3	68529	−22843
70185	−15	14037	−23395
70185	−3	70185	−23395
70971	−123	23657	−1731
71365	−35	14273	−10195
71365	−7	71365	−10195
72808	−19	18202	−958
73209	−3	73209	−24403
73923	−123	24641	−1803
75081	−3	75081	−25027
75801	−3	75801	−25267
76741	−7	76741	−10963
76773	−163	76773	−471
76817	−19	76817	−4043
77033	−11	77033	−7003
78104	−2	19526	−9763
78421	−7	78421	−11203
80168	−22	10021	−1822
80168	−11	20042	−1822
81560	−10	4078	−10195
81560	−2	20390	−10195
81580	−5	4079	−20395
81580	−1	20395	−20395
83571	−267	27857	−939
83965	−35	16793	−11995
83965	−7	83965	−11995
85217	−11	85217	−7747
85396	−37	21349	−577
85804	−19	21451	−1129
86009	−11	86009	−7819
86744	−2	21686	−10843
86817	−43	86817	−2019
86889	−3	86889	−28963
87628	−19	21907	−1153
88856	−58	766	−11107
88856	−2	22214	−11107
89053	−19	89053	−4687
89432	−2	22358	−11179
90085	−43	90085	−2095
91117	−163	91117	−559
91289	−11	91289	−8299
91372	−1	22843	−22843
91861	−7	91861	−13123
92393	−67	92393	−1379
92504	−2	23126	−11563
93261	−7	93261	−13323
93580	−5	4679	−23395
93580	−1	23395	−23395
94348	−1	23587	−23587
94385	−43	94385	−2195
94549	−7	94549	−13507
94604	−67	23651	−353
95192	−163	23798	−146
95513	−11	95513	−8683
95960	−10	4798	−11995
95960	−2	23990	−11995
96733	−7	96733	−13819
97612	−1	24403	−24403
100108	−1	25027	−25027
100184	−2	25046	−12523
101288	−22	12661	−2302
101288	−11	25322	−2302
101897	−19	101897	−5363

continued on following page

Fields where $H \simeq \mathbb{Z}_{16}$

continued from previous page

f	d_1	d_2	d_3
102657	−19	102657	−5403
103165	−235	20633	−2195
104965	−35	20993	−14995
104965	−7	104965	−14995
107821	−7	107821	−15403
108056	−2	27014	−13507
109288	−19	27322	−1438
109612	−67	27403	−409
110552	−2	27638	−13819
111148	−37	751	−27787
111148	−1	27787	−27787
113432	−11	28358	−2578
114429	−7	114429	−16347
114968	−2	28742	−14371
115611	−267	38537	−1299
115717	−7	115717	−16531
118279	−427	16897	−1939
119960	−10	5998	−14995
119960	−2	29990	−14995
120873	−43	120873	−2811
121261	−7	121261	−17323
122177	−11	122177	−11107
122969	−11	122969	−11179
123233	−11	123233	−11203
124376	−2	31094	−15547
124972	−1	31243	−31243
125333	−403	9641	−4043
126632	−22	15829	−2878
126632	−11	31658	−2878
127193	−11	127193	−11563
130328	−11	32582	−2962
131709	−43	131709	−3063
132248	−2	33062	−16531
133784	−2	33446	−16723
134029	−7	134029	−19147

f	d_1	d_2	d_3
137753	−11	137753	−12523
138472	−19	34618	−1822
140365	−67	140365	−2095
142765	−35	28553	−20395
145885	−163	145885	−895
146553	−11	146553	−13323
147065	−67	147065	−2195
147085	−115	29417	−6395
148049	−43	148049	−3443
148561	−19	148561	−7819
148577	−11	148577	−13507
149821	−7	149821	−21403
151064	−2	37766	−18883
151685	−115	30337	−6595
152009	−11	152009	−13819
154059	−267	51353	−1731
157681	−19	157681	−8299
158081	−11	158081	−14371
159271	−427	22753	−2611
163160	−10	8158	−20395
163160	−2	40790	−20395
163765	−35	32753	−23395
163765	−7	163765	−23395
164776	−43	41194	−958
165109	−7	165109	−23587
167564	−163	41891	−257
168321	−19	168321	−8859
169433	−11	169433	−15403
170661	−163	170661	−1047
171017	−11	171017	−15547
174952	−19	43738	−2302
179285	−115	35857	−7795
179767	−427	25681	−2947
180633	−19	180633	−9507
180652	−19	45163	−2377

continued on following page

Fields where $H \simeq \mathbb{Z}_{16}$

continued from previous page

f	d_1	d_2	d_3
181841	−11	181841	−16531
182744	−2	45686	−22843
183953	−11	183953	−16723
185497	−19	185497	−9763
187160	−10	9358	−23395
188337	−67	188337	−2811
190361	−43	190361	−4427
190553	−11	190553	−17323
195224	−2	48806	−24403
200216	−58	1726	−25027
200216	−2	50054	−25027
206017	−19	206017	−10843
207713	−11	207713	−18883
210617	−11	210617	−19147
211033	−19	211033	−11107
212377	−43	212377	−4939
218701	−7	218701	−31243
218728	−19	54682	−2878
219697	−19	219697	−11563
222296	−2	55574	−27787
223385	−43	223385	−5195
226393	−67	226393	−3379
230156	−163	57539	−353
231007	−427	33001	−3787
235433	−187	13849	−21403
237937	−19	237937	−12523
244165	−235	48833	−5195
247336	−43	61834	−1438
249337	−19	249337	−13123
253137	−19	253137	−13323
256633	−19	256633	−13507
256744	−67	64186	−958
270881	−67	270881	−4043
275297	−11	275297	−25027
275885	−115	55177	−11995
283585	−43	283585	−6595
296609	−67	296609	−4427
300565	−235	60113	−6395
301129	−43	301129	−7003
302572	−67	75643	−1129
302653	−403	23281	−9763
309385	−43	309385	−7195
309965	−235	61993	−6595
310593	−19	310593	−16347
313384	−43	78346	−1822
314029	−67	314029	−4687
317737	−19	317737	−16723
333121	−43	333121	−7747
335128	−163	83782	−514
335185	−43	335185	−7795
338165	−235	67633	−7195
341485	−163	341485	−2095
344885	−115	68977	−14995
347569	−43	347569	−8083
348065	−67	348065	−5195
352707	−267	117569	−3963
357785	−163	357785	−2195
363793	−19	363793	−19147
366365	−235	73273	−7795
367564	−43	91891	−2137
373369	−43	373369	−8683
385384	−67	96346	−1438
395944	−43	98986	−2302
408801	−43	408801	−9507
408844	−43	102211	−2377
418717	−403	32209	−13507
434017	−19	434017	−22843
438385	−43	438385	−10195
443416	−43	110854	−2578
455241	−43	455241	−10587

continued on following page

Fields where $H \simeq \mathbb{Z}_{16}$

continued from previous page

f	d_1	d_2	d_3
458193	−163	458193	−2811
471409	−43	471409	−10963
472417	−67	472417	−7051
475513	−19	475513	−25027
488296	−67	122074	−1822
495016	−43	123754	−2878
497209	−43	497209	−11563
515785	−43	515785	−11995
527953	−19	527953	−27787
538085	−115	107617	−23395
538489	−43	538489	−12523
541561	−67	541561	−8083
550777	−163	550777	−3379
563765	−235	112753	−11995
564289	−43	564289	−13123
572889	−43	572889	−13323
593553	−67	593553	−8859
617953	−43	617953	−14371
624616	−163	156154	−958
636969	−67	636969	−9507
644785	−43	644785	−14995
654121	−67	654121	−9763
659009	−163	659009	−4043
668521	−43	668521	−15547
704765	−235	140953	−14995
713777	−163	713777	−4379
721601	−163	721601	−4427
726481	−67	726481	−10843
734521	−67	734521	−10963
751756	−163	187939	−1153
771304	−67	192826	−2878
774721	−67	774721	−11563
793816	−67	198454	−2962

f	d_1	d_2	d_3
876985	−43	876985	−20395
880689	−163	880689	−5403
904969	−67	904969	−13507
937576	−163	234394	−1438
958565	−235	191713	−20395
962857	−67	962857	−14371
1004665	−67	1004665	−14995
1005985	−43	1005985	−23395
1042385	−163	1042385	−6395
1074985	−163	1074985	−6595
1080364	−163	270091	−1657
1086481	−43	1086481	−25267
1120441	−67	1120441	−16723
1141489	−163	1141489	−7003
1172785	−163	1172785	−7195
1274497	−163	1274497	−7819
1282849	−67	1282849	−19147
1377676	−163	344419	−2113
1444017	−163	1444017	−8859
1530481	−67	1530481	−22843
1549641	−163	1549641	−9507
1567465	−67	1567465	−23395
1661785	−163	1661785	−10195
1680856	−163	420214	−2578
1725681	−163	1725681	−10587
1810441	−163	1810441	−11107
1955185	−163	1955185	−11995
2041249	−163	2041249	−12523
2097484	−163	524371	−3217
2139049	−163	2139049	−13123
2664561	−163	2664561	−16347
2725849	−163	2725849	−16723
3844681	−163	3844681	−23587

Table 6.7: Fields where $H \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$

f	d_1	d_2	d_3
264	−33	2	−66
264	−1	66	−66
280	−5	70	−14
328	−82	2	−41
328	−2	82	−41
456	−57	2	−114
456	−1	114	−114
520	−130	2	−65
520	−2	130	−65
580	−1	145	−145
663	−3	221	−663
680	−10	170	−17
680	−5	170	−34
712	−1	178	−178
728	−13	182	−14
740	−1	185	−185
744	−93	6	−62
777	−259	21	−111
780	−15	39	−65
780	−5	195	−39
884	−13	221	−17
884	−1	221	−221
897	−39	69	−299
915	−15	305	−183
920	−5	230	−46
924	−33	21	−77
924	−21	33	−77
924	−3	231	−77
984	−6	246	−41
1032	−1	258	−258
1045	−55	209	−95
1096	−2	274	−137
1140	−5	57	−285
1140	−3	95	−285
1140	−1	285	−285

f	d_1	d_2	d_3
1155	−55	21	−1155
1160	−58	10	−145
1160	−10	58	−145
1160	−2	290	−145
1220	−5	61	−305
1221	−11	1221	−111
1240	−10	155	−62
1240	−5	310	−62
1320	−55	6	−330
1320	−15	110	−66
1320	−10	165	−66
1416	−6	59	−354
1428	−17	21	−357
1463	−19	77	−1463
1508	−13	29	−377
1540	−35	55	−77
1540	−5	385	−77
1551	−11	141	−1551
1596	−19	399	−21
1596	−7	399	−57
1608	−6	67	−402
1624	−58	7	−406
1624	−2	203	−406
1624	−1	406	−406
1640	−2	205	−410
1672	−1	418	−418
1705	−11	1705	−155
1716	−13	33	−429
1716	−11	39	−429
1716	−1	429	−429
1736	−14	217	−62
1740	−15	87	−145
1740	−3	435	−145
1767	−3	589	−1767
1785	−3	1785	−595

continued on following page

Fields where $H \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$

continued from previous page

f	d_1	d_2	d_3
1820	−7	455	−65
1848	−42	77	−66
1848	−14	33	−462
1848	−3	462	−154
1860	−5	93	−465
1860	−3	155	−465
1860	−1	465	−465
1864	−1	466	−466
1924	−1	481	−481
1932	−21	161	−69
1992	−1	498	−498
2020	−1	505	−505
2067	−3	689	−2067
2072	−37	14	−518
2072	−2	259	−518
2072	−1	518	−518
2120	−2	530	−265
2136	−6	267	−178
2244	−187	3	−561
2244	−51	11	−561
2244	−11	51	−561
2244	−3	187	−561
2248	−1	562	−562
2316	−1	579	−579
2328	−3	194	−582
2343	−11	213	−2343
2436	−3	203	−609
2436	−1	609	−609
2460	−123	15	−205
2460	−15	123	−205
2460	−3	615	−205
2504	−2	626	−313
2568	−6	107	−642
2580	−15	43	−645
2580	−3	215	−645

f	d_1	d_2	d_3
2580	−1	645	−645
2584	−19	34	−646
2584	−19	646	−34
2604	−93	21	−217
2604	−21	93	−217
2604	−1	651	−651
2632	−14	329	−94
2712	−6	339	−226
2712	−6	678	−113
2715	−3	905	−2715
2760	−115	30	−138
2760	−10	69	−690
2760	−10	345	−138
2760	−3	230	−690
2760	−2	345	−690
2856	−102	21	−238
2856	−51	42	−238
2856	−51	714	−14
2856	−6	357	−238
2856	−3	714	−238
2860	−13	715	−55
2945	−155	589	−95
2980	−5	149	−745
2996	−7	107	−749
3003	−39	77	−3003
3036	−253	33	−69
3036	−33	253	−69
3045	−15	3045	−203
3135	−15	1045	−627
3144	−6	131	−786
3160	−10	395	−158
3160	−5	158	−790
3160	−5	790	−158
3160	−2	395	−790
3160	−1	790	−790

continued on following page

Fields where $H \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$

continued from previous page

f	d_1	d_2	d_3
3172	−1	793	−793
3180	−3	795	−265
3192	−7	798	−114
3201	−11	3201	−291
3220	−115	7	−805
3220	−1	805	−805
3245	−55	649	−295
3245	−11	3245	−295
3255	−35	465	−651
3255	−15	1085	−651
3288	−6	822	−137
3336	−6	139	−834
3460	−1	865	−865
3477	−19	3477	−183
3608	−22	902	−41
3612	−21	129	−301
3640	−7	910	−130
3656	−2	914	−457
3752	−2	938	−469
3752	−1	938	−938
3756	−1	939	−939
3784	−22	43	−946
3795	−115	33	−3795
3795	−15	253	−3795
3795	−3	1265	−3795
3828	−11	87	−957
3828	−1	957	−957
3836	−7	959	−137
3864	−483	14	−138
3864	−42	69	−322
3864	−42	161	−138
3864	−6	322	−483
3864	−3	966	−322
3880	−10	970	−97
3885	−35	777	−555
3885	−15	3885	−259
3916	−1	979	−979
3944	−58	986	−17
3948	−21	329	−141
3948	−7	987	−141
3948	−1	987	−987
4008	−6	167	−1002
4008	−6	334	−501
4008	−2	1002	−501
4008	−1	1002	−1002
4017	−39	309	−1339
4017	−3	4017	−1339
4020	−15	67	−1005
4020	−3	335	−1005
4020	−1	1005	−1005
4040	−10	202	−505
4040	−2	1010	−505
4044	−3	1011	−337
4053	−7	4053	−579
4081	−11	4081	−371
4081	−7	4081	−583
4161	−19	4161	−219
4260	−15	71	−1065
4260	−15	355	−213
4260	−5	213	−1065
4260	−5	1065	−213
4260	−3	355	−1065
4260	−1	1065	−1065
4264	−13	1066	−82
4296	−2	537	−1074
4305	−35	4305	−123
4305	−3	4305	−1435
4340	−35	155	−217
4340	−5	1085	−217
4345	−55	869	−395

continued on following page

Fields where $H \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$

continued from previous page

f	d_1	d_2	d_3
4380	−15	1095	−73
4380	−5	1095	−219
4424	−14	553	−158
4452	−3	371	−1113
4452	−1	1113	−1113
4485	−115	897	−195
4485	−115	4485	−39
4488	−187	6	−1122
4488	−187	102	−66
4488	−51	22	−1122
4488	−51	374	−66
4488	−22	1122	−51
4488	−11	102	−1122
4488	−11	1122	−102
4488	−6	1122	−187
4488	−3	374	−1122
4515	−43	105	−4515
4515	−35	129	−4515
4515	−7	645	−4515
4551	−123	1517	−111
4584	−6	382	−573
4584	−2	1146	−573
4632	−6	386	−579
4669	−7	4669	−667
4692	−51	391	−69
4712	−2	1178	−589
4712	−1	1178	−1178
4740	−5	237	−1185
4740	−3	395	−1185
4740	−1	1185	−1185
4920	−15	1230	−82
4935	−235	105	−987
4935	−15	1645	−987
4953	−39	381	−1651
4956	−7	1239	−177

f	d_1	d_2	d_3
4972	−11	1243	−113
4984	−7	178	−1246
5012	−7	179	−1253
5016	−627	6	−418
5016	−627	22	−114
5016	−627	38	−66
5016	−19	1254	−66
5016	−11	1254	−114
5016	−3	1254	−418
5032	−37	1258	−34
5064	−2	633	−1266
5160	−43	30	−1290
5160	−43	1290	−30
5160	−15	86	−1290
5160	−15	430	−258
5160	−10	645	−258
5160	−3	430	−1290
5180	−5	1295	−259
5187	−91	57	−5187
5187	−19	273	−5187
5187	−7	741	−5187
5192	−2	649	−1298
5208	−42	62	−651
5208	−2	1302	−651
5215	−35	745	−1043
5236	−187	119	−77
5244	−57	437	−69
5244	−19	1311	−69
5313	−483	33	−1771
5313	−3	5313	−1771
5340	−267	15	−445
5340	−15	267	−445
5340	−3	1335	−445
5380	−5	269	−1345
5412	−123	11	−1353

continued on following page

Fields where $H \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$

continued from previous page

f	d_1	d_2	d_3
5412	−11	123	−1353
5412	−3	451	−1353
5484	−3	1371	−457
5592	−6	699	−466
5605	−19	5605	−295
5640	−235	30	−282
5640	−10	705	−282
5668	−1	1417	−1417
5704	−46	713	−62
5736	−6	478	−717
5736	−2	1434	−717
5740	−35	287	−205
5740	−7	1435	−205
5820	−15	1455	−97
5820	−5	1455	−291
5848	−43	34	−1462
5852	−133	209	−77
5860	−1	1465	−1465
5883	−3	1961	−5883
5896	−22	67	−1474
5943	−3	1981	−5943
5964	−21	497	−213
5980	−115	299	−65
5992	−1	1498	−1498
5995	−11	545	−5995
6028	−11	1507	−137
6040	−10	151	−1510
6040	−5	302	−1510
6040	−2	755	−1510
6040	−1	1510	−1510
6060	−15	303	−505
6060	−3	1515	−505
6188	−91	1547	−17
6195	−35	177	−6195
6195	−15	413	−6195

f	d_1	d_2	d_3
6195	−7	885	−6195
6195	−3	2065	−6195
6204	−33	517	−141
6279	−91	897	−483
6312	−6	263	−1578
6312	−1	1578	−1578
6315	−3	2105	−6315
6328	−7	1582	−226
6344	−2	1586	−793
6360	−2	1590	−795
6405	−427	105	−915
6405	−35	1281	−915
6405	−7	6405	−915
6420	−1	1605	−1605
6440	−115	70	−322
6440	−115	322	−70
6440	−115	1610	−14
6440	−35	230	−322
6440	−35	1610	−46
6567	−11	597	−6567
6580	−35	47	−1645
6580	−1	1645	−1645
6612	−3	551	−1653
6612	−1	1653	−1653
6636	−21	237	−553
6636	−1	1659	−1659
6692	−7	239	−1673
6744	−6	843	−562
6745	−355	1349	−95
6765	−123	6765	−55
6792	−6	283	−1698
6808	−37	1702	−46
6868	−1	1717	−1717
6888	−123	1722	−14
6924	−1	1731	−1731

continued on following page

Fields where $H \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$

continued from previous page

f	d_1	d_2	d_3
6952	−1	1738	−1738
7035	−67	105	−7035
7035	−35	201	−7035
7035	−7	1005	−7035
7052	−43	1763	−41
7084	−253	161	−77
7212	−1	1803	−1803
7293	−187	7293	−39
7320	−10	366	−915
7320	−2	1830	−915
7332	−13	1833	−141
7368	−6	307	−1842
7384	−13	1846	−142
7420	−35	371	−265
7437	−67	7437	−111
7464	−6	622	−933
7505	−19	7505	−395
7640	−10	955	−382
7640	−5	1910	−382
7656	−58	957	−66
7720	−10	1930	−193
7752	−6	1938	−323
7755	−235	33	−7755
7755	−15	517	−7755
7755	−3	2585	−7755
7761	−39	597	−2587
7780	−5	389	−1945
7832	−22	178	−979
7832	−11	1958	−178
7832	−2	1958	−979
7896	−42	94	−987
7896	−42	141	−658
7896	−42	329	−282
7896	−7	1974	−282
7896	−3	1974	−658
7896	−2	1974	−987
8008	−91	286	−154
8008	−11	182	−2002
8008	−2	1001	−2002
8040	−67	2010	−30
8052	−11	183	−2013
8052	−1	2013	−2013
8060	−403	155	−65
8060	−13	2015	−155
8120	−10	2030	−203
8216	−13	2054	−158
8265	−435	57	−2755
8265	−15	1653	−2755
8265	−3	8265	−2755
8360	−22	1045	−190
8463	−403	273	−651
8463	−91	1209	−651
8556	−93	713	−69
8648	−46	1081	−94
8652	−1	2163	−2163
8680	−35	62	−2170
8680	−10	217	−2170
8680	−7	310	−2170
8680	−2	1085	−2170
8715	−15	581	−8715
8715	−7	1245	−8715
8715	−3	2905	−8715
8760	−10	2190	−219
8844	−67	2211	−33
8855	−115	385	−1771
8855	−35	1265	−1771
8932	−11	203	−2233
9015	−15	3005	−1803
9048	−58	78	−1131
9048	−58	2262	−39

continued on following page

Fields where $H \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$

continued from previous page

f	d_1	d_2	d_3
9048	−2	2262	−1131
9145	−155	1829	−295
9192	−6	766	−1149
9228	−1	2307	−2307
9291	−3	3097	−9291
9345	−35	9345	−267
9372	−33	781	−213
9384	−51	782	−138
9384	−51	2346	−46
9412	−13	181	−2353
9436	−7	2359	−337
9548	−11	2387	−217
9560	−10	1195	−478
9560	−5	2390	−478
9579	−3	3193	−9579
9640	−2	1205	−2410
9804	−43	2451	−57
9835	−7	1405	−9835
9944	−1243	22	−226
9960	−10	1245	−498
10005	−15	10005	−667
10024	−7	358	−2506
10212	−37	2553	−69
10329	−11	10329	−939
10392	−6	866	−1299
10392	−2	2598	−1299
10412	−19	2603	−137
10472	−187	154	−238
10472	−187	238	−154
10472	−187	2618	−14
10472	−22	1309	−238
10472	−11	2618	−238
10488	−19	2622	−138
10505	−11	10505	−955
10549	−7	10549	−1507
10668	−1	2667	−2667
10696	−14	1337	−382
10696	−7	382	−2674
10716	−57	893	−141
10716	−19	2679	−141
10804	−37	2701	−73
10815	−15	3605	−2163
10824	−123	902	−66
10824	−22	2706	−123
10856	−2	2714	−1357
10945	−55	2189	−995
10945	−11	10945	−995
11060	−35	395	−553
11060	−5	2765	−553
11067	−51	3689	−651
11067	−7	1581	−11067
11076	−13	2769	−213
11240	−10	1405	−562
11284	−403	91	−217
11284	−91	403	−217
11308	−1	2827	−2827
11352	−43	2838	−66
11352	−11	2838	−258
11480	−35	2870	−82
11580	−15	2895	−193
11640	−10	2910	−291
11784	−2	1473	−2946
11816	−2	2954	−1477
11868	−43	2967	−69
12056	−2	3014	−1507
12232	−2	1529	−3058
12245	−155	2449	−395
12255	−15	4085	−2451
12376	−91	238	−442
12376	−91	3094	−34

continued on following page

Fields where $H \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$

continued from previous page

f	d_1	d_2	d_3	f	d_1	d_2	d_3
12376	−7	3094	−442	14140	−35	707	−505
12460	−35	623	−445	14140	−7	3535	−505
12460	−7	3115	−445	14168	−11	3542	−322
12616	−2	1577	−3154	14168	−2	3542	−1771
12760	−58	3190	−55	14344	−2	1793	−3586
12831	−91	1833	−987	14413	−203	497	−2059
12857	−43	12857	−299	14413	−7	14413	−2059
13048	−7	3262	−466	14424	−6	1202	−1803
13080	−10	654	−1635	14484	−51	1207	−213
13080	−2	3270	−1635	14536	−46	1817	−158
13145	−55	2629	−1195	14601	−3	14601	−4867
13160	−235	70	−658	14728	−7	526	−3682
13160	−235	658	−70	14735	−35	2105	−2947
13160	−235	3290	−14	14952	−267	3738	−14
13160	−35	470	−658	14973	−483	713	−651
13209	−51	13209	−259	14973	−7	14973	−2139
13224	−58	1653	−114	15276	−67	3819	−57
13244	−43	3311	−77	15576	−22	354	−1947
13244	−11	3311	−301	15652	−91	559	−301
13272	−2	3318	−1659	15652	−13	3913	−301
13288	−22	151	−3322	15656	−2	3914	−1957
13288	−1	3322	−3322	15715	−7	2245	−15715
13335	−35	1905	−2667	15736	−7	3934	−562
13335	−15	4445	−2667	15771	−3	5257	−15771
13384	−14	1673	−478	15953	−43	15953	−371
13384	−7	3346	−478	15980	−235	3995	−17
13452	−19	3363	−177	15996	−43	3999	−93
13468	−13	3367	−259	16008	−58	2001	−138
13629	−7	13629	−1947	16188	−57	1349	−213
13748	−7	491	−3437	16188	−19	4047	−213
13848	−2	3462	−1731	16215	−235	345	−3243
13912	−37	3478	−94	16215	−115	705	−3243
13944	−42	581	−498	16284	−177	1357	−69
13944	−7	3486	−498	16401	−3	16401	−5467
14028	−1	3507	−3507	16467	−11	1497	−16467

continued on following page

Fields where $H \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$

continued from previous page

f	d_1	d_2	d_3
16492	−133	589	−217
16492	−19	4123	−217
16492	−1	4123	−4123
16536	−2	4134	−2067
16616	−2	4154	−2077
16643	−187	1513	−979
16744	−91	598	−322
16779	−51	5593	−987
16936	−58	4234	−73
17017	−91	17017	−187
17068	−1	4267	−4267
17080	−10	4270	−427
17112	−2	4278	−2139
17121	−39	1317	−5707
17121	−3	17121	−5707
17240	−10	2155	−862
17240	−5	4310	−862
17304	−42	206	−2163
17304	−2	4326	−2163
17484	−93	1457	−141
17535	−15	5845	−3507
17608	−142	2201	−62
17841	−19	17841	−939
17841	−3	17841	−5947
17963	−11	1633	−17963
18145	−955	3629	−95
18249	−11	18249	−1659
18456	−6	1538	−2307
18460	−13	4615	−355
18696	−123	1558	−114
18705	−43	18705	−435
18907	−7	2701	−18907
18921	−51	1113	−6307
18921	−3	18921	−6307
19005	−35	3801	−2715

f	d_1	d_2	d_3
19005	−7	19005	−2715
19272	−22	2409	−438
19272	−22	4818	−219
19272	−11	4818	−438
19299	−3	6433	−19299
19352	−82	118	−2419
19448	−187	286	−442
19448	−11	4862	−442
19491	−267	6497	−219
19560	−163	4890	−30
19608	−6	1634	−2451
19745	−55	3949	−1795
19815	−15	6605	−3963
20251	−7	2893	−20251
20504	−22	2563	−466
20539	−19	1081	−20539
20636	−67	5159	−77
20811	−3	6937	−20811
20945	−355	4189	−295
21027	−3	7009	−21027
21028	−7	751	−5257
21045	−115	4209	−915
21336	−2	5334	−2667
21432	−19	5358	−282
21489	−19	21489	−1131
21516	−163	5379	−33
21720	−2	5430	−2715
21868	−1	5467	−5467
22040	−58	190	−2755
22040	−10	1102	−2755
22040	−2	5510	−2755
22083	−51	7361	−1299
22165	−403	22165	−55
22184	−2	5546	−2773
22204	−427	91	−793

continued on following page

Fields where $H \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$

continued from previous page

f	d_1	d_2	d_3
22204	−7	5551	−793
22568	−403	5642	−14
22616	−2	5654	−2827
22701	−483	329	−3243
22701	−483	1081	−987
22701	−7	22701	−3243
22940	−37	5735	−155
23124	−123	1927	−141
23144	−22	526	−2893
23144	−2	5786	−2893
23485	−427	385	−3355
23485	−427	23485	−55
23485	−35	4697	−3355
23485	−7	23485	−3355
23621	−1027	1817	−299
23736	−43	5934	−138
24072	−6	6018	−1003
24136	−14	3017	−862
24136	−7	6034	−862
24211	−11	2201	−24211
24252	−43	6063	−141
24564	−267	2047	−69
24728	−22	3091	−562
24808	−7	886	−6202
24897	−43	24897	−579
24940	−43	6235	−145
25305	−35	25305	−723
25388	−11	6347	−577
26085	−235	5217	−555
26412	−93	2201	−213
26691	−123	8897	−651
26696	−142	3337	−94
26840	−2	6710	−3355
26961	−11	26961	−2451
27020	−35	6755	−193

f	d_1	d_2	d_3
27224	−82	166	−3403
27260	−235	1363	−145
27404	−403	6851	−17
27993	−43	27993	−651
28045	−355	5609	−395
28056	−42	334	−3507
28203	−51	9401	−1659
28424	−22	7106	−323
28756	−91	1027	−553
28756	−13	7189	−553
28952	−11	7238	−658
29337	−11	29337	−2667
29393	−91	29393	−323
29427	−51	9809	−1731
29428	−7	1051	−7357
29545	−1555	5909	−95
29656	−22	7414	−337
29829	−163	29829	−183
29832	−6	7458	−1243
29928	−58	3741	−258
30107	−187	2737	−1771
30261	−7	30261	−4323
30520	−10	7630	−763
30845	−155	6169	−995
31521	−19	31521	−1659
31704	−2	7926	−3963
31960	−235	7990	−34
32116	−37	8029	−217
32712	−58	4089	−282
32889	−19	32889	−1731
32945	−55	6589	−2995
32984	−2	8246	−4123
33099	−187	3009	−1947
33099	−51	11033	−1947
33276	−177	2773	−141

continued on following page

Fields where $H \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$

continued from previous page

f	d_1	d_2	d_3
33352	−2	4169	−8338
33580	−115	8395	−73
33901	−203	1169	−4843
33943	−91	4849	−2611
34136	−34	502	−4267
34216	−91	1222	−658
34408	−187	8602	−46
34892	−11	8723	−793
34932	−123	2911	−213
35061	−403	2697	−1131
35096	−82	214	−4387
35096	−2	8774	−4387
35144	−46	4393	−382
35581	−91	2737	−5083
35581	−7	35581	−5083
35960	−58	8990	−155
36168	−6	9042	−1507
36636	−43	9159	−213
36771	−51	12257	−2163
36984	−67	9246	−138
36993	−627	1121	−1947
36993	−19	36993	−1947
37045	−155	7409	−1195
37164	−163	9291	−57
37605	−115	7521	−1635
37788	−67	9447	−141
37848	−19	9462	−498
38152	−2	4769	−9538
38248	−7	1366	−9562
38540	−235	1927	−205
38577	−11	38577	−3507
38793	−67	38793	−579
38913	−51	38913	−763
39109	−259	1057	−5587
39219	−51	13073	−2307

f	d_1	d_2	d_3
39345	−43	39345	−915
39545	−55	7909	−3595
39676	−13	9919	−763
39711	−427	5673	−651
40216	−22	10054	−457
40467	−123	13489	−987
40584	−267	3382	−114
40641	−19	40641	−2139
40940	−115	2047	−445
42028	−133	1501	−553
42028	−19	10507	−553
42380	−163	10595	−65
42441	−43	42441	−987
42945	−35	42945	−1227
43240	−235	10810	−46
43593	−11	43593	−3963
43665	−123	43665	−355
43736	−2	10934	−5467
43833	−19	43833	−2307
44776	−58	11194	−193
44872	−142	5609	−158
44988	−163	11247	−69
45353	−11	45353	−4123
45505	−19	45505	−2395
45591	−91	6513	−3507
46145	−11	46145	−4195
46248	−123	3854	−282
47357	−667	1633	−2059
47576	−2	11894	−5947
47704	−67	11926	−178
47705	−235	47705	−203
47804	−37	11951	−323
48059	−187	4369	−2827
48328	−7	1726	−12082
48433	−187	48433	−259

continued on following page

Fields where $H \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$

continued from previous page

f	d_1	d_2	d_3
48488	−58	6061	−418
48504	−43	12126	−282
49128	−267	12282	−46
49252	−7	1759	−12313
49749	−483	2369	−2163
49820	−235	2491	−265
50268	−177	4189	−213
50673	−19	50673	−2667
50808	−58	12702	−219
51688	−91	12922	−142
52948	−427	1891	−217
53537	−11	53537	−4867
53599	−403	1729	−4123
54417	−187	54417	−291
54520	−58	13630	−235
54808	−403	1054	−442
54940	−67	13735	−205
55131	−51	18377	−3243
55384	−43	13846	−322
55448	−58	6931	−478
55645	−155	11129	−1795
55913	−187	3289	−5083
55913	−11	55913	−5083
56440	−10	14110	−1411
57084	−67	14271	−213
57577	−43	57577	−1339
57939	−267	19313	−651
57964	−43	14491	−337
59619	−51	19873	−3507
61305	−67	61305	−915
61457	−11	61457	−5587
62445	−115	12489	−2715
63271	−403	2041	−4867
63624	−22	15906	−723
64077	−403	4929	−2067
65417	−11	65417	−5947
65593	−67	65593	−979
66129	−67	66129	−987
66185	−427	66185	−155
66385	−187	66385	−355
66504	−163	16626	−102
67512	−58	16878	−291
67804	−67	16951	−253
68019	−123	22673	−1659
68305	−19	68305	−3595
68385	−235	68385	−291
69153	−267	69153	−259
69864	−123	17466	−142
70252	−91	17563	−193
71020	−67	17755	−265
71071	−91	10153	−5467
72124	−13	18031	−1387
73676	−163	18419	−113
74245	−155	14849	−2395
74936	−58	18734	−323
75336	−43	18834	−438
75576	−67	18894	−282
76328	−58	9541	−658
76845	−235	15369	−1635
77165	−115	15433	−3355
78364	−13	19591	−1507
78568	−427	2806	−322
78568	−427	19642	−46
79304	−46	9913	−862
79705	−19	79705	−4195
79827	−123	26609	−1947
79849	−187	79849	−427
80661	−483	3841	−3507
80668	−67	20167	−301
81073	−323	4769	−4267

continued on following page

Fields where $H \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$

continued from previous page

f	d_1	d_2	d_3
81844	−37	20461	−553
82137	−627	2489	−4323
83145	−115	83145	−723
84760	−163	21190	−130
84845	−355	16969	−1195
85656	−43	21414	−498
86241	−267	86241	−323
86296	−67	21574	−322
87745	−115	87745	−763
88264	−22	22066	−1003
88683	−123	29561	−2163
88780	−115	22195	−193
90193	−19	90193	−4747
91180	−235	22795	−97
91273	−91	91273	−1003
91848	−267	7654	−258
92845	−155	18569	−2995
92939	−187	8449	−5467
93009	−43	93009	−2163
93849	−123	93849	−763
94540	−163	23635	−145
95029	−163	95029	−583
95116	−43	23779	−553
96664	−43	24166	−562
98332	−403	1891	−793
99201	−43	99201	−2307
99992	−58	12499	−862
105781	−1027	8137	−1339
108031	−427	15433	−1771
108488	−142	13561	−382
109545	−67	109545	−1635
111241	−43	111241	−2587
111389	−667	3841	−4843
112024	−67	28006	−418
113176	−43	28294	−658
117465	−123	117465	−955
118465	−43	118465	−2755
120801	−67	120801	−1803
122056	−22	30514	−1387
122476	−67	30619	−457
122845	−1555	24569	−395
124168	−22	31042	−1411
124684	−427	31171	−73
127445	−355	25489	−1795
127605	−235	25521	−2715
130045	−155	26009	−4195
130169	−403	130169	−323
130429	−1027	10033	−1651
130449	−67	130449	−1947
132184	−403	33046	−82
132963	−123	44321	−3243
133464	−67	33366	−498
134932	−427	4819	−553
135340	−67	33835	−505
135752	−142	16969	−478
137137	−91	137137	−1507
138489	−67	138489	−2067
140056	−427	35014	−82
141340	−37	35335	−955
143065	−403	143065	−355
143313	−67	143313	−2139
143787	−123	47929	−3507
144265	−43	144265	−3355
144921	−67	144921	−2163
147352	−163	36838	−226
148444	−37	37111	−1003
150801	−43	150801	−3507
151585	−427	151585	−355
152889	−123	152889	−1243
155176	−163	38794	−238

continued on following page

Fields where $H \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$

continued from previous page

f	d_1	d_2	d_3
156364	−403	39091	−97
164217	−67	164217	−2451
165676	−427	41419	−97
166969	−43	166969	−3883
170045	−355	34009	−2395
173283	−267	57761	−1947
177243	−123	59081	−4323
177288	−267	14774	−498
177289	−43	177289	−4123
179305	−235	179305	−763
183864	−163	45966	−282
184353	−163	184353	−1131
185889	−43	185889	−4323
190045	−955	38009	−995
190371	−267	63457	−2139
192049	−187	192049	−1027
192507	−267	64169	−2163
196252	−163	49063	−301
197449	−67	197449	−2947
204373	−1027	15721	−2587
205323	−267	68441	−2307
212524	−67	53131	−793
218139	−267	72713	−2451
221560	−58	55390	−955
228245	−955	45649	−1195
228904	−403	57226	−142
229449	−187	229449	−1227
230140	−37	57535	−1555
234969	−67	234969	−3507
238264	−58	59566	−1027
242536	−427	60634	−142
244808	−142	30601	−862
251503	−427	35929	−4123
251753	−1003	14809	−4267
255245	−355	51049	−3595
266505	−163	266505	−1635
267801	−267	267801	−1003
276241	−67	276241	−4123
282153	−163	282153	−1731
284664	−58	71166	−1227
284809	−427	284809	−667
285576	−163	71394	−438
288184	−163	72046	−442
288345	−235	288345	−1227
288376	−58	72094	−1243
290140	−163	72535	−445
290785	−187	290785	−1555
297845	−355	59569	−4195
307489	−403	307489	−763
321784	−58	80446	−1387
324696	−163	81174	−498
329260	−163	82315	−505
329644	−427	82411	−193
331585	−235	331585	−1411
333487	−427	47641	−5467
342845	−955	68569	−1795
354161	−1411	20833	−4267
360760	−58	90190	−1555
366424	−163	91606	−562
371645	−1555	74329	−1195
374329	−67	374329	−5587
376204	−163	94051	−577
376737	−267	376737	−1411
394297	−163	394297	−2419
404209	−403	404209	−1003
407785	−427	407785	−955
429016	−163	107254	−658
438529	−427	438529	−1027
442545	−163	442545	−2715
449065	−163	449065	−2755

continued on following page

Fields where $H \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$

continued from previous page

f	d_1	d_2	d_3
450853	−1027	34681	−5707
457445	−955	91489	−2395
554689	−163	554689	−3403
558245	−1555	111649	−1795
686645	−955	137329	−3595

f	d_1	d_2	d_3
704649	−163	704649	−4323
801245	−955	160249	−4195
828529	−163	828529	−5083
931445	−1555	186289	−2995
1304645	−1555	260929	−4195

Table 6.8: Fields where $H \simeq \mathbb{Z}_4 \times \mathbb{Z}_4$

f	d_1	d_2	d_3
420	−21	5	−105
420	−5	105	−21
520	−10	26	−65
840	−42	5	−210
952	−17	14	−238
952	−14	17	−238
1015	−35	145	−203
1020	−15	255	−17
1131	−39	29	−1131
1220	−1	305	−305
1240	−155	2	−310
1380	−115	15	−69
1380	−15	115	−69
1443	−39	37	−1443
1508	−1	377	−377
1560	−15	130	−78
1560	−6	130	−195
1677	−43	1677	−39
1736	−14	62	−217
1736	−2	434	−217
1752	−219	2	−438
1768	−34	13	−442
1780	−1	445	−445
1785	−35	1785	−51
1820	−91	35	−65
1820	−35	91	−65
2035	−55	37	−2035
2040	−51	170	−30
2067	−39	53	−2067
2120	−10	106	−265
2280	−15	38	−570
2280	−15	190	−114
2280	−10	285	−114
2280	−3	190	−570
2380	−35	595	−17
2392	−46	13	−598
2480	−15	870	−58
2580	−43	15	−645
2632	−1	658	−658
2755	−19	145	−2755
2788	−17	41	−697
2820	−235	15	−141
2968	−14	53	−742
2980	−1	745	−745
3080	−35	110	−154
3080	−10	385	−154
3108	−37	777	−21
3180	−15	159	−265
3192	−19	42	−798
3315	−15	221	−3315
3335	−115	145	−667
3355	−55	61	−3355
3432	−11	78	−858
3480	−2	870	−435
3640	−91	130	−70
3660	−5	183	−915
3720	−30	93	−310
3720	−15	186	−310
3720	−6	465	−310
3720	−6	930	−155
3720	−3	930	−310
3828	−3	319	−957
4004	−91	143	−77
4020	−67	15	−1005
4060	−5	1015	−203
4424	−2	1106	−553
4440	−6	370	−555
4845	−15	4845	−323
4872	−6	1218	−203
4920	−123	410	−30

continued on following page

Fields where $H \simeq \mathbb{Z}_4 \times \mathbb{Z}_4$

continued from previous page

f	d_1	d_2	d_3
5720	−22	130	−715
6020	−35	215	−301
6148	−1	1537	−1537
6307	−7	901	−6307
6420	−15	107	−1605
6420	−3	535	−1605
6540	−5	327	−1635
6612	−19	87	−1653
6916	−13	1729	−133
7735	−91	1105	−595
7780	−1	1945	−1945
7820	−115	1955	−17
7995	−123	2665	−195
8295	−35	1185	−1659
8295	−15	2765	−1659
8364	−123	51	−697
8364	−123	2091	−17
8364	−51	123	−697
8364	−3	2091	−697
8520	−15	2130	−142
8520	−6	2130	−355
8701	−7	8701	−1243
8904	−6	1113	−742
8904	−3	2226	−742
9412	−1	2353	−2353
9435	−51	3145	−555
9816	−6	818	−1227
10220	−35	2555	−73
10556	−13	2639	−203
10680	−267	890	−30
10695	−115	465	−2139
10695	−15	3565	−2139
10860	−5	543	−2715
11096	−2	2774	−1387
11256	−67	2814	−42
11305	−35	11305	−323
11544	−2	2886	−1443
12056	−22	274	−1507
12155	−187	1105	−715
12220	−235	611	−65
12792	−123	1066	−78
13340	−5	3335	−667
13515	−51	4505	−795
13580	−35	3395	−97
15045	−15	15045	−1003
15080	−58	1885	−130
15260	−5	3815	−763
16008	−6	4002	−667
16280	−10	814	−2035
16280	−2	4070	−2035
16555	−43	385	−16555
16728	−123	1394	−102
16728	−123	4182	−34
16728	−51	4182	−82
17080	−427	610	−70
17353	−67	17353	−259
17355	−267	5785	−195
17835	−123	5945	−435
17864	−22	4466	−203
18145	−19	18145	−955
18645	−15	18645	−1243
18788	−427	671	−77
19516	−7	4879	−697
19929	−91	19929	−219
22605	−15	22605	−1507
22920	−6	5730	−955
23023	−91	3289	−1771
24380	−115	1219	−265
25851	−3	8617	−25851
26312	−22	3289	−598

continued on following page

Fields where $H \simeq \mathbb{Z}_4 \times \mathbb{Z}_4$

continued from previous page

f	d_1	d_2	d_3
26312	−11	6578	−598
26481	−91	26481	−291
26572	−91	6643	−73
27363	−3	9121	−27363
27379	−19	1441	−27379
27384	−163	6846	−42
27417	−19	27417	−1443
27768	−267	2314	−78
28443	−3	9481	−28443
29643	−123	9881	−723
29883	−3	9961	−29883
30668	−187	451	−697
30668	−11	7667	−697
30705	−115	30705	−267
32403	−3	10801	−32403
35105	−35	35105	−1003
35308	−91	8827	−97
36507	−3	12169	−36507
37145	−115	37145	−323
37320	−6	9330	−1555
38643	−11	3513	−38643
41080	−10	10270	−1027
41547	−11	3777	−41547
42123	−3	14041	−42123
44620	−115	11155	−97
44689	−67	44689	−667
44733	−403	3441	−1443
45339	−51	15113	−2667
45448	−19	11362	−598
46483	−43	1081	−46483
46543	−427	6649	−763
46805	−115	9361	−2035
47163	−3	15721	−47163
48441	−67	48441	−723
48545	−35	48545	−1387
48705	−51	48705	−955
49105	−115	49105	−427
49395	−267	16465	−555
50043	−3	16681	−50043
50347	−11	4577	−50347
51843	−11	4713	−51843
53265	−67	53265	−795
53320	−43	13330	−310
53464	−163	13366	−82
54604	−187	13651	−73
56067	−11	5097	−56067
56392	−19	14098	−742
58696	−22	14674	−667
59563	−7	8509	−59563
60088	−58	15022	−259
61336	−187	15334	−82
64227	−3	21409	−64227
67144	−22	16786	−763
68620	−235	17155	−73
69843	−3	23281	−69843
70755	−267	23585	−795
70905	−163	70905	−435
71323	−7	10189	−71323
73444	−427	2623	−301
78729	−163	78729	−483
82041	−123	82041	−667
83080	−67	20770	−310
83923	−19	4417	−83923
87505	−43	87505	−2035
87576	−267	21894	−82
88257	−403	88257	−219
90376	−22	22594	−1027
96145	−67	96145	−1435
100491	−123	33497	−2451
102201	−163	102201	−627

continued on following page

Fields where $H \simeq \mathbb{Z}_4 \times \mathbb{Z}_4$

continued from previous page

f	d_1	d_2	d_3
109347	−123	36449	−2667
115345	−115	115345	−1003
115404	−163	28851	−177
117676	−403	29419	−73
122683	−19	6457	−122683
123369	−123	123369	−1003
137921	−427	137921	−323
141505	−91	141505	−1555
142945	−115	142945	−1243
144364	−187	36091	−193
147651	−267	49217	−1659
160264	−67	40066	−598
173305	−115	173305	−1507

f	d_1	d_2	d_3
181420	−235	45355	−193
198856	−67	49714	−742
200001	−163	200001	−1227
202120	−163	50530	−310
235209	−163	235209	−1443
241345	−235	241345	−1027
259369	−187	259369	−1387
291369	−403	291369	−723
292105	−235	292105	−1243
325945	−235	325945	−1387
354145	−235	354145	−1507
402369	−267	402369	−1507
523929	−427	523929	−1227

Table 6.9: Fields where $H \simeq \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

f	d_1	d_2	d_3
552	−46	6	−69
616	−14	22	−77
660	−33	5	−165
663	−51	13	−663
780	−3	195	−65
840	−70	6	−105
840	−30	14	−105
840	−10	42	−105
840	−10	210	−21
840	−6	35	−210
840	−6	70	−105
840	−5	42	−210
840	−5	210	−42
840	−2	210	−105
840	−1	210	−210
884	−13	17	−221
952	−2	119	−238
952	−1	238	−238
1015	−35	29	−1015
1092	−21	13	−273
1140	−19	15	−285
1140	−15	19	−285
1320	−30	22	−165
1320	−22	15	−330
1320	−22	30	−165
1320	−10	66	−165
1320	−10	330	−33
1320	−6	55	−330
1320	−6	110	−165
1320	−5	66	−330
1320	−1	330	−330
1428	−21	17	−357
1560	−13	390	−30
1560	−10	195	−78
1560	−6	195	−130
1560	−5	390	−78
1596	−3	399	−133
1624	−7	58	−406
1640	−10	41	−410
1640	−5	410	−82
1640	−2	410	−205
1704	−142	6	−213
1716	−3	143	−429
1752	−73	6	−438
1752	−6	438	−73
1752	−2	219	−438
1752	−1	438	−438
1768	−13	34	−442
1848	−42	33	−154
1848	−33	14	−462
1848	−22	231	−42
1848	−7	462	−66
1848	−1	462	−462
1860	−15	31	−465
1932	−7	483	−69
2040	−15	170	−102
2040	−10	255	−102
2040	−6	510	−85
2040	−5	510	−102
2072	−7	74	−518
2184	−78	14	−273
2184	−13	546	−42
2184	−6	182	−273
2184	−2	546	−273
2244	−1	561	−561
2280	−19	30	−570
2280	−10	57	−570
2280	−10	570	−57
2280	−2	285	−570
2328	−6	97	−582

continued on following page

Fields where $H \simeq \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

continued from previous page

f	d_1	d_2	d_3
2408	−14	86	−301
2436	−7	87	−609
2580	−5	129	−645
2760	−115	6	−690
2760	−15	46	−690
2760	−15	230	−138
2760	−10	138	−345
2760	−6	230	−345
2760	−2	690	−345
2856	−102	14	−357
2856	−6	238	−357
2860	−11	715	−65
3080	−22	70	−385
3080	−10	154	−385
3080	−2	770	−385
3108	−37	21	−777
3108	−7	111	−777
3108	−3	259	−777
3192	−42	133	−114
3192	−7	114	−798
3192	−6	133	−798
3192	−6	798	−133
3192	−3	266	−798
3220	−35	23	−805
3220	−7	115	−805
3220	−5	161	−805
3315	−195	17	−3315
3315	−51	65	−3315
3315	−51	1105	−195
3315	−3	1105	−3315
3432	−3	286	−858
3432	−2	429	−858
3480	−58	435	−30
3480	−6	290	−435
3560	−10	178	−445

f	d_1	d_2	d_3
3560	−2	890	−445
3588	−13	69	−897
3588	−3	299	−897
3660	−1	915	−915
3720	−10	930	−93
3864	−483	6	−322
3864	−7	966	−138
4020	−5	201	−1005
4060	−35	203	−145
4060	−7	1015	−145
4092	−11	1023	−93
4180	−19	55	−1045
4180	−11	95	−1045
4180	−5	209	−1045
4251	−3	1417	−4251
4424	−14	158	−553
4440	−37	1110	−30
4452	−7	159	−1113
4515	−15	301	−4515
4515	−3	1505	−4515
4524	−13	87	−1131
4524	−1	1131	−1131
4632	−6	1158	−193
4872	−58	1218	−21
4935	−35	705	−987
4940	−19	1235	−65
5016	−22	1254	−57
5115	−15	341	−5115
5160	−10	129	−1290
5160	−2	645	−1290
5208	−7	186	−1302
5304	−51	78	−442
5304	−3	1326	−442
5320	−10	1330	−133
5412	−1	1353	−1353

continued on following page

Fields where $H \simeq \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

continued from previous page

f	d_1	d_2	d_3
5640	−15	470	−282
5720	−22	715	−130
5772	−37	39	−1443
5772	−13	111	−1443
5964	−7	1491	−213
6052	−1	1513	−1513
6072	−11	1518	−138
6072	−6	1518	−253
6105	−555	33	−2035
6105	−3	6105	−2035
6123	−3	2041	−6123
6216	−37	1554	−42
6216	−6	1554	−259
6315	−15	421	−6315
6360	−6	530	−795
6380	−11	1595	−145
6440	−10	805	−322
6460	−5	1615	−323
6540	−1	1635	−1635
6580	−235	7	−1645
6580	−5	329	−1645
6708	−43	39	−1677
6708	−13	129	−1677
6708	−3	559	−1677
6771	−3	2257	−6771
6820	−11	155	−1705
6820	−5	341	−1705
7035	−3	2345	−7035
7080	−10	1770	−177
7161	−11	7161	−651
7176	−78	69	−598
7176	−3	1794	−598
7224	−42	301	−258
7315	−35	209	−7315
7315	−11	665	−7315
7395	−51	2465	−435
7420	−7	1855	−265
7480	−22	1870	−85
7917	−91	609	−1131
7917	−7	7917	−1131
8008	−91	22	−2002
8008	−7	286	−2002
8052	−3	671	−2013
8120	−58	1015	−70
8140	−37	55	−2035
8140	−5	407	−2035
8184	−22	2046	−93
8268	−13	159	−2067
8268	−1	2067	−2067
8680	−70	217	−310
8715	−35	249	−8715
8835	−19	465	−8835
9020	−11	2255	−205
9048	−58	1131	−78
9640	−10	241	−2410
9835	−35	281	−9835
9867	−3	3289	−9867
9880	−13	2470	−190
9960	−15	830	−498
10065	−15	2013	−3355
10065	−11	10065	−915
10065	−3	10065	−3355
10101	−7	10101	−1443
10120	−10	2530	−253
10360	−37	2590	−70
10420	−5	521	−2605
10635	−15	709	−10635
10815	−35	1545	−2163
10857	−11	10857	−987
10860	−1	2715	−2715

continued on following page

Fields where $H \simeq \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

continued from previous page

f	d_1	d_2	d_3
10948	−7	391	−2737
10948	−1	2737	−2737
11020	−5	551	−2755
11067	−51	217	−11067
11067	−3	3689	−11067
11180	−43	2795	−65
11388	−13	2847	−219
11544	−37	2886	−78
11704	−22	2926	−133
11704	−19	2926	−154
11704	−7	2926	−418
11715	−15	781	−11715
11832	−58	1479	−102
12136	−37	3034	−82
12264	−7	3066	−438
12408	−11	3102	−282
12580	−37	3145	−85
12920	−10	3230	−323
13160	−10	1645	−658
13192	−2	1649	−3298
13272	−42	158	−1659
13340	−115	667	−145
13420	−5	671	−3355
13420	−1	3355	−3355
13640	−22	3410	−155
13640	−11	3410	−310
14155	−19	745	−14155
14245	−7	14245	−2035
14469	−91	1113	−2067
14469	−7	14469	−2067
14547	−3	4849	−14547
14685	−267	14685	−55
14763	−7	2109	−14763
14892	−51	3723	−73
14973	−483	217	−2139

f	d_1	d_2	d_3
15096	−37	3774	−102
15132	−13	3783	−291
15405	−15	15405	−1027
15576	−6	1298	−1947
15580	−19	3895	−205
15715	−35	449	−15715
16215	−15	5405	−3243
16555	−11	1505	−16555
16555	−7	2365	−16555
16653	−427	16653	−39
16744	−91	4186	−46
16744	−7	4186	−598
17985	−11	17985	−1635
18088	−19	4522	−238
18156	−267	4539	−17
18312	−6	4578	−763
18655	−91	2665	−1435
18753	−19	18753	−987
19065	−123	19065	−155
19285	−35	3857	−2755
19285	−7	19285	−2755
19788	−51	4947	−97
20060	−5	5015	−1003
20140	−19	5035	−265
20332	−13	391	−5083
20332	−1	5083	−5083
20805	−15	20805	−1387
21576	−58	5394	−93
21835	−11	1985	−21835
22220	−11	5555	−505
22632	−123	5658	−46
22737	−11	22737	−2067
22792	−22	5698	−259
23560	−190	589	−310
23560	−19	5890	−310

continued on following page

Fields where $H \simeq \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

continued from previous page

f	d_1	d_2	d_3
24472	−19	6118	−322
24860	−5	6215	−1243
27740	−5	6935	−1387
28120	−37	7030	−190
30140	−5	7535	−1507
30856	−58	7714	−133
31240	−22	7810	−355
32648	−11	8162	−742
33820	−19	8455	−445
34584	−6	2882	−4323
36312	−267	9078	−34
37596	−13	9399	−723
38157	−483	1817	−1659
38665	−19	38665	−2035
38860	−67	9715	−145
39372	−51	9843	−193
40120	−10	10030	−1003
40664	−2	10166	−5083
40936	−43	10234	−238
41064	−58	10266	−177
43068	−37	10767	−291
43505	−35	43505	−1243
45580	−43	11395	−265
46460	−115	2323	−505
47724	−123	11931	−97
49720	−10	12430	−1243
50008	−19	12502	−658
50456	−2	12614	−6307
51585	−19	51585	−2715
52745	−35	52745	−1507
52972	−19	13243	−697
53599	−91	7657	−4123
55480	−10	13870	−1387
58696	−58	14674	−253
60268	−19	15067	−793

f	d_1	d_2	d_3
61341	−483	2921	−2667
61617	−19	61617	−3243
62049	−43	62049	−1443
63365	−115	12673	−2755
63784	−67	15946	−238
63804	−13	15951	−1227
73491	−51	24497	−4323
82137	−19	82137	−4323
82360	−58	20590	−355
84040	−22	21010	−955
86860	−43	21715	−505
93513	−427	93513	−219
94785	−267	94785	−355
94940	−235	4747	−505
94956	−123	23739	−193
96577	−19	96577	−5083
102856	−43	25714	−598
103873	−19	103873	−5467
107004	−37	26751	−723
118105	−115	118105	−1027
119833	−19	119833	−6307
136345	−67	136345	−2035
136396	−43	34099	−793
136840	−22	34210	−1555
139449	−43	139449	−3243
151656	−267	37914	−142
157685	−235	31537	−3355
159505	−115	159505	−1387
172780	−163	43195	−265
178585	−187	178585	−955
181596	−37	45399	−1227
183964	−37	45991	−1243
186796	−67	46699	−697
217281	−67	217281	−3243
232696	−58	58174	−1003

continued on following page

Fields where $H \simeq \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

continued from previous page

f	d_1	d_2	d_3
235705	−235	235705	−1003
268801	−403	268801	−667
274209	−267	274209	−1027
308721	−427	308721	−723
327352	−58	81838	−1411
336921	−163	336921	−2067
340561	−67	340561	−5083
384865	−403	384865	−955
415185	−267	415185	−1555
422569	−67	422569	−6307
454444	−163	113611	−697
483784	−163	120946	−742
517036	−163	129259	−793
626665	−403	626665	−1555

Table 6.10: Fields where $H \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

f	d_1	d_2	d_3
1320	−2	330	−165
1848	−21	22	−462
1848	−2	231	−462
2652	−13	663	−51
2760	−30	46	−345
2856	−2	714	−357
3080	−70	22	−385
3795	−11	345	−3795
4420	−13	1105	−85
4488	−2	561	−1122
5016	−2	1254	−627
5115	−11	465	−5115
5208	−6	217	−1302
5304	−13	1326	−102
6105	−11	6105	−555
6460	−19	1615	−85
7315	−19	385	−7315
7755	−11	705	−7755
8835	−15	589	−8835
8932	−7	319	−2233
11305	−19	11305	−595
11715	−11	1065	−11715
14763	−3	4921	−14763
18204	−37	4551	−123

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Vita

Ramona Renee Ranalli was born on January 22, 1966, in Mount Holly, New Jersey while her parents awaited visas in order to begin their missionary work. In 1967 the family moved to Costa Rica and, in 1968, to Guayaquil, Ecuador. Here they remained until 1978 when they moved to the capital city of Quito. In 1983, at the age of 17, Ramona graduated from the Alliance Academy and began attending Houghton College, near Buffalo, N.Y.. She graduated from Houghton in 1987 with a math major and minors in physics, secondary education, Bible and Spanish. That same year she headed south for warmer weather and began working towards her MA in math from Wake Forest University. Upon receiving this degree in 1989 she moved to Virginia Polytechnic Institute and State University to begin work on her doctorate, again in math. She worked at Virginia Tech as a full-time instructor for three and half of the last four years of her work towards her PhD, which may go a little way towards explaining her long sojourn there.